Chapter 3: Signed-rank charts

3.1. The Shewhart-type control chart

3.1.1. Introduction

As mentioned in Chapter 2, samples of fixed size are taken at regular intervals and the plotting statistic is then plotted. The question is: Which quality parameter should be used as the plotting statistic? In Chapter 2 the sign test statistic SN_i was described and it was mentioned that the sign test statistic is only influenced by the signs of the deviations $(x_{ij} - \theta_0)$. There is an alternative statistic that can be used to track the location of a process. The statistic is a function of both the magnitudes and signs of the $(x_{ij} - \theta_0)$'s, called the signed-rank statistic.

3.1.2. Definition of the signed-rank test statistic

The signed-rank test is a nonparametric test that can be used to test hypotheses on or construct confidence intervals (see Gibbons and Chakraborti (2003)) for the median of any symmetric continuous population distribution. Let $X_{i1}, X_{i2}, ..., X_{in}$ denote the i^{th} $(i = 1,2,...)$ sample or subgroup of independent observations of size $n > 1$ from a process with an unknown continuous distribution function denoted by F. Let θ_0 denote the known in-control location parameter (also called the target value). Let R_{ij}^+ denote the rank of the absolute deviations, $|x_{ij} - \theta_0|$, within the subgroup $(|x_{i1} - \theta_0|, |x_{i2} - \theta_0|, ..., |x_{in} - \theta_0|)$ for $i = 1,2,3...$. Then R_{ij}^+ is referred to as the within-group absolute rank of the deviations. The signed-rank test statistic is given by

$$SR_i = \sum_{j=1}^{n} sign(x_{ij} - \theta_0) R_{ij}^+ \text{ for } i = 1,2,3... \quad (3.1)$$

where $sign(x)$ = -1, 0, 1 if $x < 0$, $= 0$, > 0.

3.1.3. Plotting statistic

The signed-rank test statistic, SR_i (given in (3.1)), is used as the plotting statistic on the Shewhart signed-rank chart. If the plotting statistic SR_i falls between the two control limits, that is, $LCL < SR_i < UCL$, the process is considered to be in-control. If the plotting statistic SR_i falls on or outside one of the control limits, that is $SR_i \le LCL$ or $SR_i \ge UCL$, the process is considered to be out-of-control.

The plotting statistic is linearly related to the well-known Wilcoxon signed-rank statistic T_n^+ through the formula (see Bakir (2003), page 424, equation 2.4)

$$SR_i = 2T_n^+ - \frac{n(n+1)}{2} \tag{3.2}$$

where $T_n^+ = \sum_{j=1}^{n} \psi(x_{ij} - \theta_0) R_{ij}^+$, $\psi(x) = 0, 1$ if $x \le 0, > 0$.

Example 3.1

A two-sided Shewhart signed-rank chart for the Montgomery (2001) piston ring data

We illustrate the Shewhart-type signed-rank chart using the same set of data from Montgomery (2001) that was used in example 2.1. We assume that the underlying distribution is symmetric with a known median $\theta_0 = 74\,mm$. Panel *a* of Table 3.1 exhibits the individual observations of 15 independent samples, each of size 5 i.e. $n = 5$. The absolute deviations $|x_{ij} - \theta_0|$ and $sign(x_{ij} - \theta_0)$ are shown in panel *b* and panel *c* of Table 3.1, respectively. The known target value is taken to be 74, that is, $\theta_0 = 74$. The within-group absolute rank of the deviations R_{ij}^+ and the $sign(x_{ij} - \theta_0)R_{ij}^+$ values are shown in panel *a* and panel *b* of Table 3.2, respectively. Panel *c* of Table 3.2 holds the signed-ranks i.e. SR_i for $i = 1,2,3,...,15$.

Table 3.1. Data and calculations for the signed-rank chart[*].

Sample number	Panel *a*					Panel *b*					Panel *c*				
	x_{i1}	x_{i2}	x_{i3}	x_{i4}	x_{i5}	$\lvert x_{i1}-\theta_0\rvert$	$\lvert x_{i2}-\theta_0\rvert$	$\lvert x_{i3}-\theta_0\rvert$	$\lvert x_{i4}-\theta_0\rvert$	$\lvert x_{i5}-\theta_0\rvert$	$sign(x_{i1}-\theta_0)$	$sign(x_{i2}-\theta_0)$	$sign(x_{i3}-\theta_0)$	$sign(x_{i4}-\theta_0)$	$sign(x_{i5}-\theta_0)$
1	74.012	74.015	74.030	73.986	74.000	0.012	0.015	0.030	0.014	0.000	1	1	1	-1	0
2	73.995	74.010	73.990	74.015	74.001	0.005	0.010	0.010	0.015	0.001	-1	1	-1	1	1
3	73.987	73.999	73.985	74.000	73.990	0.013	0.001	0.015	0.000	0.010	-1	-1	-1	0	-1
4	74.008	74.010	74.003	73.991	74.006	0.008	0.010	0.003	0.009	0.006	1	1	1	-1	1
5	74.003	74.000	74.001	73.986	73.997	0.003	0.000	0.001	0.014	0.003	1	0	1	-1	-1
6	73.994	74.003	74.015	74.020	74.004	0.006	0.003	0.015	0.020	0.004	-1	1	1	1	1
7	74.008	74.002	74.018	73.995	74.005	0.008	0.002	0.018	0.005	0.005	1	1	1	-1	1
8	74.001	74.004	73.990	73.996	73.998	0.001	0.004	0.010	0.004	0.002	1	1	-1	-1	-1
9	74.015	74.000	74.016	74.025	74.000	0.015	0.000	0.016	0.025	0.000	1	0	1	1	0
10	74.030	74.005	74.000	74.016	74.012	0.030	0.005	0.000	0.016	0.012	1	1	0	1	1
11	74.001	73.990	73.995	74.010	74.024	0.001	0.010	0.005	0.010	0.024	1	-1	-1	1	1
12	74.015	74.020	74.024	74.005	74.019	0.015	0.020	0.024	0.005	0.019	1	1	1	1	1
13	74.035	74.010	74.012	74.015	74.026	0.035	0.010	0.012	0.015	0.026	1	1	1	1	1
14	74.017	74.013	74.036	74.025	74.026	0.017	0.013	0.036	0.025	0.026	1	1	1	1	1
15	74.010	74.005	74.029	74.000	74.020	0.010	0.005	0.029	0.000	0.020	1	1	1	0	1

[*] See SAS Program 5 in Appendix B for the calculation of the values in Table 3.1.

Table 3.2. Calculations for the signed-rank chart*.

Sample number	Panel *a*					Panel *b*					Panel *c*
	R_{i1}^{+}	R_{i2}^{+}	R_{i3}^{+}	R_{i4}^{+}	R_{i5}^{+}	$sign(x_{i1}-\theta_0)R_{i1}^{+}$	$sign(x_{i2}-\theta_0)R_{i2}^{+}$	$sign(x_{i3}-\theta_0)R_{i3}^{+}$	$sign(x_{i4}-\theta_0)R_{i4}^{+}$	$sign(x_{i5}-\theta_0)R_{i5}^{+}$	SR_i
1	2	4	5	3	1	2	4	5	-3	0	8
2	2	4	4	5	1	-2	4	-4	5	1	4
3	4	2	5	1	3	-4	-2	-5	0	-3	-14
4	3	5	1	4	2	3	5	1	-4	2	7
5	4	1	2	5	4	4	0	2	-5	-4	-3
6	3	1	4	5	2	-3	1	4	5	2	9
7	4	1	5	3	3	4	1	5	-3	3	10
8	1	4	5	4	2	1	4	-5	-4	-2	-6
9	3	2	4	5	2	3	0	4	5	0	12
10	5	2	1	4	3	5	2	0	4	3	14
11	1	4	2	4	5	1	-4	-2	4	5	4
12	2	4	5	1	3	2	4	5	1	3	15
13	5	1	2	3	4	5	1	2	3	4	15
14	2	1	5	3	4	2	1	5	3	4	15
15	3	2	5	1	4	3	2	5	0	4	14

Let ARL_0^{+} and FAR_0^{+} denote the in-control average run length and the false alarm rate for the upper one-sided Shewhart signed-rank control chart, respectively. For an upper one-sided chart we would take $UCL = 15$ since it is related to a false alarm rate of 0.0313 ($FAR_0^{+} = 0.0313$) and an in-control average run length of 32 ($ARL_0^{+} = 32$) - see Table 3.3. Although the in-control average run length of 32 is far from the desired value, which is generally taken to be 370 or 500, it is the best under present conditions. The false alarm rate (FAR_0) and the in-control average run length (ARL_0) for the symmetric two-sided Shewhart signed-rank chart can be obtained through the relationships $FAR_0 = 2FAR_0^{+}$ and $ARL_0 = \frac{ARL_0^{+}}{2}$, respectively (see Bakir (2003)). A symmetric two-sided chart is obtained by choosing $LCL = -UCL$. We take $UCL = 15$ for the two-sided Shewhart signed-rank chart,

* See SAS Program 5 in Appendix B for the calculation of the values in Table 3.2.

since it is related to a false alarm rate of 0.0626 ($FAR_0 = 2FAR_0^+ = 2\times0.0313 = 0.0626$) and an in-control average run length of 16 ($ARL_0 = \frac{ARL_0^+}{2} = \frac{32}{2} = 16$). The two-sided signed-rank chart is shown in Figure 3.1 with $UCL = 15$, $CL = 0$ and $LCL = -15$.

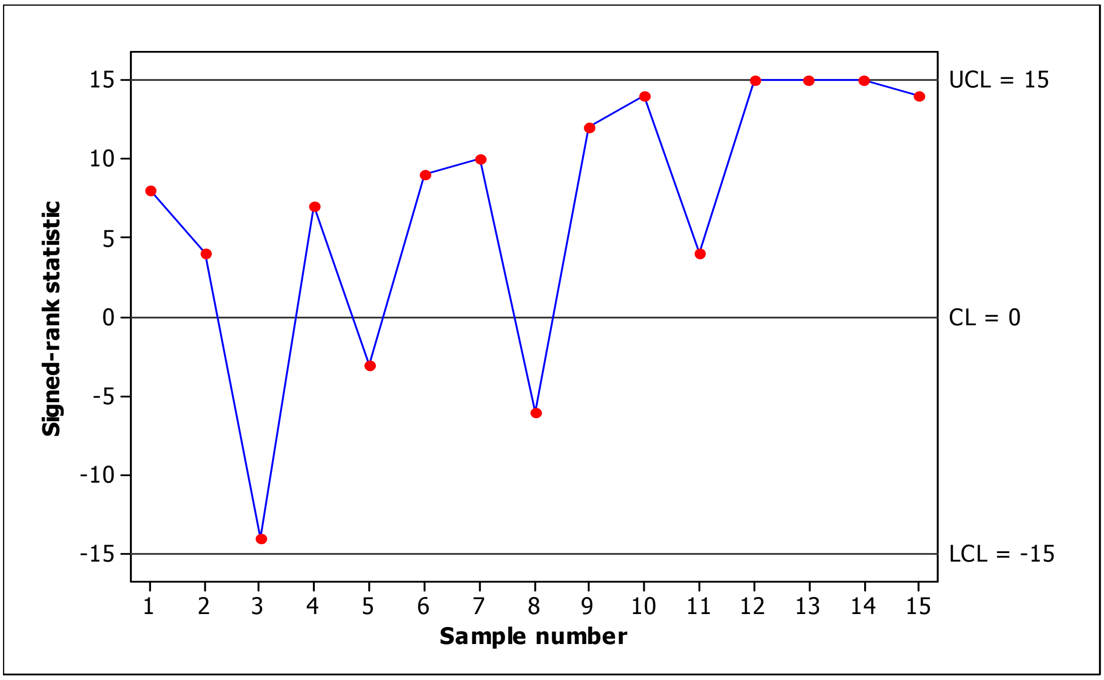

Figure 3.1. Signed-rank control chart for Montgomery (2001) piston ring data.

The chart signals at sample number 12. Therefore, a search for assignable causes is necessary. It appears most likely that the process median has shifted upwards from the target value of 74*mm*.

3.1.4. Determination of chart constants

The control limits in example 3.1 were chosen to give a certain false alarm rate or in-control ARL. Values of various control limits are given by Bakir (2003). Bakir included the following table in his article which gives the false alarm rates and the in-control average run lengths for the upper one-sided Shewhart signed-rank charts based on subgroups of sizes $n = 4,\ 5$ and 6.

Table 3.3. FAR's and ARL_0's for the upper one-sided Shewhart signed-rank chart.

UCL	***n* = 4**		***n* = 5**		***n* = 6**	
	ARL_0^+	FAR_0^+	ARL_0^+	FAR_0^+	ARL_0^+	FAR_0^+
10	16.00	0.0625	10.66	0.0938	6.40	0.1563
11	∞	0	10.66	0.0938	6.40	0.1563
12			16.00	0.0630	9.14	0.1094
13			16.00	0.0630	9.14	0.1094
14			32.00	0.0313	12.80	0.0781
15			32.00	0.0313	12.80	0.0781
16			∞	0	21.33	0.0469
17					21.33	0.0469
18					32.00	0.0312
19					32.00	0.0312
20					64.00	0.0156
21					64.00	0.0156
22					∞	0

Table 3.3 shows the false alarm rates and the in-control average run lengths for the upper one-sided Shewhart signed-rank chart as calculated using the null distribution of the Wilcoxon signed-rank statistic (see Hollander and Wolfe (1973) and Bakir (2003)).

In Table 3.3 we see that there are some duplicates in the data. We consider a specific example to shed light on the occurrence of these duplicates. Suppose $n = 5$ and $UCL = 12$. Then $FAR_0^+ = P(SR_i \geq 12 \,|\, \text{In-control}) = P(T_n^+ \geq 13.5)$ (using (3.2)). The last probability equals $P(T_n^+ \geq 14) = 0.0630$, because T_n^+ has zero probability at 13.5. When $n = 5$ and $UCL = 13$ we have that $FAR_0^+ = P(SR_i \geq 13 \,|\, \text{In-control}) = P(T_n^+ \geq 14) = 0.0630$ (by using the null distribution of the Wilcoxon signed-rank statistic). Since $FAR_0^+ = P(T_n^+ \geq 14) = 0.0630$ for two different values of the upper control limit, we have duplicates in the data. This example points out an error* in Table 1 of Bakir (2003). The probability of $P(T_n^+ \geq 13.5)$ equals $P(T_n^+ \geq 14)$ which equals 0.0630 (and not 0.0938 corresponding to $P(T_n^+ \geq 13)$ as reported by Bakir's (2003) paper). This type of correction was applied to the other entries of Bakir's (2003) Table 1 and are given in Table 3.3 of this thesis. The false alarm rates and in-control average run lengths for the two-sided Shewhart signed-rank chart were calculated using SAS (with the appropriate corrections made) and are shown in Table 3.4.

* This error is also pointed out by Chakraborti and Eryilmaz (2007).

Table 3.4. *FAR*'s and ARL_0's for the two-sided Shewhart signed-rank chart*.

UCL	***n* = 4**		***n* = 5**		***n* = 6**		***n* = 7**		***n* = 8**		***n* = 9**		***n* = 10**	
	ARL_0	*FAR*	ARL_0	*FAR*	ARL_0	*FAR*	ARL_0	*FAR*	ARL_0	*FAR*	ARL_0	*FAR*	ARL_0	*FAR*
11	8.018	0.125	3.248	0.308	2.307	0.433	2.132	0.469	1.830	0.546	1.526	0.655	1.447	0.691
12	∞	0	5.382	0.186	3.163	0.316	2.126	0.470	1.811	0.552	1.779	0.562	1.606	0.623
13			5.392	0.185	3.234	0.309	2.643	0.378	2.146	0.466	1.730	0.578	1.613	0.620
14			8.003	0.125	4.557	0.219	2.675	0.374	2.194	0.456	2.022	0.494	1.802	0.555
15			7.896	0.127	4.623	0.216	3.360	0.298	2.597	0.385	1.991	0.502	1.788	0.559
16			15.922	0.063	6.302	0.159	3.399	0.294	2.616	0.382	2.354	0.425	2.033	0.492
17			∞	0	6.483	0.154	4.552	0.220	3.201	0.312	2.346	0.426	2.032	0.492
18					10.797	0.093	4.553	0.220	3.276	0.305	2.770	0.361	2.276	0.439
19					10.762	0.093	6.514	0.154	4.004	0.250	2.778	0.360	2.345	0.426
20					16.291	0.061	6.307	0.159	3.999	0.250	3.350	0.299	2.635	0.379
21					15.982	0.063	9.164	0.109	5.053	0.198	3.309	0.302	2.664	0.375
22					29.890	0.033	9.152	0.109	5.149	0.194	4.034	0.248	3.071	0.326
23					∞	0	12.655	0.079	6.611	0.151	4.055	0.247	3.139	0.319
24							12.618	0.079	6.632	0.151	5.055	0.198	3.641	0.275
25							20.939	0.048	9.244	0.108	5.047	0.198	3.682	0.272
26							21.115	0.047	9.299	0.108	6.032	0.166	4.355	0.230
27							31.380	0.032	12.862	0.078	6.053	0.165	4.226	0.237
28							31.118	0.032	12.898	0.078	7.768	0.129	5.225	0.191
29							64.444	0.016	18.447	0.054	7.812	0.128	5.108	0.196
30							∞	0	17.947	0.056	10.180	0.098	6.286	0.159
31									25.216	0.040	10.554	0.095	6.315	0.158
32									25.285	0.040	13.573	0.074	7.670	0.130
33									42.248	0.024	13.357	0.075	7.638	0.131
34									42.872	0.023	18.499	0.054	9.505	0.105
35									63.492	0.016	18.409	0.054	9.728	0.103
36									64.492	0.016	25.763	0.039	12.004	0.083
37									129.711	0.008	25.676	0.039	11.531	0.087
38									∞	0	37.023	0.027	15.663	0.064
39											36.507	0.027	15.514	0.064
40											50.919	0.020	20.504	0.049
41											51.913	0.019	20.542	0.049

* See SAS Program 6 in Appendix B for the calculation of the values in Table 3.4. This table is an extension of Tables 1 and 2 given in Bakir (2003).

42											87.428	0.011	27.510	0.036
43											84.898	0.012	26.771	0.037
44											127.219	0.008	36.928	0.027
45											128.950	0.008	37.308	0.027
46											251.312	0.004	50.234	0.020
47											∞	0	52.249	0.019
48													73.736	0.014
49													74.261	0.013
50													104.300	0.010
51													101.973	0.010
52													165.381	0.006
53													168.821	0.006
54													251.693	0.004
55													249.627	0.004
56													443.132	0.002
57													∞	0

3.1.5. Summary

The signed-rank test is a popular nonparametric test for the median of a symmetric continuous population. The signed-rank test is more powerful than the sign test, but while the sign test is applicable for all continuous distributions, the assumption of symmetry must be made, in addition, for the signed-rank test. Furthermore, the sign test applies to all percentiles, whereas the signed-rank test is proposed only for the median. Another drawback of the signed-rank chart is that the *FAR* values for the chart are too high (in other words the ARL_0 values are too short) unless the subgroup size is 'large'. One way to remedy this problem is to use some signaling rules to enhance the sensitivity of the charts. This will be considered next.

3.2. The Shewhart-type control chart with runs-type signaling rules

3.2.1. Introduction

In addition to defining warning limits or zones on control charts (see Section 2.2), we can extend the existing charts by incorporating various signaling rules involving runs of the plotting statistic. The signaling rules considered include the following: A process is declared to be out-of-control when (a) a single point (charting statistic) plots outside the control limit(s) (*1-of-1* rule) (b) k consecutive points (charting statistics) plot outside the control limit(s) (*k-of-k* rule) or (c) exactly k of the last w points (charting statistics) plot outside the control limit(s) (*k-of-w* rule). We can consider these signaling rules where both k and w are positive integers with $1 \le k \le w$ and $w \ge 2$. Rule (a) is the simplest and is the most frequently used in the literature. Thus, the *1-of-1* rule corresponds to the usual control chart, where a signal is given when a plotting statistic falls outside the control limit(s). Rules (a) and (b) are special cases of rule (c); rules (b) and (c) have been used in the context of supplementing the Shewhart charts with warning limits and zones. Rules (a), (b) and (c) have been studied by various authors (see for example Klein (2000) and Khoo (2004)). Klein (2000) suggested two rules namely the *2-of-2* and *2-of-3* rules. Both control charts are easily implemented and have better *ARL* performance than the *1-of-1* rule. Khoo (2004) conducted a study of the *ARL* performance of the *2-of-2*, *2-of-3*, *2-of-4*, *3-of-3* and *3-of-4* charts and concluded that the *3-of-4* chart is the most sensitive scheme for detecting small process shifts.

Chakraborti and Eryilmaz (2007) considered simple alternatives to the Bakir (2004)'s class of nonparametric charts, using the signed-rank statistic but incorporating runs rules of the type discussed above to define new signaling rules. If we set k equal to 2 in rule (b) above, we obtain the simplest of the *k-of-k* type rules which are called the *2-of-2* DR and the *2-of-2* KL charts. The *2-of-2* KL chart signals, for example, when the two most recent signed-rank statistics both fall either on or above or on or below the control limits. The *2-of-2* DR chart is almost similar, but here a signal is indicated when both of the signed-rank statistics fall either both on or above or both on or below or one on or above (below) and the next one on or below (above) the control limits. It is shown that the new charts are nonparametric, have much smaller FAR (and thus larger ARL_0) than the *1-of-1* signed-rank chart of Bakir. Moreover, the new charts have better out-of-control performance than the *1-of-1* signed-rank chart for heavy-tailed and skewed distributions such as the Cauchy. We illustrate these procedures using the Montgomery (2001) piston ring data.

3.2.2. Example

Example 3.2

A two-sided Shewhart signed-rank chart with signaling rules for the Montgomery (2001) piston ring data

We illustrate the signed-rank chart with signaling rules using the Montgomery (2001) piston ring data. Recall that the dataset contains 15 samples (each of size 5). The signed-rank statistics were calculated and given in Table 3.2 and graphically represented in Figure 3.1. The symmetric two-sided control limits for the *1-of-1* and *2-of-2* signed-rank charts are given by Chakraborti and Eryilmaz (2007) for $n = 4, 5, 6$ and 10. The table for samples of size 5 is given here for reference.

Table 3.5. False alarm rates and in-control *ARL* values for the two-sided *1-of-1* and *2-of-2* signed-rank charts under DR and KL schemes, $n = 5^{*}$.

	1-of-1		***2-of-2* DR**		***2-of-2* KL**	
UCL	ARL_0	FAR_0	$ARL_{DR,0}$	$FAR_{DR,0}$	$ARL_{KL,0}$	$FAR_{KL,0}$
11	5.33	0.1876	33.74	0.0352	62.15	0.0176
13	8.00	0.1250	72.00	0.0156	136.00	0.0078
15	15.97	0.0626	271.15	0.0039	526.34	0.0019

For $n = 5$, the control limits for the *1-of-1* (Bakir's chart), the *2-of-2* DR and the *2-of-2* KL charts, based on the signed-rank statistic, are set at ± 15. These yield *FAR* values 0.0626, 0.0039, and 0.0019, respectively. If the control limits were taken to be ± 13, the *FAR* would have been much higher: 0.1250, 0.0156, and 0.0078, respectively. Although the control limits are the same, namely ± 15, the signaling rules are quite different operationally and the performance of the resulting charts turn out to be quite different. The *1-of-1* chart signals when the first signed-rank statistic falls on or outside of either of the two control limits; the *2-of-2* KL chart signals when, for the first time, two consecutive signed-rank statistics fall either on or above or on or below the two control limits, while the *2-of-2* DR chart signals when for the first time two consecutive signed-rank statistics fall on or outside the control limits, either both on or above, or both on or below, or one on or above the next on or below, or one on or below and the next on or above. On the performance side, note that the *1-of-1* SR chart has a *FAR* of 0.0626 and an ARL_0 of approximately 16. Thus many more false alarms will be signaled by this chart leading to a possible loss of time and resources. Compared to that, the *2-of-2* KL chart has a *FAR* of 0.0019 and an ARL_0 of 526.34, whereas the *2-of-2* DR chart has a *FAR* of 0.0039 and an ARL_0 of 271.15. Thus both of these run-rule-enhanced charts provide reasonable and practical false alarm rates and can be used in practice, depending on the type of shift one expects.

From Figure 3.1 we see that the DR and KL *2-of-2* signed-rank charts both signal at sample 13, indicating a most likely upward shift in the process median. The *1-of-1* signed-rank chart, on the other hand, signals earlier, at sample 12, but note the much higher *FAR* of 0.0626 (and correspondingly a much lower and less desirable ARL_0, 15.97) associated with this chart. It is interesting to note that, as shown in Montgomery (2001), for these data the

[*] Table 3.5 appears in Chakraborti and Eryilmaz (2007), Table 11.

Shewhart $\overline{X}$ chart indicates a shift in the mean at sample 11 for these data. However, the key difference is that an application of the Shewhart chart can raise several questions such as the form of the underlying distribution (small $n = 5$), and more importantly about the in-control (stable) performance of the chart in terms of the *FAR* (or the ARL_0), since it is known that the in-control performance of the Shewhart $\overline{X}$ chart is not robust in typical quality control applications. Compared to this, the proposed nonparametric charts provide a more generally applicable alternative monitoring scheme with a known (stable/robust) in-control performance and a better or equal out-of-control performance than the *1-of-1* signed-rank chart.

3.2.3. Summary

In this section we examined signed-rank control charts with runs-type signaling rules. Human, Chakraborti and Smit (2008) recently studied Shewhart-type sign charts with runs-type signaling rules. These charts are similar in spirit to the Shewhart-type signed-rank charts with runs-type signaling rules (see Section 3.2). In the paper by Human et al. they derived expressions for the run length distributions using Markov chain theory. The in-control and out-of-control performance of these charts were studied and compared to those of the existing signed-ranked charts under the normal, double exponential and Cauchy distributions, using the *ARL*, *SDRL*, *FAR* and some percentiles of the run length. These runs rules enhanced sign charts have the advantage that one does not have to assume symmetry of the underlying distribution and they can be applied in situations where the data are dichotomous.

3.3. The tabular CUSUM control chart

3.3.1. Introduction

Bakir and Reynolds (1979) investigated the CUSUM chart using the Wilcoxon signed-rank statistic. They used methods that are analogous to the methods used on the CUSUM sign chart (see Section 2.3), that is, a Markov chain approach is used to find the moments and other characteristics of the run length distribution for the CUSUM signed-rank chart.

3.3.2. One-sided control charts

3.3.2.1. Upper one-sided control charts

Fu, Spiring and Xie (2002) and Fu and Lou (2003) presented three results that must be satisfied before implementing the finite-state Markov chain approach. Let S_t^+ be a finite-state homogenous Markov chain on the state space Ω^+ with a transition probability matrix (TPM) such that (i) $\Omega^+ = \{\varsigma_0, \varsigma_1, ..., \varsigma_{r+s-1}\}$ where $0 = \varsigma_0 < \varsigma_1 < ... < \varsigma_{r+s-1} = h$ and ς_{r+s-1} is an absorbent state; (ii) the TPM is given by $TPM = [p_{ij}]$ for $i = 0,1,..., r+s-1$ and $j = 0,1,..., r+s-1$ where r denotes the number of non-absorbent* states and s the number of absorbent† states, respectively, and (iii) the starting value should equal zero with probability one, that is, $P(S_0^+ = 0) = 1$ (this is to ensure that the process starts in-control). Assume that the Markov chain S_t^+ satisfies conditions (i), (ii) and (iii), then the formulas given in (2.41) to (2.45) hold.

The time that the procedure signals is the first time such that the finite-state Markov chain S_t^+ enters the state ς_{r+s-1} where the state space is given by $\Omega^+ = \{\varsigma_0, \varsigma_1, ..., \varsigma_{r+s-1}\}$, $S_0^+ = 0$ and

$$S_t^+ = \min\{h, \max\{0, S_{t-1}^+ + SR_t - k\}\} \tag{3.3}$$

* The transient (non-absorbent) states are the states for which eventual return is uncertain.
† If a state is entered once and is never left, the state is said to be absorbent.

3.4. The EWMA control chart

3.4.1. Introduction

In this section, the approach taken by Lucas and Saccucci (1990) is extended to the use of the signed-rank statistic resulting in an EWMA signed-rank chart that accumulates the statistics $SR_1, SR_2, SR_3,...$. Section 3.4 is analogous to Section 2.4 where the approach taken by Lucas and Saccucci (1990) was extended to the use of the sign statistic resulting in an EWMA sign chart. Therefore, the reader is frequently referred back to Section 2.4 throughout this section.

3.4.2. The proposed EWMA signed-rank chart

A nonparametric EWMA-type of control chart based on the signed-rank statistic (recall that $SR_i = \sum_{j=1}^{n} sign(x_{ij} - \theta_0)R_{ij}^{+}$) can be obtained by replacing X_i in expression (2.53) of Section 2.4 with SR_i. The EWMA signed-rank chart accumulates the statistics $SR_1, SR_2, SR_3,...$ with the plotting statistics defined as

$$Z_i = \lambda SR_i + (1-\lambda)Z_{i-1} \qquad (3.10)$$

where $0 < \lambda \le 1$ is a constant called the weighting constant. The starting value Z_0 could be taken to equal zero, i.e. $Z_0 = 0$.

The EWMA signed-rank chart is constructed by plotting Z_i against the sample number i (or time). If the plotting statistic Z_i falls between the two control limits, that is, $LCL < Z_i < UCL$, the process is considered to be in-control. If the plotting statistic Z_i falls on or outside one of the control limits, that is $Z_i \le LCL$ or $Z_i \ge UCL$, the process is considered to be out-of-control.

