CHAPTER 5

A SUBSTITUTABLE TWO-PRODUCT INVENTORY SYSTEM WITH JOINT-ORDERING POLICY AND COMMON DEMAND

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5.1 INTRODUCTION

In the study of multi-product inventory systems, the concept of common demand for some products arises (Yadavalli and Hargreaves, 2003). For example, when a desired customer arrives at a shop that sells two brands of soft drinks, he/she may be satisfied by a soft drink of a particular brand with probability p_1 or by the other with probability p_2, $0 < p_i < 1$, $p_1 + p_2 = 1$. If any one of the products is out of stock, due to the desire, the customer will accept with probability 1 the other product that may be available in the shop. Also when the supplier is the same for several products under consideration, the dealer would prefer to have a simultaneous replenishment of all the products due to several reasons like cost considerations. Joint ordering policies for periodic inventory systems have been studied by several researchers (Bahadur and Acharya (1986) and Goyal and Satir (1989). Parlar and Weng (1997) and Anbazhagan (2002) developed optimal coordination policies for the supply and manufacturing departments. They considered a problem where the responsibility of the manufacturing department was to meet the random demand of a product with a short life cycle. The responsibility the supply department was to provide a sufficient amount of raw materials, so that the required production level could be achieved. Girlich (1996) and Yadavalli & Joubert (2003) studied a problem of joint coordination between manufacturing and supplying department encountered in a short life cycle multi-product environment. On the other hand, the study of continuous review multi-product inventory systems with common demand has not been considered so far in the literature. In this paper, an attempt is made to fill the gap by providing a study of a substitutable two-product inventory system with joint-ordering policy and common demand. The layout of the paper is as follows: In Section 2, the model assumptions and notation are provided. Certain auxiliary functions that characterise the occurrence of various events pertaining to the model are derived in Section 3. Section 4 gives some of the measures of the performance of the system. A cost analysis is provided in Section 5. Section 6 deals with the numerical results, which highlights the behaviour of the system.

5.2 MODEL ASSUMPTIONS AND NOTATION

5.2.1 ASSUMPTIONS

The following assumptions applies to the continuous review two-product inventory model:

(i) The maximum inventory level of product i is S_i, $i = 1, 2$.

(ii) Demands occur according to a Poisson process with parameter λ.

(iii) When both products are available, a demand is satisfied either with Product 1 with probability p_1 or with Product 2 with probability p_2, $0 < p_i < 1$, $p_1 + p_2 = 1$. When one of the products is out of stock, the demand is satisfied with the other product with probability 1. When both the products are out of stock, all demand is lost. That is, no backorders are allowed.

(iv) A re-order for both the products is placed at the epoch when the inventory level of product i reaches s_i and that of the other product j is greater than s_j, $j = 1, 2$ and $i \neq j$.

(v) The lead-time follows an arbitrary distribution with pdf $f(.)$.

(vi) When the re-order materializes, the inventory level of each product is brought to its maximum level.

5.2.2 NOTATION

The following notation is used in this chapter:

$L_i(t)$: The inventory level of product i at time t, $t = 1, 2$.

$L(t)$: The ordered pair $(L_1(t), L_2(t))$ representing the inventory level of the system at time t.

r_{ij} : Event that a replenishment of stock occurs at the epoch when the inventory level is (i, j).

γ_{ij} : Event that a re-order is placed when the product i reaches s_i and the level of the other product j, where $(i = 1, s_2 < j \leq S_2)$ or $(i = 2, s_1 < j \leq S_1)$.

$$B_i(n,t) = \frac{e^{-\lambda p_i t}(\lambda p_i t)^n}{n!}; \; i = 1, 2$$

$$B_3(n,t) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

E_0 : Initial condition representing the occurrence of an r-event.

5.3 AUXILIARY FUNCTIONS

The inventory level of each product is brought to its maximum at every epoch of replenishment. Hence the r-events constitute a renewal process. To derive the expression for the various measures of performance of the system, we proceed to study the renewal process of r-events. For this, certain auxiliary functions are defined to characterise the performance of the system in one cycle, which is the time interval between any two successive r-events.

