CHAPTER 4

TWO-COMMODITY CONTINUOUS REVIEW INVENTORY SYSTEM WITH BULK DEMAND FOR ONE COMMODITY

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4.1 INTRODUCTION

With the advent of advanced computing systems, many industries and firms deal with multi-commodity systems. In dealing with such systems, models were initially proposed with independently established reorder points. In situations where several products compete for limited storage space, or share the same transport facility, or items are produced on (procured from) the same equipment (supplier), the above strategy overlooks the potential saving associated with joint replenishment, reduction in ordering and setup costs and allowing the user to take advantage of quantity discounts.

In continuous review inventory systems, Ballintify (1964) and Silver (1974) have considered a coordinated reordering policy, which is represented by the triplet (**S, c, s**), where the three parameters S_i, c_i and s_i are specified for each item *i* with $s_i \leq c_i \leq S_i$. In this policy, if the level of *i*-th commodity at any time is below s_i, an order is placed for $S_i - s_i$ items and at the same time, for any other item $j(\neq i)$ with available inventory at or below its can-order level c_j , an order is placed so as to bring its level back to its maximum capacity S_j . Subsequently many articles have appeared with models involving the above policy. Another article of interest is due to Federgruen, Groenevelt and Tijms (1984), which deals with the general case of compound Poisson demands and non-zero lead times. A review of inventory models under joint replenishment is provided by Goyal and Satir (1989).

Kalpakam and Arivarignan (1993) have introduced (s,S) policy with a single reorder level s defined in terms of the total number of items in the stock. This policy avoids separate ordering for each commodity and hence a single processing of orders for both commodities has some advantages in situation where in procurement is made from the same supplies, items are produced on the same machine, or items have to be supplied by the same transport facility.

A natural extension of (s,S) policy to two-commodity inventory system is to have two reorder levels and to place order for each commodity independent of other. But this policy will increase the total cost, as separate processing of two orders is required. Anbazhagan and Arivarignan (2000) have considered a two commodity inventory system with independent reorder levels where a joint order for both the commodities is placed only when the levels of both commodities are less than or equal to their respective reorder levels. The demand points form an independent Poisson process and the lead-time is distributed as negative exponential. They also assumed unit demands for both commodities.

In this chapter, the above work is extended by assuming unit demand for one commodity and bulk demand for the other commodity. The number of items demanded for the latter commodity is assumed to be a random variable Y with probability function $p_k = Pr\{Y = k\}$, $k = 1, 2, 3, \ldots$. A reorder is made for both commodities when the inventory levels of these commodities are at or below the respective inventory levels.

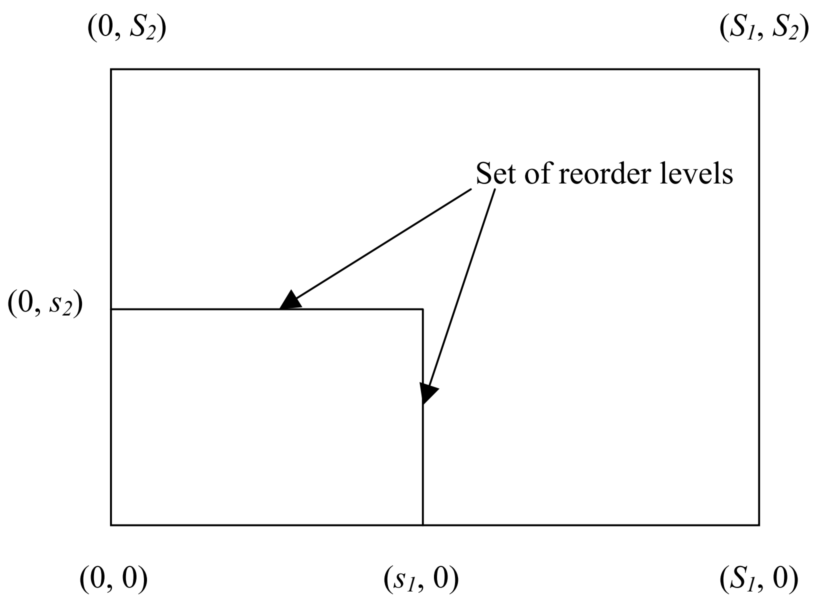

Figure 4.1: Space of Inventory Levels (*s, S*)

The joint probability distribution of the two inventory levels is obtained in both transient and steady state cases. Various measures of systems performance and the total expected cost rate in the steady state are also derived.

4.2 MODEL DESCRIPTION

Consider a two commodity inventory system with the maximum capacity S_i units for i-th commodity (i = 1, 2). We assume that demand for first commodity is for single item and demand for second commodity is for bulk items. The sequences of respective demand points for commodities 1, 2 and for both commodities are assumed to form independent Poisson processes with parameters λ_1, λ_2 and λ_{12} respectively. The number of items demanded for the second commodity at any demand point is a random variable Y with probability function $p_k = Pr\{Y = k\}$, $k = 1, 2, 3, \ldots$. The reorder level for the i-th commodity is fixed at $s_i(1 \le s_i \le S_i)$ and ordering quantity for i-th commodity is $Q_i(= S_i - s_i > s_i + 1)$ items when both inventory levels are less than or equal to their respective reorder levels. The requirement $S_i - s_i > s_i + 1$, ensure that after a replenishment the inventory level will be always above the respective reorder levels. Otherwise it may not be possible to place reorder which leads to perpetual shortage. That is if $L_i(t)$ represents inventory level of i-th commodity at time t, then a reorder is made when $L_1(t) \le s_1$ and $L_2(t) \le s_2$.