The exact control limits and the center line of the EWMA signed-rank control chart can be obtained by replacing σ and θ_0 by σ_{SR_i} and 0, respectively, in expression (2.55) of Section 2.4 to obtain

$$\begin{aligned} UCL &= L\sigma_{SR_i}\sqrt{\left(\frac{\lambda}{2-\lambda}\right)\left(1-(1-\lambda)^{2i}\right)} \\ CL &= 0 \\ LCL &= -L\sigma_{SR_i}\sqrt{\left(\frac{\lambda}{2-\lambda}\right)\left(1-(1-\lambda)^{2i}\right)} \end{aligned} \qquad (3.11)$$

Similarly, the *steady-state* control limits can be obtained by replacing σ and θ_0 by σ_{SR_i} and 0, respectively, in expression (2.56) to obtain

$$\begin{aligned} UCL &= L\sigma_{SR_i}\sqrt{\left(\frac{\lambda}{2-\lambda}\right)} \\ LCL &= -L\sigma_{SR_i}\sqrt{\left(\frac{\lambda}{2-\lambda}\right)} \end{aligned} \qquad (3.12)$$

where σ_{SR_i} denotes the in-control standard deviation of the signed-rank statistic SR_i if there are no ties within a subgroup.

The in-control standard deviation of SR_i is given by $\sigma_{SR_i} = \sqrt{\text{var}(SR_i)} = \sqrt{\text{var}\left(2T^+ - \frac{n(n+1)}{2}\right)} = \sqrt{\frac{n(n+1)(2n+1)}{6}}$. This is obtained by using the relationship between SR_i and T^+ (recall that $SR_i = 2T^+ - \frac{n(n+1)}{2}$ if there are no ties within a subgroup) and the fact that $\text{var}(T^+) = \frac{n(n+1)(2n+1)}{24}$ (see Gibbons and Chakraborti (2003) page 198).

3.4.3. Markov-chain approach

Lucas and Saccucci (1990) evaluated the properties of the *continuous state* Markov chain by *discretizing* the infinite state TPM. This procedure entails dividing the interval between the UCL and the LCL into N subintervals of width 2δ. Then the plotting statistic, Z_i, is said to be in the non-absorbing state j at time i if $S_j - \delta < Z_i \le S_j + \delta$ where S_j denotes the midpoint of the j^{th} interval. Z_i is said to be in the absorbing state if Z_i falls on or outside one of the control limits, that is, $Z_i \le LCL$ or $Z_i \ge UCL$. Let p_{ij} denote the probability of moving from state i to state j in one step, i.e. $p_{ij} = P(Moving\ to\ state\ j \mid in\ state\ i)$. To approximate this probability we assume that the plotting statistic is equal to S_i whenever it is in state i. For all j non-absorbing we obtain $p_{ij} = P(S_j - \delta < Z_k \le S_j + \delta \mid Z_{k-1} = S_i)$. By using the definition of the plotting statistic given in expression (3.10) we obtain

$$
\begin{aligned}
p_{ij} &= P(S_j - \delta < \lambda SR_k + (1-\lambda)S_i \le S_j + \delta) \\
&= P\left(\frac{(S_j - \delta) - (1-\lambda)S_i}{\lambda} < SR_k \le \frac{(S_j + \delta) - (1-\lambda)S_i}{\lambda}\right)
\end{aligned}
$$

recall that $SR_k = 2T_k^+ - \frac{n(n+1)}{2}$

$$
\begin{aligned}
p_{ij} &= P\left(\frac{(S_j - \delta) - (1-\lambda)S_i}{\lambda} < 2T_k^+ - \frac{n(n+1)}{2} \le \frac{(S_j + \delta) - (1-\lambda)S_i}{\lambda}\right) \\
&= P\left(\frac{\left(\frac{(S_j - \delta) - (1-\lambda)S_i}{\lambda} + \frac{n(n+1)}{2}\right)}{2} < T_k^+ \le \frac{\left(\frac{(S_j + \delta) - (1-\lambda)S_i}{\lambda} + \frac{n(n+1)}{2}\right)}{2}\right).
\end{aligned}
\tag{3.13}
$$

For all j absorbing we obtain

$$
\begin{aligned}
p_{ij} &= P(Z_k \le LCL \mid Z_{k-1} = S_i) + P(Z_k \ge UCL \mid Z_{k-1} = S_i) \\
&= P(\lambda SR_k + (1-\lambda)S_i \le LCL) + P(\lambda SR_k + (1-\lambda)S_i \ge UCL) \\
&= P\left(SR_k \le \frac{LCL - (1-\lambda)S_i}{\lambda}\right) + P\left(SR_k \ge \frac{UCL - (1-\lambda)S_i}{\lambda}\right) \\
&= P\left(2T_k^+ - \frac{n(n+1)}{2} \le \frac{LCL - (1-\lambda)S_i}{\lambda}\right) + P\left(2T_k^+ - \frac{n(n+1)}{2} \ge \frac{UCL - (1-\lambda)S_i}{\lambda}\right) \\
&= P\left(T_k^+ \le \frac{\frac{LCL - (1-\lambda)S_i}{\lambda} + \frac{n(n+1)}{2}}{2}\right) + P\left(T_k^+ \ge \frac{\frac{UCL - (1-\lambda)S_i}{\lambda} + \frac{n(n+1)}{2}}{2}\right).
\end{aligned}
\tag{3.14}
$$

Since the values LCL, UCL, δ, λ, n, S_i and S_j are known constants the Wilcoxon signed-rank probabilities in expressions (3.13) and (3.14) can easily be calculated. The probabilities for the Wilcoxon signed-rank statistics are given in Table H of Lehmann (1975) for samples sizes up to 20 and they are tabulated (more recently) in Table H of Gibbons and Chakraborti (2003) for sample sizes up to 15.

Once the one-step transition probabilities are calculated, the TPM can be constructed and is given by $TPM = [p_{ij}] = \begin{pmatrix} Q & | & \underline{p} \\ - & - & - \\ \underline{0}' & | & 1 \end{pmatrix}$ (written in partitioned form) where the essential transition probability sub-matrix Q is the matrix that contains all the transition probabilities of going from a non-absorbing state to a non-absorbing state, $Q:(NA \to NA)$, $\underline{p}$ contains all the transition probabilities of going from each non-absorbing state to the absorbing states, $\underline{p}:(NA \to A)$, $\underline{0}' = (0 \quad 0 \quad 0 \quad \cdots \quad 0)$ contains all the transition probabilities of going from each absorbing state to the non-absorbing states. $\underline{0}'$ is a row vector with all its elements equal to zero, because it is impossible to go from an absorbing state to a non-absorbing state, because once an absorbing state is entered, it is never left, $\underline{0}' \; :(A \to NA)$, and 1 represents the scalar value one. The probability of going of going from an absorbing state to an absorbing state is equal to one, because once an absorbing state is entered, it is never left, $1:(A \to A)$. The one-step TPM is used to calculate the expected value (ARL), the second raw moment, the variance, the standard deviation and the probability mass function (pmf) of the run-length variable N which are given in equations (2.41) to (2.45).

Example 3.10

The EWMA signed-rank chart where the sample size is even ($n = 6$)

The EWMA signed-rank chart is investigated for a smoothing constant of 0.1 ($\lambda = 0.1$) and a multiplier of 3 ($L = 3$). The *steady-state* control limits are given by

$$UCL = L\sigma_{SR_i}\sqrt{\left(\frac{\lambda}{2-\lambda}\right)}$$

$$LCL = -L\sigma_{SR_i}\sqrt{\left(\frac{\lambda}{2-\lambda}\right)}$$

where $L = 3$, $\lambda = 0.1$, and $\sigma_{SR_i} = 9.539$, since $\sigma_{SR_i} = \sqrt{\frac{n(n+1)(2n+1)}{6}} = \sqrt{\frac{6(6+1)(12+1)}{6}} = 9.539$. Clearly, we only have to calculate the UCL since $LCL = -UCL$. We obtain $UCL = 3 \times 9.539 \sqrt{\left(\frac{0.1}{2-0.1}\right)} = 6.565$. Therefore, $LCL = -6.565$.

This Markov-chain procedure entails dividing the interval between the UCL and the LCL into N subintervals of width 2δ. For this example N is taken to equal 4. Figure 3.13 illustrates the partitioning of the interval between the UCL and the LCL into subintervals.

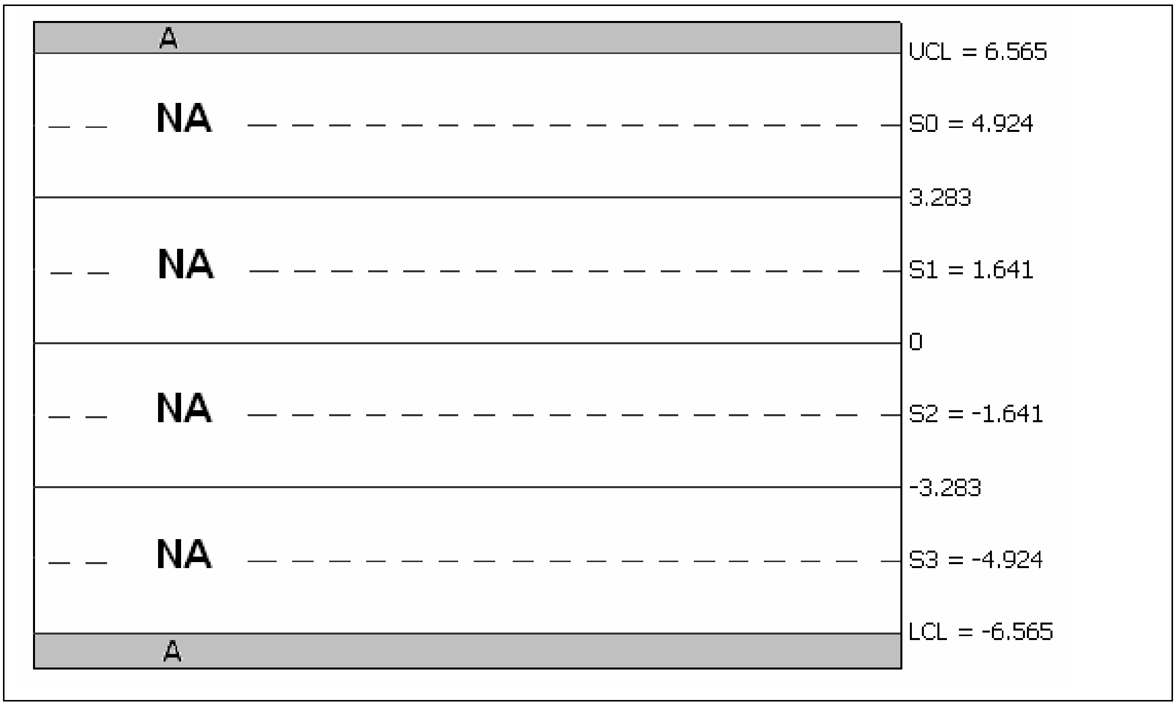

Figure 3.13. Partitioning of the interval between the UCL and the LCL into 4 subintervals.

From Figure 3.13 we see that there are 4 non-absorbing states, i.e. $r = 4$. The TPM is given by

$$TPM = \begin{pmatrix} p_{00} & p_{01} & p_{02} & p_{03} & p_{04} \\ p_{10} & p_{11} & p_{12} & p_{13} & p_{14} \\ p_{20} & p_{21} & p_{22} & p_{23} & p_{24} \\ p_{30} & p_{31} & p_{32} & p_{33} & p_{34} \\ p_{40} & p_{41} & p_{42} & p_{43} & p_{44} \end{pmatrix} = \begin{pmatrix} Q_{4\times 4} & | & \underline{p}_{4\times 1} \\ - & - & - \\ \underline{0}'_{1\times 4} & | & 1_{1\times 1} \end{pmatrix}.$$

Table 3.31. Calculation of the one-step probabilities in the first row of the TPM.

$p_{00} = P(\textit{Moving to state } 0 \mid \textit{in state } 0)$ $= P(S_0 - \delta < Z_k \le S_0 + \delta \mid Z_{k-1} = S_0)$ from (3.13) $= P\left(\frac{\left(\frac{(S_0-\delta)-(1-\lambda)S_0}{\lambda} + \frac{n(n+1)}{2}\right)}{2} < T_k^+ \le \frac{\left(\frac{(S_0+\delta)-(1-\lambda)S_0}{\lambda} + \frac{n(n+1)}{2}\right)}{2}\right)$ with $\delta = 1.641$, $\lambda = 0.1$, $L = 3$ and $S_0 = 4.924$ $= P(4.755 < T_k^+ \le 21.169)$ $= P(T_k^+ \le 21) - P(T_k^+ \le 4)$ $= \frac{57}{64}$ from Gibbons and Chakraborti (2003)
$p_{01} = P(\textit{Moving to state } 1 \mid \textit{in state } 0)$ $= P(S_1 - \delta < Z_k \le S_1 + \delta \mid Z_{k-1} = S_0)$ from (3.13) $= P\left(\frac{\left(\frac{(S_1-\delta)-(1-\lambda)S_0}{\lambda} + \frac{n(n+1)}{2}\right)}{2} < T_k^+ \le \frac{\left(\frac{(S_1+\delta)-(1-\lambda)S_0}{\lambda} + \frac{n(n+1)}{2}\right)}{2}\right)$ $= P(-11.658 < T_k^+ \le 4.755)$ $= P(T_k^+ \le 4)$ $= \frac{7}{64}$

$p_{02} = P(Moving\ to\ state\ 2 \mid in\ state\ 0)$ $= P(S_2 - \delta < Z_k \le S_2 + \delta \mid Z_{k-1} = S_0)$ from (3.13) $= P\left(\frac{\left(\frac{(S_2-\delta)-(1-\lambda)S_0}{\lambda}+\frac{n(n+1)}{2}\right)}{2} < T_k^+ \le \frac{\left(\frac{(S_2+\delta)-(1-\lambda)S_0}{\lambda}+\frac{n(n+1)}{2}\right)}{2}\right)$ $= P(-28.072 < T_k^+ \le -11.658)$ $= 0$
$p_{03} = P(Moving\ to\ state\ 3 \mid in\ state\ 0)$ $= P(S_3 - \delta < Z_k \le S_3 + \delta \mid Z_{k-1} = S_0)$ from (3.13) $= P\left(\frac{\left(\frac{(S_3-\delta)-(1-\lambda)S_0}{\lambda}+\frac{n(n+1)}{2}\right)}{2} < T_k^+ \le \frac{\left(\frac{(S_3+\delta)-(1-\lambda)S_0}{\lambda}+\frac{n(n+1)}{2}\right)}{2}\right)$ $= P(-44.486 < T_k^+ \le -28.072)$ $= 0$
$p_{04} = P(Moving\ to\ state\ 4 \mid in\ state\ 0)$ $= P(Z_k \le LCL \mid Z_{k-1} = S_0) + P(Z_k \ge UCL \mid Z_{k-1} = S_0)$ from (3.14) $= P\left(T_k^+ \le \frac{\frac{LCL-(1-\lambda)S_0}{\lambda}+\frac{n(n+1)}{2}}{2}\right) + P\left(T_k^+ \ge \frac{\frac{UCL-(1-\lambda)S_0}{\lambda}+\frac{n(n+1)}{2}}{2}\right)$ $= P(T_k^+ \le -44.486) + P(T_k^+ \ge 21.169)$ $= 0$

The one-step probabilities in the remaining rows can be calculated similarly. Therefore, the TPM is given by

$$TPM = \begin{pmatrix} 57/64 & 7/64 & 0 & 0 & 0 \\ 2/64 & 57/64 & 5/64 & 0 & 0 \\ 0 & 5/64 & 57/64 & 2/64 & 0 \\ 0 & 0 & 7/64 & 57/64 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} Q_{4\times4} & \mid & \underline{p}_{4\times1} \\ - & - & - \\ \underline{0}'_{1\times4} & \mid & 1_{1\times1} \end{pmatrix}.$$

Other values of the multiplier (L) and the smoothing constant (λ) were also considered and the results are given in Tables 3.32 and 3.33.

Table 3.32. The in-control average run length (ARL_0), standard deviation of the run length ($SDRL$), $5^{th}, 25^{th}, 50^{th}, 75^{th}$ and 95^{th} percentile values[*] for the EWMA signed-rank chart when $n = 6$ and $N = 5$, i.e. there are 5 subintervals between the lower and upper control limit[†].

	$L = 1$	$L = 2$	$L = 3$
$\lambda = 0.05$	10.45 12.32 (1, 2, 6, 15, 35)	56.69 72.45 (1, 5, 29, 82, 204)	**
$\lambda = 0.1$	7.32 8.38 (1, 1, 4, 10, 24)	33.83 40.28 (1, 4, 20, 48, 115)	330.67 369.33 (2, 63, 213, 471, 1070)
$\lambda = 0.2$	4.95 4.90 (1, 1, 3, 7, 15)	35.21 39.63 (1, 6, 22, 50, 115)	361.92 384.29 (3, 87, 243, 510, 1130)

** The inverse of the matrix ($I - Q$) does not exist and as a result the ARL (given by $E(N) = \underline{\xi}(I - Q)^{-1}\underline{1}$) can not be calculated for this combination of (λ, L).

In example 3.10 we considered a sample size that may be considered "small". The results are given for a larger sample size ($n = 10$) for various values of λ and L in Table 3.33.

Table 3.33. The in-control average run length (ARL_0), standard deviation of the run length ($SDRL$), $5^{th}, 25^{th}, 50^{th}, 75^{th}$ and 95^{th} percentile values[‡] for the EWMA signed-rank chart when $n = 10$ and $N = 5$, i.e. there are 5 subintervals between the lower and upper control limit[§].

	$L = 1$	$L = 2$	$L = 3$
$\lambda = 0.05$	11.17 13.49 (1, 2, 6, 16, 39)	67.94 83.82 (1, 7, 38, 98, 238)	1448.44 1573.37 (10, 316, 956, 2052, 4595)
$\lambda = 0.1$	6.85 7.74 (1, 1, 4, 9, 23)	48.87 57.73 (1, 6, 29, 70, 165)	352.72 384.51 (3, 76, 232, 500, 1122)
$\lambda = 0.2$	5.05 5.07 (1, 1, 3, 7, 15)	33.96 38.48 (1, 6, 21, 48, 111)	336.34 357.54 (3, 80, 226, 474, 1051)

[*] The three rows of each cell shows the ARL_0, the $SDRL$, and the percentiles ($\rho_5, \rho_{25}, \rho_{50}, \rho_{75}, \rho_{95}$), respectively.
[†] See SAS Program 8 in Appendix B for the calculation of the values in Table 3.32.
[‡] The three rows of each cell shows the ARL_0, the $SDRL$, and the percentiles ($\rho_5, \rho_{25}, \rho_{50}, \rho_{75}, \rho_{95}$), respectively.
[§] See SAS Program 8 in Appendix B for the calculation of the values in Table 3.33.

These tables can be extended by changing the sample size (*n*), the number of subintervals between the lower and upper control limit (*N*), the multiplier (*L*) and the smoothing constant (λ) in SAS Program 8 for the EWMA signed-rank chart given in Appendix B.

From Tables 3.32 and 3.33 we see that the ARL_0, $SDRL$ and percentiles increase as the value of the multiplier (*L*) increases. From Table 3.33 we find an in-control average run length of 336.34 for $n = 10$ when the multiplier is taken to equal 3 ($L = 3$) and the smoothing constant 0.2 ($\lambda = 0.2$). The chart performance is good, since the attained in-control average run length of 336.34 is in the region of the desired in-control average run length which is generally taken to be 370 or 500.

3.4.4. Summary

The EWMA control chart is one of several charting methods aimed at correcting a deficiency of the Shewhart chart - insensitivity to small shifts. Lucas and Saccucci (1990) have investigated some properties of the EWMA chart under the assumption of independent normally distributed observations, whereas in this section we have described and evaluated the nonparametric EWMA signed-rank chart. The main advantage of the nonparametric EWMA chart is that there is no need to assume a particular parametric distribution for the underlying process (see Section 1.4 for other advantages of the nonparametric EWMA chart).

where h is the decision interval and k is the reference value (see Section 2.3.1 for a detailed discussion on how the values of k and h are chosen). Equation (3.3) is obtained by replacing SN_t with SR_t in (2.46).

The distribution of SR_t can easily be obtained from the distribution of the Wilcoxon signed-rank statistic T^+ (recall that $SR_i = 2T_i^+ - \frac{n(n+1)}{2}$ $\forall i$). The probabilities for the Wilcoxon signed-rank statistics are given in Table H of Lehmann (1975) for samples sizes up to 20 and they are tabulated (more recently) in Table H of Gibbons and Chakraborti (2003) for sample sizes up to 15.

Example 3.3

An upper one-sided CUSUM signed-rank chart where the sample size is even (*n*=4)

The statistical properties of an upper one-sided CUSUM signed-rank chart with a decision interval of 6 $(h = 6)$, a reference value of 2 $(k = 2)$ and a sample size of 4 $(n = 4)$ is examined. We start by examining the pmf of the well-known Wilcoxon signed-rank statistic T^+, since the plotting statistic SR_i is linearly related to T^+.

Table 3.6. Enumeration for the distribution of T^+ for a sample size of 4.

Value of T^+	**Ranks associated with positive differences**	**Number of sample points $u(t)$**	$P(T^+ = t)$	$P(T^+ \le t)$
0		1	$1/16$	$1/16$
1	1	1	$1/16$	$2/16$
2	2	1	$1/16$	$3/16$
3	{1,2}; {3}	2	$2/16$	$5/16$
4	{1,3}; {4}	2	$2/16$	$7/16$
5	{1,4}; {2,3}	2	$2/16$	$9/16$
6	{1,2,3}; {2,4}	2	$2/16$	$11/16$
7	{1,2,4}; {3,4}	2	$2/16$	$13/16$
8	{1,3,4}	1	$1/16$	$14/16$
9	{2,3,4}	1	$1/16$	$15/16$
10	{1,2,3,4}	1	$1/16$	$16/16$

From Table 3.6 if follows that the pmf of T^+ when the sample size is 4 is

$$f_{T^+}(t) = P\left(T^+ = t\right) = \begin{cases} 1/16 & t = 0,1,2,8,9,10 \\ 2/16 & t = 3,4,5,6,7 \\ 0 & \text{otherwise} \end{cases}$$

The values of SR_t are either the even or the odd integers between (and including) $-\frac{n(n+1)}{2}$ and $\frac{n(n+1)}{2}$, depending on whether $\frac{n(n+1)}{2}$ is even or odd. In example 3.3 $\frac{n(n+1)}{2} = \frac{4(4+1)}{2} = 10$ which is even and as a result the possible values for SR_t are even integers between -10 and 10 inclusive. Thus, we have that $-10 \le SR_t \le 10$. In both cases (whether $\frac{n(n+1)}{2}$ is even or odd) the sum $\sum(SR_i - k)$ will be an integer since both SR_t and k are integers. For this example, the reference value is taken to be equal to two, because this leads to the sum $\sum(SR_i - k)$ being equal to even values which reduces the size of the state space for the Markov chain. For $h = 6$ we have that $\Omega^+ = \{\varsigma_0, \varsigma_1, \varsigma_2, \varsigma_3\} = \{0,2,4,6\}$ with

$0 = \varsigma_0 < \varsigma_1 < \varsigma_2 < \varsigma_3 = h$. The state space is calculated using equation (3.3) and the calculations are shown in Table 3.7.

Table 3.7. Calculation of the state space when $h = 6$, $k = 2$ and $n = 4$.

SR_t	$S_{t-1}^+ + SR_t - k$	$\max\{0, S_{t-1}^+ + SR_t - k\}$	$S_t^+ = \min\{h, \max\{0, S_{t-1}^+ + SR_t - k\}\}$
-10	-12*	0	0
-8	-10	0	0
-6	-8	0	0
-4	-6	0	0
-2	-4	0	0
0	-2	0	0
2	0	0	0
4	2	2	2
6	4	4	4
8	6	6	6
10	8	8	6

Table 3.8. Classification of the states.

State number	Description of the state	Absorbent (A)/ Non-absorbent (NA)
0	$S_t^+ = 0$	NA
1	$S_t^+ = 2$	NA
2	$S_t^+ = 4$	NA
3	$S_t^+ = 6$	A

From Table 3.8 we see that there are three non-absorbent states, i.e. $r = 3$, and one absorbent state, i.e. $s = 1$. Therefore, the corresponding TPM will be a $(r+s)\times(r+s) = 4\times 4$ matrix. It can be shown (see Table 3.9) that the TPM is given by

$$TPM_{4\times 4} = \begin{pmatrix} p_{00} & p_{02} & p_{04} & p_{06} \\ p_{20} & p_{22} & p_{24} & p_{26} \\ p_{40} & p_{42} & p_{44} & p_{46} \\ p_{60} & p_{62} & p_{64} & p_{66} \end{pmatrix} = \begin{pmatrix} 11/16 & 2/16 & 1/16 & | & 2/16 \\ 9/16 & 2/16 & 2/16 & | & 3/16 \\ 7/16 & 2/16 & 2/16 & | & 5/16 \\ - & - & - & - & - \\ 0 & 0 & 0 & | & 1 \end{pmatrix} = \begin{pmatrix} Q_{3\times 3} & | & \underline{p}_{3\times 1} \\ - & - & - \\ \underline{0}'_{1\times 3} & | & 1_{1\times 1} \end{pmatrix}$$

---

* Note: Since only the state space needs to be described, S_{t-1}^+ can be any value from Ω^+ and we therefore take, without loss of generality, $S_{t-1}^+ = 0$. Any other possible value for S_{t-1}^+ would lead to the same Ω^+.

where the essential transition probability sub-matrix $Q_{3\times3}:(NA \to NA)$ is an $r \times r = 3\times 3$ matrix, $\underline{p}_{3\times1}:(NA \to A)$ is an $(r+s-1)\times 1 = 3\times 1$ column vector, $\underline{0}'_{1\times3}:(A \to NA)$ is a $1\times(r+s-1) = 1\times 3$ row vector and $1_{1\times1}:(A \to A)$ represents the scalar value one.

The one-step transition probabilities are calculated by substituting SR_t in expression (3.3) by $2T^+ - \dfrac{n(n+1)}{2}$ and substituting in values for h, k, S_t^+ and S_{t-1}^+. The calculation of the one-step transition probabilities are given for illustration in Table 3.9.

The probabilities in the last column of the TPM can be calculated using the fact that $\sum_{j\in\Omega} p_{ij} = 1 \quad \forall i$ (see equation (2.18)). Therefore,

$p_{06} = 1-(p_{00}+p_{02}+p_{04}) = 1-(^{11}\!/_{16}+^{2}\!/_{16}+^{1}\!/_{16}) = ^{2}\!/_{16}$;

$p_{26} = 1-(p_{20}+p_{22}+p_{24}) = 1-(^{9}\!/_{16}+^{2}\!/_{16}+^{2}\!/_{16}) = ^{3}\!/_{16}$;

$p_{46} = 1-(p_{40}+p_{42}+p_{44}) = 1-(^{7}\!/_{16}+^{2}\!/_{16}+^{2}\!/_{16}) = ^{5}\!/_{16}$;

$p_{66} = 1-(p_{60}+p_{62}+p_{64}) = 1-(0+0+0) = 1$.