5.3.1 FUNCTION ${}_r\phi_{ij}(t)$

Defining ${}_r\phi_{ij}(t)$ as:

$$ {}_r\phi_{ij}(t) = \lim_{\Delta \to 0} \frac{P[r_{ij}\ in\ (t, t+\Delta)/E_0]}{\Delta} $$

where $(i = 1, s_2 < j \le S_2)$ or $(i = 2, s_1 < j \le S_1)$. Then the function ${}_r\phi_{ij}(t)$ has the interpretation that it represents the probability that the inventory level of product i enters the state s_i in $(t, t+dt)$, the inventory level of the other product at time t is j and a re-order is placed in $(t, t+dt)$ given that an r-event has occurred at time $t = 0$. Since a re-order is made at the epoch when the inventory level of product i reaches s_i and the inventory level of product j is greater than s_j $(j \ne i)$, where

$$ {}_r\phi_{1j}(t) = B_1(S_1 - s_1 - 1, t)B_2(S_2 - j, t)\lambda p_1, \quad s_2 < j \le S_2 \tag{5.1} $$

$$ {}_r\phi_{2j}(t) = B_1(S_1 - j, t)B_2(S_2 - s_2 - 1, t)\lambda p_2, \quad s_1 < j \le S_1 \tag{5.2} $$

5.3.2 FUNCTION ${}_r h_l(t)$

Defining ${}_r h_l(t)$ as:

$$ {}_r h_l(t) = \lim_{\Delta \to 0} \frac{P[an\ l - event\ in\ (t, t+\Delta)/E_0\}}{\Delta} $$

Then ${}_r h_l(t)dt$ represents the probability that a demand occurs in $(t, t+dt)$ and is lost given that an r-event has occurred at time $t = 0$. To derive an expression for ${}_r h_l(t)$, we characterize the occurrence of the l-event in the following diagram (Figure 1). Accordingly, we have

$$
\begin{aligned}
{}_r h_l(t) = \sum_{i=s_2+1}^{S_2} {}_r\phi_{1i}(t) © \bar{F}(t)[\{\sum_{i=1}^{i} B_1(s_1-1,t)B_2(i-k,t)\lambda p_1 © B_3(k-1,t)\lambda\} \\
+\{\sum_{r=1}^{s_1} B_1(s_1-r,t)B_2(i-1,t)\lambda p_2 © B_3(r-1,t)\lambda\}]\lambda \\
+\sum_{i=s_1+1}^{S_1} {}_r\phi_{2i}(t) © \bar{F}(t)[\{\sum_{k=1}^{S_2} B_1(i-1,t)B_2(s_2-k,t) © B_3(k-1,t)\lambda\}
\end{aligned}
$$

$$
+\{\sum_{r=1}^{i} B_1(i-r,t)B_2(s_2-1,t)\lambda p_2 © B_3(r-1,t)\lambda\}]\lambda \qquad (5.3)
$$

5.3.3 FUNCTION ${}_r\psi_{ij}(t)$

Defining ${}_r\psi_{ij}(t)$ as:

$$
{}_r\psi_{ij}(t) = \lim_{\Delta\to 0} \frac{P[an\ r_{ij}\ event\ in\ (t, t+\Delta)/E_0]}{\Delta}
$$

Then ${}_r\psi_{ij}(t)$ represents the pdf of the interval between two successive replenishments and that the replenishment which occurs in $(t, t+\Delta)$ is of r_{ij} type. Note that at the time of occurrence of the r_{ij}-event, $S_1 - i$ units of product 1 and $S_2 - j$ units of product 2 are added to the stock. Accordingly, the following cases exist:

Case 1: $i > s_1$ and $j > s_2$

$$_{r}\psi_{ij}(t) = 0 \tag{5.4}$$

Case 2: $i > s_1$ and $0 < j \le s_2$

In this case, a γ_{2k}-event should occur in $(u, u+\Delta)$, $0 < u < t$, $i \le k \le S_1$.

Consequently,

$$_{r}\psi_{ij}(t) = \sum_{k=i}^{S_1} {}_{r}\phi_{2k}(t) © [B_1(k-i, t) B_2(s_2 - j, t)] f(t) \tag{5.5}$$

Case 3: $i > s_1$ and $j = 0$

Since the inventory level of product 2 is 0 at time t, a γ_{2k} event occurs in $(u, u+\Delta)$, $0 < u < t$, and the system enters the state $(k-k', 0)$ in $(v, v+\Delta)$, $u < v < t$ and is in state $(i, 0)$ at time t. Hence

$$_{r}\psi_{ij}(t) = \sum_{k=i}^{S_1} \sum_{k'=0}^{k-i} {}_{r}\phi_{2k}(t) © [B_1(k', t) B_2(s_2 - 1, t) \lambda p_2 © B_3(k - k' - i, t)] f(t) \tag{5.6}$$

Case 4: $0 < i \le s_1$ and $j \ge B_2$

As in Case 2,

$$_{r}\psi_{ij}(t) = \sum_{k=j}^{S_2} {}_{r}\phi_{1k}(t) © [B_1(s_1 - i, t) B_2(k - j, t)] f(t) \tag{5.7}$$

