CHAPTER 6

A two unit cold standby system with non-instantaneous switchover

6.1 INTRODUCTION

Gopalan et al (1984) have analysed a single-server two-unit cold standby system subject to a slow switch and have obtained expressions for the expected switchover time of unit from standby to operative state in (0,t] and the expected repair time of a unit in (0,t]. Sharma et al (1986) modified that model by taking a two-unit warm standby system and obtained several reliability characteristics. They did not take into account the partial failure mode. The purpose of the present chapter is to study a two-unit cold standby system with three modes of the system subject to slow switch. The system fails totally only through the partial failure mode. When a unit fails partially, its repair starts immediately and the installation of a new unit in place of a partially failed unit remains operative. Regenerative point technique is used for the analysis.

6.2 MODEL ASSUMPTION

The system compromises two identical units. Initially one is operative and the other is a cold standby.

(1) Each unit is has three possible modes: normal (N), partial failure (P) and total failure (F).

(2) The system fails totally only through the partial failure mode.

(3) The failure and switchover times are distributed negative exponentially whereas the repair times of units are distributed arbitrarily.

(4) When a unit fails partially, repair of the partially failed unit starts instantaneously and installation of the standby for operation is not permitted.

(5) When a unit fails completely from the partially failed state, repair

of the failed unit and installation of the standby for operation start

simultaneously and independently.

(6) The repaired system is as good as new.

6.3 NOTATION

α, β	Constant failure rates from N to P and P to F modes
η	Constant rate of switchover time of a unit from standby state to operative state
$f(t), F(t)$	pdf and cdf of repair time of a unit from P state
$g(t), G(t)$	pdf and cdf of repair time of a unit from F state

Symbols for states of the system

N_0, N_5	system operative in N mode
P_{or}	unit operative in P mode and under repair mode
F_r	unit in F mode and under repair
F_R	unit in F mode and its repair continued from earlier state
F_w	system in F mode and waiting for repair
bso	standby being switched over

Thus the following states are possible:

$$S_0 = (N_0, N_5); S_1 = (P_{or}, N_5); S_2 = (F_r, bso);$$
$$S_3 = (N_s, bso); S_4 = (F_R, N_0); S_5 = (F_w, P_{or});$$
$$S_6 = (F_r, N_0); S_7 = (F_r, F_w).$$

Up states-$S_{0,} S_1, S_4 - S_6$; down states-$S_{2,} S_3, S_7$.

The underlined states are non-regenerative. Possible states and transitions are shown
in Figure 6.1.

6.4. TRANSITION PROBABILITIES AND SOJOURN TIMES

Let $T_0(=0), T_1, \ldots$ denote the epochs at which the system enters any state $S_i \in E$ and X_n be the state visited at time T_{n+}, i.e. just after the transition at T_n. Then $\{X_n, T_n\}$ is a Markov renewal process with state space. Let

$$Q_{ij}(t) = P\left[X_{n+1} = j, T_{n+T} T_{n \le t} X_n = i\right];$$

then the transition probability matrix of embedded Markov Chain is

$$P = \left(P_{ij}\right) = \left(\left(Q_{ij}(\infty)\right)\right) = Q(\infty),$$

with non-zero elements

$$P_{01} = P_{30} = P_{72} = 1, P_{10} = 1 - P_{12} = \widetilde{F}(\beta),$$

$$P_{23} = 1 - P_{24} = \widetilde{G}(\eta), \quad P_{20}{}^{(4)} = \eta\left[\frac{\widetilde{G}(\alpha) - \widetilde{G}(\eta)}{\eta - \alpha}\right],$$

$$P_{25}^{(4)} = 1 - \frac{\left\lfloor \eta \widetilde{G}(\alpha) - \alpha \widetilde{G}(\eta) \right\rfloor}{(\eta - \alpha)},$$

$$P_{56} = 1 - P_{57} = \widetilde{F}(\beta), \qquad P_{60} = 1 - P_{65} = \widetilde{G}(\alpha).$$

