CHAPTER 3
CLOSED-LOOP SYSTEM IDENTIFICATION THEORY

3.1 INTRODUCTION

In this chapter the basic closed-loop SID theory is explained. This chapter starts with a discussion of the SID problem in Section 3.2. Then, the closed-loop configuration, together with assumptions and notations, are discussed in Section 3.3. In Section 3.4 the correlation problem is discussed and the question of informative data is addressed in Section 3.5. The different approaches for avoiding the problems with closed-loop SID are then discussed in Section 3.6.

Section 3.7 deals with closed-loop SID in the prediction error framework. The algorithm format, different model structures, and the computation of the estimates are discussed. The relevant statistical properties of this method are also stated. The theory discussed in this section can readily be extended to the multivariable situation, as shown by Forssell and Ljung [12].

Because excitation signals have a very substantial influence on the observed data, these signals are also further discussed in Section 3.8. Lastly, Section 3.9 deals with the theory of the new inter-sampling approach to closed-loop SID, which may also ensure identifiability.

3.2 SYSTEM IDENTIFICATION PROBLEM

The system identification problem is to construct mathematical plant models from experimental data. This is an important problem in statistics and many identification methods, as well as the tools for analysing their properties, have their roots in statistics [6]. According to Ljung [11], to determine a model of a dynamic system, from observed input-output data, the following is required:

Data Record: Data might be available from normal operating records, but it may also be possible to specifically design identification experiments for a process in order to obtain specific information.

Set of Candidate Models (Model Structure): A set of candidate models is obtained by specifying within which collection of models one is going to look for a suitable model.

Criterion to Select a Particular Model in the Set: Given measurement data and a model set, one has to specify in which way the optimal model from the model set is going to be selected. The assessment of model quality is typically based on how models perform when they attempt to reproduce the measured data.

After one has arrived at a particular model, it remains to test whether this model is good enough. Such tests fall under the heading of model validation [11].

The system identification procedure has a natural logic flow: first collect data, then choose a model set, then pick the best model in this set. It is very possible that the model first selected will not pass the model validation tests. One must then go back and revise the various steps of the procedure [11].

Ljung [11] states that a model may be deficient for a variety of reasons:

- the numerical procedure can fail to find the best model according to the criteria,
- the criteria is not well chosen,
- the model set is not appropriate, in that it does not contain a good enough description of the system, or
- the data set is not informative enough to provide guidance in selecting a good model.

The major part of an identification application consists of addressing these problems, in an iterative manner, guided by prior information and the outcomes of previous attempts [11]. This iteration is depicted in Fig. 3.1. Interactive software is required to deal effectively with this problem.

According to Forssell [6], the system identification problem can therefore be divided into the following five subproblems:

- experiment design,
- data collection,
- model structure selection,
- model estimation, and
- model validation.

These steps are also applicable in closed-loop system identification and will be used in Section 4.8 to summarise the developed closed-loop SID methodology.

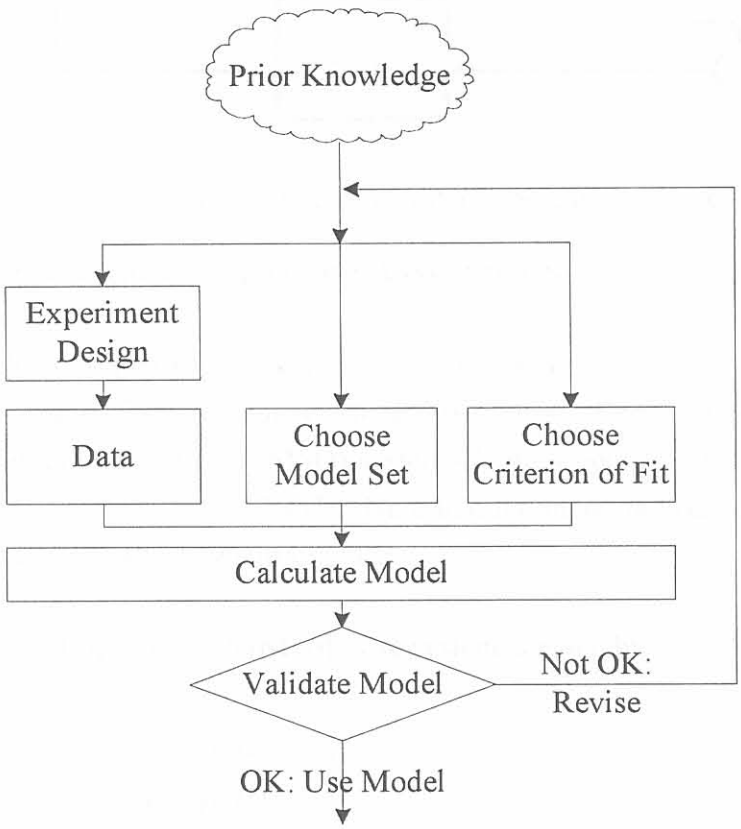

Figure 3.1: The system identification loop.

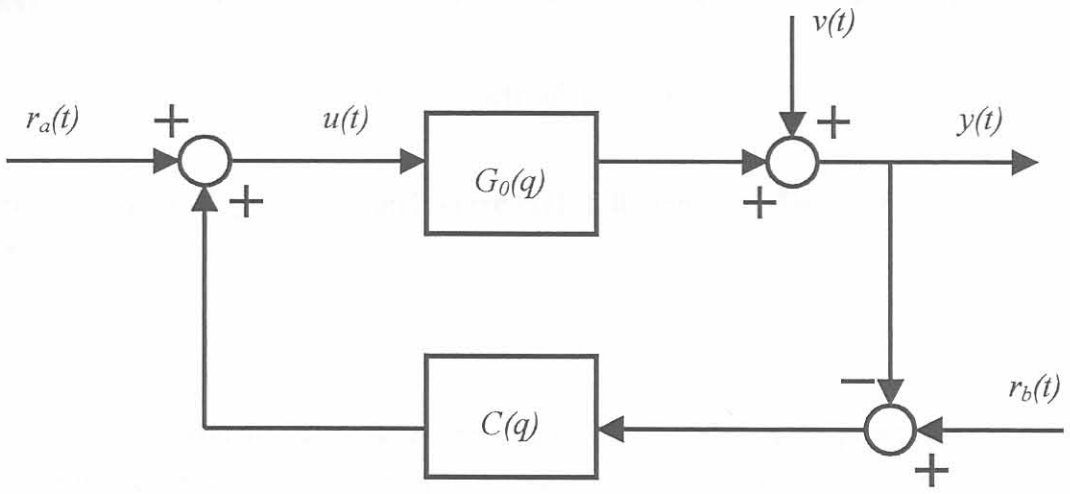

Figure 3.2: Configuration of a system operating in closed-loop

3.3 Closed-Loop Configuration: Assumptions and Notation

When an identification experiment is performed in closed-loop, i.e. the output is fed back to the input by means of some feedback mechanism, it is called closed-loop identification [6]. A typical situation is when a (SISO) system $G_0(q)$ is being controlled by some feedback controller $C(q)$, as in Fig. 3.2 [31]. Note that the input and outputs signals, used in subsequent discussions, are shown in Fig. 3.2.

