

CHAPTER 4

BUSY PERIOD ANALYSIS OF A TWO UNIT SYSTEM WITH PREVENTITIVE MAINTENANCE AND IMPERFECT SWITCH

4.1 INTRODUCTION

To increase the effectiveness of a system, a unit that has failed is renewed. The renewal can assume various forms. Several authors carried analyses of systems with two or three modes under the assumption that whenever the operative unit fails, it goes to repair immediately and after the completion of repair the server goes off. Srinivasan and Gopalan (1973) studied a two-unit system with warm standby and a single repair facility. Murari and Goyal (1984) made a comparison of two-unit cold standby reliability models; in

“Model 1” the repairman always remains with the system after the failure of the unit. Goel and Sharma (1989) analysed a two-unit standby system with two failure modes and slow switching. Makaddis (1999) considered a system with three modes and an administrative delay in repair. However, in practice, there may be occasions when the repairman appears in and disappears from the system randomly with some probabilities. In this Chapter, two models have been studied. In both the models, a single-unit repairable system with three possible modes of the unit- normal (N), partial failure (P) and total failure (F) – is examined.

In “Model 1”, if the repairman finds the unit in P-mode, then he takes the unit under repair while the unit is operating, whereas in “Model 2”, the partially failed unit does not go under repair but repair is started only when the unit fails completely. Using the regeneration point technique, the various measures of system performance such as MTSF, steady state availability (A_{∞}), busy period analysis of the repair facility, expected number of visits by a repairman, and the profit analysis, are studied, for each model. Numerical example illustrated some of the results obtained. Two models have been compared on the basis of

numerical results by carrying out MTSF and profit analysis for a particular case when repair time distributions are exponential.

4.2 SYSTEM DESCRIPTION

1. The system consists of a single unit. Initially, we assume that the unit is operating. The unit fails through a partial failure.

2. There is a single repair facility which appears in and disappears from the system randomly.
3. The life time of a unit and time of appearance and disappearance of the repairman are negative exponential, whereas the repair times are arbitrarily distributed.
4. The repairman cannot leave the system while repairing the unit.
5. Switch is perfect and the switchover is instantaneous.

4.3 NOTATION

E_0	state of the system at $t=0$
E	set of regeneration states ($S_0 - S_5$) for each model
\bar{E}	set of non-regenerative states for each model
a, b	The rates of appearance and disappearance of repairman
$g_i(t), G_i(t)$	pdf and cdf of the repair time in phase $i=1, 2$.
$q_{ij}(t), Q_{ij}(t)$	pdf and cdf of time for one- step transition from regenerative states S_i to S_j .

$\Phi_i(t)$	cdf of time to system failure where starting state $E_0 = S_i \in E$
$A_i(t)$ = 0]	P [system is up at t S_i at t
$B_i(t)$ repair at	P [repairman is busy in t S_i at $t=0$]
$N_i(t)$	Expected number of visits by the repairman to state i in $(0, t]$
A, NA	repair facility is available/not available.
λ_1	failure rate from N -mode to P -mode
$M_i(t)$	P [system is up at t without passing through any regenerative state or returning to itself S_i at $t=0$]
μ_i	mean sojourn time in state S_i
L.S.T	Laplace-Stieltjes transform

L.T

Laplace transforms

$$\tilde{Q}_{ij}(s) = \int_0^{\infty} e^{-st} dQ_{ij}(t)$$

$$q_{ij}^{*'}(s) = \int_0^{\infty} e^{-st} q_{ij}(t) dt$$

$$\begin{aligned} \mu_i &= \sum_j t dQ_{ij}(t) = -\sum_j q_{ij}^{*'}(0) = -\sum_j Q_{ij}'(0) \\ &= \sum_j m_{ij} \quad i=1,2,\dots \end{aligned}$$

