CHAPTER 3

AN n UNIT SYSTEM OPERATING IN A RANDOM ENVIRONMENT

A modified version of this chapter has been published in OPSEARCH, Vol.42, No.3, 169-176, 2005.

3.1 INTRODUCTION

In the probabilistic analysis of multi-unit redundant systems it is usually assumed that a constant number of units perform the system operation at all times. However, we have situations in which this assumption is not true. For example, to increase the thermal power plant availability, an additional induced draft fan (ID fan) may be installed in 200 MW sets, though two ID fans are normally used to handle flue gas and fly ash during full load operation of the plant, i.e., the load on a system may change randomly (see Das and Acharya, 1988). Again, in a telecommunication network, the success of sending a message from an origin to a destination depends upon the existence of at least one path connecting the origin to a destination depends upon the existence of at least one path connecting the origin with the destination with all units determining the path in the operable state. Therefore, the number of units required for sending the message successfully at any time is determined by the availability of units in the intermediate stations and the locations of the origin and destination. Hence the number of units required for the satisfactory performance of the system may depend on the environment in which the system is functioning and the environment is also changing with time.

Sharafali et al (1988) have considered a two-unit n system with similar assumption and obtained expressions for the mean time to the first disappointment and expected number of disappointments in an interval. (see Limnios and Cocozza (1992)).

An attempt is made in this chapter to study a system consisting of n units with the assumption that the number of units required for the satisfactory performance of the system at any time t is prescribed by

the state of a randomly changing environment described by a Markov process $\{Y(t): t \geq 0\}$. The model is discussed in detail in the following section.

3.2 THE MODEL AND ASSUMPTIONS

The system under consideration is an n unit system with a single repair facility. Precisely; the assumptions of the model are as follows:

(i) There are n identical units in the system, which are statistically independent. The failure rate of an operable unit during the need period is a constant.

(ii) The environment determining the number of units required for the satisfactory performance of the system at any time t is a Markov process $\{Y(t): t \geq 0\}$ with the state space $\{0,1,2,\ldots,n\}$.It may be noted that the environment process is independent of the system behaviour.

(iii) The infinitesimal generator of the environment process $\{ Y(t) : t \geq 0 \}$ is given by:

$$A = \begin{array}{c|c} & \begin{array}{ccccc} 0 & 1 & 2 & \dots & n \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ . \\ . \\ . \\ \\ n \end{array} & \begin{bmatrix} -\lambda_0 & \lambda_{01} & \lambda_{02} & \dots & \lambda_{0n} \\ \lambda_{10} & -\lambda_1 & \lambda_{12} & \dots & \lambda_{1n} \\ \dots & \dots & \dots & \dots & \dots \\ & & & & \\ & & & & \\ & & & & \\ \dots & \dots & \dots & \dots & \dots \\ \lambda_{n0} & \lambda_{n1} & \lambda_{n2} & \dots & -\lambda_n \end{bmatrix} \end{array}$$

(iv) If any time t, Y (t) = i, then i (i = 0,1,….,n) of the n identical units are online (if operable) and the remaining operable units will be kept as warm standbys. These i units which online behave like a series system.

(v) Whenever an online unit fails a standby unit if operable is switched online instantaneously.

(vi) A unit in standby can also fail and its failure rate is a constant.

(vii) The failed units are taken up for repair in FIFO order. However, a repair for a failed unit cannot commence, when the environment process is in state zero. Repair is perfect and the repair rate is a constant ‘μ’.

(viii) Whenever the number of units in the operable state is less than the number of units required at that instant of time for the satisfactory performance of the system, the system enters the down state.

(ix) When the system is in the down state, an operable unit cannot fail.

3.3 THE NOTATION

n	number of units in the system
a	constant failure rate of an operable unit during the need period
b	constant failure rate of a standby unit (b < a)
μ	constant repair rate of a failed unit
$\{Y(t): t \geq 0\}$	state of the environment process that determines the number of units required for the satisfactory performance of the system
Matrix A	the infinitesimal generator of $\{Y(t): t \geq 0\}$
$X(t)$	number of failed units in the system at time t
E and Q	respectively the state space and infinitesimal generator of the Markov Process $\{(X(t), Y(t)): t \geq 0\}$

3.4 STATE OF THE SYSTEM

Let X (t) represent the number of failed units at time t and Y(t), the number of units required for the satisfactory performance of the system at time t. Clearly {(X (t), Y (t)): t≥0} is a Markov Process on the state space:

$$E = E_0 \cup E_1 \cup E_2 \cup \ldots \cup E_n$$

where

E_i = {(i , 0),(i , 1),…,(i , n)}, i = 0, 1, 2,. . ., n.