Table 3.9. The calculation of the transition probabilities when $h = 6$, $k = 2$ and $n = 4$.

p_{00} $= P(S_t = 0 \mid S_{t-1} = 0)$ $= P(\min\{6, \max\{0, 0 + SR_t - 2\}\} = 0)$ $= P(\max\{0, SR_t - 2\} = 0)$ $= P(SR_t - 2 \le 0)$ $= P(SR_t \le 2)$ $= P(2T^+ - 10 \le 2)$ $= P(T^+ \le 6)$ $= \frac{11}{16}$	p_{02} $= P(S_t = 2 \mid S_{t-1} = 0)$ $= P(\min\{6, \max\{0, 0 + SR_t - 2\}\} = 2)$ $= P(\max\{0, SR_t - 2\} = 2)$ $= P(SR_t - 2 = 2)$ $= P(SR_t = 4)$ $= P(2T^+ - 10 = 4)$ $= P(T^+ = 7)$ $= \frac{2}{16}$	p_{04} $= P(S_t = 4 \mid S_{t-1} = 0)$ $= P(\min\{6, \max\{0, 0 + SR_t - 2\}\} = 4)$ $= P(\max\{0, SR_t - 2\} = 4)$ $= P(SR_t - 2 = 4)$ $= P(SR_t = 6)$ $= P(2T^+ - 10 = 6)$ $= P(T^+ = 8)$ $= \frac{1}{16}$
p_{20} $= P(S_t = 0 \mid S_{t-1} = 2)$ $= P(\min\{6, \max\{0, 2 + SR_t - 2\}\} = 0)$ $= P(\max\{0, SR_t\} = 0)$ $= P(SR_t \le 0)$ $= P(2T^+ - 10 \le 0)$ $= P(T^+ \le 5)$ $= \frac{9}{16}$	p_{22} $= P(S_t = 2 \mid S_{t-1} = 2)$ $= P(\min\{6, \max\{0, 2 + SR_t - 2\}\} = 2)$ $= P(\max\{0, SR_t\} = 2)$ $= P(SR_t = 2)$ $= P(2T^+ - 10 = 2)$ $= P(T^+ = 6)$ $= \frac{2}{16}$	p_{24} $= P(S_t = 4 \mid S_{t-1} = 2)$ $= P(\min\{6, \max\{0, 2 + SR_t - 2\}\} = 4)$ $= P(\max\{0, SR_t\} = 4)$ $= P(SR_t = 4)$ $= P(2T^+ - 10 = 4)$ $= P(T^+ = 7)$ $= \frac{2}{16}$
p_{40} $= P(S_t = 0 \mid S_{t-1} = 4)$ $= P(\min\{6, \max\{0, 4 + SR_t - 2\}\} = 0)$ $= P(\max\{0, SR_t + 2\} = 0)$ $= P(SR_t + 2 \le 0)$ $= P(SR_t \le -2)$ $= P(2T^+ - 10 \le -2)$ $= P(T^+ \le 4)$ $= \frac{7}{16}$	p_{42} $= P(S_t = 2 \mid S_{t-1} = 4)$ $= P(\min\{6, \max\{0, 4 + SR_t - 2\}\} = 2)$ $= P(\max\{0, SR_t + 2\} = 2)$ $= P(SR_t + 2 = 2)$ $= P(SR_t = 0)$ $= P(2T^+ - 10 = 0)$ $= P(T^+ = 5)$ $= \frac{2}{16}$	p_{44} $= P(S_t = 4 \mid S_{t-1} = 4)$ $= P(\min\{6, \max\{0, 4 + SR_t - 2\}\} = 4)$ $= P(\max\{0, SR_t + 2\} = 4)$ $= P(SR_t + 2 = 4)$ $= P(SR_t = 2)$ $= P(2T^+ - 10 = 2)$ $= P(T^+ = 6)$ $= \frac{2}{16}$
p_{60} $= P(S_t = 0 \mid S_{t-1} = 6)$ $= 0$ *	p_{62} $= P(S_t = 2 \mid S_{t-1} = 6)$ $= 0$	p_{64} $= P(S_t = 4 \mid S_{t-1} = 6)$ $= 0$

Using the TPM the *ARL* can be calculated using $ARL = \underline{\xi}(I - Q)^{-1}\underline{1}$. A well-known concern is that important information about the performance of a control chart can be missed when only examining the *ARL* (this is especially true when the process distribution is skewed). Various authors, see for example, Radson and Boyd (2005) and Chakraborti (2007), have

* The probability equals zero, because it is impossible to go from an absorbent state to a non-absorbent state.

suggested that one should examine a number of percentiles, including the median, to get the complete information about the performance of a control chart. Therefore, we now also consider percentiles. The $100\rho^{th}$ percentile is defined as the smallest integer l such that the cdf is at least $(100\times\rho)\%$. Thus, the $100\rho^{th}$ percentile l is found from $P(N\leq l)\geq\rho$. The median (50^{th} percentile) will be considered, since it is a more representative performance measure than the *ARL*. The first and third quartiles (25^{th} and 75^{th} percentiles) will also be considered, since it contains the middle half of the distribution. The 'tails' of the distribution should also be examined and therefore the 5^{th} and 95^{th} percentiles are calculated. The calculation of these percentiles is shown below for illustration purposes.

Table 3.10. Calculation of the percentiles when $h=6$, $k=2$ and $n=4$ *.

N	$P(N\leq l)$	**The 5th, 25th, 50th, 75th and 95th percentiles**
1	0.125	$\rho_{0.05}=1$ (smallest integer such that the cdf is at least 0.05)
2	0.254	$\rho_{0.25}=2$ (smallest integer such that the cdf is at least 0.25)
3	0.366	
4	0.462	
5	0.544	$\rho_{0.5}=5$ (smallest integer such that the cdf is at least 0.5)
6	0.613	
7	0.671	
8	0.721	
9	0.763	$\rho_{0.75}=9$ (smallest integer such that the cdf is at least 0.75)
10	0.799	
11	0.829	
12	0.855	
13	0.877	
14	0.896	
15	0.912	
16	0.925	
17	0.936	
18	0.946	
19	0.954	$\rho_{0.95}=19$ (smallest integer such that the cdf is at least 0.95)
20 †	0.961	

* See SAS Program 7 in Appendix B for the calculation of the values in Table 3.10.
† The value of the run length variable is only shown up to $N = 20$ for illustration purposes.

The formulas of the moments and some characteristics of the run length distribution have been studied by Fu, Spiring and Xie (2002) and Fu and Lou (2003) – see equations (2.41) to (2.45). By substituting $\underline{\xi}_{1\times3} = (1 \quad 0 \quad 0)$, $Q_{3\times3} = \begin{pmatrix} 11/16 & 2/16 & 1/16 \\ 9/16 & 2/16 & 2/16 \\ 7/16 & 2/16 & 2/16 \end{pmatrix}$ and $\underline{1}_{3\times1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ into these equations, we obtain the following:

$$ARL = E(N) = \underline{\xi}(I - Q)^{-1}\underline{1} = 6.81$$

$$E(N^2) = \underline{\xi}(I + Q)(I - Q)^{-2}\underline{1} = 83.64$$

$$SDRL = \sqrt{Var(N)} = \sqrt{E(N^2) - (E(N))^2} = 6.11$$

5^{th} percentile $= \rho_5 = 1$

25^{th} percentile $= \rho_{25} = 2$

Median $= 50^{th}$ percentile $= \rho_{50} = 5$

75^{th} percentile $= \rho_{75} = 9$

95^{th} percentile $= \rho_{95} = 19$

Other values of h, k and n were also considered and the results are given in Table 3.11.

Table 3.11. The in-control average run length (ARL_0^+), standard deviation of the run length ($SDRL$), 5^{th}, 25^{th}, 50^{th}, 75^{th} and 95^{th} percentile values* for the upper one-sided CUSUM signed-rank chart when $n = 4$ †.

k	**h**				
	2	**4**	**6**	**8**	**10**
0	2.29 1.71 (1, 1, 2, 3, 6)	3.05 2.44 (1, 1, 2, 4, 8)	4.27 3.50 (1, 2, 3, 6, 11)	5.49 4.55 (1, 2, 4, 7, 15)	7.24 5.98 (1, 3, 5, 10, 19)
2	3.20 2.65 (1, 1, 2, 4, 8)	4.92 4.31 (1, 2, 4, 7, 14)	6.81 6.11 (1, 2, 5, 9, 19)	10.17 9.21 (1, 4, 7, 14, 29)	
4	5.33 4.81 (1, 2, 4, 7, 15)	7.74 7.19 (1, 3, 6, 11, 22)	13.28 12.58 (1, 4, 9, 18, 38)		
6	8.00 7.48 (1, 3, 6, 11, 23)	15.06 14.49 (1, 5, 11, 21, 44)			
8	16.00 15.49 (1, 5, 11, 22, 47)				

In order to allow for the possibility of stopping after one group, the values of h is taken to satisfy $h \leq \frac{n(n+1)}{2} - k$. For example, for $n = 4$ and $k = 0$, the reference value h is taken to be smaller than or equal to 10, since $\frac{n(n+1)}{2} - k = \frac{4(4+1)}{2} - 0 = 10$.

The five percentiles are displayed in boxplot-like‡ graphs in Figure 3.2 for all the (h,k)-combinations that are shaded in Table 3.11. It clearly shows the effects of h and k on the run length distribution. Figure 3.2 describes the run-length distribution when the process is in-control. We would prefer a "boxplot" with a high valued (large) in-control average run length and a small spread. The "boxplots" are classified into 3 categories, namely, small ($h + k \leq 4$),

*The three rows of each cell shows the ARL_0^+, the $SDRL$, and the percentiles (ρ_5, ρ_{25}, ρ_{50}, ρ_{75}, ρ_{95}), respectively.

† See SAS Program 7 in Appendix B for the calculation of the values in Table 3.11.

‡ It should be noted that these boxplot-like graphs differ from standard box plots. In the latter case the whiskers are drawn from the ends of the box to the smallest and largest values inside specified limits, whereas, in the case of the boxplot-like graphs, the whiskers are drawn from the ends of the box to the 5^{th} and 95^{th} percentiles, respectively. In this thesis "boxplot" will refer to a boxplot-like graph from this point forward.

moderate ($5 \le h + k \le 8$) and large ($h + k \ge 9$). If the sum of the reference value, *k*, and the decision interval, *h*, is small (moderate or large), the corresponding "boxplot" is classified under small (moderate or large). For example, where $h + k = 4$, the "boxplot" is classified as small, since the ARL_0^+, *SDRL* and percentile values are small for $n = 4$. In contrast, where $h + k = 10$, the "boxplot" is classified as large, since the ARL_0^+, *SDRL* and percentile values are large for $n = 4$.

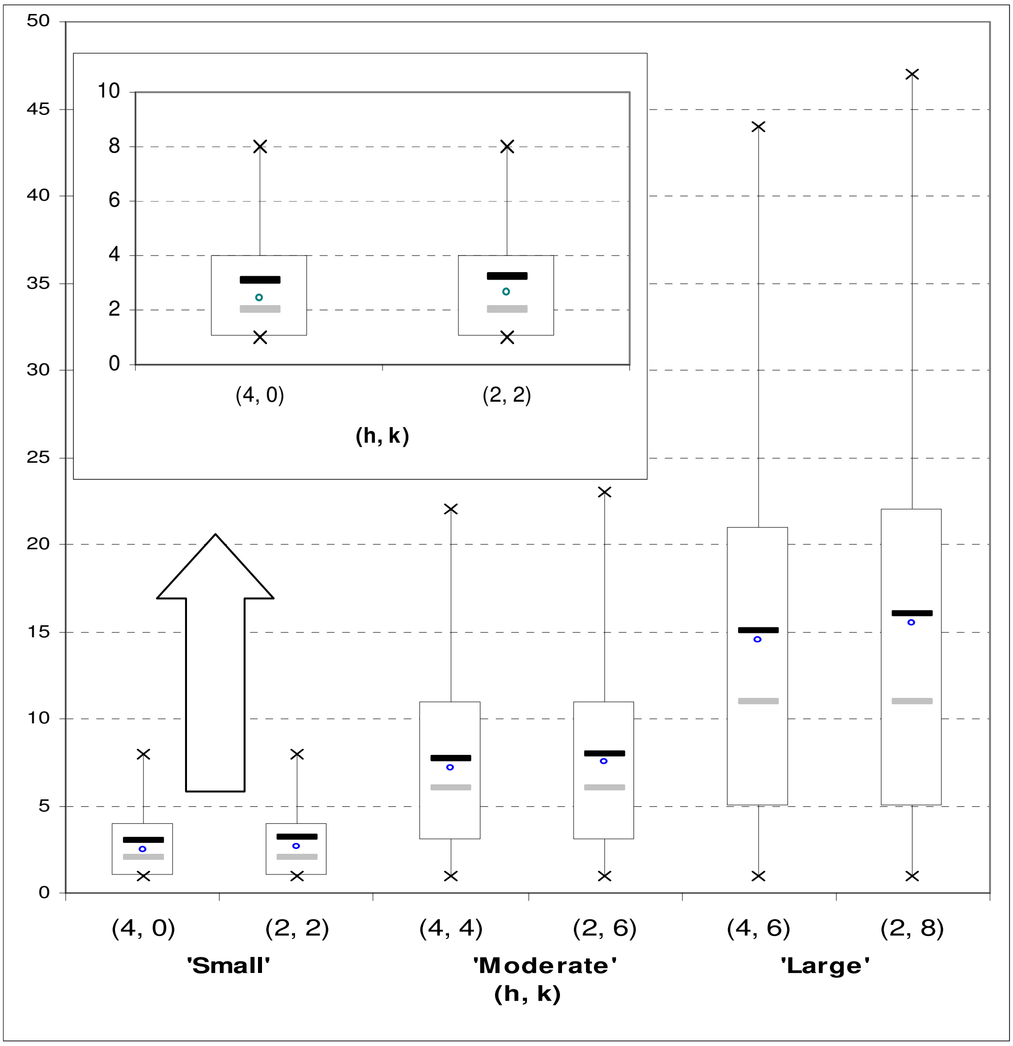

Figure 3.2. Boxplot-like graphs for the in-control run length distribution of various upper one-sided CUSUM signed-rank charts when $n = 4$. The whiskers extend to the 5th and the 95th percentiles. The symbols "▬", "○" and "▬" denote the *ARL*, *SDRL*[*] and *MRL*, respectively.

[*] For ease of interpretation, the standard deviation (as measure of spread) is included in the (location) measures of percentiles.

Example 3.4

An upper one-sided CUSUM signed-rank chart where the sample size is odd (*n*=5)

The statistical properties of an upper one-sided CUSUM signed-rank chart with a decision interval of 6 $(h=6)$, a reference value of 3 $(k=3)$ and a sample size of 5 $(n=5)$ is examined. We start by examining the pmf of the well-known Wilcoxon signed-rank statistic T^+, since the plotting statistic SR_i is linearly related to T^+ (see equation (3.2)).

Table 3.12. Enumeration for the distribution of T^+ for a sample size of 5.

Value of T^+	Ranks associated with positive differences	Number of sample points $u(t)$	$P(T^+ = t)$	$P(T^+ \le t)$
0		1	$1/32$	$1/32$
1	1	1	$1/32$	$2/32$
2	2	1	$1/32$	$3/32$
3	{1,2}; {3}	2	$2/32$	$5/32$
4	{1,3}; {4}	2	$2/32$	$7/32$
5	{1,4}; {2,3}; {5}	3	$3/32$	$10/32$
6	{1,2,3}; {1,5}; {2,4}	3	$3/32$	$13/32$
7	{1,2,4}; {2,5}; {3,4}	3	$3/32$	$16/32$
8	{1,2,5}; {1,3,4}; {3,5}	3	$3/32$	$19/32$
9	{1,3,5}; {2,3,4}; {4,5}	3	$3/32$	$22/32$
10	{1,2,3,4}; {1,4,5}; {2,3,5}	3	$3/32$	$25/32$
11	{1,2,3,5}; {2,4,5}	2	$2/32$	$27/32$
12	{1,2,4,5}; {3,4,5}	2	$2/32$	$29/32$
13	{1,3,4,5}	1	$1/32$	$30/32$
14	{2,3,4,5}	1	$1/32$	$31/32$
15	{1,2,3,4,5}	1	$1/32$	$32/32$

From Table 3.12 if follows that the pmf of T^+ when the sample size is 5 is

$$f_{T^+}(t) = P(T^+ = t) = \begin{cases} 1/32 & t = 0,1,2,13,14,15 \\ 2/32 & t = 3,4,11,12 \\ 3/32 & t = 5,6,7,8,9,10 \\ 0 & \text{otherwise} \end{cases}$$

The reference value was taken to be equal to three, because this leads to the sum $\sum(SR_i - k)$ being equal to even values which reduces the size of the state space for the Markov chain. For $h = 6$ we have that $\Omega^+ = \{\varsigma_0, \varsigma_1, \varsigma_2, \varsigma_3\} = \{0,2,4,6\}$ with $0 = \varsigma_0 < \varsigma_1 < \varsigma_2 < \varsigma_3 = h$. The state space is calculated using equation (3.3) and the calculations are shown in Table 3.13.

Table 3.13. Calculation of the state space when $h = 6$, $k = 3$ and $n = 5$.

SR_t	$S_{t-1}^+ + SR_t - k$	$\max\{0, S_{t-1}^+ + SR_t - k\}$	$S_t^+ = \min\{h, \max\{0, S_{t-1}^+ + SR_t - k\}\}$
-15	-18*	0	0
-13	-16	0	0
-11	-14	0	0
-9	-12	0	0
-7	-10	0	0
-5	-8	0	0
-3	-6	0	0
-1	-4	0	0
1	-2	0	0
3	0	0	0
5	2	2	2
7	4	4	4
9	6	6	6
11	8	8	6
13	10	10	6
15	12	12	6

* Note: Since only the state space needs to be described, S_{t-1}^+ can be any value from Ω^+ and we therefore take, without loss of generality, $S_{t-1}^+ = 0$. Any other possible value for S_{t-1}^+ would lead to the same Ω^+.

Table 3.14. Classification of the states.

State number	Description of the state	Absorbent (A)/ Non-absorbent (NA)
0	$S_t^+ = 0$	NA
1	$S_t^+ = 2$	NA
2	$S_t^+ = 4$	NA
3	$S_t^+ = 6$	A

From Table 3.14 we see that there are three non-absorbent states, i.e. $r = 3$, and one absorbent state, i.e. $s = 1$. Therefore, the corresponding TPM will be a $(r+s)\times(r+s) = 4\times 4$ matrix. It can be shown (see Table 3.15) that the TPM is given by

$$TPM_{4\times 4} = \begin{pmatrix} p_{00} & p_{02} & p_{04} & p_{06} \\ p_{20} & p_{22} & p_{24} & p_{26} \\ p_{40} & p_{42} & p_{44} & p_{46} \\ p_{60} & p_{62} & p_{64} & p_{66} \end{pmatrix} = \begin{pmatrix} 22/32 & 3/32 & 2/32 & \vert & 5/32 \\ 19/32 & 3/32 & 3/32 & \vert & 7/32 \\ 16/32 & 3/32 & 3/32 & \vert & 10/32 \\ - & - & - & - & - \\ 0 & 0 & 0 & \vert & 1 \end{pmatrix} = \begin{pmatrix} Q_{3\times 3} & \vert & \underline{p}_{3\times 1} \\ - & - & - \\ \underline{0}'_{1\times 3} & \vert & 1_{1\times 1} \end{pmatrix}$$

where the essential transition probability sub-matrix $Q_{3\times 3} : (NA \to NA)$ is an $r \times r = 3\times 3$ matrix, $\underline{p}_{3\times 1} : (NA \to A)$ is an $(r+s-1)\times 1 = 3\times 1$ column vector, $\underline{0}'_{1\times 3} : (A \to NA)$ is a $1\times(r+s-1) = 1\times 3$ row vector and $1_{1\times 1} : (A \to A)$ represents the scalar value one. The calculation of the one-step transition probabilities are given for illustration in Table 3.15.

Recall that the probabilities in the last column of the TPM are calculated using the fact that $\sum_{j\in\Omega} p_{ij} = 1 \quad \forall i$ (see equation (2.18)). Therefore,

$p_{06} = 1 - (p_{00} + p_{02} + p_{04}) = 1 - (22/32 + 3/32 + 2/32) = 5/32$;

$p_{26} = 1 - (p_{20} + p_{22} + p_{24}) = 1 - (19/32 + 3/32 + 3/32) = 7/32$;

$p_{46} = 1 - (p_{40} + p_{42} + p_{44}) = 1 - (16/32 + 3/32 + 3/32) = 10/32$;

$p_{66} = 1 - (p_{60} + p_{62} + p_{64}) = 1 - (0 + 0 + 0) = 1$.

Table 3.15. The calculation of the transition probabilities when $h=6$, $k=3$ and $n=5$.

p_{00} $=P(S_t=0 \mid S_{t-1}=0)$ $=P(\min\{6,\max\{0,0+SR_t-3\}\}=0)$ $=P(\max\{0,SR_t-3\}=0)$ $=P(SR_t-3\leq 0)$ $=P(SR_t\leq 3)$ $=P(2T^+-15\leq 3)$ $=P(T^+\leq 9)$ $=\frac{22}{32}$	p_{02} $=P(S_t=2 \mid S_{t-1}=0)$ $=P(\min\{6,\max\{0,0+SR_t-3\}\}=2)$ $=P(\max\{0,SR_t-3\}=2)$ $=P(SR_t-3=2)$ $=P(SR_t=5)$ $=P(2T^+-15=5)$ $=P(T^+=10)$ $=\frac{3}{32}$	p_{02} $=P(S_t=4 \mid S_{t-1}=0)$ $=P(\min\{6,\max\{0,0+SR_t-3\}\}=4)$ $=P(\max\{0,SR_t-3\}=4)$ $=P(SR_t-3=4)$ $=P(SR_t=7)$ $=P(2T^+-15=7)$ $=P(T^+=11)$ $=\frac{2}{32}$
p_{20} $=P(S_t=0 \mid S_{t-1}=2)$ $=P(\min\{6,\max\{0,2+SR_t-3\}\}=0)$ $=P(\max\{0,SR_t-1\}=0)$ $=P(SR_t-1\leq 0)$ $=P(SR_t\leq 1)$ $=P(2T^+-15\leq 1)$ $=P(T^+\leq 8)$ $=\frac{19}{32}$	p_{22} $=P(S_t=2 \mid S_{t-1}=2)$ $=P(\min\{6,\max\{0,2+SR_t-3\}\}=2)$ $=P(\max\{0,SR_t-1\}=2)$ $=P(SR_t-1=2)$ $=P(SR_t=3)$ $=P(2T^+-15=3)$ $=P(T^+=9)$ $=\frac{3}{32}$	p_{24} $=P(S_t=4 \mid S_{t-1}=2)$ $=P(\min\{6,\max\{0,2+SR_t-3\}\}=4)$ $=P(\max\{0,SR_t-1\}=4)$ $=P(SR_t-1=4)$ $=P(SR_t=5)$ $=P(2T^+-15=5)$ $=P(T^+=10)$ $=\frac{3}{32}$
p_{40} $=P(S_t=0 \mid S_{t-1}=4)$ $=P(\min\{6,\max\{0,4+SR_t-3\}\}=0)$ $=P(\max\{0,SR_t+1\}=0)$ $=P(SR_t+1\leq 0)$ $=P(SR_t\leq -1)$ $=P(2T^+-15\leq -1)$ $=P(T^+\leq 7)$ $=\frac{16}{32}$	p_{42} $=P(S_t=2 \mid S_{t-1}=4)$ $=P(\min\{6,\max\{0,4+SR_t-3\}\}=2)$ $=P(\max\{0,SR_t+1\}=2)$ $=P(SR_t+1=2)$ $=P(SR_t=1)$ $=P(2T^+-15=1)$ $=P(T^+=8)$ $=\frac{3}{32}$	p_{44} $=P(S_t=4 \mid S_{t-1}=4)$ $=P(\min\{6,\max\{0,4+SR_t-3\}\}=4)$ $=P(\max\{0,SR_t+1\}=4)$ $=P(SR_t+1=4)$ $=P(SR_t=3)$ $=P(2T^+-15=3)$ $=P(T^+=9)$ $=\frac{3}{32}$
p_{60} $=P(S_t=0 \mid S_{t-1}=6)$ $=0$ [*]	p_{62} $=P(S_t=2 \mid S_{t-1}=6)$ $=0$	p_{64} $=P(S_t=4 \mid S_{t-1}=6)$ $=0$

[*] The probability equals zero, because it is impossible to go from an absorbent state to a non-absorbent state.

The formulas of the moments and some characteristics of the run length distribution have been studied by Fu, Spiring and Xie (2002) and Fu and Lou (2003) – see equations (2.41) to (2.45). By substituting $\underline{\xi}_{1\times 3} = (1 \quad 0 \quad 0)$, $Q_{3\times 3} = \begin{pmatrix} 22/32 & 3/32 & 2/32 \\ 19/32 & 3/32 & 3/32 \\ 16/32 & 3/32 & 3/32 \end{pmatrix}$ and $\underline{1}_{3\times 1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ into these equations, we obtain the following:

$$ARL = E(N) = \underline{\xi}(I - Q)^{-1}\underline{1} = 5.79$$

$$E(N^2) = \underline{\xi}(I + Q)(I - Q)^{-2}\underline{1} = 60.14$$

$$SDRL = \sqrt{Var(N)} = \sqrt{E(N^2) - (E(N))^2} = 5.16$$

$$5^{th} \text{ percentile} = p_5 = 1$$

$$25^{th} \text{ percentile} = p_{25} = 2$$

$$\text{Median} = 50^{th} \text{ percentile} = p_{50} = 4$$

$$75^{th} \text{ percentile} = p_{75} = 8$$

$$95^{th} \text{ percentile} = p_{95} = 16$$

Other values of *h*, *k* and *n* were also considered and the results are given in Table 3.16.

Table 3.16. The in-control average run length (ARL_0^+), standard deviation of the run length ($SDRL$), 5^{th}, 25^{th}, 50^{th}, 75^{th} and 95^{th} percentile values* for the upper one-sided CUSUM signed-rank chart when $n = 5$ †.

k	***h***						
	2	**4**	**6**	**8**	**10**	**12**	**14**
1	2.46 1.90 (1, 1, 2, 3, 6)	3.11 2.53 (1, 1, 2, 4, 8)	4.08 3.42 (1, 2, 3, 5, 11)	5.14 4.38 (1, 2, 4, 7, 14)	6.71 5.73 (1, 3, 5, 9, 18)	8.29 7.13 (1, 3, 6, 11, 22)	10.46 8.99 (2, 4, 8, 14, 28)
3	3.20 2.65 (1, 1,2, 4, 8)	4.39 3.82 (1, 2, 3, 6, 12)	5.79 5.16 (1, 2, 4, 8, 16)	8.13 7.34 (1, 3, 6, 11, 23)	10.68 9.75 (1, 4, 8, 14, 30)	14.78 13.56 (2, 5, 11, 20, 42)	
5	4.57 4.04 (1, 2, 3, 6, 13)	6.24 5.69 (1, 2, 4, 8, 18)	9.44 8.79 (1, 3, 7, 13, 27)	13.04 12.31 (1, 4, 9, 18, 38)	20.16 19.22 (2, 6, 14, 28, 59)		
7	6.40 5.88 (1, 2, 5, 9, 18)	10.24 9.68 (1, 3, 7, 14, 30)	14.77 14.18 (1, 5, 10, 20, 43)	25.17 24.43 (2, 8, 18, 35, 74)			
9	10.67 10.15 (1, 3, 8, 15, 31)	15.75 15.22 (1, 5, 11, 22, 46)	29.15 28.55 (2, 9, 20, 40, 86)				
11	16.00 15.49 (1, 5, 11, 22, 47)	31.03 30.50 (2, 9, 22, 43, 92)					
13	32.00 31.50 (2, 10, 22, 44, 95)						

* The three rows of each cell shows the ARL_0^+, the $SDRL$, and the percentiles (ρ_5, ρ_{25}, ρ_{50}, ρ_{75}, ρ_{95}), respectively.

† See SAS program 7 in Appendix B for the calculation of the values in Table 3.16.

The five percentiles are displayed in boxplot-like graphs in Figure 3.3 for all the (h,k)-combinations that are shaded in Table 3.16. It clearly shows the effects of h and k on the run length distribution. Figure 3.3 describes the run-length distribution when the process is in-control. We would prefer a "boxplot" with a high valued (large) in-control average run length and a small spread. The "boxplots" are classified into 3 categories, namely small ($h+k \leq 5$), moderate ($6 \leq h+k \leq 10$) and large ($h+k \geq 11$).

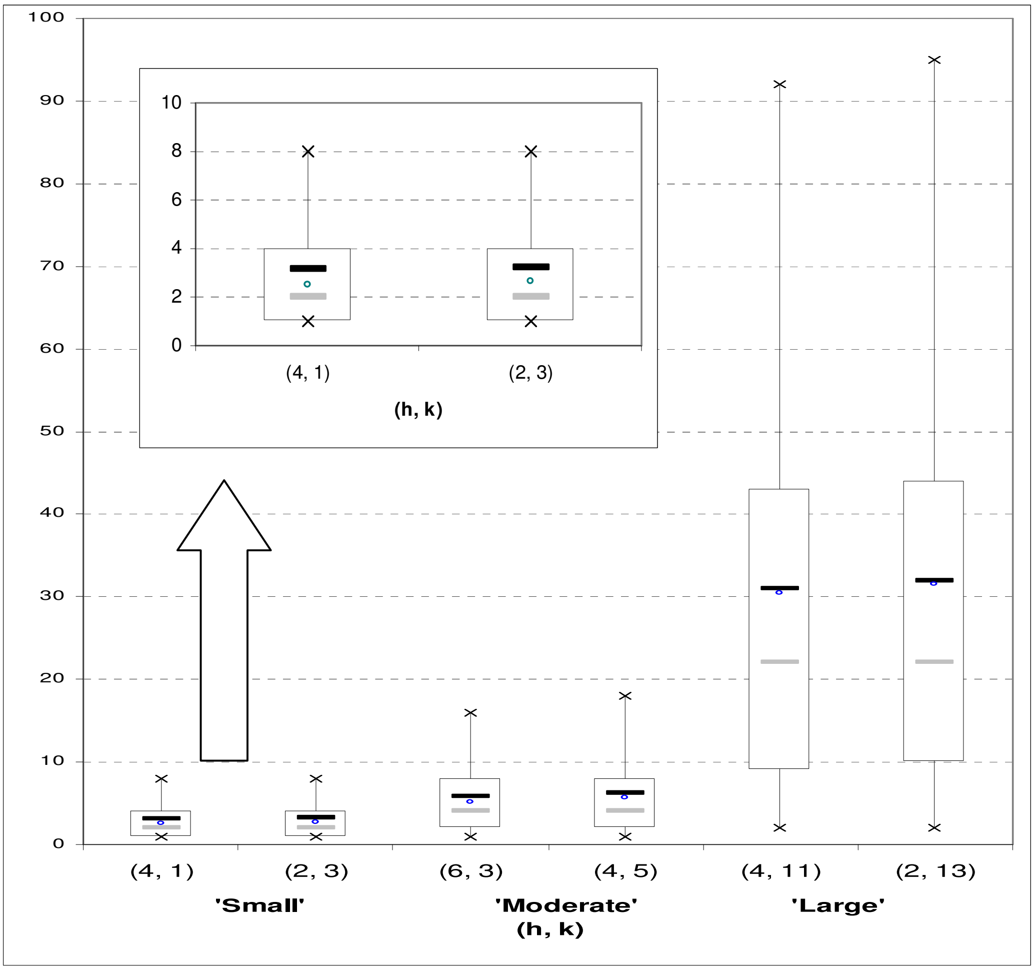

Figure 3.3. Boxplot-like graphs for the in-control run length distribution of various upper one-sided CUSUM signed-rank charts when $n=5$. The whiskers extend to the 5th and the 95th percentiles. The symbols "▬", "○" and "▬" denote the *ARL*, *SDRL*[*] and *MRL*, respectively.