m_{ij} = contribution to mean sojourn time in state S_i when the transition to

$$S_j = -\tilde{Q}_{ij}(0) = -q_{ij}^{*'}(0)$$

⊕ Laplace-Stieltjes convolution

⊙ Laplace convolution

4.4 SYMBOLS FOR STATES OF THE SYSTEM

N_0 : Unit in N-mode, and operative

P : Unit in partial failure mode

P_r : Unit in partial failure mode and under repair

F_r : Unit in complete failure and under repair

F : Unit in failure mode

A, NA : repairman is available / not available

W_r : Unit waiting for repair

Using the above symbols the system may be in one of the following states:

$$S_0 = (N_0, NA), S_1 = (N_0, A), S_2 = (P, NA).$$

$$S_3 = (P, A), S_4 = (W_r, NA).$$

$$S_5 = (F_r, A) \text{ for "Model 1"}.$$

$$S_0 = (N_0, NA),$$

$$S_1 = (N_0, A),$$

$$S_2 = (P, NA),$$

$$S_3 = (P, A),$$

$$S_4 = (W_r, NA),$$

$$S_5 = (F_r, A) \text{ (for "Model 2")}$$

4.5 RELIABILITY ANALYSIS (MODEL 1)

Here the repairman repairs the unit only when it is P-mode (Figure 4.1). Meantime to system failure analysis gives:

$$\Phi_0(t) = Q_{01}(t) \otimes \Phi_1(t) + Q_{02}(t) \otimes \Phi_2(t)$$

$$\Phi_1(t) = Q_{10}(t) \otimes \Phi_0(t) + Q_{13}(t) \otimes \Phi_3(t)$$

$$\Phi_2(t) = Q_{23}(t) \otimes \Phi_3(t) + Q_{24}(t)$$

$$\Phi_3(t) = Q_{31}(t) \otimes \Phi_1(t) + Q_{35}(t) \tag{4.1}$$

$$d Q_{01}(t) = a e^{-at} e^{-\lambda_1 t} dt ;$$

$$d Q_{02}(t) = \lambda_1 e^{-\lambda_1 t} dt .$$

$$dQ_{10}(t) = b e^{-bt} e^{-\lambda_1 t} dt;$$

$$dQ_{23}(t) = \lambda_2 e^{-\lambda_2 t} dt.$$

$$dQ_{24}(t) = \lambda_2 e^{-\lambda_2 t} e^{-at} dt;$$

$$dQ_{35}(t) = \lambda_2 e^{-\lambda_2 t} \overline{G}(t) dt.$$

$$dQ_{31}(t) = g_1(t) e^{-\lambda_2 t} dt;$$

$$dQ_{45}(t) = a e^{-at} dt;$$

$$dQ_{51}(t) = g_2(t) dt \tag{4.2}$$

Letting $t \rightarrow \infty$, using $Q_{ij}(\infty) = P_{ij}$,

$$p_{01} = \frac{a}{a + \lambda_1}; \quad p_{02} = \frac{\lambda_1}{a + \lambda_1}.$$

$$p_{10} = \frac{b}{b + \lambda_1}; \quad p_{13} = \frac{\lambda_1}{b + \lambda_1}.$$

$$p_{23} = \frac{a}{a + \lambda_2}; \quad p_{24} = \frac{\lambda_2}{a + \lambda_2}$$

$$p_{35} = 1 - g_1^*(\lambda_2); \quad p_{31} = g_1^*(\lambda_2);$$

$$p_{45} = 1 = P_{51} \quad (4.3)$$

$$\mu_0 = \int_0^{\infty} P(T < t) dt = \frac{1}{a + \lambda_1}$$

$$\mu_1 = \frac{1}{b + \lambda_1}$$

Figure 4.1 (Model 1)