The lead-time is assumed to be distributed as negative exponential with parameter μ (> 0). The demands that occur during stock out periods are lost. The stochastic process $\{(L_1(t), L_2(t)), t \ge 0\}$ has the state space,

$$E = E_1 \times E_2$$

where $E_1 = \{0, 1, 2, \ldots, S_1\}$ and $E_2 = \{0, 1, \ldots, S_2\}$.

Notation:

0: zero matrix

I_N' : $(1, 1, \ldots, 1)_{1\times N}$

I : an identity matrix

$$< \mathrm{x} > = \begin{cases} x \; if \;\; x > 0 \\ 0 \; if \;\; x \le 0 \end{cases}$$

From the assumptions made on demand and on replenishment processes, it follows that $\{(L_1(t), L_2(t)), t \ge 0\}$ is a Markov process. To determine the infinitesimal generator $A = ((a\,(i, q; j, r))), (i, q), (j, r) \in E$, we use the following arguments:

The demand for the first commodity takes the state of the process from (i, q) to $(i-1, q)$ and the intensity of transition $a(i, q; i-1, q)$ is given by λ_1, $i = 1, 2, \ldots, S_1$. A bulk demand of k items for second commodity takes the state from (i, q) to $(i, < q-k >)$, $i = 0, 1, \ldots, S_1$, $q = 0, 1, \ldots, S_2$, $k = 1, 2, \ldots,$ and the respective intensity of transitions are given by $\lambda_2 p_k$ and $\lambda_2 \sum_{u=k}^{\infty} p_u$. A joint demand for single item of first commodity and for k items of second commodity takes the system from the state (i, q) to $(i-1, < q-k >)$, $i = 0, 1, 2, \ldots, S_1$, $q = 0, 1, 2, \ldots, S_2$, $k = 1, 2, \ldots,$ and the respective intensity of transition are given by $\lambda_{12}\, p_k$ and $\lambda_{12} \sum_{u=k}^{\infty} p_k$. From the state (i, q) $(\le (s_1, s_2))$ a replenishment takes the joint inventory level to $(i + Q_1, q + Q_2)$ and the intensity of transition for this is given by μ. For other transition from (i, q) to (j, r), when $(i, q) \ne (j, r)$, is zero. To obtain the intensity of passage, $a\,(i, q; i, q)$ of state (i, q), we note that the entries in any row of this matrix add to zero. Hence the diagonal entry is equal to the negative of the sum of the other entries in that row. More explicitly

$$a(i,q;i,q) = -\sum_{j}\sum_{r} a(i,j;j,r)$$
$$(j,r) \ne (i,q)$$

Hence we have,

$$a(i,q;j,r)=\begin{cases}
\lambda_1 & r=q; & j=i-1 \\
 & q=0,1,2,\dots,S_2; & i=1,2,3,\dots,S_1, \\
\lambda_2 p_k & r=q-k; & j=i \\
 & k=1,2,\dots,q-1; & i=0,1,2,\dots,S_1, \\
 & q=2,3,\dots,S_2, & \\
\lambda_2 p'_q & r=0; & j=i, \\
 & q=1,2,3,\dots,S_2; & i=0,1,2,\dots,S_1, \\
\lambda_{12} p_k & r=q-k; & j=i-1 \\
 & k=1,2,\dots,q-1; & i=1,2,\dots,S_1, \\
 & q=2,3,\dots,S_2, & \\
\lambda_{12} p'_q & r=0; & j=i-1, \\
 & q=1,2,3,\dots,S_2; & i=0,1,2,\dots,S_1, \\
\mu & r=q+Q_2; & j=i+Q_1, \\
 & q=0,1,2,\dots,s_2; & i=0,1,2,\dots,s_1, \\
-(\lambda_1+\lambda_2+\lambda_{12}) & r=q; & j=i, \\
 & q=1,2,3,\dots,S_2; & i=s_1+1,s_1+2,\dots,S_1, \\
-\lambda_1 & r=q; & j=i, \\
 & q=0; & i=s_1+1,s_1+2,\dots,S_1, \\
-(\lambda_1+\lambda_2+\lambda_{12}) & r=q; & j=i, \\
 & q=s_2+1,s_2+2,\dots,S_2, & i=0,1,2,\dots,s_1, \\
-(\lambda_1+\lambda_2+\lambda_{12}+\mu) & r=q; & j=i, \\
 & q=1,2,3,\dots,s_2; & i=0,1,2,\dots,s_1, \\
-(\lambda_1+\mu) & r=q; & j=i, \\
 & q=0; & i=1,2,\dots,s_1, \\
-\lambda_2 & r=q; & j=i, \\
 & q=s_2+1,s_2+2,\dots,S_2, & i=0 \\
-(\lambda_2+\mu) & r=q; & j=i, \\
 & q=1,2,3,\dots,s_2; & i=0, \\
-\mu & r=q; & j=i, \\
 & q=0; & i=0, \\
0 & \mathit{Otherwise}, &
\end{cases}$$

where $p'_q = \sum_{k=q}^{\infty} p_k$.

