PROBABILISTIC ANALYSIS OF REPAIRABLE REDUNDANT SYSTEMS

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PROBALISTIC ANALYSIS OF REPAIRABLE REDUNDANT SYSTEMS

by

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submitted in accordance with the requirements

for the degree of

DOCTOR OF PHILOSOPHY

in

Systems Engineering,

Faculty of Engineering, Built Environment and Information Technology,

UNIVERSITY OF PRETORIA

PRETORIA

PROMOTER: PROF. V. S. S. YADAVALLI

July 2005

ACKNOWLEDGEMENTS

My sincere gratitude goes to my promoter, Professor Yadavalli, for his excellent guidance and patience.

My gratitude also to all the members of the Department of Industrial and Systems Engineering who gave their support, guidance and friendship in abundance.

A word of thanks to the Department of Statistics, University of Namibia, who granted me leave to complete the degree.

Without the help of friends, like Dr. Mauritz van Schalkwyk, Anna and Emil Boshoff, this thesis would not have been possible. I particularly thank them.

I also appreciate the interest and support of my children and my family.

My sincere gratitude goes especially to my husband, Gert, who granted me leave of absence from home for such a long period to complete the final product.

Soli Deo Gloria

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SUMMARY

Two well-known methods of improving the reliability of a system are

(i) provision of redundant units, and

(ii) repair maintenance.

In a redundant system more units are made available for performing the system function when fewer are actually required. There are two major types of redundancy – parallel and standby.

Some of the typical assumptions made in the study of standby redundant systems are:

(a) the repair facility can take up a failed unit for repair at any time, if no other unit is undergoing repair

(b) the state of the standby unit is either cold or warm throughout

(c) the random variables like failure times and repair times are independent

(d) the failures can be in one mode

(e) estimation of operating characteristics.

In this testis, an attempt is made to study a few complex and novel models of standby redundant repairable systems by relaxing one or more of these assumptions.

A number of interesting and important characteristics useful for reliability practioners and system designers are obtained for several models. Further, emphasis is also laid on the construction of comprehensive cost functions and their numerical optimization. We give below the conclusions and the possible extensions for future work. These conclusions are drawn from a limited but reasonably exhaustive numerical work carried out.

The thesis contains six chapters. Chapter 1 is introductory in nature and contains a brief description of various types of systems and the mathematical techniques used in the analysis of redundant systems.

In Chapter 2, a stochastic model of an urea decomposition system in the fertilizer industry is studied. A set of difference-differential equations for the state probabilities are formulated under suitable conditions. The state probabilities are obtained explicitly and the steady state availability of the system is obtained analytically as well as illustrated numerically. Confidence limits for the steady state availability are also obtained.

A two dissimilar unit system with different modes of failure is studied in Chapter 3. The system is a priority system in which one of the units is a priority unit and the one other unit is an ordinary unit. The concept of 'dead time' is introduced with the assumption that the 'dead time' is an arbitrarily distributed random variable. The operating characteristics like MTSF, Expected up time, Expected down time, and the busy period analysis, as well as the cost benefit analysis is studied. These characteristics have been demonstrated numerically.

Chapter 4 is a study of a two unit cold standby system with varying physical conditions of the repair facility. The system measures like MTSF, Availability, Busy period of the repairman, etc. are studied. Confidence limits, the steady state availability and the busy period of the repairman in the steady state are also obtained.

In most of the available literature on n-unit standby systems, many of the associated distributions are taken to be exponential, one of the main reasons for this assumption is the number of built-in difficulties otherwise faced while analysing such systems. In Chapter 5, this exponential nature of the distributions is relaxed and a general model of a three unit cold standby redundant system, where the failure and repair time distributions are arbitrary, is studied.

In Chapter 6, a stochastic model of a reliability system which is operated by a human operator is studied. The system fails due to the failure of the human operator. Once again, it is assumed that the human operator can be in any one of the three states; namely, normal stress, moderate stress or extreme stress. Different operating characteristics like availability, mean number of visits to a particular state and the expected profit are obtained. The results are illustrated numerically.

CHAPTER 1

INTRODUCTION

1.1 INTRODUCTION

The manufacturing of tools and special equipment is part of human nature. In earlier days faults and accidents were the only way of learning to make safer and more reliable equipment. Before structural design became an engineering science, the reliability of a bridge was tested with a team of elephants. If it collapsed, a stronger bridge was built and tested again! Obviously, these methods could not continue and as human skills developed a wide variety of very reliable items and structures were designed and manufactured. One example is the undersea telephone cables built by Bell Telephone Laboratories.

Man's earliest preoccupation with reliability was undoubtedly related to weaponry. Interest flowered as a result of the terrible non-reliability of electronic weapons systems used during World War II. Increasingly complex systems, such as the first missiles, also emphasized the importance of successful operation of equipment in a specific environment during a certain time period. The V-1 missile, developed in Germany with high-quality parts and careful attention, was catastrophic: the first 10 missiles either exploded on the launching pad, or landed short of their targets.

Technological developments lead to an increase in the number of complicated systems as well as an increase in the complexity of the systems themselves. With remarkable advancements made in electronics and communications, systems became more and more sophisticated. Because of their varied nature, these problems have attracted the attention of scientists from various disciplines especially the systems engineers, software engineers and the applied probabilists. An overall scientific discipline, called *reliability theory*, that deals with the methods and techniques to ensure the maximum effectiveness of systems (from known qualities of their component parts) has developed. *'Reliability theory introduces quantitative indices of the quality of production'* (Gnedenko et al. (1969)) and these are carried through from the design and subsequent manufacturing process to the use and storage of technological devices. Engineers, Scientists and Government leaders are all concerned with increasing the reliability of manufactured goods and operating systems. As '*Unreliability has consequences in cost, time wasted, the psychological effect of inconvenience, and in certain instances personal and national security'* (Lloyd & Lipow (1962)). In 1963 the first journal on reliability, IEEE-Transactions on Reliability saw the light.

Due to the very nature of the subject, the methods of Probability theory and Mathematical statistics (information theory, queuing theory, linear and nonlinear programming, mathematical logic, the methods of statistical simulation on electronic computers, demography, manufacturing, etc.), play an important role in the problem solving of reliability theory. Other areas include contemporary medicine, reliable software systems, geoastronomy, irregularities in neuronal activity, interactions of physiological growth, fluctuations in business investments, and many more. In human behaviour mathematical models based on probability theory and stochastic processes are helpful in rendering realistic modelling for social mobility of individuals, industrial mobility of labour, educational advancements, diffusion of information and social networks. In the biological sciences stochastic models were first used by Watson and Galton (1874) in a study of extinction of families. Research on population genetics, branching process, birth and death processes, recovery, relapse, cell survival after irradiation, the flow of particles through organs, etc. then followed. In business management, analytical models evolved for the purchasing behaviour of the individual consumer, credit risk and term structure, income determination under uncertainty and many more related subjects. Traffic flow theory is a well known field for stochastic models and studies have been developed for traffic of pedestrians, freeways, parking lots, intersections, etc.

Problems encountered in the design of highly reliable technical systems have led to the development of high-accuracy methods of reliability analysis. Two major problems can be identified, namely:

- creating classes of probability-statistical models that can be used in the description of the reliability behaviour of the system, and
- developing mathematical methods for the examination of the reliability characteristic of a class of systems.

Considering only redundant systems the classical examples are the models of Markov processes with a finite set of states (in particular birth and death processes) (Gnedenko et al. (1969)), Barlow (1984), Gertsbakh (1989) and Kovalenko et al. (1997)), the renewal process method (Cox (1962)), the semi-Markov process method and its generalizations (Cinlar (1975a, b)), generalized semi-Markov process (GSMP) method (Rubenstein (1981)), spacial models for coherent systems (Aven (1996)) and systems in random and variable environment (Ozekici (1996)) and Finkelstein (1999a, b, c)).

Depending on the nature of the research, the applicable form of reliability theory can be introduced to each. A stochastic analysis is made based on some good probability characteristics. It is, however, not simply a case of changing terminology in standard probability theory (say, “random variable” changes to “lifetime”), but reliability distinguishes itself by providing answers and solutions to a series of new problems not solved in the "standard" probability theory framework. Gertsbakh (1989) points out that reliability,

- of a system is based on the information regarding the reliability of the system's *components*
- gives a mathematical description of the *ageing process* with the introduction of several formal notations of ageing (failure rate, etc.)
- introduces well-developed techniques of *renewal theory*
- introduces *redundancy* to achieve optimal use of standby components (an excellent introduction to redundant systems is given in Gnedenko et al. (1969)
- includes the theory of *optimal preventative maintenance* (Beichelt and Fischer (1980))
- is a study of *statistical inference* (often from censored data)

Generally, the mathematical problems of lifetime studies of technical objects (reliability theory) and of biological entities (survival analysis) are similar, differing only in the notation. The term "lifetime" therefore does not apply to lifetimes in the strictest literal sense, but can be used in the figurative sense. The idea is that the statistical analysis done in this thesis should be true in any of the applicable disciplines, although the notation is mostly as for engineering (systems, components, units, etc.). With minor modifications the discipline can be changed to biological, or financial, etc.

1.2 FAILURE

'A failure is the result of a joint action of many unpredictable, random processes going on inside the operating system as well as in the environment in which the system is operating.' (Gertsbakh (1989)). Functioning is therefore seriously impeded or completely stopped at a certain moment in time and all failures have a stochastic nature. In some cases the time of failure is easily observed. But if units deteriorate continuously, determination of the moment of failure is not an easy task. In this study we assume that failure of a unit can be obtained exactly. Failure of a system is called a *disappointment* or a *death* and failure results in the system being in the *down state*. This can also be referred to as a *breakdown* (Finkelstein (1999a)).

Zacks (1992) points out that there are two types of data to consider, namely:

- data from continuous monitoring of a unit until failure is observed
- data from observations made at discrete time points, therefore failure counts

Villemeur (1992) gives an extensive list of possible failures and inter-dependent failures. There are catastrophic failures, determined by a sharp change in the parameters and drift failures (the result of wear or fatigue), arising as a result of a gradual change in the values of the parameters.

1.3 REPAIRABLE SYSTEMS

Failed units of a system may be replaced by new ones, but this may prove to be expensive. To repair the failed units at a repair facility is usually a more cost-effective option than replacement. A *repairable* (or *renewable*) *system* can be described as one where the system can be made operable again. If a system can be renewed, the reliability is increased, resulting in an increase in its time of service. If no repair facility is free, failed units queue up for repair. The life time of a unit while on-line, while in standby as well as the repair times, are all independent random variables. It is assumed that the distributions of these random variables are known and that they have probability density functions.

Repairable systems have been the subject of intensive investigation for a long time. Different random variables can form the basis for research, such as

- availability (or non-availability) and reliability
- time necessary for repair
- number of repairs that can be handled
- switch over time to and from the repair facility
- possibility of a vacation time for the repair facility, and many more.

Barlow (1962) considered some ‘repairman’ (or repair-facility) problems and they have much in common with queuing problems while Rau (1964) analyzed the problem of finding the optimum value of an *k-out-of*-n: G system for maximum reliability. Ascher (1968) has pointed out some inconsistencies in modelling of repairable systems by renewal theory. Several authors, notably Buzacott (1970), Shooman (1968) have used continuous time discrete state Markov process models for describing the behaviour of a repairable system. These models, although conceptually simple, are not practically feasible in the case of a large number of states. Gaver (1964), Gnedenko et al. (1969), Srinivasan (1966) and Osaki (1970a) have used semi-Markov processes for calculation of the reliability of a system with exponential failures. Osaki (1969) has used signal flow graphs to discuss a two-unit system. With the use of semi-Markov processes Kumagi (1971) studied the effect of different failure distributions on the availability through numerical calculations. Branson and Shah (1971) also used semi-Markov process analysis to study repairable systems with arbitrary distributions. Srinivasan and Subramanian (1980), Venkatakrishnan (1975), Ravichandran (1979), Natarajan (1980) and Sarma (1982) have used regeneration point techniques to analyze repairable systems with arbitrary distributions. More references in this and related topics can be found in various papers by Subba Rao and Natarajan (1970), Osaki and Nakagawa (1976), Pierskalla and Voelker (1976), Lie et al. (1977), Kumar and Agarwal (1980), Birolini (1985) and Yearout et al. (1986) and Finkelstein (1993a, 1993b). Jain and Jain (1994) have considered the regulation of 'up' and 'down' times of a repairable system to improve the efficiency of the system.

1.4 REDUNDANCY AND DIFFERENT TYPES OF REDUNDANT SYSTEMS

In a *redundant system* more units are built into it than is actually necessary for proper system performance. Redundancy can be applied in more than one way and a definite distinction can be made between *parallel* and *standby* (sequential) redundancy. In parallel redundancy the redundant units form part of the system from the start, whereas in a standby system, the redundant units do not form part of the system from the start (until they are needed).

1.4.1 Parallel systems

A parallel redundant system with n units is one in which all units operate simultaneously, although system operation requires at least one unit to be in operation. Hence a system failure only occurs when all the components have failed.

Let k be a non-negative integer, such that $k \leq n$, counting the number of units in an n-unit system. It is customary to refer to such a system as *k-out-of-n* system.

1.4.2 k-out-of-n: F system

If *k-out-of-n* system fails, that is when k units fail, it is called an F-system. The functioning of a minimum number of units ensures that the system is up (Sfakianakis and Papastavridis (1993)).

1.4.3 k-out-of-n: G-system

A G-system is operational if and only if at least k units out of n units of the system are operational. Recent work related to this topic can be seen in Zhang and Lam (1998) and Liu (1998). Suppose a radar network has n radar control stations covering a certain area: the system can be operable if and only if at least k of these stations are operable. In other words, to ensure functioning of the system it is essential that a minimum number of units, k, are functioning.

Lately attention moved to load-sharing *k-out-of-n*: G systems, where

- the serving units share the load
- the failure rate of a component is affected by the magnitude of the load it shares.

1.4.4 n-out-of-n: G system

A series that consists of *n* units and when the failure of any one unit causes the system to fail. Although this type of system is not a redundant system, as all the units are in series and have to be operational, it can still be considered as a special vas of a *k-out-of-n* system. There are many papers on the reliability of these systems. Scheuer (1988) studied reliability for *shared-load k-out-of-n*: *G systems*, where there is an increasing failure rate in survivors, assuming identically distributed components with constant failure rates. Shao and Lamberson (1991) considered the same scenario, but with imperfect switching. Then Huamin (1998) published a paper on the influence of work-load sharing in non-identical, non-repairable components, each having an arbitrary failure time distribution. He assumed that the failure time distribution of the components can be represented by the accelerated failure time model, which is also a proportional hazards model when base-line reliability is Weibull.

1.4.5 Standby redundancy

Standby redundancy consists in attaching to an operating unit one or more redundant (*standby*) units, which can, on failure of the operating unit, be switched *on-line* (if operable). Gnedenko et al. (1969) classifies standby units as *cold, warm or hot*.

1. A *cold standby* is completely inactive and because it is not hooked up, it cannot (in theory) fail until it is replacing the primary unit. Also assume that, having been in a non-operating state its reliability will not change when it is put into an operating state.

2. A *warm standby* has a diminished load because it is only partially energized. The standby unit is not subject to the same loading conditions as the on-line unit and failure is generally due to some extraneous random influence. So, although such warm standby can fail, the probability of it failing is smaller than the probability of the unit on-line failing. This the most general type of standby because of hot standby's failure rate and cold standby's possible time lapse before it is operable.
3. A *hot standby* is fully active in the system (although redundant) and the probability of loss of operational ability of a hot standby is the same as that of an operating unit in the standby state. The reliability of a hot standby is independent of the instant at which it takes the place of the operable unit.

1.4.6 Priority redundant systems

A priority system consists of n (≥ 2) units in which some of the units are given *priority* (p-units) and the other units are termed as *ordinary* units (o-units). The operating on-line unit must be the p-unit and this p-unit is never used in the status of a standby and, in the event of a failure, it is immediately taken up for repair – if the repair facility is available. On the other hand, the o-unit only operates on-line when the p-unit has failed and is under repair. Different policies can be adopted (Jaiswal (1968)) if the p-unit fails during the repair of an o-unit, namely pre-emptive and non-pre-emptive priorities.

1.4.6.1 Pre-emptive priority

The repair of the o-unit will be interrupted by the p-unit if the p-unit fails when the repair for *o*-units is on. After completion of the repair of the *p*-unit, the repair of the *o*-unit is continued in one of two ways:

(i) *pre-emptive resume*, where the repair of the *o*-unit continues from the previous point of interruption

(ii) *pre-emptive repeat*, where repair of the *o*-unit is started afresh after completion of the previous interruption. This implies that the time spent by the I-unit before it was pre-empted from the repair has no influence on the re-started repair time.

1.4.6.2 Non-pre-emptive priority

The repair of the o-unit continues and the repair of the *p*-unit is entertained only after completion of the repair of the *o*-unit.

1.5 INTERMITTENTLY USED SYSTEMS

When a system is turned on and off intermittently for the purpose of performing a certain function it is referred to as an *intermittently used system*. It is obvious that for such a system continuous failure free performance is not so absolutely necessary. In such cases consideration has to be given to the fact that the system can be in the down state during certain time intervals without any real consequence. The probability that the system is in the up state is not an important measure; what is really important is the probability that the system is available when needed. *Operational reliability* is thus a function of the readiness and the probability of continuous functioning over a specified period of time and it can grow or decline with age, depending on the nature of the system.

Gaver (1964) pointed out that it is pessimistic to evaluate the performance of an intermittently used system solely on the basis of the distribution of the time to failure. Srinivasan (1966), Nakagawa et al. (1976), Srinivasan and Bhaskar (1979a, 1979b, 1979c), Kapur and Kapoor (1978, 1980), Sarma (1982) and Yadavalli and Hines (1991) extended Gaver's results for two-unit and n-unit systems, and, obtained various system measures.

1.6 MEASURES OF SYSTEM PERFORMANCE

In the previous sections a brief discussion was given of the various types of redundant systems as discussed in the literature. In this section the discussion is about measures of system performance as applicable in different contexts (Barlow & Proschan (1965) and also Gnedenko et al. (1969)).

1.6.1 Reliability

Reliability engineering has developed, and advanced substantially during the past 50 years, mainly due to the use of high risk and complex systems (Beichelt (1997)). Reliability is a quantitative measure to ensure operational efficiency. *'The reliability of a product is the measure of its ability to perform its function, when required, for a specific time, in a particular environment. It is measured as a probability.'* (Leitch (1995)). This implies that reliability contains four parts, namely

- the *expected function* of a system
- the *environment* of a system (climate, packaging, transportation, storage, installation, pollution, etc.)
- *time*, which is often negatively correlated with reliability

- *probability*, which is time-dependent, thus causing the need for a statistical analysis.

One can distinguish between *mission reliability*, when a device is constructed for the performance of one mission only and *operational reliability*, when a system is turned on and off intermittently for the purpose of performing a certain function. In the latter case we refer to an *intermittently used system*.

Ordinarily the period of time intended is $(0, t]$.

Let $\{\phi(t), t \geq 0\}$ be the performance process of the system.

For fixed t this $\phi(t)$ is a binary variable, defined as follows:

$$\phi(\mathrm{t}) = \begin{cases} 0 & \text{if the system is functioning at time t} \\ 1 & \text{if the system is in a failed state at time t.} \end{cases}$$

1.6.1.1 The reliability function

The *reliability function*, $R(t)$ gives the probability that the system does not fail up to t, that is

$$\begin{aligned} R(t) &= P[\text{system is functioning in } (0, t]] \\ &= P[\phi(u) = 0 \ \forall \ u \text{ such that } 0 < u \leq t]. \end{aligned}$$

1.6.1.2 Interval reliability

If the number of system failures in the interval $(t, t + x]$ is considered, the performance measure

$$R(\mathrm{t}, x) = P[\phi(\mathrm{u}) = 0 \ \forall \ u \text{ such that } 0 < u \leq t + x]$$

is referred to as the *interval reliability*.

If t = 0 the interval reliability becomes the reliability $R(x)$.

1.6.1.3 Limiting interval reliability

Limiting interval reliability is defined as the limit of $R(t, x)$ as $t \to \infty$, and is denoted $R_\infty(x)$.

1.6.1.4 Mean time to system failure

The expectation of the random variable representing the duration of time, measured from the point the system starts operating, till the instant it fails for the *first* time is called *mean time to system failure* (MTSF). This is obtained from the relation

$$\text{MTSF} = \int_0^\infty R(u)du\,.$$

1.6.2 Availability

This measure of system performance '*…denotes the probability that the system is available for use (in operable condition) at any arbitrary instant t'*. Availability is therefore the probability that, at the given time t, the system will be operational. It combines aspects of reliability, maintainability and maintenance support and implies that the system is either in active operation or is able to operate if required.

Availability pertains only to systems which undergo repair and are restored after failure, or to intermittently used systems. As such, it is eminently reasonable to introduce an *availability function* $A(t)$. In theory $A(0)$ should be 100%, but even equipment coming directly out of storage may be defective. A high availability can be obtained either by increasing the average operational time until the next failure, or by improving the maintainability of the system. Gnedenko and Usnakov (1995) defines different coefficients of availability for one-unit systems.

1.6.2.1 Instantaneous or pointwise availability

This is a point function which describes the probability that a system will be able to operate at a given instant of time (Klaassen and Van Peppen (1989) and Beasley (1991)). In symbols:

$$A(t) = P[\phi(t) = 0].$$

1.6.2.2 Interval availability

Given an interval of time (and with given tolerances), interval availability is the expected fraction of this time that the system will be able to operate.

1.6.2.3 Average availability

If a failed unit is repaired and is then ‘as good as new’, the average availability is defined as

$$\text{Average Availability} = \frac{MTSF}{MTSF + MTSR}$$

where MTSF and MTSR are the Mean Time to System Failure and Mean Time to System Repair respectively.

1.6.2.4 Asymptotic or steady-state or limiting availability

The limiting availability, A_∞, is the expected fraction of time that the system operates satisfactorily in the long run (Barlow and Proschan (1965)): it is the probability that the system will be in an operational state at time t, when t is considered to be infinitely large

$$A_\infty = \lim_{t\to\infty} A(t)\,.$$

1.6.3 Time to first disappointment

The system is said to be in a state of *disappointment* if the number of operable units at any time is less than the number of units required for the satisfactory performance of the system at that instant of time. For an intermittently used system, Gaver (1964) pointed out that a disappointment realizes in one of two possible ways: the system enters the down state during a need period, or a need for the system arises and at that time the system is in the down state. The event 'disappointment' is very useful as it renders the distribution of the time to the first disappointment, the mean number of disappointments over an arbitrary interval and also the mean duration of the disappointments.

1.6.4 Mean number of events in (0, *t*]

Let $N(a,\ t)$ denote the number of a particular type of a event (e.g. a disappointment, system recovery, system down, etc.) in (0, t]. The *mean number of events* in (0, t] is then given by

$$E[N(a, t)] = \frac{1}{t}\int_0^t h_1(u)du$$

where $h_1(u)$ is the first order product density of the events (product densities are defined in a subsequent section of this chapter).

The *mean stationary rate of occurrence* of these events is given by

$$E[N(a)] = \lim_{t\to\infty} \frac{\mathrm{E[N(a,\ t)]}}{\mathrm{t}}$$

1.6.5 Confidence limits for the steady state availability

A 100(1 – α)% *confidence interval* for A_∞ is defined by

$$P[a < A_\infty < b] = 1 - \alpha$$

where the numbers a and b ($a < b$) are determined using the appropriate statistical tables. It may be noted that A_∞ is a function of the parameters of operating time distribution, repair time, need and no-need period distributions, etc.

1.7 STOCHASTIC PROCESSES USED IN THE ANALYSIS OF REDUNDANT SYSTEMS

Previous sections briefly looked at different types of redundant systems and the various measures of system performance. In this section the techniques used in the analysis of redundant repairable systems will be summarized.

1.7.1 Renewal theory

In renewal theory there exists times, usually random, from which onward the future of the process is a probabilistic replica of the original process and interest is in the lifetime (a stochastic variable) of a unit. At time $t = 0$ a repairable unit is put into operation and is functioning. At each failure the unit is replaced by a new one of the same type, or subjected to maintenance that completely restores it to an 'as good as new' condition. This process is repeated and replacement time is taken as negligible. The result is a sequence of lifetimes, and the study is restricted to these *renewal points*. The probability object in these sums of non-negative independently identically distributed random variables lies in the number of renewals N_t up to some time t.

Renewal processes are extensively used by many researchers to study specific reliability problems. The homogeneous Poisson process is the simplest renewal process and has received considerable attention. As in all other processes, the time parameter can be considered as either discrete or continuous. Feller (1950) gave a proper lead for the discrete and this was followed by the very lucid account of Cox (1962) for the continuous case (he provided an introduction to renewal theory in the case of a repair facility not being available and failed units queuing up for repair). Barlow (1962) applied queuing theory in his research on repairable systems. Srinivasan (1971) studied some operating characteristics of a one unit system, Gnedenko et al. (1969) obtained the mean life time to system failure of a two-unit standby system, Buzacott (1970) studied some priority redundant systems, etc.

Although renewals can take on different forms, the system starts a new cycle after each renewal (which is independent of the previous ones). If repair time is not negligible, each cycle consists of a lifetime and a repair time and both are random variables with individual distributions (repair time can also be considered as a fixed time). The process is called

- an *ordinary renewal process* if the time origin is the initial installation of the system and the repair time is considered negligibly small in comparison with the lifetime of the unit – renewal is taken as instantaneous, or

- *a general renewal process* if the time origin is some point subsequent to the initial installation of the system (Cox (1962)). Høyland and Rausand (1994) calls this a *modified* renewal process, while Feller (1957) refers to such a process considering the residual life time of a system at an arbitrary chosen time origin as a *delayed renewal* process.

1.7.1.1 Ordinary renewal process: instantaneous renewal

Consider a basic model of continuous operation where a unit begins operating at instant $t = 0$ and stays operational for a random time T_1 and then fails. At this instant it is replaced by a new and statistically identical unit, which operates for a length of time T_2, then fails and is again replaced etc. These random component life lengths T_1, T_2, …, T_r … of the identical units are independent, non-negative and identically distributed random variables that constitute a random flow or *ordinary renewal process*.

Let $P[T_i \leq t] = F(t)$; $t > 0$, $i = 1, 2, \ldots$ be the underlying distribution of the renewal process.

The time until the rth renewal is given by

$$t_r = T_1 + T_2 + \ldots + T_r = \sum_{i=1}^{r} T_i \, .$$

Let the random variable $N(t) = \max\,\{r;\, R_r \leq t\}$ indicate the number of times a renewal takes place in the interval $(0; t]$, then the number of renewals in an arbitrary time interval $(t_1, t_2]$ is equal to $N(t_2) - N(t_1)$.

A *renewal function* $H(t)$, which is the expected value of $N(t)$ in the time interval $(0; t]$, can be defined as

$$H(t) = E[N(t)] = \sum_{r=1}^{\infty} F^{(r)}(t)$$

where $F^{(r)}(\cdot)$ is the r-fold convolution of F.

Furthermore, (Cox (1962)),

$$H(t) = F(t) + \int_0^t H(t-x)dF(x) .$$

The renewal density function h(t) satisfies the equation

$$h(t) = \sum_{n=1}^{\infty} f^{(r)}(t)$$

and the renewal density function $h(t)$ satisfies the equation

$$h(t) = f(t) + \int_0^t h(t-x) f(x)dx .$$

Seeing that $h(t)\Delta t = P$ [exactly one renewal point in $(t, t+\Delta]$],

which implies that the renewal density $h(t)$ basically differs from the hazard rate $h^0(t)$, as

$$h^0(t) = \frac{f(t)}{R(t)} = \frac{f(t)}{(1 - F(t))} .$$

1.7.1.2 Random renewal time

Suppose the time for renewal is not instantaneous but considered as a random variable that is included in the consecutive time-periods, or cycles, of the systems' performance. Each cycle then consists of a *time to failure* and a *time to repair* and both are stochastic variables. Instants of failure and cycles of renewal can be identified.

Let $F(t)$ be the life distribution and $G(x)$ be the repair length function with respective probability density functions $f(t)$ and $g(x)$, then the density function of the cycles C of the life time and repair time, say $k(t)$ is obtained by the convolution formula

$$k(t) = \int_0^t f(x)g(t-x)dx .$$

If $N_F(t)$ counts the number of failures and $N_R(t)$ the number of repairs in $(0; t]$, define

$$W(t) = E[N_F(t)]$$

and

$$V(t) = E[N_R(t)]$$

and let $Q(t) = W(t) - V(t);\ \forall\, t$, assuming that $w(t) = W'(t)$ and $v(t) = V'(t)$.

The *failure and repair intensities* can be then respectively be defined as

$$\lambda(t) = \frac{w(t)}{A(t)}$$

where $A(t)$ is the availability function

$$\mu(t) = \frac{v(t)}{Q(t)}; \qquad Q(t) \neq 0.$$

1.7.1.3 Alternating renewal processes

Alternating renewal processes were first studied in detail by Takács (1957) and are discussed in many textbooks (Birolini (1994) and Ross (1970)). A generalization of the ordinary renewal process discussed previously where the state of the unit is given by the binary variable

$$X(t) = \begin{cases} 0 & \text{if the system is functioning at time t} \\ 1 & \text{otherwise.} \end{cases}$$

The two alternating states may be ‘system up’ and ‘system down’. If these alternating independent renewal processes are distributed according to $F(x)$ and $G(x)$, there are two renewal processes embedded in them for the different transitions from ‘system up’ to ‘system down’.

One-item repairable structures are generally described by alternating renewal processes with the assumption that after each repair the item is like new.

1.7.1.4 The age and remaining lifetime of a unit

In the notation of 1.7.1(a), let t_r indicate the random component life lengths, that is

$$t_r = \sum_{i=1}^{r} T_i \, .$$

Let R_r, $r \in N$, represent the length of the rth repair time, then the sequence $T_1, R_1, T_2, R_2, \ldots$ forms an alternating renewal process. Define

$$t_n = \sum_{r=1}^{n-1} (R_r + T_{r+1}) ; \qquad n \in \mathrm{N}$$

and set $t_0 = t_0^o = 0$.

This sequence t_n generates a delayed renewal process.

If $B_1(t)$ denotes the *forward recurrence time at time t*, then

$$B_1(t) = t_{N_t+1} - t \qquad \text{or} \qquad B_1(t) = t_{N_t^o+1} - t$$

Hence,

- $B_1(t)$ equals the *time to the next failure time* if the system is up at time t, or
- $B_1(t)$ equals the *time to complete the repair* if the system is down at time t.

Hence,

- $B_2(t)$ equals the *age* of the unit if the system is up at time t, or
- $B_2(t)$ equals the *duration of the repair* if the system is down at time t.

Returning to the renewal function $H(t)$, define the elementary renewal theorem (Feller (1949)), stating that, for an ordinary renewal process with underlying exponential distribution (parameter λ and $H(t) = \lambda t$)

$$\lim_{t \to \infty} \frac{\mathrm{H(t)}}{\mathrm{t}} = \frac{1}{\mu}$$

with $\mu = E(T_i) = 1/\lambda$, the mean lifetime.

If the renewals correspond to component failures, the mean number of failures in $(0, t]$ is approximately (for t large)

$$H(t) = E[N(t)] \approx \frac{1}{\mu} = \frac{1}{MTSF}.$$

1.7.2 Semi-Markov and Markov renewal processes

Consider a general description of a process where a system

- moves from one state to another with random sojourn times in between
- the successive states visited form a Markov chain
- the sojourn times have a distribution which depend on the present state as well as the next state to be entered.

This describes a Markov chain if all sojourn states are equal to one, a Markov process if the distribution of the sojourn times are all exponential and independent of the next state and a renewal process if there is only one state (then allowing an arbitrary distribution of the sojourn times).

Denote the state space by the set of non-negative integers {0, 1, 2 ...} and the transition probabilities by p_{ij}, $i, j = 0, 1, 2$ ... If $F_{ij}(t)$, $t > 0$ is the conditional distribution function of the sojourn time in state i, given that the next transition will be into state j, let

$$Q_{ij}(t) = p_{ij} F_{ij}(t), \quad i, j = 0, 1, 2 \ldots$$

denote the probability that the process makes a transition into state j in an amount of time less than or equal to t, given that it just entered state i at $t = 0$. The functions $Q_{ij}(t)$ satisfy the following conditions

$$Q_{ij}(0) = 0, \quad Q_{ij}(\infty) = p_{ij}$$

$$Q_{ij}(t) \geq 0, \quad i, j = 0, 1, 2 \ldots$$

$$\sum_{j=0}^{\infty} Q_{ij}(t) = 1$$

Let J_0 and J_n respectively denote the initial state and the state after the nth transition occurred. The embedded Markov chain $\{J_n, n = 0, 1, 2 \ldots\}$ then describes a Markov chain with transition probabilities p_{ij}.

Let $N_i(t)$ denote the number of transitions into state i in $(0, t]$ and

$$N(t) = \sum_{i=0}^{\infty} N_i(t)$$

The stochastic process $\{X(t), t \geq 0\}$ with $X(t) = i$ denoting the process is in state i at time t is called a semi-Markov process (SMP) and it is clear that $X(t) = J_{N(t)}$. A SMP is a pure jump process and all states are regeneration states. The consecutive states form a time-homogeneous Markov chain, but it is a process without memory at the transition point from one state to the next.

The vector stochastic process $\{N_1(t)\ _,\ N_2(t)\ ...\}$ for $t \geq 0$ is called a Markov renewal process (MRP). This implies that the SMP records the state of the process at each time point, while the MRP is a counting process keeping track of the number of visits to each state.

