

CHAPTER 5

AN M/G/1 QUEUEING SYSTEM WITH TWO MODES OF FAILURE

5.1 Introduction

In the previous chapter, queueing systems with an exponential service distribution were considered. However in real life situations the service distribution need not be exponential. It may be of phase type or k-Erlang as in the case of buying cosmetics and provisions in a default mental state, taking X-rays, blood test etc. in a hospital; receiving cash from a bank. Besides, exponential service is found in industry or in production or in mechanical devices. Hence the study of such systems is absolutely essential. In this chapter, an M/G/1 queueing system where the service facility is subject to failure in two modes is considered, partial and total.

5.2 Model description

In this model the inter-arrival time of entities follows a negative exponential distribution i.e. the arrival process is Poisson. The service time X_n of the n th entities follows a general distribution by with an average service rate of μ_1 when the service channel is in the normal working condition. The service rate of the n th customer follows a general distribution and the average service time is denoted by $\mu_2 (< \mu_1)$ when the service channel is in a partial failure mode. After completion of the repair of the total failure mode the channel directly changes to the normal mode without passing through the partial failure mode. If the service channel repair in the partial failure mode is completed, the system enters the normal working mode; otherwise it goes into the total failure mode. The repair times of the partial failure mode and the total failure mode are exponentially distributed with different densities. The failure times from normal to partial, and partial to total failure mode are also exponentially distributed with different densities. If the repairs in the partial failure mode are in the process of being completed, the system will not enter the total failure mode. Further, it is assumed that the repair process starts instantaneously after the completion of repair.

5.3 System description

λ : Arrival rate of entities ($\lambda > 0$).

$W_n(n,t)$: The joint probability that at time t , there are n ($n > 0$) entities in the system with elapsed service time lying between x and $x+dx$ and the service channel is in the normal working condition.

$S_n(x,t)$: The joint probability that at time t , there are n ($n > 0$) entities in the system with elapsed service time lying x and $x+dx$ and the service channel is in the partial failure mode.

$R_n(t)$: The probability that at time t , there are n ($n > 0$) entities in the system and the service channel is in the total failure mode.

$\alpha_1(\alpha_2)dt$: The first order probability that a total (partial) failure occurs during the short interval of time dt .

$\beta_1(\beta_2)dt$: The first order probability that the repair of the service channel is in the total (partial) failure mode will be complete during the short interval of time dt . As soon as the service channel is subject to total failure, it ceases to provide service instantaneously.

$\mu_1(x)dx$: The first order probability that the service will be complete in time x and $x+dx$, when the service channel is in normal working condition, given that the same was not complete till time x and is related to the density function $B_1(x)$ by the relation.

$$B_1(x) = \mu_1(x) \exp \left[- \int_0^x \mu_1(y) dy \right]$$

$\mu_2(x)dx$: The first order probability that the service will be complete in time x and $x+dx$, when the service channel is in the partial failure mode, given that the same was not complete till time x and is related to the density function $B_2(x)$ by the relation.

$$B_2(x) = \mu_2(x) \exp \left[- \int_0^x \mu_2(y) dy \right]$$

5.4 Equations governing the system

Using probability arguments, the following partial difference-differential equations are obtained.

$$\frac{\partial}{\partial x} W_n(x,t) + \frac{\partial}{\partial t} W_n(x,t) + [\lambda + \mu_1(x) + \alpha_2] W_n(x,t) = \lambda W_{n-1}(x,t) \quad ; n > 1 \quad (5.4.1)$$

$$\frac{\partial}{\partial x} W_1(x,t) + \frac{\partial}{\partial t} W_1(x,t) + [\lambda + \mu_1(x) + \alpha_2] W_0(x,t) = 0 \quad (5.4.2)$$

$$\frac{d}{dt} W_0(t) + [\lambda + \alpha_2] W_0(t) = \int_0^{\infty} W_1(x,t) \mu_1(x) dx + \beta_1 R_0(t) + \beta_2 S_0(t) \quad (5.4.3)$$