Let $\Delta(\underline{e}_j)$ be diagonal matrix of order (n + 1) with the first leading (j + 1) diagonal elements being the integers 0, 1, 2,…, j and the remaining elements zero. That is:

$$\Delta(\underline{e}_j) = \text{diag}\,(0, 1, 2, j-1, j, \ldots, 0, 0, 0),\ j = 1,2,\ldots,n$$

Also, let $\Delta(\underline{f}_i)$ be a diagonal matrix of order (n + 1) with the first leading (n-i+1) diagonal elements being the integers
(n – i) , (n – i – 1), (n – i – 2), . . ., 2, 1, 0 and the remaining elements
(n-i). that is:

$$\Delta(\underline{f}_i) = \text{diag}\,(n-i, n-i-1, n-i-2, \ldots, 2, 1, 0)$$

Then, the infinitesimal generator of this process is easily seen to be:

$$E_o \quad E_1 \quad E_2 \quad \ldots \quad E_j \quad \ldots \quad E_n$$

$$\mathbf{Q} = \begin{array}{c} E_0 \\ E_1 \\ E_2 \\ \cdot \\ \cdot \\ \cdot \\ E_i \\ \cdot \\ \cdot \\ \cdot \\ E_n \end{array} \begin{bmatrix} \tilde{Q}_{00} & \tilde{Q}_{01} & \tilde{Q}_{02} & \cdots & \tilde{Q}_{oj} & \cdots & \tilde{Q}_{on} \\ \tilde{Q}_{10} & \tilde{Q}_{11} & \tilde{Q}_{12} & \cdots & \tilde{Q}_{1j} & \cdots & \tilde{Q}_{1n} \\ \tilde{Q}_{20} & \tilde{Q}_{21} & \tilde{Q}_{22} & \cdots & \tilde{Q}_{2j} & \cdots & \tilde{Q}_{2n} \\ \cdots & & & & & & \\ & & & & & & \\ & & & & & & \\ \tilde{Q}_{i0} & \tilde{Q}_{i1} & \tilde{Q}_{i2} & \cdots & \tilde{Q}_{ij} & \cdots & \tilde{Q}_{in} \\ \cdots & & & & & & \\ & & & & & & \\ & & & & & & \\ \tilde{Q}_{n0} & \tilde{Q}_{n1} & \tilde{Q}_{n2} & \cdots & \tilde{Q}_{nj} & \cdots & Q_{nn} \end{bmatrix} \qquad (3.1)$$

where the partitioned matrices $\tilde{Q}_{ij}$ are given by:

$\tilde{Q}_{00} = \mathrm{A} - \mathrm{a}\,\Delta(\underline{e}_n) - \mathrm{b}\,\Delta(\underline{f}_0)$,

$\tilde{Q}_{nn} = \mathrm{A} - \mu\mathrm{I}$

$\tilde{Q}_{i,i} = \mu\mathrm{I}$ for $i = 1, 2, \ldots, n$,

$\tilde{Q}_{i,i} = \mathrm{A} - \mu\mathrm{I} - \mathrm{a}\,\Delta(\underline{e}_{n-i}) - \mathrm{b}\,\Delta(\underline{f}_i)$,

$\tilde{Q}_{i,i+1} = a\Delta(\underline{e}_{n-i}) + b\Delta(\underline{f}_i)$

and

$\tilde{Q}_{i,j} = 0$, for other values of i and j.