Case 5: $i = 0$ and $j > s_2$

This case is similar to Case 3 and

$$ {}_r\psi_{ij}(t) = \sum_{k=j}^{S_2} \sum_{k'=0}^{k-i} {}_r\phi_{1k}(t) \copyright [B_1(s_1 - i, t) B_2(k', t) \lambda p_1 \copyright B_3(k - k' - j, t)] f(t) \tag{5.8} $$

Case 6: $0 < i \leq s_1$ and $0 < j \leq s_2$

In this case, either a γ_{1k} event or a γ_{2k} event should occur in $(u, u + \Delta)$, $0 < u < t$. Hence

$$ \begin{aligned} {}_r\psi_{ij}(t) = & \sum_{k=s_2+1}^{S_2} {}_r\phi_{1k}(t) \copyright [B_1(s_1 - i, t) B_2(k - j, t)] f(t) \\ & + \sum_{k=s_1+1}^{S_1} {}_r\phi_{2k}(t) \copyright [B_1(k - i, t) B_2(s_2 - j, t)] f(t) \end{aligned} \tag{5.9} $$

Case 7: $0 < i \leq s_1$ and $j = 0$

At time $t = 0$, the system is in state (S_1, S_2) and enters the state (s_1, k) $k > s_1$ or the state (k, s_2), $k > s_2$ in $(u, u + \Delta)$ when a re-order is placed.

Then the system enters the state $(k, 0)$ in $(v, v + \Delta)$, $u < v < t$ and the inventory level is in state $(i, 0)$ at time t and the re-order materializes in $(t, t + \Delta)$. Hence

$$ {}_r\psi_{ij}(t) = \sum_{k=s_2+1}^{S_2} \sum_{k'=0}^{s_1-i} {}_r\phi_{1k}(t) \copyright [B_1(s_1 - k', t) B_2(k - 1, t) \lambda p_2 \tag{5.10} $$

$$© B_3(s_1 - k' - i, t)] f(t)$$

Case 8: $i = 0$ and $0 < j \le s_2$

This case is similar to Case 7. Hence

$$_r\psi_{ij}(t) = \sum_{k=s_2+1}^{S_2} \sum_{k'=0}^{k-j} {}_r\phi_{1k}(t) © [B_1(s_1 - 1, t) B_2(k', t) \lambda p_1$$

$$© B_3(k - k' - j, t)] f(t) + \sum_{k=s_1+1}^{S_1} \sum_{k'=0}^{s_2-j} {}_r\phi_{2k}(t)$$

$$© [B_1(k-1, t) B_2(k', t) \lambda p_1 © B_3(s_2 - k' - j, t)] f(t) \qquad (5.11)$$

Case 9: $i = 0$ and $j = 0$

At time $t = 0$, the inventory level is (S_1, S_2) and it enters the state (s_1, k), $k > s_2$ or the state (k, s_2), $k > s_1$ in $(u, u + \Delta)$, where a re-order is also placed. That re-order does not materialize in $(0, t)$ and the system enters the state $(r,0)$ or the state $(0, r)$ in $(v, v + \Delta)$, $0 < u < v < w < t$. The system then enters the state $(0,0)$ in $((w, w + \Delta)$, $0 < u < v < w < t$, and is in state $(0,0)$ at time t and the re-order materializes in $(t, t + \Delta)$.

Consequently,

$$_r\psi_{ij}(t) = \sum_{k=s_2+1}^{S_2} {}_r\phi_{1k}(t) © [\{\sum_{k'=0}^{k-i} B_1(s_1 - 1, t) B_2(k', t) \lambda p_1 © B_3(k, k' - 1, t) \lambda © 1]$$

$$+\{\sum_{k'=0}^{s_1-1} B_1(k',t)B_2(k-1,t)\lambda p_2 © B_3(s_1-k'-1,t)\lambda ©1\}]f(t)$$

$$+\sum_{k=s_1+1}^{S_1} {}_r\phi_{2k}(t) ©[\{\sum_{k'=0}^{s_2-1} B_1(k-1,t)B_2(k',t)\lambda p_1$$

$$© B_3(k-k'-1,t)\lambda ©1\}+\{\sum_{k'=0}^{k-1} B_1(k',t)B_2(s_2-1,t)\lambda p_2$$

$$© B_3(k-k'-1,t)\lambda ©1\}]f(t) \qquad (5.12)$$

5.3.4 FUNCTION ${}_r p_{ij}(t)$

Defining ${}_r p_{ij}(t)$ as:

$${}_r p_{ij}(t) = P[Y(t)=j, \mathrm{N}(r,t)=0/E_0] \;\; i=0,1,...,S_1;\;\; j=0,1,...,S_2$$

The following cases exists:

Case 1: $i > s_1$ and $j > s_2$

$${}_r p_{ij}(t) = B_1(S_1-i,t)B_2(S_2-j,t) \qquad (5.13)$$

Case 2: $0 < i < s_1$ and $j > s_2$

In this case a γ_{1k}-event, $j \le k < S_2$, should occur in $(u, u+\Delta)$, $0 < u < t$. Hence

$${}_r p_{ij}(t) = \sum_{k=i}^{S_1} {}_r\phi_{2k}(t) ©[B_1(s_1-i,t)B_2(k-j,t)]\overline{F}(t) \qquad (5.14)$$

Case 3: $i > s_1$ and $0 < j \le s_2$

This case is similar to case 2. Thus

$$ {}_r p_{ij}(t) = \sum_{k=i}^{S_1} {}_r \phi_{2k}(t) © [B_1(k-i, t) B_2(s_2 - j, t)] \overline{F}(t) \tag{5.15} $$

Case 4: $i > s_1$ and $j = 0$

Since the inventory level of product 2 is zero at t, a γ_{2k}-event occurs at $(u, u+\Delta)$, $0 < u < t$, and the inventory level enters the state $(k-k', 0)$ in $(v, v+\Delta)$, $0 < u < v < t$ and is in state $(i, 0)$ at time t. Hence

$$ {}_r p_{ij}(t) = \sum_{k=i}^{S_1} \sum_{k'=0}^{k-j} {}_r \phi_{2k}(t) © [B_1(k', t) B_2(s_1 - 1, t) \lambda p_2 \\ © B_3(k - k' - i, t)] \overline{F}(t) \tag{5.16} $$

Case 5: $i = 0$ and $j > s_2$

This case is similar to case 4. So, we obtain

$$ {}_r p_{ij}(t) = \sum_{k=i}^{S_1} \sum_{k'=0}^{k-j} {}_r \phi_{1k}(t) © [B_1(s_1 - 1, t) B_2(k' - 1, t) \lambda p_1 \\ © B_3(k - k' - j, t)] \overline{F}(t) \tag{5.17} $$

Case 6: $0 < i < s_1$ and $j = 0$

At time $t = 0$, the system is in state (S_1, S_2) and either enters the state $(s_1, k), k > s_2$ or enters the state (k, s_2), $k > s_1$, a re-order is placed in $(u, u+\Delta)$, $0 < u < t$. And the inventory level enters the state $(m,0), m \geq i$ in $(v, v+\Delta)$, $0 < u < v < t$, and the inventory is in state $(i,0)$ at time t. Hence

$$
\begin{aligned}
{}_r p_{ij}(t) = & \sum_{k=s_2+1}^{S_2} {}_r\phi_{1k}(t) © [\sum_{k'=0}^{s_1-i} B_1(k', t) B_2(k-1, t) \lambda p_2 © B_3(s_1 - k' - i, t)] \bar{F}(t) \\
& + \sum_{k=s_1+1}^{S_1} {}_r\phi_{2k} © [\sum_{k'=0}^{k-i} B_1(k', t) B_2(s_2 - 1, t) \lambda p_2 © B_3(k - k' - i, t)] \bar{F}(t)
\end{aligned}
$$

(5.18)

Case 7: $i = 0$ and $0 < j \leq s_2$

This is similar to Case 6. Hence

$$
\begin{aligned}
{}_r p_{ij}(t) = & \sum_{k=s_2+1}^{S_2} {}_r\phi_{1k}(t) © [\sum_{k'=0}^{k-j} B_1(s_1 - 1, t) B_2(k', t) \lambda p_1 © B_3(k - k' - j, t)] \bar{F}(t) \\
& + \sum_{k=s_1+1}^{S_1} {}_r\phi_{2k} © [\sum_{k'=0}^{s_2-j} B_1(k-1, t) B_2(k', t) \lambda p_1 © B_3(s_2 - k' - j, t)] \bar{F}(t)
\end{aligned}
$$

(5.19)

Case 8: $0 < i \leq s_1$ and $0 < j \leq s_2$

In this case either a γ_{1k}-event or a γ_{2k}-event should occur in $(u, u+\Delta)$, $0 < u < t$. The following equation is obtained

$$
{}_r p_{ij}(t) = \sum_{k=s_2+1}^{S_2} {}_r\phi_{1k}(t) © [B_1(s_1 - i, t) B_2(k - j, t) \bar{F}(t) + \sum_{k=s_1+1}^{S_1} {}_\Omega\Phi_{2k}(t)
$$