[*] For ease of interpretation, the standard deviation (as measure of spread) is included in the (location) measures of percentiles.

Examples 3.3 and 3.4 illustrated the Markov chain approach used to calculate run length characteristics for n even and odd, respectively. On the performance side, note that the largest in-control average run length that the upper one-sided CUSUM signed-rank can obtain is 2^n. Therefore, for a sample size of 4 the largest ARL_0^+ equals $2^4 = 16$ (this is obtained when $h = 2$ and $k = 8$). Thus, a large number of false alarms will be signaled by this chart leading to a possible loss of time and resources. Compared to this, for a sample of size 5 the largest ARL_0^+ equals $2^5 = 32$ (this is obtained when $h = 2$ and $k = 13$). Both examples considered sample sizes that may be considered "small". Some results will be given for larger sample sizes ($n = 6$ and 10).

Table 3.17. The in-control average run length (ARL_0^+), standard deviation of the run length ($SDRL$), $5^{th}, 25^{th}, 50^{th}, 75^{th}$ and 95^{th} percentile values* for the upper one-sided CUSUM signed-rank chart when $n = 6$†.

k	***h***								
	2	**4**	**6**	**8**	**10**	**12**	**14**	**16**	**18**
1	2.37 1.80 (1, 1, 2, 3, 6)	2.86 2.28 (1, 1, 2, 4, 7)	3.39 2.79 (1, 1, 3, 4, 9)	4.08 3.42 (1, 2, 3, 5, 11)	5.03 4.25 (1, 2, 4, 7, 13)	6.10 5.17 (1, 2, 5, 8, 16)	7.24 6.17 (1, 3, 5, 10, 20)	8.72 7.41 (2, 3, 6, 12, 23)	10.21 8.69 (2, 4, 8, 14, 28)
3	2.91 2.36 (1, 1, 2, 4, 8)	3.51 2.95 (1, 1, 3, 5, 9)	4.33 3.74 (1, 2, 3, 6, 12)	5.55 4.87 (1, 2, 4, 7, 15)	7.00 6.21 (1, 3, 5, 9, 19)	8.63 7.72 (1, 3, 6, 12, 24)	10.99 9.86 (2, 4, 8, 15, 31)	13.43 12.12 (2, 5, 10, 18, 38)	16.78 15.18 (2, 6, 12, 23, 47)
5	3.56 3.01 (1, 1, 3, 5, 10)	4.49 3.94 (1, 2, 3, 6, 12)	5.95 5.35 (1, 2, 4, 8, 17)	7.82 7.14 (1, 3, 6, 11, 22)	10.02 9.25 (1, 3, 7, 14, 28)	13.55 12.60 (2, 5, 10, 18, 39)	17.39 16.29 (2, 6, 12, 24, 50)	23.44 22.07 (2, 8, 17, 32, 67)	
7	4.57 4.04 (1, 2, 3, 6, 13)	6.24 5.70 (1, 2, 4, 8, 18)	8.50 7.91 (1, 3, 6, 12, 24)	11.26 10.61 (1, 4, 8, 15, 32	16.17 15.39 (2, 5, 11, 22, 47)	21.79 20.90 (2, 7, 15, 30, 63)	32.01 30.88 (3, 10, 23, 44, 94)		
9	6.40 5.88 (1, 2, 5, 9, 18)	8.96 8.42 (1, 3, 6, 12, 26)	12.16 11.60 (1, 4, 9, 17, 35)	18.48 17.83 (2, 6, 13, 25, 54)	25.89 25.17 (2, 8, 18, 36, 76)	41.56 40.64 (2, 13, 29, 57, 123)			
11	9.14 8.63 (1, 3, 6, 12, 26)	12.64 12.12 (1, 4, 9, 17, 37)	20.05 19.48 (2, 6, 14, 28, 59)	28.88 28.27 (2, 9, 20, 40, 85)	50.26 49.52 (3, 15, 35, 69, 149)				
13	12.80 12.29 (1, 4, 9, 18, 37)	20.90 20.37 (2, 6, 15, 29, 62)	30.76 30.22 (2, 9, 21, 42, 91)	56.62 55.99 (3, 17, 39, 78, 168)					
15	21.33 20.83 (2, 6, 15, 29, 63)	31.75 31.24 (2, 9, 22, 44, 94)	61.08 60.53 (4, 18, 43, 84, 182)						
17	32.00 31.50 (2, 10, 22, 44, 95)	63.02 62.50 (4, 18, 44, 87, 188)							
19	64.00 63.50 (4, 19, 45, 89, 191)								

* The three rows of each cell shows the ARL_0^+, the $SDRL$, and the percentiles (ρ_5, ρ_{25}, ρ_{50}, ρ_{75}, ρ_{95}), respectively.

† See SAS Program 7 in Appendix B for the calculation of the values in Table 3.17.

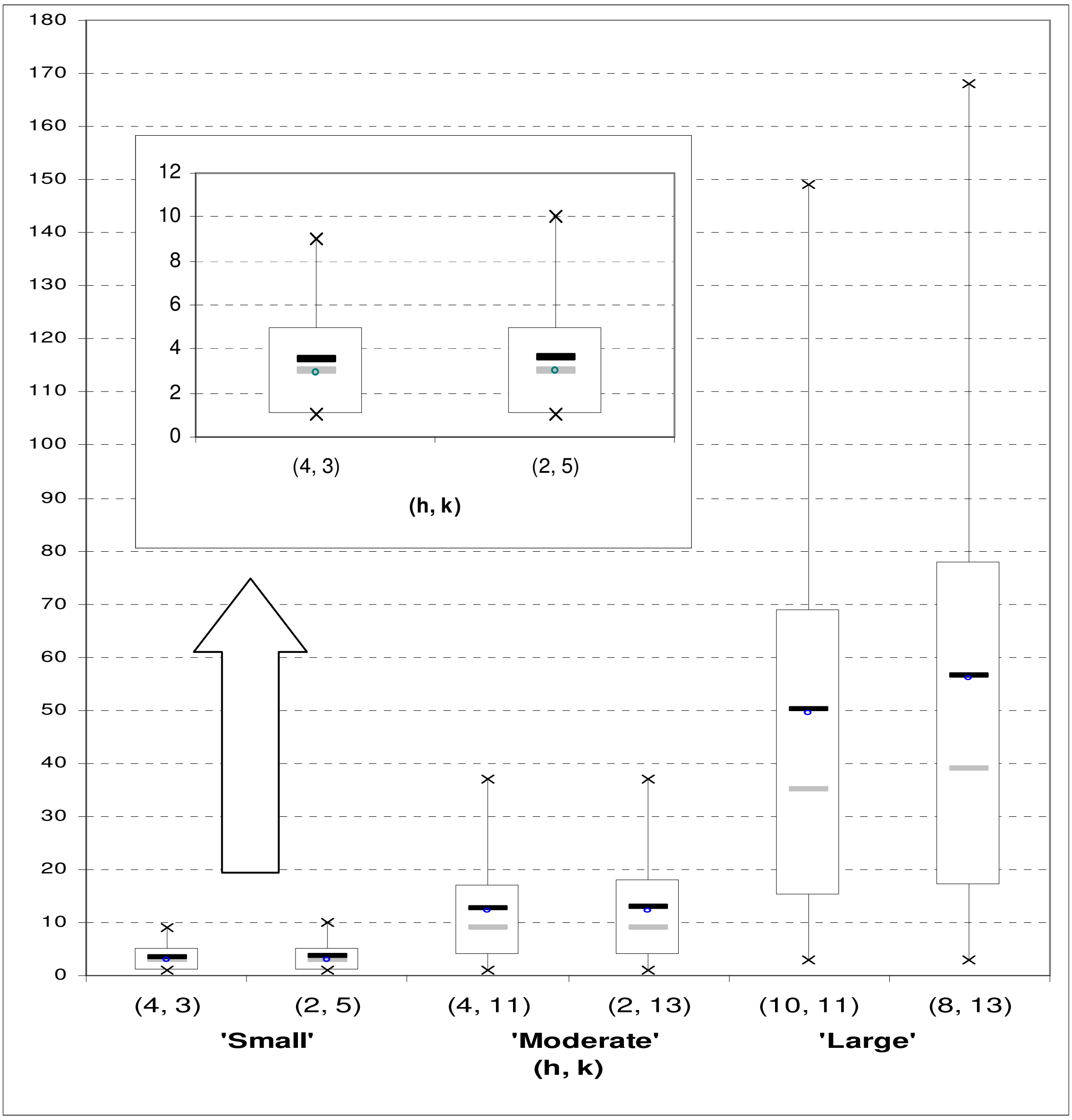

Figure 3.4. Boxplot-like graphs for the in-control run length distribution of various upper one-sided CUSUM signed-rank charts when $n = 6$. The whiskers extend to the 5th and the 95th percentiles. The symbols "▬", "◦" and "▬" denote the *ARL*, *SDRL*[*] and *MRL*, respectively[†].

[*] For ease of interpretation, the standard deviation (as measure of spread) is included in the (location) measures of percentiles.

[†] The "boxplots" are classified into 3 categories, namely small ($h + k \le 7$), moderate ($8 \le h + k \le 16$) and large ($h + k \ge 17$).

Table 3.18. The in-control average run length (ARL_0^+), standard deviation of the run length ($SDRL$), 5^{th}, 25^{th}, 50^{th}, 75^{th} and 95^{th} percentile values* for samples of size $n = 10$ for h = 2,4,...,14 and k = 1,3,...,23 for the upper one-sided CUSUM signed-rank chart†.

k	***h***						
	2	**4**	**6**	**8**	**10**	**12**	**14**
1	2.17 1.59 (1, 1, 2, 3, 5)	2.36 1.78 (1, 1, 2, 3, 6)	2.57 1.99 (1, 1, 2, 3, 7)	2.81 2.22 (1, 1, 2, 4, 7)	3.07 2.47 (1, 1, 2, 4, 8)	3.36 2.73 (1, 1, 3, 4, 9)	3.68 3.02 (1, 2, 3, 5, 10)
3	2.36 1.80 (1, 1, 2, 3, 6)	2.59 2.02 (1, 1, 2, 3, 7)	2.84 2.27 (1, 1, 2, 4, 7)	3.12 2.54 (1, 1, 2, 4, 8)	3.44 2.84 (1, 1, 3, 5, 9)	3.79 3.17 (1, 2, 3, 5, 10)	4.18 3.52 (1, 2, 3, 6, 11)
5	2.60 2.04 (1, 1, 2, 3, 7)	2.87 2.31 (1, 1, 2, 4, 7)	3.16 2.60 (1, 1, 2, 4, 8)	3.50 2.93 (1, 1, 3, 5, 9)	3.88 3.29 (1, 2, 3, 5, 10)	4.31 3.69 (1, 2, 3, 6, 12)	4.79 4.14 (1, 2, 4, 6, 13)
7	2.88 2.32 (1, 1, 2, 4, 8)	3.19 2.64 (1, 1, 2, 4, 8)	3.55 2.99 (1, 1, 3, 5, 10)	3.96 3.39 (1, 2, 3, 5, 11)	4.42 3.83 (1, 2, 3, 6, 12)	4.95 4.34 (1, 2, 4, 7, 14)	5.55 4.91 (1, 2, 4, 7, 15)
9	3.20 2.65 (1, 1, 2, 4, 8)	3.58 3.03 (1, 1, 3, 5, 10)	4.01 3.46 (1, 2, 3, 5, 11)	4.51 3.94 (1, 2, 3, 6, 12)	5.08 4.50 (1, 2, 4, 7, 14)	5.75 5.14 (1, 2, 4, 8, 16)	6.49 5.85 (1, 2, 5, 9, 18)
11	3.59 3.05 (1, 1, 3, 5, 10)	4.05 3.51 (1, 2, 3, 5, 11)	4.57 4.03 (1, 2, 3, 6, 13)	5.19 4.63 (1, 2, 4, 7, 14)	5.91 5.33 (1, 2, 4, 8, 17)	6.73 6.12 (1, 2, 5, 9, 19)	7.69 7.05 (1, 3, 6, 10, 22)
13	4.06 3.53 (1, 2, 3, 5, 11)	4.61 4.08 (1, 2, 3, 6, 13)	5.26 4.72 (1, 2, 4, 7, 15)	6.03 5.48 (1, 2, 4, 8, 17)	6.92 6.35 (1, 2, 5, 9, 20)	7.97 7.37 (1, 3, 6, 11, 23)	9.24 8.60 (1, 3, 7, 13, 26)
15	4.63 4.10 (1, 2, 3, 6, 13)	5.31 4.78 (1, 2, 4, 7, 15)	6.12 5.59 (1, 2, 4, 8, 17)	7.06 6.52 (1, 2, 5, 10, 20)	8.20 7.63 (1, 3, 6, 11, 23)	9.58 8.99 (1, 3, 7, 13, 28)	11.19 10.56 (1, 4, 8, 15, 32)
17	5.33 4.81 (1, 2, 4, 7, 15)	6.18 5.65 (1, 2, 4, 8, 17)	7.17 6.64 (1, 2, 5, 10, 20)	8.37 7.83 (1, 3, 6, 11, 24)	9.86 9.30 (1, 3, 7, 13, 28)	11.60 11.02 (1, 4, 8, 16, 34)	13.74 13.12 (1, 4, 10, 19, 40)
19	6.21 5.68 (1, 2, 4, 8, 18)	7.23 6.71 (1, 2, 5, 10, 21)	8.50 7.97 (1, 3, 6, 12, 24)	10.07 9.53 (1, 3, 7, 14, 29)	11.93 11.37 (1, 4, 8, 16, 35)	14.24 13.66 (1, 5, 10, 20, 42)	17.12 16.50 (1, 5, 12, 23, 50)
21	7.26 6.74 (1, 2, 5, 10, 21)	8.57 8.05 (1, 3, 6, 12, 25)	10.21 9.69 (1, 3, 7, 14, 30)	12.18 11.64 (1, 4, 9, 17, 35)	14.64 14.08 (1, 5, 10, 20, 43)	17.73 17.15 (1, 6, 12, 24, 52)	21.60 21.00 (2, 7, 15, 30, 64)
23	8.61 8.09 (1, 3, 6, 12, 25)	10.30 9.79 (1, 3, 7, 14, 30)	12.34 11.82 (1, 4, 9, 17, 36)	14.93 14.39 (1, 5, 11, 20, 44)	18.20 17.65 (1, 6, 13, 25, 53)	22.36 21.79 (2, 7, 16, 31, 66)	28.16 27.56 (2, 9, 20, 39, 83)

* The three rows of each cell shows the ARL_0^+, the $SDRL$, and the percentiles (ρ_5, ρ_{25}, ρ_{50}, ρ_{75}, ρ_{95}) , respectively.

† See SAS Program 7 in Appendix B for the calculation of the values in Table 3.18.

Table 3.18 continued for h = 2, 4, ..., 14 and k = 25, 27, ..., 53.

k	***h***						
	2	**4**	**6**	**8**	**10**	**12**	**14**
25	10.34 9.83 (1, 3, 7, 14, 30)	12.44 11.93 (1, 4, 9, 17, 36)	15.12 14.60 (1, 5, 11, 21, 44)	18.55 18.02 (1, 6, 13, 26, 55)	22.93 22.38 (2, 7, 16, 32, 68)	29.14 28.57 (2, 9, 20, 40, 86)	37.30 36.70 (2, 11, 26, 51, 111)
27	12.49 11.98 (1, 4, 9, 17, 36)	15.23 14.72 (1, 5, 11, 21, 45)	18.77 18.25 (1, 6, 13, 26, 55)	23.33 22.80 (2, 7, 16, 32, 69)	29.87 29.33 (2, 9, 21, 41, 88)	38.56 38.00 (3, 11, 27, 53, 114)	49.52 48.94 (3, 15, 35, 68, 147)
29	15.28 14.78 (1, 5, 11, 21, 45)	18.91 18.40 (1, 6, 13, 26, 56)	23.59 23.08 (2, 7, 17, 33, 70)	30.39 29.87 (2, 9, 21, 42, 90)	39.50 38.96 (3, 12, 28, 55, 117)	51.09 50.53 (3, 15, 36, 71, 152)	67.68 67.10 (4, 20, 47, 94, 202)
31	18.96 18.46 (1, 6, 13, 26, 56)	23.75 23.24 (2, 7, 17, 33, 70)	30.74 30.22 (2, 9, 21, 42, 91)	40.17 39.64 (3, 12, 28, 55, 119)	52.23 51.70 (3, 15, 36, 72, 155)	69.70 69.14 (4, 20, 48, 96, 208)	95.33 94.76 (5, 28, 66, 132, 284)
33	23.81 23.31 (2, 7, 17, 33, 70)	30.94 30.43 (2, 9, 22, 43, 92)	40.60 40.09 (3, 12, 28, 56, 121)	53.02 52.51 (3, 16, 37, 73, 158)	71.14 70.61 (4, 21, 49, 98, 212)	98.00 97.46 (6, 9, 68, 136, 292)	137.20 136.63 (8, 40, 95, 190, 410)
35	31.03 30.53 (2, 9, 22, 43, 92)	40.86 40.35 (3, 12, 28, 56, 121)	53.53 53.02 (3, 16, 37, 74, 159)	72.12 71.61 (4, 21, 50, 100, 215)	99.90 99.37 (6, 29, 69, 138, 298)	140.75 140.21 (8, 41, 98, 195, 421)	194.51 193.96 (11, 56, 135, 269, 582)
37	40.96 40.46 (3, 12, 29, 57, 122)	53.80 53.29 (3, 16, 37, 74, 160)	72.71 72.20 (4, 21, 51, 101, 217)	101.12 100.60 (6, 29, 70, 140, 302)	143.15 142.63 (8, 42, 99, 198,428)	198.67 198.14 (11, 58, 138, 275, 594)	323.14 322.58 (17, 93, 224, 448, 967)
39	53.89 53.39 (3, 16, 38, 75, 160)	73.02 72.51 (4, 21, 51, 101, 218)	101.84 101.33 (6, 30, 71, 141, 304)	144.68 144.17 (8, 42, 100, 200, 432)	201.42 200.90 (11, 58, 140, 279, 602)	330.31 329.78 (17, 95, 229, 458, 988)	490.25 489.71 (26, 141, 340, 679, 1468)
41	73.14 72.64 (4, 21, 51, 101, 218)	102.24 101,74 (6, 30, 71, 142, 305)	145.61 145.11 (8, 42, 101, 202, 435)	203.16 203.65 (11, 59, 141, 281, 608)	335.17 334.65 (18, 97, 232, 464, 1003)	499.40 498.88 (26, 144, 346, 692, 1495)	973.74 973.19 (50, 281, 675, 1350, 2916)
43	102.40 101.90 (6, 30, 71, 142, 306)	146.10 145.60 (8, 42, 101, 202, 437)	204.16 203.66 (11, 59, 142, 283, 611)	338.24 337.73 (18, 98, 235, 469, 1012)	505.29 504.77 (26, 146, 350, 700, 1513)	994.57 994.05 (52, 286, 690, 1379, 2978)	
45	146.29 145.78 (8, 42, 102, 203, 437)	204.64 204.14 (11, 59, 142, 283, 612)	340.00 339.50 (18, 98, 236, 471, 1018)	508.76 508.25 (27, 147, 353, 705, 1523)	1008.16 1007.64 (52, 290, 699, 1397, 3019)		
47	204.80 204.30 (11, 59, 142, 284, 613)	340.89 340.39 (18, 98, 236, 472, 1020)	510.75 510.25 (27, 147, 354, 708, 1529)	1016.04 1015.53 (53, 293, 704, 1408, 3043)			
49	341.33 340.83 (18, 99, 237, 473, 1022)	511.75 511.25 (27, 148, 355, 709, 1532)	1021.00 1020.50 (53, 294, 708, 1415, 3058)				
51	512.00 511.50 (27, 148, 355, 710, 1533)	1023.00 1022.50 (53, 295, 709, 1418, 3064)					
53	1024.00 1023.50 (53, 295, 710, 1419, 3067)						

Table 3.18 continued for *h* = 16, 18, ..., 28 and *k* = 1, 3, ..., 25.

k	***h***						
	16	**18**	**20**	**22**	**24**	**26**	**28**
1	4.03 3.33 (1, 2, 3, 5, 11)	4.41 3.67 (1, 2, 3, 6, 12)	4.83 4.03 (1, 2, 4, 6, 13)	5.27 4.41 (1, 2, 4, 7, 14)	5.76 4.82 (1, 2, 4, 8, 15)	6.28 5.26 (1, 3, 5, 8, 17)	6.83 5.72 (1, 3, 5, 9, 18)
3	4.61 3.91 (1, 2, 3, 6, 12)	5.09 4.34 (1, 2, 4, 7, 14)	5.60 4.80 (1, 2, 4, 7, 15)	6.18 5.31 (1, 2, 5, 8, 17)	6.80 5.86 (1, 3, 5, 9, 18)	7.47 6.44 (1, 3, 6, 10, 20)	8.19 7.07 (1, 3, 6, 11, 22)
5	5.33 4.64 (1, 2, 4, 7, 15)	5.93 5.18 (1, 2, 4, 8, 16)	6.59 5.79 (1, 2, 5, 9, 18)	7.34 6.47 (1, 3, 5, 10, 20)	8.14 7.19 (1, 3, 6, 11, 22)	9.03 7.99 (1, 3, 7, 12, 25)	10.00 8.87 (1, 4, 7, 13, 28)
7	6.23 5.54 (1, 2, 5, 8, 17)	6.99 6.25 (1, 3, 5, 9, 19)	7.86 7.06 (1, 3, 6, 11, 22)	8.82 7.95 (1, 3, 6, 12, 25)	9.90 8.95 (1, 4, 7, 13, 28)	11.10 10.06 (1, 4, 8, 15, 31)	12.43 11.30 (2, 4, 9, 17, 35)
9	7.36 6.68 (1, 3, 5, 10, 21)	8.37 7.63 (1, 3, 6, 11, 24)	9.49 8.69 (1, 3, 7, 13, 27)	10.77 9.90 (1, 4, 8, 15, 31)	12.23 11.28 (1, 4, 9, 17, 35)	13.88 12.85 (2, 5, 10, 19, 40)	15.83 14.69 (2, 5, 11, 22, 45)
11	8.83 8.15 (1, 3, 6, 12, 25)	10.12 9.38 (1, 3, 7, 14, 29)	11.62 10.83 (1, 4, 8, 16, 33)	13.36 12.50 (1, 4, 10, 18, 38)	15.38 14.44 (2, 5, 11, 21, 44)	17.82 16.78 (2, 6, 13, 24, 51)	20.61 19.47 (2, 7, 15, 28, 59)
13	10.69 10.01 (1, 4, 8, 15, 31)	12.41 11.69 (1, 4, 9, 17, 36)	14.45 13.67 (1, 5, 10, 20, 42)	16.85 16.00 (2, 5, 12, 23, 49)	19.85 18.91 (2, 6, 14, 27, 58)	23.36 22.32 (2, 7, 17, 32, 68)	27.32 26.19 (2, 9, 19, 37, 80)
15	13.13 12.46 (1, 4, 9, 18, 38)	15.46 14.74 (1, 5, 11, 21, 45)	18.26 17.48 (2, 6, 13, 25, 53)	21.85 21.00 (2, 7, 15, 30, 64)	26.15 25.21 (2, 8, 18, 36, 76)	31.11 30.09 (3, 10, 22, 43, 91)	37.23 36.10 (3, 12, 26, 51, 109)
17	16.36 15.70 (1, 5, 12, 22, 48)	19.54 18.84 (2, 6, 14, 27, 57)	23.74 22.97 (2, 7, 17, 33, 70)	28.88 28.03 (2, 9, 20, 40, 85)	34.92 34.00 (3, 11, 24, 48, 103)	42.59 41.57 (3, 13, 30, 59, 126)	52.23 51.12 (4, 16, 37, 72, 154)
19	20.67 20.02 (2, 6, 15, 28, 61)	25.45 24.74 (2, 8, 18, 35, 75)	31.43 30.67 (2, 10, 22, 43, 93)	38.60 37.77 (3, 12, 27, 53, 114)	47.93 47.02 (3, 14, 34, 66, 142)	60.03 59.03 (4, 18, 42, 83, 178)	75.33 74.23 (5, 22, 53, 104, 223)
21	26.93 26.28 (2, 8, 19, 37, 79)	33.72 33.02 (2, 10, 24, 46, 100)	41.99 41.23 (3, 13, 29, 58, 124)	53.02 52.20 (3, 16, 37, 73, 157)	67.73 66.84 (4, 20, 47, 94, 201)	86.89 85.90 (5, 26, 61, 120, 258)	110.04 108.97 (7, 32, 77, 152, 327)
23	35.69 35.04 (2, 11, 25, 49, 106)	44.98 44.29 (3, 13, 31, 62, 133)	57.66 57.66 (4, 17, 40, 80, 171)	75.02 74.21 (5, 22, 52, 104, 223)	98.22 97.34 (6, 29, 68, 136, 292)	126.80 125.85 (7, 37, 88, 175, 378)	175.50 174.41 (10, 51, 122, 243, 524)
25	47.50 46.87 (3, 14, 33, 66, 141)	61.69 61.02 (4, 18, 43, 85, 183)	81.56 80.84 (5, 24, 57, 113, 243)	108.78 107.99 (6, 32, 76, 150, 324)	142.88 142.04 (8, 42, 99, 198, 426)	204.49 203.53 (11, 60, 142, 283, 611)	275.50 274.45 (15, 80, 191, 382, 823)

Table 3.18 continued for h = 16, 18, ..., 28 and k = 27, 29, ..., 53.

k	***h***						
	16	**18**	**20**	**22**	**24**	**26**	**28**
27	65.03 64.41 (4, 19, 45, 90, 194)	87.16 86.50 (5, 26, 61, 121, 260)	118.11 117.40 (7, 34, 82, 163, 352)	157.47 156.72 (9, 46, 109, 218, 470)	232.48 231.62 (13, 67, 161, 322, 695)	320.44 319.51 (17, 93, 222, 444, 958)	500.06 498.96 (27, 145, 347, 693, 1496)
29	91.75 91.14 (5, 27, 64, 127, 274)	126.00 125.35 (7, 37, 88, 174, 376)	170.09 169.40 (9, 49, 118, 236, 508)	258.15 257.38 (14, 75, 179, 358, 772)	363.03 362.21 (19, 105, 252, 503, 1086)	594.47 593.50 (31, 172, 412, 824, 1779)	
31	132.34 131.74 (7, 38, 92, 183, 395)	180.44 179.81 (10, 52, 125, 250, 539)	280.37 279.68 (15, 81, 195, 388, 839)	400.92 400.19 (21, 116, 278, 556, 1200)	686.75 685.89 (36, 198, 476, 952, 2056)		
33	188.52 187.93 (10, 55, 131, 261, 564)	298.55 297.92 (16, 86, 207, 414, 893)	432.84 432.18 (23, 125, 300, 600, 1295)	770.90 770.13 (40, 222, 535, 1068, 2308)			
35	312.77 312.18 (17, 90, 217, 433, 936)	458.19 457.58 (24, 132, 318, 635, 1371)	824.79 842.11 (44, 243, 584, 1168, 2523)				
37	476.93 476.36 (25, 138, 331, 661, 1428)	899.99 899.36 (47, 259, 624, 1247, 2695)					
39	942.81 942.23 (49, 272, 654, 1307, 2823)						
41							
43							
45							
47							
49							
51							
53							

Table 3.18 continued for $h = 30, 32, ..., 42$ and $k = 1, 3, ..., 53$.

k	**h**						
	30	**32**	**34**	**36**	**38**	**40**	**42**
1	7.41 6.21 (1, 3, 6, 10, 20)	8.03 6.72 (1, 3, 6, 11, 21)	8.68 7.26 (2, 4, 7, 12, 23)	9.38 7.84 (2, 4, 7, 12, 25)	10.10 8.44 (2, 4, 8, 13, 27)	10.84 9.05 (2, 4, 8, 14, 29)	11.61 9.70 (2, 5, 9, 15, 31)
3	8.79 7.75 (1, 3, 7, 12, 24)	9.80 8.47 (2, 4, 7, 13, 27)	10.71 9.26 (2, 4, 8, 14, 29)	11.66 10.09 (2, 5, 9, 16, 32)	12.65 10.96 (2, 5, 9, 17, 34)	13.71 11.89 (2, 5, 10, 18, 37)	14.82 12.87 (2, 6, 11, 20, 40)
5	11.05 9.82 (2, 4, 8, 15, 31)	12.24 10.89 (2, 5, 9, 16, 34)	13.50 12.04 (2, 5, 10, 18, 37)	14.84 13.25 (2, 5, 11, 20, 41)	16.28 14.57 (2, 6, 12, 22, 45)	17.84 15.99 (3, 7, 13, 24, 50)	19.49 17.50 (3, 7, 14, 26, 54)
7	13.96 12.71 (2, 5, 10, 19, 39)	15.63 14.26 (2, 6, 11, 21, 44)	17.41 15.93 (2, 6, 13, 24, 49)	19.39 17.79 (2, 7, 14, 26, 55)	21.57 19.83 (3, 7, 16, 29, 61)	23.92 22.04 (3, 8, 17, 32, 68)	26.43 24.42 (3, 9, 19, 36, 75)
9	18.01 16.76 (2, 6, 13, 24, 51)	20.38 19.01 (2, 7, 15, 28, 58)	23.07 21.58 (3, 8, 16, 31, 66)	26.10 24.48 (3, 9, 19, 36, 75)	29.44 27.69 (3, 10, 21, 40, 85)	33.06 31.17 (3, 11, 24, 45, 95)	37.30 35.26 (4, 12, 27, 51, 108)
11	23.71 22.46 (2, 8, 17, 32, 69)	27.32 25.95 (3, 9, 19, 37, 79)	31.48 29.99 (3, 10, 22, 43, 91)	36.20 34.57 (3, 12, 26, 50, 105)	41.38 39.62 (4, 13, 29, 57, 120)	47.70 45.79 (4, 15, 34, 65, 139)	54.39 52.34 (5, 17, 38, 75, 159)
13	32.08 30.83 (3, 10, 23, 44, 94)	37.72 36.35 (3, 12, 27, 52, 110)	44.27 42.77 (4, 14, 31, 61, 130)	51.58 49.97 (4, 16, 36, 71, 151)	60.93 59.16 (5, 19, 43, 84, 179)	70.90 69.00 (5, 22, 50, 98, 209)	83.34 81.28 (6, 25, 58, 115, 246)
15	44.70 43.46 (3, 14, 31, 61, 131)	53.62 52.26 (4, 16, 38, 74, 158)	63.77 62.30 (5, 19, 45, 88, 188)	77.41 75.78 (6, 23, 54, 107, 229)	92.09 90.34 (6, 28, 64, 127, 272)	111.35 109.43 (8, 33, 78, 154, 330)	
17	64.08 62.85 (4, 19, 45, 88, 190)	77.84 76.51 (5, 23, 54, 107, 231)	97.39 95.89 (6, 29, 68, 134, 289)	118.64 117.02 (8, 35, 83, 164, 352)	148.17 146.38 (9, 44, 103, 205, 440)		
19	93.47 92.27 (6, 28, 65, 129, 278)	120.80 119.44 (7, 36, 84, 167, 359)	150.87 149.40 (9, 44, 105, 209, 449)	195.52 193.87 (12, 57, 136, 270, 582)			
21	147.14 145.92 (9, 43, 102, 204, 438)	188.54 187.22 (11, 55, 131, 261, 562)	254.71 253.20 (14, 74 , 177, 353, 760)				
23	230.72 229.53 (13, 67, 160, 319, 689)	326.23 324.86 (18, 95, 227, 452, 975)					
25	409.08 407.84 (22, 119, 284, 567, 1223)						
27							
29							
⋮							
53							

Table 3.18 continued for h = 44, 46, ..., 54 and k = 1, 3, ..., 53.

k	**h**					
	44	**46**	**48**	**50**	**52**	**54**
1	12.42 10.37 (2, 5, 9, 16, 33)	13.25 11.06 (2, 5, 10, 18, 35)	14.09 11.78 (3, 6, 11, 19, 38)	14.99 12.53 (3, 6, 11, 20, 40)	15.90 13.30 (3, 7, 12, 21, 42)	16.85 14.10 (3, 7, 13, 22, 45)
3	15.99 13.90 (3, 6, 12, 21, 44)	17.19 14.96 (3, 7, 13, 23, 47)	18.49 16.11 (3, 7, 14, 25, 51)	19.82 17.30 (3, 8, 15, 27, 54)	21.23 18.56 (3, 8, 16, 28, 58)	
5	21.22 19.10 (3, 8, 15, 29, 59)	23.14 20.86 (3, 8, 17, 31, 65)	25.10 22.69 (3, 9, 18, 34, 70)	27.25 24.67 (4, 10, 20, 37, 76)		
7	29.27 27.11 (3, 10, 21, 40, 83)	32.22 29.92 (4, 11, 23, 44, 92)	35.53 33.07 (4, 12, 25, 48, 101)			
9	41.75 39.57 (4, 14, 30, 57, 121)	46.88 44.55 (5, 15, 33, 64, 136)				
11	62.38 60.18 (5, 20, 44, 86, 182)					
13						
15						
⋮						
53						

Recall that the reason why there are so many open cells is because the values of h is taken to satisfy $h \leq \frac{n(n+1)}{2} - k$. For example, for $k = 11$ the reference value h is taken to be smaller than or equal to 44, since $\frac{n(n+1)}{2} - k = \frac{10(10+1)}{2} - 11 = 55 - 11 = 44$.