It may be noted that Q is a square matrix of order $(n + 1)^2 \times (n + 1)^2$. Let

$$p_{ij}(t) = \Pr\{X(t) = i;\ Y(t) = j\};\ i, j = 0,\ 1,\ 2, \ldots, n$$

represent the probability that the system is in state (i, j) at time t. Also let:

$$\underline{\mathrm{p}}(\mathrm{t}) = (p_{00}(t),\ p_{01}(t),\ p_{nm}(t))$$

To derive an expression for $\underline{\mathrm{p}}(\mathrm{t})$, we note that $\underline{\mathrm{p}}(\mathrm{t})$ satisfies the Kolmogorov equation which leads to:

$$\frac{\mathrm{d}\underline{\mathrm{p}}(\mathrm{t})}{\mathrm{dt}} = \underline{\mathrm{p}}(\mathrm{t}).\mathrm{Q}$$

Solving this differential equation, we obtain:

$$\underline{\mathrm{p}}(\mathrm{t}) = \underline{\mathrm{p}}(0).\mathrm{e}^{\mathrm{Qt}} \tag{3.2}$$

where

$\underline{\mathrm{p}}(0)$ is the vector of initial state probabilities.

3.5 STATIONARY DISTRIBUTION

Let $\underline{\pi} = (\underline{\pi}_0, \underline{\pi}_1, \underline{\pi}_2, \ldots, \underline{\pi}_n)$ where $\underline{\pi}_{\mathrm{k}} = (\pi_{k0,}\ \pi_{k1,}\ \ldots,\ \pi_{kn})$ for k = 0, 1, 2, n is the stationary distribution corresponding to the Markov process {(X (t), Y (t)): t ≥ 0}. This is the solution of the equation:

$$\underline{\pi}\ \mathbf{Q} = \underline{0} \tag{3.3}$$

with

$$\sum_{k=0}^{n} \underline{\pi}_k \underline{e} = 1 \tag{3.4}$$

where

$\underline{\mathbf{e}} = (1,1,1)^T$.

Equation (3.3) gives:

$$\underline{\pi}_0 \left(A - a\Delta\left(\underline{e}_n\right) - b\Delta\left(\underline{f}_0\right)\right) + \mu\underline{\pi}_1 = \underline{0} \tag{3.5}$$

$$\underline{\pi}_0 \left(a\Delta\left(\underline{e}_n\right) + b\Delta\left(\underline{f}_0\right)\right) + \underline{\pi}_1 \left(A - \mu I - a\Delta\left(\underline{e}_{n-1}\right) - b\Delta\left(\underline{f}_1\right) + \mu\underline{\pi}_2\right) = \underline{0} \tag{3.6}$$

$$\underline{\pi}_1 \left(a\Delta\left(\underline{e}_{n-1}\right) + b\Delta\left(\underline{f}_1\right)\right) + \underline{\pi}_2 \left(A - \mu I - a\Delta\left(\underline{e}_{n-2}\right) - b\Delta\left(\underline{f}_2\right)\right) + \mu\underline{\pi}_3 = 0 \tag{3.7}$$

$$\ldots$$

$$\underline{\pi}_{n-2} \left(a\Delta\left(\underline{e}_2\right) + b\Delta\left(\underline{f}_{n-2}\right)\right) + \underline{\pi}_{n-1} \left(A - \mu I - a\Delta\left(\underline{e}_1\right) - b\Delta\left(\underline{f}_{n-1}\right)\right) + \mu\underline{\pi}_n = \underline{0} \tag{3.8}$$

$$\underline{\pi}_{n-1} \left(a\Delta\left(\underline{e}_1\right) + b\Delta\left(\underline{f}_{n-1}\right)\right) + \underline{\pi}_n \left(A - \mu I\right) = \underline{0} \tag{3.9}$$

Addition of all these equations in (3.5) – (3.9) yields $(\underline{\pi}_0 + \underline{\pi}_1 + ... + \underline{\pi}_n)$ A = 0. These, together with equation (3.4) implies that $(\underline{\pi}_0 + \underline{\pi}_1 + ... + \underline{\pi}_n)$ must be the invariant measure of the Markov process {(X(t),Y(t)):t ≥ 0} with the generator A. Assume that a possess the invariant measure and let it be η. Hence:

$$\underline{\pi}_0 + \underline{\pi}_1 + ... + \underline{\pi}_n = \underline{\eta} \tag{3.10}$$

We can express $\underline{\pi}_0, \underline{\pi}_1, . . ., \underline{\pi}_n$ in terms of $\underline{\pi}_0$ by solving (3.5) – (3.9). Using equation (3.10), we can get explicit expression for ($\underline{\pi}_0, \underline{\pi}_1, . . ., \underline{\pi}_n$).

3.6 TIME TO THE FIRST DISAPPOINTMENT

The system is said to be in a state of disappointment if the number of operable units at any time is less than the number of units required for the satisfactory performance of the system at that instant of time. i.e.,
n – X (t) < Y (t). In other words, X (t) + Y (t) > n.