$$©[B_1(k-i,t)B_2(s_2-j,t)]\bar{F}(t) \tag{5.20}$$

Case 9: $i=0$ and $j=0$

At time $t=0$, the system is in state (S_1,S_2) and enters the state (s_1,k), $k>s_2$ or enters the state (k,s_2), $k>s_1$ and corresponding re-order is placed in $(u,u+\Delta)$. And the re-order does not materialize in $(0,t)$ and the system enters the state $(0,m)$ or $(m,0)$, $m>0$ in $(v,v+\Delta)$, $0<u<v<t$ and then enters the state $(0,0)$ in $(w,w+\Delta)$, $0<u<v<w<t$ and is in state $(0,0)$ at time t. Accordingly,

$$\begin{aligned}
{}_r p_{ij}(t) = \sum_{k=s_2+1}^{S_2} {}_r\phi_{1k}(t) ©[\{\sum_{k'=0}^{k-1} B_1(s_1-1,t)B_2(k',t)\lambda p_1 © B_3(k-k'-1,t)\lambda ©1\} \\
+\{\sum_{k'=0}^{s_1-1} B_1(k',t)B_2(k-1,t)\lambda p_2 © B_3(s_1-k'-1,t)\lambda ©1\}] \\
+\sum_{k=s_1+1}^{S_1} {}_r\phi_{2k}(t) ©[\{\sum_{k'=0}^{s_2-1} B_1(k-1,t)B_2(k',t)\lambda p_1 © B_3(k-k'-1,t)\lambda ©1\} \\
+\{\sum_{k'=0}^{s_1-1} B_1(k',t)B_2(s_2-1,t)\lambda p_2 © B_3(k-k'-1,t)\lambda ©1\}]\bar{F}(t)
\end{aligned} \tag{5.21}$$

Based on the above auxiliary functions (5.1) to (5.21), some measures of system performance are presented in the next section.

5.4 MEASURES OF SYSTEM PERFORMANCE

5.4.1 MEAN NUMBER OF REPLENISHMENTS

The r-events correspond to the epoch of replenishments, and as such they constitute a renewal process. The first-order product density $h_r(t)$ corresponding to the r-events is given by

$$h_r(t) = \sum_{n=1}^{\infty} g^{(n)}(t)$$

where $g(t)$ is the pdf of the interval between two successive occurrences of r-events.

To obtain an expression for $g(t)$, an expression for the survivor function $\overline{G}(t)$ corresponding to $g(t)$ is defined. Since $\overline{G}(t)$ is the probability that replenishment has not occurred up to time t, the following probabilities exist:

(i) A re-order is not placed up to time t

(ii) A re-order is placed in $(u, u+\Delta)$, $0 < u < t$, but it has not materialized until t

$$\overline{G}(t) = \sum_{i=0}^{S_1 - s_1 - 1} \sum_{j=0}^{S_2 - s_2 - 1} B_1(i,t) B_2(j,t) + \sum_{j=s_2+1}^{S_2} {}_r\phi_{ij}(t) \copyright \overline{F}(t) + \sum_{j=s_1+1}^{S_1} {}_r\phi_{2j}(t) \copyright \overline{F}(t) \quad (5.22)$$

Consequently, the mean number of replenishments is given by

$$E[\mathrm{N}(r,t)] = \int_0^t h_r(u)du$$

and the expected stationary rate of replenishments is given by

$$E(r) = \lim_{t\to\infty} h_r(t)$$

5.4.2 MEAN NUMBER OF RE-ORDERS PLACED

Defining $h_{\gamma_{ij}}(t)$ as:

$$h_{\gamma_{ij}}(t) = \lim_{\Delta\to 0} \frac{P[a\ \gamma_{ij} - event\ in\ (t, t+\Delta)/E_0]}{\Delta}$$

Since an epoch of re-order corresponds to the occurrence of a γ-event the first-order product density $h_\gamma(t)$ corresponding to re-orders is given by

$$h_\gamma(t) = \sum_{j=s_2+1}^{S_2} h_{\gamma ij}(t) + \sum_{j=s_1+1}^{S_1} h_{\gamma 2j}(t) \qquad (5.23)$$

To derive an expression for $h_{\gamma ij}(t)$, consider the following mutually exclusive and exhaustive possibilities

(i) No r-event has occurred up to time t

(ii) At least one r-event has occurred in $(0, t)$

$$h_{\gamma ij}(t) = {}_r\phi_{ij}(t) + h_r(t) © {}_r\phi_{ij}(t) \qquad (5.24)$$

Hence the mean number of re-orders placed in $(0, t)$ is given by

$$E[\mathrm{N}(\gamma,t)] = \sum_{j=s_2+1}^{S_2} \int_0^t h_{\gamma ij}(u)du + \sum_{j=s_1+1}^{S_1} \int_0^t h_{\gamma 2j}(u)du$$

The mean stationary rate of re-ordering is given by

$$\begin{aligned} E(\gamma) &= \lim_{t\to\infty} h_\gamma(t) \\ &= E(r)[\sum_{j=s_2+1}^{S_2} {}_r\phi_{1j}^*(0) + \sum_{j=s_1+1}^{S_1} {}_r\phi_{2j}^*(0)] \end{aligned} \tag{5.25}$$

where ${}_r\phi_{ij}^*(.)$ is the Laplace transform of ${}_r\phi_{ij}(.)$, (see Girlich, 2003).