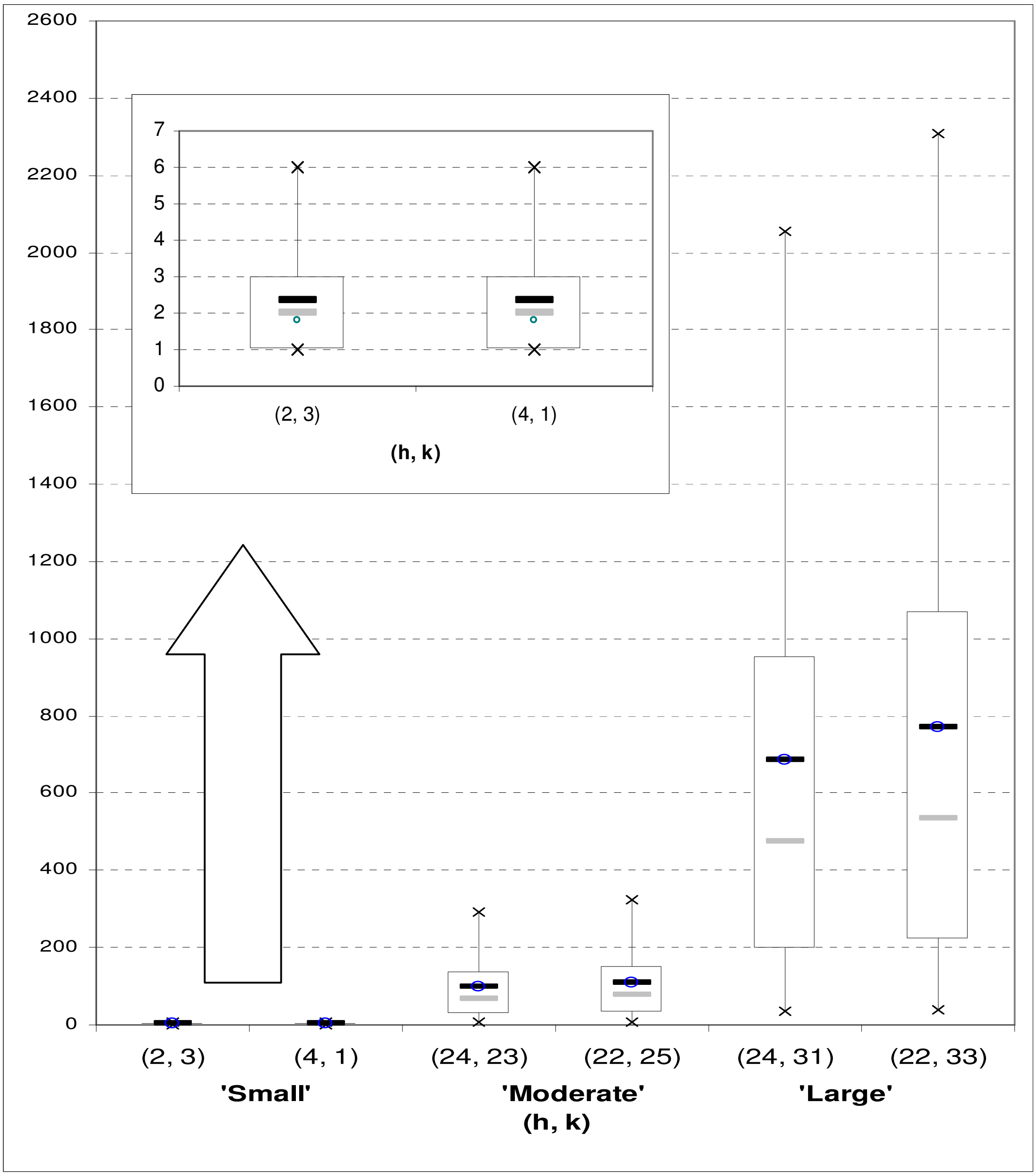

Figure 3.5. Boxplot-like graphs for the in-control run length distribution of various upper one-sided CUSUM signed-rank charts when $n = 10$. The whiskers extend to the 5^{th} and the 95^{th} percentiles. The symbols "▬", "◦" and "▬" denote the *ARL*, *SDRL** and *MRL*, respectively†.

* For ease of interpretation, the standard deviation (as measure of spread) is included in the (location) measures of percentiles.

† The "boxplots" are classified into 3 categories, namely small ($h + k \le 25$), moderate ($25 < h + k \le 50$) and large ($h + k > 50$).

Example 3.5

An upper one-sided CUSUM signed-rank chart for the Montgomery (2001) piston ring data

We conclude this sub-section by illustrating the upper one-sided CUSUM signed-rank chart using the Montgomery (2001) piston ring data. Recall that the dataset contains 15 samples (each of size 5). For illustration take $k = 3$ and $h = 8$. From Table 3.16 it can be seen that the in-control average run length equals 8.13 when $(h,k) = (8,3)$. Generally, one chooses the chart constants so that a specified in-control average run length, such as 500, or 370, is obtained. Taking this into consideration, an in-control average run length of 8.13 is considered small. Recall that unless the sample size n is 10 or more, the signed-rank chart is somewhat unattractive (from an operational point of view) in SPC applications. The plotting statistics for the Shewhart signed-rank chart (SR_i for $i = 1,2,...,15$) are given in the second row of Table 3.19. The upper one-sided CUSUM plotting statistics (S_i^+ for $i = 1,2,...,15$) are given in the last row of Table 3.19.

Table 3.19. SR_i and S_i^+ values for the piston ring data in Montgomery (2001)*.

Sample No:	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
$\boldsymbol{SR_i}$	8	4	-14	7	-3	9	10	-6	12	14	4	15	15	15	14
$\boldsymbol{S_i^+}$	5	6	0	4	0	6	13	4	13	24	25	37	49	61	72

To illustrate the calculations, consider sample number 1. The equation for the plotting statistic is $S_1^+ = \max[0, S_0^+ + SR_1 - k] = \max[0, 0 + 8 - 3] = \max[0,5] = 5$ where a signaling event occurs for the first i such that $S_i^+ \geq h$, that is, $S_i^+ \geq 8$. The graphical display of the upper one-sided CUSUM signed-rank chart is shown in Figure 3.6.

* The values in Table 3.19 were generated using Microsoft Excel.

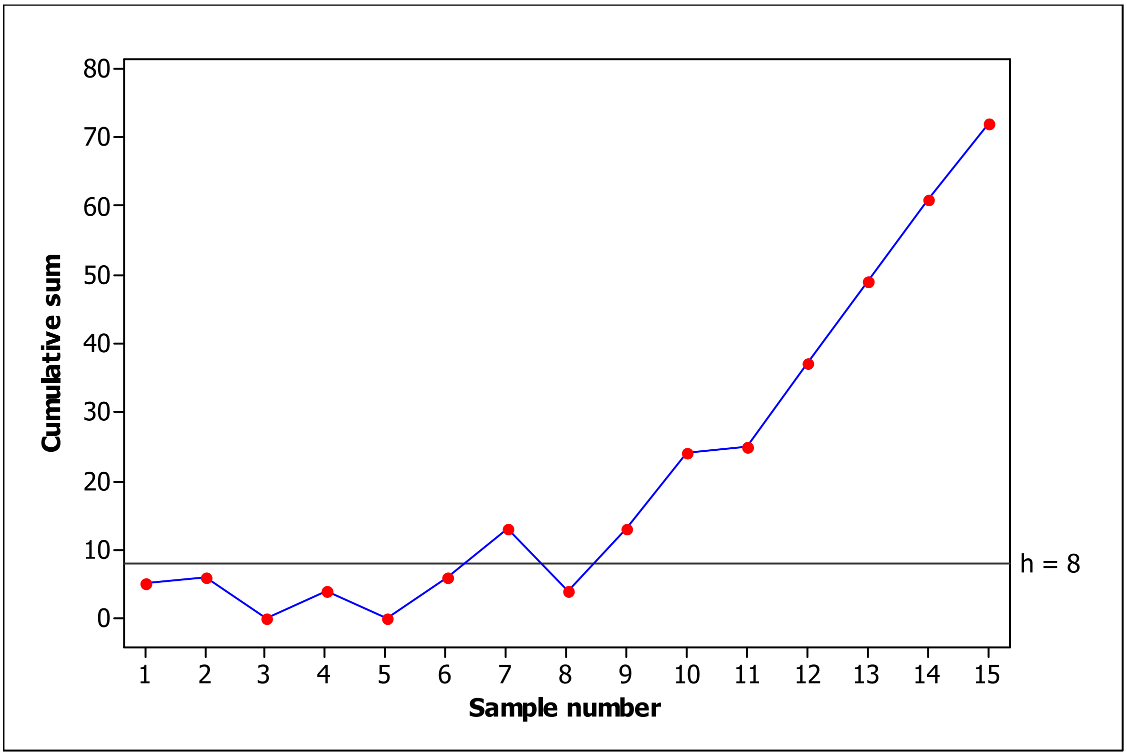

Figure 3.6. The upper one-sided CUSUM signed-rank chart for the Montgomery (2001) piston ring data.

The upper one-sided CUSUM signed-rank chart signals at sample 7, indicating a most likely positive deviation from the known target value θ_0. The action taken following an out-of-control signal on a CUSUM chart is identical to that with any control chart. A search for assignable causes should be done, corrective action should be taken (if required) and, following this, the CUSUM is reset to zero.

3.3.2.2. Lower one-sided control charts

The time that the procedure signals is the first time such that the finite-state Markov chain S_t^- enters the state ς_0 where the state space is given by $\Omega^- = \{\varsigma_0, \varsigma_1, ..., \varsigma_{r+s-1}\}$ with $-h = \varsigma_0 < ... < \varsigma_{r+s-1} = 0$, $S_0^- = 0$ and

$$S_t^- = \max\{-h, \min\{0, S_{t-1}^- + SR_t + k\}\}. \quad (3.4)$$

Example 3.6

A lower one-sided CUSUM signed-rank chart where the sample size is even (*n*=4)

The statistical properties of a lower one-sided CUSUM signed-rank chart with a decision interval of 6 ($h = 6$), a reference value of 2 ($k = 2$) and a sample size of 4 ($n = 4$) is examined. For n even, the reference value is taken to be even, because this leads to the sum $\sum(SR_i - k)$ being equal to even values which reduces the size of the state space for the Markov chain. For $h = 6$ we have $\Omega^- = -h(2)0 = \{-6,-4,-2,0\}$. The state space is calculated using equation (3.4) and the calculations are shown in Table 3.20.

Table 3.20. Calculation of the state space when $h = 6$, $k = 2$ and $n = 4$.

SR_t	$S_{t-1}^- + SR_t + k$	$\min\{0, S_{t-1}^- + SR_t + k\}$	$S_t^- = \max\{-h, \min\{0, S_{t-1}^- + SR_t + k\}\}$
-10	-8*	-8	-6
-8	-6	-6	-6
-6	-4	-4	-4
-4	-2	-2	-2
-2	0	0	0
0	2	0	0
2	4	0	0
4	6	0	0
6	8	0	0
8	10	0	0
10	12	0	0

Table 3.21. Classification of the states.

State number	Description of the state	Absorbent (A)/ Non-absorbent (NA)
0	$S_t^+ = 0$	NA
1	$S_t^+ = -2$	NA
2	$S_t^+ = -4$	NA
3	$S_t^+ = -6$	A

* Note: Since only the state space needs to be described, S_{t-1}^- can be any value from Ω^- and we therefore take, without loss of generality, $S_{t-1}^- = 0$. Any other possible value for S_{t-1}^- would lead to the same Ω^-.

From Table 3.21 we see that there are three non-absorbent states, i.e. $r=3$, and one absorbent state, i.e. $s=1$. Therefore, the corresponding TPM will be a $(r+s)\times(r+s)=4\times4$ matrix. It can be shown (see Table 3.22) that the TPM is given by

$$TPM_{4\times4}=\begin{pmatrix} p_{00} & p_{0(-2)} & p_{0(-4)} & p_{0(-6)} \\ p_{(-2)0} & p_{(-2)(-2)} & p_{(-2)(-4)} & p_{(-2)(-6)} \\ p_{(-4)0} & p_{(-4)(-2)} & p_{(-4)(-4)} & p_{(-4)(-6)} \\ p_{(-6)0} & p_{(-6)(-2)} & p_{(-6)(-4)} & p_{(-6)(-6)} \end{pmatrix}=\begin{pmatrix} 11/16 & 2/16 & 1/16 & | & 2/16 \\ 9/16 & 2/16 & 2/16 & | & 3/16 \\ 7/16 & 2/16 & 2/16 & | & 5/16 \\ - & - & - & - & - \\ 0 & 0 & 0 & | & 1 \end{pmatrix}=\begin{pmatrix} Q_{3\times3} & | & \underline{p}_{3\times1} \\ - & - & - \\ \underline{0}'_{1\times3} & | & 1_{1\times1} \end{pmatrix}$$

where the essential transition probability sub-matrix $Q_{3\times3}:(NA\rightarrow NA)$ is an $r\times r=3\times3$ matrix, $\underline{p}_{3\times1}:(NA\rightarrow A)$ is an $(r+s-1)\times1=3\times1$ column vector, $\underline{0}'_{1\times3}:(A\rightarrow NA)$ is a $1\times(r+s-1)=1\times3$ row vector and $1_{1\times1}:(A\rightarrow A)$ represents the scalar value one.

The one-step transition probabilities are calculated by substituting SR_t in expression (3.4) by $2T^+-\frac{n(n+1)}{2}$ and substituting values for h, k, S_t^- and S_{t-1}^-. The calculation of the one-step transition probabilities are given for illustration in Table 3.22.

Recall that the probabilities in the last column of the TPM are calculated using the fact that $\sum_{j\in\Omega}p_{ij}=1 \quad \forall i$ (see equation (2.18)). Therefore,

$p_{0(-6)}=1-(p_{00}+p_{0(-2)}+p_{0(-4)})=1-(11/16+2/16+1/16)=2/16$;

$p_{(-2)(-6)}=1-(p_{(-2)0}+p_{(-2)(-2)}+p_{(-2)(-4)})=1-(9/16+2/16+2/16)=3/16$;

$p_{(-4)(-6)}=1-(p_{(-4)0}+p_{(-4)(-2)}+p_{(-4)(-4)})=1-(7/16+2/16+2/16)=5/16$;

$p_{(-6)(-6)}=1-(p_{(-6)0}+p_{(-6)2}+p_{(-6)4})=1-(0+0+0)=1$.

Table 3.22. The calculation of the transition probabilities when $h = 6$, $k = 2$ and $n = 4$.

p_{00} $= P(S_t = 0 \mid S_{t-1} = 0)$ $= P(\max\{-6, \min\{0, 0 + SR_t + 2\}\} = 0)$ $= P(\min\{0, 0 + SR_t + 2\} = 0)$ $= P(SR_t + 2 \geq 0)$ $= P(SR_t \geq -2)$ $= P(2T^+ - 10 \geq -2)$ $= P(T^+ \geq 4)$ $= 1 - P(T^+ \leq 3)$ $= \frac{11}{16}$	$p_{0(-2)}$ $= P(S_t = -2 \mid S_{t-1} = 0)$ $= P(\max\{-6, \min\{0, 0 + SR_t + 2\}\} = -2)$ $= P(\min\{0, SR_t + 2\} = -2)$ $= P(SR_t + 2 = -2)$ $= P(SR_t = -4)$ $= P(2T^+ - 10 = -4)$ $= P(T^+ = 3)$ $= \frac{2}{16}$	$p_{0(-4)}$ $= P(S_t = -4 \mid S_{t-1} = 0)$ $= P(\max\{-6, \min\{0, 0 + SR_t + 2\}\} = -4)$ $= P(\min\{0, SR_t + 2\} = -4)$ $= P(SR_t + 2 = -4)$ $= P(SR_t = -6)$ $= P(2T^+ - 10 = -6)$ $= P(T^+ = 2)$ $= \frac{1}{16}$
$p_{(-2)0}$ $= P(S_t = 0 \mid S_{t-1} = -2)$ $= P(\max\{-6, \min\{0, -2 + SR_t + 2\}\} = 0)$ $= P(\min\{0, -2 + SR_t + 2\} = 0)$ $= P(SR_t \geq 0)$ $== P(2T^+ - 10 \geq 0)$ $= P(T^+ \geq 5)$ $= 1 - P(T^+ \leq 4)$ $= \frac{9}{16}$	$p_{(-2)(-2)}$ $= P(S_t = -2 \mid S_{t-1} = -2)$ $= P(\max\{-6, \min\{0, -2 + SR_t + 2\}\} = -2)$ $= P(\min\{0, SR_t\} = -2)$ $= P(SR_t = -2)$ $== P(2T^+ - 10 = -2)$ $= P(T^+ = 4)$ $= \frac{2}{16}$	$p_{(-2)(-4)}$ $= P(S_t = -4 \mid S_{t-1} = -2)$ $= P(\max\{-6, \min\{0, -2 + SR_t + 2\}\} = -4)$ $= P(\min\{0, SR_t\} = -4)$ $= P(SR_t = -4)$ $= P(2T^+ - 10 = -4)$ $= P(T^+ = 3)$ $= \frac{2}{16}$
$p_{(-4)0}$ $= P(S_t = 0 \mid S_{t-1} = -4)$ $= P(\max\{-6, \min\{0, -4 + SR_t + 2\}\} = 0)$ $= P(\min\{0, -4 + SR_t + 2\} = 0)$ $= P(SR_t \geq 2)$ $== P(2T^+ - 10 \geq 2)$ $= P(T^+ \geq 6)$ $= 1 - P(T^+ \leq 5)$ $= \frac{7}{16}$	$p_{(-4)(-2)}$ $= P(S_t = -2 \mid S_{t-1} = -4)$ $= P(\max\{-6, \min\{0, -4 + SR_t + 2\}\} = -2)$ $= P(\min\{0, SR_t - 2\} = -2)$ $= P(SR_t - 2 = -2)$ $== P(2T^+ - 10 = 0)$ $= P(T^+ = 5)$ $= \frac{2}{16}$	$p_{(-4)(-4)}$ $= P(S_t = -4 \mid S_{t-1} = -4)$ $= P(\max\{-6, \min\{0, -4 + SR_t + 2\}\} = -4)$ $= P(\min\{0, SR_t - 2\} = -4)$ $= P(SR_t - 2 = -4)$ $= P(SR_t = -2)$ $= P(2T^+ - 10 = -2)$ $= P(T^+ = 4)$ $= \frac{2}{16}$
$p_{(-6)0}$ $= P(S_t = 0 \mid S_{t-1} = -6)$ $= 0$ *	$p_{(-6)(-2)}$ $= P(S_t = -2 \mid S_{t-1} = -6)$ $= 0$	$p_{(-6)(-4)}$ $= P(S_t = -4 \mid S_{t-1} = -6)$ $= 0$

* The probability equals zero, because it is impossible to go from an absorbent state to a non-absorbent state.

The formulas of the moments and some characteristics of the run length distribution have been studied by Fu, Spiring and Xie (2002) and Fu and Lou (2003) – see equations (2.41) to (2.45). By substituting $\underline{\xi}_{1\times 3} = (1 \quad 0 \quad 0)$, $Q_{3\times 3} = \begin{pmatrix} 11/16 & 2/16 & 1/16 \\ 9/16 & 2/16 & 2/16 \\ 7/16 & 2/16 & 2/16 \end{pmatrix}$ and $\underline{1}_{3\times 1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ into these equations, we obtain the following:

$$ARL = E(N) = \underline{\xi}(I - Q)^{-1}\underline{1} = 6.81$$

$$E(N^2) = \underline{\xi}(I + Q)(I - Q)^{-2}\underline{1} = 83.64$$

$$SDRL = \sqrt{Var(N)} = \sqrt{E(N^2) - (E(N))^2} = 6.11$$

5^{th} percentile $= \rho_5 = 1$

25^{th} percentile $= \rho_{25} = 2$

Median $= 50^{th}$ percentile $= \rho_{50} = 5$

75^{th} percentile $= \rho_{75} = 9$

95^{th} percentile $= \rho_{95} = 19$

The in-control average run length (ARL_0^-) values, standard deviation of the run length ($SDRL$) values and percentiles for the lower one-sided CUSUM signed-rank chart are exactly the same as for the upper one-sided CUSUM signed-rank chart, since the one-step transition probabilities matrices are the same. Therefore, the in-control average run length (ARL_0), standard deviation of the run length ($SDRL$), 5^{th}, 25^{th}, 50^{th}, 75^{th} and 95^{th} percentile values for the upper one-sided CUSUM signed-rank chart will also hold for the lower one-sided CUSUM signed-rank chart.

Example 3.7

A lower one-sided CUSUM signed-rank chart for the Montgomery (2001) piston ring data

We conclude this sub-section by illustrating the lower one-sided CUSUM signed-rank chart using the Montgomery (2001) piston ring data. Recall that the dataset contains 15 samples (each of size 5). For illustration take $k = 3$ and $h = 8$. From Table 3.16 it can be seen that the in-control average run length equals 8.13 when $(h,k) = (8,3)$. Generally, one chooses the chart constants so that a specified in-control average run length, such as 500, or 370, is obtained. Taking this into consideration, an in-control average run length of 8.13 is considered small. Recall that unless the sample size n is 10 or more, the signed-rank chart is somewhat unattractive (from an operational point of view) in SPC applications.

The plotting statistics for the Shewhart signed-rank chart (SR_i for $i = 1,2,...,15$) are given in the second row of Table 3.23. The lower one-sided CUSUM plotting statistics (S_i^- for $i = 1,2,...,15$) are given in the last row of Table 3.23.

Table 3.23. SR_i and S_i^- values for the piston ring data in Montgomery (2001)*.

Sample No:	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
$\boldsymbol{SR_i}$	8	4	-14	7	-3	9	10	-6	12	14	4	15	15	15	14
$\boldsymbol{S_i^-}$	0	0	-11	-1	-1	0	0	-3	0	0	0	0	0	0	0

To illustrate the calculations, consider sample number 1. The equation for the plotting statistic S_1^- is

$$S_1^{-*} = \max[0, S_0^{-*} - SR_1 - k] = \max[0, 0 - 8 - 3] = \max[0, -11] = 0 \qquad (3.5)$$

or

$$S_1^- = \min[0, S_0^- + SR_1 + k] = \min[0, 0 + 8 + 3] = \min[0, 11] = 0 \qquad (3.6)$$

* The values in Table 3.23 were generated using Microsoft Excel.

A signaling event occurs for the first i such that $S_i^{-*} \geq h$, that is, $S_i^{-*} \geq 8$ if expression (3.5) is used or $S_i^{-} \leq -h$, that is, $S_i^{-} \leq -8$ if expression (3.6) is used. The graphical display of the lower one-sided CUSUM signed-rank chart is shown in Figure 3.7.

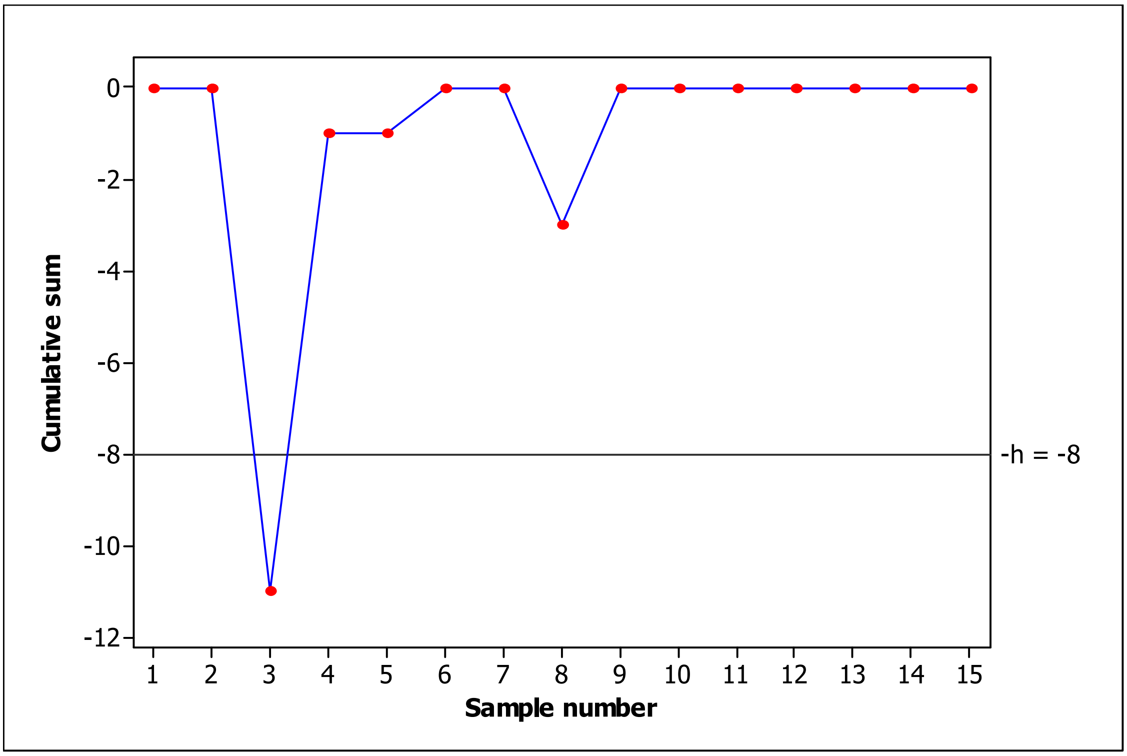

Figure 3.7. The lower one-sided CUSUM signed-rank chart for the Montgomery (2001) piston-ring data.