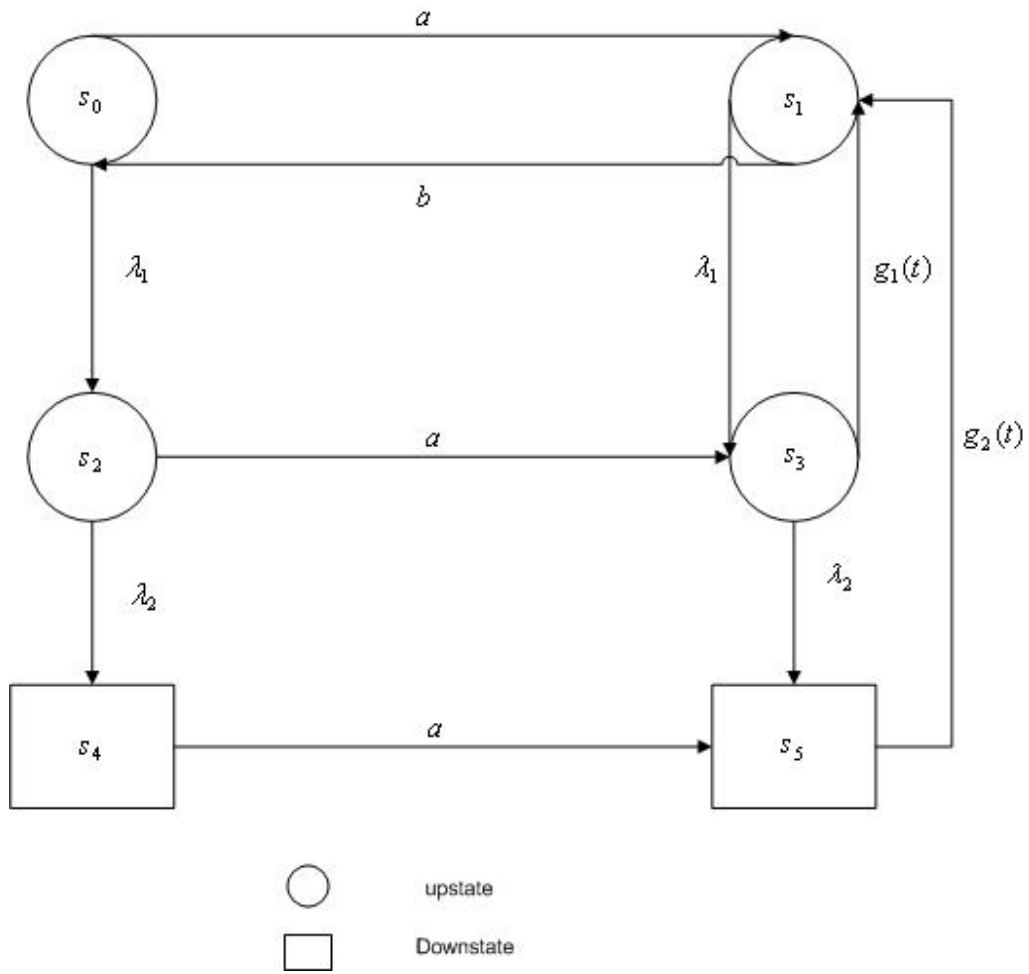
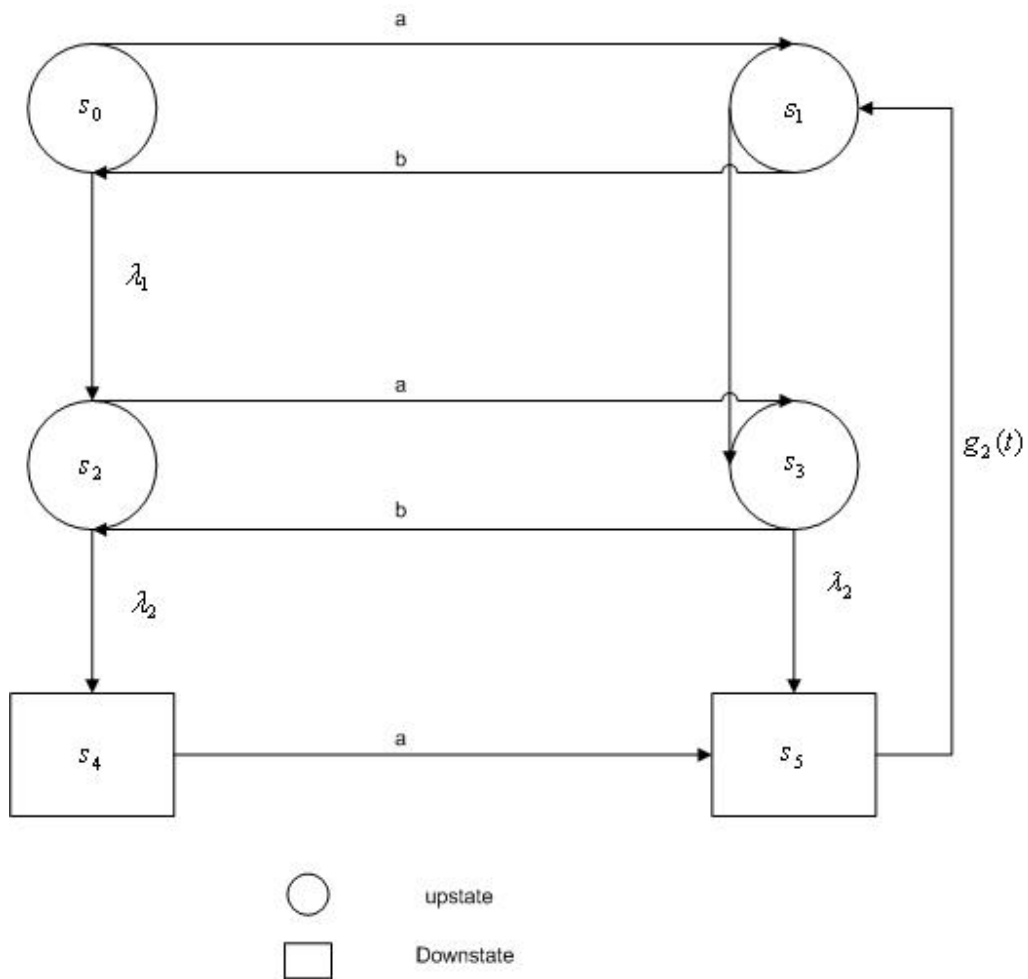