The following set-up is considered [6]: The system is given by

$$\begin{aligned} y(t) &= G_0(q)u(t) + v(t), \\ v(t) &= H_0(q)e(t), \\ y(t) &\in \Re^{Nx1},\ u(t) \in \Re^{Nx1},\ v(t) \in \Re^{Nx1},\ e(t) \in \Re^{Nx1}. \end{aligned} \tag{3.1}$$

Here $y(t)$ is the output, $u(t)$ the input, $v(t)$ the additive output noise, and $e(t)$ a white noise signal with zero mean and variance λ_0. N is the number of data samples. The symbol q donates the discrete-time shift operator:

$$q^{-1}u(t) = u(t-1). \tag{3.2}$$

Without loss of generality, it is assumed that $G_0(q)$ contains a delay and that the noise model $H_0(q)$ is monic, i.e. $H_0(q) = \sum_{k=0}^{\infty} h(k)q^{-k}$ and $h(0) = 1$ [11], and inversely stable, i.e.

$(H_0(q))^{-1}$ is stable. Furthermore, it can be assumed that the input is generated as

$$u(t) = k(t, y^t, u^{t-1}, r(t)), \tag{3.3}$$

where $y^t = [y(1), ..., y(t)]$, etc., and where $r(t)$, following Van Den Hof [31], is defined as follows:

$$r(t) := r_a(t) - C(q)r_b(t). \tag{3.4}$$

In Fig. 3.2 the signals $r_a(t)$ can be either a reference value, a set-point or a noise disturbance on the regulator output. Similarly, signal $r_b(t)$ can be either a set-point or a measured noise on the output signal. $r(t)$ is a quasi-stationary signal, independent of $v(t)$. $C(q)$ is a given deterministic function such that the closed-loop system is stable.

For now, it can be assumed that the feedback is linear and given by

$$u(t) = r(t) - C(q)y(t). \tag{3.5}$$

The closed-loop equations then become

$$\begin{aligned} y(t) &= S_0(q)G_0(q)r(t) + S_0(q)v(t), \\ u(t) &= S_0(q)r(t) - C(q)S_0(q)v(t), \end{aligned} \tag{3.6}$$

where $S_0(q)$ is the sensitivity function

$$S_0(q) = \frac{1}{1 + G_0(q)C(q)}. \tag{3.7}$$

The closed-loop system can also be rewritten as

$$\begin{aligned} y(t) &= G_0^c(q)r(t) + v_c(t), \\ v_c(t) &= H_0^c(q)e(t), \end{aligned} \tag{3.8}$$

where $G_0^c(q) = S_0(q)G_0(q)$ and $H_0^c = S_0(q)H_0(q)$.

The total expectation operator $\bar{E}$ [11] will also be used in subsequent sections:

$$\bar{E}x(t) = \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} Ex(t), \tag{3.9}$$

where Ex, the mathematical expectation of the random vector x, is defined as

$$Ex = \int_{\Re^N} x f_x(t) dt, \tag{3.10}$$

where $f_x(t)$ is the Probability Density Function (PDF) of x.

Lastly, the power spectrum of a signal $s(t)$ will be denoted by $\Phi_s(\omega)$ and the cross spectrum between two signals $s(t)$ and $w(t)$ will be denoted by $\Phi_{sw}(\omega)$.

3.4 Closed-Loop Correlation Problem

Many of the well-known identification methods, which work well in open-loop, fail when applied directly to input-output data obtained under closed-loop control. The reason why these methods fail is the non-zero correlation between the input $u(t)$ and the additive output noise $v(t)$ [6].

These well-known identification methods, which cannot be applied directly to closed-loop data, are listed below with some short explanations of why they fail [6, 11].

Instrumental Variable Method: In open-loop it is common to let the instruments, used in the IV method, be filtered and delayed versions of the input. However, this straightforward approach will fail in closed-loop, because the input is correlated with the output noise.

Subspace Method: An important part of the subspace method is the choice of a suitable multiplication matrix, which is uncorrelated with the noise, in order to eliminate the noise term in the Hankel Matrix equation. This matrix is usually built up of delayed inputs. Again, this approach will fail because of the correlation of the input with the output noise.

Spectral Analysis: When the spectral analysis method is applied in a straightforward fashion, as described by Ljung [11], it will give erroneous results. The estimate of $G(e^{i\omega}, \theta)$ will not converge to $G_0(e^{i\omega})$, but to

$$G_*(e^{i\omega}, \theta) = \frac{G_0(e^{i\omega})\Phi_r(\omega) - C(e^{-i\omega})\Phi_v(\omega)}{\Phi_r(\omega) + \mid C(e^{i\omega}) \mid^2 \Phi_v(\omega)}. \tag{3.11}$$

With no external reference signal present, i.e. $\Phi_r(\omega) = 0$, $G_*(e^{i\omega}, \theta)$ will equal the negative inverse of the controller, i.e. $G_*(e^{i\omega}, \theta) = -(C(e^{i\omega}))^{-1}$.

Correlation Analysis: The correlation analysis method will give a biased estimate of the impulse response, because the assumption $\bar{E}u(t)v(t-\tau) = 0$ is violated.

Output Error Models: The OE model will only give a consistent estimate of $G(q)$ if the additive noise is white in the case of closed-loop data, because the noise model is assumed fixed and equal to one.

Thus, the above-mentioned methods are not directly applicable to closed-loop data. Luckily, the Prediction Error Method is directly applicable to closed-loop data, as long as the parameterizing is flexible enough. Ljung [11] states that the PEM estimation method will consistently estimate the system if

- the data are informative, and
- the model set contains the true system (both noise model and dynamics model),

irrespective of whether the data set $\{u(t), y(t)\}$ has been collected under feedback control. It is concluded by Ljung [11] that under the above assumptions, the PEM estimate will have optimal accuracy. The PEM method is described in Section 3.7.

3.5 Informative Closed-Loop Experiments

Forssell [6] says that in order to be able to identify a system it is necessary for the data to be sufficiently rich in such a way that it is possible to uniquely determine the system parameters.

In open-loop the situation is rather transparent and it can be shown that the data set is *informative enough* if the input is persistently exciting of sufficiently high order [11]. This means that the power spectrum of the input signal, $\Phi_u(\omega)$, should be non-zero on at least n points in the interval $-\pi < \omega < \pi$ [11]. Here n is the necessary order of the PE input, and this equals the number of parameters to be estimated.

In closed-loop, the situation is less transparent. The following standard example shows what can happen [6, 11]:

Example: *Proportional Feedback*

Consider the first-order model structure

$$y(t) + ay(t-1) = bu(t-1) + e(t), \tag{3.12}$$

and suppose that the system is under proportional control during the experiment:

$$u(t) = -fy(t). \tag{3.13}$$

Inserting the feedback law into the model gives

$$y(t) + (a + bf)y(t-1) = e(t), \tag{3.14}$$

which is the model of the closed-loop system. From Eqn. (3.14) it can be concluded that *all models* $(\hat{a}, \hat{b})$, subject to

$$\begin{aligned} \hat{a} &= a + \gamma f, \\ \hat{b} &= b - \gamma, \end{aligned} \tag{3.15}$$

with γ an arbitrary scalar, give the same input-output description of the system as the model (a, b) under proportional feedback, as described in Eqn. (3.13). There is consequently no way to distinguish between model (a, b) and all models $(\hat{a}, \hat{b})$. It is of no help to know the regulator parameter f. The experimental condition, as described in Eqn. (3.13), is not informative enough with respect to the model structure of Eqn. (3.12). It is true, though, that the input signal $u(t)$ is PE, since it consists of filtered white noise. Persistence of excitation of the inputs signal $u(t)$ is therefore not a sufficient condition in closed-loop experiments.