Denoting $\boldsymbol{m} = ((m, S_2), (m, S_2{-}1), \ldots, (m, 1), (m, 0))$ for $m = 0, 1, 2, \ldots, S_1$, the infinitesimal generator A can be conveniently expressed as a block partitioned matrix:

$$A = \begin{array}{c} S_1 \\ S_1 - 1 \\ \vdots \\ s_1 + 1 \\ s_1 \\ s_1 - 1 \\ \vdots \\ 1 \\ 0 \end{array} \begin{pmatrix} A_1 & B & & & & & & & & \\ & A_1 & B & & & & & & & \\ & & \cdots & & & & & & & \\ & & & \cdots & A_1 & B & & & & \\ C & & & & & A_2 & B & & & \\ & C & & & & & A_2 & & & \\ & & \cdots & & & & \cdots & & & \\ & & & & & & & \cdots & A_2 & B \\ & & & C & & & & & & D \end{pmatrix}$$

where,

$$B = \begin{array}{cc} & \begin{array}{cccccc} S_2 & S_2 - 1 & S_2 - 2 & \cdots & 1 & 0 \end{array} \\ \begin{array}{c} S_2 \\ S_2 - 1 \\ \vdots \\ \vdots \\ 1 \\ 0 \end{array} & \begin{pmatrix} \lambda_1 & \lambda_{12}p_1 & \lambda_{12}p_2 & \cdots & \lambda_{12}p_{S_2-1} & \lambda_{12}p'_{S_2} \\ 0 & \lambda_1 & \lambda_{12}p_1 & \cdots & \lambda_{12}p_{S_2-2} & \lambda_{12}p'_{S_2-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \lambda_1 & \lambda_{12}p'_1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda_1 \end{pmatrix} \end{array}$$

$$C = \begin{array}{c} S_2 \\ S_2 - 1 \\ \vdots \\ s_2 \\ s_2 - 1 \\ \vdots \\ 0 \end{array} \begin{pmatrix} 0 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \mu & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \mu & \cdots & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \mu & \cdots & 0 \end{pmatrix}$$

$$A_1 = \begin{pmatrix} -(\lambda_1+\lambda_2+\lambda_{12}) & \lambda_2 p_1 & \lambda_2 p_2 & \cdots & \lambda_2 p_{S_2-1} & \lambda_2 p'_{S_2} \\ 0 & -(\lambda_1+\lambda_2+\lambda_{12}) & \lambda_2 p_1 & \cdots & \lambda_2 p_{S_2-2} & \lambda_2 p'_{S_2-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -(\lambda_1+\lambda_2+\lambda_{12}) & \lambda_2 p'_1 \\ 0 & 0 & 0 & \cdots & 0 & -\lambda_1 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} -(\lambda_1+\lambda_2+\lambda_{12}) & \cdots & \lambda_2 p_{Q_2-1} & \lambda_2 p_{Q_2} & \cdots & \cdots & \lambda_2 p'_{S_2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & -(\lambda_1+\lambda_2+\lambda_{12}) & \lambda_2 p_1 & \cdots & \cdots & \lambda_2 p'_{s_2+1} \\ 0 & \cdots & 0 & d & \lambda_2 & \cdots & \lambda_2 p'_{s_2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & 0 & \cdots & d & \lambda_2 p'_1 \\ 0 & \cdots & 0 & 0 & \cdots & \cdots & -(\lambda_1+\mu) \end{pmatrix}$$

with $d = -(\lambda_1 + \lambda_2 + \lambda_{12} + \mu)$,

and

$$D = \begin{pmatrix} -\lambda_2 & \cdots & \lambda_2 p_{Q_2-1} & \lambda_2 p_{Q_2} & \cdots & \cdots & \lambda_2 p'_{S_2} \\ 0 & \cdots & \lambda_2 p_{Q_2-2} & \lambda_2 p_{Q_2-1} & \cdots & \cdots & \lambda_2 p'_{S_2-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & -\lambda_2 & \lambda_2 p_1 & \cdots & \cdots & \lambda_2 p'_{s_2+1} \\ 0 & \cdots & 0 & d_1 & \cdots & \cdots & \lambda_2 p'_{s_2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & 0 & \cdots & d_1 & \lambda_2 p'_1 \\ 0 & \cdots & 0 & 0 & \cdots & \cdots & -\mu \end{pmatrix}$$

with $d_1 = -(\lambda_2 + \mu)$.