5.4.3 MEAN NUMBER OF LOST DEMANDS

Let $h_l(t)$ be the first-order product density corresponding to the epochs of occurrences of lost demands. Then the following expression can be derived:

$$h_l(t) =_r \phi_1(t) + h_r(t) © {}_r h_l(t)\,. \tag{5.26}$$

Hence the mean number of lost demands in $[0,t]$ is given by

$$E[\mathrm{N}(l,t)] = \int_0^t h_1(u)du$$

and the mean stationary rate of lost demands is given by

$$\begin{aligned} E(l) &= \lim_{t\to\infty} h_1(t) \\ &= E(r)_r h_l^*(0) \end{aligned} \tag{5.27}$$

5.4.4 MEAN NUMBER OF UNITS REPLENISHED

At the occurrence of each r_{ij}-event $S_1 - i$ units of product 1 and $S_2 - j$ units of product 2 are replenished. Also note that $E(r_{ij})$ is the mean stationary state of r_{ij}-events and it is given by

$$E(r_{ij}) = \lim_{t \to \infty} h_{r_{ij}}(t)$$

where $h_{r_{ij}}(t)$ is the first order product density corresponding to r_{ij}-events. Then

$$h_{r_{ij}}(t) = {}_r\psi_{ij}(t) + h_r(t) © {}_r\psi_{ij}(t) \tag{5.28}$$

Consequently,

$$\begin{aligned} E(r_{ij}) &= \lim_{s \to 0} s[1 + h_r^*(s)]_r\, \psi_{ij}^*(s) \\ &= E(r)_r\, \psi_{ij}^*(0) \end{aligned} \tag{5.29}$$

Thus, the mean number of Product 1 that may be added to the inventory in unit time in the long run is given by

$$\sum_{i=0}^{s_1} \sum_{j=0}^{S_2} E(r_{ij})(S_1 - i) + \sum_{i=s_1+1}^{S_1} \sum_{j=0}^{s_2} E(r_{ij})(S_1 - i)$$

and, in the same manner, the mean number of Product 2 that may be added to the inventory in unit time in the long run is given by

$$\sum_{i=0}^{s_1}\sum_{j=0}^{S_2} E(r_{ij})(S_2 - j) + \sum_{i=s_1+1}^{S_1}\sum_{j=0}^{s_2} E(r_{ij})(S_2 - j) \tag{5.30}$$

5.4.5 DISTRIBUTION OF THE INVENTORY LEVEL

The probability distribution of the inventory level is defined by

$$p_{ij}(t) = P[Y(Z) = (i, j) / E_0]$$

where $0 \le i \le S_1$ and $0 \le j \le S_2$.

Using renewal theoretic arguments,

$$p_{ij}(t) = {}_r p_{ij}(t) + h_r(t) © {}_r\, p_{ij}(t) \tag{5.31}$$

Consequently, the stationary distribution of the inventory level is given by

$$\Pi_{ij} = \lim_{t\to\infty} p_{ij}(t) = E(r)_r p_{ij}^{*}(0)$$

(5.32)

5.5 COST ANALYSIS

We have two types of re-orders, namely

(i) the re-order is placed when the inventory level of Product 1 reaches s_1 or

(ii) the re-order is placed when the inventory level of Product 2 reaches s_2

It can be assumed that the two types of re-orders placed are with two different suppliers and hence that the corresponding costs are different. Let CR_i be the cost corresponding to a re-order due to the inventory level of product i reaching s_i, $i = 1, 2$. Let CL be the cost corresponding to a lost demand. Since $E(r)_r h_1^*(0)$ is the mean rate of the lost demand, the cost due to lost demands is given by $E(r)_r h_1^*(0) CL$. In the same way, the cost corresponding to re-orders placed is given by

$$E(r)[\sum_{j=s_2+1}^{S_2} {}_r\phi_{1j}^*(0)CR_1 + \sum_{j=s_1+1}^{S_1} {}_r\phi_{2j}^*(0)CR_2]$$

Hence the total cost is given by

$$Total\ Cost = E(r)[{}_r h_1^*(0)CL + \sum_{j=s_2+1}^{S_2} {}_r\phi_{1j}^*(0)CR_1 + \sum_{j=s_1+1}^{S_1} {}_r\phi_{2j}^*(0)CR_2] \tag{5.33}$$

The total cost can be considered as a function of s_1 and its optimal value can be obtained.