Evidently,

$$P_{10} + P_{12} = 1, P_{23} + P_{24} = 1, P_{23} + P^{(4)}{}_{20} + P^{(4)}{}_{25} = 1,$$

$$P_{56} + P_{57} = 1, P_{60} + P_{65} = 1.$$

Mean sojourn times μ_i in state S_i are

$$\mu_0 = \frac{1}{\alpha},\ \mu_1 = \mu_5 = \frac{\left\lfloor 1 - \widetilde{F}(\beta) \right\rfloor}{\beta},$$

$$\mu_2 = \frac{\left\lfloor 1 - \widetilde{G}(\eta) \right\rfloor}{\eta}, \mu_3 = \frac{1}{\eta},$$

$$\mu_6 = \frac{\left\lfloor 1 - \widetilde{G}(\alpha) \right\rfloor}{\alpha},$$

$$\mu_7 = \int_0^\infty \overline{G}(t)\,dt$$

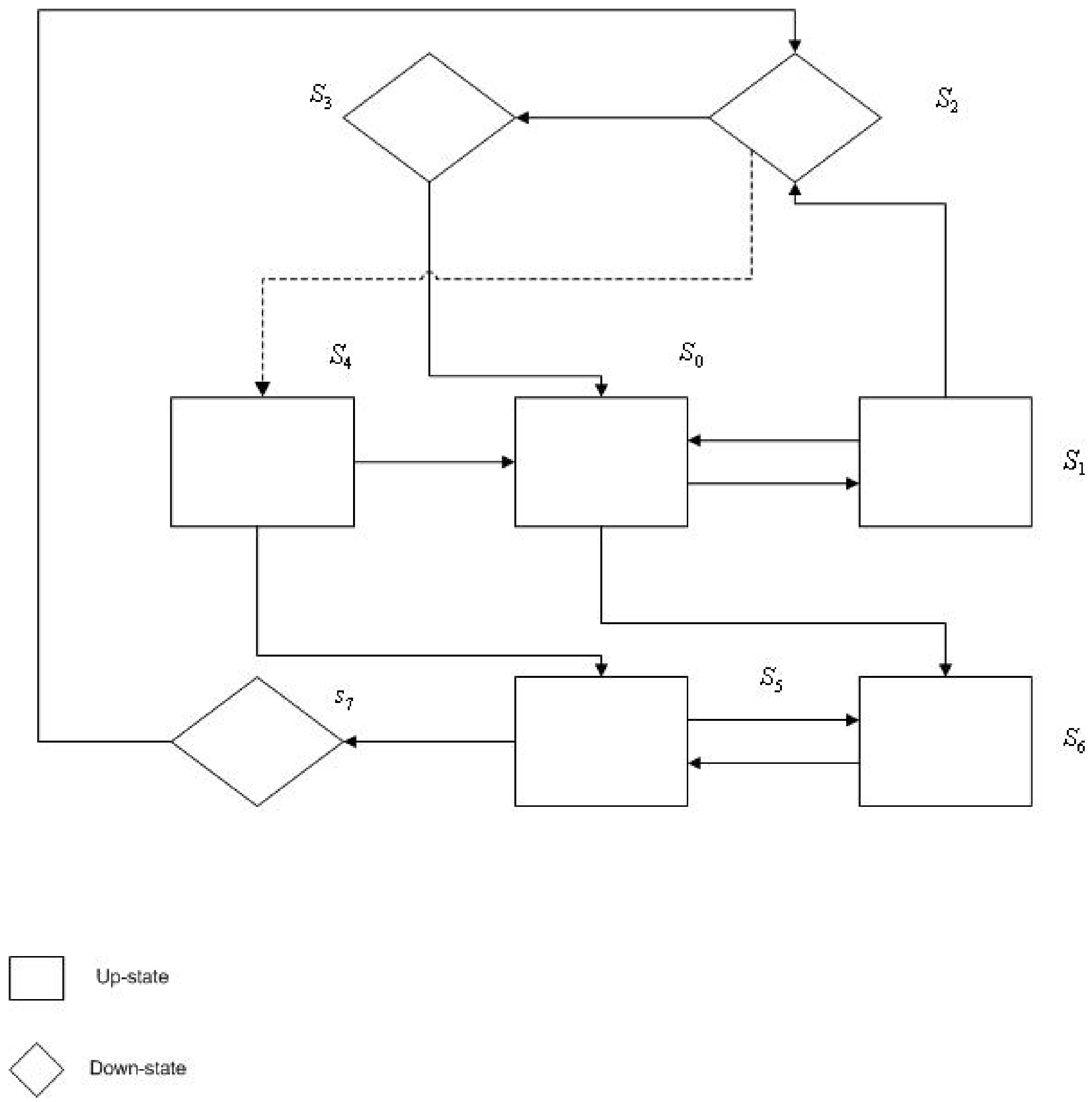

Figure 6.1

6.5 TIME TO SYSTEM FAILURE

Time to system failure can be regarded as the first passage to the failed state. To obtain it we consider down states as absorbing. We obtain the following recursive relations for $\pi_i(t)$, the cdf of time to system failure when the system starts from state S_i

$$\pi_0(t) = Q_{01}(t) \circledS \pi_1(t) \tag{6.1}$$

$$\pi_1(t) = Q_{10}(t) \circledS \pi_0(t) + Q_{12}(t) \tag{6.2}$$

$$\pi_5(t) = Q_{56}(t) \circledS \pi_6(t) + Q_{57}(t) \tag{6.3}$$

$$\pi_6(t) = Q_{60}(t) \circledS \pi_0(t) + Q_{65}(t) \copyright \pi_5(t) \tag{6.4}$$

Taking Laplace-Stieltjies transforms of equations (1)-(4) and solving for $\tilde{\pi}_0(s)$, we have

$$\tilde{\pi}_0(s) = \frac{\tilde{Q}_{01}\tilde{Q}_{12}\left(1 - \tilde{Q}_{56}\tilde{Q}_{65}\right)}{\left(1 - \tilde{Q}_{01}\tilde{Q}_{10}\right)\left(1 - \tilde{Q}_{56}\tilde{Q}_{65}\right)}$$