4.3 TRANSIENT ANALYSIS

Define

$$\phi(i, q; j, r, t) = Pr\,[L(t) = j, X(\mathrm{t}) = r | L(0) = i, X(0) = q], (i, q), (j, \mathrm{r}) \in E.$$

Let $\phi_{\mathrm{ij}}(t)$ denote a matrix whose $(q, r)^{\text{th}}$ element is $\phi(i, q; j, r, t)$ and $\Phi(t)$ denote a block partitioned matrix with the sub matrix $\phi_{\mathrm{ij}}(\mathrm{t})$ at (i, j)th position. The Kolmogorov's differential equation can be written as

$$\Phi'(t) = \Phi(t)A,$$

the solution of which is given by

$$\Phi(t) = e^{At}$$

where e^{At} represents

$$I + \frac{At}{1!} + \frac{A^2t^2}{2!} + \ldots$$

Alternatively, if we use the notation A* (α) to denote the Laplace transform of the function (or matrix) A (t) then we have

$$\Phi^*_\alpha = (\alpha I - A)^{-1}$$

The matrix $(\alpha I - A)$ has the following block partitioned form

$$(\alpha I - A) = \begin{array}{c} S_1 \\ S_1 - 1 \\ \cdots \\ s_1 \\ s_1 - 1 \\ \cdots \\ 1 \\ 0 \end{array} \begin{pmatrix} D_{S_1} & -B & 0 & \cdots & 0 & 0 \\ 0 & D_{S_1 - 1} & -B & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -C & 0 & 0 & \cdots & 0 & 0 \\ 0 & -C & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & D_1 & -B \\ 0 & 0 & 0 & \cdots & 0 & D_0 \end{pmatrix}$$

Where

$$D_i = \begin{cases} \alpha\, I - D, & i = 0, \\ \alpha\, I - A_2, & i = 1,2,\ldots,s_1, \\ \alpha\, I - A_1, & i = s_1 + 1,\ldots,S_1 \end{cases}$$

(Note that the rows and columns have been numbered in decreasing order if magnitude.)

It may be observed that $(\alpha\, I - A)$ is an almost lower triangular matrix in block partitioned form. That is, if we denote the $(i, j)^{\text{th}}$ sub matrix of $P(= \alpha\, I - A)$ by P_{ij}, then we have

$$P_{ij} = 0 \qquad i = 1, 2, \ldots, S_1;\ j > i - 1.$$

To compute $P^{-1} = (\alpha\, I - A)^{-1}$ we proceed as described below:

Consider a lower triangular matrix

$$U = \begin{array}{c} S_1 \\ S_1 - 1 \\ \cdots \\ 1 \\ 0 \end{array} \begin{pmatrix} U_{S_1 S_1} & 0 & \cdots & 0 & 0 \\ U_{(S_1-1)S_1} & U_{(S_1-1)(S_1-1)} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \\ U_{1S_1} & U_{1(S_1-1)} & \cdots & U_{11} & 0 \\ U_{0S_1} & U_{0(S_1-1)} & \cdots & U_{01} & U_{00} \end{pmatrix}$$

with $U_{ii} = 1$, $i = 0, 1, 2, \ldots, S_1$ and an almost lower triangular matrix

$$R = \begin{array}{c} S_1 \\ S_1-1 \\ \cdots \\ 1 \\ 0 \end{array} \begin{pmatrix} \mathbf{0} & -B & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -B & \cdots & \mathbf{0} & \mathbf{0} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & -B \\ R_{0S_1} & R_{0(S_1-1)} & R_{0(S_1-2)} & \cdots & R_{01} & R_{00} \end{pmatrix}$$

such that $PU = R$. We find the sub matrices U_{ij} and R_{0j} by computing the product PU and equating it to R. The $(i, j)^{\text{th}}$ sub matrix of PU, denoted by $[PU]_{ij}$ is given by

By equating the sub matrices of PU to the corresponding elements of R, we get

$$U_{ij} = \begin{cases} (B^{-1}D_{i+1})(B^{-1}D_{i+2})\ldots(B^{-1}D_j), & i = 0,1,2,\ldots,j+1 \\ & j = 1,2,\ldots,Q_1 \\ & \text{or} \\ & i = j-Q_1, j-Q_1+1,\ldots,S_1 \\ & j+Q_1+1, Q_1+2,\ldots,S_1 \\ B^{-1}CU_{(i+Q_1+1)j} + B^{-1}D_{i+1}U_{(i+1)j}, & i = 0,1,\ldots,j-Q_1-1 \\ & j = Q_1+1, Q_1+2,\ldots,S_1 \end{cases}$$

and

$$R_{0j} = \begin{cases} D_0, & j = 0 \\ D_0U_{0,j}, & j = 1,2,\ldots,Q_1-1 \\ CU_{Q_1j} + D_0U_{0,j}, & j = Q_1, Q_1+1,\ldots,S_1. \end{cases}$$

The equation PU = R implies

$$\begin{aligned} (PU)^{-1} &= R^{-1} \\ U^{-1}P^{-1} &= R^{-1} \\ P^{-1} &= UR^{-1}. \end{aligned}$$

It can be verified that the inverse of R is given by,

$$R^{-1} = \begin{pmatrix} R_{0S_1}^{-1} R_{0(S_1-1)} B^{-1} & R_{0S_1}^{-1} R_{0(S_1-2)} B^{-1} & \cdots & R_{0S_1}^{-1} R_{00} B^{-1} & R_{0S_1}^{-1} \\ -B^{-1} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -B^{-1} & \cdots & \mathbf{0} & \mathbf{0} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{0} & \mathbf{0} & \cdots & -B^{-1} & \mathbf{0} \end{pmatrix}$$

Since the expression for R^{-1} involves the term $R_{0S_1}^{-1}$, it is demonstrated that the latter exists.