Figure 4.2 (Model 2)



$$\mu_2 = \frac{1}{a + \lambda_2} ,$$

$$\mu_3 = \frac{1 - g^*(\lambda_2)}{\lambda_2}$$

$$\mu_4 = \frac{1}{a} ,$$

$$\mu_5 = \int_0^{\infty} g_2(t) dt \quad (4.4)$$

It can easily be verified that

$$p_{01} + p_{02} = p_{13} + p_{10} = p_{23} + p_{24} = p_{31} + p_{35} = p_{45} = p_{51}.$$

$$\mu_0 = m_{01} + m_{02}; \mu_1 = m_{10} + m_{13}; \mu_2 = m_{24} + m_{23}.$$

$$\mu_3 = m_{31} + m_{35}; \mu_4 = m_{45}; \mu_5 = m_{51}.$$

Taking Laplace-Stieltjes transform of these relations and solving for

$\tilde{\Phi}_0(s)$, we have

$$\int \tilde{\Phi}_0(s) = \frac{N_1(s)}{D_1(s)} \quad (4.5)$$

where

$$N_1(s) = \tilde{Q}_{01}(s)\tilde{Q}_{13}(s)\tilde{Q}_{35}(s) + \tilde{Q}_{02}(s)\{\tilde{Q}_{24}(s)(1 - \tilde{Q}_{13}(s)\tilde{Q}_{31}(s) + \tilde{Q}_{23}(s)\tilde{Q}_{35}(s))\}.$$

$$D_1(s) = 1 - \tilde{Q}_{13}(s)\tilde{Q}_{31}(s) - \tilde{Q}_{01}(s)\tilde{Q}_{10}(s) - \tilde{Q}_{23}(s)\tilde{Q}_{31}(s)\tilde{Q}_{10}(s)\tilde{Q}_{02}(s)$$