If the model structure, given by Eqn. (3.12), is restricted by, for example, constraining b to be 1:

$$y(t) + a(t-1) = u(t-1) + e(t), \tag{3.16}$$

then it is clear that the data, generated as in Eqn. (3.13), are sufficiently informative to distinguish between values of the a-parameter.

Consequently, it could be problematic to obtain relevant information from closed-loop experiments. Conditions for informative data sets must also involve the feedback mechanisms

[11, 31]. An explicit condition, which must be satisfied for the data to be informative, is given in Section 3.7.5.

3.6 Different Approaches to Closed-Loop Identification

In Section 3.4 some examples were given of methods that fail when applied directly to closed-loop data. There are some alternative ways to implement the methods to make them applicable to closed-loop situations. According to Forssell [6], a large number of methods have been developed, in particular a number of PEM methods, that are applicable to closed-loop identifications. Depending on what assumptions are made on the nature of the feedback, all closed-loop identification methods can be classified as *direct*, *indirect* and *joint input-output* methods [6, 11]:

Direct Approach: In the direct approach, the basic PEM estimation method is applied in a straightforward manner. No assumptions whatsoever are made on how the data set was generated. The output $y(t)$ of the process and the input $u(t)$ are used in the same way as for open-loop identification.

Indirect Approach: The methods used in the indirect approach assume perfect knowledge of the controller used in the identification experiment. The closed-loop system is identified from reference input $r(t)$ to output $y(t)$. From this identified closed-loop system the open-loop system (plant) is retrieved making use of the known controller $C(q)$. Ljung [11] states that any error in the assumed regulator will directly cause a corresponding error in the estimate of $G(q)$. Because most regulators contain nonlinearities, the indirect identification approach could yield incorrect plant models.

Joint Input-Output Approach: In the joint input-output approach the output $y(t)$ of the process and the input $u(t)$ of the plant are considered as outputs of a system driven by the reference input $r(t)$ and unmeasured noise $v(t)$. Knowledge of the system and the controller is recovered from this joint model. Therefore, in this approach knowledge of the controller is not required, but the controller structure must be known.

The indirect and the joint input-output approaches are typically only used when the feedback law is linear, as in Eqn. (3.5), but can also be applied when the feedback is nonlinear. Of course, in the case of nonlinear feedback, the estimation problems become much more involved [6].

Forssell [6] emphasizes the fact that the direct approach is only applicable when the PEM method and some of the subspace methods are used. The reason for this is the unavoidable correlation between the input and the additive output noise, which, as was shown, rules out most other methods. With the indirect and joint input-output approaches the closed-loop problem is converted into an open-loop problem, because the reference signal, which serves as the input, is uncorrelated with the output noise. Therefore, these approaches are similar to the open-loop methods [6].

In Section 3.7 the PEM methods that fall into these three categories, will be discussed.

3.7 Closed-Loop Identification in the Prediction Error Framework

3.7.1 Assumptions and Notation

The notation, as used by Forssell and Ljung [12], will be used to discuss closed-loop SID in the PEM framework. A parametrised set of models like Eqn. (3.17) is called a model structure and is denoted by M [11].

$$y(t) = G(q,\theta)u(t) + H(q,\theta)e(t). \tag{3.17}$$

$G(q,\theta)$ will is called the *dynamic model* and $H(q,\theta)$ the *noise model*. $H(q,\theta)$ is assumed monic. The parameter vector θ ranges over a set $D_M \subset \Re^d$ ($d =$ dimension of θ), which is assumed compact and connected [32].

The model structure given by Eqn. (3.17), where $\theta \in D_M$, describes a model set. The true system is contained in the model set if, for some $\theta_0 \in D_M$,

$$G(q,\theta_0) = G_0(q), \quad H(q,\theta_0) = H_0(q). \tag{3.18}$$

This can also be written as $S \in M$, where S is the true system (noise and plant transfer functions). The case were the true noise properties cannot be described by any noise model found within the model set, but were there exists a $\theta_0 \in D_M$ such that

$$G(q,\theta_0) = G_0(q), \tag{3.19}$$

can be denoted as $G_0 \in \mathcal{G}$, where $\mathcal{G}$ is a set of transfer functions obtained in a given structure.

The following notation is used [6]:

$$T(q,\theta) = \begin{bmatrix} G(q,\theta) \\ H(q,\theta) \end{bmatrix}, \tag{3.20}$$

$$T_0(q) = \begin{bmatrix} G_0(q) \\ H_0(q) \end{bmatrix}. \tag{3.21}$$

The notation requires the introduction of the combined signal consisting of $u(t)$ and $e(t)$ in the following way:

$$\chi_0(t) = [u(t) \;\; e(t)]^T. \tag{3.22}$$

The spectrum of $\chi_0(t)$ is given by

$$\Phi_{\chi_0}(\omega) = \begin{bmatrix} \Phi_u(\omega) & \Phi_{ue}(\omega) \\ \Phi_{eu}(\omega) & \lambda_0 \end{bmatrix}. \tag{3.23}$$

The part of the input spectrum that originates from the reference signal $r(t)$ can be written as

$$\Phi_u^r(\omega) \;\; = \;\; \Phi_u(\omega) - \left|\Phi_{ue}(\omega)\right|^2 / \lambda_0. \tag{3.24}$$

Using Eqn. (3.6), $\Phi_u^r(\omega)$ can also be expressed as follows,

$$\Phi_u^r(\omega) \;\; = \;\; \left|S_0(e^{i\omega})\right|^2 \Phi_r(\omega), \tag{3.25}$$

were $\Phi_r(\omega)$ is the spectrum of the reference signal. The cross spectrum between $u(t)$ and $e(t)$ is

$$\Phi_{ue}(\omega) \;\; = \;\; -C(e^{i\omega})S_0(e^{i\omega})H_0(e^{i\omega})\lambda_0. \tag{3.26}$$

The spectrum of $v(t)$ is

$$\Phi_v(\omega) = \left|H_0(e^{i\omega})\right|^2 \lambda_0. \tag{3.27}$$

Furthermore, the component of the input spectrum that originates from the noise can be written as [11]

$$\Phi_u^e(\omega) = \left|C(e^{i\omega})\right|^2 |S_0|^2 \Phi_v(\omega). \tag{3.28}$$

3.7.2 Prediction Error Estimation Method

Ljung [11] shows that the one-step ahead predictor for the model structure, given by Eqn. (3.17), can be written as

$$\hat{y}(t|\theta) = H^{-1}(q,\theta)G(q,\theta)u(t) + (1 - H^{-1}(q,\theta))y(t), \tag{3.29}$$

and the prediction error as

$$\varepsilon(t,\theta) = y(t) - \hat{y}(t|\theta) = H^{-1}(q,\theta)(y(t) - G(q,\theta)u(t)). \tag{3.30}$$

Given the model in Eqn. (3.29) and the measured data

$$Z^N = \{y(1), u(1), ..., y(N), u(N)\}, \tag{3.31}$$

the predictor estimate can be determined from [6, 11]:

$$\hat{\theta}_N = \arg\min_{\theta \in D_M} V_N(\theta, Z^N), \tag{3.32}$$

$$V_N(\theta, Z^N) = \frac{1}{N}\sum_{t=1}^{N} l(\varepsilon(t,\theta)). \tag{3.33}$$