From $PU = R$, we get

$$\det(PU) = \det(R)$$
$$\det(P)\det(U) = \det(R_{0S_1})\det(-B)\det(-B) \cdots \det(-B).$$

Since U is a lower triangular matrix and B is a upper triangular matrix, their determinant values are not equal to zero. Hence $\det(R_{0S_1})$ is not equal to zero. This proves the existence of the inverse of $R_{0S_1}^{-1}$. From $P^{-1} = UR^{-1}$, we can compute the (i, j)th sub matrix (denoted by Pij) of $P^{-1} = (\alpha I - A)^{-1}$ and it is given by,

$$P^{ij} = \begin{cases} U_{iS_1} R_{0S_1}^{-1} R_{0(j-1)} B^{-1} & i = j, j+1, \ldots, S_1; \quad j = 1, 2, \ldots, S_1 \\ U_{iS_1} R_{0S_1}^{-1} & i = 0, 1, 2, \ldots, S_1; \quad j = 0 \\ U_{iS_1} R_{0S_1}^{-1} R_{0(j-1)} - U_{i(j-1)} B^{-1} & i = 0, 1, \ldots, S_1 - 1; \quad j = i+1, \ldots, S_1. \end{cases}$$

4.4 Steady State Analysis

It can be seen from the structure of A that the homogeneous Markov process $\{(L_1(t), L_2(t)), t \geq 0\}$ on the state space E is irreducible. Hence the limiting distribution

$$\mathbf{\Phi} = (\phi^{S_1}, \phi^{S_1-1}, \ldots, \phi^1, \phi^0) \tag{4.1}$$

with $\phi^m = (\phi^{(m,S_2)}, \phi^{(m,S_2-1)}, \ldots, \phi^{(m,0)})$, where $\phi^{(i,j)}$ denotes the steady state probability for the state (i, j) of the inventory level process, exists and is given by

$$\Phi A = 0 \quad and \quad \sum_{(i,j)\in E} \sum \phi^{(i,j)} = 1. \tag{4.2}$$

The first equation of the above yields the following set of equations:

$$\begin{aligned}
\phi^1 B + \phi^0 D &= 0 \\
\phi^i B + \phi^{i-1} A_2 &= 0, & i &= 2, \ldots, s_1 + 1 \\
\phi^i B + \phi^{i-1} A_1 &= 0, & i &= s_1 + 2, \ldots, Q_1 \\
\phi^i B + \phi^{i-1} A_1 + \phi^{i-1-Q_1} C &= 0, & i &= Q_1 + 1, \ldots, S_1 \\
\phi^{S1} A_1 + \phi^{s_1} C &= 0.
\end{aligned}$$

Simplification yields the following:

$$\begin{aligned}
\phi^i &= (-1)^i \phi^0 D B^{-1} (A_2 B^{-1})^{i-1} & i &= 1, 2, \ldots, s_1 + 1 \\
&= (-1)^i \phi^0 D B^{-1} (A_2 B^{-1})^{s_1} (A_1 B^{-1})^{i-s_1-1} & i &= s_1 + 2, \ldots, Q_1 \\
&= (-1)^i \phi^0 D B^{-1} (A_2 B^{-1})^{s_1} (A_1 B^{-1})^{i-s_1-1} - \phi_0 C B^{-1} & i &= Q_1 + 1 \\
&= (-1)^i \phi^0 D B^{-1} (A_2 B^{-1})^{s_1} (A_1 B^{-1})^{i-s_1-1} \\
&\quad + (-1)^{i-Q_1} \phi_0 C B^{-1} (A_1 B^{-1})^{i-Q_1-1} + \\
&\quad (-1)^{i-Q_1} \phi^0 D B^{-1} \sum_{k=0}^{i-Q_1-2} (A_2 B^{-1})^k C B^{-1} (A_1 B^{-1})^{i-Q_1-2-k} & i &= Q_1 + 2, Q_1 + 2, \ldots, S_1.
\end{aligned}$$

where ϕ^0 can be obtained by solving,

$$\phi^{S1} A_1 + \phi^{s_1} C = 0 \quad \textit{and} \quad \sum_{i=0}^{S_1} \phi^i 1_{(S_2+1)\times 1} = 1,$$

that is

and

$$\begin{aligned}&\phi^{\mathbf{0}}\Big[(-1)^{S_1} DB^{-1}(A_2B^{-1})^{s_1}(A_1B^{-1})^{Q_1-1} + (-1)^{s_1} CB^{-1}(A_1B^{-1})^{s_1-1} + \\ &(-1)^{s_1} DB^{-1}\sum_{k=0}^{s_1-2}(A_2B^{-1})^k CB^{-1}(A_1B^{-1})^{i-Q_1-2-k} + \\ &(-1)^{s_1} DB^{-1}(A_2B^{-1})^{s_1-1}\Big] = \mathbf{0}.\end{aligned}$$

$$\begin{aligned}&\phi^0\Bigg[\sum_{i=1}^{s_1+1}(-1)^i DB^{-1}\left(A_2B^{-1}\right)^{i-1} + \sum_{i=s_1+2}^{Q_1}(-1)^i DB^{-1}\left(A_2B^{-1}\right)^{s_1}\left(A_1B^{-1}\right)^{i-s_1-1} + (-1)^{Q_1+1}DB^{-1}\left(A_2B^{-1}\right)^{s_1}\left(A_1B^{-1}\right)^{Q_1-s_1} \\ &-CB^{-1} + \sum_{i=Q_1+2}^{S_1}\Big\{(-1)^i DB^{-1}\left(A_2B^{-1}\right)^{s_1}\left(A_1B^{-1}\right)^{i-s_1-1} + (-1)^{i-Q_1} DB^{-1}\sum_{k=0}^{i-Q_1-2}\left(A_2B^{-1}\right)^k CB^{-1}\left(A_1B^{-1}\right)^{i-Q_1-2-k}\Big\}\Bigg]1_{(S_2+1)\times 1} = 1\end{aligned}$$