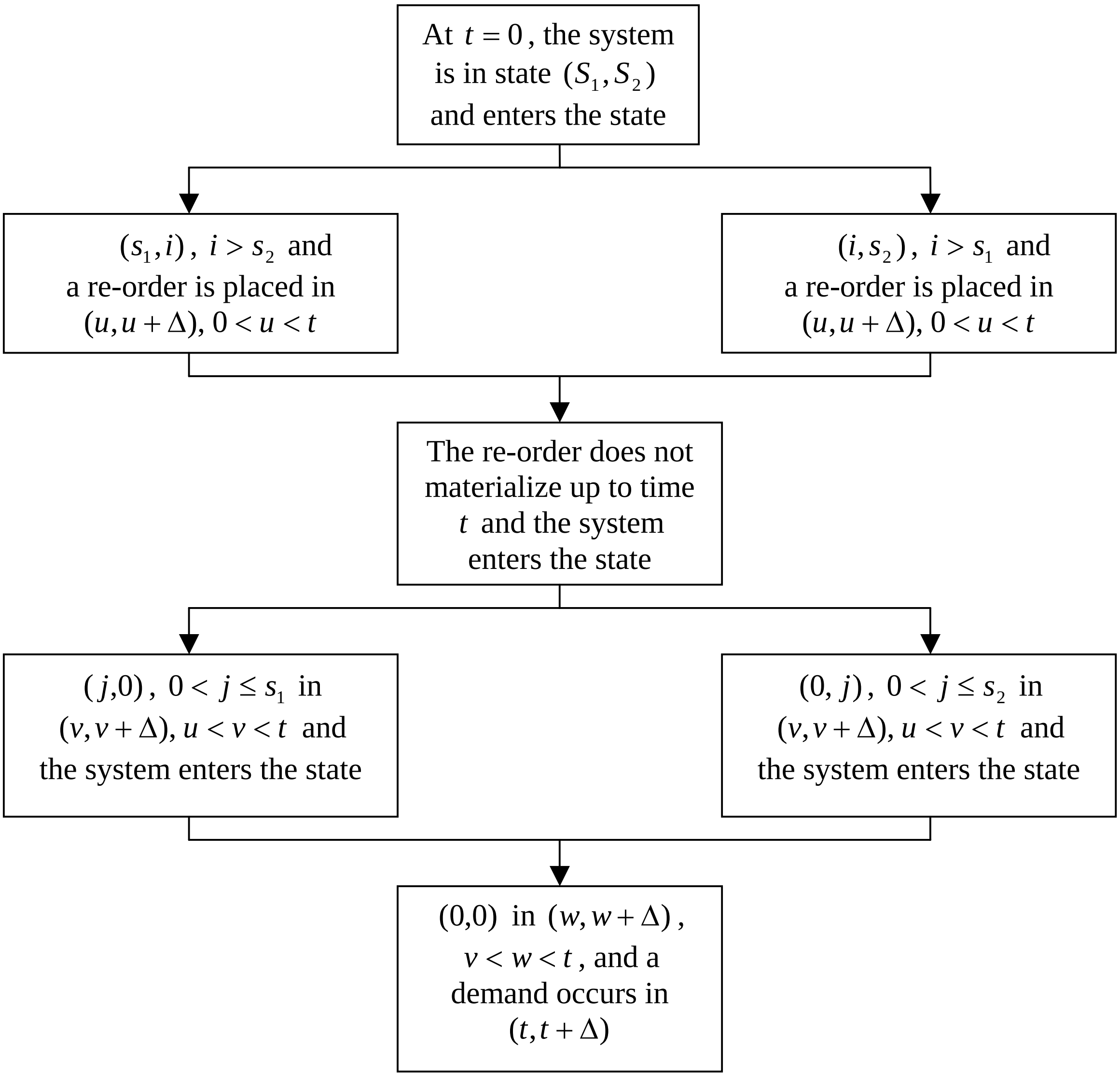

Figure 5.1: System State for Cost Function

5.6 NUMERICAL ILLUSTRATION

For the purpose of illustration, we assume that f(t) = a exp{-at} and the values of various parameters as follows:

λ = 1.2,
a = 0.5,
S_1 = 8,
S_2 = 5,
s_1 = 1,
CL =10,
CR_1 = 200,
CR_2 = 300

First, the re-order level for Product 1 is fixed as $s_1 = 2$ and the value of p_1 increased from 0.1 to 0.9 to obtain the behaviour of the mean rates of

(i) r-events,
(ii) γ_{ij}-events
(iii) Lost demands
(iv) Unit 1 replenished
(v) Unit 2 replenished
(vi) Total cost

From Table 5.1, it can be observed that, as p_1, the probability of demand for Product 1, increases,