Assuming that the time-intervals in which the random variables X(t) continues to remain in the n-point state are independently distributed such that

$$\lim_{t \to \infty} P[X(t + x) = j, X(t + u) = i: \forall u \leq x \,|X(t) = i, X(t - \Delta) \neq i]$$

$$= f_{ij}(x) ; i, j = 0, 1, 2 ...$$

If the transition of $X(t)$ is characterized by a change of state, then the quantities $f_{ii}(\cdot)$ are zero functions. Such a process which is a Markov chain with a randomly transformed time scale is called a MRP.

To remove the consequence that $f_{ii}(\cdot) = 0$, another function of a MRP can be given, namely defining it as a regenerative stochastic process $\{X(t)\}$ in which the epoch*s* at which $X(t)$ visits any member of a certain countable set of states are regeneration points; the visits being regenerative events.

In a combination of a Markov chain and a renewal process to form a SMP, the purpose is to create a tool that is more powerful than what either could provide individually. SMP were independently introduced by Lévy (1954) and Smith (1955). Detailed use of SMP and MRP can be found in Pyke (1961a, 1961b), Cinlar (1975) and Ross (1970). Barlow and Proschan (1965) used these processes to determine the MTSF in a two-unit system. Cinlar (1975), Osaki (1970a, 1970b), Arora (1976), Nakagawa & Osaki (1974, 1976) and Nakagawa (1974) have used the theory of SMP to discuss certain reliability problems.

1.7.3 Regenerative processes

In a regenerative stochastic process $X(t)$ there exists a sequence t_0, t_1, ... of stopping times such that $t = \{t_n;\ n \in N\}$ is a renewal process. If a point of regeneration happens at $t = t_1$, then the knowledge of the history of the process prior to t_1 loses its predictive value; the future of the process is totally independent of the past. Thus $X(t)$ regenerates itself repeatedly at these stopping times and the times between consecutive renewals are called *regeneration times*. The application of renewal theory to regenerative processes makes renewal theory such an important tool in elementary probability theory.

The *delayed renewal process* is defined as follows: if $\hat{t} = \{t_n - t_{0;} n \in N\}$ is a renewal process such that $t_0 \geq 0$ is independent of $\hat{t}$, (implying that the time t_0 of the first renewal is not necessarily the time origin) it is called a delayed renewal process. A delayed regenerative process is a process with a sequence $t = \{t_n; n \in N\}$ of stopping times which form a delayed renewal process. As an example: for any initial state i, the times of successive entrances to a fixed state j in a Markov process form a delayed renewal process.

In some cases non-exponentially distributed repair times and/or failure free operating times may lead to semi-Markov processes, but in general it leads to processes with only a few states (or even to non-regenerative processes). Recent research in this field is concerned with Brownian motion with the interest on the random set of all regeneration times and on the excursions of the process between generations.

1.7.4 Stochastic point processes

Among discrete stochastic processes, point processes are widely used in reliability theory to describe the appearance of events in time. A *renewal process* is a well known type of point process, used as a mathematical model to describe the flow of failures in time. It is a point process with restricted memory and each event is a regeneration point. In practical reliability problems, the interest is often in the behaviour of a renewal process in a stationary regime, i.e., when $t \to \infty$, as repairable systems enter an 'almost stationary' regime very quickly. A generalization of a renewal process is the so-called *alternating renewal process*, which consists of two types of independently identically distributed random variables alternating with each other in turn.

This theory of recurrent events has a huge variety of applications ranging from classical physics, biology, management sciences, cybernetics and many other areas. The result is that point processes have been defined differently by individuals in the different fields of application. The properties of stationary point processes were first studied by Wold (1948) and Bartlett (1954), to whom we owe the current terminology. Moyal (1962) gave a formal and well-knit theory of the subject that even provides an extension to cover non-Euclidean spaces. Srinivasan (1974), Srinivasan and Subramanian (1980) and Finkelstein (1998, 1999c) extensively used point processes in reliability theory and applications.

Our interest in point processes lies in those applications which, in general, lead to the development of multivariate point processes. For this purpose we can define a point process as a stochastic process '*whose realizations are related to a series of point events occurring in a continuous one-dimensional parameter space (such as time, etc)*'. The sequence of times $\{t_n\}$ are the "renewal" epochs which generates the point process and the two random variables of interest are

- the number of points that fall in the interval $(t; t + x]$
- the time that has lapsed since the nth point after (or before) t.

The characterization property of *stationarity* applies to certain point processes, namely that the density function of observed events in a time interval does not depend on its position on the time axis, but only on the length of the interval. There are different types of stationarity that can be defined, namely simply stationary, weakly stationary and completely stationary (Srinivasan and Subramanian (1980)).

Furthermore, define $p(n; t, x) = P[N(t, x) = n]$ and if $\sum_{n\geq 2} P(n; t, t+\Delta) = o(\Delta)$ for small Δ, the point process is said to be *orderly* or *regular* (there are no multiple events, or clusters of events with probability one).

1.7.4.1 Multivariate point processes

Applications for multivariate stationary point processes can be found in many fields and the properties of these processes have been studied in depth by Cox and Lewis (1970).

If the constraint of independence of the intervals in a stationary renewal process is relaxed, a *stationary point process* is obtained; if the same constraint is removed in the case of a Markov renewal process a *multivariate point process* is obtained.

1.7.4.2 Product densities

Ramakrishnan (1954) developed, analyzed and perfected the *product density* technique as a sophisticated tool for the study of point processes. A point process is described by the triplet (Φ, **B,** P), where P is a probability distribution on some σ-field **B** of subsets of the space Φ of all states. Describe the state of a set of objects by a point x of a fixed set of points X. Assume for this discussion that X is the real number line. Define A_k as intervals and $N(.)$ as a counting measure which is uniquely associated with a sequence of points $\{t_i\}$ such that:

$N(A)$ = the number of points of the sequence $\{ t_i : t_i \in A\}$

$N(t, x)$ = the number of points (events) in the interval $(t; t + x]$

$N'(t, x)$ = the number of points (events) in $(t + x; t + x + \Delta]$.

The central quantity of interest in the product density technique is $\mathrm{N}'(\mathrm{t, x})$, denoting the number of entities with parametric values between x and $x + \Delta$ at time t.

From the factorial moment distribution the product density of order , which represents the probability of an event in each of the intervals

$$(x_1, x_1 + \Delta_1), (x_2, x_2 + \Delta_2), ..., (x_n, x_n + \Delta_n),$$

can be defined. It is expressed as the product of the density of expectation measures at different points, namely

$$h_n(x_1, x_2, ..., x_n) = \lim_{\Delta_1, \Delta_2 ..., \Delta_n \to 0} \frac{E[\prod_{i=1}^{n} N(x_i, \Delta_i)]}{\Delta_1 \Delta_2 \ldots \Delta_n} \; ; \; x_1 \neq x_2 \neq \ldots \neq x_n$$

or equivalently

$$h_n(x_1, x_2, ..., x_n) = \lim_{\Delta_1, \Delta_2 ..., \Delta_n \to 0} \frac{P[N(x_i, \Delta_i) \geq 1, i = 1,2\ldots,n]}{\Delta_1 \Delta_2 \ldots \Delta_n} \; ; \; x_1 \neq x_2 \neq \ldots \neq x_n$$

Since $h_n(...)$ is a product of the density of expectation measures at different points, the density is aptly called the *product density*.

Considering the *ordinary renewal process* as defined in 1.7.1(a), the renewal function $H(t)$ is the expected number of random points in the interval $(0; t]$. Modify the process by allocation of all integral values to $\{ t_i \}$ and consider a corresponding sequence of points on the real line. In the point process then generated by the random variables $\{ t_i \}$, the counting process $N(t, x)$ represents the number of points in the interval $(t, t + x]$ and the product density is

$$h_m(t, t_1, t_2, ..., t_m) = \mathrm{E}[\mathrm{N}'(\mathrm{t}, \mathrm{t}_1)\, \mathrm{N}'(\mathrm{t}, \mathrm{t}_2) \ldots \mathrm{N}'(\mathrm{t}, \mathrm{t}_\mathrm{m})]$$

The product density of degree m is

$$h_m(t, t_1, t_2, ..., t_m) = h_1(t, t_1)\, h(t_2 - t_1)\, h(t_3 - t_2) \ldots h(t_m - t_{m-1});$$

$$(t_1 < t_2 < \ldots < t_m).$$

1.8 SCOPE OF THE WORK

A stochastic model of an urea decomposition system in the fertilizer industry is studied in Chapter 2. A set of difference-differential equations for the state probabilities are formulated under suitable conditions. The state probabilities are obtained explicitly and the steady state availability of the system is obtained analytically as well as illustrated numerically. Confidence limits for the steady state availability are also obtained.

In Chapter 3, a dissimilar unit system with different modes of failure is studied. The system is a priority system in which one of the units is a priority unit and the other unit one is an ordinary unit. The concept of 'dead time' is introduced with the assumption that the 'dead time' is an arbitrarily distributed random variable. The operating characteristics like MTSF, Expected up time, Expected down time, and the busy period analysis, as well as the cost benefit analysis is studied.

A two unit priority redundant system is studied in Chapter 4. The main aim of this chapter is to consider the physical conditions of the repair facility since the repair time distribution is affected by such conditions. Various system measures are studied, and the confidence limits for the availability and busy period are obtained in the steady state case.

In most of the available literature on *n*-unit standby systems, many of the associated distributions are taken to be exponential, one of the main reasons for this assumption is the number of built-in difficulties otherwise faced while analysing such systems. In Chapter 5 this exponential nature of the distributions is relaxed and a general model of a three unit cold standby redundant system, where the failure and repair time distributions are arbitrary, is studied.

In Chapter 6, a stochastic model of a reliability system which is operated by a human operator is studied. The system fails due to the failure of the human operator. Once again, it is assumed that the human operator can be in any one of the three states; namely, normal stress, moderate stress or extreme stress. Different operating characteristics like availability, mean number of visits to a particular state and the expected profit are obtained.

Results are illustrated numerically at the end of the chapters.

1.9 GENERAL NOTATION

$X(\cdot)$	A stochastic process describing the state of a system
p.d.f.	Probability density function
r. v.	Random variable
$f(\cdot)$	The p.d.f. of the lifetime of a unit while on-line
$g(\cdot)$	The p.d.f. of the repair time of a unit
©	Convolution symbol
$f^{(n)}(\cdot)$	n-fold convolution of a function $f(\cdot)$ with itself, where $f(\cdot)$ is arbitrary
$f^*(s)$	Laplace transform of the function $f(t)$
$F(t)$	Cumulative distribution function: $\int_0^t f(u)du$
$\overline{F}(t)$	Survivor function: $1 - F(t)$
E_i	Regenerative event of type i
A	Availability
$A_i(t)$	P(system is up at t / E_i at $t = 0$)
A_∞	Steady state availability
R	Reliability
$R_i(t)$	P(system is up in $(0, t]$ / E_i at $t = 0$)
MLE	Maximum likelihood estimator
MTSF	Mean time to system failure (also MTTF)

MTSR	Mean time to first appointment
SMP	Semi-Markov process
MRP	Markov renewal process

CHAPTER 2

CONFIDENCE LIMITS FOR THE STEADY-STATE AVAILABILITY OF A STOCHASTIC MODEL OF UREA DECOMPOSITION SYSTEM IN THE FERTILIZER INDUSTRY

2.1 INTRODUCTION:

The role and importance of reliability has been a core issue in any Engineering industry for the last three decades. Reliability is of importance to both manufacturers and consumers. From the consumers and manufacturers point of view reliability provides quality and vice versa. So, the reliability measure is very important, as the improvement in reliability is achieved through quality. While this measure of reliability assumes great importance in industry there are many situations where continuous failure free performance of the system, though desirable, may not be absolutely necessary.

In such situations it may be eminently reasonable to introduce another measure called 'availability', which denotes the probability that the system is functioning at any time point. In the process industry like the fertilizer industry, we come across many processes like synthesis decomposition, crystallization, prilling and recovery [see U.N. Fertiliser Manual (1967), Kumar et al. (1991)].

The gas liquid mixture (urea, NH_3, CO_2, Biuret) flows from the reactor at 126°C into the upper part of a high-pressure decomposer where the flushed gases are separated. The liquid falls through a sieve plate, which comes in contact with high temperature gas available from the boiler and the falling film heater. The process is repeated in a low-pressure absorber. The solution is further heated to 165°C in the falling film heater, which reduces the Biuret formation and hydrolysis of urea (see figure 1).

The overhead gases from the high-pressure decomposition go to the high-pressure absorber cooler. The liquid flows to the top of the low-pressure absorber and is cooled in a heat exchanger. Additional flushing of the solution takes place in the upper part of the low-pressure absorber to reduce the solution pressure from 17.5 to 2.5 kg/cm^2. The low-pressure absorber has four sieve trays and a packed bed. In the packed bed, the remaining ammonia is stripped off by CO_2 gas.

The overhead gases go to the low-pressure absorber cooler, in which the pressure is controlled at 2.2 kg/cm^2. Most of the excess ammonia and carbonate is separated from the solution flowing to the gas separator. The gas separator has two parts:

(i) the upper part is at 105°C and 0.3 kg/cm^2 and here the remaining small amounts of ammonia and CO_2 are recovered by reducing the pressure; the sensible heat of the solution is enough to vaporize these gases.

(ii) The lower part has a packed section at 110°C and atmospheric pressure.

Air containing a small amount of ammonia and CO_2 is fed off from the gas absorber by an on/off gas blower, to remove the remaining small amounts of ammonia and CO_2 present on the solution. Off gases from the lower and upper parts are mixed and led to the off-gas condenser. The urea solution concentrated to 70-75% is fed to a crystallizer.

It is well known that the steady state availability is a satisfactory measure for systems, which are operated continuously (e.g. a detection radar system).

A point estimator of steady state availability is usually the only statistic calculated, although decisions about the true steady state availability of the system should take uncertainty into account. Since

$$A_\infty = \frac{MTBF}{MTBF + MTTR},$$

the uncertainties in the values of the MTBF and MTTR reflect an uncertainty in the values of the point steady state availability (where MTBF stand for Mean Time Between Failures and MTTR for Mean Time To Repair).

By treating these uncertain parameters as random variables, we can obtain the distribution of point steady state availability by combining the distribution of operation and repair times. Hence we can construct estimators and confidence limits for the steady state availability, which are consistent with equivalent statements on the operating time and repair time parameters. Thomson (1966) has derived techniques for placing a lower confidence limit on the system's steady state availability that differ significantly from a specified value, when MTBF and MTTR are estimated from test data.

Gray and Lewis (1967) established the exact confidence interval for steady state availability of systems assuming that the time between failures is described by an exponential random variable and that the time to repair is described by a lognormal random variable.

Butterworth and Nikolaisen (1973) have obtained the bounds on the availability function for the general repair time distribution. Masters and Lewis (1987) have derived exact confidence limits for the system steady state availability with Gamma life time and lognormal repair time. Masters et al. (1992) have proposed a method of establishing exact confidence limits for steady state availability of systems when the time between failure and time to repair are independent Weibull and lognormal random variables respectively.

Abu-Salih et al. (1990) have derived 100(1- α)% confidence limits for the steady state availability of a two unit parallel system with the assumption that the failure time distribution is exponential and the repair time has a two stage Erlangian distribution. They have also assumed that an upstate unit will not fail when the other unit is in the second stage of repair.

Chandrasekhar and Natarajan (1994a, b) have considered and *n*-unit parallel system with the assumption that the failure time distribution is exponential and the repair time has a two stage Erlangian distribution. Further they have assumed that an operable unit can also fail while the other unit is in the second stage of repair. In particular they have derived a 100(1-α)% confidence limits for the steady state availability of a two unit parallel system. Yadavalli et al. (2001, 2002, 2005) have studied the 100(1- α)% confidence limits for different types of systems (parallel and standby) with the assumption that the repair facility is not available for a random time.

The organisation of this chapter is as follows: Section 2.1 is introductory in nature, the system description and notation is given in Section 2.2. The availability analysis of the system is studied in Section 2.3. In Section 2.4, the interval estimation for A_∞ is studied, and subsequently the numerical results for Sections 2.3 and 2.4 are shown in Section 2.5.

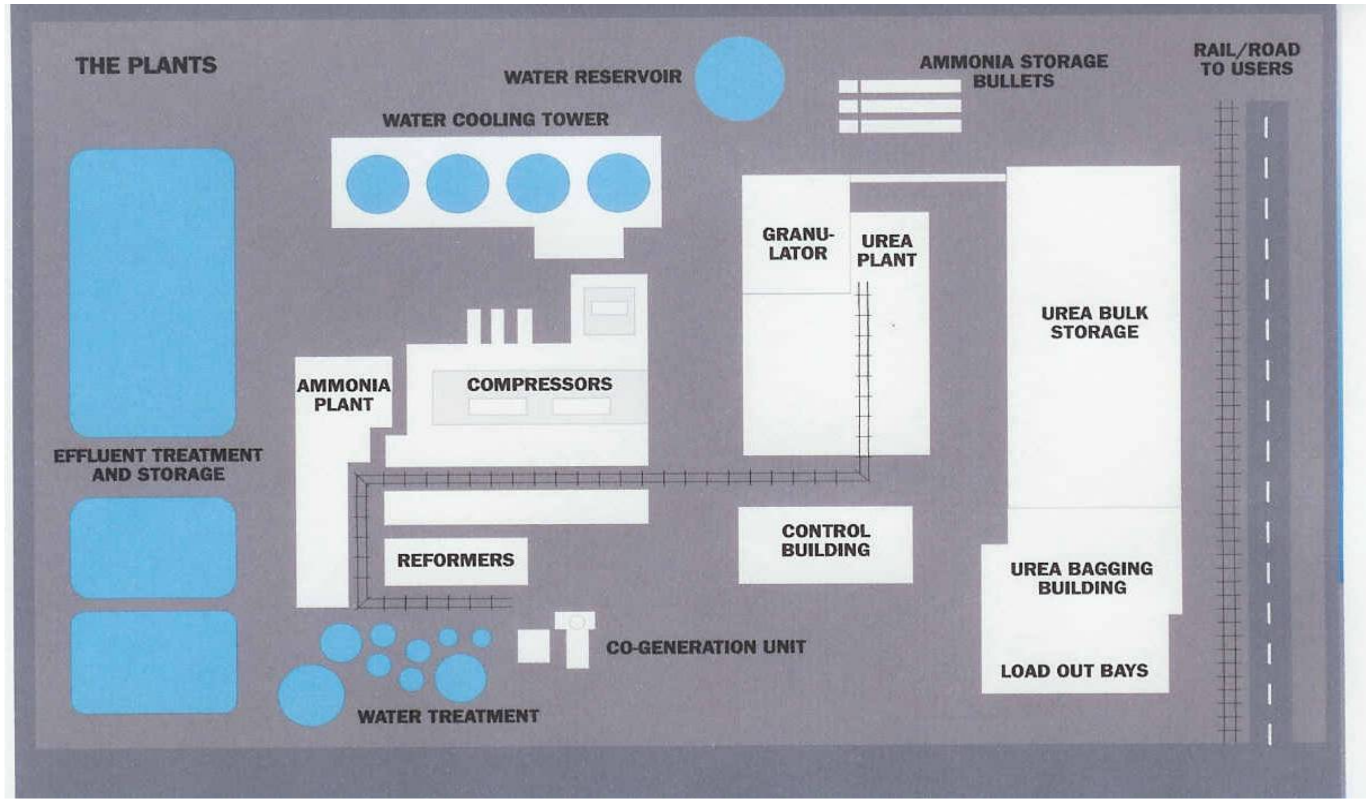

Figure 2.1 Urea plant (by courtesy of Balance Kapuni, South Taranaki, New Zealand)

2.2 SYSTEM DESCRIPTION AND NOTATION

The complex system described above consisting of four subsystems connected in series.

1. Subsystem (A_i) has two units. Unit A_1 is the boiler for the high-pressure absorber and A_2 is the falling filter heater for the low-pressure absorber. This subsystem (A_i) fails by failure of A_1 or A_2.
2. Subsystem B_i has two units in series. Unit B_1 is called the high-pressure absorber and unit B_2 is called the low-pressure absorber. Failure of either causes complete failure of the system.
3. Subsystem D, the gas separator, has one unit only, arranged in series with B_1 and B_2. Failure of unit D causes complete failure of the system.
4. Subsystem E_i the heat exchanger has one unit in standby. Failure occurs only when both units fail.
5. The life time of the units (A_i, B_i, D, E; i = 1,2) are exponentially distributed random variables with parameters λ_i; i=1,2,3,4,5,6.
6. The repair time of the units are exponentially distributed random variables with parameters $\mu_{j;}$ j = 1,2,3,4,5,6.
7. Each unit is as good as new after the repair.
8. Spare parts and the repair facility are always available.
9. The standby unit in E is of the same nature and capacity as the operating active unit.

10. The repair is done at regular time interval or at the time of failure. The repair includes the replacement as well.
11. There is no simultaneous failure among subsystems.
12. State O indicates the operating state without using standby unit and state 6 indicates the operating state using the standby state in subsystem E.
13. E_1 is the state of the system running at full capacity with a standby unit in subsystem E.

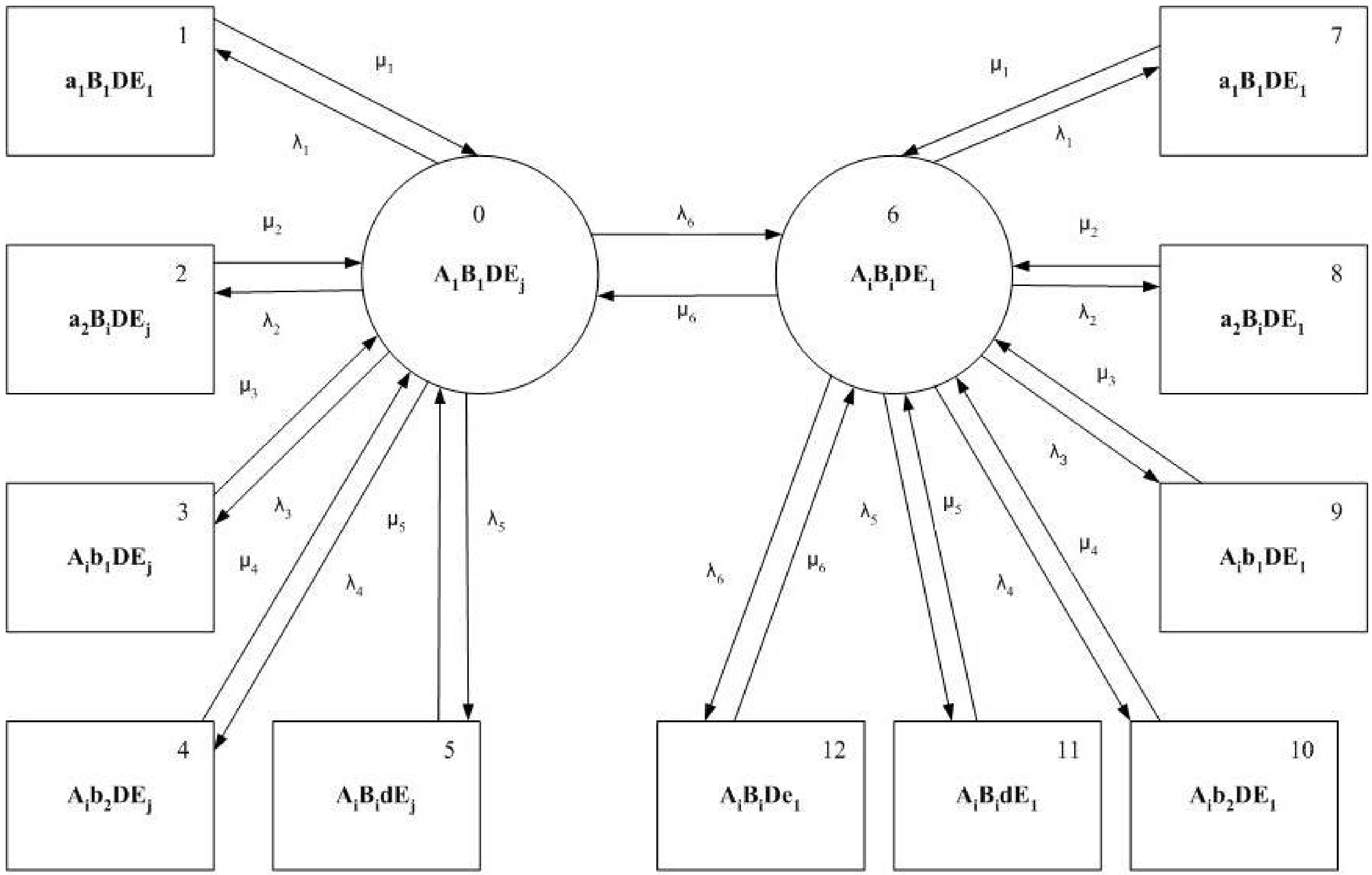
1
$a_1B_1DE_1$
2
$a_2B_iDE_j$
3
$A_ib_1DE_j$
4
$A_ib_2DE_j$
5
$A_iB_idE_j$
0
$A_1B_1DE_j$
6
$A_iB_iDE_1$
7
$a_1B_1DE_1$
8
$a_2B_iDE_1$
9
$A_ib_1DE_1$
10
$A_ib_2DE_1$
11
$A_iB_idE_1$
12
$A_iB_iDe_1$
μ_1
λ_1
μ_2
λ_2
μ_3
λ_3
μ_4
λ_4
μ_5
λ_5
μ_6
λ_6

Figure 2.2

2.3 AVAILABILITY ANALYSIS OF THE SYSTEM

Let $P_i(t)$ = P[system is in state i, with only failure at time t]

$$P_i = \lim_{t \to \infty} P_i(t).$$

Writing the application of flow balance (Ravindran et al. (1982)), the steady state probability can be determined from the following equations:

With the help of Figure 2.2, we obtain the following differential equations describing the state probabilities:

$$P_0'(t) = -(\sum_{i=1}^{6} \lambda_i) P_0(t) + \sum_{i=1}^{6} \mu_i P_i(t) \tag{2.3.1}$$

$$P_6'(t) = -(\sum_{i=1}^{6} \lambda_i + \mu_6) P_6(t) + \sum_{i=1}^{6} \mu_i P_{i+6}(t) + \lambda_6 P_0(t) \tag{2.3.2}$$

$$\sum_{i=1}^{5} \mu_i P_i'(t) = -\sum_{i=1}^{5} \mu_i P_i(t) + \sum_{i=1}^{5} \mu_i P_0(t) \qquad i = 1, 2, \dots, 5 \tag{2.3.3}$$

$$\sum_{i=1}^{6} \mu_i P_{i+6}'(t) = -\sum_{i=1}^{5} \mu_i P_{i+6}(t) + \sum_{i=1}^{6} \mu_i P_6(t) \qquad i = 1, 2, \dots, 6 \tag{2.3.4}$$

$$\sum_{i=1}^{6} P_i(t) = 1 \tag{2.3.5}$$

In the steady state, the equations (2.3.1) – (2.3.5) become:

$$(\sum_{i=1}^{6} \lambda_i)\, p_0 = \sum_{i=1}^{6} \mu_i\, p_i \tag{2.3.6}$$

$$(\sum_{i=1}^{6} \lambda_i + \mu_6)\, p_6 = \sum_{i=1}^{6} \mu_i\, p_{i+6} + \lambda_6\, p_0 \tag{2.3.7}$$

$$\sum_{i=1}^{5} \mu_i\, p_i = \sum_{i=1}^{5} \lambda_i\, p_0\, ; i = 1, 2, \ldots, 5 \tag{2.3.8}$$

$$\sum_{i=1}^{6} \mu_i\, p_{i+6} = \sum_{i=1}^{6} \lambda_i\, p_6\, ; i = 1, 2, \ldots, 6 \tag{2.3.9}$$

$$\sum_{i=1}^{6} p_i = 1\,. \tag{2.3.10}$$

Solving the system of simultaneous equations (2.3.6) - (2.3.10), the steady state availability A_∞ can be obtained as

$$A_\infty = p_0 + p_6 = \frac{1 + \frac{\lambda_6}{\mu_6}}{1 + (1 + \frac{\lambda_6}{\mu_6}) \sum_{i=1}^{6} \frac{\lambda_i}{\mu_i}}\,.$$

For different parameters, Tables 2.3.1(a) – 2.3.1 (e) and the Figure 2.3 explain the availability function.

2.4. INTERVAL ESTIMATION FOR A_∞

Let $X_{i1}, X_{i2}, \ldots, X_{in},; (i = 1, 2, \ldots, 6)$ be random samples of size n, each drawn from different exponential populations with failure rates λ_i, similarly $Y_{i1}, Y_{i2}, \ldots, Y_{in},; (i = 1, 2, \ldots, 6)$ be random samples each drawn from exponential populations with parameters μ_i. Since λ_i 's are the parameters of the exponential distribution, then an estimate can be found for λ_i or for $1/\lambda_i = \alpha_i$ (say), which is equal to the mean value of the time of failure-free operation.

For the analysis, let

$$\alpha_i = \frac{1}{\lambda_i}, \beta_i = \frac{1}{\mu_i}.$$

Then the maximum likelihood estimates (MLE) of α_i and β_i are given by

$$\frac{1}{n}\sum_{j=1}^{n} X_{ij} = \overline{X}_i, \qquad \frac{1}{n}\sum_{j=1}^{n} Y_{ij} = \overline{Y}_i.$$

Hence

$$\hat{A}_\infty = \frac{1+\dfrac{\overline{y}_6}{\overline{x}_6}}{1+\left(1+\dfrac{\overline{y}_6}{\overline{x}_6}\right)\displaystyle\sum_{i=1}^{6}\frac{\overline{y}_i}{\overline{x}_i}}.$$

By an application of the multivariate central limit theorem (Rao (1973)), it follows that

$\sqrt{n}(\overline{x} - \theta) \xrightarrow{D} N_6(0, \Sigma)$ as $n \to \infty$ where $\overline{X} = (\overline{X}_1, \overline{X}_2, \overline{X}_3, \overline{X}_4, \overline{X}_5, \overline{X}_6, \overline{Y}_1, \overline{Y}_2, \overline{Y}_3, \overline{Y}_4, \overline{Y}_5, \overline{Y}_6)$

We know that $\hat{A}_\infty$ is a real-valued function in $\overline{X}_i$ and $\overline{Y}_i$; $i = 1, 2, ..., 6$.

$$\theta = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6).$$

The dispersion matrix $\Sigma = (\sigma_{ij})_{12x12}$ is given by

$$\Sigma = diag(\alpha_1^2, \ldots, \alpha_6^2, \beta_1^2, \ldots, \beta_6^2).$$

From Rao (1973), as $n \to \infty$, i.e. using the multivariate central limit theorem,

$$\sqrt{n}(\hat{A}_\infty - A_\infty) \xrightarrow{D} N_6(0, \sigma^2(\theta)) \text{ where}$$

$$\sigma^2(\theta) = \sum_{i=1}^{6}\left(\frac{\partial A_\infty}{\partial \alpha_i}\right)^2 \sigma_{ii} + \sum_{i=1}^{6}\left(\frac{\partial A_\infty}{\partial \beta_i}\right)^2 \sigma_{ii}$$

Replacing by its consistent estimator

$\hat{\theta} = (\overline{X}_1, \overline{X}_2, \ldots, \overline{X}_6, \overline{Y}_1, \overline{Y}_2, \ldots, \overline{Y}_6)$, it follows that $\hat{\sigma}^2 = \sigma^2(\hat{\theta})$ is a consistent estimator of $\sigma^2(\theta)$. Since $\sigma^2(\theta)$ is a consistent estimator of θ, we know that $\sigma^2(\hat{\theta})$ is a consistent estimator of θ (see Wackerly et al. (2002)).

Then by Slutsky's theorem (Slutsky (1928))

$$\frac{\sqrt{n}\,(\hat{A}_\infty - A_\infty)}{\hat{\sigma}} \xrightarrow{D} N(0,1) \text{ as } n \to \infty.$$

This implies that

$$P\left[-k_{\frac{\alpha}{2}} \le \frac{\sqrt{n}\,(\hat{A}_\infty - A_\infty)}{\hat{\sigma}} \le k_{\frac{\alpha}{2}}\right] = 1 - \alpha,$$

where $k_{\alpha/2}$ is obtained from normal tables, i.e. 100(1 – α)% confidence interval is given by

$$\hat{A}_\infty \pm k_{\alpha/2}\hat{\sigma}(\theta).$$

2.5 NUMERICAL ILLUSTRATION

For different values of the parameters, the numerical computations for A_∞ are shown in Tables 2.5.1(a) – 2.5.1(e) and Figure 2.3.

The confidence limits for A_∞ were also obtained and shown Table 2.5.2.