The marginal probability distribution $\{\phi_{i1}, i = 0, 1, 2, \ldots, S_1\}$ of the first commodity is given by

$$\phi_{i1} = \sum_{q=0}^{S_2}\phi^{(i,q)}, \qquad i = 0,1,2,\ldots,S_1,$$

and the marginal probability distribution $\{\phi_{2q}, q = 0, 1, 2, \ldots, S_2\}$ of the second commodity is given by

$$\phi_{2j} = \sum_{i=0}^{S_1}\phi^{(i,q)}, \qquad i = 0,1,2,\ldots,S_2,$$

The expected inventory level in the steady state, for the i-th commodity is given by

$$E[L_i] = \sum_{k=0}^{S_i} k\phi_{ik}, \qquad i = 1,2. \tag{4.3}$$

4.5 REORDERS AND SHORTAGES

In this section the reorder and shortages are studied. This requires the study of time points at which a transition occurs in the inventory level process.

Let $0 = T_0 < T_1 < T_2 < \cdots$ be the instances of transitions of the process. Let $(L_n^{(1)}, L_n^{(2)}) = (L_1(T_n+), L_2(T_n+))$, $n = 0, 1, 2, \ldots$. From the well known theory of Markov processes, $\{(L_n^{(1)}, L_n^{(2)}), n = 0, 1, 2, \ldots\}$ is a Markov chain and with the transition probability matrix (*tpm*)

$$P = ((\, p(i, j; k, l)\,))_{(i,j) \in E, (k,l) \in E},$$

where,

$$p(i, j; k, l) = \begin{cases} 0 & (i, j) = (k, l) \\ -a(i, j; k, l)/\theta_{ij}, & (i, j) \neq (k, l) \end{cases}$$

Here $\theta_{ij} = a(i, j; i, j)$ which is a negative value. Moreover for all n, we also have,

$$Pr\,[(L_1(T_{n+1}+), L_2(T_{n+1}+)) = (k, l), T_{n+1} - T_n > t \mid (L_1(T_n+), L_2(T_n+)) = (i, j)]$$

$$= p(i, j; k, l)e^{\theta_{ij} t}.$$

4.5.1 Reorders

A reorder for both commodities is made when the joint inventory level at any time t, drops to either (s_1, s_2) or $(s_1, j), j < s_2$ or $(i, s_2), i < s_1$.

We associate with a reorder a counting process $N(t)$. Define

$$\beta_1(i, j, t) = \lim_{\Delta \to 0} P_{ij}\,[N(t + \Delta) - N(t) = 1]\,\frac{1}{\Delta}$$

where $\bar{P}_{ij}\,[\cdot\cdot\cdot]$ represents $Pr[\cdot\cdot\cdot \mid (L_0^{(1)}, L_0^{(2)}) = (i, j)]$. The fact that the reorder at time t is either due to the first transition or due to a subsequent one, gives the following equations:

$$\begin{aligned}
\beta_1(i,j,t) &= \lim_{\Delta\to 0}\frac{1}{\Delta}\{P_{ij}\,[N(t+\Delta)-N(t)=1, t<T_1<t+\Delta]+ \\
&\quad \sum_{(k,l)\in E}\int_0^t P_{ij}\,[(L_1(T_1+),L_2(T_1+))=(k,l), u<T_1<u+\Delta] \\
&\quad Pr\,[N(t-u+\Delta)-N(t-u)=1 \mid (L_1(T_1+),L_2(T_1+))=(k,l)]\} \\
&= \tilde{\beta}_1(i,j,t)+\sum_{(k,l)\in E}\int_0^t p(i,j;k,l)\theta_{ij}e^{\theta_{ij}u}\beta_1(k,l,t-u)du
\end{aligned}$$

where $\tilde{\beta}_1(i,j,t)$ is given by

$$\tilde{\beta}_1(i,j,t)=\begin{cases}
\lambda_1 e^{\theta_{ij}t} & i=s_1+1, \quad j=0,1,\ldots,s_2 \\
\lambda_2 e^{\theta_{ij}t}\sum\limits_{u=j-s_2}^{\infty}p_u & i=0,1,\ldots,s_1 \quad j=s_2+1,s_2+2,\ldots,S_2 \\
\lambda_{12} e^{\theta_{ij}t}\sum\limits_{u=j-s_2}^{\infty}p_u & i=1,2,\ldots,s_1+1 \quad j=s_2+1,s_2+2,\ldots,S_2 \\
0 & \text{otherwise.}
\end{cases}$$

In the above expression we have used the fact that, when $j = 0, 1, 2, \ldots, s_2$ and $i = s_1 + 1$, then the next demand for commodity 1 will trigger a reorder. When $i = 0, 1, 2, \ldots, s_1$ and $j = s_2 + k$, ($k = 1, 2, \ldots, Q_2$) then k or more than k demands for commodity 2 alone will trigger a reorder. A demand for both commodities will trigger a reorder if the number of demanded items for the second commodity is k ($k = 1, 2, \ldots,$) when $i = 1, 2, \ldots, s_1 + 1$ and $j = s_2 + k$.