Clearly, the set of states of disappointment is:

D ={(1,n), (2,n-1), (2,n), (3,n-2), (3,n-1), (3,n), (n,1), (n,2), (n,n-1), (n,n)}

Let U be the set of upstates, which is the complement of D. By suitably altering the rows and coloumns, we can partition the matrix Q as:

$$\mathbf{Q} = \begin{array}{c c} & \begin{array}{cc} U & D \end{array} \\ \begin{array}{c} U \\ D \end{array} & \begin{bmatrix} Q_U & B_D \\ B_U & Q_D \end{bmatrix} \end{array} \tag{3.11}$$

Let T_D represent the time to the first disappointment. To obtain the distribution of the random variable T_D, we lump together the states of disappointment of the Markov Process {(X (t), Y (t)): t ≥ 0} into a single absorbing state D. We obtain the absorbing Markov Process with generator:

$$\mathbf{Q}' = \begin{bmatrix} Q_U & B_D \underline{e} \\ \underline{0} & 0 \end{bmatrix} \tag{3.12}$$

Let us assume that the process starts in a state in U and so let $\tilde{P}_U(0)$ be the row vector of the initial state probabilities corresponding to this situation. Now the time to the first disappointment is the same as the time to absorption in the Markov process with the generator Q′ given in (3.12). If $G_D(t)$ is the distribution function of the random variables T_D, then:

$$G_D(t) = 1 - \tilde{P}_U(0)\, e^{Q_U T}\ \underline{e}, t>0 \tag{3.13}$$

It may be noted that the distribution function $G_D(t)$ given in (3.13) corresponds to the distribution function of a continuous PH-distribution with representation ($\tilde{P}_U(0), Q_U$) (See Neuts, 1981).

The raw moments are given by:

$$E\left(T_D^K\right) = (-1)\ k!\,\tilde{P}_U(0)\ Q_U^{-k}\ \underline{e},\ k = 0,1,2 \tag{3.14}$$

3.7 MEAN NUMBER OF DISAPPOINTMENTS

To derive an expression for the mean number of disappointments in an arbitrary interval (0, t], we consider the point process generated by the events corresponding to the occurrence of a disappointment. Let $h_D(.)$ be the first order product density of a disappointment (See Srinivasan, 1974). Then $h_D(t)dt$ is the probability that a disappointment occurs in
(t, t + dt). By considering the various possibilities of entering into the states of disappointment, we have:

$$h_D(t) = a\sum_{i=1}^{n} i p_{(n-i),i}(t) + \sum_{j=1}^{n}\sum_{i=0}^{n-j}\sum_{k=1}^{j} \lambda_{i,n-j+k} P_{ji}(t) \tag{3.15}$$

where $P_{ij}(t)$ can be obtained from (3.2).

The above result is in agreement with Chandrasekhar and Natarajan (1999). It may be noted that $h_D(t)$ given in (3.15) is independent of the constant failure rate b of the standby unit.

The expected value of N (D, t), the number of disappointments in (0, t] is given by :

$$E[N(D,t)] = \int_0^t h_D(u)du \tag{3.16}$$

3.8 MEAN STATIONARY RATE OF DISAPPOINTMENTS

The mean stationary rate of disappointments is given by:

$$E[N(D,t)] = \lim_{t\to\infty} h_D(t)$$

and can be easily obtained from (15) by replacing $p_{ki}(t)$ by π_{ki}.

3.9 CONCLUSIONS

This chapter is a study of a more general system in the sense that the results corresponding to several systems can be deduced as special cases as shown below.

3.9.1 Two unit system

For n = 2, we have:

$$h_D(t) = 2ap_{02}(t) + \lambda p_{10}(t) + (\lambda_{12} + a)p_{11}(t) + \lambda_0 p_{20}(t)$$

This result is in agreement with Sharafali *et al*. (1988).

3.9.2 Intermittently used n unit standby redundant system (Yadavalli,1982)

We observe that the results corresponding to an intermittently used n unit standby redundant system can be obtained as a particular case of the model discussed in this paper by taking the state space of the stochastic process $\{Y(t): t \geq 0\}$ to be consisting of only two states 0 and 1 representing the 'need' and 'no need' states respectively.