$$\frac{\partial}{\partial x} S_n(x,t) + \frac{\partial}{\partial t} S_n(x,t) + [\lambda + \mu_2(x) + \alpha_1 + \beta_2] S_n(x,t) = \lambda S_{n-1}(x,t) \quad ; n > 1 \quad (5.4.4)$$

$$\frac{\partial}{\partial x} S_1(x,t) + \frac{\partial}{\partial t} S_1(x,t) + [\lambda + \mu_2(x) + \alpha_1 + \beta_2] S_1(x,t) = 0 \quad (5.4.5)$$

$$\frac{d}{dt} S_0(t) + [\lambda + \alpha_1 + \beta_2] S_0(t) = \int_0^{\infty} S_1(x,t) \mu_2(x) dx + \alpha_2 W_0(t) \quad (5.4.6)$$

$$\frac{d}{dt} R_n(t) + [\lambda + \beta_1] R_n(t) = \lambda R_{n-1}(t) + \alpha_1 S_n(t) \quad ; n > 0 \quad (5.4.7)$$

$$\frac{d}{dt} R_0(t) + [\lambda + \beta_1] R_0(t) = \alpha_1 S_0(t) \quad (5.4.8)$$

subject to boundary conditions

$$W_n(0,t) = \int_0^{\infty} W_{n+1}(x,t)\mu_1(x)dx + \beta_2 S_n(t) + \beta_1 R_n(t) \quad ; n > 1 \quad (5.4.9)$$

$$W_1(0,t) = \int_0^{\infty} W_2(x,t)\mu_1(x)dx + \beta_2 S_1(t) + \lambda W_0(t) \quad (5.4.10)$$

$$S_n(0,t) = \int_0^{\infty} S_{n+1}(x,t)\mu_1(x)dx + \alpha_2 W_n(t) \quad ; n > 1 \quad (5.4.11)$$

$$S_1(0,t) = \int_0^{\infty} S_2(x,t)\mu_2(x)dx + \alpha_2 W_1(t) + \lambda S_0(t) \quad (5.4.12)$$

Without loss of generality it may be assumed that the system is initially empty and the server is in an idle period when the service channel is in the normal working condition $W_0(0) = 1$ and all other initial probabilities are zero. i.e.

$W_n(0) = \delta_{n,0}$ where $\delta_{n,0}$ is Kronecker's delta function

$S_n(0) = 0, \text{ for } n \geq 0$

$R_n(0) = 0, \text{ for } n \geq 0 \quad (5.4.13)$

5.5 Time dependant solution

Denoting the Laplace transform of a function $f(t)$ by $f^*(s)$, and taking the Laplace transform of equations (5.4.1) to (5.4.12) and using (5.4.13), it follows that

$$\frac{\partial}{\partial x} W_n^*(x, s) + [s + \lambda + \mu_1(x) + \alpha_2] W_n^*(x, s) = \lambda W_{n-1}^*(x, s) \quad , n > 1 \quad (5.5.1)$$

$$\frac{\partial}{\partial x} W_1^*(x, s) + [s + \lambda + \mu_1(x) + \alpha_2] W_1^*(x, s) = 0 \quad (5.5.2)$$

$$(s + \lambda + \alpha_2) W_0^*(s) = \int_0^{\infty} W_1^*(x, s) \mu_1(x) dx + \beta_1 R_0^*(s) + \beta_2 S_0^*(s) + 1 \quad (5.5.3)$$

$$\frac{\partial}{\partial x} S_n^*(x, s) + [s + \lambda + \mu_2(x) + \alpha_1 + \beta_2] S_n^*(x, s) = \lambda S_n^*(x, s) \quad , n > 1 \quad (5.5.4)$$

$$\frac{\partial}{\partial x} S_1^*(x, s) + [s + \lambda + \mu_2(x) + \alpha_2 + \beta_2] S_1^*(x, s) = 0 \quad (5.5.5)$$