		A_∞									
$A_1=\alpha_2$	$A_3=\alpha_4=\alpha_5$	$\alpha_6=$ 0.0	$\alpha_6=$ 0.001	$\alpha_6=$ 0.002	$\alpha_6=$ 0.003	$\alpha_6=$ 0.004	$\alpha_6=$ 0.005	$\alpha_6=$ 0.006	$\alpha_6=$ 0.007	$\alpha_6=$ 0.008	$\alpha_6=$ 0.009
0.0	0.0	1.00000	0.99998	0.99994	0.99985	0.99974	0.99961	0.99944	0.99924	0.99901	0.99875
	0.005	0.90909	0.90907	0.90903	0.90897	0.90888	0.90877	0.90863	0.90846	0.90827	0.90806
	0.010	0.83333	0.83332	0.83328	0.83323	0.83316	0.83306	0.83294	0.83280	0.83264	0.83247
0.001	0.0	0.99602	0.99600	0.99595	0.99587	0.99577	0.99563	0.99546	0.99526	0.99503	0.99478
	0.005	0.90578	0.90578	0.90574	0.90568	0.90559	0.90548	0.90534	0.90517	0.90498	0.90477
	0.010	0.83056	0.83055	0.83052	0.83046	0.83039	0.83029	0.83018	0.83004	0.82988	0.82970
0.005	0.0	0.98037	0.98037	0.98033	0.98025	0.98015	0.98002	0.97985	0.97966	0.97944	0.97919
	0.005	0.89284	0.89284	0.89280	0.89274	0.89266	0.89254	0.89241	0.89255	0.89207	0.89186
	0.010	0.81967	0.81966	0.81962	0.81957	0.81950	0.81941	0.81929	0.81916	0.81901	0.81883

Table 2.5.1 (a): Effect of Failure Rate (taking $\beta_1 = \beta_2 = 0.5$; $\beta_3 = \beta_4 = 0.2$; $\beta_5 = 0.1$; $\beta_6 = 0.25$)

		A_∞									
$\alpha_1=\alpha_2$	$A_3=\alpha_4=\alpha_5$	$\alpha_6=$ 0.0	$\alpha_6=$ 0.001	$\alpha_6=$ 0.002	$\alpha_6=$ 0.003	$\alpha_6=$ 0.004	$\alpha_6=$ 0.005	$\alpha_6=$ 0.006	$\alpha_6=$ 0.007	$\alpha_6=$ 0.008	$\alpha_6=$ 0.009
0.0	0.0	1	0.99998	0.99994	0.99986	0.99975	0.99961	0.99944	0.99924	0.99901	0.99875
	0.005	0.9090	0.90908	090904	0.90894	0.90888	0.90877	0.90863	0.90846	0.90827	0.90806
	0.010	0.83333	0.83332	0.83329	0.83323	0.83316	0.83316	0.83306	0.83294	0.83280	0.83247
0.001	0.0	0.99668	0.99666	0.99661	0.99654	0.99643	0.99629	0.99612	0.99592	0.99569	0.99544
	0.005	0.90634	0.90633	0.90629	0.90623	0.90614	0.90602	0.90588	0.90572	0.90553	0.90532
	0.010	0.83102	0.83101	0.83098	0.83093	0.83085	0.83075	0.83064	0.83050	0.83034	0.83016
0.005	0.0	0.98361	0.98359	0.98355	0.98347	0.98336	0.98323	0.98306	0.98287	0.98265	0.98240
	0.005	0.89552	0.89551	0.89547	0.89541	0.89532	0.89521	0.89507	0.89491	0.89473	0.89452
	0.010	0.82192	0.82191	0.82187	0.82182	0.82175	0.82165	0.82154	0.82140	0.82125	0.82107

Table 2.5.1 (b): Effect of Failure Rate (taking $\beta_1 = \beta_2 = 0.6$; $\beta_3 = \beta_4 = 0.2$; $\beta_5 = 0.1$; $\beta_6 = 0.25$)

		A_∞									
$\alpha_1=\alpha_2$	$A_3=\alpha_4=\alpha_5$	$\alpha_6=$ 0.0	$\alpha_6=$ 0.001	$\alpha_6=$ 0.002	$\alpha_6=$ 0.003	$\alpha_6=$ 0.004	$\alpha_6=$ 0.005	$\alpha_6=$ 0.006	$\alpha_6=$ 0.007	$\alpha_6=$ 0.008	$\alpha_6=$ 0.009
0.0	0.0	1	0.99998	0.99994	0.99986	0.99975	0.99961	0.99944	0.99924	0.99901	0.99875
	0.005	0.92308	0.92306	0.92302	0.92296	0.92286	0.92274	0.92260	0.92243	0.92223	0.92201
	0.010	0.85714	0.85713	0.85710	0.85704	0.85696	0.85685	0.92260	0.92243	0.92223	0.92201
0.001	0.0	0.99715	0.99714	0.99709	0.99701	0.99690	0.99676	0.99659	0.99639	0.99617	0.99591
	0.005	0.92065	0.92064	0.92060	0.92053	0.92044	0.92032	0.92017	0.92000	0.91981	0.91969
	0.010	0.85505	0.85504	0.85500	0.85494	0.85486	0.85476	0.84637	0.84623	0.84606	0.84588
0.005	0.0	0.98592	0.98590	0.98585	0.98578	0.98567	0.98553	0.98537	0.98517	0.98495	0.98470
	0.005	0.91106	0.91105	0.91101	0.91094	0.91085	0.91074	0.91060	0.91043	0.91024	0.91003
	0.010	0.84677	0.84676	0.84673	0.84667	0.84659	0.84649	0.84637	0.84623	0.84606	0.84588

Table 2.5.1 (c): Effect of Failure Rate (taking $\beta_1 = \beta_2 = 0.7$; $\beta_3 = \beta_4 = 0.3$; $\beta_5 = 0.1$; $\beta_6 = 0.25$)

		A_∞									
$\alpha_1=\alpha_2$	$\alpha_3=\alpha_4=\alpha_5$	$\alpha_6=$ 0.0	$\alpha_6=$ 0.001	$\alpha_6=$ 0.002	$\alpha_6=$ 0.003	$\alpha_6=$ 0.004	$\alpha_6=$ 0.005	$\alpha_6=$ 0.006	$\alpha_6=$ 0.007	$\alpha_6=$ 0.008	$\alpha_6=$ 0.009
0.0	0.0	0.99998	0.99994	0.99986	0.99975	0.99975	0.99961	0.99944	0.99924	0.99901	0.99875
	0.005	0.93023	0.93022	0.93018	0.93011	0.93001	0.92989	0.92975	0.92957	0.92937	0.92915
	0.010	0.86957	0.86955	0.86952	0.86946	086937	0.8627	0.86914	0.86899	0.86882	0.86862
0.001	0.0	0.99751	0.99749	0.99744	0.99736	0.99726	0.99712	0.99695	0.99675	0.99652	0.99626
	0.005	0.92807	0.92806	0.92802	0.92795	0.92786	0.92774	0.92759	0.92742	0.92722	0.92700
	0.010	0.86768	0.86767	0.86763	0.86757	0.86749	0.86738	0.86726	0.86711	0.86893	0.86674
0.005	0.0	0.98765	0.98764	0.98759	0.98752	0.98741	0.98727	0.98711	0.98691	0.98669	0.98644
	0.005	0.91954	0.91953	0.91949	0.91942	0.91933	0.91921	0.91906	0.91890	0.91870	0.91848
	0.010	0.86022	0.86020	0.86017	0.86011	0.86003	0.85992	0.85800	0.85970	0.85948	0.85929

Table 2.5.1 (d): Effect of Failure Rate (taking $\beta_1 = \beta_2 = 0.8$; $\beta_3 = \beta_4 = 0.4$; $\beta_5 = 0.1$; $\beta_6 = 0.25$)

		A_∞									
$\alpha_1=\alpha_2$	$\alpha_3=\alpha_4=\alpha_5$	$\alpha_6=$ 0.0	$\alpha_6=$ 0.001	$\alpha_6=$ 0.002	$\alpha_6=$ 0.003	$\alpha_6=$ 0.004	$\alpha_6=$ 0.005	$\alpha_6=$ 0.006	$\alpha_6=$ 0.007	$\alpha_6=$ 0.008	$\alpha_6=$ 0.009
0.0	0.0	1	0.99998	0.99994	0.99986	0.99975	0.99961	0.99944	0.99924	0.99901	0.99875
	0.001	0.93458	0.93457	0.93452	0.93446	0.93436	0.93424	0.93409	0.93391	0.93371	0.93349
	0.005	0.87719	0.87718	0.87714	0.87708	0.87100	0.87689	0.87676	0.87661	0.87643	0.87623
0.001	0.0	0.99778	0.99777	0.99772	0.99764	0.99753	0.99739	0.99722	0.99702	0.99680	0.99654
	0.001	0.93264	0.93263	0.93259	0.93252	0.93242	0.93230	0.93215	0.93198	0.93178	0.93156
	0.005	0.87549	0.87547	0.87544	0.87538	0.87529	0.87519	0.87506	0.87490	0.87473	0.87453
0.005	0.0	0.98901	0.98900	0.98895	0.98887	0.98876	0.98863	0.98846	0.98827	0.98804	0.98779
	0.001	0.92497	0.92496	0.92492	0.92485	0.92476	0.92464	0.92449	0.92432	0.92413	0.92391
	0.005	0.86873	0.86871	0.86868	0.86862	0.86854	0.86843	0.86830	0.86815	0.86798	0.86778

Table 2.5.1 (e): Effect of Failure Rate (taking $\beta_1 = \beta_2 = 0.9$; $\beta_3 = \beta_4 = 0.5$; $\beta_5 = 0.1$; $\beta_6 = 0.25$)

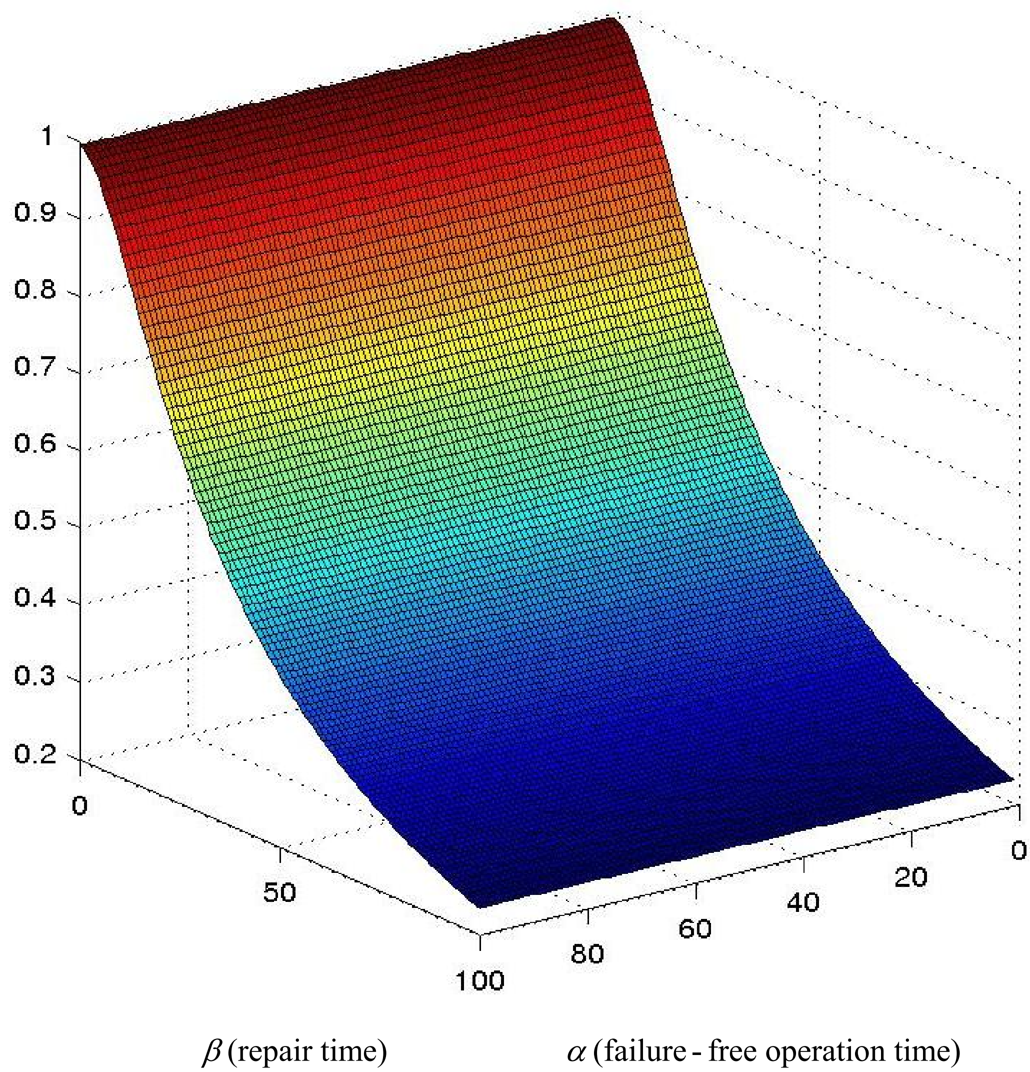

Figure 2.3: Availability for different α (failure-free operation time) and β (repair time) values

Table 2.5.2 presents the α = 95% and α = 99% confidence intervals for different simulated samples.

For $\alpha_1=\alpha_2 = 0$; $\alpha_3=\alpha_4=\alpha_5 = 0$; $\alpha_6 = 0.001$ $\beta_1 = \beta_2 = 0.5$; $\beta_3 = \beta_4 = 0.2$; $\beta_5 = 0.1$; $\beta_6 = 0.25$			
		α = 95%	**α = 99%**
n = 100	20	(0.79414; 0.96586)	(0.76702; 0.99298)
	40	(0.62674; 0.78366)	(0.60196; 0.80824)
	60	(0.54593 ; 0.68317)	(0.52533; 0.69537)
	80	(0.50775; 0.61515)	(0.48552; 0.62088)
	100	(0.47816; 0.56924)	(0.45556; 0.57544)
n = 200	20	(0.81928; 0.94072)	(0.80008; 0.95992)
	40	(0.64986; 0.76074)	(0.63234; 0.77826)
	60	(0.57289; 0.66421)	(0.55843; 0.67867)
	80	(0.52347; 0.59943)	(0.51147; 0.61143)
	100	(0.49148; 0.55592)	(0.48128; 0.56612)
n = 2000	20	(0.86080; 0.89920)	(0.85468; 0.90532)
	40	(0.68790; 0.72270)	(0.67998; 0.73062)
	60	(0.60409; 0.63301)	(0.59953; 0.63757)
	80	(0.54945; 0.57345)	(0.54567; 0.57723)
	100	(0.51356; 0.53384)	(0.51032; 0.53708)

Table 2.5.2

It can be observed that, as n increases, the steady state availability decreases.

2.6 CONCLUSION

The availability of equipment used for de-composition process in the urea production system is discussed. The system consisted of four subsystems, with a standby unit in one of the sub-systems. The failure and repair rates in each subsystem are taken to be constants. The log-run availability of the system is calculated, and the asymptotic confidence limits are obtained for the steady-state availability. The results are illustrated numerically for different measures. In tables 2.5.1(a) – (e), Figure 2.3 and table 2.5.2 shows that, as the repair time increases, the steady state availability decreases. This has been noticed in point availability and in the confidence limits.

CHAPTER 3

TWO-UNIT PRIORITY REDUNDANT SYSTEM WITH 'DEADTIME' FOR THE OPERATOR

3.1 INTRODUCTION

Two-unit standby redundant systems have attracted the attention of many applied probabilists and reliability engineers. A bibliography of the work done has been prepared by Osaki and Nakagawa (1976), Lie et al. (1977), Kumar and Agarwal (1980), Sarma (1982). Goel et al. (1985) analysed a two-unit cold standby system under the assumption that the operator of the system does not need rest, i.e. he is capable to work on the system without any rest. The literature available so far has the assumption that the operator is continuously available to repair the failed units. But it is reasonable to expect that a preparation time or rest period might be needed to get the operator ready before the next repair could be taken up. If this preparation is started only when a unit arrives for repair, it is easy to solve the problem, since the preparation time plus the actual repair time of the operator must be taken as the total repair time. But this preparation time usually starts immediately after each repair completion, so that the operator becomes available at the earliest. In our daily life the situations come about when a person needs such a preparation time. This preparation time of the operator is similar to the 'Dead time' in the counter models Ramakrishnan and Mathews (1953), Ramakrishnan (1954), Takács (1956, 1957). Yadavalli et al. (2002) studied several Markovian and non-Markovian models by introducing the 'Dead time'. Cold standby redundant systems in which the 'priority of units' and 'dead time' are introduced in this chapter.

The organisation of this chapter is as follows: Section 3.1 is introductory in nature describing the model considered in this chapter. In section 3.2, the basic assumptions and notation are presented. Various auxiliary functions (transition probabilities and sojourn times) are derived in section 3.3. The important system measures, Reliability and MTSF, are presented in section 3.4. The other important measures like mean up time in a particular interval, mean down time, expected number of visits by a repairman are studied in section 3.5. In sectin 3.6, the profit analysis is studied. Some special cases are presented in section 3.7. The system considered in this section is illustrated numerically in section 3.8.

3.2. SYSTEM DESCRIPTION AND NOTATION

1. The system consists of two dissimilar units each having two modes- Normal (N) and Total Failure (F).
2. Initially one unit of the system is operative, called the priority (P) unit and the other is kept as cold standby, called the non-priority or ordinary unit (O).
3. P-unit gets preference for both operation and repair over O-unit. When P-unit fails, the standby unit is switched to operate with a perfect switching device.
4. There is only one operator. Each unit is new after repair.
5. After each repair completion, the operator is not available for a random time. This corresponds to the 'dead time' in counter models and will be interpreted here as the 'rest time' or 'preparation time' needed before another repair could be taken up.
6. Switch is perfect and switchover is instantaneous. When the P-unit fails, it will be instantaneously switched over to the O-unit from standby state to online.

7. The lifetime of a unit, while online for P-unit and O-unit is arbitrarily distributed with pdf's $f_1(\cdot)$ and $f_2(\cdot)$.
8. The repair time of units (P-unit and O-unit) are exponentially distributed random variables with parameters β_1 and β_2 respectively.
9. The 'Dead time' of the operator is an arbitrarily distributed random variable with pdf $k(\cdot)$.

NOTATION:

$F_1(\cdot)$ and $F_2(\cdot)$	The c.d.f of the life time of P-unit and O-unit respectively
E	Set of regenerative events ≡ ($E_0, E_1, E_2, E_3, E_4, E_5, E_6$)
η	Constant rate of working time of the operator
$K(\cdot)$	The c.d.f of the 'dead time' of the operator
P_{ij}	Transition probability from regenerative event E_i to E_j
$q_{ij}(\cdot)$, $Q_{ij}(\cdot)$	The p.d.f. and c.d.f. of transition time from regenerative event E_i to E_j
ψ_i	Mean sojourn time in event E_i
$R_i(t)$	Reliability of the system when $E_i \in E$ (i = 0, 1, 2, 3, 4, 5, 6)
$U_i(t)$	Probability that the system is up when the events are E_0, E_1 or E_5 at epoch given that E_i;(i = 0, 1, 2, 3, 4, 5, 6)
$D_i(t)$	Probability that the system is down when the events are E_2, E_4 or E_6 at epoch given that E_i;(i = 0, 1, 2, 3, 4, 5, 6)
$B_i(t)$	Probability that the system is busy at epoch starting from $E_i \in E$.
$V_i(t)$	Expected number of visits by the repairman in (0,t] given that $E_i \in E$.

$\widetilde{Q}_{ij}(s) = \int_0^\infty e^{-st} dQ_{ij}(t)$, where ~ is the symbol for Laplace-Stieltjes transform

$q_{ij}^*(s) = \int_0^\infty e^{-st} q_{ij} dt$, the symbol * for Laplace transform

$\psi_i = \sum_j \int dQ_{ij}(t) = -\sum_j q_{ij}^{*'}(0) = \sum_j Q_{ij}^{'}(0)$

© Symbol for ordinary convolution

A(t) © B(t) = $\int_0^t A(t-u)B(u)du$

ⓢ Symbol for Stieltjes convolution

A(t)ⓢB(t) = $\int_0^t A(t-u)dB(u)$

<u>Symbols for the Events of the System:</u>

For the study of this system, we need to define the following states (see EL-Said & EL-Sherbeny (2005)). The reliability with dependent repair modes was also studied by Lim & Lie (2000).

N_a : unit in N-mode and operative
N_s : unit in N-mode and standby
F_r : unit in F-mode and under repair
F_w : unit in F-mode and waiting for repair
N_d : unit in N-mode when operator is in 'dead time'

We make use of the events given in Table 3.1 for the reliability analysis.

	State of	
Event	**P-unit**	**O-unit**
$E_0(N_o,N_s)$	operative	operable standby
$E_1(F_r,N_o)$	failed and under repair	operable
$E_2(N_o,N_s)$	not operating due to operator in 'dead time'	operable
$E_3(F_r,F_w)$	failed and under repair	failed and waiting for repair
$E_4(F_w,N_d)$	failed and waiting for repair	not operating due to operator in 'dead time'
$E_5(N_o,F_r)$	operative	under repair
$E_6(N_d,F_w)$	not operating due to operator in 'dead time'	failed and waiting for repair

Table 3.1

Transitions between events are shown in Figure 3.1

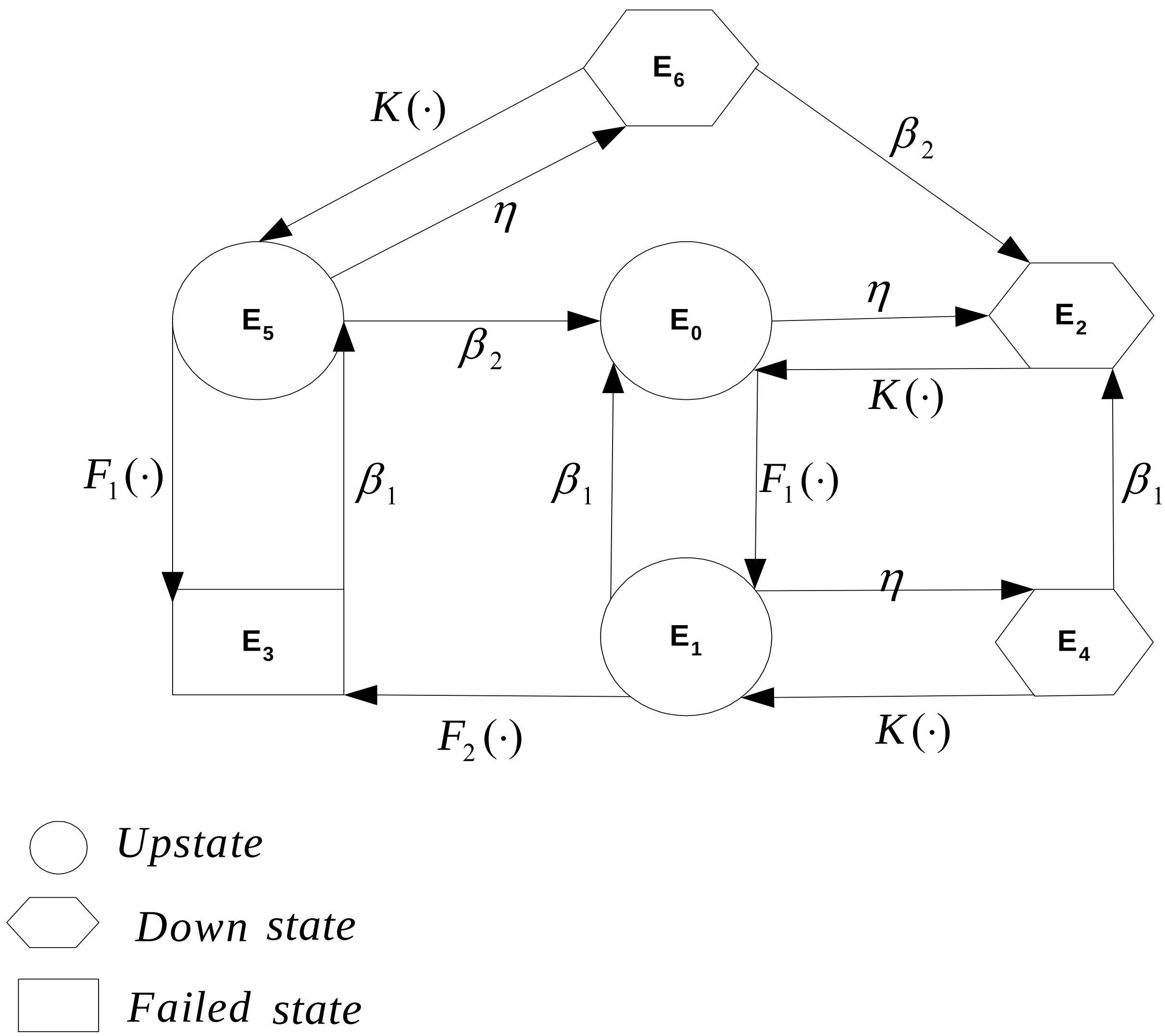
E6
K(·)
β2
η
E5
E0
η
E2
β2
K(·)
F1(·)
β1
β1
F1(·)
β1
η
E3
E1
E4
F2(·)
K(·)
Upstate
Down state
Failed state

Figure 3.1

3.3 AUXILIARY FUNCTIONS (TRANSITION PROBABILITIES AND SOJOURN TIMES)

Let $O = T_0, T_1, \dots$ denote the epochs at which the system enters any state $E_i \in E$. Let X_n denote the state visited at epoch T_n+, i.e. just after the transition at T_n. Then

$$Q_{ij}(t) = P[X_{n+1} = j, T_{n+1} - T_n \leq t \mid X_n = i].$$

The transition probability matrix is given by

$P = [P_{ij}] = [Q_{ij}(\infty)] = Q(\infty)$ with non-zero elements.

Further,

$$P_{01} = \tilde{F}_1(\eta), \; P_{02} = 1 - \tilde{F}_1(\eta)$$

$$P_{10} = \frac{[1-\tilde{F}_2(\beta_1+\eta)]\beta_1}{\beta_1+\eta}$$

$$P_{13} = \tilde{F}_2(\beta_1+\eta), \; P_{14} = \frac{[1-\tilde{F}_2(\beta_1+\eta)]\eta}{\beta_1+\eta}$$

$$P_{20} = p_{35} = 1, \; p_{41} = \tilde{K}(\beta_1), \; P_{40}^{(2)} = 1 - \tilde{K}(\beta_1)$$

$$P_{53} = \tilde{F}_1(\beta_2+\eta), \; P_{51}^{(0)} = \tilde{F}_1(\eta) - \tilde{F}_1(\beta_2+\eta)$$

$$P_{56} = \frac{[1-\tilde{F}_2(\beta_2+\eta)]\eta}{\beta_2+\eta}$$

$$P_{52}^{(0)} = \frac{1-\tilde{F}_1(\eta)-[1-\tilde{F}_1(\beta_2+\eta)]\eta}{\beta_2+\eta}$$

and $\quad P_{60}^{(2)} = 1 - \tilde{K}(\beta_2), \; P_{65} = \tilde{K}(\beta_2)$.

It can easily be verified that

$P_{01} + P_{02} = 1, \quad P_{10} + P_{13} + P_{14} = 1, \quad P_{20} = P_{35} = 1$

$P_{40}^{(2)} + P_{41} = 1, \quad P_{51}^{(0)} + P_{52}^{(0)} + P_{53} + P_{56} = 1$

$P_{60}^{(2)} + P_{65} = 1.$

To calculate mean sojourn time ψ_0 in state E_0, there is no transition to E_1 and E_2. Hence if T_0 denotes the sojourn time in E_0 then

$$\psi_0 = \int_0^\infty P[T_0 > t]dt = \frac{1-\tilde{F}_1(\eta)}{\eta}.$$

Similarly

$$\psi_1 = \frac{[1-\tilde{F}_2(\beta_1+\eta)]}{\beta_1+\eta}$$

$$\psi_2 = \int_0^\infty \bar{K}(t)dt = m_1 \text{ (say) where } \bar{K}(t) = 1-K(t)$$

$$\psi_3 = \frac{1}{\beta_1}$$

$$\psi_4 = \frac{1-\tilde{K}(\beta_1)}{\beta_1}$$

$$\psi_5 = \frac{1-\tilde{F}_1(\beta_2+\eta)}{\beta_2+\eta}$$

and $$\psi_6 = \frac{1-\tilde{K}(\beta_2)}{\beta_2}.$$

3.4 RELIABILITY ANALYSIS

Let the random variable T_i denote time to system failure from event E_i

(i = 0, 1, …,6).

The reliability of the system is given by

$$R_i(t) = P[T_i > t]$$

To determine the reliability of the system we regard the failed state of the system (E_3) as absorbing. By probabilistic arguments

$$R_0(t) = e^{-\eta t}\,\overline{F}_1(t) + q_{01}(t) © R_1(t) + q_{02}(t) © R_2(t) \tag{3.4.1}$$

$$R_1(t) = e^{-(\eta+\beta_1)t}\,\overline{F}_2(t) + q_{10}(t) © R_0(t) + q_{14}(t) © R_4(t) \tag{3.4.2}$$

$$R_2(t) = \overline{K}(t) + q_{20}(t) © R_0(t) \tag{3.4.3}$$

$$R_4(t) = \overline{K}(t) + q_{40}^{(2)}(t) © R_0(t) + q_{41}(t) © R_1(t) \tag{3.4.4}$$

$$R_5(t) = e^{-(\eta+\beta_2)t}\,\overline{F}_1(t) + q_{51}^{(0)}(t) © R_1(t) + q_{52}^{(0)}(t) © R_2(t) + q_{56}(t) © R_6(t) \tag{3.4.5}$$

$$R_6(t) = \overline{K}(t) + q_{60}^{(2)}(t) © R_0(t) + q_{65}(t) © R_5(t). \tag{3.4.6}$$

Taking Laplace transforms for the equations (3.4.1) – (3.4.6) and simplify for $R_0^*(s)$ and omitting the argument 's' for brevity, we get

$$R_0^*(s) = \frac{N_1(s)}{D_1(s)} \tag{3.4.7}$$

where

$$N_1(s) = (1 - q_{56}^*\, q_{65}^*)\,[\,\overline{F}_1^*(\eta)(1 - q_{14}^*\, q_{41}^*) + \overline{K}^*(s)\, q_{02}^*\,(1 - q_{14}^*\, q_{41}^*)$$

$$+ \overline{F}_2^*(\eta+\beta_1)\, q_{01}^* + \overline{K}^*(s)\, q_{01}^*\, q_{14}^*]$$

and

$$D_1(s) = (1 - q_{56}^*\, q_{65}^*)\,[1 - q_{14}^*\, q_{41}^* - q_{01}^*\, q_{10}^* - q_{40}^{*(2)}\, q_{01}^* q_{14}^* - q_{02}^*\, q_{20}^*$$

$+ q_{02}^* q_{20}^* q_{14}^* q_{41}^*]$.

Note: For simplicity in this chapter, $q_{ij}^*(s)$ is written as q_{ij}^*.

From (3.4.7), the Mean Time to System Failure (MTSF) can be obtained

$$E(T_0) = \lim_{t\to\infty} R_0(t) = \lim_{s\to 0} sR_0^*(s)$$

$$= \frac{(1-p_{14}p_{41})(\psi_0 + \psi_2 p_{02}) + \psi_1 p_{01} + m_1 p_{01} p_{14}}{p_{01}p_{13}}. \qquad (3.4.8)$$

3.5 SYSTEM MEASURES

3.5.1 MEAN UP TIME IN (0, t]

As defined earlier $U_i(t)$ is the probability that the system is up in E_0, E_1 or E_5 at t given that $E_i \in E$. Hence we get

$$U_0(t) = e^{-\eta t}\ \overline{F}_1(t) + q_{01}(t) © U_1(t) + q_{02}(t) © U_2(t) \qquad (3.5.1)$$

$$U_1(t) = e^{-(\eta+\beta_2)t}\ \overline{F}_2(t) + q_{01}(t) © U_0(t) + q_{13}(t) © U_3(t) + q_{14}(t) © U_4(t) \qquad (3.5.2)$$

$$U_2(t) = q_{20}(t) © U_0(t) \qquad (3.5.3)$$

$$U_3(t) = q_{35}(t) © U_5(t) \qquad (3.5.4)$$

$$U_4(t) = q_{40}^{(2)}(t) © U_0(t) + q_{41}(t) © U_1(t) \qquad (3.5.5)$$

$$U_5(t) = e^{-(\eta+\beta_2)t}\ \overline{F}_1(t) + q_{51}^{(0)}(t) © U_1(t) + q_{52}^{(0)}(t) © U_1(t) + q_{53}(t) © U_3(t)$$

$$+ q_{56}(t) © U_6(t) \qquad (3.5.6)$$

and $$U_6(t) = q_{60}^{(2)}(t) © U_0(t) + q_{65}(t) © U_5(t). \qquad (3.5.7)$$

Taking Laplace transforms for (3.5.1) – (3.5.7), we get

$$U_0^* = \frac{N_2(s)}{D_2(s)} \tag{3.5.8}$$

where

$$N_2(s) = \overline{F}_1^*(\eta)[(1 - q_{56}^* q_{65}^* - q_{35}^* q_{53}^* - q_{51}^{(0)} q_{13}^* q_{35}^* - q_{14}^* q_{41}^* - q_{14}^* q_{41}^* q_{56}^* q_{65}^*$$
$$+ q_{14}^* q_{41}^* q_{35}^* q_{53}^*] + \overline{F}_2^*(\eta+\beta_1)[q_{01}^* - q_{01}^* q_{56}^* q_{65}^* - q_{01}^* q_{35}^* q_{53}^*]$$
$$+ \overline{F}_1^*(\eta+\beta_1)[q_{01}^* q_{13}^* q_{35}^*]$$

$$D_2(s) = [1 - q_{56}^* q_{65}^*][1 - q_{14}^* q_{41}^* - q_{40}^{*(2)} q_{01}^* q_{14}^* - q_{02}^* q_{20}^*$$
$$+ q_{02}^* q_{20}^* q_{14}^* q_{41}^* - q_{01}^* q_{10}^*] - q_{35}^* q_{53}^*[1 - q_{14}^* q_{41}^*$$
$$- q_{40}^{*(2)} q_{01}^* q_{14}^* - q_{02}^* q_{20}^* + q_{02}^* q_{20}^* q_{14}^* q_{41}^* - q_{01}^* q_{10}^*]$$
$$-q_{13}^* q_{35}^* [q_{51}^{*(0)} + q_{52}^{*(0)} q_{20}^* q_{01}^* + q_{60}^{*(2)} q_{01}^* q_{56}^* q_{02}^* q_{20}^* q_{51}^{*(2)}].$$

The steady-state availability U_0 is given by

$$U_0 = \lim_{s\to 0} sU_0^*(s) = \frac{N_2(0)}{D_2'(0)} \tag{3.5.9}$$

where

$$N_2(0) = [(1-P_{14}P_{41})(1-P_{53}-P_{56}P_{65}) - P_{13}(P_{51}^{(0)} - P_{01})]\psi_0 + P_{01}(1-P_{53}-P_{56}P_{65})\psi_1$$

and

$$D_2'(0) = N_2(0) + [P_{01}P_{14}(1-P_{53}-P_{56}P_{65}) - P_{01}P_{13}P_{56}]m_1 + \psi_3[P_{01}P_{13}(1-P_{56}P_{65})]$$
$$+\psi_2[P_{02}(1-P_{14}P_{41})](1-P_{53}-P_{56}P_{65}) - P_{02}P_{13}(1-P_{53}-P_{56}) + P_{13}P_{52}^{(0)}].$$

Mean up time of the system during (0,t] is

$$\mu_{up}(t) = \int_0^t U_0(u)du \text{ so that}$$

$$\mu_{up}^{*}(s) = \frac{U_0^{*}(s)}{s}. \tag{3.5.10}$$

3.5.2. MEAN DOWN TIME DURING (0, t]

To obtain mean down-time during (0, t], we consider $D_i(t)$ as the probability that the system is in state E_2, E_4 or E_6 at epoch t given that E_i has occurred at t = 0.