Here $l(\cdot)$ is a suitable positive (norm) function. The standard choice is the quadratic norm [6, 11]:

$$l(\varepsilon(t,\theta)) = \frac{1}{2}\varepsilon^2(t,\theta). \tag{3.34}$$

This choice can be combined with some linear, monic and possibly parameterised prefilter $L(q,\theta)$:

$$\begin{aligned} l(\varepsilon(t,\theta)) &= \frac{1}{2}\varepsilon_F^2(t,\theta), \\ \varepsilon_F(t,\theta) &= L(q,\theta)\varepsilon(t,\theta). \end{aligned} \tag{3.35}$$

Without loss of generality the prefilter can be included in the noise model and $L(q,\theta) = 1$ can be assumed, since

$$\epsilon_F(t,\theta) = L(q,\theta)H^{-1}(q,\theta)(y(t) - G(q,\theta)u(t)). \tag{3.36}$$

The above choice of $l(\cdot)$ results in the Least Square Error method [6]. If the criterion function $l(\cdot)$ is equal to $f_e(\cdot)$, the PDF of $e(t)$, the maximum likelihood function is obtained [6]. In the following analysis it is assumed that the quadratic criterion is used.

The estimates $G(q,\hat{\theta}_N)$, and $H(q,\hat{\theta}_N)$ will be written as

$$\hat{G}_N = G(q,\hat{\theta}_N) \text{ and } \hat{H}_N = H(q,\hat{\theta}_N). \tag{3.37}$$

3.7.3 Family of Model Structures

With the PEM approach, the models may have an arbitrary parameterisation [11]. The most simple input-output relationship is obtained by describing it as a linear difference equation:

$$\begin{aligned} & y(t) + a_1 y(t-1) + ... + a_{n_a} y(t-na) \\ = \; & b_1 u(t-1) + ... + b_{n_b} u(t-nb) + e(t). \end{aligned} \tag{3.38}$$

Since the white-noise term $e(t)$ enters as a direct error in the difference equation, the model, given by Eqn. (3.38), is often called an *equation error model* structure [11]. The adjustable parameters are

$$\theta = [a_1 \;\; a_2 \;\; ... \;\; a_{n_a} \;\; b_1 \;\; ... \;\; b_{n_b}]^T. \tag{3.39}$$

The linear difference equation, given by Eqn. (3.38), can be written in the form of Eqn. (3.17), by introducing the following two operators:

$$\begin{aligned} A(q) &= 1 + a_1 q^{-1} + ... + a_{n_a} q^{-n_a}, \text{ and} \\ B(q) &= b_1 q^{-1} + ... + b_{n_b} q^{-n_b}. \end{aligned} \tag{3.40}$$

Eqns. (3.38), (3.17) and (3.40) leads the following expression for $G(q,\theta)$ and $H(q,\theta)$:

$$G(q,\theta) = \frac{B(q)}{A(q)}, \quad H(q,\theta) = \frac{1}{A(q)}. \tag{3.41}$$

This model is called an Auto-Regressive with External Input (ARX) model. In the case when n_a in Eqn.(3.39) is 0, $y(t)$ is a Finite Impulse Response (FIR) model.

The predictor for Eqn. (3.38) can now be given as follows [11]:

$$\hat{y}(t|\theta) = B(q)u(t) + [1 - A(q)]y(t). \tag{3.42}$$

The following vector, called a *linear regressor vector*, is introduced:

$$\varphi(t) = [-y(t-1) \; -y(t-2)... - y(t-n_a) \; u(t-1) \; u(t-2)...u(t-n_b)]^T \tag{3.43}$$

With this vector, the predictor can be written as a scalar product between a known vector $\varphi(t)$ and the parameter vector θ. Such a model is called a *linear regression* in statistics [11]:

$$\hat{y}(t|\theta) = \theta^T\varphi(t) = \varphi^T(t)\theta. \tag{3.44}$$

Consider the following generalised model structure [6, 11]:

$$A(q)y(t) = \frac{B(q)}{F(q)}u(t) + \frac{C(q)}{D(q)}e(t), \tag{3.45}$$

where the $C(q)$-, $D(q)$- and $F(q)$-polynomials are similar to $A(q)$. Sometimes the dynamics from $u(t)$ to $y(t)$ contains a delay of n_k samples, so that some leading coefficients of $B(q)$ are zero; that is, according to Ljung [11]:

$$B(q) = b_{n_k}q^{-n_k} + b_{n_k+1}q^{-n_k-1} + ... + b_{n_k+n_b-1}q^{-n_k-n_b+1}. \tag{3.46}$$

The model structure, given by Eqn. (3.45), contains several common special cases, some of which are listed in Table 3.1.

For the model structures other than ARX and FIR, there is no linear regression. This can be demonstrated for the ARMAX model. Here the regressor vector is

$$\varphi(t,\theta) = [-y(t-1)... - y(t-n_a) \; u(t-1)...u(t-n_b) \; \varepsilon(t-1,\theta) \; ...\varepsilon(t-n_c,\theta)]^T \tag{3.47}$$

Table 3.1: Common model structures.

Polynomials Used in Eqn. (3.45)	Name of Model Structure	
$B(q)$	Finite Impulse Response	FIR
$A(q), B(q)$	Auto-Regressive with External input	ARX
$A(q), B(q), C(q)$	Auto-Regressive Moving Average with External input	ARMAX
$B(q), F(q)$	Output Error	OE
$B(q), C(q), D(q), F(q)$	Box-Jenkins	BJ

and the predictor becomes

$$\hat{y}(t|\theta) = \varphi^T(t,\theta)\theta. \tag{3.48}$$

Equation (3.48) is pseudolinear regression, due to the nonlinear effect of θ on the vector $\varphi(t,\theta)$ [11].

3.7.4 Computing the Estimate

Forssell [6] sates that if an FIR or an ARX model is used together with a quadratic criterion function, the standard LSE method is obtained. Then, to find the estimate $\hat{\theta}_N$, one only has to solve a standard LSE problem, which can be done analytically.

For the LSE method the prediction error, given by Eqn. (3.30), becomes

$$\varepsilon(t,\theta) = y(t) - \varphi^T(t)\theta, \tag{3.49}$$

and the criterion function, given by Eqn. (3.33), becomes

$$V_N(\theta, Z^N) = \frac{1}{N}\sum_{t-1}^{N}\frac{1}{2}\left[y(t) - \varphi^T(t)\theta\right]^2. \tag{3.50}$$

Eqn. (3.50) can be minimised analytically, which gives the LSE, provided the inverse of the term between the square brackets exists [11]:

$$\hat{\theta}_N^{LS} = \arg\min_{\theta\in D_M} V_N(\theta, Z^N) = \left[\frac{1}{N}\sum_{t-1}^{N}\varphi(t)\varphi^T(t)\right]^{-1}\frac{1}{N}\sum_{t-1}^{N}\varphi(t)y(t). \tag{3.51}$$

For other parameterisations and criteria, one has to rely on some iterative scheme to find $\hat{\theta}$. The standard choice is to use a search routine of the form

$$\hat{\theta}_N^{(i+1)} = \hat{\theta}_N^{(i)} - \mu_N^{(i)} \left[R_N^{(i)}\right]^{-1} V_N'(\hat{\theta}_N^{(i)}, Z^N), \tag{3.52}$$

where $V_N'(\hat{\theta}_N^{(i)}, Z^N)$ denotes the gradient with respect to θ of the criterion function, given by Eqn. (3.33). $R_N^{(i)}$ is a matrix that modifies the search direction, and $\mu_N^{(i)}$ is a scaling factor that determines the step length.