The mean time to system failure is found to be (MTSF)

$$T_1 = \lim_{s \rightarrow 0} \frac{1 - \tilde{\Phi}_0(s)}{s} \quad (4.6)$$

$$T_1 = \frac{N_1}{D_1}$$

$$N_1 = \mu_0(1 - p_{13}p_{31}) + \mu_1(p_{01} + p_{23}p_{31}p_{02}) + \mu_2\{p_{02}(1 - p_{13}p_{23})\} + \mu_3(p_{23}p_{02} + p_{01}p_{13})$$

$$D_1 = 1 - p_{13}p_{31} - p_{01}p_{10} - p_{23}p_{31}p_{10}p_{02}$$

4.6 AVAILABILITY ANALYSIS

$$A_i(t) = P [\text{system is up at } t \mid S_i \text{ at } t = 0],$$

then

$$A_0(t) = M_0(t) + q_{01}(t) \odot A_1(t) + q_{02}(t) \odot A_2(t)$$

$$A_1(t) = M_1(t) + q_{10}(t) \odot A_0(t) + q_{13}(t) \odot A_3(t)$$

$$A_2(t) = M_2(t) + q_{23}(t) \odot A_3(t) + q_{24}(t) \odot A_4(t)$$

$$A_3(t) = M_3(t) + q_{31}(t) \odot A_1(t) + q_{35}(t) \odot A_5(t)$$

$$A_4(t) = q_{45}(t) \odot A_5(t)$$

$$A_5(t) = q_{51}(t) \odot A_1(t) \quad (4.7)$$

where

$$M_0(t) = e^{-(a+\lambda_1)t}$$

$$M_1(t) = e^{-(b+\lambda_1)t}$$

$$M_2(t) = e^{-(a+\lambda_2)t}$$

$$M_3(t) = e^{-\lambda_2 t} \bar{G}(t)$$

Using the Laplace-transforms, we obtain

$$A_0^*(s) = \frac{N_2(s)}{D_2(s)}$$

$$N_2(s) = M_0^*(s) [1 - q_{13}^*(s)q_{31}^*(s) - q_{13}^*(s)q_{35}^*(s)q_{51}^*(s)]$$

$$+ M_1^*(s) \left[\begin{array}{l} q_{02}^*(s)q_{23}^*(s)q_{35}^*(s)q_{51}^*(s) + q_{02}^*(s)q_{24}^*(s)q_{45}^*(s)q_{51}^*(s) + q_{01}^*(s) + q_{01}^*(s) \\ + q_{02}^*(s)q_{23}^*(s)q_{31}^*(s) \end{array} \right]$$

$$+ M_2^*(s) [q_{02}^*(s) \{1 - q_{13}^*(s)q_{31}^*(s)q_{13}^*(s)q_{35}^*(s)q_{51}^*(s)\}]$$

$$+ M_3^*(s) [q_{02}^*(s)q_{24}^*(s)q_{45}^*(s)q_{51}^*(s)q_{13}^*(s) + q_{01}^*(s)q_{13}^*(s) + q_{03}^*(s)q_{23}^*(s)]$$

and

$$\begin{aligned} D_2(s) = & 1 - q_{13}^*(s)q_{31}^*(s) - q_{35}^*(s)q_{51}^*(s)q_{13}^*(s) - q_{10}^*(s)q_{01}^*(s) - \\ & q_{10}^*(s)q_{02}^*(s)q_{23}^*(s) \{q_{31}^*(s) + q_{35}^*(s)q_{51}^*(s)\} \\ & - q_{45}^*(s)q_{51}^*(s)q_{10}^*(s)q_{01}^*(s)q_{24}^*(s) \end{aligned}$$

For $A_0^*(s)$, we can obtain the steady state availability, A_∞

$$A_\infty = \lim_{s \rightarrow 0} s A_0^*(s) = \frac{N_2}{D_2}$$

where

$$N_2 = (\mu_0 + \mu_2 p_{02}) p_{10} + \mu_1 + \mu_3 [p_{02} \{p_{24} p_{13} + p_{23}\} + p_{01} p_{13}]$$

$$D_2 = (\mu_0 + \mu_2 p_{02}) p_{10} + \mu_1 + \mu_2 (p_{13} + p_{10} p_{02} p_{23}) \\ + \mu_4 p_{10} p_{02} p_{24} + \mu_5 [p_{13} p_{35} + p_{10} p_{02} (1 - p_{23} p_{31})]$$