Here we have

$$D_0(t) = q_{01}(t) © D_1(t) + q_{02}(t) © D_2(t) \tag{3.5.11}$$

$$D_1(t) = q_{10}(t) © D_0(t) + q_{13}(t) © D_3(t) + q_{14}(t) © D_4(t) \tag{3.5.12}$$

$$D_2(t) = \overline{K}(t) + q_{20}(t) + D_0(t) \tag{3.5.13}$$

$$D_3(t) = q_{35}(t) © D_5(t) \tag{3.5.14}$$

$$D_4(t) = \overline{K}(t) + q_{40}^{(2)}(t) © D_0(t) + q_{41}(t) © D_1(t) \tag{3.5.15}$$

$$D_5(t) = q_{51}^{(0)}(t) © D_1(t) + q_{52}^{(0)}(t) © D_2(t) + q_{53}(t) © D_3(t)$$
$$+ q_{56}(t) © D_6(t) \tag{3.5.16}$$

and $$D_6(t) = \overline{K}(t) + q_{60}^{(2)}(t) © D_0(t) + q_{65}(t) © D_5(t). \tag{3.5.17}$$

Taking Laplace transforms for the equations (3.5.11) - (3.5.17) and simplifying for $D_0^{*}(s)$ we get

$$D_0^{*}(s) = \frac{N_3(s)}{D_2(s)}$$

(3.5.18)

where

$$N_3(s) = \overline{F}_2^{*}(\eta + \beta_1)\,[q_{01}^{*}\, q_{13}^{*} q_{35}^{*}\; q_{52}^{(0)} + q_{02}^{*}(1 - q_{56}^{*}\, q_{65}^{*} - q_{35}^{*}\, q_{53}^{*}) - q_{02}^{*}\, q_{13}^{*} q_{35}^{*}\; q_{51}^{(0)}$$

$- q_{02}^* q_{14}^* q_{41}^* (1 - q_{56}^* q_{65}^* - q_{35}^* q_{53}^*)] + \overline{K}(s) q_{01}^* q_{14}^* (1 - q_{56}^* q_{65}^* - q_{35}^*$

$q_{53}^*)$

$+ \overline{K}(s) q_{01}^* q_{13}^* q_{35}^* q_{56}^*$

The value of $D_0(t)$ can be obtained on taking the inverse Laplace transform of $D_0^*(s)$.

The steady-state probability of the system being down is given by

$$D_0 = \lim_{s \to 0} \frac{sN_3(s)}{D_2(s)} = \frac{N_3(0)}{D_2'(0)} \tag{3.5.19}$$

where

$$N_3(0) = m_1[1 - p_{01}p_{10} - p_{13}p_{51}^{(0)} - p_{02}p_{14}p_{41} + p_{56}p_{65}(1 - p_{14}p_{41}) + p_{01}p_{14}p_{56}p_{65}].$$

Now the mean down-time of the system during (0, t] is

$$\mu_{dn}(t) = \int_0^t D_0(u)du$$

$$\mu_{dn}^*(s) = \frac{D_0^*(s)}{s} \tag{3.5.20}$$

and the mean failed time in (0, t] is

$$\mu_f(t) = t - \mu_{up}(t) - \mu_{dn}(t)$$

so that

$$\mu_f^*(s) = \frac{1}{s^2} - \mu_{up}^*(s) - \mu_{dn}^*(s). \tag{3.5.21}$$

3.5.3 BUSY PERIOD ANALYSIS

$B_i(t)$ is defined as the probability that the system is busy at epoch t starting from state E_i, $E_i \in E$. We have the following recursive relations

$$B_0(t) = q_{01}(t) © B_1(t) + q_{02}(t) © B_2(t) \tag{3.5.22}$$

$$B_1(t) = e^{-(\eta+\beta_1)t}\ \bar{F}_2(t) + q_{01}(t) © B_0(t) + q_{13}(t) © B_3(t) + q_{14}(t) © B_4(t) \tag{3.5.23}$$

$$B_2(t) = q_{20}(t) + B_0(t) \tag{3.5.24}$$

$$B_3(t) = e^{-\beta_1 t} + q_{35}(t) © B_5(t) \tag{3.5.25}$$

$$B_4(t) = e^{-\beta_1 t}\ \bar{K}(t) + q_{40}^{(2)}(t) © B_0(t) + q_{41}(t) © B_1(t) \tag{3.5.26}$$

$$B_5(t) = e^{-(\eta+\beta_1)t}\ \bar{F}_1(t) + q_{51}^{(0)}(t) © B_1(t) + q_{52}^{(0)}(t) © B_2(t) + q_{53}(t) © B_3(t)$$
$$+ q_{56}(t) © B_6(t) \tag{3.5.27}$$

and $$B_6(t) = e^{-\beta_2 t}\ \bar{K}(t) + q_{60}^{(2)}(t) © B_0(t) + q_{65}(t) © B_5(t)\ . \tag{3.5.28}$$

Taking Laplace transforms for the equations (3.5.22) to (3.5.28) and simplifying for $B_0^*(s)$, we get

$$B_0^*(s) = \frac{N_4(s)}{D_2(s)} \tag{3.5.29}$$

where

$$N_4(s) = \bar{F}_2^*(\eta+\beta_1)\, q_{01}^*\, [1 - q_{56}^* q_{65}^* - q_{35}^*\, q_{53}^*] + \frac{1}{\beta_1 + s} q_{01}^*\, q_{13}^*\, [1 - q_{56}^*\, q_{65}^*]$$
$$+ \bar{K}^*(\beta_2+s)\, q_{01}^*\, q_{14}^* [1 - q_{56}^*\, q_{65}^* - q_{35}^*\, q_{53}^*] + \bar{F}_1^*(\eta+\beta_2+s)\, q_{01}^*\, q_{13}^*\, q_{35}^*$$
$$+ \bar{K}^*(\beta_2+s)\, q_{01}^*\, q_{13}^* q_{35}^*\, q_{56}^*\ .$$

The steady-state probability that the system is under repair starting from state E_0, i.e. probability that in the long run the repairman will be busy is given by

$$B_0 = \lim_{s\to 0} sB_0^*(s) = \frac{N_4(0)}{D_2^{'}(0)} \tag{3.5.30}$$

where

$$N_4(0) = P_{01}[1 - P_{53} - P_{56}P_{65}](\Psi_1 + \Psi_4 P_{14}) + P_{01}P_{13}[\Psi_5 + \Psi_6 P_{56} + \Psi_3(1 - P_{56}P_{65})]\ .$$

The expected duration of busy time of repairman in (0, t] is

$$\mu_b(t) = \int_0^t B_0(u)du \, ,$$

so that

$$\mu_b^*(s) = \frac{B_0^*(s)}{s} \tag{3.5.31}$$

and the expected idle time of repairman in (0, t] is

$$\mu_I(t) = t - \mu_b(t)$$

so that

$$\mu_I^*(s) = \frac{1}{s^2} - \mu_b^*(s) \, . \tag{3.5.32}$$

3.5.4 EXPECTED NUMBER OF VISITS BY THE REPAIRMAN IN (0, t]

According to the definition of $V_i(t)$, by elementary probability arguments we have the following relations:

$$V_0(t) = Q_{01}(t) \circledS [1 + V_1(t)] + Q_{02}(t) \circledS V_2(t) \tag{3.5.33}$$

$$V_1(t) = Q_{01}(t) \circledS V_0(t) + Q_{13}(t) \circledS V_3(t) + Q_{14}(t) \circledS V_4(t) \tag{3.5.34}$$

$$V_2(t) = Q_{20}(t) \circledS V_0(t) \tag{3.5.35}$$

$$V_3(t) = Q_{35}(t) \circledS V_5(t) \tag{3.5.36}$$

$$V_4(t) = Q_{40}^{(2)}(t) \circledS V_0(t) + Q_{41}(t) \circledS V_1(t) \tag{3.5.37}$$

$$V_5(t) = Q_{51}^{(0)}(t) \circledS [1 + V_1(t)] + Q_{52}^{(0)}(t) \circledS V_2(t) + Q_{53}(t) \circledS V_3(t) + Q_{56}(t) \circledS V_6(t) \tag{3.5.38}$$

and $V_6(t) = Q_{60}^{(2)}(t) \circledS V_0(t) + Q_{65}(t) \circledS V_5(t)$. (3.5.39)

Taking Laplace-Stieljes transforms and simplifying $\widetilde{V}_0(s)$, we get

$$\widetilde{V}_0(s) = \frac{\widetilde{N}_5(s)}{\widetilde{D}_2(s)} \qquad (3.5.40)$$

where

$$\widetilde{N}_5(s) = \widetilde{Q}_{01}(1-\widetilde{Q}_{14}\widetilde{Q}_{41})[1-\widetilde{Q}_{56}\widetilde{Q}_{65}-\widetilde{Q}_{35}\widetilde{Q}_{53}].$$

In the steady state, the number of visits per unit time is given by

$$V_0 = \lim_{t\to\infty}\frac{V_0(t)}{t} = \frac{\widetilde{N}_5(0)}{\widetilde{D}_2'(0)} \qquad (3.5.41)$$

$$\widetilde{N}_5(0) = P_{01}\,[1-P_{14}P_{41}][1-P_{35}-P_{56}P_{65}].$$

3.6 COST BENEFIT ANALYSIS

We are now in the position to obtain the profit function by the system considering mean up time, mean down time in (0, t], busy period and expected number of visits by the repairman in (0, t]. The next expected profit incurred in (0, t] is

C(t) = expected total revenue in (0, t] – expected total repair cost in (0, t]

– expected cost of visit by the repairman in (0, t]

$$= (C_0 - C_1)\,\mu_{up}(t) - C_1\mu_{dn}(t) - c_2\mu_b(t) - c_3V_0(t). \qquad (3.6.1)$$

The expected total profit per unit of time in steady state is

$$C = \lim_{t\to\infty}\frac{C(t)}{t} = \lim_{s\to 0} s^2 C^*(s).$$

That is,

$$C = (C_0 - C_1)\,V_0 - C_1\,D_0 - C_2\,B_0 - C_3V_0 \qquad (3.6.2)$$

where C_0 is the revenue per unit uptime, C_1 is the salary of the operator per unit time, C_2 is the cost per unit for which the system is under repair and C_3 is the cost per visit by the repairman.

3.7 SPECIAL CASES

CASE I

When the 'dead time' of the operator is zero, i.e. η = 0, then the results are as follows:

$$E(T_0) = \frac{n_1 + \phi_1}{P_{13}}$$

$$U_0 = \frac{n_1(1 - P_{10}P_{53}) + \phi_1 P_{51}^{(0)}}{X}$$

$$B_0 = \frac{P_{13}(\phi_3 + \phi_5) + \phi_1 P_{51}^{(0)}}{X}$$

and $$V_0 = \frac{P_{51}^{(0)}}{X}$$

where

$$X = P_{13}(\phi_3 + \phi_5) + n_1 P_{10} P_{51}^{(0)} + \phi_1 P_{51}^{(0)}$$

and

$$n_1 = \int F_1(t)dt \, ; \qquad \varphi_1 = \frac{1 - \widetilde{F}_2(\beta_1)}{\beta_1}$$

$$\varphi_3 = \frac{1}{\beta_1} \, ; \qquad \varphi_5 = \frac{1 - \widetilde{F}_2(\beta_2)}{\beta_2}$$

$$P_{10} = 1 - \widetilde{F}_2(\beta_2) \, ; \; P_{13} = \widetilde{F}_2(\beta_2)$$

$$P_{53} = \widetilde{F}_1(\beta_2) \, , \; P_{51}^{(0)} = 1 - \widetilde{F}_1(\beta_2) \, .$$

CASE II

When failure time distributions of both units in case I are negative exponential i.e.

$F_1(t) = 1 - e^{-\lambda_1 t}$; $F_2(t) = 1 - e^{-\lambda_2 t}$

then the results are as follows:

$$E(T_0) = \frac{\beta_1 + \lambda_1 + \lambda_2}{\lambda_1 \lambda_2}$$

$$U_0 = \frac{\beta_1(\beta_2(\lambda_1 + \beta_1) + \lambda_2(\lambda_1 + \beta_2))}{Y}$$

$$B_0 = \frac{\lambda_1(\lambda_1\lambda_2 + \beta_1\beta_2 + \beta_1\lambda_2 + \lambda_2\beta_2)}{Y}$$

$$V_0 = \frac{\lambda_1\beta_1\beta_2(\beta_1 + \lambda_2)}{Y}$$

where

$$Y = \beta_1\beta_2(\lambda_1 + \beta_1) + \lambda_1\lambda_2(\lambda_1 + \beta_1 + \beta_2).$$

3.8 NUMERICAL ANALYSES

Figure 3.2(i) shows graphically the change for β_1 versus $E(T_0)$

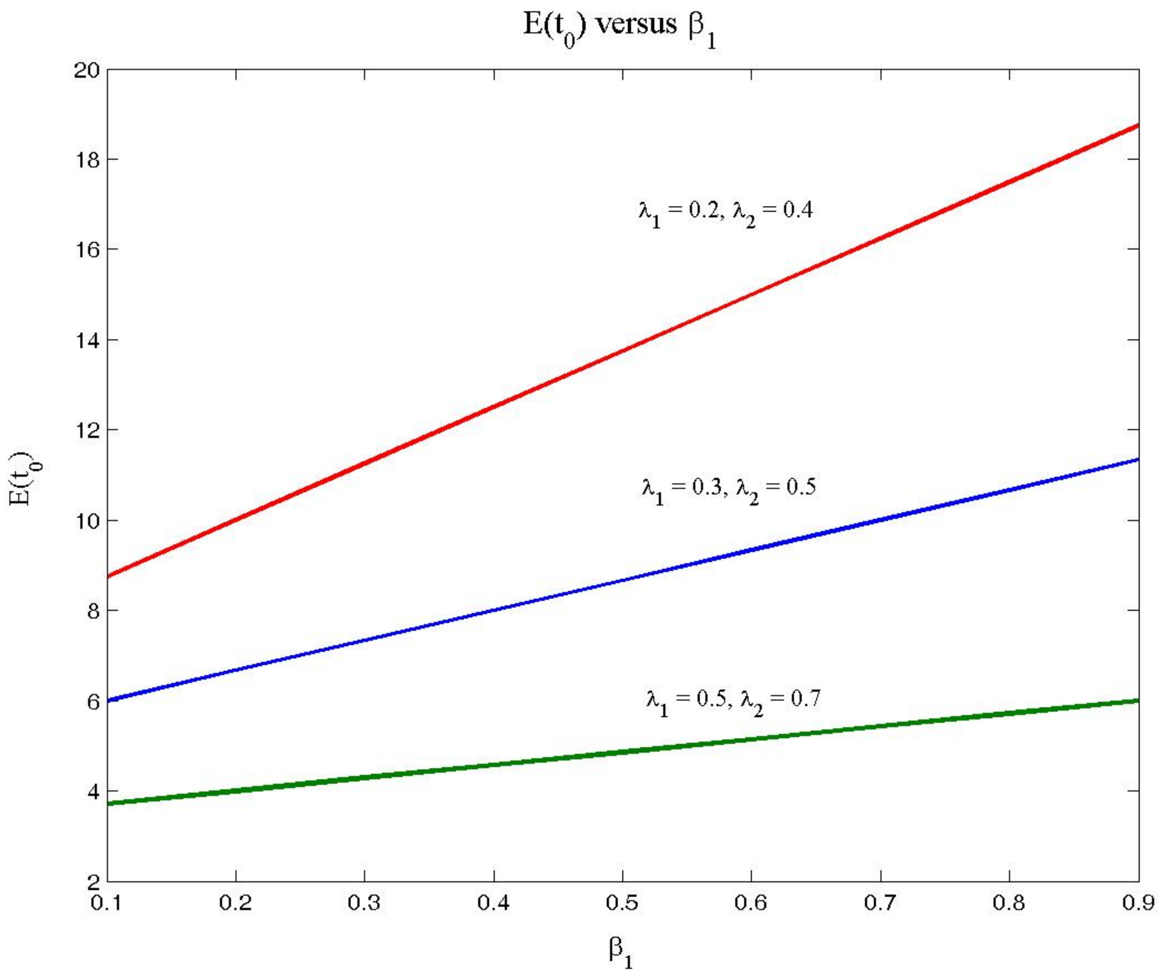

Figure 3.2

As the repair time of the priority unit, β_1, increases the mean expected time to failure $E(t_0)$ is an increasing function of β_1 (for different values of λ_1 and λ_2).

Figure 3.3 shows graphically the change for β_2 versus U_0

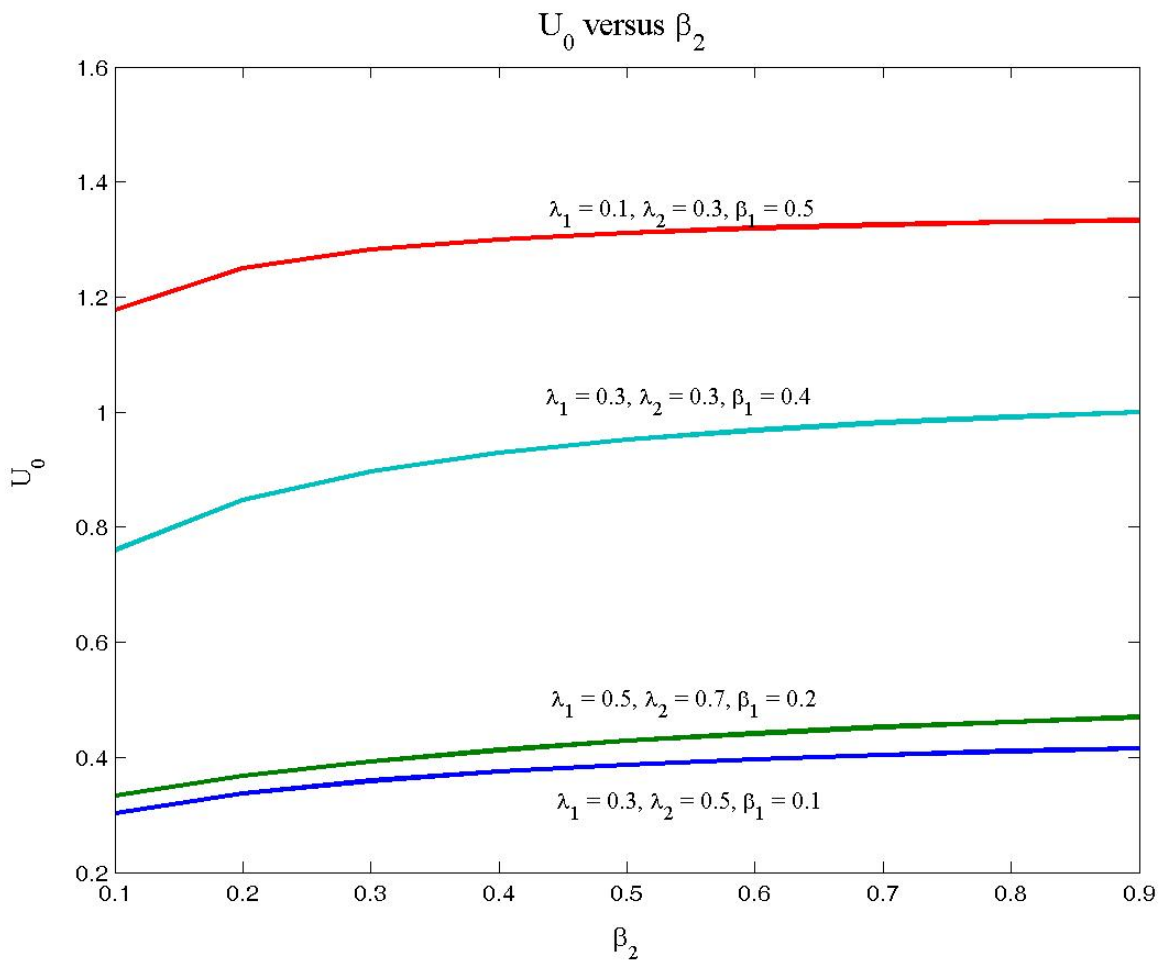

Figure 3.3

As the repair time of the ordinary unit, β_2, increases the steady-state availability U_0 is an increasing function of β_2 (for different values of λ_1, λ_2 and β_1).

Figure 3.4 shows graphically the change for β_2 versus B_0

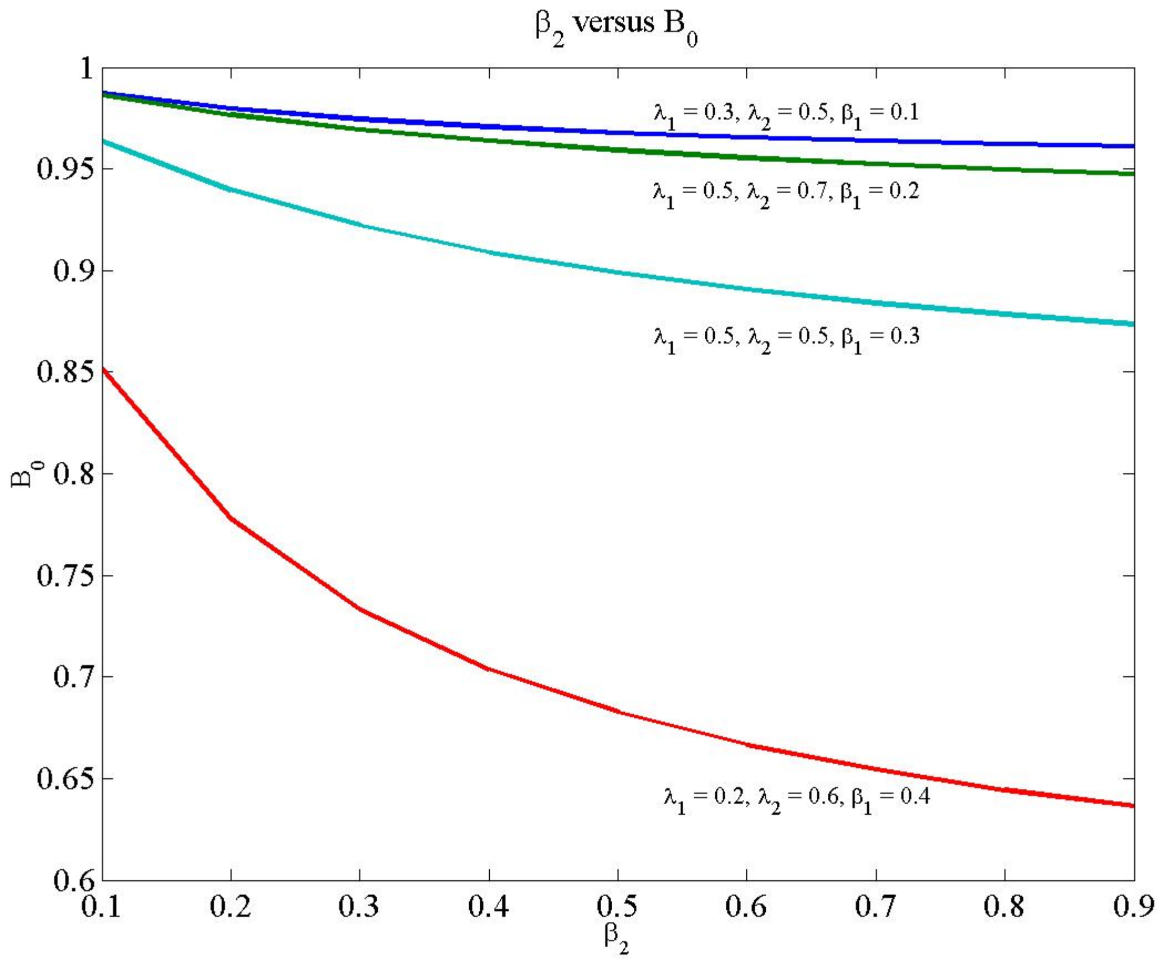

Figure 3.4

As β_2 increases the probability that the system is busy, B_0, is a decreasing function of β_2 (for different values of λ_1, λ_2 and β_1).

Figure 3.5 shows graphically the change for β_2 versus V_0

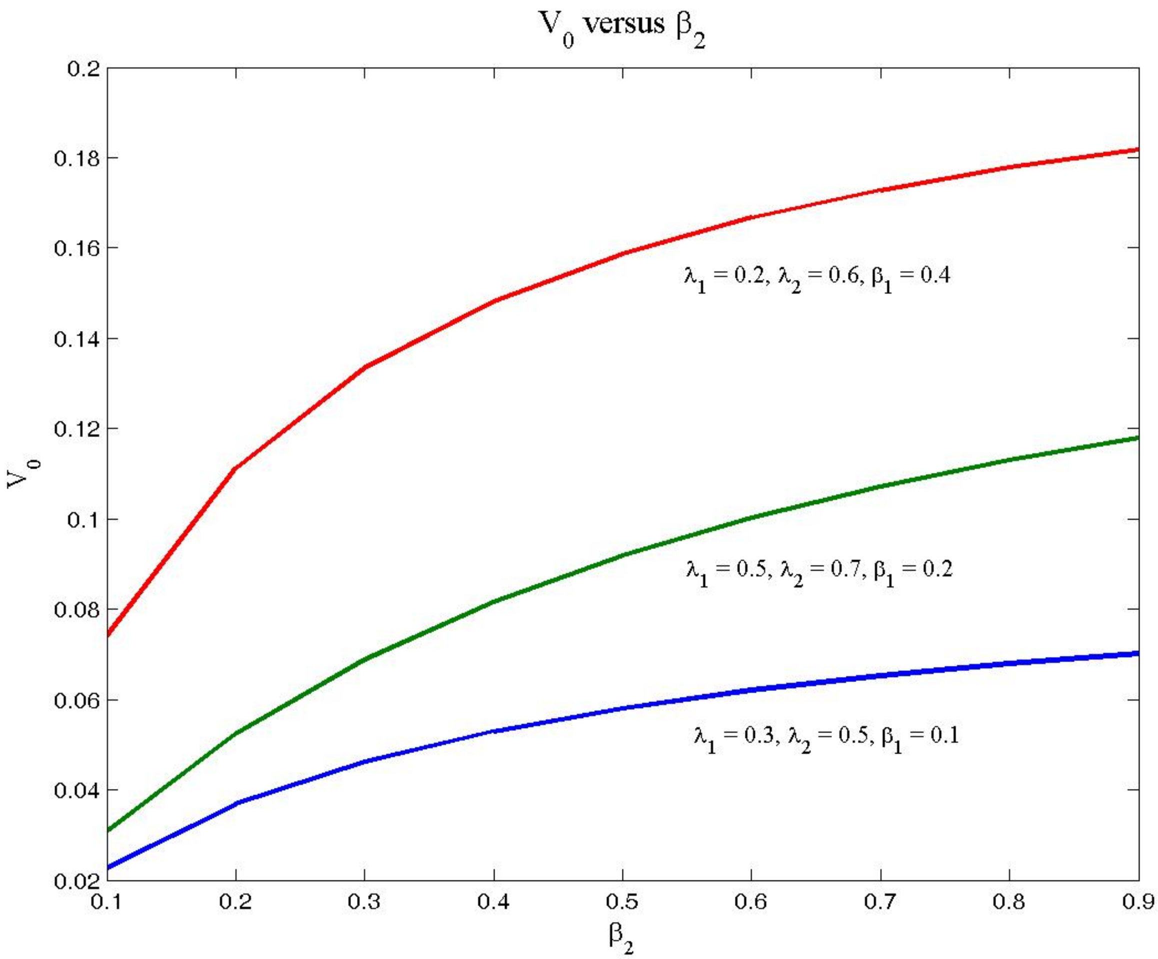

Figure 3.5

As β_2 increases the expected number of visits by the repairman, V_0, is an increasing function of β_2 (for different values of λ_1 , λ_2 and β_1).

3.9 CONCLUSION

A two-unit single server priority redundant repairable system with two modes – normal and total failure has been studied. The priority unit got preference both in operation and repair. It is assumed that the repair facility is not available for a random time (Dead time). The system fails when both units are in total failure mode. Identifying the regeneration point technique, various operating characteristics of the system are obtained. The cost-benefit analysis is studied, and the results are illustrated numerically. The numerical results as shown in Figures 3.2 – 3.5 justify the results.

CHAPTER 4

CONFIDENCE LIMITS FOR A TWO-UNIT COLD STANDBY PRIORITY SYSTEM WITH VARYING PHYSICAL CONDITIONS OF THE REPAIR FACILITY AND WITH IMPERFECT SWITCHING DEVICE

4.1 INTRODUCTION

In the literature of reliability extensive studies have been made on different types of two-unit standby systems owing to their frequent use in modern business and industrial systems. Nakagawa and Osaki (1974) have studied the behaviour of a two-unit (priority and ordinary) standby system with two modes for each unit. They have taken exponential failure and repair time distributions for the ordinary unit, while the distributions for the priority unit are arbitrary. Much work related to the switching device in standby systems has been done by various authors including Goel and Gupta (1984a, b). The cost analysis of such systems has also been discussed by Murari and Goel (1984) and Goel et al. (1985).

Goel et al (1985) have discussed a man-machine system considering the physical conditions of the repair facility, namely poor and good. The physical conditions of the repair facility also affect the operation of the system. However, no previous work has considered the physical conditions of the repair facility. It is reasonable to expect the repair facility to work with a higher repair rate if it is in a poor physical condition. Consequently the repair time distribution will be different in these two situations. The purpose of the present chapter is to analyze such a system. The system under consideration is a two dissimilar unit cold standby system with an imperfect switch. Initially, one unit is operative and is called a priority unit (p) and the other is a cold standby or ordinary unit (o). The p-unit gets priority for both operation and repair (Shi and Liu (1996)). When the p-unit fails the standby unit is switched to operate with the help of a switching device. The switch may be available at the time of need with known probability $p\ (1-q)$.

The distribution of random variables denoting time to failure and time to repair are taken to be arbitrary. Depending on the physical conditions (good or poor) of the repair facility, there are two different repair time distributions to be considered. The probability that at any time the repairman's condition will be good is $p_1(1-q_1)$. We analyze the system by using the regenerative point technique and obtain various operating characteristics. The confidence limits for the standby state availability and the busy period in steady-state are obtained.

The organisation of this chapter is as follows: Section 4.1 is introductory in nature, and the notation of this chapter is discussed in section 4.2. Various auxiliary functions (transition probabilities and sojourn times) are derived in section 4.3. The reliability analysis is discussed in section 4.4. In section 4.5, availability analysis is discussed. The busy period analysis and the cost benefit analysis have been studied in sections 4.6 and 4.7 respectively. The confidence limits, for the steady state availability, are studied in section 4.8, under the assumption that all the underlying distributions are exponential, with different parameters. In section 4.9, the system is illustrated numerically.

4.2 NOTATION

E_0	State of the system at t=0
E	Set of regenerative states
$\overline{E}$	Set of non-regenerative states
p_1	P[the switch is good at the time of need]; $p_1 = 1 - q_1$
$f_1(t)$, $F_1(t)$	The p.d.f. and c.d.f. of the life time of the *p*-unit
$f_2(t)$, $F_2(t)$	The p.d.f. and c.d.f. of the life time of the *o*-unit
$g_i(t)$, $G_i(t)$	The p.d.f. and c.d.f. of the repair of the *p*-unit ($i = 1, 2$)

$k_i(t)$, $K_i(t)$	The p.d.f. and c.d.f. of the repair of the *o*-unit ($i = 1, 2$)
$h_i(t)$, $H_i(t)$	The p.d.f. and c.d.f. of the time to repair of the switching device; $i = 1, 2$
i	$= \begin{cases} 1 & \text{if the repair facility is in good condition} \\ 2 & \text{if the repair facility is in bad condition} \end{cases}$
p_2	P[the repair facility's condition is good]; $p_2 = 1 - q_2$
$q_{ij}(t)$, $Q_{ij}(t)$	The p.d.f. and c.d.f. of direct transition time from one regenerative state S_i to another regenerative state S_j
p_{ij}	P[the system transits from regenerative state S_i to regenerative state S_j] $= Q_{ij}(\infty)$
$q_{ij}^{(k)}(t)$, $Q_{ij}^{(k)}(t)$	The p.d.f. and c.d.f. of transition time from regenerative state S_i to S_j via non-regenerative state S_k
$p_{ij}^{(k)}$	Steady-state probability that the system transits from state S_i to S_j via non-regenerative states S_k; $Q_{ij}^{(k)}(\infty)$
$\pi_i(\cdot)$	The c.d.f. of the time to system failure when the starting state $E_0 = S_i \in E$
$A_i(t)$	P[System is up at time $t \mid E_0 = S_i \in E$]
$B_i(t)$	P[System is under repair at time $t \mid E_0 = S_i \in E$]
μ_i	Mean sojourn time in states $S_i \in E$
$\tilde{Q}_{ij}(s)$	$\int_0^\infty e^{-st} dQ_{ij}(t)$
$q_{ij}^*(s)$	$\int_0^\infty e^{-st} q_{ij}(t) dt$

μ_i $$= \sum_j \int_0^\infty t \, dQ_{ij}(t) = -\sum_j \tilde{Q}'_{ij}(0) = -\sum_j q^{*'}_{ij}(0)$$

© Symbol for ordinary convolution

Ⓢ Symbol for Stieltjes convolution

4.3 AUXILIARY FUNCTIONS

For the reliability and unavailability analyses, and the busy period analysis, we need to derive various auxiliary functions (transition probabilities and sojourn times). We need to define first the following states (see EL-Said & EL-Sherbeny (2005)):

Up states: S_0 (N_0, N_s) ; S_2 (F_r, N_0); S_4 (N_0, F_r),

Down states: S_1 (F_w, N_s, S_r) ; S_3 (F_r, F_w),

where

N_0 : unit in normal mode and operative

N_s : unit in normal mode and standby

F_r : unit in failure mode and repair from the epoch of entry into the state

F_w : unit in failure mode and waiting for repair

S_r : switching device under repair

F_r : unit in failure mode and under repair with the repair continued from the earlier state.