The lower one-sided CUSUM signed-rank chart signals at sample 3. Recall that the lower one-sided CUSUM sign chart did not signal at all. This emphasizes the fact that the signed-rank test is more powerful than the sign test. The question arises: Why not always use the signed-rank test if it is more powerful than the sign test? The sign test is applicable for all continuous distributions, while the assumption of symmetry must be made, in addition, for the signed-rank test. Also, the sign test applies to all percentiles while the signed-rank test is proposed only for the median.

3.3.3. Two-sided control charts

Recall that for the upper one-sided CUSUM signed-rank chart we use

$$S_t^+ = \min\{h, \max\{0, SR_t - k + S_{t-1}^+\}\} \text{ for } t = 1,2,... \quad (3.7)$$

For a lower one-sided CUSUM signed-rank chart we use

$$S_t^- = \max\{-h, \min\{0, SR_t + k + S_{t-1}^-\}\} \text{ for } t = 1,2,... \quad (3.8)$$

For the two-sided scheme the two one-sided schemes are performed simultaneously. The corresponding two-sided CUSUM chart signals for the first n at which either one of the two inequalities is satisfied, that is, either $S_t^+ \geq h$ or $S_t^- \leq -h$. Starting values are typically chosen to equal zero, that is, $S_0^+ = S_0^- = 0$. The two-sided scheme signals at N where

$$N = \min_t\{t : S_t^+ \geq h \quad \text{or} \quad S_t^- \leq -h\} \quad (3.9)$$

where h is a positive integer.

The two-sided CUSUM scheme can be represented by a Markov chain with states corresponding to the possible combinations of values of S_t^+ and S_t^-. The states corresponding to values for which a signal is given by the CUSUM scheme are called absorbing states. Clearly, there are two absorbing states ($s = 2$) since the chart signals when S_t^+ falls on or above h or when S_t^- falls on or below $-h$. The probability of going from an absorbing state to the same absorbing state is equal to one, because once an absorbing state is entered, it is never left. The transient states are the remaining states for which eventual return is uncertain. Let r denote the number of remaining states, i.e. r denotes the number of transient (non-absorbing) states. Clearly, in total there are $r + s$ states and therefore the corresponding TPM will be an $(r+s)\times(r+s)$ matrix.

The time that the procedure signals is the first time such that the finite-state Markov chain enters the state ς_0 or ς_{r+s-1} where the state space is given by $\Omega = \Omega^+ \cup \Omega^- = \{\varsigma_0, \varsigma_1, ..., \varsigma_{r+s-1}\}$ with $-h = \varsigma_0 < ... < \varsigma_{r+s-1} = h$.

Example 3.8

A two-sided CUSUM signed-rank chart where the sample size is even (*n*=4)

The statistical properties of a two-sided CUSUM signed-rank chart with a decision interval of 4 ($h=4$), a reference value of 2 ($k=2$) and a sample size of 4 ($n=4$) is examined. Let Ω denote the state space for the two-sided chart. Ω is the union of the state space for the upper one-sided chart $\Omega^+=\{0,2,4\}$ and the state space for the lower one-sided chart $\Omega^-=\{-4,-2,0\}$. Therefore, $\Omega = \Omega^+ \cup \Omega^- = \{-4,-2,0\} \cup \{0,2,4\} = \{-4,-2,0,2,4\} = \{\varsigma_0,\varsigma_1,\varsigma_2,\varsigma_3,\varsigma_4\}$ with $-h=\varsigma_0<\varsigma_1<\varsigma_2<\varsigma_3<\varsigma_4=h$.

Table 3.24. Classification of the states.

State number	Values of the CUSUM statistic(s)	Absorbing (A) Non-absorbing (NA)
0	$S_t^-=0$ and $S_t^+=0$	NA
1	$S_t^-=2$ or $S_t^+=2$ *	NA
2	$S_t^-=-2$ or $S_t^+=-2$ †	NA
3	$S_t^-=4$ or $S_t^+=4$ ‡	A
4	$S_t^-=-4$ or $S_t^+=-4$ §	A

From Table 3.24 we see that there are three non-absorbing states, i.e. $r=3$, and two absorbing states, i.e. $s=2$. Therefore the corresponding TPM will be a (5×5) matrix. The layout of the TPM is as follows. There are three transient states and two absorbing states. By

* Moving from state 0 to state 1 can happen when either the upper cumulative sum or the lower cumulative sum equals 2. But the lower cumulative sum can not equal 2 since by definition the lower cumulative sum can only take on integer values smaller than or equal to zero. Therefore, we only use the probability that the upper cumulative sum equals 2 in the calculation of the probabilities in the TPM.

† Moving from state 0 to state 2 can happen when either the upper cumulative sum or the lower cumulative sum equals -2. But the upper cumulative sum can not equal -2 since by definition the upper cumulative sum can only take on integer values greater than or equal to zero. Therefore, we only use the probability that the lower cumulative sum equals -2 in the calculation of the probabilities in the TPM.

‡ A similar argument to the argument in the first footnote on this page holds. Therefore, we only use the probability that the upper cumulative sum equals 4 in the calculation of the probabilities in the TPM.

§ A similar argument to the argument in the second footnote on this page holds. Therefore, we only use the probability that the lower cumulative sum equals -4 in the calculation of the probabilities in the TPM.

convention we first list the non-absorbing states and then we list the absorbing states. In column one we compute the probability of moving from state i to state 0, for all i. Note that the process reaches state 0 when both the upper and the lower cumulative sums equal zero. In columns two and three, we compute the probabilities of moving from state i to the remaining non-absorbing states, for all i. Finally, in the remaining two columns we compute the probabilities of moving from state i to the absorbing states, for all i. Thus, the TPM can be conveniently partitioned into 4 sections given by

$$TPM_{5\times 5} = \begin{pmatrix} p_{00} & p_{01} & p_{02} & p_{03} & p_{04} \\ p_{10} & p_{11} & p_{12} & p_{13} & p_{14} \\ p_{20} & p_{21} & p_{22} & p_{23} & p_{24} \\ p_{30} & p_{31} & p_{32} & p_{33} & p_{34} \\ p_{40} & p_{41} & p_{42} & p_{43} & p_{44} \end{pmatrix} = \begin{pmatrix} 6/16 & 2/16 & 2/16 & \mid & 3/16 & 3/16 \\ 4/16 & 2/16 & 2/16 & \mid & 5/16 & 3/16 \\ 4/16 & 2/16 & 2/16 & \mid & 3/16 & 5/16 \\ - & - & - & - & - & - \\ 0 & 0 & 0 & \mid & 1 & 0 \\ 0 & 0 & 0 & \mid & 0 & 1 \end{pmatrix} = \begin{pmatrix} Q_{3\times 3} & \mid & C_{3\times 2} \\ - & - & - \\ Z_{2\times 3} & \mid & I_{2\times 2} \end{pmatrix}$$

where $Q_{3\times 3} : (NA \to NA)$ is an $r \times r = 3\times 3$ matrix, $C_{3\times 2} : (NA \to A)$ is an $r \times s = 3\times 2$ matrix, $Z_{2\times 3} : (A \to NA)$ is an $s \times r = 2\times 3$ matrix and $I_{2\times 2} : (A \to A)$ is an $s \times s = 2\times 2$ matrix .

The calculation of the elements of the TPM is illustrated next. Note that this essentially involves the calculation of the matrices Q and C. First consider the transient states. Note that the process moves to state 0, i.e., $j = 0$, when both the upper *and* the lower cumulative sums equal 0. Thus the required probability of moving to 0, from any other transient state, is the probability of an *intersection* of two sets involving values of the upper and the lower CUSUM statistics, respectively. On the other hand, the probability of moving to any state $j \neq 0$, from any other state, is the probability of a *union* of two sets involving values of the upper and the lower CUSUM statistics, respectively. However, one of these two sets is empty so that the required probability is the probability of only the non-empty set.

The calculation of the entry in the first row and the first column of the TPM, p_{00}, will be discussed in detail. This is the probability of moving from state 0 to state 0 in one step at time *t*. As we just described, this can happen only when the upper *and* the lower cumulative sums both equal 0 at time t. For the upper one-sided CUSUM p_{00} is the probability that the upper CUSUM equals 0 at time t, given that the upper CUSUM equaled 0 at time $t-1$, that is, $P(S_t^+ = 0 \mid S_{t-1}^+ = 0)$. For the lower one-sided procedure p_{00} is the probability that the lower CUSUM equals 0 at time t, given that the lower CUSUM equaled 0 at time $t-1$, that is, $P(S_t^- = 0 \mid S_{t-1}^- = 0)$. For the two-sided procedure the two one-sided procedures are performed simultaneously. Therefore we have that $p_{00} = P(\{S_t^+ = 0 \mid S_{t-1}^+ = 0\} \cap \{S_t^- = 0 \mid S_{t-1}^- = 0\})$. We have that

$$p_{00}$$
$$= P(\{S_t^+ = 0 \mid S_{t-1}^+ = 0\} \cap \{S_t^- = 0 \mid S_{t-1}^- = 0\})$$

this is computed by substituting in values for h, k, S_t^+, S_{t-1}^+, S_t^- and S_{t-1}^- into (3.7) and (3.8)

$$= P(\{\min\{4, \max\{0, SR_t - 2 + 0\}\} = 0\} \cap \{\max\{-4, \min\{0, SR_t + 2 + 0\}\} = 0\})$$
$$= P(\{\max\{0, SR_t - 2 + 0\} = 0\} \cap \{\min\{0, SR_t + 2 + 0\} = 0\})$$
$$= P((SR_t - 2 \le 0) \cap (SR_t + 2 + 0 \ge 0))$$
$$= P((SR_t \le 2) \cap (SR_t \ge -2))$$

recall that $SR_t = 2T^+ - \frac{n(n+1)}{2}$ where T^+ is the Wilcoxon signed-rank statistic

$$= P((2T^+ - 10 \le 2) \cap (2T^+ - 10 \ge -2))$$
$$= P((T^+ \le 6) \cap (T^+ \ge 4))$$
$$= P(T^+ = 4) + P(T^+ = 5) + P(T^+ = 6)$$
$$= 6/16.$$

The remaining entries of the TPM can be calculated similarly. In doing so, we find that

$$TPM = \begin{pmatrix} 6/16 & 2/16 & 2/16 & 3/16 & 3/16 \\ 4/16 & 2/16 & 2/16 & 5/16 & 3/16 \\ 4/16 & 2/16 & 2/16 & 3/16 & 5/16 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Using the TPM the ARL can be calculated using $ARL = \underline{\xi}(I - Q)^{-1}\underline{1}$. A well-known concern is that important information about the performance of a control chart can be missed when only examining the ARL (this is especially true when the process distribution is skewed). Various authors, see for example, Radson and Boyd (2005) and Chakraborti (2007), have suggested that one should examine a number of percentiles, including the median, to get the complete information about the performance of a control chart. Therefore, we now also consider percentiles. The calculation of these percentiles is shown in Table 3.25 for illustration purposes. The first column of Table 3.25 contains the values that the run length variable (N) can take on.

Table 3.25. Calculation of the percentiles when $h = 4$, $k = 2$ and $n = 4$ *.

N	$P(N \leq l)$	The 5th, 25th, 50th, 75th and 95th percentiles
1	0.3750000	$\rho_{0.25} = 1$ (smallest integer such that cdf is at least 0.05 and 0.25)
2	0.6406250	$\rho_{0.5} = 2$ (smallest integer such that cdf is at least 0.50)
3	0.7949219	$\rho_{0.75} = 3$ (smallest integer such that cdf is at least 0.75)
4	0.8830566	
5	0.9333191	
6	0.9619789	$\rho_{0.95} = 6$ (smallest integer such that cdf is at least 0.95)
7†	0.9783206	

The formulas of the moments and some characteristics of the run length distribution have been studied by Fu, Spiring and Xie (2002) and Fu and Lou (2003) – see equations (2.41) to (2.45). By substituting $\underline{\xi}_{1\times 3} = (1 \quad 0 \quad 0)$, $Q_{3\times 3} = \frac{1}{16}\begin{pmatrix} 6/16 & 2/16 & 2/16 \\ 4/16 & 2/16 & 2/16 \\ 4/16 & 2/16 & 2/16 \end{pmatrix}$ and $\underline{1}_{3\times 1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ into these equations, we obtain the following:

$$ARL = E(N) = \underline{\xi}(I - Q)^{-1}\underline{1} = 2.46$$

$$E(N^2) = \underline{\xi}(I + Q)(I - Q)^{-2}\underline{1} = 9.28$$

$$SDRL = \sqrt{Var(N)} = \sqrt{E(N^2) - (E(N))^2} = 1.79$$

$$5^{th} \text{ percentile} = p_{0.05} = 1$$

$$25^{th} \text{ percentile} = p_{0.25} = 1$$

$$\text{Median} = 50^{th} \text{ percentile} = p_{0.5} = 2$$

* See SAS Program 7 in Appendix B for the calculation of the values in Table 3.25.
† The value of the run length variable is only shown up to $N = 7$ for illustration purposes.

75^{th} percentile $= p_{0.75} = 3$

95^{th} percentile $= p_{0.95} = 6$

Other values of *h*, *k* and *n* were also considered and the results are given in Table 3.26.

Table 3.26. The in-control average run length (ARL_0), standard deviation of the run length ($SDRL$), $5^{th}, 25^{th}, 50^{th}, 75^{th}$ and 95^{th} percentile values* for samples of size $n = 4$ and various values of *h* and *k* for the two-sided CUSUM signed-rank chart†.

k	***h***				
	2	**4**	**6**	**8**	**10**
0	1.14 0.40 (1, 1, 1, 1, 2)	1.52 0.81 (1, 1, 1, 2, 3)	2.14 1.51 (1, 1, 1, 3, 5)	2.74 2.19 (1, 1, 2, 3, 7)	3.62 2.80 (1, 1, 2, 4, 10)
2	1.60 0.98 (1, 1, 1, 2, 4)	2.46 1.79 (1, 1, 2, 3, 6)	3.41 2.66 (1, 1, 2, 4, 9)	5.09 4.07 (1, 2, 4, 7, 13)	
4	2.67 2.11 (1, 1, 2, 3, 7)	3.87 3.29 (1, 1, 3, 5, 10)	6.64 5.92 (1, 2, 5, 9, 18)		
6	4.00 3.46 (1, 1, 3, 5, 11)	7.53 6.95 (1, 3, 5, 10, 21)			
8	8.00 7.45 (1, 2, 5, 11, 23)				

Values of k and h are restricted to be integers so that the Markov chain approach could be employed to obtain exact values for the average run length. In order to allow for the possibility of stopping after one group, the values of h is taken to satisfy $h \le \frac{n(n+1)}{2} - k$. For example, for $n = 4$ and $k = 0$, the reference value h is taken to be smaller than or equal to 10, since $\frac{n(n+1)}{2} - k = \frac{4(4+1)}{2} - 0 = 10$. The five percentiles are displayed in boxplot-like graphs in Figure 3.8 for all the (h,k)-combinations that are shaded in Table 3.26. The "boxplots" are

* The three rows of each cell shows the ARL_0, the $SDRL$, and the percentiles (ρ_5, ρ_{25}, ρ_{50}, ρ_{75}, ρ_{95}), respectively.

† See SAS Program 7 in Appendix B for the calculation of the values in Table 3.26.

classified into 3 categories, namely small ($h+k \leq 4$), moderate ($5 \leq h+k \leq 8$) and large ($h+k \geq 9$).

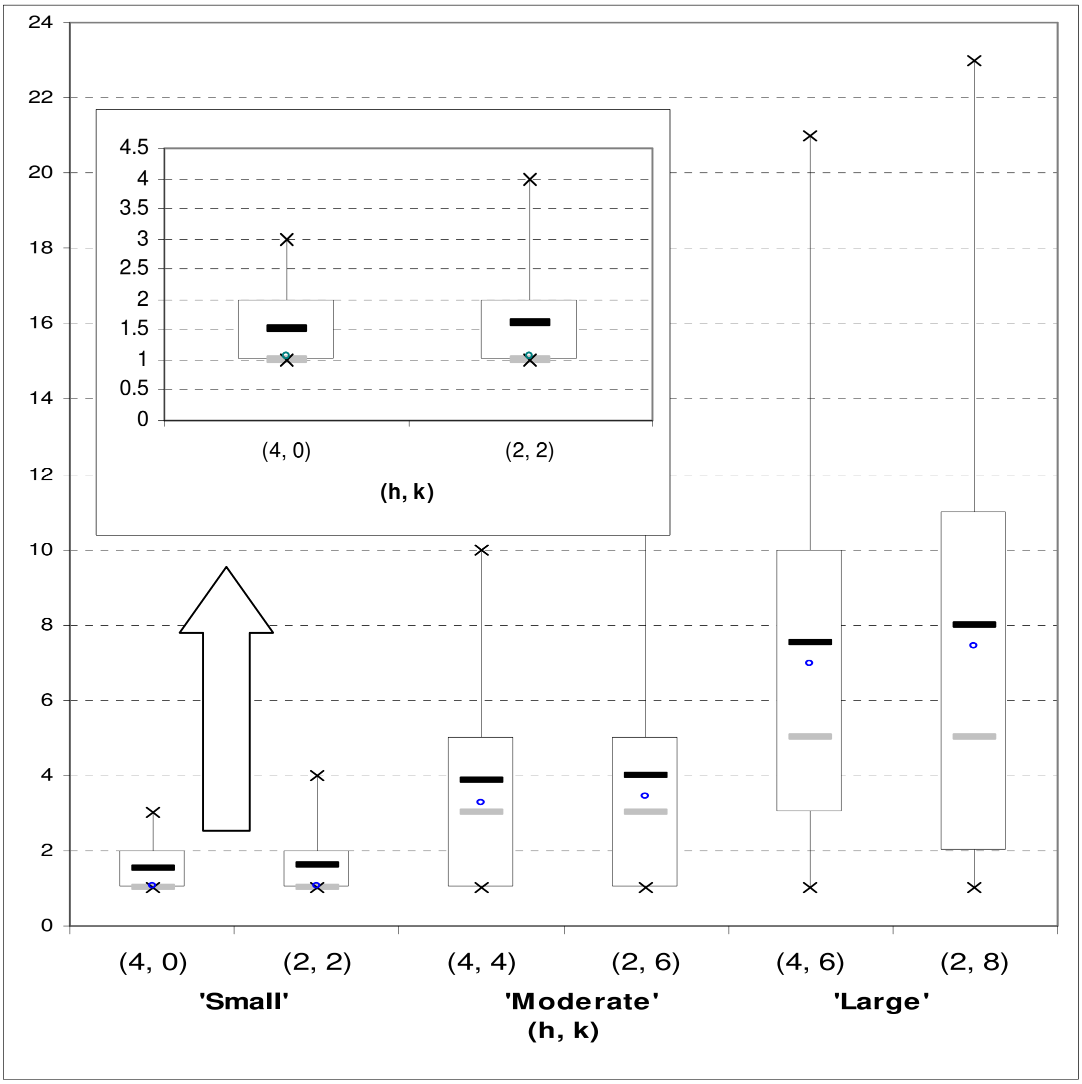

Figure 3.8. Boxplot-like graphs for the in-control run length distribution of various two-sided CUSUM signed-rank charts when $n = 4$. The whiskers extend to the 5th and the 95th percentiles. The symbols "▬", "○" and "▬" denote the *ARL*, *SDRL** and *MRL*, respectively.

* For ease of interpretation, the standard deviation (as measure of spread) is included in the (location) measures of percentiles.

Table 3.27. The in-control average run length (ARL_0), standard deviation of the run length ($SDRL$), $5^{th}, 25^{th}, 50^{th}, 75^{th}$ and 95^{th} percentile values[*] for samples of size $n = 5$ and various values of h and k for the two-sided CUSUM signed-rank chart[†].

k	**h**						
	2	**4**	**6**	**8**	**10**	**12**	**14**
1	1.23 0.53 (1, 1, 1, 1, 2)	1.56 0.88 (1, 1, 1, 2, 3)	2.04 1.30 (1, 1, 2, 3, 5)	2.57 1.87 (1, 1, 2, 3, 7)	3.36 2.97 (1, 1, 3, 5, 11)	4.15 3.46 (1, 1, 3, 5, 11)	5.23 4.38 (1, 2, 4, 7, 14)
3	1.60 0.98 (1, 1, 1, 2, 4)	2.20 1.57 (1, 1, 2, 3, 5)	2.90 2.22 (1, 1, 2, 4, 7)	4.07 3.23 (1, 2, 3, 5, 10)	5.34 4.36 (1, 2, 4, 7, 14)	7.39 6.61 (1, 2, 5, 10, 21)	
5	2.29 1.71 (1, 1, 2, 3, 6)	3.12 2.54 (1, 1, 2, 4, 8)	4.72 4.05 (1, 2, 3, 6, 13)	6.52 5.77 (1, 2, 5, 9, 18)	10.09 9.11 (1, 4, 7, 14, 28)		
7	3.20 2.65 (1, 1, 2, 4, 8)	5.12 4.55 (1, 2, 4, 7, 14)	7.39 6.78 (1, 3, 5, 10, 21)	12.58 11.83 (1, 4, 9, 17, 36)			
9	5.33 4.81 (1, 2, 4, 7, 15)	7.87 7.34 (1, 3, 6, 11, 23)	14.57 13.97 (1, 5, 10, 20, 42)				
11	8.00 7.48 (1, 3, 6, 11, 23)	15.52 14.98 (1, 5, 11, 21, 45)					
13	16.00 15.49 (1, 5, 11, 22, 47)						

The five percentiles are displayed in boxplot-like graphs in Figure 3.9 for all the (h, k)-combinations that are shaded in Table 3.27. The "boxplots" are classified into 3 categories, namely small ($h + k \leq 5$), moderate ($6 \leq h + k \leq 10$) and large ($h + k \geq 11$).

[*] The three rows of each cell shows the ARL_0, the $SDRL$, and the percentiles (ρ_5, ρ_{25}, ρ_{50}, ρ_{75}, ρ_{95}), respectively.

[†] See SAS Program 7 in Appendix B for the calculation of the values in Table 3.27.

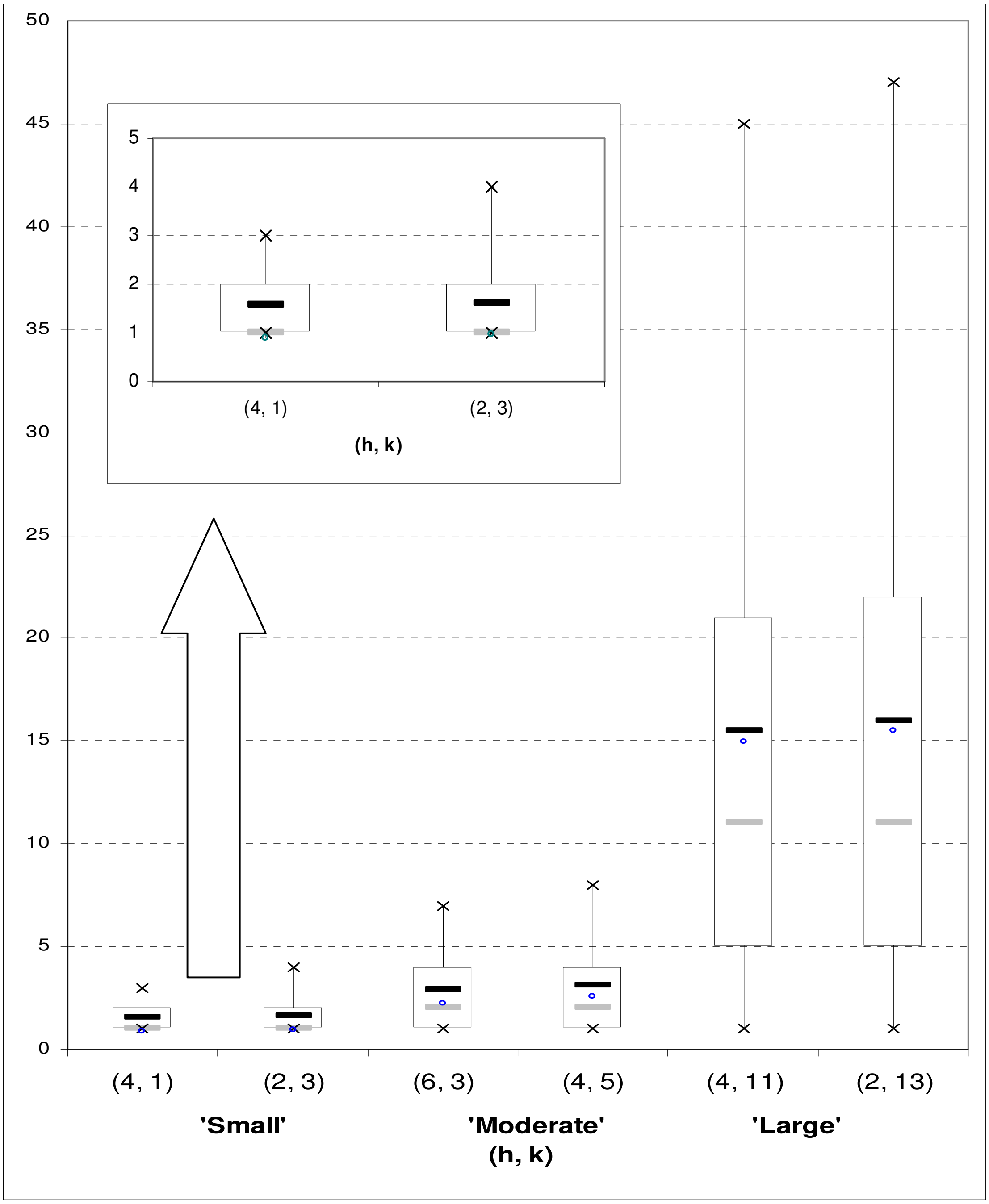

Figure 3.9. Boxplot-like graphs for the in-control run length distribution of various two-sided CUSUM signed-rank charts when $n = 5$. The whiskers extend to the 5th and the 95th percentiles. The symbols "▬", "∘" and "▬" denote the *ARL*, *SDRL*[*] and *MRL*, respectively.

[*] For ease of interpretation, the standard deviation (as measure of spread) is included in the (location) measures of percentiles.

Table 3.28. The in-control average run length (ARL_0^+), standard deviation of the run length ($SDRL$), $5^{th}, 25^{th}, 50^{th}, 75^{th}$ and 95^{th} percentile values* for samples of size $n = 6$ and various values of h and k for the two-sided CUSUM signed-rank chart†.

k	***h*** 2	4	6	8	10	12	14	16	18
1	1.19 0.47 (1, 1, 1, 1, 2)	1.43 0.75 (1, 1, 1, 2, 3)	1.69 1.01 (1, 1, 1, 2, 4)	2.08 1.37 (1, 1, 1, 2, 5)	2.52 2.10 (1, 1, 2, 3, 6)	3.05 2.48 (1, 1, 2, 4, 8)	3.62 3.09 (1, 1, 2, 5, 10)	4.36 3.32 (1, 1, 3, 6, 11)	5.11 4.24 (1, 2, 4, 7, 14)
3	1.45 0.81 (1, 1, 1, 2, 3)	1.75 1.13 (1, 1, 1, 2, 4)	2.17 1.52 (1, 1, 2, 3, 5)	2.77 2.04 (1, 1, 2, 4, 7)	3.50 2.65 (1, 2, 3, 5, 9)	4.32 3.36 (1, 1, 3, 6, 12)	5.49 4.72 (1, 2, 4, 8, 15)	6.72 5.96 (1, 2, 5, 9, 19)	8.39 7.49 (1, 3, 6, 11, 24)
5	1.78 1.18 (1, 1, 1, 2,4)	2.25 1.65 (1, 1, 1, 2, 3, 6)	2.98 2.34 (1, 1, 2, 4, 8)	3.91 3.20 (1, 2, 3, 5, 10)	5.01 4.21 (1, 2, 4, 7, 13)	6.77 5.79 (1, 3, 5, 9, 18)	8.69 7.56 (1, 3, 6, 12, 24)	11.72 10.54 (1, 5, 11, 22, 47)	
7	2.29 1.71 (1, 1, 2, 3, 6)	3.12 2.54 (1, 1, 2, 4, 8)	4.25 3.63 (1, 2, 3, 6, 11)	5.63 4.96 (1, 2, 4, 8, 16)	8.09 7.28 (1, 3, 6, 11, 23)	10.90 9.98 (1, 4, 8, 15, 31)	16.01 14.85 (2, 5, 11, 22, 46)		
9	3.20 2.65 (1, 1, 2, 4, 8)	4.48 3.93 (1, 2, 3, 6, 12)	6.08 5.50 (1, 2, 4, 8, 17)	9.24 8.57 (1, 3, 7, 13, 26)	12.95 12.21 (1, 4, 9, 18, 37)	20.78 19.85 (2, 7, 15, 28, 60)			
11	4.57 4.04 (1, 2, 3, 6, 13)	6.32 5.78 (1, 2, 5, 9, 18)	10.02 9.44 (1, 3, 7, 14, 29)	14.44 13.82 (1, 5, 10, 20, 24)	25.13 24.38 (2, 8, 18, 35, 74)				
13	6.40 5.88 (1, 2, 5, 9, 18)	10.45 9.91 (1, 3, 7, 14, 30)	15.38 14.83 (1, 5, 11, 21, 45)	28.31 27.68 (2, 9, 20, 39, 84)					
15	10.67 10.15 (1, 3, 8, 15, 31)	15.87 15.36 (1, 5, 11, 22, 47)	30.54 29.99 (2, 9, 21, 42, 90)						
17	16.00 15.49 (1, 5, 11, 22, 47)	31.51 30.99 (2, 9, 22, 43, 93)							
19	32.00 31.50 (2, 10, 22, 44, 95)								

* The three rows of each cell shows the ARL_0, the $SDRL$, and the percentiles (ρ_5 , ρ_{25} , ρ_{50} , ρ_{75} , ρ_{95}) , respectively.