4.7 BUSY PERIOD ANALYSIS

$$B_i(t) = P [\text{the repairman is busy at } t \mid S_i \text{ at } t = 0]$$

$$B_0(t) = q_{01}(t) \odot B_1(t) + q_{02}(t) \odot B_2(t)$$

$$B_1(t) = q_{10}(t) \odot B_0(t) + q_{13}(t) \odot B_3(t)$$

$$B_2(t) = q_{23}(t) \odot B_3(t) + q_{24}(t) \odot B_4(t)$$

$$B_3(t) = M_3(t) + q_{31}(t) \odot B_1(t) + q_{35}(t) \odot B_5(t)$$

$$B_4(t) = q_{45}(t) \odot B_5(t)$$

$$B_5(t) = M_5(t) + q_{51}(t) \odot B_1(t)$$

where

$$M_3(t) = e^{-\lambda_2 t} \overline{G}_1(t)$$

$$M_5(t) = \overline{G}_2(t)$$

Using the Laplace transforms, we can find

$$B_0^*(s) = \frac{N_3(s)}{D_2(s)}$$

where

$$N_3(s) = M_3^*(s) [q_{01}^*(s)q_{01}^*(s)q_{13}^*(s) + q_{02}^*(s)q_{45}^*(s)q_{13}^*(s)]$$

$$+ M_5^*(s) [q_{35}^*(s)(q_{13}^*(s)q_{01}^*(s) + q_{02}^*(s)q_{23}^*(s)) + q_{02}^*(s)q_{24}^*(s)q_{45}^*(s)\{1 - q_{13}^*(s)q_{31}^*(s)\}]$$

and $D_2(s)$ is already given earlier. In the long run, the function of time for which the system is under repair is

$$\begin{aligned} B_\infty &= \lim_{t \rightarrow 0} B_0(t) = \lim_{s \rightarrow 0} s B_0^*(s) \\ &= \frac{N_3}{D_2} \end{aligned}$$

$$N_3 = \mu_3 [p_{01}p_{13} + p_{02}(p_{23} + p_{24}p_{13})] + \mu_5 [p_{35}(p_{01}p_{13} + p_{02}p_{23}) + p_{02}p_{24}(1 - p_{13}p_{31})]$$

4.8 EXPECTED NUMBER OF VISITS BY REPAIR FACILITY

The equations for $N_0(t), N_1(t), N_2(t), N_3(t)$ and $N_4(t)$ are given by

$$N_0(t) = Q_{01}(t) \otimes [1 + N_1(t)] + Q_{02}(t) \otimes N_2(t)$$

$$N_1(t) = Q_{10}(t) \otimes N_0(t) + Q_{13}(t) \otimes N_3(t)$$

$$N_2(t) = Q_{23}(t) \otimes [1 + N_3(t)] + Q_{24}(t) \otimes N_4(t)$$

$$N_3(t) = Q_{31}(t) \otimes N_1(t) + Q_{35}(t) \otimes N_5(t)$$

$$N_4(t) = Q_{45}(t) \otimes [1 + N_5(t)]$$

$$N_5(t) = Q_{51}(t) \otimes N_1(t)$$

Taking L.S.T of these equations, and solving for $\tilde{N}_0(s)$, we get

$$\tilde{N}_0(s) = \frac{N_4(s)}{D_2(s)}.$$

$$N_4(s) = [\tilde{Q}_{01}(s) + \tilde{Q}_{02}(s) \{ \tilde{Q}_{23}(s) + \tilde{Q}_{24}(s) \tilde{Q}_{45}(s) \}] [1 - \tilde{Q}_{13}(s) \tilde{Q}_{31}(s) - \tilde{Q}_{51}(s) \tilde{Q}_{13}(s) \tilde{Q}_{35}(s)]$$

In steady state the number of visits per unit time is

$$N_\infty = \lim_{t \rightarrow \infty} \frac{N_0(t)}{t} = \frac{\tilde{N}_4}{D_2}$$

where $\tilde{N}_4 = p_{10}$.