The order of the position of units in the states specifies the type of unit. Possible transitions between states, with the failure and repair time c.d.f's, are shown in Figure 4.1 .

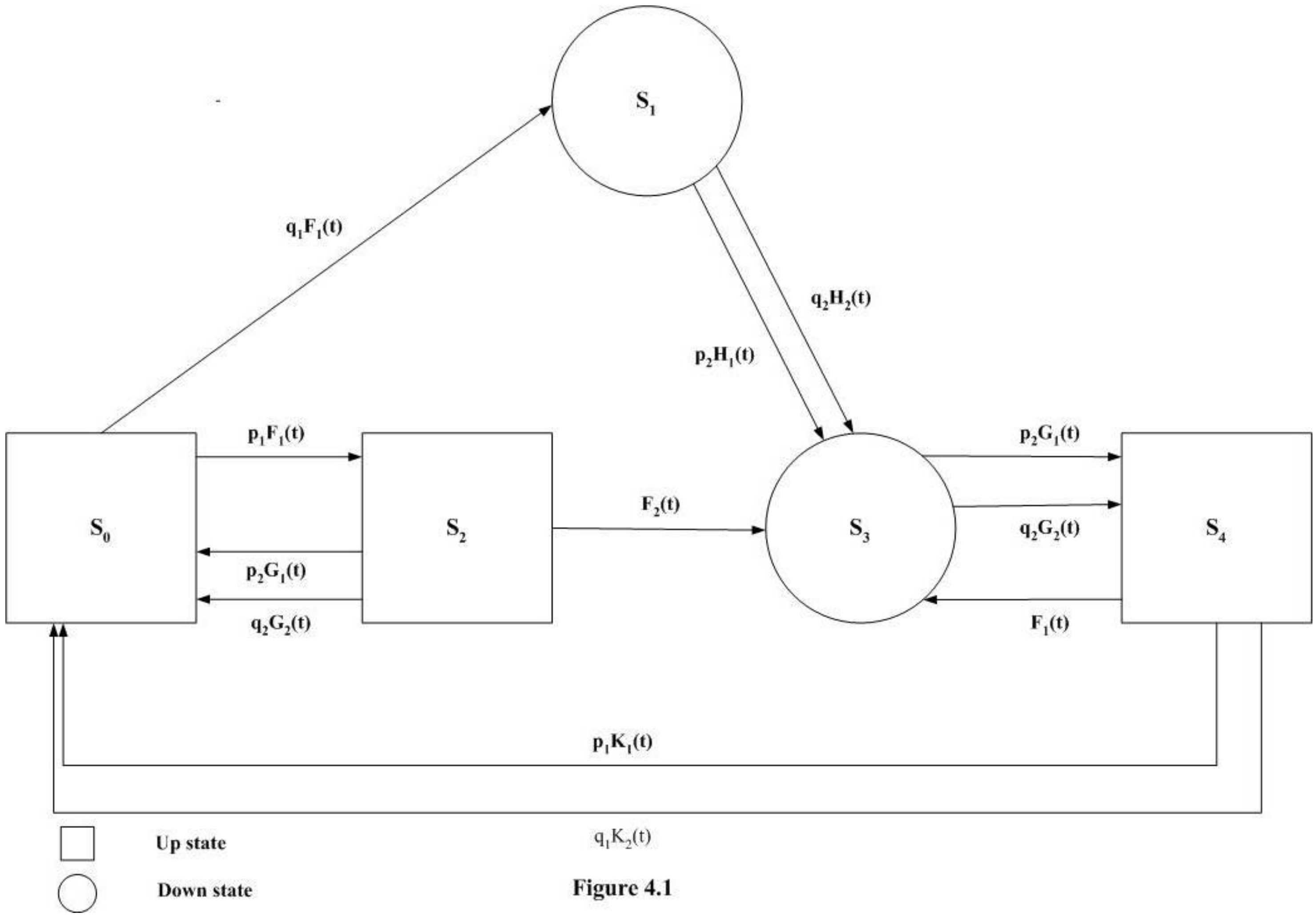
S1
q1F1(t)
q2H2(t)
p2H1(t)
p1F1(t)
S0
S2
F2(t)
S3
p2G1(t)
q2G2(t)
S4
p2G1(t)
q2G2(t)
F1(t)
p1K1(t)
q1K2(t)
Up state
Down state

Figure 4.1

It is observed that the epoch of entry into the states S_1, S_2 and S4 are degenerative points and therefore these states are regenerative states. E denotes the set of these states. Furthermore, the epochs of entry into the states S_3 from S_4 and S_0 from S_2 are regenerative and the epoch of entry into S_3 from S_2 and S_0 from S_4 are non-regenerative. Therefore these states will behave as regenerative states only with respect to S_4 and S_2 respectively.

Let 0 = T_0, T_1, ... denote the epochs of entry into the states $S_i \in E$ and X_n denote the state visited at epoch T_n^+, i.e. just after the transition at T_n. Then $\{X_n, T_n\}$ is a Markov renewal process with state space E.

Further

$$Q_{ij}(t) = P[X_{n+1} = j, T_{n+1} - T_n \le t \mid X_n = i]$$

where

$$Q_{01}(t) = q_1 \int_0^t f_1(u)du = q_1 F_1(t)$$

$$Q_{02}(t) = p_1 \int_0^t f_1(u)du = p_1 F_1(t)$$

$$Q_{12}(t) = p_2 \int_0^t dH_1(t) + q_2 \int_0^t dH_2(t)$$

$$= p_2 H_1(t) + q_2 H_2(t)$$

$$Q_{20}(t) = p_2 \int_0^t \bar{F}_2(t)dG_1(u) + q_2 \int_0^t \bar{F}_2(u)dG_2(u)$$

$$Q_{24}^{(3)}(t) = p_2 \int_0^t F_2(u)dG_1(u) + q_2 \int_0^t F_2(u)dG_2(u)$$

$$= Q_{23}(t)$$

$$Q_{34}(t) = p_2 G_1(t) + q_2 G_2(t)$$

$$Q_{41}^{(0)}(t) = q_1[p_2 \int_0^t k_1(u)dF_1(u) + q_2 \int_0^t k_2(u)dF_1(u)]$$

$$Q_{42}^{(0)}(t) = p_1[p_2 \int_0^t k_1(u)dF_1(u) + q_2 \int_0^t k_2(u)dF_1(u)]$$

and $$Q_{43}(t) = p_2 \int_0^t \bar{K}_1(u)dF_1(u) + q_2 \int_0^t \bar{K}_2(u)dF_1(u).$$

Letting $t \to \infty$ and using $p_{ij} = Q_{ij}(\infty)$, we get the transition probability matrix $P = \left[p_{ij}\right]$ with the following non-zero elements

$$p_{01} = q_1\,;\; p_{02} = p_1\,;\; p_{12} = p_{34} = 1 \tag{4.1}$$

$$p_{20} = p_2 \int_0^\infty \bar{F}_2(t)dG_1(t) + q_2 \int_0^\infty \bar{F}_2(t)dG_2(t)$$

$$p_{24}^{(3)} = p_2 \int_0^\infty F_2(t)dG_1(t) + q_2 \int_0^\infty F_2(t)dG_2(t)$$

$$p_{41}^{(0)} = q_1[p_2 \int_0^\infty k_1(t)dF_1(t) + q_2 \int_0^\infty k_2(t)dF_1(t)]$$

$$p_{42}^{(0)} = p_1[p_2 \int_0^\infty k_1(t)dF_1(t) + q_2 \int_0^\infty k_2(t)dF_1(t)]$$

and $$p_{43} = p_2 \int_0^\infty \bar{K}_1(t)dF_1(t) + q_2 \int_0^\infty \bar{K}_2(t)dF_1(t).$$

We can easily verify that

$$p_{01} + p_{02} = 1 \tag{4.2}$$

$$p_{20} + p_{24}^{(3)} + p_{23} = 1 \tag{4.3}$$

and

$$p_{41}^{(0)} + p_{42}^{(0)} + p_{43} = 1\,. \tag{4.4}$$

To calculate the mean sojourn time μ_0 in state S_0, we observe that so long as the system is in S_0, there is no transition on to S_1 or S_2. Hence if T denotes the sojourn time in state S_0, then

$$\mu_0 = \int_0^\infty P[T > t]dt$$

$$= \int_0^\infty p_1 \bar{F}_1(t)dt + \int_0^\infty q_1 \bar{F}_1(t)dt$$

$$= \int_0^\infty \bar{F}_1(t)dt \tag{4.5}$$

$$\mu_1 = p_2 \int_0^\infty \bar{H}_1(t)dt + q_2 \int_0^\infty \bar{H}_2(t)dt \tag{4.6}$$

$$\mu_2 = p_2 \int_0^\infty \bar{G}_1(t)\bar{F}_2 dt + q_2 \int_0^\infty \bar{G}_2(t)\bar{F}_2 dt \tag{4.7}$$

$$\mu_3 = p_2 \int_0^\infty \bar{G}_1(t)dt + q_2 \int_0^\infty \bar{G}_2(t)dt \tag{4.8}$$

and

$$\mu_4 = p_2 \int_0^\infty \bar{K}_1(t)\bar{F}_1 dt + q_2 \int_0^\infty \bar{K}_2(t)\bar{F}_1 dt\,. \tag{4.9}$$

4.4 RELIABILITY ANALYSIS

The time to system failure (TSF) can be regarded as the first passage time to either of the failed states S_1 or S_3. To obtain it we regard these states as absorbing. Employing the arguments used for regenerative processes we obtain the following

$$\pi_0(t) = Q_{01}(t) + Q_{02}(t) \circledS \pi_1(t) \tag{4.10}$$

$$\pi_2(t) = Q_{23}(t) + Q_{20}(t) \circledS \pi_0(t) \tag{4.11}$$

and
$$\pi_4(t) = Q_{41}^{(0)}(t) + Q_{42}^{(0)}(t) \circledS \pi_2(t) + Q_{43}(t)\,. \tag{4.12}$$

Taking the Laplace-Stieljes transform of the equations (4.10) to (4.12), the solution of $\pi_i(s)$, $(i = 0,2,4)$ can be written in the following form

$$\begin{bmatrix} \tilde{\pi}_0 \\ \tilde{\pi}_2 \\ \tilde{\pi}_4 \end{bmatrix} = \begin{bmatrix} 1 & -\tilde{Q}_{02} & 0 \\ -\tilde{Q}_{20} & 1 & 0 \\ 0 & -\tilde{Q}_{42} & 1 \end{bmatrix}^{-1} \begin{bmatrix} \tilde{Q}_{01} \\ \tilde{Q}_{23} \\ \tilde{Q}_{41} + \tilde{Q}_{43} \end{bmatrix} \tag{4.13}$$

We have omitted the argument 's' for simplicity from $\tilde{Q}_{ij}(s)$ and $\tilde{\pi}_{ij}(s)$. Simplifying (4.13), we get

$$\tilde{\pi}_0(s) = \frac{N_1(s)}{D_1(s)} \tag{4.14}$$

where

$$N_1(s) = \tilde{Q}_{01} + \tilde{Q}_{02}\tilde{Q}_{23}$$

$$D_1(s) = 1 - \widetilde{Q}_{02}\widetilde{Q}_{20}\,.$$

Making use of relations (4.1) – (4.4), it can be shown that $\widetilde{\pi}_0(0) = 1$, which implies that $\pi_0(t)$ is a proper distribution. Now, the mean time to system failure, given that the system started from S_0,

$$E(t) = -\frac{d}{ds}\widetilde{\pi}_0(s)\Big|_{s=0}$$

$$= \frac{\mu_0 + p.\mu_2}{1 - p.p_{20}}\,. \tag{4.15}$$

4.5 AVAILABILITY ANALYSIS

Let $M_i(t)$ be the probability that the system, having started from S_i, is up at time t, without making any transition to any other regenerative state belonging to E.

By simple probabilistic arguments we have

$$M_0(t) = p_1\overline{F}_1(t) + q_1\overline{F}_1(t) = \overline{F}_1(t) \tag{4.16}$$

$$M_2(t) = \overline{F}_2(t)[p_2\overline{G}_1(t) + q_2\overline{G}_2(t)] \tag{4.17}$$

and

$$M_4(t) = \overline{F}_1(t)[p_2K_1(t) + q_2K_2(t)]\,. \tag{4.18}$$

From the arguments used in the theory of regenerative process, the pointwise availabilities $A_i(t)$ are seen to satisfy the following relations:

$$A_0(t) = q_{01}(t) © A_1(t) + q_{02}(t) © A_2(t) + M_0(t) \tag{4.19}$$

$$A_1(t) = q_{12}(t) © A_2(t) \tag{4.20}$$

$$A_2(t) = q_{20}(t) © A_0(t) + q_{24}^{(0)}(t) © A_4(t) + M_2(t) \tag{4.21}$$

$$A_3(t) = q_{34}(t) © A_4(t) \tag{4.22}$$

and
$$A_4(t) = q_{43}(t) © A_3(t) + q_{41}^{(0)}(t) © A_1(t) + q_{42}^{(0)}(t) © A_2(t) + M_4(t)\,. \tag{4.23}$$

Taking Laplace transforms of (4.19) – (4.23), the solution for $A_i^*(s)$ can be written in the matrix form

$$\begin{bmatrix} A_0^*(s) \\ A_1^*(s) \\ A_2^*(s) \\ A_3^*(s) \\ A_4^*(s) \end{bmatrix} = \begin{bmatrix} 1 & -q_{01}^*(s) & -q_{02}^*(s) & 0 & 0 \\ 0 & 1 & -q_{12}^*(s) & 0 & 0 \\ -q_{20}^*(s) & 0 & 1 & 0 & -q_{24}^*(s) \\ 0 & 0 & 0 & 1 & -q_{34}^*(s) \\ 0 & -q_{41}^*(s) & -q_{42}^*(s) & -q_{43}^*(s) & 1 \end{bmatrix}^{-1} \begin{bmatrix} M_0^*(s) \\ 0 \\ M_2^*(s) \\ 0 \\ M_4^*(s) \end{bmatrix}. \tag{4.24}$$

Simplifying (4.24) for $A_0^*(s)$, the Laplace transform of pointwise availability when the system started operation from state S_0, we get

$$A_0^*(s) = \frac{N_2(s)}{D_2(s)}$$

where $N_2(s) = (1 - q_{34}^* q_{43}^*)[M_0^* + M_2^*(q_{01}^* q_{12}^* + q_{02}^*)]$

$$-q_{24}^*[M_0^*(q_{41}^* q_{12}^* + q_{42}^*) + M_4^*(q_{01}^* q_{12}^* + q_{02}^*)]$$

and

$$D_2^*(s) = (1 - q_{34}^* q_{43}^*)[1 - q_{20}^*(q_{01}^* q_{12}^* + q_{02}^*)] - q_{24}^{(3)*}(q_{42}^{(0)*} + q_{41}^{(0)*} q_{12}^*)\,.$$

Here $q_{ij}^*(s) = q_{ij}^*$

The steady state availability A_∞, is given by

$$A_\infty = \lim_{s \to \infty} sA_0^*(s) = \frac{N_2}{D_2}$$

where

$$N_2 = (1 - p_{43})[p_{20}\mu_0 + \mu_2)] + p_{24}^{(3)}\mu_0$$

$$D_2 = (1 - p_{43})[p_{20}\mu_0 + q_1 p_{20}\mu_1 + m]$$

$$+ p_{24}^{(3)}[p_{41}^{(0)}\mu_1 + p_{43}\mu_3 + n]$$

$$m = p_2 \int_0^\infty t\,dG_1(t) + q_2 \int_0^\infty t\,dG_2(t)$$

and $$n = \int_0^\infty t\,dF_1(t)\,.$$

Now the expected up-time of the system in (0, *t*] is

$$\mu_u(t) = \int_0^t A_0(u)du$$

so that

$$\mu_u^*(s) = \frac{A_0^*(s)}{s}$$

and the expected down-time of the system in (0, *t*] is

$$\mu_d(t) = t - \mu_u(t)$$

so that

$$\mu_d^*(s) = \frac{1}{s^2} - \mu_u^*(s)\,.$$

Since $A_0^*(s)$ is known explicitly, the above quantities can be computed easily.

4.6 BUSY PERIOD ANALYSIS

Let $B_i(t)$ be the probability that the repair facility is busy given that the system entered state S_i at $t = 0$.

By probabilistic arguments, we have

$$B_0(t) = q_{01}(t) © B_1(t) + q_{02}(t) © B_2(t) \tag{4.29}$$

$$B_1(t) = q_{12}(t) © B_2(t) + v_1(t) \tag{4.30}$$

$$B_2(t) = q_{20}(t) © B_0(t) + q_{24}^{(3)}(t) © B_4(t) + v_2(t) \tag{4.31}$$

$$B_4(t) = q_{41}(t) © B_1(t) + q_{42}^{(0)}(t) © B_2(t) + q_{43}(t) © B_3(t) + v_4(t) \tag{4.32}$$

where

$$v_1(t) = p_2\overline{H}_1(t) + q_2\overline{H}_2(t)$$

$$v_2(t) = p_2\overline{G}_1(t) + q_2\overline{G}_2(t)$$

$$v_3(t) = p_2\overline{G}_1(t) + q_2\overline{G}_2(t)$$

and $$v_4(t) = \overline{F}_1(t)[p_2\overline{K}_1(t) + q_2\overline{K}_2(t)]. \tag{4.33}$$

Taking Laplace transforms of equations (4.29) – (4.33) and solving for $B_0^*(s)$,

$$B_0^*(s) = \frac{N_3(s)}{D_2(s)}$$

$$N_3(s) = v_1^*[q_{01}^*(1 - q_{43}^* q_{34}^*) + q_{24}^{(3)*} q_{41}^{(0)*} - q_{01}^*(1 - q_{43}^*)]$$

$$+(q_{01}^* q_{12}^* + q_{02}^*)[(1 - q_{43}^* q_{34}^*)v_2^* + q_{42}^* q_{24}^{(3)*} v_3^* + q_{24}^{(3)*} v_4^*].$$

In the long run, the fraction of time for which the system is under repair is given by

$$B_\infty = \lim_{t\to\infty} B_0(t) = \lim_{s\to 0} sB_0^*(s) = \frac{N_3}{D_2}$$

$$N_3 = \mu_1(1-p_{43})q_1[p_{20} + p_{24}^{(3)}p_4^{(0)}] + p_{24}^{(3)}\mu_4 + m(1-p_{43}p_{20})]$$

The expected busy period of the repair facility in (0, *t*] is

$$\mu_b(t) = \int_0^t B_0(u)du$$

so that

$$\mu_b^*(s) = \frac{B_0^*(s)}{s}.$$

4.7 COST ANALYSIS

We now obtain the cost function of the system considering the mean up-time of the system and the expected busy period of the repair facility.

Let us define C_1 as the revenue per unit-time and C_2 as the cost of repairs per unit time. Then the expected total profit earned in (0, *t*] is

G(t) = expected total revenue in (0, *t*] – expected repair cost in (0, *t*]

$$= C_1\mu_u(t) - C_2\mu_0(t).$$

The expected profit per unit time is

$$g(t) = \frac{G(t)}{t}$$

4.8 CONFIDENCE LIMITS

When failure and repair time distributions are exponentially distributed and the switch is perfect, i.e. $p_1 = p_2 = 1$; $q_1 = q_2 = 0$

$$f_1(t) = \alpha_1 e^{-\alpha_1 t} \;\; ; \;\; f_2(t) = \alpha_2 e^{-\alpha_2 t}$$

$$g(t) = \beta_1 e^{-\beta_1 t} \;\; ; \;\; k(t) = \beta_2 e^{-\beta_2 t}$$

then

$$MTSF = \frac{\beta_1 + \alpha_1 + \alpha_2}{\alpha_1 \alpha_2} \tag{4.34}$$

$$A_\infty = \frac{\beta_1[\beta_2(\beta_1 + \alpha_1 + \alpha_2) + \alpha_1\alpha_2]}{\beta_1\beta_2(\beta_1 + \alpha_1 + \alpha_2) + \alpha_1\alpha_2(\beta_1 + \beta_2 + \alpha_1)} \tag{4.35}$$

and

$$B_\infty = \frac{\alpha_1[\beta_1\beta_2 + \alpha_2(\beta_1 + \beta_2 + \alpha_1) + \alpha_1\alpha_2]}{\beta_1\beta_2(\beta_1 + \alpha_1 + \alpha_2) + \alpha_1\alpha_2(\beta_1 + \beta_2 + \alpha_1)}\,. \tag{4.36}$$

4.8.1 CONFIDENCE LIMITS FOR A_∞

Let $X_{i1}, X_{i2}, \ldots, X_{in}$; ($i$ = 1, 2) be random samples of size n, each drawn from exponential populations with failure rates, (α_1, α_2) respectively.

Similarly $Y_{i1}, Y_{i2}, \ldots, Y_{in}$; ($i$ = 1, 2) be random samples of size n, each drawn from exponential populations with repair rates (both p-unit and o-unit) (β_1, β_2) respectively.

If α_1 is the parameter of the exponential distribution, then an estimate can be found for either α_1, or for the parameter $\theta_1 = \frac{1}{\alpha_1}$, which is equal to the mean value of the time of failure-free operation of the *p*-unit.

For the sake of analysis, let

$$\theta_1 = \frac{1}{\alpha_1},\ \theta_2 = \frac{1}{\alpha_2},\ \theta_3 = \frac{1}{\beta_1}, \theta_4 = \frac{1}{\beta_2}.$$

The maximum likelihood estimator (MLE) of θ_1 is given by $\overline{X}_1 = \frac{1}{n}\sum_{i=1}^{n} X_{1i}$. Similarly $\overline{X}_2, \overline{X}_3$ and $\overline{X}_4$ are the MLE's of θ_2, θ_3 and θ_4 respectively.

$$\hat{A}_\infty = \frac{\overline{X}_1[(\overline{X}_1\overline{X}_2 + \overline{Y}_1\overline{X}_2 + \overline{X}_2\overline{Y}_2) + \overline{Y}_1\overline{Y}_2]}{\overline{X}_1(\overline{X}_1\overline{X}_2 + \overline{X}_1\overline{Y}_1 + \overline{Y}_1\overline{X}_2) + \overline{Y}_1(\overline{X}_1\overline{Y}_2 + \overline{X}_1\overline{Y}_1 + \overline{Y}_1\overline{Y}_2)}$$

$\hat{A}_\infty$ is a real-valued function in $\overline{X}_1, \overline{X}_2, \overline{X}_3, \overline{X}_4$, which is also differentiable.

By an application of the central limit theorem [Rao (1973)], it follows that

$$\sqrt{n}\,(\overline{X} - \theta) \xrightarrow{D} N_4\,(0, \Sigma) \text{ as n} \to \infty.$$

where

$$\overline{X} = (\overline{X}_{1,} \overline{X}_2, \overline{Y}_1, \overline{Y}_2)$$

$$\theta = (\theta_1, \theta_2, \theta_3, \theta_4)\ .$$

The dispersion matrix $\Sigma = (\sigma_{ij})_{4\text{x}4}$ is given by

$$\Sigma = diag(\theta_1^{\ 2}, \theta_2^{\ 2}, \theta_3^{\ 2}, \theta_4^{\ 2})\ .$$

From (Rao (1973)), as n $\to \infty$

$$\sqrt{n}\,(\hat{A}_\infty - A_\infty) \xrightarrow{D} N(0,\ \sigma^2(\theta)) \text{ where}$$

$$\sigma^2(\theta) = \sum_{i=1}^{4} \left(\frac{\partial A_\infty}{\partial \theta_1} \right)^2 \sigma_{ii}$$

$$= \sum_{i=1}^{4} \left(\frac{\partial A_\infty}{\partial \theta_i} \right)^2 \theta_i^2 .$$

Replacing θ by its consistent estimator $\hat{\theta} = (\overline{X}_1, \overline{X}_2, \overline{Y}_1, \overline{Y}_2)$, it follows that

$\hat{\sigma}^2 = \sigma^2(\hat{\theta})$ is a consistent estimator of $\sigma^2(\theta)$ (see Wackerly et al (2002).

Then by Slutzky's theorem, (Slutsky (1928)),

$$\frac{\sqrt{n}\,(\hat{A}_\infty - A_\infty)}{\hat{\sigma}} \xrightarrow{D} N(0,\ D) \text{ as n} \to \infty.$$

This implies

$$P\left[-k_{\frac{\alpha}{2}} \le \frac{\sqrt{n}\,(\hat{A}_\infty - A_\infty)}{\hat{\sigma}} \le k_{\frac{\alpha}{2}} \right] = 1 - \alpha$$

where $k_{\alpha/2}$ is obtained from normal tables, i.e. the 100(1 – α)% confidence interval is given by

$$\hat{A}_\infty \pm k_{\alpha/2} \frac{\hat{\sigma}}{\sqrt{n}} .$$

4.8.2 CONFIDENCE LIMITS FOR B_∞

The procedure is identical to section 4.8.1 except

$$\hat{B}_\infty = \frac{\overline{Y}_1[(\overline{X}_1\overline{X}_2 + \overline{Y}_1\overline{X}_2 + \overline{X}_1\overline{Y}_1) + \overline{Y}_1\overline{Y}_2]}{\overline{X}_1(\overline{X}_1\overline{X}_2 + \overline{X}_1\overline{Y}_1 + \overline{Y}_1\overline{X}_2) + \overline{Y}_1(\overline{X}_1\overline{Y}_2 + \overline{X}_1\overline{Y}_1 + \overline{Y}_1\overline{Y}_2)} .$$

When we follow the procedure as in section 4.8.1, we get the confidence limits for $\hat{\beta}_\infty$.

The confidence limits for $\hat{\beta}_\infty$ are

$$\hat{B}_\infty \pm k_{\alpha/2} \frac{\hat{\sigma}}{\sqrt{n}}.$$

4.9 NUMERICAL ILLUSTRATION

Assuming that the units are identical, the switch is perfect and failure and repair rates are constant, that is

$$f_1(t) = f_2(t) = \alpha e^{-\alpha t}$$

$$g_1(t) = g_2(t) = \beta e^{-\beta t}$$

$$k_1(t) = k_2(t) = \gamma e^{-\gamma t}.$$

The expressions for MTSF and A_∞ reduce to the following forms:

$$MTSF = \frac{(\alpha+\beta)(\alpha+\gamma)+\alpha[p_2(\alpha+\gamma)+q_2(\alpha+\beta)]}{\alpha[(\alpha+\beta)(\alpha+\gamma)-p_2\beta(\alpha+\gamma)-q_2\gamma(\alpha+\beta)]}$$

and $$A_\infty = \frac{A}{B+C}$$

where $$A = \beta\gamma(\alpha+\beta)^2(\alpha+\gamma)^2$$

$$B = [\alpha\{(\beta+\gamma)-(p_2\gamma+q_2\beta)\}+\beta\gamma][\beta\gamma\{\alpha(p_2\gamma+q_2\beta)+\beta\gamma\}$$
$$+\alpha^3+\alpha^2(\beta+\gamma)+\alpha\beta\gamma(p_2\gamma+q_2\beta)]$$

and

$$C = [\alpha^2+\alpha(p_2\gamma+q_2\beta)][\{\alpha^3+\alpha^2(p_2\gamma+q_2\beta)\}(p_2\gamma+q_2\beta)+\beta\gamma(\alpha+\beta)(\alpha+\gamma)].$$

Taking β = 4, γ = 1 and β = 15, γ = 5, the values for MTSF and steady state availability corresponding to p_2 = 1, 0.5 and 0 and for different values of α can be calculated.

Figures 4.2 and 4.3 represent the values for A_∞ and MTSF respectively.

These graphs clearly indicate that the better the physical condition of the repair facility the better the performance of the system.

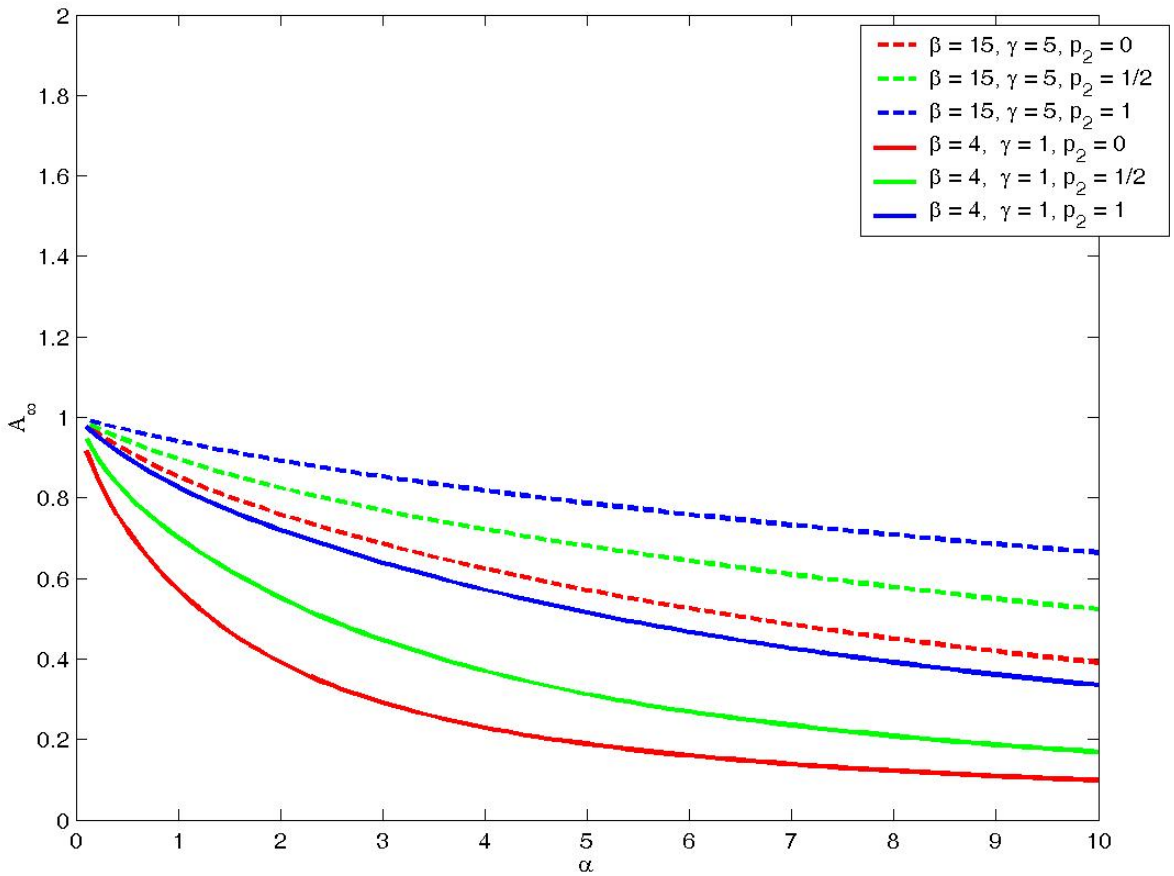

Figure 4.2

As α increases the steady-state availabity, A_∞, is a decreasing function of α (for different values of β, γ and p_2).

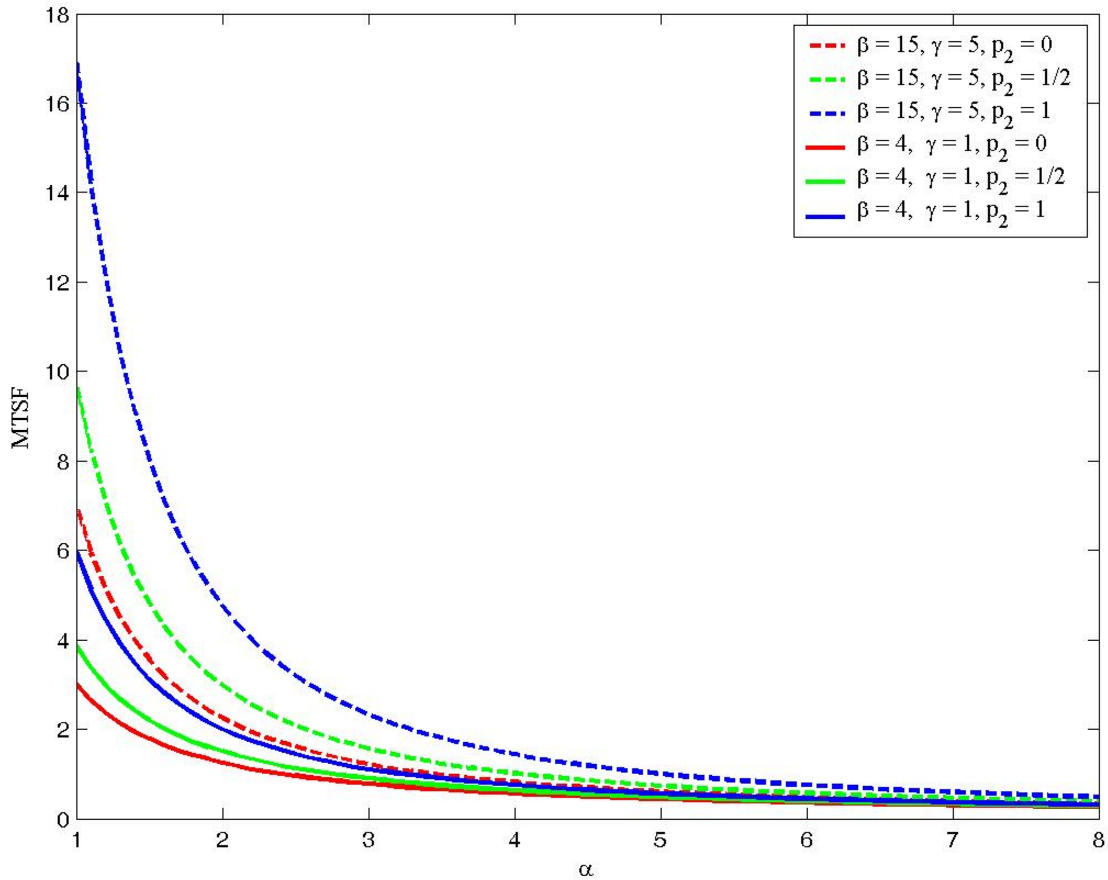

Figure 4.3

As α increases the Mean Time to System Failure (MTSF) is a decreasing function of α (for different values of β, γ and p_2).