3.7.5 Consistency and Identifiability

The following consistency and identifiability results for the direct closed-loop SID approach can be found in Forssell's dissertation [6]. Consider the PEM method applied with quadratic criterion

$$V_N(\theta, Z^N) = \frac{1}{N}\sum_{t=1}^{N}\frac{1}{2}\varepsilon^2(t,\theta). \tag{3.53}$$

Under mild conditions it is true that

$$V_N(\theta, Z^N) \to \bar{V}(\theta) = \bar{E}\frac{1}{2}\varepsilon^2(t,\theta) \quad \text{with probability (w.p). 1 as } N \to \infty \quad \text{and} \tag{3.54}$$

$$\hat{\theta}_N \to D_c = \arg\min_{\theta \in D_M} \bar{V}(\theta) \quad \text{w.p. 1 as } N \to \infty. \tag{3.55}$$

Using Parseval's relationship [33], one can then write

$$\bar{V}(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{2}\Phi_\varepsilon(\omega)d\omega, \tag{3.56}$$

where $\Phi_\varepsilon(\omega)$ is the spectrum of the prediction error. If the data were generated by

$$y(t) = G_0(q)u(t) + H_0(q)e(t), \tag{3.57}$$

it can be shown that

$$\begin{aligned}\varepsilon(t,\theta) &= H^{-1}(q,\theta)(y(t) - G(q,\theta)u(t)) \\ &= H^{-1}(q,\theta)[G_0(q) - G(q,\theta)u(t) + (H_0(q) - H(q,\theta)e(t)) + e(t)]\end{aligned} \tag{3.58}$$

$$= H^{-1}(q,\theta)\tilde{T}^T(q,\theta)\chi_0(t) + e(t),$$

with $\tilde{T}(q,\theta) = T_0(q) - T(q,\theta)$.

If $G(q,\theta)u(t)$ and $G_0(q)u(t)$ depend only on $e(s)$ for $s < t$, and $H(q,\theta)$ and $H_0(q)$ are monic, then the last term of Eqn. (3.58) is independent of the other terms and

$$\Phi_\varepsilon(\omega) = \frac{1}{|\, H(e^{i\omega},\theta)\,|^2}\tilde{T}^T(e^{i\omega},\theta)\Phi_{\chi_0}(\omega)\tilde{T}(e^{-i\omega},\theta) + \lambda_0. \tag{3.59}$$

Together with Eqns. (3.55) and (3.56)

$$\hat{\theta}_N \to D_c = \arg\min_{\theta\in D_M}\frac{1}{2\pi}\int_{-\pi}^{\pi}\tilde{T}^T(e^{i\omega},\theta)\Phi_{\chi_0}(\omega)\tilde{T}(e^{-i\omega},\theta)\frac{1}{|\, H(e^{i\omega},\theta)\,|^2}d\omega$$

$$\text{w.p. } 1 \text{ as} N \to \infty. \tag{3.60}$$

Forssell [6] states that the result of Eqn. (3.60) shows that the PEM estimate $\hat{T}(q,\hat{\theta}_N)$ will converge to the true transfer function $T_0(q)$, if the parameterisation is flexible enough so that $S \in M$, and if the structure $\tilde{T}^T(e^{i\omega},\theta)$ does not lie in the left nul space of $\Phi_{\chi_0}(\omega)$. Then it can be said that the data are *informative enough* with respect to the chosen model structure [11]. An *informative* data set is one for which the matrix $\Phi_{\chi_0}(\omega)$ is positive definite for almost all frequencies. In the informative case, the limit estimate will always be such that $\tilde{T}^T(e^{i\omega},\theta) = 0$, i.e. $\hat{T}(e^{i\omega})$ will tend to $T_0(q)$ as the number of data points tends to infinity.

If the following factorization is considered:

$$\Phi_{\chi_0}(\omega) = \begin{bmatrix} 1 & \Phi_{ue}(\omega)/\lambda_0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \Phi_u^r(\omega) & 0 \\ 0 & \lambda_0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ \Phi_{eu}(\omega)/\lambda_0 & 1 \end{bmatrix}, \tag{3.61}$$

it follows that $\Phi_{\chi_0}(\omega)$ is positive definite for almost all frequencies if and only if $\Phi_u^r(\omega) > 0$ for almost all frequencies. This condition is ensured when there is a nonlinear relationship between the input and the output (see Section 3.5). With a linear feedback law $\Phi_u^r(\omega) = |S_0(e^{i\omega})|^2\,\Phi_r(\omega)$ and because the analytic function $S_0(e^{i\omega})$ can have only a finite number of zeros, the condition becomes $\Phi_r(\omega) > 0$ for almost all frequencies. That is the reference signal $r(t)$ should be persistently exciting [11].

The following will ensure $\Phi_u^r(\omega) > 0$ for almost all ω, i.e. identifiability [31]:

- $r_a(t)$ or $r_b(t)$ should be persistently exciting, i.e. $\Phi_a(\omega) > 0$ or $\Phi_b(\omega) > 0$ for almost all ω, or

- $C(q)$ should be a controller of sufficiently high order, or
- $C(q)$ should be a controller that switches between several settings during the experiment.

3.7.6 Bias Distribution

Direct Approach: Again, for the direct approach, Forssell [6] shows that by inserting the factorization

$$\Phi_{\chi_0}(\omega) = \begin{bmatrix} 1 & 0 \\ \Phi_{ue}(\omega)/\Phi_u(\omega) & 1 \end{bmatrix} \begin{bmatrix} \Phi_u(\omega) & 0 \\ 0 & \Phi_e^r(\omega) \end{bmatrix} \begin{bmatrix} 1 & \Phi_{eu}(\omega)/\Phi_u(\omega) \\ 0 & 1 \end{bmatrix} \quad (3.62)$$

into Eqn. (3.60), gives the following estimate $\hat{\theta}_N$ as the number of data tends to infinity:

$$\hat{\theta}_N \to D_c = \arg\min_{\theta \in D_M} \frac{1}{2\pi} \int_{-\pi}^{\pi} \{ \left| G_0(e^{i\omega}) + B(e^{i\omega},\theta) - G(e^{i\omega},\theta) \right|^2 \Phi_u(\omega) \quad (3.63)$$

$$+ \left| H_0(e^{i\omega}) - H(e^{i\omega},\theta) \right|^2 \Phi_e^r(\omega) \} \frac{1}{|H(e^{i\omega},\theta)|^2} d\omega \quad \text{w.p. 1 as } N \to \infty,$$

where

$$\begin{aligned} B(e^{i\omega},\theta) &= (H_0(e^{i\omega},\theta) - H(e^{i\omega},\theta)) \frac{\Phi_{eu}(\omega)}{\Phi_u(\omega)} \\ &= \frac{\lambda_0}{\Phi_u(\omega)} \cdot \frac{\Phi_u^e(\omega)}{\Phi_u(\omega)} \cdot \left| H_0(e^{i\omega},\theta) - H(e^{i\omega},\theta) \right|^2. \end{aligned} \quad (3.64)$$

A variant of this result is given in [12, 31]. Ljung [11] states that the bias $B(e^{i\omega},\theta)$ will be small in the frequency ranges where either or all of the following holds:

- the noise model is good, i.e. $H_0(e^{i\omega},\theta) - H(e^{i\omega},\theta)$ is small,
- the feedback contribution to the input spectrum is small, i.e. $\Phi_u^e(\omega)/\Phi_u(\omega)$ is small, and
- the Signal-to-Noise Ratio (SNR) is good, i.e. $\lambda_0/\Phi_u(\omega)$ is small.