4.9 MODEL 2

Here the repairman repairs the unit only when it is in the F-mode (figure 4.2). The equations for $\Phi_0(t)$, $\Phi_1(t)$ and $\Phi_2(t)$ are the same as in model 1.

The additional equation is

$$\Phi_3(t) = \Phi_{32}(t) \oplus \Phi_1(t) + Q_{35}(t)$$

Transition probabilities

$Q_{01}(t)$, $Q_{02}(t)$, $Q_{13}(t)$, $Q_{10}(t)$, $Q_{23}(t)$, $Q_{24}(t)$ and $Q_{45}(t)$ are the same as in model 1. The additional probabilities are

$$dQ_{35}(t) = \lambda_2 e^{-(b+\lambda_2)t} dt$$

$$dQ_{32}(t) = b e^{-(b+\lambda_2)t} dt$$

$$dQ_{51}(t) = g(t)dt$$

Letting $t \rightarrow \infty$ and using

$$Q_{ij}(\infty) = p_{ij}, \text{ we get}$$

$$p_{35} = \frac{\lambda_2}{b + \lambda_2}, \quad p_{32} = \frac{b}{b + \lambda_2}$$

It can be easily verified that

$$p_{01} + p_{02} = 1 = p_{10} + p_{13} = p_{23} + p_{24} = p_{35} + p_{32} = p_{45} = p_{51}$$

The mean sojourn times (μ_0, μ_1, μ_2) are the same as in model 1.

The additional times are

$$\mu_3 = \frac{1}{b + \lambda_2}, \quad \mu_5 = \int_0^{\infty} \bar{G}(t)dt.$$

Now, proceeding in a similar manner as in model 1, we have the MTSF as:

$$MTSF = \frac{N}{D}$$

where

$$N = (\mu_0 + \mu_1 p_{01})(1 - p_{32} p_{23}) + \mu_2 (p_{02} + p_{01} p_{13} p_{32}) + \mu_1 (p_{01} p_{13} + p_{02} p_{23})$$

$$D = (1 - p_{01}p_{10})(1 - p_{32}p_{23})$$

4.10 AVAILABILITY ANALYSIS (MODEL 2)

The equations for $A_0(t), A_1(t)$ and $A_2(t)$ are the same as in model

1. The additional equation is

$$A_3(t) = M_3(t) + q_{32}(t) \odot A_2(t) + q_{35}(t) \odot A_5(t)$$

The steady state availability A_∞ for “Model 2” is

$$A_\infty = \frac{N_2}{D_2} \quad \text{where}$$

$$N_2 = \mu_2 [1 - p_{13}p_{35} - p_{35}(p_{13}p_{24} + p_{23})] + \mu_1 [p_{02}(p_{23}p_{35} + p_{24}) + p_{01}(1 - p_{23}p_{32})] \\ + \mu_2 [p_{01}p_{13}p_{32} + p_{02}(1 - p_{13}p_{35})] + \mu_3 p_{02}p_{24}p_{13}$$

and

$$D_2 = [\mu_0 p_{10} + \mu_1 + \mu_5 \{p_{13} + p_{02}p_{10}\}] [1 - p_{23}p_{32}] + \\ [\mu_2 + \mu_4 p_{24}] [p_{13}p_{32} + p_{10}p_{02}] + \mu_3 [p_{13} + p_{10}p_{02}p_{23}]$$