† See SAS Program 7 in Appendix B for the calculation of the values in Table 3.28.

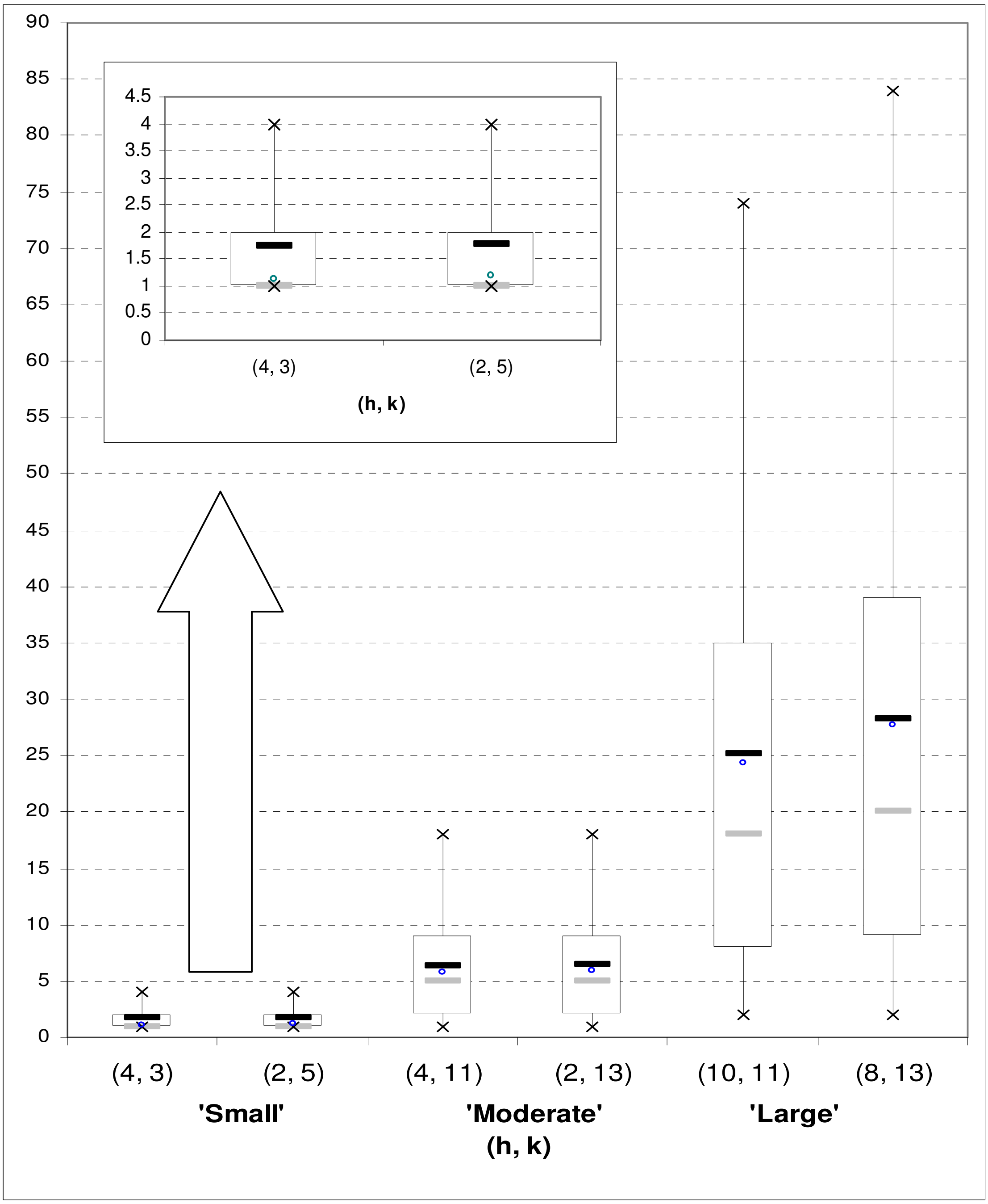

Figure 3.10. Boxplot-like graphs for the in-control run length distribution of various two-sided CUSUM signed-rank charts when $n = 6$. The whiskers extend to the 5th and the 95th percentiles. The symbols "▬", "∘" and "▬" denote the *ARL*, *SDRL* and *MRL*, respectively*.

* The "boxplots" are classified into 3 categories, namely small ($h + k \le 7$), moderate ($8 \le h + k \le 16$) and large ($h + k \ge 17$).

Table 3.29. The in-control average run length (ARL_0), standard deviation of the run length ($SDRL$), 5^{th}, 25^{th}, 50^{th}, 75^{th} and 95^{th} percentile values* for samples of size $n = 10$ for h = 2,4,...,14 and k = 1,3,...,23 for the two-sided CUSUM signed-rank chart†.

k	**h**						
	2	**4**	**6**	**8**	**10**	**12**	**14**
1	1.09 0.80 (1, 1, 1, 1, 2)	1.18 0.89 (1, 1, 1, 1, 3)	1.29 1.00 (1, 1, 1, 1, 3)	1.41 1.11 (1, 1, 1, 2, 3)	1.54 1.24 (1, 1, 1, 2, 4)	1.68 1.37 (1, 1, 1, 2, 4)	1.84 1.51 (1, 1, 1, 2, 5)
3	1.18 0.90 (1, 1, 1, 1, 3)	1.30 1.01 (1, 1, 1, 1, 3)	1.42 1.14 (1, 1, 1, 2, 3)	1.56 1.27 (1, 1, 1, 2, 4)	1.72 1.42 (1, 1, 1, 2, 4)	1.90 1.59 (1, 1, 1, 2, 5)	2.09 1.76 (1, 1, 1, 3, 5)
5	1.30 1.02 (1, 1, 1, 1, 3)	1.44 1.16 (1, 1, 1, 2, 3)	1.58 1.30 (1, 1, 1, 2, 4)	1.75 1.47 (1, 1, 1, 2, 4)	1.94 1.65 (1, 1, 1, 2, 5)	2.16 1.85 (1, 1, 1, 3, 6)	2.40 2.07 (1, 1, 2, 3, 6)
7	1.44 1.16 (1, 1, 1, 2, 4)	1.60 1.32 (1, 1, 1, 2, 4)	1.78 1.50 (1, 1, 1, 2, 5)	1.98 1.70 (1, 1, 1, 2, 5)	2.21 1.92 (1, 1, 1, 3, 6)	2.48 2.17 (1, 1, 2, 3, 7)	2.78 2.46 (1, 1, 2, 3, 7)
9	1.60 1.33 (1, 1, 1, 2, 4)	1.79 1.52 (1, 1, 1, 2, 5)	2.01 1.73 (1, 1, 1, 2, 5)	2.26 1.97 (1, 1, 1, 3, 6)	2.54 2.25 (1, 1, 2, 3, 7)	2.88 2.57 (1, 1, 2, 4, 8)	3.25 2.93 (1, 1, 2, 4, 9)
11	1.80 1.53 (1, 1, 1, 2, 5)	2.03 1.76 (1, 1, 1, 2, 5)	2.29 2.02 (1, 1, 1, 3, 6)	2.60 2.32 (1, 1, 2, 3, 7)	2.96 2.67 (1, 1, 2, 4, 8)	3.37 3.06 (1, 1, 2, 4, 9)	3.85 3.53 (1, 1, 3, 5, 11)
13	2.03 1.77 (1, 1, 1, 2, 5)	2.31 2.04 (1, 1, 1, 3, 6)	2.63 2.36 (1, 1, 2, 3, 7)	3.02 2.74 (1, 1, 2, 4, 8)	3.46 3.18 (1, 1, 2, 4, 10)	3.99 3.69 (1, 1, 3, 5, 11)	4.62 4.30 (1, 1, 3, 6, 13)
15	2.32 2.05 (1, 1, 1, 3, 6)	2.66 2.39 (1, 1, 2, 3, 7)	3.06 2.80 (1, 1, 2, 4, 8)	3.53 3.26 (1, 1, 2, 5, 10)	4.10 3.82 (1, 1, 3, 5, 11)	4.79 4.50 (1, 1, 3, 6, 14)	5.60 5.28 (1, 2, 4, 7, 16)
17	2.67 2.41 (1, 1, 2, 3, 7)	3.09 2.83 (1, 1, 2, 4, 8)	3.59 3.32 (1, 1, 2, 5, 10)	4.19 3.92 (1, 1, 3, 5, 12)	4.93 4.65 (1, 1, 3, 6, 14)	5.80 5.51 (1, 2, 4, 8, 17)	6.87 6.56 (1, 2, 5, 9, 20)
19	3.11 2.84 (1, 1, 2, 4, 9)	3.62 3.36 (1, 1, 2, 5, 10)	4.25 3.99 (1, 1, 3, 6, 12)	5.04 4.77 (1, 1, 3, 7, 14)	5.97 5.69 (1, 2, 4, 8, 17)	7.12 6.83 (1, 2, 5, 10, 21)	8.56 8.25 (1, 2, 6, 11, 25)
21	3.63 3.37 (1, 1, 2, 5, 10)	4.29 4.03 (1, 1, 3, 6, 12)	5.11 4.85 (1, 1, 3, 7, 15)	6.09 5.82 (1, 2, 4, 8, 17)	7.32 7.04 (1, 2, 50, 10, 21)	8.87 8.58 (1, 3, 6, 12, 26)	10.80 10.50 (1, 3, 7, 15, 32)
23	4.31 4.05 (1, 1, 3, 6, 12)	5.15 4.90 (1, 1, 3, 7, 15)	6.17 5.91 (1, 2, 4, 8, 18)	7.47 7.20 (1, 2, 5, 10, 22)	9.10 8.83 (1, 3, 6, 12, 26)	11.18 10.90 (1, 3, 8, 15, 33)	14.08 13.78 (1, 4, 10, 19, 41)

* The three rows of each cell shows the ARL_0^+, the $SDRL$, and the percentiles (ρ_5, ρ_{25}, ρ_{50}, ρ_{75}, ρ_{95}), respectively.

† See SAS Program 7 in Appendix B for the calculation of the values in Table 3.29.

Table 3.29 continued for $h = 2, 4, ..., 14$ and $k = 25, 27, ..., 53$.

k	h						
	2	**4**	**6**	**8**	**10**	**12**	**14**
25	5.17 4.92 (1, 1, 3, 7, 15)	6.22 5.97 (1, 2, 4, 8, 18)	7.56 7.30 (1, 2, 5, 10, 22)	9.28 9.01 (1, 3, 6, 13, 27)	11.47 11.19 (1, 3, 8, 16, 34)	14.57 14.29 (1, 4, 10, 20, 43)	18.65 18.35 (1, 5, 13, 25, 55)
27	6.25 5.99 (1, 2, 4, 8, 18)	7.62 7.36 (1, 2, 5, 10, 22)	9.39 9.13 (1, 3, 6, 13, 27)	11.67 11.40 (1, 3, 8, 16, 34)	14.94 14.67 (1, 4, 10, 20, 44)	19.28 19.00 (1, 5, 13, 26, 57)	24.76 24.47 (1, 7, 17, 34, 73)
29	7.64 7.39 (1, 2, 5, 10, 22)	9.46 9.20 (1, 3, 6, 13, 28)	11.80 11.54 (1, 3, 8, 16, 35)	15.20 14.94 (1, 4, 10, 21, 45)	19.75 19.48 (1, 6, 14, 27, 58)	25.55 25.27 (1, 7, 18, 35, 76)	33.84 33.55 (2, 10, 23, 47, 101)
31	9.48 8.97 (1, 3, 7, 13, 27)	11.87 11.36 (1, 4, 8, 16, 35)	15.37 14.85 (1, 5, 11, 21, 45)	20.08 19.56 (2, 6, 14, 28, 59)	26.12 25.58 (2, 8, 18, 36, 77)	34.85 34.29 (2, 10, 24, 48, 103)	47.67 47.38 (2, 14, 33, 66, 142)
33	11.91 11.40 (1, 4, 8, 16, 35)	15.47 14.96 (1, 5, 11, 21, 45)	20.30 19.79 (2, 6, 14, 28, 60)	26.51 25.99 (2, 8, 19, 37, 78)	35.57 35.04 (2, 11, 25, 49, 105)	49.00 48.45 (3, 14, 34, 68, 146)	68.60 68.32 (4, 20, 47, 95, 205)
35	15.52 15.01 (1, 5, 11, 21, 45)	20.43 19.92 (2, 6, 14, 28, 60)	26.76 26.25 (2, 8, 19, 37, 79)	36.06 35.54 (2, 11, 25, 50, 107)	49.95 49.42 (3, 15, 35, 69, 149)	70.38 69.83 (4, 21, 49, 97, 210)	97.26 96.98 (5, 28, 67, 134, 291)
37	20.48 19.97 (2, 6, 14, 28, 60)	26.90 26.39 (2, 8, 19, 37, 80)	36.35 35.85 (2, 11, 25, 50, 108)	50.56 50.04 (3, 15, 35, 70, 150)	71.58 71.05 (4, 21, 50, 99, 213)	99.34 98.80 (6, 29, 69, 138, 297)	161.57 161.29 (8, 46, 112, 224, 483)
39	26.95 26.44 (2, 8, 19, 37, 80)	36.51 36.00 (2, 11, 25, 50, 108)	50.92 50.41 (3, 15, 35, 70, 152)	72.34 71.83 (4, 21, 50, 100, 216)	100.71 100.19 (6, 29, 70, 139, 301)	165.16 164.62 (9, 48, 115, 229, 494)	245.13 244.86 (13, 70, 170, 339, 734)
41	36.57 36.07 (2, 11, 26, 51, 109)	51.12 50.62 (3, 15, 36, 71, 152)	72.81 72.30 (4, 21, 51, 101, 217)	101.58 101.07 (6, 30, 71, 141, 303)	167.59 167.07 (9, 49, 116, 232, 501)	249.70 249.18 (13, 72, 173, 346, 747)	486.87 486.60 (25, 140, 337, 675, 1458)
43	51.20 50.70 (3, 15, 36, 71, 152)	73.05 72.55 (4, 21, 51, 101, 218)	102.08 101.58 (6, 30, 71, 141, 305)	169.12 168.61 (9, 49, 117, 234, 506)	252.64 252.13 (13, 73, 175, 350, 756)	497.29 496.76 (26, 143, 345, 689, 1489)	
45	73.14 72.64 (4, 21, 51, 101, 218)	102.32 101.82 (6, 30, 71, 142, 306)	170.00 169.50 (9, 49, 118, 235, 508)	254.38 253.87 (14, 74, 176, 352, 761)	504.08 503.56 (26, 145, 350, 699, 1509)		
47	102.40 101.90 (6, 30, 71, 142, 306)	170.44 169.94 (9, 49, 118, 236, 510)	255.38 254.87 (14, 74, 177, 354, 764)	508.02 507.51 (27, 147, 352, 704, 1521)			
49	170.67 170.17 (9, 49, 118, 236, 510)	255.87 255.37 (14, 74, 178, 355, 766)	510.50 510.00 (27, 147, 354, 708, 1528)				
51	256.00 255.50 (14, 74, 178, 355, 766)	511.50 511.00 (27, 148, 355, 709, 1531)					
53	512.00 511.50 (27, 148, 355, 710, 1533)						

Table 3.29 continued for h = 16, 18, ..., 28 and k = 1, 3, ..., 25.

k	***h***						
	16	**18**	**20**	**22**	**24**	**26**	**28**
1	2.02 1.67 (1, 1, 1, 2, 5)	2.21 1.84 (1, 1, 1, 3, 6)	2.42 2.02 (1, 1, 2, 3, 6)	2.64 2.21 (1, 1, 2, 3, 7)	2.88 2.41 (1, 1, 2, 4, 7)	3.14 2.63 (1, 1, 2, 4, 8)	3.42 2.86 (1, 1, 2, 4, 9)
3	2.31 1.96 (1, 1, 1, 3, 6)	2.55 2.17 (1, 1, 2, 3, 7)	2.80 2.40 (1, 1, 2, 3, 7)	3.09 2.66 (1, 1, 2, 4, 8)	3.40 2.93 (1, 1, 2, 4, 9)	3.74 3.22 (1, 1, 3, 5, 10)	4.10 3.54 (1, 1, 3, 5, 11)
5	2.67 2.32 (1, 1, 2, 3, 7)	2.97 2.59 (1, 1, 2, 4, 8)	3.30 2.90 (1, 1, 2, 4, 9)	3.67 3.24 (1, 1, 2, 5, 10)	4.07 3.60 (1, 1, 3, 5, 11)	4.52 4.00 (1, 1, 3, 6, 12)	5.00 4.44 (1, 2, 3, 6, 14)
7	3.12 2.77 (1, 1, 2, 4, 8)	3.50 3.13 (1, 1, 2, 4, 9)	3.93 3.53 (1, 1, 3, 5, 11)	4.41 3.98 (1, 1, 3, 6, 12)	4.95 4.48 (1, 2, 3, 6, 14)	5.55 5.03 (1, 2, 4, 7, 15)	6.22 5.65 (1, 2, 4, 8, 17)
9	3.68 3.34 (1, 1, 2, 5, 10)	4.19 3.82 (1, 1, 3, 5, 12)	4.75 4.35 (1, 1, 3, 6, 13)	5.39 4.95 (1, 2, 4, 7, 15)	6.12 5.64 (1, 2, 4, 8, 17)	6.94 6.43 (1, 2, 5, 9, 20)	7.92 7.35 (1, 2, 5, 11, 22)
11	4.42 4.08 (1, 1, 3, 6, 12)	5.06 4.69 (1, 1, 3, 7, 14)	5.81 5.42 (1, 2, 4, 8, 16)	6.68 6.25 (1, 2, 5, 9, 19)	7.69 7.22 (1, 2, 5, 10, 22)	8.91 8.39 (1, 3, 6, 12, 25)	10.31 9.74 (1, 3, 7, 14, 29)
13	5.35 5.01 (1, 2, 4, 7, 15)	6.21 5.85 (1, 2, 4, 8, 18)	7.23 6.84 (1, 2, 5, 10, 21)	8.43 8.00 (1, 2, 6, 11, 24)	9.93 9.46 (1, 3, 7, 13, 29)	11.68 11.16 (1, 3, 8, 16, 34)	13.66 13.10 (1, 4, 9, 18, 40)
15	6.57 6.23 (1, 2, 4, 9, 19)	7.73 7.37 (1, 2, 5, 10, 22)	9.13 8.74 (1, 3, 6, 12, 26)	10.93 10.50 (1, 3, 7, 15, 32)	13.08 12.61 (1, 4, 9, 18, 38)	15.56 15.05 (1, 5, 11, 21, 45)	18.62 18.05 (1, 6, 13, 25, 54)
17	8.18 7.85 (1, 2, 6, 11, 24)	9.77 9.42 (1, 3, 7, 13, 28)	11.87 11.49 (1, 3, 8, 16, 35)	14.44 14.02 (1, 4, 10, 20, 42)	17.46 17.00 (1, 5, 12, 24, 51)	21.30 20.79 (1, 6, 15, 29, 63)	26.12 25.56 (2, 8, 18, 36, 77)
19	10.34 10.01 (1, 3, 7, 14, 30)	12.73 12.37 (1, 4, 9, 17, 37)	15.72 15.34 (1, 5, 11, 21, 46)	19.30 18.89 (1, 6, 13, 26, 57)	23.97 23.51 (1, 7, 17, 33, 71)	30.02 29.52 (2, 9, 21, 41, 89)	37.67 37.12 (2, 11, 26, 52, 111)
21	13.47 13.14 (1, 4, 9, 18, 39)	16.86 16.51 (1, 5, 12, 23, 50)	21.00 20.62 (1, 6, 14, 29, 62)	26.51 26.10 (1, 8, 18, 36, 78)	33.87 33.42 (2, 10, 23, 47, 100)	43.45 42.95 (2, 13, 30, 60, 129)	55.02 54.49 (3, 16, 38, 76, 163)
23	17.85 17.52 (1, 5, 12, 24, 53)	22.49 22.15 (1, 6, 15, 31, 66)	28.83 28.50 (2, 8, 20, 40, 85)	37.51 37.11 (2, 11, 26, 52, 111)	49.11 48.67 (3, 14, 34, 68, 146)	63.40 62.93 (3, 18, 44, 87, 189)	87.75 87.21 (5, 25, 61, 121, 262)
25	23.75 23.44 (1, 7, 16, 33, 70)	30.85 30.51 (2, 9, 21, 42, 91)	40.78 40.42 (2, 12, 28, 56, 121)	54.39 54.00 (3, 16, 38, 75, 162)	71.44 71.02 (4, 21, 49, 99, 213)	102.25 101.77 (5, 30, 71, 141, 305)	137.75 137.23 (7, 40, 95, 191, 411)

Table 3.29 continued for *h* = 16, 18, ..., 28 and *k* = 27, 29, ..., 53.

k	***h***						
	16	**18**	**20**	**22**	**24**	**26**	**28**
27	32.52 32.21 (2, 9, 22, 45, 97)	43.58 43.25 (2, 13, 30, 60, 130)	59.06 58.70 (3, 17, 41, 81, 176)	78.74 78.36 (4, 23, 54, 109, 235)	116.24 115.81 (6, 33, 80, 161, 347)	160.22 159.76 (8, 46, 111, 222, 479)	250.03 249.48 (13, 72, 173, 346, 748)
29	45.88 45.57 (2, 13, 32, 63, 137)	63.00 62.68 (3, 18, 44, 87, 188)	85.05 84.70 (4, 24, 59, 118, 254)	129.08 128.69 (7, 37, 79, 179, 386)	181.52 181.11 (9, 52, 126, 251, 543)	297.24 296.75 (15, 86, 206, 412, 889)	
31	66.17 65.87 (3, 19, 46, 91, 197)	90.22 89.91 (5, 26, 62, 125, 269)	140.19 139.84 (7, 40, 97, 194, 419)	200.46 200.10 (10, 58, 139, 278, 600)	343.38 342.95 (18, 99, 238, 476, 1028)		
33	94.26 93.97 (5, 27, 65, 130, 282)	149.28 148.96 (8, 43, 103, 207, 446)	216.42 216.09 (11, 62, 150, 300, 647)	385.45 385.06 (20, 111, 267, 534, 1154)			
35	156.39 156.09 (8, 45, 108, 216, 468)	229.10 228.79 (12, 66, 159, 317, 685)	412.40 421.06 (22, 121, 292, 584, 1261)				
37	238.47 238.18 (12, 69, 165, 330, 714)	450.00 449.68 (23, 129, 312, 623, 1347)					
39	471.21 471.12 (24, 136, 327, 653, 1411)						
41							
43							
45							
47							
49							
51							
53							

Table 3.29 continued for h = 30, 32, ..., 42 and k = 1, 3, ..., 53.

k	***h***						
	30	**32**	**34**	**36**	**38**	**40**	**42**
1	3.71 3.11 (1, 1, 3, 5, 10)	4.02 3.36 (1, 1, 3, 5, 10)	4.34 3.63 (1, 2, 3, 6, 11)	4.69 3.92 (1, 2, 3, 6, 12)	5.05 4.22 (1, 2, 4, 6, 13)	5.41 4.53 (1, 2, 4, 7, 14)	5.81 4.85 (1, 2, 4, 7, 15)
3	4.40 3.88 (1, 1, 3, 6, 12)	4.90 4.24 (1, 2, 3, 6, 13)	5.36 4.63 (1, 2, 4, 7, 14)	5.83 5.05 (1, 2, 4, 8, 16)	6.33 5.48 (1, 2, 4, 8, 17)	6.86 5.95 (1, 2, 5, 9, 18)	7.41 6.44 (1, 3, 5, 10, 20)
5	5.53 4.91 (1, 2, 4, 7, 15)	6.12 5.45 (1, 2, 4, 8, 17)	6.75 6.02 (1, 2, 5, 9, 18)	7.42 6.63 (1, 2, 5, 10, 20)	8.14 7.29 (1, 3, 6, 11, 22)	8.92 8.00 (1, 3, 6, 12, 25)	9.75 8.75 (1, 3, 7, 13, 27)
7	6.98 6.36 (1, 2, 5, 9, 19)	7.82 7.13 (1, 3, 5, 10, 22)	8.71 7.97 (1, 3, 6, 12, 24)	9.70 8.90 (1, 3, 7, 13, 27)	10.79 9.92 (1, 3, 8, 14, 30)	11.96 11.02 (1, 4, 8, 16, 34)	13.22 12.21 (1, 4, 9, 18, 37)
9	9.01 8.38 (1, 3, 6, 12, 25)	10.19 9.51 (1, 3, 7, 14, 29)	11.54 10.79 (1, 4, 8, 15, 33)	13.05 12.24 (1, 4, 9, 18, 37)	14.72 13.85 (1, 5, 10, 20, 42)	16.53 15.59 (1, 5, 12, 22, 47)	18.65 17.63 (2, 6, 13, 25, 54)
11	11.86 11.23 (1, 4, 8, 16, 34)	13.66 12.98 (1, 4, 9, 18, 39)	15.74 15.00 (1, 5, 11, 21, 45)	18.10 17.29 (1, 6, 13, 25, 52)	20.69 19.81 (2, 6, 14, 28, 60)	23.85 22.90 (2, 7, 17, 32, 69)	27.20 26.17 (2, 8, 19, 37, 79)
13	16.04 15.42 (1, 5, 11, 22, 47)	18.86 18.18 (1, 6, 13, 26, 55)	22.14 21.29 (2, 7, 15, 30, 65)	25.79 24.99 (2, 8, 18, 35, 75)	30.47 29.58 (2, 9, 21, 42, 89)	35.45 34.50 (2, 11, 25, 49, 104)	41.67 40.64 (3, 12, 29, 57, 123)
15	22.35 21.73 (1, 7, 15, 30, 65)	26.81 26.13 (2, 8, 19, 37, 79)	31.89 31.15 (2, 9, 22, 44, 94)	38.71 37.89 (3, 11, 27, 53, 114)	46.05 45.17 (3, 14, 32, 63, 136)	55.68 54.71 (4, 16, 39, 77, 165)	
17	32.04 31.43 (2, 9, 22, 44, 95)	38.92 38.26 (2, 11, 27, 53, 115)	48.70 47.95 (3, 14, 34, 67, 144)	59.32 58.51 (4, 17, 41, 82, 176)	74.09 73.19 (4, 22, 51, 102, 220)		
19	46.74 46.14 (3, 14, 32, 64, 139)	60.40 59.72 (3, 18, 42, 83, 179)	75.44 74.70 (4, 22, 52, 104, 224)	97.76 96.94 (6, 28, 68, 135, 291)			
21	73.57 72.96 (4, 21, 51, 102, 219)	94.27 93.61 (5, 27, 65, 130, 281)	127.36 126.60 (7, 37 , 88, 176, 380)				
23	115.36 114.77 (6, 33, 80, 159, 344)	163.12 162.43 (9, 47, 113, 226, 487)					
25	204.54 203.92 (11, 59, 142, 283, 611)						
27							
29							
⋮							
53							

Table 3.29 continued for $h = 44, 46, ..., 54$ and $k = 1, 3, ..., 53$.

k	***h***					
	44	**46**	**48**	**50**	**52**	**54**
1	6.21 5.19 (1, 2, 4, 8, 16)	6.63 5.53 (1, 2, 5, 9, 17)	7.05 5.89 (1, 3, 5, 9, 19)	7.50 6.27 (1, 3, 5, 10, 20)	7.95 6.65 (1, 3, 6, 10, 21)	8.43 7.05 (1, 3, 6, 11, 22)
3	8.00 6.95 (1, 3, 6, 10, 22)	8.60 7.48 (1, 3, 6, 11, 23)	9.25 8.06 (1, 3, 7, 12, 25)	9.91 8.65 (1, 4, 7, 13, 27)	10.62 9.28 (1, 4, 8, 14, 29)	
5	10.61 9.55 (1, 4, 7, 14, 29)	11.57 10.43 (1, 4, 8, 15, 32)	12.55 11.35 (1, 4, 9, 17, 35)	13.63 12.34 (2, 5, 10, 18, 38)		
7	14.64 13.56 (1, 5, 10, 20, 41)	16.11 14.96 (2, 5, 11, 22, 46)	17.77 16.54 (2, 6, 12, 24, 50)			
9	20.88 19.79 (2, 7, 15, 28, 60)	33.44 22.28 (2, 7, 16, 32, 68)				
11	31.19 30.09 (2, 10, 22, 43, 91)					
13						
15						
⋮						
53						

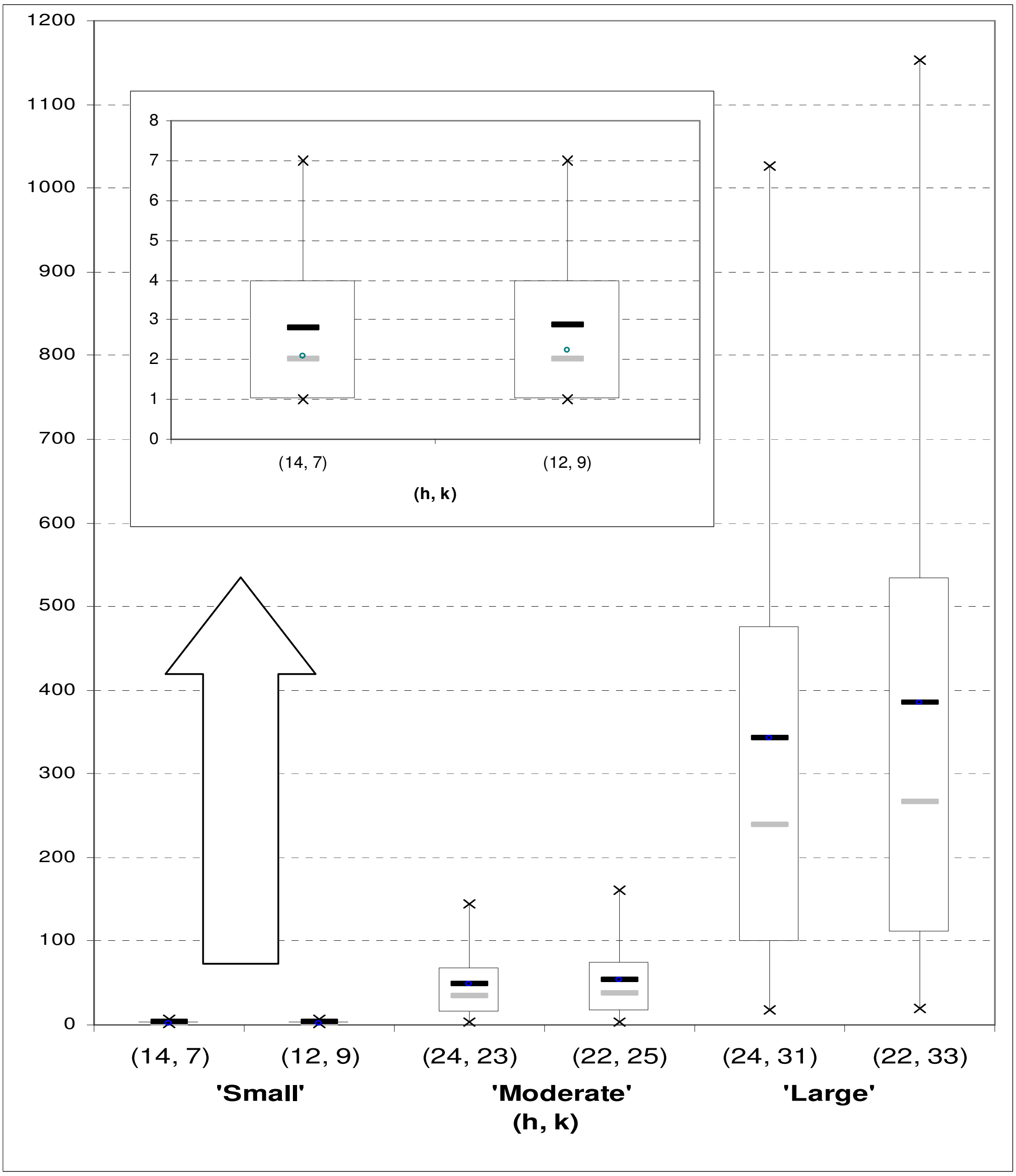

Figure 3.11. Boxplot-like graphs for the in-control run length distribution of various two-sided CUSUM signed-rank charts when $n = 10$. The whiskers extend to the 5th and the 95th percentiles. The symbols "▬", "◇" and "▬" denote the *ARL*, *SDRL*[*] and *MRL*, respectively[†].