$$= \frac{\tilde{Q}_{01}\tilde{Q}_{12}}{\left(1 - \tilde{Q}_{01}\tilde{Q}_{10}\right)}, \tag{6.5}$$

where, for brevity, the argument ‘s’ is omitted.

The mean time to system failure (MTSF), when the system starts from S_0, is

$$MTSF = E(T) = \frac{[D_r'(0) - N_r'(0)]}{D_r(0)} = \frac{(\mu_0 + \mu_1)}{P_{12}}. \quad (6.6)$$

6.6 AVAILABILITY ANALYSIS

Let $M_i(t)$ be the probability that the system which started from state S_i has reached time t without making any transition into any other regenerative state belonging to E. By probabilistic arguments, we have

$$M_0(t) = e^{-\alpha t},$$

$$M_1(t) = M_5(t) = e^{-\beta t}\overline{F}(t),$$

$$M_6(t) = e^{-\alpha t}\overline{G}(t).$$

From then theory of regenerative process, the pointwise availabilities $A_i(t)$ of a system which has started from a given regenerative point are seen to satisfy the following recursive relations:

$$A_0(t) = M_0(t) + q_{01}(t) © A_1(t) \quad (6.7)$$

$$A_1(t) = M_1(t) + q_{10}(t) © A_0(t) + q_{12}(t) © A_2(t) \tag{6.8}$$

$$A_2(t) = q_{20}{}^{(4)}(t) © A_0(t) + q_{23}(t) © A_3(t) + q_{25}{}^{(4)}(t) © A_5(t) \tag{6.9}$$

$$A_3(t) = q_{30}(t) © A_0(t) \tag{6.10}$$

$$A_5(t) = M_5(t) + q_{56}(t) © A_6(t) + q_{57}(t) © A_7(t) \tag{6.11}$$

$$A_6(t) = M_6(t) + q_{60}(t) © A_0(t) + q_{65}(t) © A_5(t) \tag{6.12}$$

$$A_7(t) = q_{72}(t) © A_2(t). \tag{6.13}$$

Taking Laplace-transforms of equations (6.7)-(6.12) and solving for $A_0{}^*(s)$ we have