4.11 BUSY PERIOD ANALYSIS (MODEL 2)

The equations for $B_0(t), B_1(t), B_2(t)$ and $B_4(t)$ are the same as in model 1. The additional equations are

$$B_3(t) = q_{32}(t) \odot B_2(t) + q_{35}(t) \odot B_5(t),$$

$$B_5(t) = M_5(t) + q_{51}(t) \odot B_1(t)$$

where $M_5(t) = \overline{G}(t)$

In the long run, the function of time of which the system is under repair is given by

$$B_\infty = \lim_{t \rightarrow \infty} B_0(t) = \frac{N_3}{D_2}$$

where

$$N_3 = \mu_5 p_{01} p_{13} (p_{35} + p_{32} p_{24}) - p_{02} (p_{23} p_{35} + p_{24})$$

4.12 EXPECTED NUMBER OF VISITS BY REPAIR FACILITY

(Model 2)

The equations for $N_0(t)$, $N_1(t)$, $N_2(t)$ and $N_4(t)$ are the same as in “Model 1”. The additional equation is

$$N_3(t) = Q_{32}(t) \odot N_2(t) + Q_{35}(t) \odot N_5(t).$$

In the steady state the number of visits per unit time is given by

$$N_\infty = \lim_{t \rightarrow \infty} \left| \frac{N_0(t)}{t} \right| = \frac{N_4}{D_2}$$

where

$N_4 = p_{01} \{p_{13} p_{32} - p_{10} (1 - p_{23} p_{32})\} + p_{02} (1 - p_{13} p_{35})$ and D_2 is already specified.

4.13 PROFIT ANALYSIS

The expected up time and down time of the system and the busy period of the repairman in $(0, t]$ are

$$\mu_{up}(t) = \int_0^t A_0(\mu) d\mu$$

$$\mu_{dn}(t) = t - \mu_{up}(t) = \int_0^t B_0(u) du$$

so that
$$\mu_{up}^*(s) = \frac{A_0^*(s)}{s} = \frac{B_0^*(s)}{s}$$

$$\mu_{dn}^*(s) = \frac{1}{s^2} - \mu_{up}^*(s)$$

Now expected profit incurred in $(0, t]$

= Expected total revenue in $(0, t]$

- Expected total repair in $(0, t]$

- Expected cost of visits by repairman in $(0, t]$.

For “Model 1” and “Model 2”, we have the profit functions as follows:

$$p_1 = k_1 A_\infty - k_2 B_\infty - k_3 N_\infty$$

$$p_2 = k_1 A_\infty - k_2 B_\infty - k_3 N_\infty$$

k_1 = revenue per unit up time of the system

k_2 = cost per unit time for which the repairman is busy

k_3 = cost per unit visits by the repair facility

Table 4.1

$\lambda_2 = 1.2, \mu_1 = 1.1, \mu_2 = 1.2, a = 0.3, b = 0.7, \mu = 1.1, k_1 = 400, k_2 = 30, k_3 = 100$

MEAN TIME TO SYSTEM FAILURE

Failure rate	Model I	Model 2
0.16	9.6123	8.6617
0.17	9.4888	8.2501
0.18	8.7677	7.9101
0.19	8.4711	7.5223

0.20	8.2506	7.2551
0.21	7.7569	6.9915
0.22	7.5116	6.7962

Table 4.2

$$a = 0.3, b = 0.7, \mu_1 = 1.1, \mu_2 = 1.2, \mu = 1.1, k_1 = 400, k_2 = 20, \lambda_1 = 1.2, \lambda_2 = 1$$

PROFIT

Failure rate	Model 1	Model 2
0.16	245.0015	272.5101
0.17	244.1121	269.3112
0.18	239.8162	262.8716

0.19	236.6617	259.9867
0.20	233.3351	256.6226
0.21	230.6318	254.1512
0.22	226.8813	251.8664

4.14 NUMERICAL ILLUSTRATION

In these models, λ_i, μ_i are taken from (Yadavalli et al, 2005).

From Table 4.1, we conclude that as failure rate increases the mean time to system failure decreases. For both models as the failure rate increases the MTSF of the system decreases.

From Table 4.2 we conclude that for both models as the failure rate increases the profit of the system decreases. It is clear that “Model 2” is more beneficial than “Model 1”.