If the system was operating in open-loop, so that $\Phi_{eu}(\omega) = 0$, then the bias term $B(e^{i\omega},\theta)$ equals zero regardless of the noise model $H(e^{i\omega},\theta)$ [11].

Indirect Approach: For the indirect approach the closed-loop system can be parameterised in terms of the open-loop system as follows

$$y(t) = \frac{G(q,\theta)}{1+G(q,\theta)C(q)}r(t) + H_*(q)e(t), \tag{3.65}$$

where $H_*(q)$ is a fixed noise model. The indirect method is equivalent to the direct method with the noise model $H(q,\theta) = (1+G(q,\theta)C(q))H_*(q)$, as covered in [12]. If the model structure is as given by Eqn. (3.65) then Forssell [6] shows that

$$\hat{\theta}_N \to D_c = \arg\min_{\theta\in D_M} \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\frac{G_0(e^{i\omega}) - G(e^{i\omega},\theta)}{1+G(e^{i\omega},\theta)C(e^{i\omega})}\right|^2 \frac{|S_0(e^{i\omega})|^2\,\Phi_r(\omega)}{|H_*(e^{i\omega})|^2}d\omega \tag{3.66}$$

$$\text{w.p. 1 as } N \to \infty$$

This means that the indirect method can give unbiased estimates of $G_0(e^{i\omega})$ if the parameterisation of $G(e^{i\omega},\theta)$ is flexible enough, that is, $G_0 \in G$, even with a fixed noise model. This is the main advantage of the indirect method.

Joint Input-Output Approach: It is shown in [12], that, for the joint input-output approach, if the model structure is:

$$y(t) = G(q,\theta)\hat{u}(t) + H_*(q)e(t), \quad \hat{u}(t) = \hat{S}_N(q)r(t), \quad \tilde{u}(t) = u(t) - \hat{u}(t) \tag{3.67}$$

then

$$\hat{\theta}_N \to D_c = \arg\min_{\theta\in D_M} \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|G_0(e^{i\omega}) + \tilde{B}(e^{i\omega}) - G(e^{i\omega},\theta)\right|^2 \frac{\Phi_{\hat{u}}(\omega)}{|H_*(e^{i\omega})|^2}d\omega \tag{3.68}$$

$$\text{w.p. 1 as } N \to \infty,$$

where

$$\tilde{B}(e^{i\omega}) = G_0(e^{i\omega})\Phi_{\tilde{u}\hat{u}}(\omega)\Phi_{\hat{u}}^{-1}(\omega). \tag{3.69}$$

Thus, an unbiased estimation of $G_0(e^{i\omega})$ can be obtained, regardless of the noise model used, as long as the correlation between $\hat{u}$ and $\tilde{u}$ is negligible.

3.7.7 Asymptotic Variance Distribution

Direct Approach: For the direct approach, Ljung [11] derives the following equations for asymptotic variance of the estimated transfer function:

$$Cov\,\hat{G}_N \sim \frac{n}{N}\frac{\Phi_v(\omega)}{\Phi_u^r(\omega)}, \tag{3.70}$$

where n is the model order and N is the number of data samples. The denominator of Eqn. (3.70) contains the spectrum of that part of the input that originates from the reference signal $r(t)$. The open-loop expression contains the *total* input spectrum in the denominator [11]. It is the signal-to-noise ratio, where *signal* is the part of the plant input that derives from the injected reference, that determines the quality of the open-loop transfer function estimate. From this perspective, the part of the input that originates from the feedback has no information value when estimating $G(q)$ [11].

The purpose of feedback is to make the sensitivity function S_0 small, especially at frequencies with disturbances. Feedback will thus decrease the information content of measured data at these frequencies, but it will also allow one to inject more input power at certain frequency ranges, without increasing the output power [11].

Ljung [11] stresses that the result of Eqn. (3.70) is asymptotic when the orders of both $G(q)$ and $H(q)$, as well as N, tends to infinity.

Indirect and Joint Input-Output Approaches: Suppose that the input is generated as in Eqn. (3.5), i.e. linear feedback is present. Forssell and Ljung [12] show that when the transfer function estimate $\hat{G}_N$ is found using either the indirect or the joint input-output approach, Eqn. (3.70) also holds for these approaches.

Forssell [6] shows that as far as the *asymptotic variance of the parameter estimates* is concerned, i.e. in the finite model order case, the indirect and joint input-output approaches are typically less precise than the optimal direct method that meets the Cramèr-Rao bound. In the direct approach the noise in the loop is used to reduce the variance, while in the other closed-loop approaches, the part of the input signal that originates from the noise is not used in the identification process. However, this effect of the noise vanishes as more and more parameters are used to estimate the noise model $H(q)$ [12].

Please note that if the *variance of* a model is small the model is *precise* and if the bias of a model is small the model is *accurate*.

3.8 Excitation Signals

An experiment is *informative enough* if it generates a data set that is informative enough. In open-loop this is achieved when the input $u(t)$ is PE of a certain order, i.e. so that it contains sufficiently many distinct frequencies. As mentioned in Section 3.5, in closed-loop the experiment may still be non-informative for the above-mentioned situation, and the easiest way to ensure an *informative* experiment is to ensure that $r_a(t)$ or $r_b(t)$ is PE [31].

The excitation signals determine the operating point of the system, as well as which parts and modes of the system are excited during the experiment. The user's freedom in choosing these signals' characteristics may vary considerably according to the application. In the process industry it is not always possible to manipulate a system during normal production [11].

Two different aspects are associated with the choice of excitation signals. One concern is the second-order properties of the signal, such as its spectrum. The other concern is the *shape* of the signal. Excitation signals can be sums of sinusoids, filtered white noise, pseudorandom signals, or binary signals (assuming only two values), and so on [34]. Typical signals are discussed below.

Step or Pulses: The step and pulse signals are often used for transient analysis. Typical information acquired, are the largest and smallest frequencies and the static gain of the process. These signals can be used to determine which type of excitation signals to consider in the final experiment. However, when a plant has a low frequency bandwidth, these signals will suffice, since the frequency components that make up the step excite the plant sufficiently to obtain an accurate model with low uncertainty around the crossover frequency region [34].

White Noise: General use is made of signals that are broad-banded, i.e., they have a spectrum that covers a wide frequency range. This generally guarantees that the signals are sufficiently exciting. One such signal is discrete white noise. White noise has a flat frequency spectrum. Such a signal can be generated from white Gaussian noise, filtered through a linear filter. Virtually any signal spectrum can be achieved in this way [34].

Random Binary Signals: A Random Binary Signal (RBS) is a random process that assumes only two values. It has the same spectral properties as white noise and can be generated in a number of different ways. Since the signal is binary, it has maximum signal power under amplitude bound constraints [35].

Pseudo-Random Binary Signals: A Pseudo-Random Binary Signal (PRBS) is a periodic, deterministic signal with white-noise-like properties. Typically the clock frequency of the PRBS should be about 2.5 times faster than the bandwidth to be covered by the signal [35].