$$A^*{}_0 = \frac{N_1(s)}{D_1(s)} \tag{6.14}$$

$$M_0(t) = e^{-\alpha t},$$

$$M_1(t) = M_5(t) = e^{-\beta t}\overline{F}(t),$$

$$M_6(t) = e^{-\alpha t}\overline{G}(t).$$

Where

$$N_1(s) = \left(1 - q_{56}^* q_{65}^* - q_{25}^{*(4)} q_{57}^* q_{72}^*\right) \times \left(M_0^* + q_{01}^* M_1^*\right)$$
$$+ q_{01}^* q_{12}^* q_{25}^{*(4)} \times \left(M_5^* + q_{56}^* M_6^*\right)$$

$$D_1(s) = \left(1 - q_{56}^* q_{65}^* - q_{25}^{*(4)} q_{57}^* q_{72}^*\right) \times \left(1 - q_{01}^* q_{10}^*\right)$$
$$- q_{01}^* q_{12}^* \left[\left(1 - q_{56}^* q_{65}^*\right) \times \left(q_{20}^{*(4)} + q_{23}^* q_{30}^*\right) + q_{25}^* q_{56}^* q_{60}^*\right]$$

The steady-state availability of the system is

$$A_0 = \lim_{s \to 0} s \ A_0^*(s) = \frac{N_1}{D_1'} . \tag{6.15}$$

$$N_1 = \left(1 - P_{56} P_{65} - P_{25}^{(4)} P_{57}\right)\left(\mu_0 + P_{01}\mu_1\right) + P_{01} P_{12} P_{25}^{(4)} \left(\mu_1 + P_{56}\mu_6\right)$$

$$D_1' = \left(\mu_0 + \mu_1\right)\left(1 - P_{56} P_{65} - P_{25}^4 P_{57}\right) + P_{12}\left(1 - P_{56} P_{65}\right)\left[\frac{\eta\mu_6 - \alpha\mu_2}{(\eta - \alpha)} + P_{23}\mu_3\right] +$$
$$P_{12} P_{25}^{(4)} \left(\mu_1 + P_{56}\mu_6 + P_{57}\mu 7\right)$$

6.7 BUSY PERIOD ANALYSIS

As defined earlier, $B_i(t)$ is the probability that the system is under repair at time t given that the system entered regenerative state s_i at $t = 0$. By probabilistic arguments we have

$$B_0(t) = q_{01}(t) © B_1(t) \tag{6.16}$$

$$B_1(t) = W_1(t) + q_{10}(t) © B_0(t) + q_{12}(t) © B_2(t) \tag{6.17}$$

$$B_2(t) = W_2(t) + q_{20}{}^{(4)} © B_0(t) + q_{23}(t) © B_3(t) \tag{6.18}$$
$$+ q_{25}^{(4)} © B_5(t)$$

$$B_3(t) = q_{30}(t) © B_0(t) \tag{6.19}$$

$$B_5(t) = W_5(t) + q_{56}(t) © B_6(t) + q_{57}(t) © B_7(t) \tag{6.20}$$

$$B_6(t) = W_6(t) + q_{60}(t) © B_0(t) + q_{65}(t) © B_5(t) \tag{6.21}$$

$$B_7(t) = W_7(t) + q_{72}(t) © B_2(t) \tag{6.22}$$

where

$W_1(t) = W_5(t) = e^{-\beta t}\overline{F}(t)$,

$W_6(t) = e^{-\alpha t}\overline{G}(t)$,

$W_7(t) = \overline{G}(t)$,

$$W_2(t) = \left(\eta e^{-\alpha t} - \alpha e^{-\eta t}\right)\frac{\overline{G}(t)}{(\eta - \alpha)}.$$