(i) The mean rate of replenishments decreases and then increases
(ii) The mean rate of γ_{1j}-events increase and that of γ_{2j} decreases
(iii) The mean rate of lost demands increases

(iv) The mean rate of unit 1 replenished increases

(v) The mean rate of unit 2 replenished decreases

(vi) The mean rate of total cost decreases and then increases. The total cost is a minimum when $p_1 = 0.7$

Next, as p_1 is fixed and the re-order level for Product 1 increased, the results presented in Table 2 is obtained. The result is that, as s_1 increases with $p_1 = 0.7$,

(i) The mean rate of replenishments increases

(ii) The mean rate of γ_{1j} increases and that of γ_{2j} decreases

(iii) The mean rate of lost demands decreases

(iv) The mean rate of unit 1 replenished increases

(v) The mean rate of unit 2 replenished increases

(vi) The mean rate of total cost increases.

Re-Order Level for Product 1 Fixed at 2

p_1	p_2	LL1	RR	ERO1	ERO2	RLD	U1RR	U2RR	TCOST
0.1	0.9	2	2.852	0.000	2.852	0.003	0.908	7.534	855.613
0.2	0.8	2	1.496	0.005	1.492	0.004	1.113	4.107	448.500
0.3	0.7	2	1.022	0.026	0.996	0.008	1.362	2.909	304.097
0.4	0.6	2	0.790	0.078	0.711	0.014	1.689	2.300	229.198
0.5	0.5	2	0.675	0.171	0.503	0.026	2.117	1.934	185.494
0.6	0.4	2	0.643	0.310	0.333	0.056	2.683	1.679	162.366
0.7	0.3	2	0.699	0.510	0.189	0.141	3.498	1.464	160.152
0.8	0.2	2	0.908	0.831	0.078	0.421	4.944	1.251	193.664
0.9	0.1	2	1.633	1.620	0.014	1.668	9.013	1.030	344.680

p_1 : Probability of Demand for Product 1
p_2 : Probability of Demand for Product 2
LL1 : Re-Order Level for Product 1
RR : Rate of Replenishment
ERO1 : Rate of Type 1 Re-Order
ERO2 : Rate of Type 2 Re-Order
RLD : Rate of Lost Demand
U1RR : Rate of Units of Product 1 Replenishment
U2RR : Rate Of Units Of Product 2 Replenishment
TCOST : Rate of Total Cost

Table 5.1: Variation of Measures of System Performance Against the Probability of Demand for Product 1

Probability of Demand for Product 1 Fixed at Various Levels

p_1	p_2	LL1	RR	ERO1	ERO2	RLD	U1RR	U2RR	TCOST
0.3	0.7	1	1.014	0.011	1.003	0.104	1.330	2.889	303.227
0.3	0.7	2	1.022	0.026	0.996	0.008	1.362	2.909	304.097
0.3	0.7	3	1.041	0.060	0.981	0.005	1.408	2.946	306.296
0.5	0.5	1	0.634	0.109	0.525	0.116	1.888	1.826	180.451
0.5	0.5	2	0.675	0.171	0.503	0.026	2.117	1.934	185.494
0.5	0.5	3	0.741	0.269	0.472	0.009	2.284	2.041	195.454
0.7	0.3	1	0.600	0.390	0.210	0.942	2.690	1.224	150.363
0.7	0.3	2	0.699	0.510	0.189	0.141	3.498	1.464	160.152
0.7	0.3	3	0.848	0.684	0.165	0.019	3.977	1.633	186.306

p_1 : Probability of Demand for Product 1
p_2 : Probability of Demand for Product 2
LL1 : Re-Order Level for Product 1
RR : Rate of Replenishment
ERO1 : Rate of Type 1 Re-Order
ERO2 : Rate of Type 2 Re-Order
RLD : Rate of Lost Demand
U1RR : Rate of Units of Product 1 Replenishment
U2RR : Rate Of Units Of Product 2 Replenishment
TCOST : Rate of Total Cost

Table 5.2: Variation of Measures of System Performance Against Re-Order Level for Product 1

5.7 CONCLUSION

A substitutable two-product inventory system with joint-ordering policy is considered in this chapter. Common demands occur according to a Poisson process. A demand is satisfied either with an item of Product 1 with probability p_1 or with an item of Product 2 with probability p_2 ($p_1 + p_2 = 1$). When one of the products is out of stock, the demand is satisfied with the other available product with probability 1. Analyzing the imbedded renewal process describing the system, expressions for the stationary distribution of the inventory level and the stationary rates of the replenishments, the re-orders placed, the lost demands, and the units replenished are obtained. A cost analysis is also provided and a numerical example illustrates the results obtained.