Multisinusoidal Signal: Instead of distributing the input power over a relatively wide range of frequencies, one can also choose to concentrate on a relatively small number of frequency points. This can be done by using a periodic excitation signal, e.g. a multisinusoidal signal. This kind of periodic excitation is beneficial in, e.g. nonparametric model identification, and in situations where accurate knowledge of the plant is required at a limited number of frequencies [34].

Periodic Excitation: Forssell [6] shows that there are several advantages in using periodic excitation signals, such as PRBS or multisinusoidal signals. The most obvious advantages are the data reduction resulting from the averaging of the measured signals over the periods. Another advantage is that with periodic excitation it is possible to separate the *signal* from the output *noise* and therefore the noise properties can be estimated separately.

Theoretically, in the absence of noise, any arrangement of n non-zero points in the frequency domain of the excitation signal will be sufficient to uniquely determine n parameters. However, with the slightest bit of noise, the whole fit can become disastrous. In the more practical case were there is noise and only a finite amount of data points, a *good* arrangement of n non-zero points in the frequency domain is required for a proper fit. It will also help to include more than n frequency points in the spectrum. Even then, the arrangement of these points is still relevant, e.g. 20 points around the resonance frequency of a second order system are much better than 100 points in the roll-off region. Also, because of correlation, the effective number of points might be less [36].

In both the bias and variance expressions, the input spectrum appears as a weighting term [11]. The bias and variance will be less where the input spectrum is large. Therefore, it is best to put the energy of the input in that region where one wants to model the plant accurately. It is also advisable to choose a signal with a spectrum that is non-zero at frequencies where the system has a good SNR [36].

In off-line situations it is always a good idea to check the spectrum of a signal before using it as an excitation signal, to see if it is acceptable.

3.9 Inter-Sampling Approach

3.9.1 Introduction

In the restrictive situation were the reference signal is not PE, it usually would mean that the controller should be of a sufficient high order or time-varying in order to ensure identifiability of the plant. However, Sun *et al.* [18] show in their paper that the information obtained by inter-sampling the plant output during the control interval can extend the identification capability.

Output inter-sampling, which is also referred to as output over-sampling, is a very useful technique in blind identification problems [37]. It is also effective in closed-loop SID. The inter-sampling approach is actually the direct closed-loop SID approach, which uses the output inter-sampling technique to address the problem of identifiability [18]. Inter-sampling the output in this context possibly has the same effect as a nonlinear controller [36].

3.9.2 Model Representation

In their proof Sun *et al.* [18] consider a strictly proper linear plant with time delay τ, which is managed by a digital controller whose holding period is T. The control input to the plant is then a piecewise signal with a holding period T. The closed-loop configuration is as shown in Fig. 3.2, where $y(t)$ is the plant output sampled at a sampling interval T. The discrete model of the plant $G(q)$, identified from data sampled at an interval of T, can be expressed as

$$G(q) = \frac{B(q)}{A(q)} = \frac{b_{\tau_T} q^{-\tau_T} + \ldots + b_{\tau_T+n_b-1} q^{-\tau_T-n_b+1}}{1 + a_1 q^{-1} + \ldots + a_{n_a} q^{-n_a}}, \tag{3.71}$$

and the controller $C(q)$ can be expressed as

$$C(q) = \frac{L(q)}{P(q)} = \frac{l_{\tau_l} q^{-\tau_T} + \ldots + l_{\tau_l+n_l-1} q^{-\tau_l-n_l+1}}{1 + p_1 q^{-1} + \ldots + p_{n_p} q^{-n_p}}, \tag{3.72}$$

where τ_T is the discrete-time delay. The model in Eqn. (3.71) is referred to as the T-model. Please note that the notation used in the proof of Sun *et al.* [18] has been changed to match the notation, used in this work.

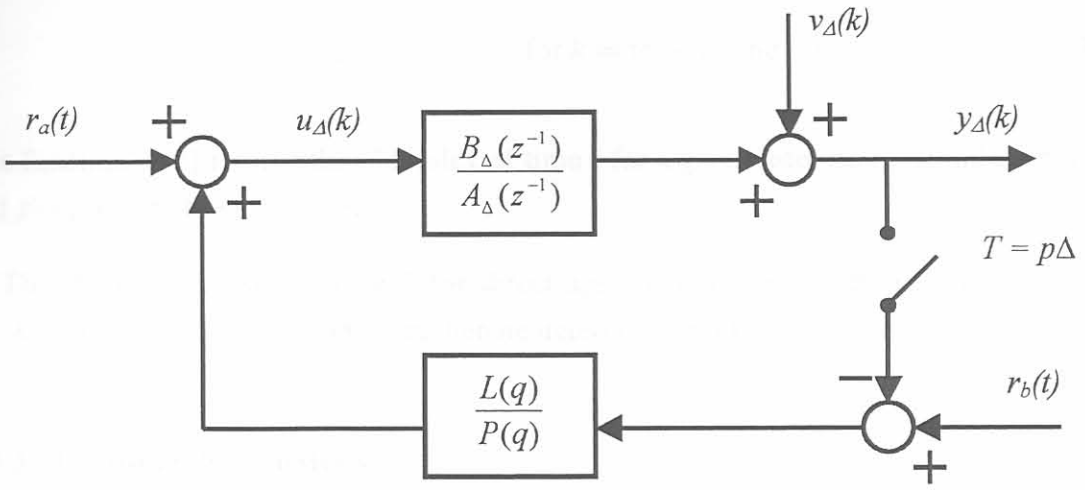

Figure 3.3: Closed-loop identification by output inter-sampling.

Let $y_\Delta(k)$ in Fig. 3.3 be the plant output sampled at a short sampling interval $\Delta = T/p$, where p is an integer, while the control interval is still T. Therefore, the output is sampled p times faster than the rate at which control action is taken. The controller only uses the plant output of every $T = p\Delta$ for control, i.e. $u_\Delta(k) = u_\Delta(k+1\Delta) = ... = u_\Delta(k+(p-1)\Delta)$. The plant Δ-model with sampling interval Δ is given by

$$\frac{B_\Delta(z^{-1})}{A_\Delta(z^{-1})} = \frac{b_{\Delta,\tau_\Delta} z^{-\tau_\Delta} + ... + b_{\Delta,\tau_\Delta+n_{b\Delta}-1} z^{-\tau_\Delta-n_{b\Delta}+1}}{1 + a_{\Delta,1} z^{-1} + ... + a_{\Delta,n_{a\Delta}} z^{-n_{a\Delta}}}, \tag{3.73}$$

where z^{-1} is a shift operator, like q, and τ_Δ is the corresponding delay time.

The polynomials $A(q)$ and $B(q)$ of the T-model can be uniquely determined from the Δ-model as follows [18]:

$$\begin{aligned} A(q) &= \det(I - A^p q^{-1}) \\ B(q) &= q^{-\alpha(\tau_\Delta-1)} c^T \mathrm{adj}(I - A^p q^{-1}) \left(\sum_{i=0}^{\beta(-\tau_\Delta)} A^i b q^{-1} + \sum_{i=\beta(-\tau_\Delta)+1}^{p-1} A^i b q^{-2} \right), \end{aligned} \tag{3.74}$$

where A, b, and c are the state-space realization of the Δ-model and *det* and *adj* are the matrix determinant and adjoint matrix respectively. The adj$U$ is defined as the matrix formed from the transpose of matrix U after all elements have been replaced by their cofactors [39].