Taking Laplace-transforms of relations (6.16)-(6.22) we have

$$B_0(s) = \frac{N_2(s)}{D_1(s)} \tag{6.23}$$

where

$$N_2(s) = q_{01}^*\left(1 - q_{56}^* q_{65}^* - q_{25}^* q_{57}^* q_{72}^*\right) \times W_1^* + q_{01}^* q_{12}^* \left[\left(1 - q_{56}^* q_{65}^*\right) W_2^* + q_{25}^{*(4)}\left(W_5^* + q_{56}^* W_6^* + q_{57}^* W_7^*\right)\right]$$

In the long run, the fraction of time for which the system is under repair is given by

$$B_0 = \lim_{t \to \infty} B_0(t) = \lim_{s \to 0} s\, B_0^*(s) = \frac{N_2}{D_1'} \tag{6.24}$$

where, in terms of

$$W_1^*(0) = W_5^*(0) = \mu_1$$

$$W_6^*(0) = \mu_6,$$

$$W_7^*(0) = \mu_{7,}$$

$$W_2^*(0) = \frac{(\eta\mu_6 - \alpha\mu_2)}{(\eta - \alpha)},$$

we have

$$N_2 = \left(1 - P_{56}P_{65} - P_{25}^{(4)}P_{57}\right)\mu_1$$
$$+ \left[\left(P_{25}^{(4)}\left(\mu_1 + P_{56}\mu_6 + P_{57}\mu_7\right) + \left(1 - P_{56}P_{65}\right)\left(\eta\mu_6 - \alpha\mu_2\right)\right)\right]P_{12.}$$

6.8 EXPECTED NUMBER OF VISITS BY THE REPAIRFACILITY

We define $V_i(t)$ as the expected number of visits by the repairman in $(0,t]$ given that the system initially starts from regenerative states S_i. By probabilistic arguments, we have the following recursive relations:

$$V_0(t) = Q_{01}(t) © \left[1 + V_1(t)\right] \tag{6.25}$$

$$V_1(t) = Q_{10}(t) © V_0(t) + Q_{12}(t) © V_2(t) \tag{6.26}$$

$$V_2(t) =$$
$$Q_{20}{}^{(4)} © V_0(t) + Q_{23}(t) © V_3(t) + Q^{(4)}{}_{25}(t) © V_5(t) \tag{6.27}$$

$$V_3(t) = Q_{30}(t) © V_0(t) \tag{6.28}$$

$$V_5(t) = Q_{56}(t) © V_6(t) + Q_{57}(t) © V_5(t) \tag{6.29}$$

$$V_6(t) = Q_{60}(t) © V_0(t) + Q_{65}(t) © V_5(t) \tag{6.30}$$

$$V_7(t) = Q_{72}(t) © V_2(t) \tag{6.31}$$

Taking the Laplace-Stieltjes transforms of the above equations and solving for $\widetilde{V}_0(s)$, we have

$$\widetilde{V}_0(s) = \frac{N_3(s)}{D_2(s)} \tag{6.32}$$

where

$$N_3(s) = \widetilde{Q}_{01}\left(1 - \widetilde{Q}_{56}\widetilde{Q}_{65} - \widetilde{Q}_{25}{}^{(4)}\widetilde{Q}_{57}\widetilde{Q}_{72}\right)$$

$$\begin{aligned} D_2(s) &= \left(1 - \widetilde{Q}_{56}Q_{65} - \widetilde{Q}_{25}{}^{(4)}\widetilde{Q}_{57}\widetilde{Q}_{72}\right)\left(1 - \widetilde{Q}_{01}\widetilde{Q}_{10}\right) \\ &- \widetilde{Q}_{01}\widetilde{Q}_{12}\left[\left(1 - \widetilde{Q}_{56}\widetilde{Q}_{65}\right)\left(\widetilde{Q}_{20}{}^{(4)} + \widetilde{Q}_{23}\widetilde{Q}_{30}\right) + \widetilde{Q}^{(4)}{}_{25}Q_{56}\widetilde{Q}_{60}\right] \end{aligned}$$