4.10 CONCLUSION

A single server two-unit priority cold standby system is studied with varying physical conditions for the repairman, since the repair time's distribution is affected by such conditions. It is assumed that the switching device (the device which transfers the unit from cold standby state to operating online state) is not perfect, i.e. the switch can also fail. Identifying the regeneration points, various operating characteristics are obtained, both analytically and numerically. Explicit expressions for the steady state availability and the busy period in the steady state are obtained, when all underlying distributions are exponential. For these two measures, the asymptotic confidence limits are also obtained. These results were shown in Figure 4.2 (For an increasing α the steady-state availability (A_∞) decreases for different values of β, γ and p_2) and Figure 4.3 (For an increasing α the Mean Time to System Failure (MTSF) decreases for different values of β, γ and p_2).

CHAPTER 5

COST ANALYSIS OF A THREE-UNIT STANDBY REDUNDANT SYSTEM

5.1 INTRODUCTION

In the study of standby redundant systems, two unit systems have been examined extensively in the past. However, the study of *n*-unit redundant systems has received much less attention because of the built-in difficulties in the analysis. Kistner and Subramanian (1974) considered an *n*-unit warm standby system with a single repair facility. In this case, the pdf of the life time of the online unit was taken to be arbitrary while all the other distributions were exponential; these results were later extended to cover the case of several repair facilities by Subramanian, Venkatakrishnan and Kistner (1976). In the dual problem, viz., the *n*-unit system in which the pdf's of the repair time is arbitrary has been studied by Gopalan (1975). Gupta, Bajaj and Singh (1986) have studied the cost-benefit analysis of a single three unit redundant system with inspection, delayed replacement and two types of repair. Kalpakam et al. (1987) have considered a multi-component system in which *n* identical units connected in series; one needed for the system function, the units being supported by *m* spares and a single repair facility. Subramanian et al. (1987) studied a *n*-unit system in which the pdf of the life time is arbitrary and with the varying repair rate. Gupta and Bansal (1991) have analysed a cost function for a three unit standby system subject to random shocks and linear failure rates. It can be seen that in almost all the articles on standby systems in which the number of units is greater than two, at least one of the associated distributions is taken to be exponential.

The study of *n*-unit systems, even in the case of cold standbys, appears to be rather complicated when the pdf of both the life time of the online unit and that of the repair time are arbitrary. However, in this chapter, we study the case when n=3 and obtain elegantly many interesting performance measures.

The organisation of the chapter is as follows: In section 5.2, assumptions and notation are given and in section 5.3, the various system measures are obtained. In section 5.4 special cases are considered and in section 5.5 a comprehensive cost function is constructed. In section 5.6 numerical results are given to illustrate some of the results obtained.

5.2 ASSUMPTIONS AND NOTATION

5.2.1 ASSUMPTIONS

1. The system consists of three identical units with a single repair facility. Each individual unit performs the system function satisfactorily.
2. Initially at t=0, one unit is switched online and the other two units are installed as cold standbys. The initial condition is denoted by E_0.
3. After the completion of repair, a unit is installed back into the system as cold standby if at the epoch another unit functions online; else it is installed as the online unit.

The following table 5.1 describes all possible events:

Event	N(t-0)	N(t)	The system is
E_1	0	1	operable
E_2	1	2	operable
E_3	2	3	just not operable
E_4	2	1	operable
E_5	3	2	just operable
E_6	1	0	operable

Table 5.1

5.2.2 NOTATION

The following functions are defined only for regenerative events E_i.

$N_i(t)$ = Number of events of the type E_i in (0, t]

$U_i(t)$ = P[System unavailable at time t | E_i at t= 0]

$$M_i(t) = \lim_{\Delta \to 0} \frac{P[\text{a repair commencement in (t, t } + \Delta) \,|\, E_i \; at \; t = 0]}{\Delta}$$

$$\varphi(t) = \lim_{\Delta \to 0} \frac{P[\text{first system failure in (t, t } + \Delta) \,|\, E_i \; at \; t = 0]}{\Delta}$$

$$P_{ij}(t) = \lim_{\Delta \to 0} \frac{P[N_j(t+\Delta) = 1, N_k(t) = 0, k = 1{,}5 | E_i \; at \; t = 0]}{\Delta} \quad , \quad \text{for i, j = 1, 5}$$

$$P_{i3}(t) = \lim_{\Delta \to 0} \frac{P[N_3(t+\Delta) = 1, N_k(t) = 0, k = 1{,}3{,}5 | E_i \; at \; t = 0]}{\Delta}$$

$\Pi_{i3}(t)$ = P[System in down state at time t, no E_1 or E_5 events in (0, t]$|$ E_i at t= 0]

$f(\cdot)$ = pdf of the life time of a unit, while operating online

$g(\cdot)$ = pdf of the repair time of a failed unit.

5.3 ANALYSIS

It is noted that the events E_1 and E_5 listed in table 5.1 are regenerative, while the rest are not.

5.3.1 AUXILLIARY FUNCTIONS

The following expressions for the probabilities $P_{ij}(t)$'s and Π_{ij} (t)'s can be obtained:

By definition, $P_{ij}(t)$ (i, j = 1, 5) denote the probability that an E_j event occurring in (t, t + dt) given that an event E_i had occurred at time t = 0 and that no E_1 or E_5 event occurs in (0, t]. Similarly $P_{i3}(t)dt$ refers to the probability of a system breakdown on E_i event had occurred at time t = 0 and that no E_1 or E_5 event occurs in (0, t]. Hence,

$$P_{11}(t) = f(t)G(t) + \sum_{n=1}^{\infty} \int\int \ldots \int f(u_1)g(v_1)f(u_2-u_1)g(v_2-v_1)\ldots g(v_n-v_{n-1})$$
$$f(t-u_n)G(t-v_n)du_1dv_1\ldots du_ndv_n,$$
$$0 \le u_1 \le v_1 \le \ldots \le u_n \le v_n \le t. \quad (5.1)$$

This equation is obtained by considering the following two mutually exclusive and exhaustive cases:

(a) The repair of a unit commenced at t = 0 is completed before the online unit fails

(b) The online unit fails before this repair completion.

In this case, since by definition of $P_{11}(t)$, no E_3 event (since no E_5 event can occur) can occur in (0, t], the third unit cannot fail before the repair completion. This way a sequence of E_2 and E_4 can occur any number of times before the repair completion. This way a sequence of E_2 and E_4 can occur any number of times before the ultimate occurrence of an E_1 event.

By similar arguments, we get

$$P_{15}(t) = \int_0^t f(u)F(t-u)g(t)du + \sum_{n=1}^{\infty} \int \cdots \int f(u_1)g(v_1)f(u_2-u_1)g(v_2-v_1)\ldots f(u_{n+1}-u_n)$$
$$F(t-u_{n+1})g(t-v_n)du_1dv_1\ldots du_ndv_ndu_{n+1} \quad (5.2)$$

$$P_{13}(t) = \int_0^t f(u)f(t-u)G(t)du + \sum_{n=1}^{\infty} \int \cdots \int f(u_1)g(v_1)f(u_2-u_1)g(v_2-v_1)\ldots f(u_{n+1}-u_n)$$
$$f(t-u_{n+1})G(t-v_n)du_1dv_1\ldots du_ndv_ndu_{n+1}\,. \quad (5.3)$$

Also, by its definition, $\Pi_{i3}(t)$ refer to the state probabilities of the system being down given that at time t = 0, E_i had occurred and that no E_1 or E_5 event occurred in (0, t]. We have

$$\Pi_{i3}(t) = \int_0^t f(u)F(t-u)G(t)du + \sum_{n=1}^{\infty} \int \cdots \int f(u_1)g(v_1)f(u_2-u_1)g(v_2-v_1)\ldots f(u_{n+1}-u_n)$$
$$F(t-u_{n+1})G(t-v_n)du_1dv_1\ldots du_ndv_ndu_{n+1} \quad (5.4)$$

$$P_{51}(t) = \int_0^t g(u)G(t-u)f(t)du + \sum_{n=1}^{\infty} \int \cdots \int g(u_1)f(v_1)g(u_2-u_1)f(v_2-v_1)\ldots g(u_{n+1}-u_n)$$
$$f(t-v_n)G(t-v_{n+1})du_1dv_1\ldots du_ndv_ndu_{n+1}\,. \quad (5.5)$$

For all these expressions, the integrations have to be performed for $0 \le u_1 \le v_1 \le \ldots \le u_n \le v_n \le u_{n+1} < t$ while for the following expressions, it has to be performed for $0 \le u_1 \le v_1 \le \ldots \le u_n \le v_n < t$.

$$P_{55}(t) = g(t)F(t) + \sum_{n=1}^{\infty} \int \cdots \int g(u_1)f(v_n)g(u_2-u_1)f(v_2-v_1)\ldots g(u_n-u_{n-1})$$
$$f(v_n-v_{n-1})F(t-v_n)g(t-u_n)du_1dv_1\ldots du_ndv_n \quad (5.6)$$

$$P_{53}(t) = f(t)G(t) + \sum_{n=1}^{\infty} \int \cdots \int g(u_1)f(v_1)g(u_2-u_1)f(v_2-v_1)\ldots g(u_n-u_{n-1})$$

$$f(v_n - v_{n-1}) f(t - v_n) G(t - u_n) du_1 dv_1 \dots du_n dv_n \qquad (5.7)$$

$$\Pi_{53}(\mathrm{t}) = F(t)G(t) + \sum_{n=1}^{\infty} \int\int \dots \int g(u_1) f(v_1) g(u_2 - u_1) f(v_2 - v_1) \dots g(u_n - u_{n-1})$$
$$f(v_n - v_{n-1}) F(t - v_n) G(t - u_n) du_1 dv_1 \dots du_n dv_n \ . \qquad (5.8)$$

5.3.2 RELIABILITY ANALYSIS

We have

$$\varphi_0(t) = f(t) \copyright \varphi_1(t)$$

$$\varphi_1(t) = P_{11}(t) \copyright \varphi_1(t) + P_{13}(t) \ .$$

(5.9)

The equation $\varphi(t)$ is derived by observing the fact that the online unit has to fail before t, if there is to be a system failure in (t, t + Δ). The equation for $\varphi_1(t)$ is obtained by considering the following mutually exclusive and exhaustive cases:

(a) E_1 event occurs in (u, u + du), u < t

(b) no E_1 event occurs before t and the system fails in (t, t + Δ).

Hence the reliability of the system is given by

$$R_0(t) = \int_t^{\infty} \varphi_0(u) du \ . \qquad (5.10)$$

5.3.3 AVAILABILITY ANALYSIS

It is easier to write the equations governing the unavailability of the system. We have, by arguments similar to those in reliability analysis:

$$U_0(t) = f(t) \copyright U_1(t)$$

$$U_1(t) = P_{11}(t) \copyright U_1(t) + P_{15}(t) \copyright U_1(t) + \pi_{13}(t)$$

$$U_5(t) = P_{51}(t) \copyright U_1(t) + P_{55}(t) \copyright U_5(t) + \pi_{53}(t).$$
(5.11)

Solving the equations in (5.11), by using the Laplace transform technique, we get

$$U_0^*(s) = f^*(s) \frac{P_{15}^*(s)\Pi_{53}^*(s) + \Pi_{13}^*(s)[1 - P_{55}^*(s)]}{[1 - P_{11}^*(s)][1 - P_{55}^*(s)] - P_{15}^*(s)P_{51}^*(s)}.$$

By inverting $U_0^*(s)$, wet get $U_0(t)$.

The steady state availability is given by $A_0 = 1 - U_0$ where U_0 is the steady state value of $U_0(t)$ obtained by using the relation

$$\lim_{s \to 0} sU_0^*(s) = U_0.$$

5.3.4 MEASURES OF SYSTEM PERFORMANCE

5.3.4.1 EXPECTED NUMBER OF TRANSITIONS FROM STATE 0 TO STATE 1 in (0, t]

The expected number of visits by the repairman in (0, t] is given by $\int_0^t V_0(u)du$.

The equations governing $V_i(t)$ are:

$$V_0(t) = f(t) \copyright V_1(t) + f(t)$$

$$V_1(t) = P_{11}(t) + P_{11}(t) \copyright V_1(t) + P_{15}(t) \copyright V_5(t)$$

and $$V_5(t) = P_{51}(t) \copyright V_1(t) + P_{55}(t) \copyright V_5(t).$$
(5.12)

$V_0(t)$ can be obtained using the Laplace transform technique.

5.3.4.2 EXPECTED NUMBER OF REPAIRS COMMENCED IN (0, t]

The expected number of repairs commenced in (0, t] is given by $\int_0^t M_0(u)du$. The governing equations for $M_0(t)$ are:

$$M_0(t) = f(t) + f(t) © M_1(t)$$

$$M_1(t) = P_{11}(t) + P_{11}(t) © M_1(t) + P_{15}(t) + P_{15}(t) © M_5(t) + \xi_{121}(t)$$

$$M_5(t) = P_{51}(t) © M_1(t) + P_{55}(t) + P_{55}(t) © M_5(t) + \xi_{21}(t) \qquad (5.13)$$

where

$$\xi_{121}(t) = \int_0^t f(u)F(t-u)g(t)du + \sum_{n=2}^{\infty} \int \ldots \int f(u_1)g(v_1)f(u_2-u_1)g(v_2-v_1)\ldots f(u_n-u_{n-1}) F(t-u_n)g(t-v_{n-1})du_1dv_1\ldots dv_{n-1}dv_n$$

$$\xi_{21}(t) = g(t)F(t) + \sum_{n=1}^{\infty} \int \ldots \int g(u_1)f(v_1)g(u_2-u_1)f(v_2-v_1)\ldots g(u_n-u_{n-1}) f(v_n-v_{n-1})F(t-v_n)g(t-v_n)du_1\ldots du_n dv_n .$$

These integrals are to be evaluated for $0 \le u_1 \le v_1 \le \ldots \le u_n \le v_n \le u_{n+1} \le t$.

5.3.4.3 EXPECTED NUMBER OF REPAIRS COMPLETED IN (0, t]

The expected number of repairs completed in (0, t] is given by $\int_0^t M_0(u)du$. We have

$$M_0(t) = f(t) © M_1(t)$$

$$M_1(t) = P_{11}(t) © M_1(t) + P_{15}(t) © M_5(t) + \eta_1(t)$$

$$M_2(t) = P_{51}(t) © M_1(t) + \quad P_{55}(t) + P_{55}(t) © M_5(t) + \eta_2(t)$$
(5.14)

where

$$\eta_1(t) = g(t)F(t) + \sum_{n=1}^{\infty} \int\int \dots \int f(u_1)g(v_1)f(u_2-u_1)f(v_2-v_1)\dots f(u_n-u_{n-1})$$
$$\{g(t-v_{n-1}) + g(v_n-v_{n-1})g(t-v_n)F(t-u_n)du_1dv_1\dots du_ndv_n\,;$$
$$0 \le u_1 \le v_1 \le \dots \le u_n \le v_n \le t$$

$$\eta_2(t) = g(t)F(t) + \int_0^t g(u)g(t-u)F(t)du + \sum_{n=1}^{\infty} \int\int \dots \int g(u_1)f(v_1)g(u_2-u_1)$$
$$f(v_2-v_1)\dots f(v_n-v_{n-1})\{g(t-u_n) + g(u_{n+1}-u_n)g(t-v_{n+1})$$
$$F(t-v_n)du_1dv_1\dots du_ndv_ndv_{n+1}.$$

$M_0(t)$ can be solved from this set of equations.

5.3.4.4 EXPECTED NUMBER OF SYSTEM BREAKDOWNS IN (0, t]

The expected number of system breakdowns is given by $\int_0^t D_0(u)du$. We have

$$D_0(t) = \ f(t) © D_1(t)$$

$$D_1(t) = P_{11}(t) © D_1(t) + \ P_{15}(t) © D_5(t) + P_{13}(t)$$

$$D_2(t) = P_{51}(t) © D_1(t) + \quad P_{55}(t) © D_5(t) + P_{53}(t)$$
(5.15)

By Laplace transforms technique, we get $D_0(t)$.

5.3.4.5 EXPECTED NUMBER OF SYSTEM RECOVERIES IN (0, t]

The expected number of system recoveries is given by $\int_0^t S_0(u)du$. In this case, the govening equations for $S_0(t)$ are:

$$S_0(t) = f(t) © S_1(t)$$

$$S_1(t) = P_{11}(t) © S_1(t) + P_{15}(t) © S_5(t) + P_{15}(t)$$

and $\quad S_2(t) = P_{51}(t) © S_1(t) + P_{55}(t) © S_5(t) + P_{55}(t)$.
(5.16)

Equations (5.16) can be solved for $S_0(t)$.

REMARK:

It is noted that the steady state value of these expected numbers also represent the respective expected numbers per unit time in the steady state. Further, in the steady state, the expected number per unit time of the system breakdowns and recoveries are equal while that of the repair commencements is equal to repair completions.

5.4 SPECIAL CASES:

In this section, we consider two important special cases of the general model studied.

5.4.1 MODEL 1

All the results obtained in section 5.3 are deduced for the special case where the life time distribution of the online unit is general and the repair time distribution is exponential.

By setting g(t) $= \mu e^{-\mu t}$, the various P_{ij}(t) and Π_{i3}(t)'s reduce to simpler form. For illustration purposes, we consider P_{11}(t):

Substituting g(t) $= \mu e^{-\mu t}$ in equation (5.1), we have

$$\mathrm{P}_{11}(\mathrm{t}) = f(t)[1-e^{-\mu t}] + \sum_{n=1}^{\infty} \int_0^t \int_0^{v_n} \ldots \int_0^{v_1} f(u_1) f(u_2-u_1) f(u_n-u_{n-1}) f(t-u_n) \mu^{n-1}$$

$$[\mu e^{-\mu v_n} - \mu e^{-\mu t}] du_1 dv_1 \ldots du_n dv_n \, .$$

Changing the order of integration, we have

$$\mathrm{P}_{11}(\mathrm{t}) = \quad f(t)[1-e^{-\mu t}] + \sum_{n=1}^{\infty} \int_0^t \int_0^{u_n} \ldots \int_0^{u_2} f(u_1) \mu(u_2-u_1) f(u_2-u_1) \ldots f(u_n-u_{n-1})$$

$$f(u_n-u_{n-1}) f(t-u_n) e^{-u\mu_n}[1-\mu e^{-\mu(t-u_n)} - f(t-u_n) e^{-\mu(t-\mu_n)}] du_1 dv_1 \ldots du_n$$

which gives

$$P_{11}(t) = f(t)[1-e^{-\mu t}] + f(t)e^{-\mu t} © \sum_{n=1}^{\infty} \{f(t)\mu t e^{-\mu t}\}^{(n-1)} © f(t)[1-e^{-\mu t} - \mu t e^{-\mu t}] \, .$$

The simplified expressions for the other $\mathrm{P}_{ij}(\mathrm{t})$'s and $\Pi_{i3}(\mathrm{t})$'s are obtained by similar arguments. By substituting these expressions in the corresponding integral equations and solving them, we get the results for the various system characteristics as:

$$\mathrm{MTSF} = \frac{-f^{*'}(0)[1+f^{*}(\mu)+\mu f^{*'}(\mu)+\{f^{*}(\mu)\}^2]}{\{f^{*}(\mu)\}^2}$$

$$\mathrm{A}_0 = \frac{-\mu f^{*'}(0)[1+\mu f^{*'}(\mu)]}{\{f^{*}(\mu)\}^2 - \mu^2 f^{*'}(0) f^{*'}(\mu) - \mu f^{*'}(0)}$$

$$\mathrm{V}_0 = \frac{\mu^2[1+\mu f^{*'}(\mu) - f^{*}(\mu)]}{\{f^{*}(\mu)\}^2 - \mu^2 f^{*'}(0) f^{*'}(\mu) - \mu f^{*'}(0)}$$

$$\mathrm{m}_0 = \mathrm{M}_0 = \frac{\mu[1+\mu f^{*'}(\mu)]}{\{f^{*}(\mu)\}^2 - \mu^2 f^{*'}(0) f^{*'}(\mu) - \mu f^{*'}(0)}$$

$$\mathrm{D}_0 = \mathrm{S}_0 = \frac{\mu\{f^{*'}(\mu)\}^2}{\{f^{*}(\mu)\}^2 - \mu^2 f^{*'}(0) f^{*'}(\mu) - \mu f^{*'}(0)} \, .$$

As is to be expected, m_0 and D_0 are equal to M_0 and S_0 respectively.

5.4.2 MODEL 2

In this section, the various system characteristics are deduced for the special case when the life time distributions of the online unit is exponential with parameter λ and the repair time distribution is general. By following the same procedure as in Model 1, all $P_{ij}(t)$'s and $\Pi_{i3}(t)$'s reduce to simpler form and the system measures become

$$\text{MTSF} = \frac{3 + 2\lambda g^*(\lambda) - 2g^*(\lambda)}{\lambda[1 - g^*(\lambda) + \lambda g^{*'}(\lambda)]}$$

$$A_0 = \frac{[1 + \lambda g^*(\lambda)]}{\{g^*(\lambda)\}^2 - \lambda^2 g^{*'}(0) f^{*'}(\lambda) - \lambda g^{*'}(0)}$$

$$V_0 = \frac{\lambda \{g^{*'}(\lambda)\}^2}{\{g^*(\lambda)\}^2 - \lambda^2 g^{*'}(0) g^{*'}(\lambda) - \lambda g^{*'}(0)}$$

$$m_0 = M_0 = \frac{\lambda[1 + g^{*'}(\lambda)]}{\{g^*(\lambda)\}^2 - \lambda^2 g^{*'}(0) g^{*'}(\lambda) - \lambda g^{*'}(0)}$$

$$D_0 = S_0 = \frac{\lambda[1 - g^*(\lambda) + \lambda g^{*'}(\lambda)]}{\{g^*(\lambda)\}^2 - \lambda^2 g^{*'}(0) g^{*'}(\lambda) - \lambda g^{*'}(0)}.$$

5.5 COST ANALYSIS

In this section, we construct a comprehensive cost function per unit time in the steady state.

1. The costs due to the visits by the repairman to the repair facility per unit time is βV_0, where β is the cost per visit.

2. The cost associated with the repair rate is $r\ (\frac{1}{MRT})$, where r (> 0) is the constant of proportionality associated with the mean repair rate.

$$CF = \alpha U_0 + \beta V_0 + V(\frac{1}{MRT}) + \eta D_0 \tag{5.17}$$

This cost function is to be optimised with respect to the control parameter MRT within some known bounds.

5.6 NUMERICAL RESULTS

In this section, some of the results obtained for models 1 and 2 are illustrated with numerical examples. We consider the following special cases for this purpose

MODEL 1

We assume that

$$f(t) = \frac{a_1 a_2}{a_2 - a_1}(e^{-a_1 t} - e^{-a_2 t}); \qquad a_1 > 0, a_2 > 0$$

In figures 5.1 to 5.6, three cases are considered for each characteristic corresponding to three different mean failure times to three different failure times of the online unit; (a_1, a_2) were chosen randomly in increasing order, namely

(a_1, a_2):(0.058,0.2),(0.067,0.2),(0.076,0.2) .

MODEL 2

In this model, we assume that

$$g(t) = \mu^2 t e^{-\mu t} .$$

Then MRT is $\frac{2}{\mu}$. For this model also, three cases are considered for each characteristic corresponding to the three values of mean failure times, viz., 0.058, 0.067 and 0.076. Figures 5.7 to 5.12 gives the variation of the various characteristics when MRT is varied. The results demonstrate the following results, viz., as the MRT of a failed unit increase, for the assumed parametric structure thereby giving a unique optimal.

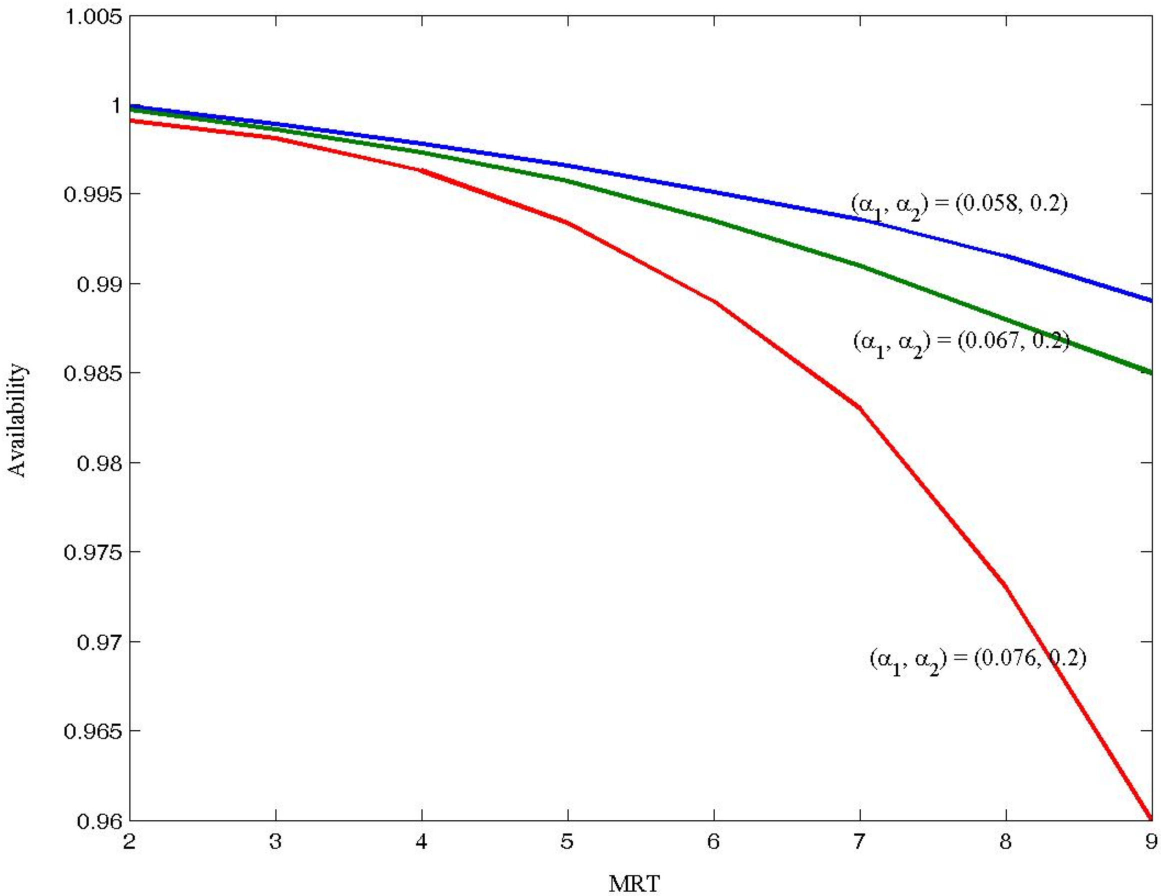

Figure 5.1 (Model 1)

As the Mean Repair Time (MRT) increases the steady-state availability is a decreasing function of MRT (for the different values of α_1 and α_2).

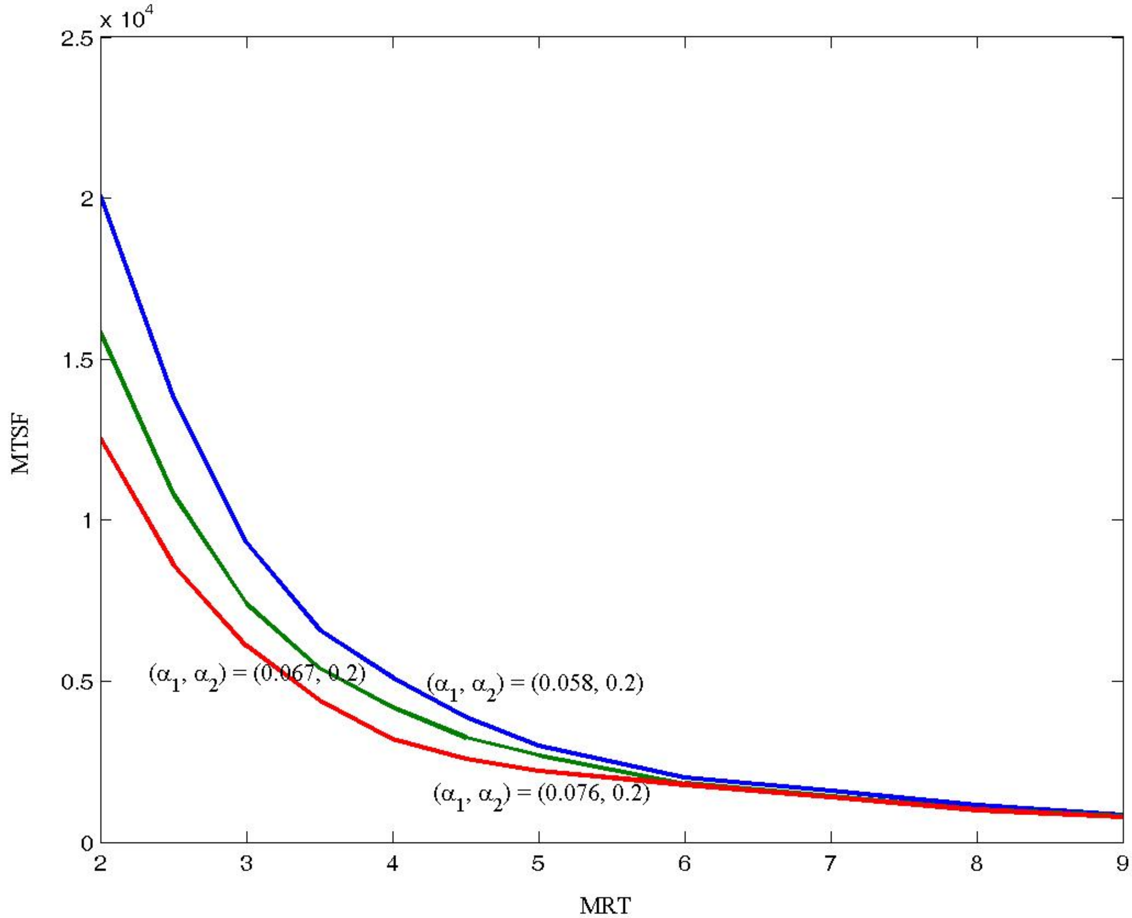

Figure 5.2 (Model 1)

As the Mean Repair Time (MRT) increases the Mean Time to System Failure (MTSF) is a decreasing function of MRT (for different values of α_1 and α_2) with almost convergence of MTSF at MRT = 7.

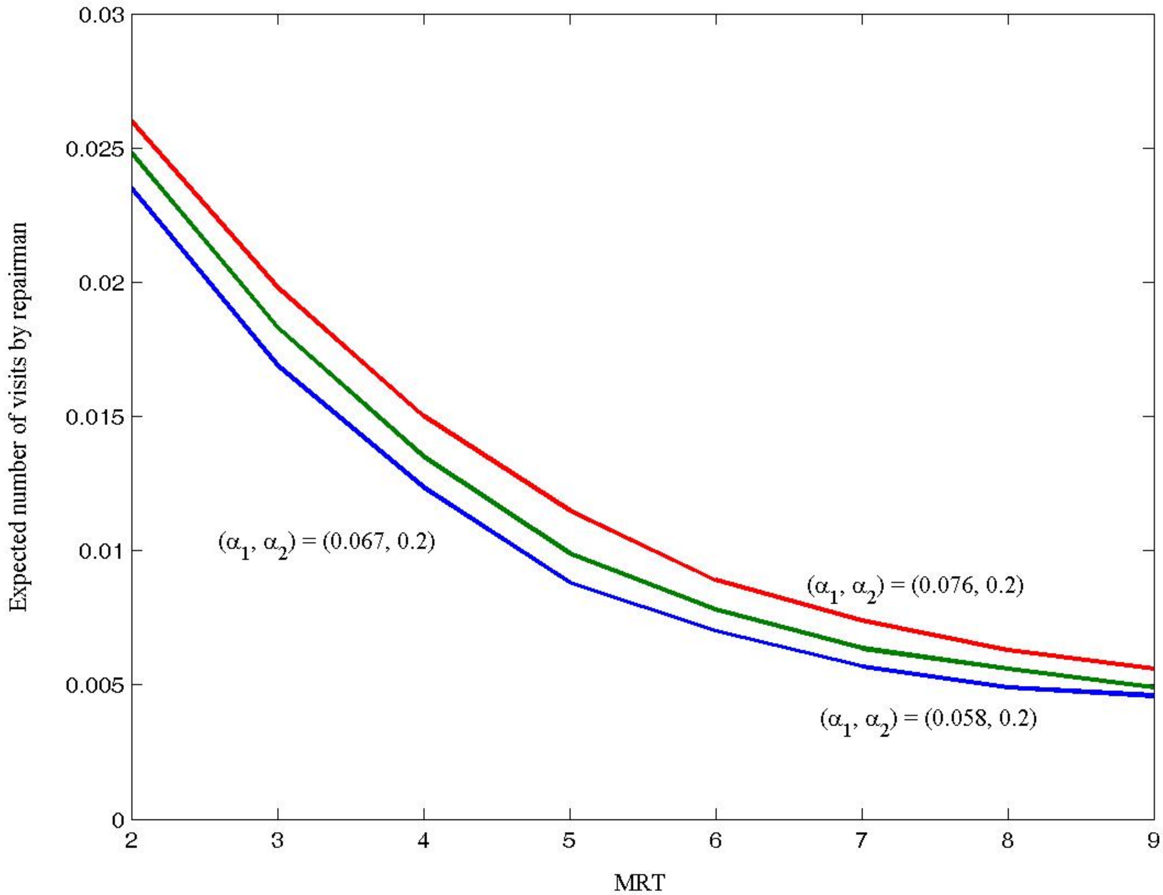

Figure 5.3 (Model 1)

As the Mean Repair Time (MRT) increases the Expected number of visits of the repairman is a decreasing function of MRT (for different values of α_1 and α_2).