As the Markov process $\{(L_1(t), L_2(t)), t \geq 0\}$ is irreducible and recurrent (due to finite state space),

$$\beta_1 = \lim_{t\to\infty}\beta_1(i,j,t)$$

exists and will be equal to the steady state mean reorder rate. Moreover, we have from Cinlar(1975),

$$\beta_1 = \sum_{(i,j)\in E} \pi^{(i,j)} \int_0^\infty \widetilde{\beta}_1(i,j,t)dt \Big/ \sum_{(i,j)\in E} \pi^{(i,j)} m_{ij}, \tag{4.4}$$

where m_{ij} is the mean sojourn time in the inventory level (i, j) and is given by $1/\theta_{ij}$, and $\pi^{(i,j)}$ is the stationary distribution of the Markov chain

$$\{(L_n^{(1)}, L_n^{(2)}), n = 0, 1, 2, \ldots\}.$$

Since for a Markov process,

$$\phi^{(i.j)} = \pi^{(i,j)} m_{ij} \Big/ \sum_{(k,l)\in E} \pi^{(k,l)} m_{kl}, \tag{4.5}$$

we have from (4.5)

$$\begin{aligned}
\beta_1 &= \sum_{(i,j)\in E} \left(\frac{\phi^{(i,j)}}{m_{ij}} \right) \int_0^\infty \tilde{\beta}_1(i,j,t)dt \\
&= \sum_{(i,j)\in E} \phi^{(i,j)} \theta_{ij} \int_0^\infty \tilde{\beta}_1(i,j,t)dt. \\
&= \lambda_2 \sum_{k=0}^{s_1} \sum_{j=1}^{Q_2} \phi^{(k,s_2+j)} \sum_{u=j}^{\infty} p_u + \lambda_1 \sum_{k=0}^{s_2} \phi^{(s_1+1,k)} + \lambda_{12} \sum_{k=1}^{s_1+1} \sum_{j=1}^{Q_2} \phi^{(k,s_2+j)} \sum_{u=j}^{\infty} p_u
\end{aligned}$$

4.5.2 Shortages

A shortage for a commodity occurs when a demand occurs during a stockout period. We associate with a shortage a counting process $M(t)$. Define

$$\beta_2(i,j,t) = \lim_{\Delta\to 0}\frac{1}{\Delta}Pr[M(t+\Delta)-M(t)=1 \mid (I_0^{(1)}, I_0^{(2)}) = (i,j)] \tag{4.6}$$

which satisfies the equation,

$$\beta_2(i,j,t) = \widetilde{\beta}_2(i,j,t) + \sum_{(k,l)\in E}\int_0^t p(i,j;k,l)\theta_{ij}e^{\theta_{ij}u}\beta_2(k,l,t-u)du.$$

We have used the fact that the shortage at time t is due to the first demand or a subsequent one. Hence

$$\tilde{\beta}_2(i,j,t) = \lim_{\Delta\to 0}\frac{1}{\Delta}Pr[M(t+\Delta)-M(t) = 1, t < T_1 < t+\Delta \mid (L_0^{(1)}, L_0^{(2)}) = (i,j)]$$

and is given by,

$$\tilde{\beta}(i,j,t) = \begin{cases} (\lambda_1+\lambda_{12})e^{\theta_{ij}t}, & i=0, \quad j=1,2,\ldots,S_2 \\ (\lambda_2+\lambda_{12})e^{\theta_{ij}t}\sum_{u=j+1}^{\infty}p_u, & i=1,2,\ldots,S_1, \quad j=1,2,\ldots,S_2 \\ 0, & \text{Otherwise.} \end{cases} \tag{4.7}$$

Derivations similar to the one used to derive β_1 (refer subsection Reorder) yields,

$$\begin{aligned}\beta_2 &= \lim_{t\to\infty}\beta_2(i,j,t) \\ &= (\lambda_2+\lambda_{12})\sum_{i=1}^{S_1}\sum_{j=0}^{S_2}\phi^{(i,j)}\sum_{l=j+1}^{\infty}p_l + (\lambda_1+\lambda_{12})\sum_{j=0}^{S_2}\phi^{(0,j)}.\end{aligned}$$