In the steady state, the number of visits per unit time is given by

$$V_0 = \lim_{t \to \infty} \frac{V_0(t)}{t} = \lim_{s \to 0} s^2\, \widetilde{V}_0(s) = \frac{N_3}{D_1'}, \tag{6.33}$$

where

$$N_3 = \left(1 - P_{56}P_{65} - P_{25}{}^{(4)}P_{57}\right).$$

6.9 SWITCHOVER ANALYSIS

We define $I_i(t)$ as the probability that the standby unit being switched is under switching device in $(0, t]$, given that the system entered regenerative state S_i at $t = 0$. By probabilistic arguments, we have

$$I_0(t) = q_{01}(t) \copyright I_1(t) \tag{6.34}$$

$$I_1(t) = q_{10}(t) © I_0(t) + q_{12}(t) © I_2(t) \tag{6.35}$$

$$I_2(t) = H_2(t) + q_{20}^{(4)}(t) © I_0(t) + q_{23}(t) © I_3(t) + q_{25}^{(4)}(t) © I_5(t) \tag{6.36}$$

$$I_3(t) = H_3(t) + q_{30}(t) © I_0(t) \tag{6.37}$$

$$I_5(t) = q_{56}(t) © I_6(t) + q_{57}(t) © I_7(t) \tag{6.38}$$

$$I_6(t) = q_{60}(t) © I_0(t) + q_{65}(t) © I_5(t) \tag{6.39}$$

$$I_7(t) = q_{72}(t) © I_2(t) \tag{6.40}$$

where

$$H_2(t) = e^{-\eta t}\overline{G}(t),$$

$$H_3(t) = e^{-\eta t}.$$

Taking the Laplace-transforms of relations (6.34) – (6.40), we have

$$I^*{}_0(s) = \frac{N_4(s)}{D_1(s)}, \tag{6.41}$$

$$N_4(s) = q^*{}_{01} q^*{}_{12}\left(1 - q_{56}{}^* q_{65}{}^*\right)\left(H^*{}_2 + q^*{}_{23} H_3{}^*\right)$$

In the steady-state, the fraction of time for which the system is under switch activation is given by

$$I_0 = \lim_{t\to\infty} I_0(t) = \lim_{s\to 0} sI^*{}_0(s) = \frac{N_4}{D_1'} \tag{6.42}$$

where, in terms of

$$H_2{}^*(0) = \mu_2,$$

$$H^*{}_3(0) = \mu_3,$$

we have

$$N_4 = P_{12}(1 - P_{56}P_{65})(\mu_2 + P_{23}\mu_3).$$

6.10 COST ANALYSIS

(1) The expected uptime of the system in $(0,\ t]$ is

$$\mu^*{}_{up} = \int_0^t A_0(u)du$$

so that

$$\mu^*{}_{up}(s) = \frac{A^*{}_0(s)}{s} \tag{6.43}$$

(2) The expected duration of the repairman's busy time in $(0,t]$ is

$$\mu_b(t) = \int_0^t B_0(u)du$$

so that

$$\mu^*{}_b(s) = \frac{B^*{}_0(s)}{s} \tag{6.44}$$

(3) The expected switchover time of the standby unit in $(0,t]$ is

$$\mu_1(t) = \int_0^t I_0(\mu)du\,,$$

so that

$$\mu^*{}_I(s) = \frac{I^*{}_0(s)}{s} \tag{6.45}$$

The expected total cost (gain) incurred in $(0,t]$ is

$$G(t) = C_1\mu_{up}(t) - C_2\mu_b(t) - C_3V_0(t) - C_4\mu_1(t) \tag{6.46}$$

where C_1 is the revenue per unit up time, C_2 is the cost per unit for which the system is under

repair, C_3 is the cost per visit by the repairman and C_4 is the cost per unit time for which the

system is under switch activation device.

The expected profit per unit time in the steady state is

$$G = \lim_{t \to \infty} \frac{G(t)}{t} = \lim_{s \to 0} s^2 G^*(s)$$

$$= C_1 A_0 - C_2 B_0 - C_3 V_0 - C_4 I_0 \tag{6.47}$$