[*] For ease of interpretation, the standard deviation (as measure of spread) is included in the (location) measures of percentiles.
[†] The "boxplots" are classified into 3 categories, namely small ($h + k \le 25$), moderate ($25 < h + k \le 50$) and large ($h + k > 50$).

Example 3.9

A two-sided CUSUM signed-rank chart for the Montgomery (2001) piston ring data

We conclude this sub-section by illustrating the two-sided CUSUM signed-rank chart using the piston ring data set from Montgomery (2001). We assume that the underlying distribution is symmetric with a known target value of $\theta_0 = 74\,mm$. For illustration take $k = 3$ and $h = 8$. From Table 3.27 it can be seen that the in-control average run length equals 4.07 when $(h,k) = (8,3)$. Generally, one chooses the chart constants so that a specified in-control average run length, such as 500, or 370, is obtained. Taking this into consideration, an in-control average run length of 4.07 is considered small. Table 3.30 shows the upper and lower signed-rank CUSUM statistics, respectively.

Table 3.30. One-sided signed-rank (S_i^+ and S_i^-) statistics*.

Sample number	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
S_i^+	5	6	0	4	0	6	13	4	13	24	25	37	49	61	72
S_i^-	0	0	-11	-1	-1	0	0	-3	0	0	0	0	0	0	0

* The values in Table 3.30 we generated using Microsoft Excel.

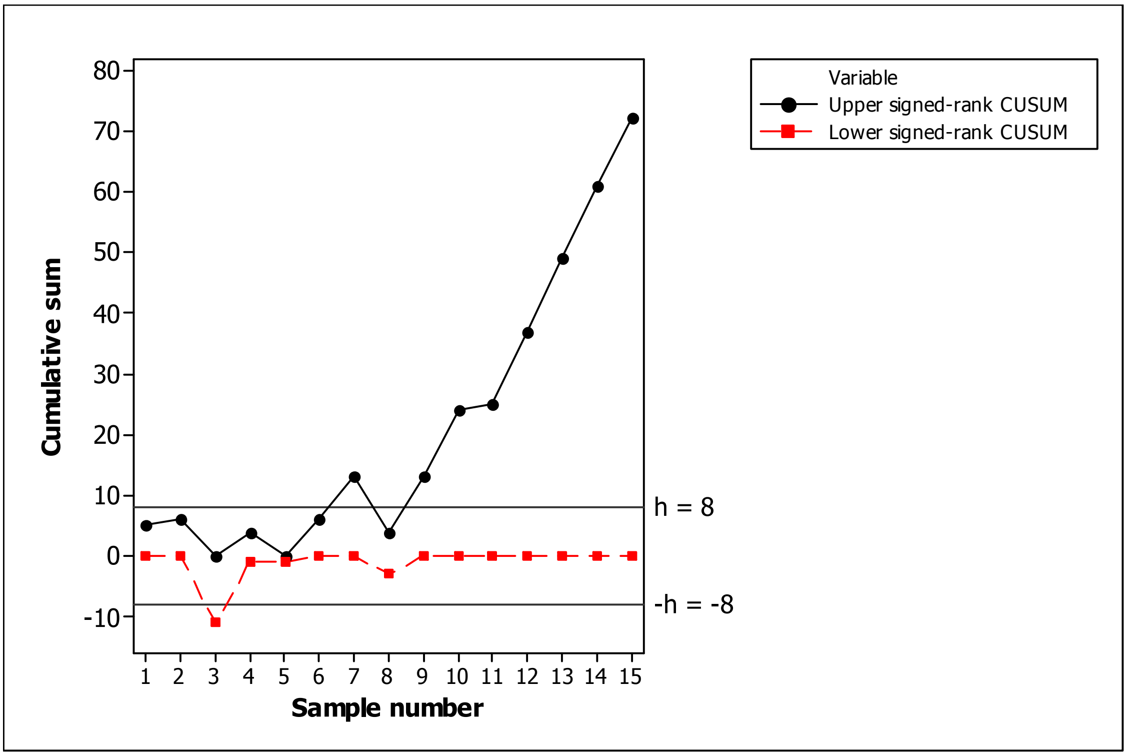

Figure 3.12. The two-sided CUSUM signed-rank chart for the Montgomery (2001) piston ring data.

The two-sided CUSUM signed-rank chart signals at sample number 3, indicating a most likely upward shift in the process median. The action taken following an out-of-control signal on a CUSUM chart is identical to that with any control chart. A search for assignable causes should be done, corrective action should be taken (if required) and, following this, the CUSUM is reset to zero.

3.3.4. Summary

While the Shewhart-type charts are the most widely known and used control charts in practice because of their simplicity and global performance, other classes of charts, such as the CUSUM charts are useful and sometimes more naturally appropriate in the process control environment in view of the sequential nature of data collection. In this chapter we have described the properties of the CUSUM signed-rank chart and given tables for its implementation. Detailed calculations have been given to help the reader to understand the subject more thoroughly.

3.4. The EWMA control chart

3.4.1. Introduction

In this section, the approach taken by Lucas and Saccucci (1990) is extended to the use of the signed-rank statistic resulting in an EWMA signed-rank chart that accumulates the statistics $SR_1, SR_2, SR_3,...$. Section 3.4 is analogous to Section 2.4 where the approach taken by Lucas and Saccucci (1990) was extended to the use of the sign statistic resulting in an EWMA sign chart. Therefore, the reader is frequently referred back to Section 2.4 throughout this section.

3.4.2. The proposed EWMA signed-rank chart

A nonparametric EWMA-type of control chart based on the signed-rank statistic (recall that $SR_i = \sum_{j=1}^{n} sign(x_{ij} - \theta_0) R_{ij}^+$) can be obtained by replacing X_i in expression (2.53) of Section 2.4 with SR_i. The EWMA signed-rank chart accumulates the statistics $SR_1, SR_2, SR_3,...$ with the plotting statistics defined as

$$Z_i = \lambda SR_i + (1-\lambda) Z_{i-1} \tag{3.10}$$

where $0 < \lambda \le 1$ is a constant called the weighting constant. The starting value Z_0 could be taken to equal zero, i.e. $Z_0 = 0$.

The EWMA signed-rank chart is constructed by plotting Z_i against the sample number i (or time). If the plotting statistic Z_i falls between the two control limits, that is, $LCL < Z_i < UCL$, the process is considered to be in-control. If the plotting statistic Z_i falls on or outside one of the control limits, that is $Z_i \le LCL$ or $Z_i \ge UCL$, the process is considered to be out-of-control.

The exact control limits and the center line of the EWMA signed-rank control chart can be obtained by replacing σ and θ_0 by σ_{SR_i} and 0, respectively, in expression (2.55) of Section 2.4 to obtain

$$\begin{aligned} UCL &= L\sigma_{SR_i}\sqrt{\left(\frac{\lambda}{2-\lambda}\right)\left(1-(1-\lambda)^{2i}\right)} \\ CL &= 0 \\ LCL &= -L\sigma_{SR_i}\sqrt{\left(\frac{\lambda}{2-\lambda}\right)\left(1-(1-\lambda)^{2i}\right)} \end{aligned} \quad . \tag{3.11}$$

Similarly, the *steady-state* control limits can be obtained by replacing σ and θ_0 by σ_{SR_i} and 0, respectively, in expression (2.56) to obtain

$$\begin{aligned} UCL &= L\sigma_{SR_i}\sqrt{\left(\frac{\lambda}{2-\lambda}\right)} \\ LCL &= -L\sigma_{SR_i}\sqrt{\left(\frac{\lambda}{2-\lambda}\right)} \end{aligned} \tag{3.12}$$

where σ_{SR_i} denotes the in-control standard deviation of the signed-rank statistic SR_i if there are no ties within a subgroup.

The in-control standard deviation of SR_i is given by $\sigma_{SR_i} = \sqrt{\text{var}(SR_i)} = \sqrt{\text{var}\left(2T^+ - \frac{n(n+1)}{2}\right)} = \sqrt{\frac{n(n+1)(2n+1)}{6}}$. This is obtained by using the relationship between SR_i and T^+ (recall that $SR_i = 2T^+ - \frac{n(n+1)}{2}$ if there are no ties within a subgroup) and the fact that $\text{var}(T^+) = \frac{n(n+1)(2n+1)}{24}$ (see Gibbons and Chakraborti (2003) page 198).

3.4.3. Markov-chain approach

Lucas and Saccucci (1990) evaluated the properties of the *continuous state* Markov chain by *discretizing* the infinite state TPM. This procedure entails dividing the interval between the UCL and the LCL into N subintervals of width 2δ. Then the plotting statistic, Z_i, is said to be in the non-absorbing state j at time i if $S_j - \delta < Z_i \le S_j + \delta$ where S_j denotes the midpoint of the j^{th} interval. Z_i is said to be in the absorbing state if Z_i falls on or outside one of the control limits, that is, $Z_i \le LCL$ or $Z_i \ge UCL$. Let p_{ij} denote the probability of moving from state i to state j in one step, i.e. $p_{ij} = P(Moving\ to\ state\ j \mid in\ state\ i)$. To approximate this probability we assume that the plotting statistic is equal to S_i whenever it is in state i. For all j non-absorbing we obtain $p_{ij} = P(S_j - \delta < Z_k \le S_j + \delta \mid Z_{k-1} = S_i)$. By using the definition of the plotting statistic given in expression (3.10) we obtain

$$p_{ij} = P(S_j - \delta < \lambda SR_k + (1-\lambda)S_i \le S_j + \delta)$$
$$= P\left(\frac{(S_j - \delta) - (1-\lambda)S_i}{\lambda} < SR_k \le \frac{(S_j + \delta) - (1-\lambda)S_i}{\lambda}\right)$$

recall that $SR_k = 2T_k^+ - \dfrac{n(n+1)}{2}$

$$p_{ij} = P\left(\frac{(S_j - \delta) - (1-\lambda)S_i}{\lambda} < 2T_k^+ - \frac{n(n+1)}{2} \le \frac{(S_j + \delta) - (1-\lambda)S_i}{\lambda}\right)$$
$$= P\left(\frac{\left(\frac{(S_j - \delta) - (1-\lambda)S_i}{\lambda} + \frac{n(n+1)}{2}\right)}{2} < T_k^+ \le \frac{\left(\frac{(S_j + \delta) - (1-\lambda)S_i}{\lambda} + \frac{n(n+1)}{2}\right)}{2}\right). \qquad (3.13)$$

For all j absorbing we obtain

$$p_{ij} = P(Z_k \le LCL \mid Z_{k-1} = S_i) + P(Z_k \ge UCL \mid Z_{k-1} = S_i)$$
$$= P(\lambda SR_k + (1-\lambda)S_i \le LCL) + P(\lambda SR_k + (1-\lambda)S_i \ge UCL)$$
$$= P\left(SR_k \le \frac{LCL - (1-\lambda)S_i}{\lambda}\right) + P\left(SR_k \ge \frac{UCL - (1-\lambda)S_i}{\lambda}\right)$$
$$= P\left(2T_k^+ - \frac{n(n+1)}{2} \le \frac{LCL - (1-\lambda)S_i}{\lambda}\right) + P\left(2T_k^+ - \frac{n(n+1)}{2} \ge \frac{UCL - (1-\lambda)S_i}{\lambda}\right)$$
$$= P\left(T_k^+ \le \frac{\frac{LCL - (1-\lambda)S_i}{\lambda} + \frac{n(n+1)}{2}}{2}\right) + P\left(T_k^+ \ge \frac{\frac{UCL - (1-\lambda)S_i}{\lambda} + \frac{n(n+1)}{2}}{2}\right). \qquad (3.14)$$

Since the values LCL, UCL, δ, λ, n, S_i and S_j are known constants the Wilcoxon signed-rank probabilities in expressions (3.13) and (3.14) can easily be calculated. The probabilities for the Wilcoxon signed-rank statistics are given in Table H of Lehmann (1975) for samples sizes up to 20 and they are tabulated (more recently) in Table H of Gibbons and Chakraborti (2003) for sample sizes up to 15.

Once the one-step transition probabilities are calculated, the TPM can be constructed and is given by $TPM = [p_{ij}] = \begin{pmatrix} Q & | & \underline{p} \\ - & - & - \\ \underline{0}' & | & 1 \end{pmatrix}$ (written in partitioned form) where the essential transition probability sub-matrix Q is the matrix that contains all the transition probabilities of going from a non-absorbing state to a non-absorbing state, $Q:(NA \rightarrow NA)$, $\underline{p}$ contains all the transition probabilities of going from each non-absorbing state to the absorbing states, $\underline{p}:(NA \rightarrow A)$, $\underline{0}' = (0 \quad 0 \quad 0 \quad \cdots \quad 0)$ contains all the transition probabilities of going from each absorbing state to the non-absorbing states. $\underline{0}'$ is a row vector with all its elements equal to zero, because it is impossible to go from an absorbing state to a non-absorbing state, because once an absorbing state is entered, it is never left, $\underline{0}' \; :(A \rightarrow NA)$, and 1 represents the scalar value one. The probability of going of going from an absorbing state to an absorbing state is equal to one, because once an absorbing state is entered, it is never left, $1:(A \rightarrow A)$. The one-step TPM is used to calculate the expected value (ARL), the second raw moment, the variance, the standard deviation and the probability mass function (pmf) of the run-length variable N which are given in equations (2.41) to (2.45).

Example 3.10

The EWMA signed-rank chart where the sample size is even ($n = 6$)

The EWMA signed-rank chart is investigated for a smoothing constant of 0.1 ($\lambda = 0.1$) and a multiplier of 3 ($L = 3$). The *steady-state* control limits are given by

$$UCL = L\sigma_{SR_i}\sqrt{\left(\frac{\lambda}{2-\lambda}\right)}$$

$$LCL = -L\sigma_{SR_i}\sqrt{\left(\frac{\lambda}{2-\lambda}\right)}$$

where $L = 3$, $\lambda = 0.1$, and $\sigma_{SR_i} = 9.539$, since $\sigma_{SR_i} = \sqrt{\frac{n(n+1)(2n+1)}{6}} = \sqrt{\frac{6(6+1)(12+1)}{6}} = 9.539$. Clearly, we only have to calculate the UCL since $LCL = -UCL$. We obtain $UCL = 3 \times 9.539 \sqrt{\left(\frac{0.1}{2-0.1}\right)} = 6.565$. Therefore, $LCL = -6.565$.

This Markov-chain procedure entails dividing the interval between the UCL and the LCL into N subintervals of width 2δ. For this example N is taken to equal 4. Figure 3.13 illustrates the partitioning of the interval between the UCL and the LCL into subintervals.

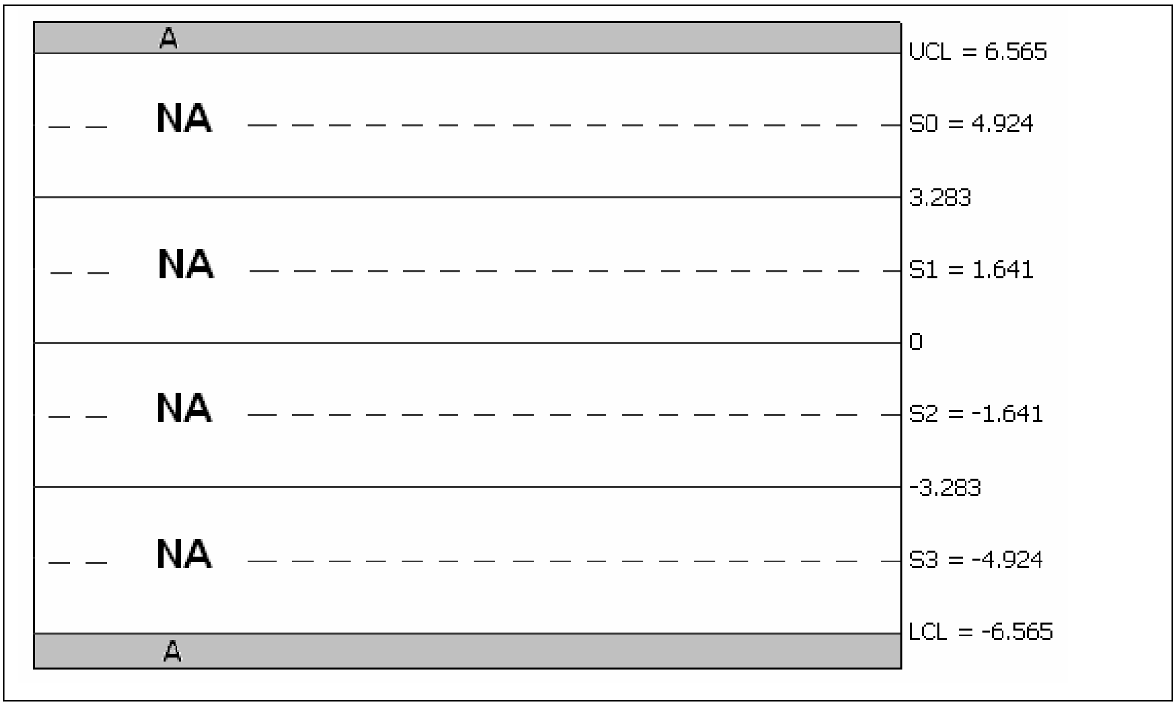

Figure 3.13. Partitioning of the interval between the UCL and the LCL into 4 subintervals.

From Figure 3.13 we see that there are 4 non-absorbing states, i.e. $r = 4$. The TPM is given by

$$TPM = \begin{pmatrix} p_{00} & p_{01} & p_{02} & p_{03} & p_{04} \\ p_{10} & p_{11} & p_{12} & p_{13} & p_{14} \\ p_{20} & p_{21} & p_{22} & p_{23} & p_{24} \\ p_{30} & p_{31} & p_{32} & p_{33} & p_{34} \\ p_{40} & p_{41} & p_{42} & p_{43} & p_{44} \end{pmatrix} = \begin{pmatrix} Q_{4\times4} & | & \underline{p}_{4\times1} \\ - & - & - \\ \underline{0}'_{1\times4} & | & 1_{1\times1} \end{pmatrix}.$$

Table 3.31. Calculation of the one-step probabilities in the first row of the TPM.

$p_{00} = P(Moving\ to\ state\ 0 \mid in\ state\ 0)$ $= P(S_0 - \delta < Z_k \leq S_0 + \delta \mid Z_{k-1} = S_0)$ from (3.13) $= P\left(\dfrac{\left(\dfrac{(S_0-\delta)-(1-\lambda)S_0}{\lambda} + \dfrac{n(n+1)}{2}\right)}{2} < T_k^+ \leq \dfrac{\left(\dfrac{(S_0+\delta)-(1-\lambda)S_0}{\lambda} + \dfrac{n(n+1)}{2}\right)}{2}\right)$ with $\delta = 1.641$, $\lambda = 0.1$, $L = 3$ and $S_0 = 4.924$ $= P(4.755 < T_k^+ \leq 21.169)$ $= P(T_k^+ \leq 21) - P(T_k^+ \leq 4)$ $= \dfrac{57}{64}$ from Gibbons and Chakraborti (2003)
$p_{01} = P(Moving\ to\ state\ 1 \mid in\ state\ 0)$ $= P(S_1 - \delta < Z_k \leq S_1 + \delta \mid Z_{k-1} = S_0)$ from (3.13) $= P\left(\dfrac{\left(\dfrac{(S_1-\delta)-(1-\lambda)S_0}{\lambda} + \dfrac{n(n+1)}{2}\right)}{2} < T_k^+ \leq \dfrac{\left(\dfrac{(S_1+\delta)-(1-\lambda)S_0}{\lambda} + \dfrac{n(n+1)}{2}\right)}{2}\right)$ $= P(-11.658 < T_k^+ \leq 4.755)$ $= P(T_k^+ \leq 4)$ $= \dfrac{7}{64}$

$$p_{02} = P(Moving\ to\ state\ 2 \mid in\ state\ 0)$$
$$= P(S_2 - \delta < Z_k \le S_2 + \delta \mid Z_{k-1} = S_0) \text{ from (3.13)}$$
$$= P\left(\frac{\left(\frac{(S_2 - \delta) - (1-\lambda)S_0}{\lambda} + \frac{n(n+1)}{2} \right)}{2} < T_k^+ \le \frac{\left(\frac{(S_2 + \delta) - (1-\lambda)S_0}{\lambda} + \frac{n(n+1)}{2} \right)}{2} \right)$$
$$= P(-28.072 < T_k^+ \le -11.658)$$
$$= 0$$

$$p_{03} = P(Moving\ to\ state\ 3 \mid in\ state\ 0)$$
$$= P(S_3 - \delta < Z_k \le S_3 + \delta \mid Z_{k-1} = S_0) \text{ from (3.13)}$$
$$= P\left(\frac{\left(\frac{(S_3 - \delta) - (1-\lambda)S_0}{\lambda} + \frac{n(n+1)}{2} \right)}{2} < T_k^+ \le \frac{\left(\frac{(S_3 + \delta) - (1-\lambda)S_0}{\lambda} + \frac{n(n+1)}{2} \right)}{2} \right)$$
$$= P(-44.486 < T_k^+ \le -28.072)$$
$$= 0$$

$$p_{04} = P(Moving\ to\ state\ 4 \mid in\ state\ 0)$$
$$= P(Z_k \le LCL \mid Z_{k-1} = S_0) + P(Z_k \ge UCL \mid Z_{k-1} = S_0) \text{ from (3.14)}$$
$$= P\left(T_k^+ \le \frac{\frac{LCL - (1-\lambda)S_0}{\lambda} + \frac{n(n+1)}{2}}{2} \right) + P\left(T_k^+ \ge \frac{\frac{UCL - (1-\lambda)S_0}{\lambda} + \frac{n(n+1)}{2}}{2} \right)$$
$$= P(T_k^+ \le -44.486) + P(T_k^+ \ge 21.169)$$
$$= 0$$

The one-step probabilities in the remaining rows can be calculated similarly. Therefore, the TPM is given by

$$TPM = \begin{pmatrix} 57/64 & 7/64 & 0 & 0 & 0 \\ 2/64 & 57/64 & 5/64 & 0 & 0 \\ 0 & 5/64 & 57/64 & 2/64 & 0 \\ 0 & 0 & 7/64 & 57/64 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} Q_{4\times4} & \mid & \underline{p}_{4\times1} \\ - & - & - \\ \underline{0}'_{1\times4} & \mid & 1_{1\times1} \end{pmatrix}.$$

Other values of the multiplier (L) and the smoothing constant (λ) were also considered and the results are given in Tables 3.32 and 3.33.

Table 3.32. The in-control average run length (ARL_0), standard deviation of the run length ($SDRL$), $5^{th}, 25^{th}, 50^{th}, 75^{th}$ and 95^{th} percentile values[*] for the EWMA signed-rank chart when $n = 6$ and $N = 5$, i.e. there are 5 subintervals between the lower and upper control limit[†].

	$L = 1$	$L = 2$	$L = 3$
$\lambda = 0.05$	10.45 12.32 (1, 2, 6, 15, 35)	56.69 72.45 (1, 5, 29, 82, 204)	**
$\lambda = 0.1$	7.32 8.38 (1, 1, 4, 10, 24)	33.83 40.28 (1, 4, 20, 48, 115)	330.67 369.33 (2, 63, 213, 471, 1070)
$\lambda = 0.2$	4.95 4.90 (1, 1, 3, 7, 15)	35.21 39.63 (1, 6, 22, 50, 115)	361.92 384.29 (3, 87, 243, 510, 1130)

** The inverse of the matrix ($I - Q$) does not exist and as a result the ARL (given by $E(N) = \underline{\xi}(I - Q)^{-1}\underline{1}$) can not be calculated for this combination of (λ, L).

In example 3.10 we considered a sample size that may be considered "small". The results are given for a larger sample size ($n = 10$) for various values of λ and L in Table 3.33.

Table 3.33. The in-control average run length (ARL_0), standard deviation of the run length ($SDRL$), $5^{th}, 25^{th}, 50^{th}, 75^{th}$ and 95^{th} percentile values[‡] for the EWMA signed-rank chart when $n = 10$ and $N = 5$, i.e. there are 5 subintervals between the lower and upper control limit[§].

	$L = 1$	$L = 2$	$L = 3$
$\lambda = 0.05$	11.17 13.49 (1, 2, 6, 16, 39)	67.94 83.82 (1, 7, 38, 98, 238)	1448.44 1573.37 (10, 316, 956, 2052, 4595)
$\lambda = 0.1$	6.85 7.74 (1, 1, 4, 9, 23)	48.87 57.73 (1, 6, 29, 70, 165)	352.72 384.51 (3, 76, 232, 500, 1122)
$\lambda = 0.2$	5.05 5.07 (1, 1, 3, 7, 15)	33.96 38.48 (1, 6, 21, 48, 111)	336.34 357.54 (3, 80, 226, 474, 1051)

[*] The three rows of each cell shows the ARL_0, the $SDRL$, and the percentiles ($\rho_5, \rho_{25}, \rho_{50}, \rho_{75}, \rho_{95}$), respectively.
[†] See SAS Program 8 in Appendix B for the calculation of the values in Table 3.32.
[‡] The three rows of each cell shows the ARL_0, the $SDRL$, and the percentiles ($\rho_5, \rho_{25}, \rho_{50}, \rho_{75}, \rho_{95}$), respectively.
[§] See SAS Program 8 in Appendix B for the calculation of the values in Table 3.33.

These tables can be extended by changing the sample size (*n*), the number of subintervals between the lower and upper control limit (*N*), the multiplier (*L*) and the smoothing constant (λ) in SAS Program 8 for the EWMA signed-rank chart given in Appendix B.

From Tables 3.32 and 3.33 we see that the ARL_0, $SDRL$ and percentiles increase as the value of the multiplier (*L*) increases. From Table 3.33 we find an in-control average run length of 336.34 for $n = 10$ when the multiplier is taken to equal 3 ($L = 3$) and the smoothing constant 0.2 ($\lambda = 0.2$). The chart performance is good, since the attained in-control average run length of 336.34 is in the region of the desired in-control average run length which is generally taken to be 370 or 500.

3.4.4. Summary

The EWMA control chart is one of several charting methods aimed at correcting a deficiency of the Shewhart chart - insensitivity to small shifts. Lucas and Saccucci (1990) have investigated some properties of the EWMA chart under the assumption of independent normally distributed observations, whereas in this section we have described and evaluated the nonparametric EWMA signed-rank chart. The main advantage of the nonparametric EWMA chart is that there is no need to assume a particular parametric distribution for the underlying process (see Section 1.4 for other advantages of the nonparametric EWMA chart).