4.5.3 Expected Cost

The long run expected cost rate $C(S_1, S_2, s_1, s_2)$, is given by,

$$C(S_1, S_2, s_1, s_2) = h_1 E[L_1] + h_2 E[L_2] + K\beta_1 + b\beta_2$$

where h_1 and h_2 are holding cost for first and second commodity respectively, K is the

$$\begin{aligned} C(S_1, S_2, s_1, s_2) &= h_1 \sum_{i=0}^{S_1} i\phi^{(i,k)} + h_2 \sum_{i=0}^{S_1} i\phi^{(i,k)} + K \left\{ \lambda_2 \sum_{k=0}^{s_1} \sum_{j=1}^{Q_2} \phi^{(k,s_2+j)} \sum_{u=j}^{\infty} p_u \right. \\ &\left. + \lambda_1 \sum_{k=0}^{s_2} \phi^{(s_1+1,k)} + \lambda_{12} \sum_{k=1}^{s_1+1} \sum_{j=1}^{Q_2} \phi^{(k,s_2+j)} \sum_{u=j}^{\infty} p_u \right\} \\ &+ b \left\{ (\lambda_2 + \lambda_{12}) \sum_{i=1}^{S_1} \sum_{j=0}^{S_2} \phi^{(i,j)} \sum_{l=j+1}^{\infty} p_l + (\lambda_1 + \lambda_{12}) \sum_{j=0}^{S_2} \phi^{(0,j)} \right\} \end{aligned}$$

fixed cost per order and b is the shortage cost. Then we have

4.6 NUMERICAL ILLUSTRATIONS

The limiting probability distribution of inventory level is computed for specific values of parameters. For the first example we have assumed,

$S_1 = 6$, $S_2 = 7$, $s_1 = 2$, $s_2 = 2$, $\lambda_1 = 1.5$, $\lambda_2 = 2$, $\lambda_{12} = 1$, $\mu = 1.5$, $p_1 = 0.25$, $p_2 = 0.20$, $p_3 = 0.15$, $p_4 = 0.05$, $p_5 = 0.01$, $p_6 = 0.005$, $p_7 = 0.001$, $h_1 = 0.2$, $h_2 = 0.3$, $b = 0.7$, $K = 50$.

		Commodity II			
		0	1	2	3
Commodity I	0	0.075012	0.001401	0.001621	0.00252
	1	0.065147	0.001181	0.001587	0.00204
	2	0.117119	0.002746	0.003406	0.00405
	3	0.205454	0.005726	0.007193	0.00704
	4	0.152802	0.005999	0.008525	0.00894
	5	0.086886	0.004373	0.006852	0.00754
	6	0.031357	0.001949	0.003524	0.00413

		Commodity II			
		4	5	6	7
Commodity I	0	0.00164	0.001108	0.000059	0.000025
	1	0.00155	0.001436	0.000064	0.000034
	2	0.00340	0.004236	0.000166	0.000103
	3	0.00656	0.012514	0.000411	0.000311
	4	0.00894	0.037037	0.000974	0.000935
	5	0.00792	0.035197	0.001012	0.001185
	6	0.00460	0.039340	0.001134	0.001967

Table 4.1: Limiting probability distribution of the inventory level – Example 1

This example gives the following results:

Expected reorder rate = 0.39466.

Expected shortage rate = 2.41792.

Expected inventory level for the commodity I = 3.25862.

Expected inventory level for the commodity II = 1.04528.

Total Expected Cost rate = 22.39090.

As a second example, the following values have been considered and the calculated joint probability distribution of the inventory level is given in Table 4.2:

$S_1 = 5$, $S_2 = 6$, $s_1 = 1$, $s_2 = 2$, $\lambda 1 = 1.2$, $\lambda_2 = 1.5$, $\lambda_{12} = 0.8$, $\mu = 1$, $p_1 = 0.3$, $p_2 = 0.20$, $p_3 = 0.15$, $p_4 = 0.05$, $p_5 = 0.01$, $p_6 = 0.005$, $h_1 = 0.3$, $h_2 = 0.3$,$b = 0.9$, $K = 75$.

		Commodity II			
		0	1	2	3
Commodity I	0	0.168866	0.004688	0.007036	0.005056
	1	0.116626	0.003385	0.004312	0.003621
	2	0.188656	0.006302	0.007149	0.006626
	3	0.145235	0.007318	0.009887	0.010186
	4	0.082216	0.005493	0.008781	0.009814
	5	0.021723	0.001687	0.002995	0.003492

		Commodity II		
		4	5	6
Commodity I	0	0.003363	0.002014	0.000053
	1	0.002929	0.002484	0.000067
	2	0.006133	0.007181	0.000196
	3	0.010919	0.020758	0.000573
	4	0.011697	0.059996	0.001671
	5	0.004383	0.033446	0.000967

Table 4.2: Joint probability distribution of the inventory level – Example 2

This example gives the following results:

Expected reorder rate = 0.288526.

Expected shortage rate = 1.718531.

Expected inventory level for the commodity I = 2.254718.

Expected inventory level for the commodity II = 1.033886.

Total Expected Cost rate = 24.172744.

4.7 CONCLUISION

This article analyses a two-commodity inventory system under continuous review. The maximum storage capacity for the *i*-th item is S_i (i = 1, 2). The demand points for each commodity are assumed to form an independent Poisson process. We also assume that unit demand for one item and bulk demand for the other. The reorder level is fixed as s_i for the *i*-th commodity (i = 1, 2) and the ordering policy is to place order for Q_i (= $S_i - s_i$, i = 1, 2) items for the *i*-th commodity when both the inventory levels are less than or equal to their respective reorder levels. The lead-time is assumed to be exponential. The joint probability distribution for both commodities is obtained in both transient and steady state cases. Various measures of systems performance and the total expected cost rate in the steady state are derived. The results are illustrated with numerical examples.