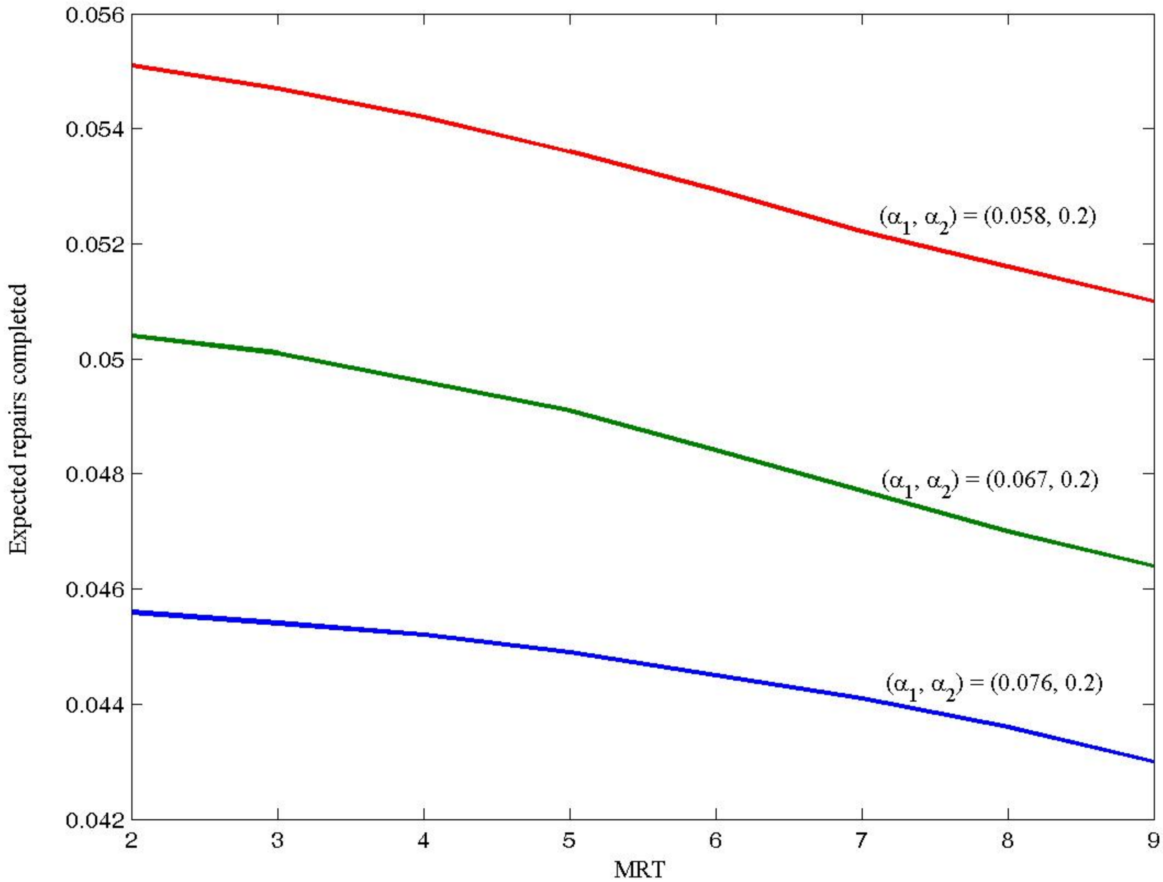

Figure 5.4 (Model 1)

As the Mean Repair Time (MRT) increases the Expected repairs completed is a decreasing function of MRT (for different values of α_1 and α_2).

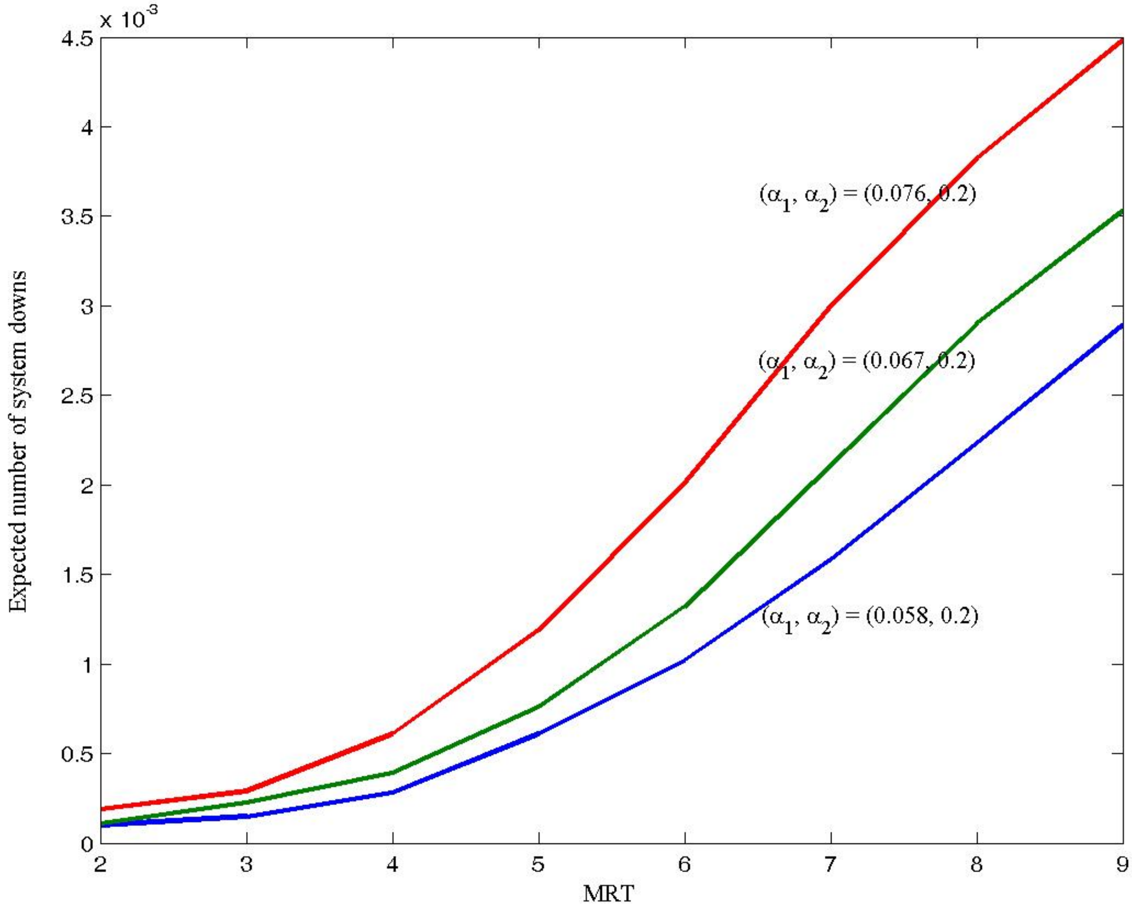

Figure 5.5 (Model 1)

As the Mean Repair Time (MRT) increases the Expected number of system downs is an increasing function of MRT (for different values of α_1 and α_2).

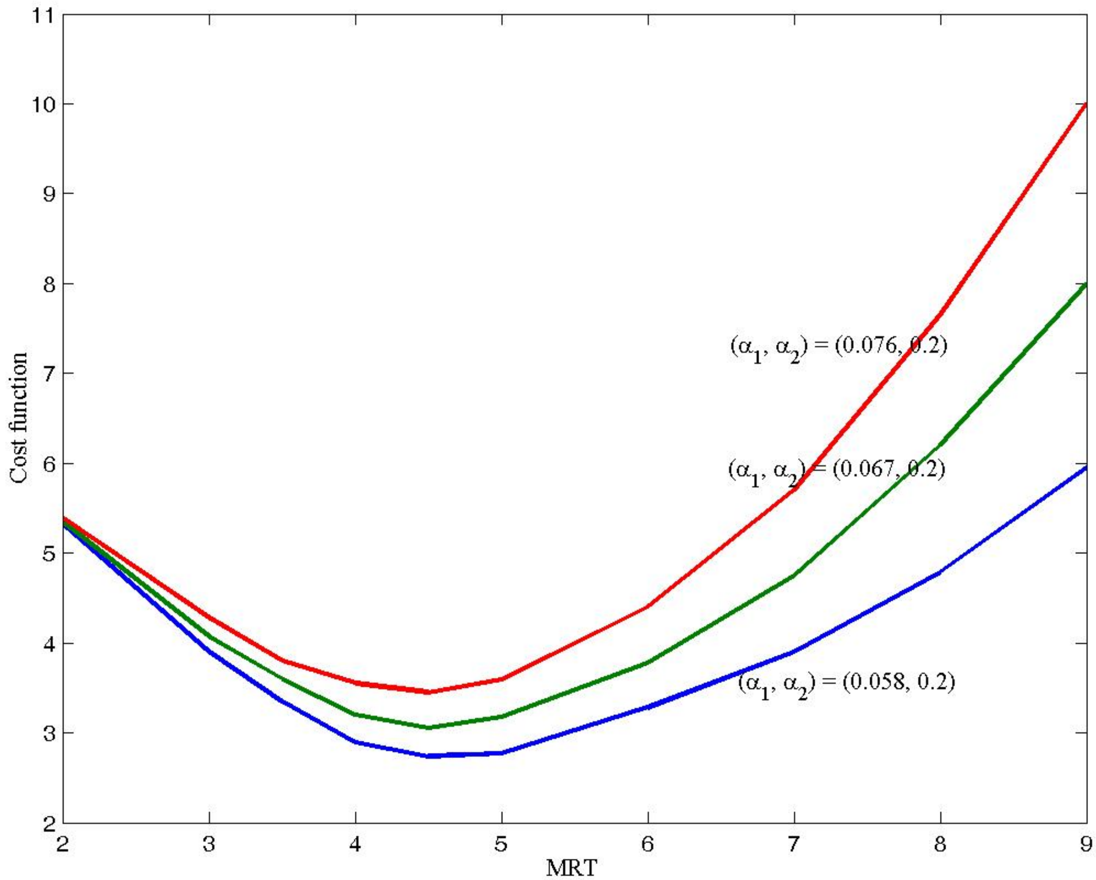

Figure 5.6 (Model 1)

As the Mean Repair Time (MRT) increases the Cost function first decreases but then increases after MRT = 4.5 (for different values of α_1 and α_2).

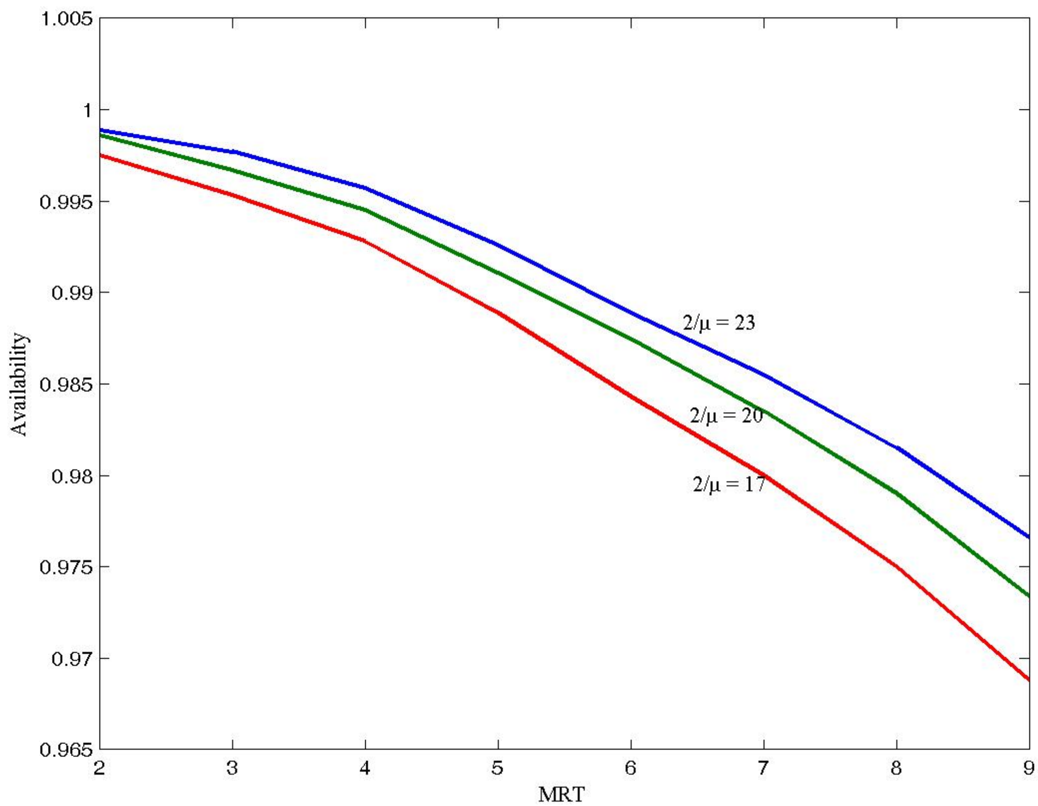

Figure 5.7 (Model 2)

As the Mean Repair Time (MRT) increases the steady-state availability is a decreasing function of MRT (for the different values of $2/\mu$).

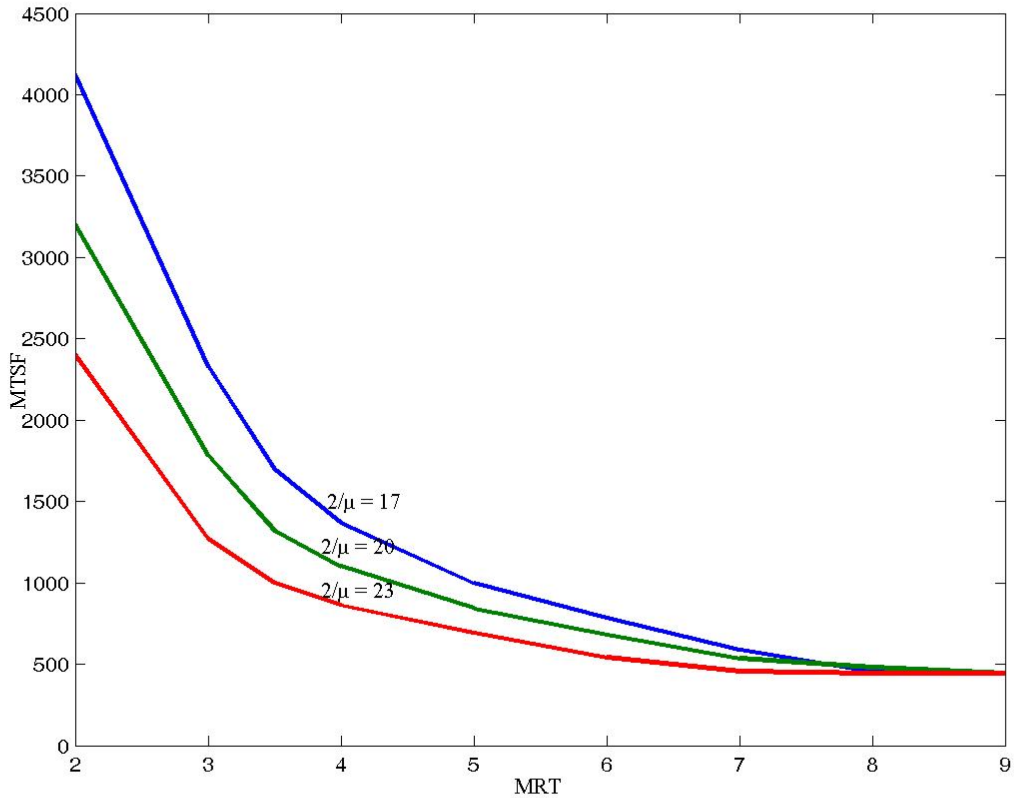

Figure 5.8 (Model 2)

As the Mean Repair Time (MRT) increases the Mean Time to System Failure (MTSF) is a decreasing function of MRT (for different values of $2/\mu$) with almost convergence of MTSF at MRT = 8.

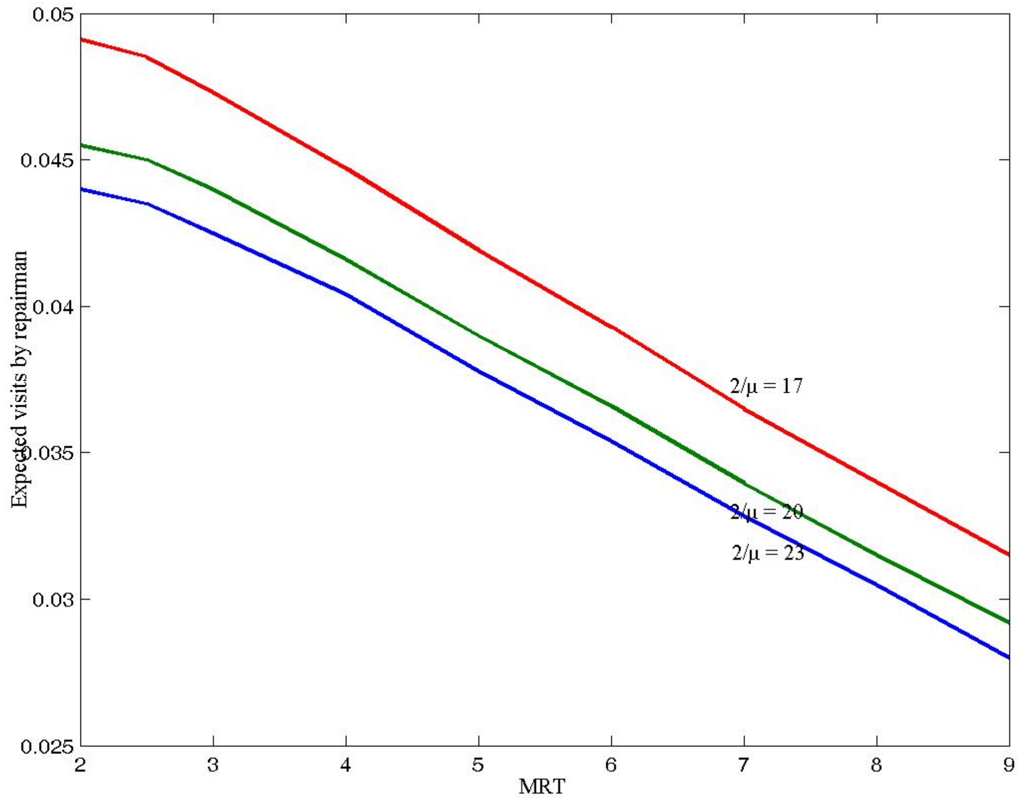

Figure 5.9 (Model 2)

As the Mean Repair Time (MRT) increases the Expected number of visits of the repairman is a decreasing function of MRT (for different values of $2/\mu$).

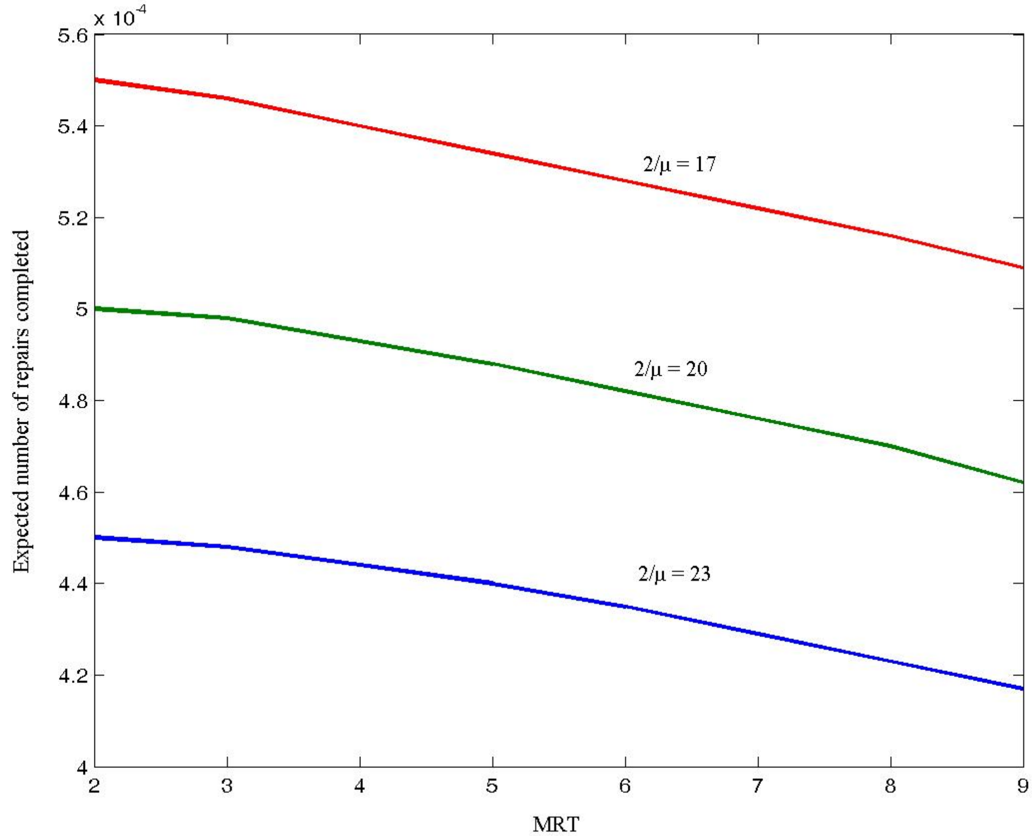

Figure 5.10 (Model 2)

As the Mean Repair Time (MRT) increases the Expected repairs completed is a decreasing function of MRT (for different values of $2/\mu$).

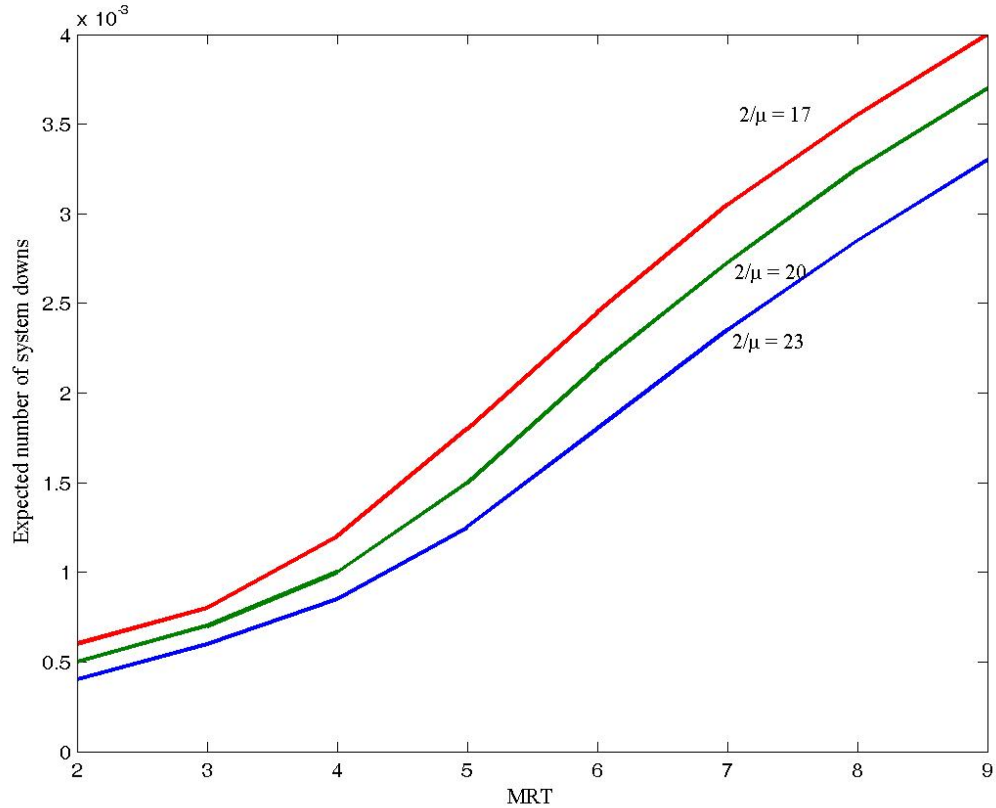

Figure 5.11 (Model 2)

As the Mean Repair Time (MRT) increases the Expected number of system downs is an increasing function of MRT (for different values of $2/\mu$).

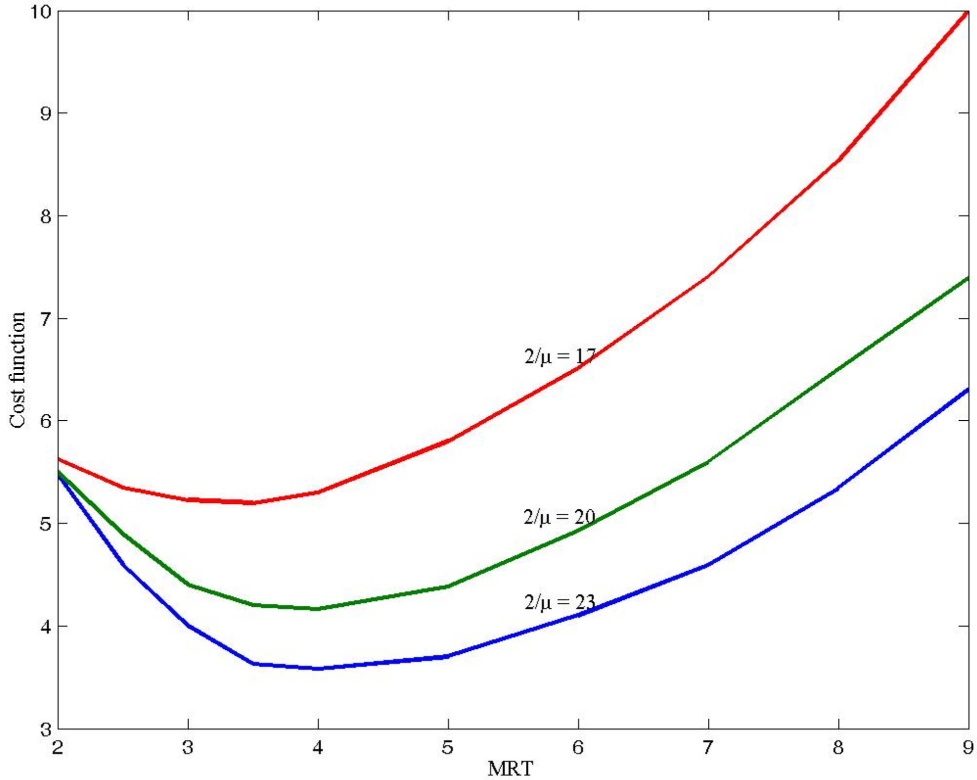

Figure 5.12 (Model 2)

As the Mean Repair time (MRT) increases the Cost function first decreases but then increases after MRT = 4.

5. 7 CONCLUSION

Contrary to the previous chapters a three unit system is considered in this chapter. The life time distributions are all assumed to be non-Markovian. The problem is very challenging when we assume that all the distributions are arbitrary. The system measures, like expected number of transitions from different states, expected number of repairs commenced, expected number of repairs completed, expected number of system breakdowns, expected number of recoveries, are obtained. Results are shown in Figures 5.1 – 5.12.

CHAPTER 6

A STOCHASTIC MODEL OF A RELIABILITY SYSTEM WITH A HUMAN OPERATOR

6.1 INTRODUCTION

With the advancement of methods and developments in artificial intelligence, computer systems and electronic systems, we find a high degree of automation all around. Nevertheless, it cannot be overlooked that human beings are inseparable parts of systems. Radars, motor vehicles, aircrafts and ships are examples of systems which are continuously monitored by human operators. Errors that are caused by human operators are referred to as 'human-errors' (Dhillon, 1980, 1984; Yadavalli & Bekker, 2005).

Dhillon has analysed several models incorporating the concepts of human reliability (1981, 1984). Yadavalli & Bekker (2005) studied a stochastic model of a two unit system with human error and common-cause failures. Their main focus was on the estimation of the steady-state availability (both classical and Bayesian) with the assumption of human error and common-cause failures. One common feature of all the models is that they are Markovian in nature. Kumar et al. (1986) analysed systems operating in fluctuating weather conditions and subject to critical human error, the models are being Markovian. Dhillon and Rayapati (1985) studied five models for a system which needs a human operator. In their models, a human operator is assumed to be working in one of the three states: normal, moderate stress and extreme stress. In all three states, the system is assumed to be subject to two failure modes, i.e. from each state, the system may fail because of self-corrected-human error and non-self-corrected human error. The system can recover from a self-corrected-human failure state, whereas it remains in a failed state when this occurs because of non-self-corrected human error. All the underlying failure distributions are assumed to be exponential. The organization of this chapter is as follows: Section 6.2 presents the system description and relevant notation; in Section 6.3, we represent transition probability functions and sojourn times, which are used in the subsequent analyses; Section 6.4 is a study of reliability analysis and the mean time to system failure; the availability analysis is presented in Section 6.5; the study of expected number of visits to a state and the profit analysis are presented in Section 6.6.

6.2 SYSTEM DESCIPTION AND NOTATION

1. The system is operated by a human operator. The system may fail because of its built-in nature or because of the human operator.
2. The human operator working on the system can be in one of three states: normal, moderate stress or extreme stress. The human operator is more prone to commit errors while in extreme stress state than in the other two states.
3. The system is subject to two failure modes irrespective of the state of the human operator: (a) failure because of self-corrected human-error, (b) failure because of non-self-corrected human error.
4. The rates of change of human operator condition from normal to moderate stress, to extreme stress and vice versa are all exponential. Further, human errors also occur at an exponential rate.
5. Repair time distributions for the system failed from the three human operator conditions are arbitrary and different from one another.
6. Failures are statistically independent.
7. The repaired system is as good as new.

NOTATION

λ_{ij} Constant rate of occurrence of human error, where i denotes the state of human operator and j denotes the system states

α_{lm} Constant rate of change of the state of human operator from l^{th} state to m^{th} state; $l = 1, 2, 3$ and $m = 1, 2, 3$ $(l \neq m)$

$G_{ij}(t)$ Repair time distribution for the system, where i denotes the state of human operator from which the system failed and j denotes the system state; $i = 1, 2, 3$ and $j = 1, 2$.

P_{ij} One-step transition probability from state S_i to S_j

P Transition probability matrix

μ_i Mean unconditional sojourn time of the system in S_i

$\mu^{(1)}$ The diag (μ_0, μ_3, μ_6)

ξ $(1, \quad 1, \quad 1)'$, column vector

N_{ij} Total time spent in a transit state S_j before the system failure, given that the system starts in S_i

$N^{(1)}$ $\left[N_{ij}\right]$, matrix

t_i' Total time spent in up-states given that the system starts to S_i

v_i $E(t_i')$

$V^{(1)}$ $(v_0, \quad v_3, \quad v_6)'$, column vector

b_{ij} P[System is absorbed in S_j $|$ system started in S_i]

B $\left[b_{ij}\right]$

E $\{0, 1, 2, \ldots, 8\}$

E_1 $\{0, 3, 6\}$, up-states

E_2 $\{1, 2, 4, 5, 7, 8\}$, down-states

Π_i Limiting probability that the Markov-Chain in S_i, ($i \in E$)

d_i Determinant of the minor of D

ψ_i Limiting probability of the system being in S_i, ($i \in E$)

Y_i Earning rate of the system per unit time in S_j

g Expected profit per unit time in steady state

r_{ij} Fixed transition reward for a transition from S_i to S_j

6.3 TRANSITION PROBABILITIES AND SOJOURN TIMES

At any instant, the system can be in one of the following states: (See figure 6.1)

(i) Up-states:

$S_0\,(1, 0)$, $S_3\,(2, 0)$, $S_6\,(3, 0)$,

(ii) Down-states:

$S_1\,(1, 1)$, $S_2\,(1, 2)$, $S_4\,(2, 1)$,

$S_5\,(2, 2)$, $S_7\,(3, 1)$, $S_8\,(3, 2)$.

The first symbol in parenthesis denotes the human operator state and the second symbol denotes the state of the system.