The functions $\alpha(\cdot)$ and $\beta(\cdot)$ are defined as

$$\alpha(k) = t, \quad \beta(k) = j, \quad \text{for } k = tp + j \quad \text{and } 0 \leq j < p. \tag{3.75}$$

The function $\alpha(k)$ returns the old index of time t for a given inter-sampled index of time k and $\beta(k)$ returns the remainder.

The Δ-model is estimated with the direct approach, as described in Section 3.7, from $u_\Delta(k)$ and $y_\Delta(k)$. The T-model can then be determined making use of Eqn. (3.74).

3.9.3 Identifiability Analysis

The identifiability can now be considered under the restrictive situation where $r_a(t) = 0$ and $r_b(t) = 0$. $u(t)$ can then be expressed by

$$u(t) = \frac{-L(q)A(q)}{P(q)A(q) + L(q)B(q)} v(t). \tag{3.76}$$

Let the transfer function of the plant Δ-model be given by

$$y_\Delta(k) = \frac{B_\Delta(z^{-1})}{A_\Delta(z^{-1})} u_\Delta(k) + v_\Delta(k). \tag{3.77}$$

The prediction error, given by Eqn. (3.30), then becomes

$$\varepsilon_\Delta(k, \hat{\theta}_\Delta) = \frac{\hat{\Gamma}_\Delta(z^{-1})}{\hat{\Pi}_\Delta(z^{-1})} (\hat{A}_\Delta(z^{-1}) y_\Delta(k) - \hat{B}_\Delta(z^{-1}) u_\Delta(k)). \tag{3.78}$$

Here $\Gamma_\Delta(z^{-1})$ and $\Pi_\Delta(z^{-1})$ are coprime, i.e. have no common factors [11], minimum phase polynomials. The gradient vector of $\varepsilon_\Delta(k, \hat{\theta}_\Delta)$, the regressor, is then

$$\varphi_\Delta(k, \hat{\theta}_\Delta) = -\frac{d\varepsilon_\Delta(k, \hat{\theta}_\Delta)}{d\hat{\theta}_\Delta}. \tag{3.79}$$

The following conditions are assumed:

- the true system model is in the model set defined by Eqn. (3.77) and Eqn. (3.78),
- the closed-loop is stable and $\tau_T \geq 1$,
- $\varepsilon_\Delta(k)$ is a stationary signal with zero mean and finite fourth-order moments [38], and is independent of $r_1(t)$ and $r_2(t)$,
- $A_\Delta(z^{-1})$, $\Gamma_\Delta(z^{-1})$ and $\Pi_\Delta(z^{-1})$ are minimum phase and any pair of them is coprime,

- $\Pi_\Delta(z^{-1})/\Gamma_\Delta(z^{-1})$ and $B_\Delta(z^{-1})/A_\Delta(z^{-1})$ are smooth functions with respect to the true parameters.

By making use of the fact that $k = tp + j$ and that $v_\Delta(k)$ is a stationary signal Sun *et al.* [18] prove that $\varphi_\Delta(k, \hat{\theta}_\Delta)$ has cyclostationarity in k. With the aid of cyclostationarity, the correlation matrix of the regressor $\varphi_\Delta(k, \hat{\theta}_\Delta)$ has full-rank, even in closed-loop.

By making use of the convergence theorem of the PEM algorithm [11] for $p \geq 2$, and the fact that the correlation matrix of the regressor has full-rank, Sun *et al.* [18] prove that $\hat{\theta}_\Delta$ converges to a minimum such that

$$V_N(\hat{\theta}_\Delta) = \frac{1}{p}\sum_{j=0}^{p-1} E\varepsilon_\Delta^2(tp + j, \hat{\theta}_\Delta) = E\varepsilon_\Delta^2(k) \quad \text{w.p. 1 as } N \to \infty. \tag{3.80}$$

In other words the closed-loop Δ-model is identifiability. Moreover, following Eqn. (3.74), the T-model is also identifiable [18]. The proof does not depend on the property of $r_a(t)$, $r_b(t)$, or the controller structure.

Sun *et al.* [18] also mention that by increasing p, more information can be acquired, but may contribute little when Δ is very small. Therefore, the estimation error of the Δ-model will not always decrease much by increasing p. On the other hand, the convergence error from the Δ-model to the T-model increases with p. Both the estimate variance of the Δ-model and the model conversion error should, therefore, be considered.

According to Sun *et al.* [18] the main drawback of this method is still, as with the normal direct approach, that the estimation accuracy depends on the noise model estimation.

3.10 Conclusion

In this chapter it is shown that the basic problem with closed-loop data is that it is typically less informative about the open-loop system, and that many estimation methods may fail when applied in a direct way to closed-loop data, because of the correlation between the additive output noise and the input to the plant. Researchers have studied different ways of avoiding the problems associated with closed-loop identification, which lead to a characterization of the possible closed-loop SID methods into three categories: the direct, the indirect and the joint input-output approach.

From analysing the literature it is concluded that the PEM estimation method, applied in the direct fashion with a noise model that can describe the true noise properties, gives

consistent estimates and optimal accuracy. Therefore, the PEM method is currently regarded as the prime estimation method and this estimation method will, thus, also be used in the proposed methodology.

The *identifiability* analysis, found in literature, shows that the open-loop condition for informative experiments, namely that the input should be PE of sufficiently high order, does not ensure identifiability in closed-loop. In closed-loop only a part of the input originates from the reference signal. The other part of the input, which originates from the output, is correlated with noise and PE of this part will not ensure identifiability. So, it is concluded that, in closed-loop, the spectrum of that part of the input that originates from the reference signal should be non-zero. This means that there should be either a nonlinear relationship between the input and the output or the reference signal should be PE.

From the identifiability analysis of the new *inter-sampling method*, where the plant output is sampled at a higher rate than the control input, it can be concluded that this approach can also ensure identifiability. Thus, when the traditional identifiability conditions are not satisfied, the direct closed-loop identification scheme can also be considered. The precision of this method is discussed in the next chapter.

By reviewing the available bias and variance analysis of the different approached the following can be concluded:

Bias: In the direct approach, if the noise model cannot describe the true noise properties, a bias will result, but will be small in frequency ranges where either or all of the following holds:

- the noise model is good, i.e. $H_0(e^{i\omega}, \theta) - H(e^{i\omega}, \theta)$ is small,
- the feedback contribution to the input spectrum is small, i.e. $\Phi_u^e(\omega)/\Phi_u(\omega)$ is small, and
- the SNR is good, i.e. $\lambda_0/\Phi_u(\omega)$ is small.

However, the indirect approach can give unbiased estimates of $G_0(e^{i\omega})$ if the parameterisation of $G(e^{i\omega}, \theta)$ is flexible enough, even with a fixed noise model. Also the bias in the joint input-output approach does not depend on the noise model.

Variance of Estimated Transfer Function: The asymptotic variance of the estimated transfer function is equal for the direct, indirect and joint input-output approaches, if the feedback is linear. This variance is, however, worse than the variance obtained in open-loop SID, since the denominator of the variance expression contains the spectrum of only that part

of the input that originates from the reference signal, while the open-loop expression has the total input spectrum in the denominator.

Variance of Parameter Estimates: When the asymptotic variance of the parameter estimates, i.e. the finite model order case, is considered, the direct approach is optimal, since it meets the Cramèr-Rao bound. The other closed-loop approaches are less precise, since they do not meet this bound.

In the next chapter, the reviewed closed-loop SID theory, as well as other criteria, is considered when a closed-loop SID methodology for MPC controlled plants is chosen.