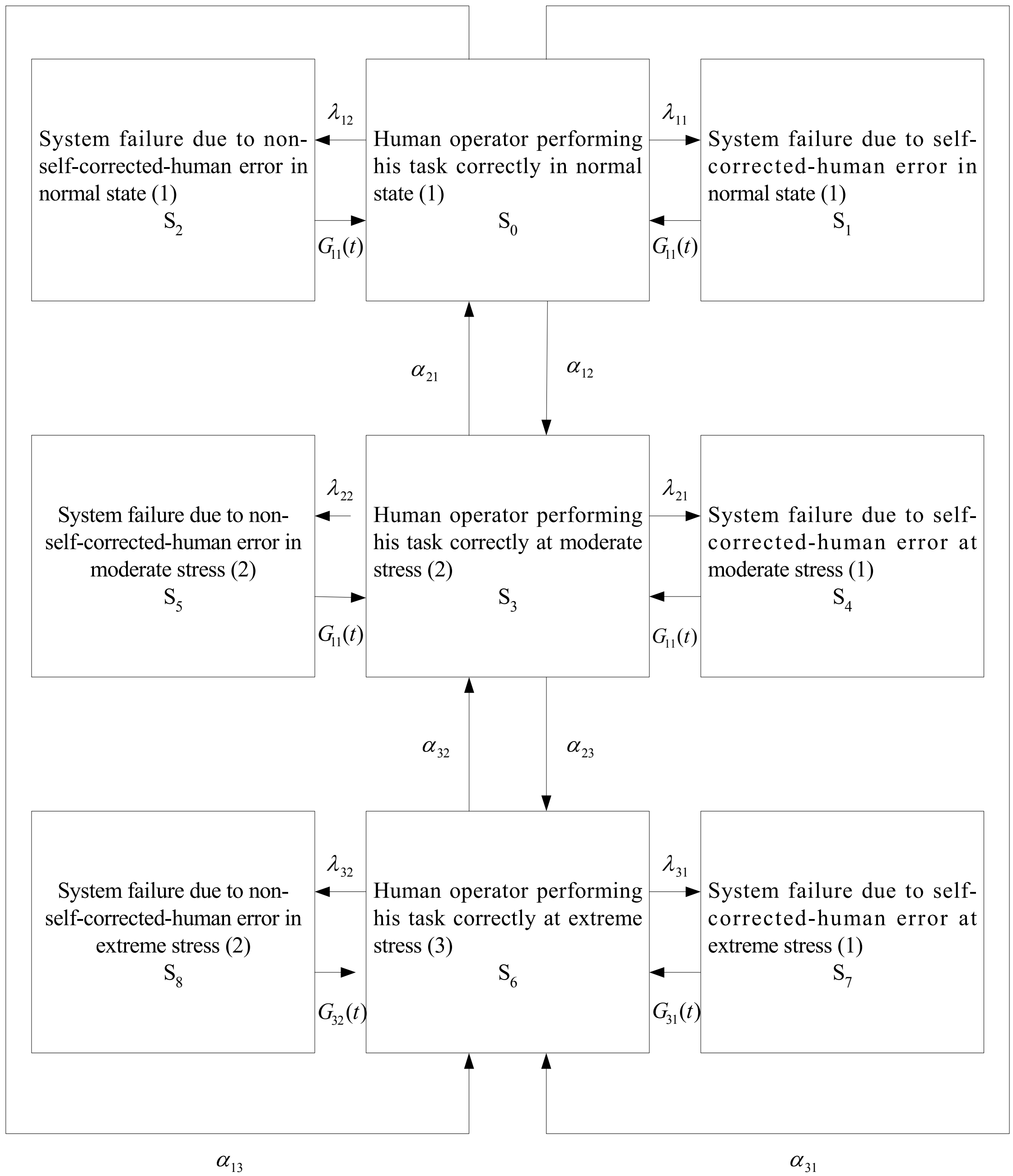

Figure 6.1: Transition Diagram

As indicated earlier, the states of human operator are (1) normal (2) moderate stress and (3) extreme stress. Similarly, the system states are 0: operative, 1: failed because of self-corrected-human error and 2: failed because of non-self-corrected-human error.

The system behavior can be described by a stochastic process $\{Z(t), t \geq 0\}$ with state space $E = \{0,1,\ldots,8\}$, where Z(*t*) denotes the state of the system at time *t*. It may be noted that the process Z(*t*), is a semi-Markovian and as such the well-known properties of Semi-Markovian Process (SMP) (see Cinlar (1975)) are applied to study the system behavior in detail.

The transition probabilities are given by:

$$P_{01} = \int_0^\infty \lambda_{11} e^{-(\lambda_{11}+\lambda_{12}+\alpha_{12}+\alpha_{13})t} dt$$

$$= \frac{\lambda_{11}}{\lambda_{11}+\lambda_{12}+\alpha_{12}+\alpha_{13}} \tag{6.1}$$

$$P_{02} = \int_0^\infty \lambda_{12} e^{-(\lambda_{12}+\lambda_{11}+\alpha_{12}+\alpha_{13})t} dt$$

$$= \frac{\lambda_{12}}{\lambda_{11}+\lambda_{12}+\alpha_{12}+\alpha_{13}} \tag{6.2}$$

$$P_{03} = \int_0^\infty \alpha_{12} e^{-(\alpha_{12}+\lambda_{11}+\lambda_{12}+\alpha_{13})t} dt$$

$$= \frac{\alpha_{12}}{\lambda_{11}+\lambda_{12}+\alpha_{12}+\alpha_{13}} \tag{6.3}$$

$$P_{06} = \int_0^\infty \alpha_{13} e^{-(\alpha_{13}+\alpha_{12}+\lambda_{11}+\lambda_{12})t} dt$$

$$= \frac{\alpha_{13}}{\alpha_{12} + \alpha_{13} + \lambda_{11} + \lambda_{12}} \tag{6.4}$$

$$P_{10} = P_{20} = 1 \tag{6.5}$$

$$P_{30} = \int_0^\infty \alpha_{21} e^{-(\alpha_{21}+\lambda_{12}+\lambda_{22}+\lambda_{23})t} dt$$

$$= \frac{\alpha_{21}}{\lambda_{21} + \lambda_{22} + \alpha_{21} + \alpha_{23}} \tag{6.6}$$

$$P_{34} = \int_0^\infty \lambda_{21} e^{-(\lambda_{21}+\lambda_{22}+\alpha_{21}+\alpha_{23})t} dt$$

$$= \frac{\lambda_{21}}{\lambda_{21} + \lambda_{22} + \alpha_{21} + \alpha_{23}} \tag{6.7}$$

$$P_{35} = \int_0^\infty \lambda_{22} e^{-(\lambda_{22}+\lambda_{21}+\alpha_{21}+\alpha_{23})t} dt$$

$$= \frac{\lambda_{22}}{\lambda_{21} + \lambda_{22} + \alpha_{21} + \alpha_{23}} \tag{6.8}$$

$$P_{36} = \int_0^\infty \alpha_{23} e^{-(\alpha_{23}+\lambda_{22}+\lambda_{21}+\alpha_{21})t} dt$$

$$= \frac{\alpha_{23}}{\lambda_{21} + \lambda_{22} + \alpha_{21} + \alpha_{23}} \tag{6.9}$$

$$P_{43} = P_{53} = 1 \tag{6.10}$$

$$P_{67} = \int_0^\infty \lambda_{31} e^{-(\lambda_{31}+\lambda_{32}+\alpha_{32}+\alpha_{31})t} dt$$

$$= \frac{\lambda_{31}}{\lambda_{31}+\lambda_{32}+\alpha_{31}+\alpha_{32}} \tag{6.11}$$

$$P_{68} = \int_0^\infty \lambda_{32} e^{-(\lambda_{32}+\lambda_{31}+\alpha_{32}+\alpha_{31})t} dt$$

$$= \frac{\lambda_{32}}{\lambda_{31}+\lambda_{32}+\alpha_{31}+\alpha_{32}} \tag{6.12}$$

$$P_{60} = \int_0^\infty \alpha_{31} e^{-(\alpha_{31}+\lambda_{31}+\lambda_{32}+\alpha_{32})t} dt$$

$$= \frac{\alpha_{31}}{\alpha_{31}+\lambda_{31}+\lambda_{32}+\alpha_{32}} \tag{6.13}$$

$$P_{63} = \int_0^\infty \alpha_{32} e^{-(\alpha_{32}+\alpha_{31}+\lambda_{31}+\lambda_{32})t} dt$$

$$= \frac{\alpha_{32}}{\lambda_{31}+\lambda_{32}+\alpha_{31}+\alpha_{32}} \tag{6.14}$$

and $\quad P_{76} = P_{86} = 1.$ $\quad$ (6.15)

The unconditional mean sojourn time of the system in state S_i are given below:

$$\mu_0 = \int_0^\infty e^{-(\lambda_{11}+\lambda_{12}+\alpha_{12}+\alpha_{13})t} dt$$

$$= \frac{1}{\lambda_{11}+\lambda_{12}+\alpha_{12}+\alpha_{13}} \tag{6.16}$$

$$\mu_1 = \int_0^\infty \bar{G}_{11}(t) dt = m_1 \text{ (say)} \tag{6.17}$$

$$\mu_2 = \int_0^\infty \bar{G}_{12}(t)dt = m_2 \text{ (say)} \tag{6.18}$$

$$\mu_3 = \int_0^\infty e^{-(\lambda_{21}+\lambda_{22}+\alpha_{21}+\alpha_{23})t}dt$$

$$= \frac{1}{\lambda_{21}+\lambda_{22}+\alpha_{21}+\alpha_{23}} \tag{6.19}$$

$$\mu_4 = \int_0^\infty \bar{G}_{22}(t)dt = m_4 \text{ (say)} \tag{6.20}$$

$$\mu_5 = \int_0^\infty \bar{G}_{22}(t)dt = m_5 \text{ (say)} \tag{6.21}$$

$$\mu_6 = \int_0^\infty e^{-(\lambda_{31}+\lambda_{32}+\alpha_{32}+\alpha_{31})t}dt$$

$$= \frac{1}{\lambda_{31}+\lambda_{32}+\alpha_{32}+\alpha_{31}} \tag{6.22}$$

$$\mu_7 = \int_0^\infty \bar{G}_{31}(t)dt = m_7 \text{ (say)} \tag{6.23}$$

$$\text{and } \mu_8 = \int_0^\infty \bar{F}_{32}(t)dt = m_8 \text{ (say).} \tag{6.24}$$

6.4 RELIABILITY ANALYSIS

We need to find the MTSF using reliability analysis. To obtain MTSF, we convert the down-states of the system into absorbing states, so that the transition probability matrix

(TPM) is steadily available in a canonical form and to obtain MTSF by the SMP approach (Agarwal et al. (1986)).

$$\boldsymbol{P} = \begin{array}{c} \\ 1 \\ 2 \\ 4 \\ 5 \\ 7 \\ 8 \\ 0 \\ 3 \\ 6 \end{array} \begin{array}{c} 1\ 2\ 4\ 5\ 6\ 7\ 8 \quad 0\ 3\ 6 \\ \left[\begin{array}{c|c} I & O \\ \hline R & Q \end{array}\right] \end{array}$$

$$\boldsymbol{R} = \begin{array}{c|cccccc} & 1 & 2 & 4 & 5 & 7 & 8 \\ \hline 0 & P_{01} & P_{02} & 0 & 0 & 0 & 0 \\ 3 & 0 & P_{34} & P_{35} & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 & 0 & P_{67} & P_{68} \end{array}$$

$$\boldsymbol{Q} = \begin{array}{c|ccc} & 0 & 3 & 6 \\ \hline 0 & 0 & P_{03} & P_{06} \\ 3 & P_{30} & 0 & P_{36} \\ 6 & P_{60} & P_{63} & 0 \end{array}$$

Then we know that (See Agarwal et al. (1986))

$$\boldsymbol{N}^{(1)} = (\boldsymbol{I} - \boldsymbol{Q})^{-1} \boldsymbol{M}^{(1)}$$

$$\boldsymbol{V}^{(1)} = \boldsymbol{N}^{(1)}$$

The elements of $\boldsymbol{V}^{(1)}$ produces the MTSF, given that the system starts in a transient state.

Specifically,

$$\boldsymbol{N}^{(1)} = \frac{1}{D}\begin{bmatrix} (1-P_{36}P_{63})\mu_0 & (P_{03}+P_{06}P_{63})\mu_3 & (P_{06}+P_{03}P_{36})\mu_6 \\ (P_{30}+P_{60}P_{36})\mu_0 & (1-P_{06}P_{60})\mu_3 & (P_{36}+P_{06}P_{30})\mu_6 \\ (P_{60}+P_{30}P_{63})\mu_0 & (P_{63}+P_{03}P_{60})\mu_3 & (1-P_{03}P_{30})\mu_6 \end{bmatrix} = V^{(1)} \tag{6.25}$$

$$\boldsymbol{V}^{(1)} = \frac{1}{D}\begin{bmatrix} (1-P_{36}P_{63})\mu_0 + (P_{03}+P_{06}P_{63})\mu_3 + (P_{06}+P_{03}P_{36})\mu_6 \\ (P_{30}+P_{60}P_{36})\mu_0 + (1-P_{06}P_{60})\mu_3 + (P_{36}+P_{06}P_{30})\mu_6 \\ (P_{60}+P_{30}P_{63})\mu_0 + (P_{63}+P_{03}P_{60})\mu_3 + (1-P_{03}P_{30})\mu_6 \end{bmatrix} \tag{6.26}$$

where

$$D = \left[(1-P_{36}P_{63}) - P_{30}(P_{03}+P_{06}P_{63}) + P_{60}(P_{06}+P_{03}P_{36})\right].$$

After substituting the required values into (6.25) and (6.26), we obtain

$$V_0 = \frac{B_0}{B} \tag{6.27}$$

$$V_3 = \frac{B_3}{B} \tag{6.28}$$

$$V_6 = \frac{B_6}{B} \tag{6.29}$$

where

$$B_0 = M_1 + (\alpha_{21}+\alpha_{12}+\alpha_{23})M_1M_2 + (\alpha_{32}+\alpha_{31}+\alpha_{13})M_1M_3 + PM_1M_2M_3$$

$$B_3 = M_2 + (\alpha_{31}+\alpha_{32}+\alpha_{23})M_2M_3 + (\alpha_{21}+\alpha_{12}+\alpha_{13})M_2M_3 + PM_1M_2M_3$$

$$B_6 = M_3 + (\alpha_{31}+\alpha_{12}+\alpha_{13})M_1M_3 + (\alpha_{32}+\alpha_{23}+\alpha_{21})M_2M_3 + PM_1M_2M_3$$

$$B = 1 + (\alpha_{12}+\alpha_{13})M_1 + (\alpha_{23}+\alpha_{31})M_2$$
$$+ (\alpha_{31}+\alpha_{32})M_3 + K_1M_2M_3 + K_2M_1M_3 + K_3M_1M_2$$

$$\begin{aligned} K_1 &= \alpha_{31}(\alpha_{23}+\alpha_{21})+\alpha_{21}\alpha_{32} \\ K_2 &= \alpha_{12}(\alpha_{32}+\alpha_{31})+\alpha_{32}\alpha_{13} \\ K_3 &= \alpha_{23}(\alpha_{12}+\alpha_{13})+\alpha_{21}\alpha_{32} \end{aligned} \tag{6.30}$$

$$\begin{aligned} P = {} & (\alpha_{12}\alpha_{23}+\alpha_{23}\alpha_{31}+\alpha_{31}\alpha_{12})+(\alpha_{23}\alpha_{13}+\alpha_{32}\alpha_{21}+\alpha_{21}\alpha_{13}) \\ & +(\alpha_{12}\alpha_{32}+\alpha_{31}\alpha_{21}+\alpha_{23}\alpha_{13}) \end{aligned} \tag{6.31}$$

$$M_1 = \frac{1}{\lambda_{11}+\lambda_{12}}; \quad M_2 = \frac{1}{\lambda_{21}+\lambda_{22}} \quad ; \; M_3 = \frac{1}{\lambda_{31}+\lambda_{32}}.$$

In fact, M_1, M_2 and M_3 are MTSF's of respective decomposed subsystems, corresponding to normal, moderate and extreme stress states. It can be noted that (6.27) is in agreement with Dhillon and Rayapati (1985).

According to Agarwal et al. (1986), the absorption probabilities, i.e. the probabilities that the process starting from S_i (i = 0, 3, 6) enters the absorbing states S_j (j = 1, 2, 4, 5, 7, 8) are given by the matrix B = [b_{ij}], namely

$$B = [I-Q]^{-1}R$$

$$= \frac{1}{D} \begin{array}{c} \\ 0 \\ 3 \\ 6 \end{array} \!\!\! \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 4 & 5 & 7 & 8 \end{array} \\ \begin{bmatrix} P_{01}C_{01} & P_{02}C_{01} & P_{34}C_{02} & P_{35}C_{02} & P_{67}C_{03} & P_{68}C_{03} \\ P_{01}C_{31} & P_{02}C_{31} & P_{34}C_{34} & P_{35}C_{32} & P_{67}C_{03} & P_{68}C_{33} \\ P_{01}C_{61} & P_{02}C_{61} & P_{34}C_{62} & P_{35}C_{62} & P_{67}C_{63} & P_{68}C_{63} \end{bmatrix} \end{array}$$

where

$$C_{01} = (1-P_{36}P_{63}); \quad C_{02} = (P_{03}+P_{06}P_{63});$$

$$C_{03} = (P_{06}+P_{03}P_{36}); \; C_{04} = (P_{30}+P_{60}P_{36});$$

$$C_{32} = (1-P_{06}P_{60}); \quad C_{33} = (P_{36}+P_{06}P_{30});$$

$$C_{61} = (P_{60}+P_{30}P_{63}); \; C_{62} = (P_{63}+P_{03}P_{60});$$

$$C_{63} = (1-P_{03}P_{30}). \tag{6.32}$$

6.5 AVAILABILITY ANALYSIS

To obtain the steady-state availability A_∞, the limiting probabilities α_i's are required

$$\psi_i = \frac{\mu_i \pi_i}{\sum_{i \in E} \mu_i \pi_i}$$

$$= \frac{\mu_i d_i}{\sum_{i \in E} \mu_i d_i} \quad ; \quad i \in E_1 \tag{6.33}$$

$$A_\infty = \psi_0 + \psi_3 + \psi_6 . \tag{6.34}$$

Since d_i's are the determinants of the minors of D,

$$d_0 = (1 - P_{34} - P_{35})(1 - P_{67} - P_{68}) - P_{63} P_{36} \tag{6.35}$$

$$d_1 = (1 - P_{02})[(1 - P_{34} - P_{35})(1 - P_{67} - P_{68}) - P_{63} P_{36}]$$

$$- P_{30}[P_{03}(1 - P_{67} - P_{68}) + P_{63} P_{36}]$$

$$- P_{60}[P_{03} P_{36} + P_{06}(1 - P_{34} - P_{35})] \tag{6.36}$$

$$d_2 = (1 - P_{01})[(1 - P_{34} - P_{35})(1 - P_{67} - P_{68}) - P_{63} P_{36}]$$

$$- P_{30}[P_{03}(1 - P_{67} - P_{68}) + P_{63} P_{36}]$$

$$- P_{60}[P_{03} P_{36} + P_{06}(1 - P_{34} - P_{35})] \tag{6.37}$$

$$d_3 = (1 - P_{01} - P_{02})(1 - P_{67} - P_{68}) - P_{60} P_{06} \tag{6.38}$$

$$d_4 = (1 - P_{01} - P_{02})[(1 - P_{35})(1 - P_{67} - P_{68}) - P_{63} P_{36}]$$

$$- P_{30}[P_{03}(1 - P_{67} - P_{68}) + P_{63} P_{06}]$$

$$- P_{60} P_{06}(1 - P_{34}) - P_{60} P_{03} P_{36} \tag{6.39}$$

$$d_5 = (1 - P_{01} - P_{02})[(1 - P_{34})(1 - P_{67} - P_{68}) - P_{63} P_{36}]$$

$$- P_{30}[P_{03}(1 - P_{67} - P_{68}) + P_{63} P_{06}]$$

$$-P_{60}P_{06}(1-P_{34})-P_{60}P_{03}P_{36} \tag{6.40}$$

$$d_6=(1-P_{01}-P_{02})(1-P_{34}-P_{35})-P_{30}P_{04} \tag{6.41}$$

$$d_7=(1-P_{01}-P_{02})[(1-P_{34}-P_{35})(1-P_{68})-P_{63}P_{36}]$$

$$-P_{30}[P_{03}(1-P_{68})+P_{06}P_{63}]$$

$$-P_{60}[P_{03}P_{36}+P_{06}(1-P_{34}-P_{35})] \tag{6.42}$$

$$d_8=(1-P_{01}-P_{02})[(1-P_{34}-P_{35})(1-P_{67})-P_{63}P_{36}]$$

$$-P_{30}[P_{03}(1-P_{67})+P_{06}P_{63}]$$

$$-P_{60}[P_{03}P_{36}+P_{06}(1-P_{34}-P_{35})]\,. \tag{6.43}$$

Using the d_i's, the steady-state availability A_∞, can be obtained as

$$A_\infty=\frac{K_1+K_2+K_3}{\frac{K_1}{A_1}+\frac{K_2}{A_2}+\frac{K_3}{A_3}} \tag{6.44}$$

where $$A_1=\frac{1}{1+a_{11}+a_{12}}$$

$$A_2=\frac{1}{1+a_{21}+a_{22}}$$

$$A_3=\frac{1}{1+a_{31}+a_{32}}$$

$$a_{11}=\lambda_{11}M_1\ ;\quad a_{12}=\lambda_{12}M_2\ ;$$

$$a_{21}=\lambda_{21}M_4\ ;\quad a_{22}=\lambda_{22}M_5\ ;$$

$$a_{31}=\lambda_{31}M_7\ ;\quad a_{32}=\lambda_{32}M_8\,.$$

In fact, A_1, A_2 and A_3 are the steady-state availabilities of the decomposed sub-systems corresponding to normal, moderate and extreme stress states respectively. It may be noted that A_∞ is the harmonic mean of A_1, A_2 and A_3 with the weights K_1, K_2 and K_3 which are functions of α_i's. Also

(i) $\underset{1\le i\le 3}{Min}\, A_i \le A_\infty \le \underset{1\le i\le 3}{Max}\, A_i$

(ii) The above range of A_∞ is independent of α_i's.

6.6 EXPECTED NUMBER OF VISITS TO A STATE AND EXPECTED PROFIT

$$\delta_i = \frac{\mu_i}{\sum_{i\in E} \mu_i d_i}$$

Hence, from equations 6.16 – 6.24 and 6.35 – 6.43,

$$\delta_0 = \frac{K}{(\lambda_{11} + \lambda_{12} + \alpha_{12} + \alpha_{13})T}$$

$$\delta_1 = \frac{M_1 K}{T}$$

$$\delta_2 = \frac{M_2 K}{T}$$

$$\delta_3 = \frac{K}{(\lambda_{21} + \lambda_{22} + \alpha_{21} + \alpha_{23})T}$$

$$\delta_4 = \frac{m_4 K}{T}$$

$$\delta_5 = \frac{m_5 K}{T}$$

$$\delta_6 = \frac{K}{(\lambda_{31} + \lambda_{32} + \alpha_{32} + \alpha_{31})T}$$

$$\delta_7 = \frac{M_7 K}{T}$$

and $$\delta_8 = \frac{M_8 K}{T}$$

where

$$K = (\lambda_{11} + \lambda_{12} + \alpha_{12} + \alpha_{13})(\lambda_{21} + \lambda_{22} + \alpha_{21} + \alpha_{23})(\lambda_{31} + \lambda_{32} + \alpha_{31} + \alpha_{32})$$

$$T = \frac{K_1}{A_1} + \frac{K_2}{A_2} + \frac{K_3}{A_3}.$$

We follow the same approach as in Agarwal (1988), to find the expected profit

$$g = \frac{\sum_i \pi_i \mu_i \mathrm{V}_i}{\sum_i \pi_i \mathrm{V}_i}$$

$$\mu_i \mathrm{V}_i = \sum_i P_{ij} r_{ij} + y_i \mu_i \ .$$

Hence g can be calculated as

$$g = \frac{Z}{T}$$

where

$$g = \frac{K_1}{(\lambda_{11} + \lambda_{12} + \alpha_{12} + \alpha_{13})}[\lambda_{11} r_{01} + \lambda_{12} r_{02} + \alpha_{12} r_{03} + \alpha_{13} r_{06} + y_0]$$

$$+ \frac{\lambda_{11} K_1}{M_1}[M_1 r_{10} + y_1] + \frac{\lambda_{12} K_1}{M_2}[M_2 r_{20} + y_2]$$

$$+ \frac{K_2}{(\lambda_{21} + \lambda_{22} + \alpha_{23} + \alpha_{21})}[\lambda_{21} r_{34} + \lambda_{22} r_{35} + \alpha_{23} r_{36} + \alpha_{21} r_{30} + y_3]$$

$$+ \frac{\lambda_{21} K_2}{M_4}[M_4 r_{43} + y_4] + \frac{\lambda_{22} K_2}{M_5}[M_5 r_{53} + y_5]$$

$$+ \frac{K_3}{(\lambda_{31} + \lambda_{32} + \alpha_{32} + \alpha_{31})}[\lambda_{31} r_{67} + \lambda_{32} r_{68} + \alpha_{32} r_{63} + \alpha_{32} r_{60} + y_6]$$

$$+\frac{\lambda_{31}K_3}{M_7}[M_7r_{76}+y_7]+\frac{\lambda_{32}K_3}{M_8}[M_8r_{86}+y_8].$$

6.7 NUMERICAL ILLUSTRATION

As a numerical illustration, the behaviour of q_0, the mean time to system filure, has been studied with respect to changes in M_1, M_2 and M_3, the mean times to failure of decomposed sub-systems. It may be noted that MTSF increases as M_1 increases but there is always an upper bound on the MTSF when other factors M_2 and M_3 are kept constant. This is convincing, as, after a certain change, any improvement in M_1 is not likely to improve the MTSF value. In fact, for fixed values of M_2 and M_3, q_0 takes the form

$$q_0=\frac{M_1}{a+bM_2}\le\frac{1}{b}$$

where *a* and *b* are functions of M_2 and M_3 and thus may be treated as constants as long as M_2 and M_3 are fixed. The above inequality gives the least upper bound for q_0.

For example, for $M_2 = 100$, least upper bounds for q_0 are given as follows for varying M_3:

M_3	25	40	60	100	...	∞
q_0	97.03	126.48	155.65	194.09	...	324.80

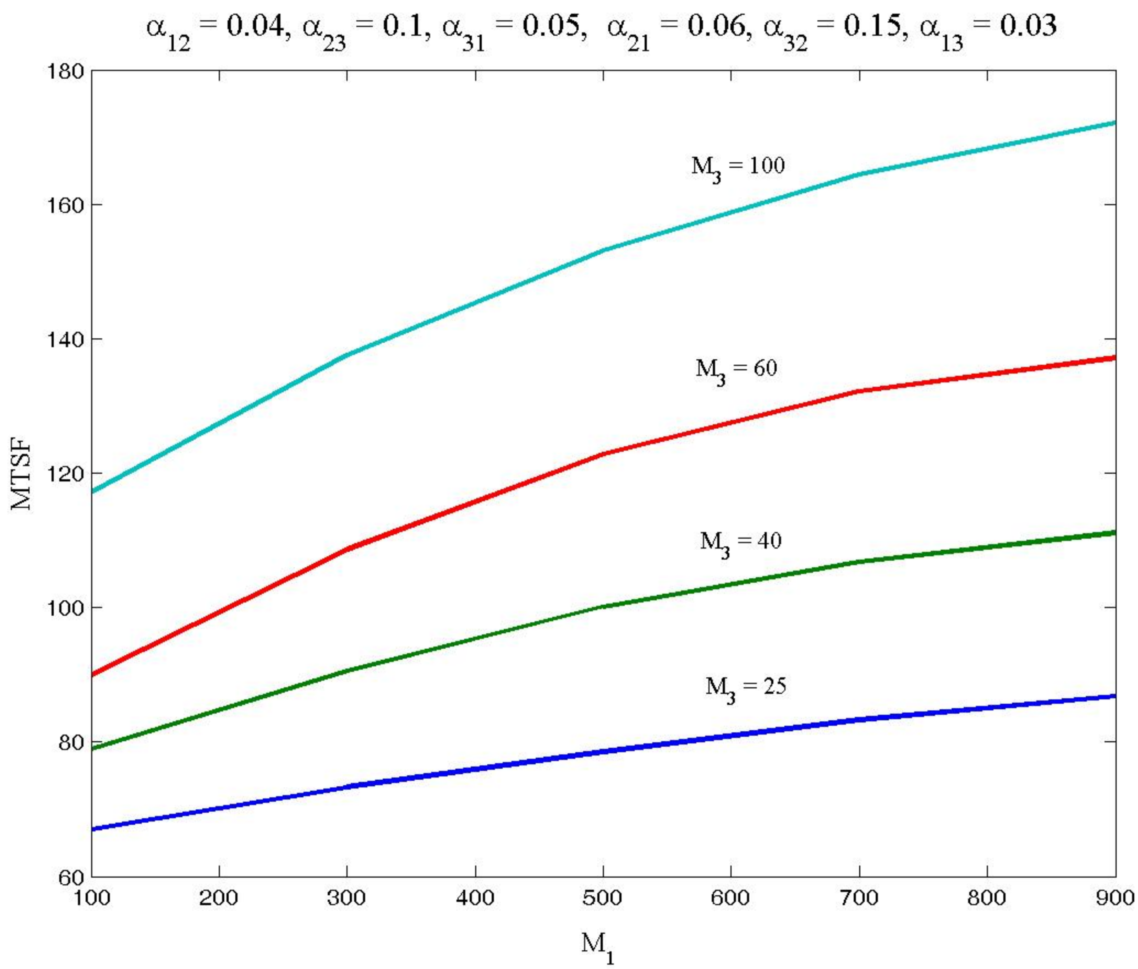

Figure 6.1

As the Mean time to failure of decomposed subsystem 1 (M_1) increases Mean Time to System Failure (MTSF) is an increasing function of M_1 (for different values of M_3 and fixed values of α_{12}, α_{23}, α_{31}, α_{21}, α_{32} and α_{13}).

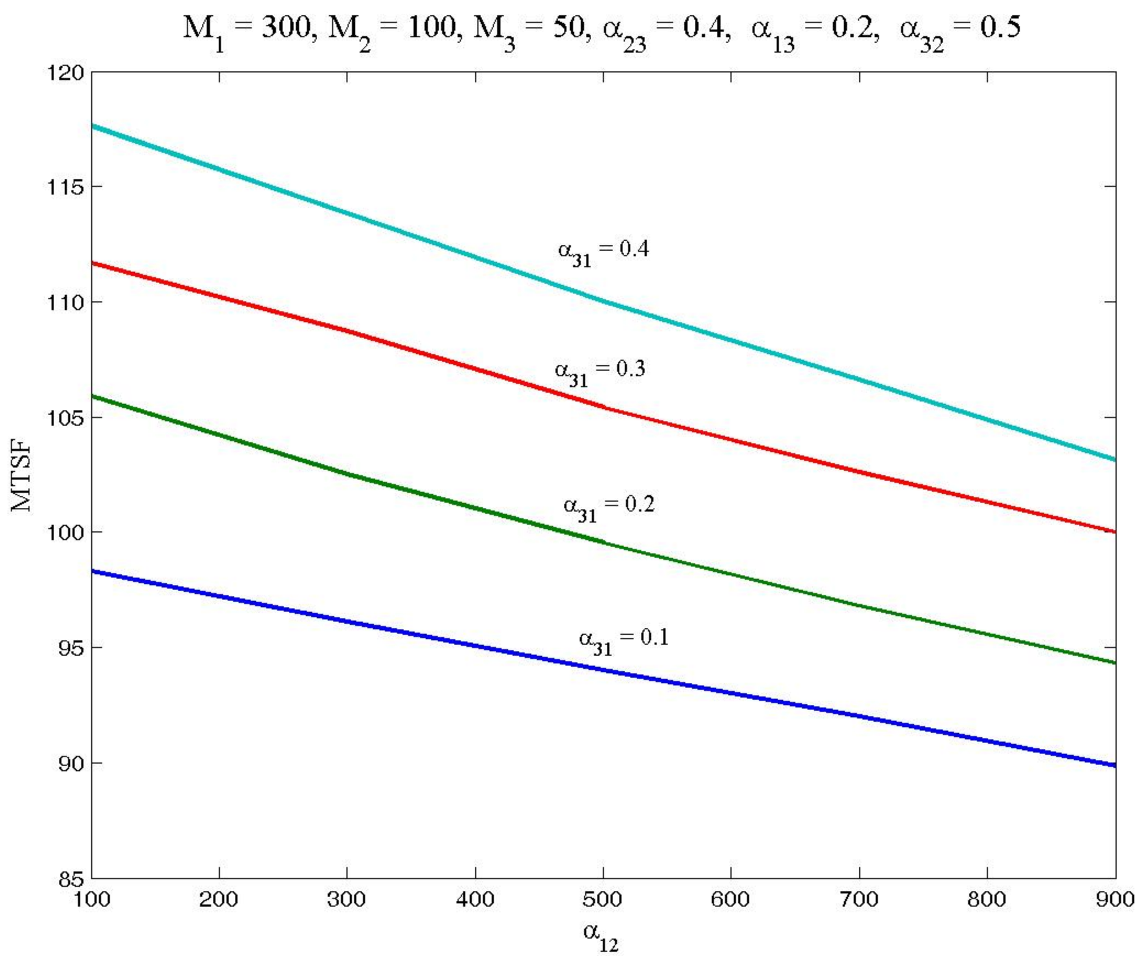

Figure 6.2

As the constant rate of change from first to second state, α_{12}, increases Mean Time to System Failure (MTSF) is a decreasing function of α_{12} (for different values of α_{31} and fixed values of M_1, M_2, M_3, α_{23}, α_{13}, α_{21}, and α_{32}).

6.8 CONCLUSION

A repairable system with a human operator is considered. The operator could be in one of the three states – normal, moderate stress and extreme stress. The system can fail due to self-corrected and non-self-corrected errors. With the help of a semi-Markovian process and a regeneration point technique various characteristics, like availability and MTSF are obtained (results in Figures 6.1 – 6.2). A cost-benefit analysis is also obtained for such a system.

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