$$(s + \lambda + \alpha_1 + \beta_2) S_0^*(s) = \int_0^{\infty} S_1^*(x, s) \mu_2(x) dx + \alpha_2 W_0^*(s) \quad (5.5.6)$$

$$(s + \lambda + \beta_1) R_n^*(s) = \lambda R_{n-1}^*(s) + \alpha_1 S_n^*(s) \quad , n > 0 \quad (5.5.7)$$

$$(s + \lambda + \beta_1) R_0^*(s) = \alpha_1 S_0^*(s) \quad , n > 0 \quad (5.5.8)$$

subject to boundary conditions

$$W_n^*(0, s) = \int_0^{\infty} W_{n+1}^*(x, s) \mu_1(x) dx + \beta_2 S_n^*(s) + \beta_1 R_n^*(s) \quad ; n > 1$$

(5.5.9)

$$W_1^*(0, s) = \int_0^{\infty} W_2^*(x, s) \mu_1(x) dx + \beta_2 S_1^*(s) + \beta_1 R_1^*(s) + \lambda W_0^*(s)$$

(5.5.10)

$$S_n^*(0, s) = \int_0^{\infty} S_{n+1}^*(x, s) \mu_2(x) dx + \alpha_2 W_n^*(s) \quad ; n > 1$$

(5.5.11)

$$S_1^*(0, s) = \int_0^{\infty} S_2^*(x, s) \mu_2(x) dx + \alpha_2 W_1^*(s) + \lambda S_0^*(s)$$

(5.5.12)

Defining the following probability generating functions:

$$W^*(x, s, z) = \sum_{n=1}^{\infty} W_n^*(x, s) z^n$$

$$W^*(0, s, z) = \sum_{n=1}^{\infty} W_n^*(0, s) z^n$$

$$W^*(s, z) = W_0^*(s) + \int_0^{\infty} W^*(x, s, z) dx$$

$$S^*(x, s, z) = \sum_{n=1}^{\infty} S_n^*(x, s) z^n$$

$$S^*(0, s, z) = \sum_{n=1}^{\infty} S_n^*(0, s) z^n$$

$$S^*(s, z) = S_0^*(s) + \int_0^{\infty} S^*(x, s, z) dx$$

$$R^*(s, z) = \sum_{n=0}^{\infty} R_n^*(s) z^n$$

(5.5.13)

$\sum_{n=2}^{\infty} z^n$ * (5.5.1) + z * (5.5.1) and using (5.5.13), it follows that

$$\frac{\partial}{\partial x} W^*(x, s, z) + [s + \lambda + \lambda z + \mu_1(x) + \alpha_2] W^*(x, s, z) = 0$$

Integrating this from 0 to x gives

$$W^*(x, s, z) = W^*(0, s, z) \exp \left[-(s + \lambda + \lambda z + \alpha_2)x - \int_0^x \mu_1(x) dx \right]$$

(5.5.14)

$\sum_{n=2}^{\infty} z^n$ * (5.5.4) + z * (5.5.5) and using (5.5.13), it follows that

$$\frac{\partial}{\partial x} S^*(x, s, z) + [s + \lambda - \lambda z + \mu_2(x) + \alpha_1 + \beta_2] S^*(x, s, z) = 0$$

Integrating this from 0 to x

$$S^*(x, s, z) = S^*(0, s, z) \exp \left[-(s + \lambda + \lambda z + \alpha_1 + \beta_2)x - \int_0^x \mu_2(x) dx \right] \quad (5.5.15)$$

$\sum_{n=1}^{\infty} z^n$ *(5.5.7) + z^* (5.5.8) and using (5.5.13), it follows that

$$(s + \lambda + \lambda z + \beta_1)R^*(s, z) = \alpha_1 S^*(s, z) \quad (5.5.16)$$

$\sum_{n=2}^{\infty} z^{n+1}$ *(5.5.9) + z^2 *(5.5.10)+ z^* (5.5.3) and using (5.5.13), it further follows that

$$zW^*(0, s, z) + (s + \lambda + \alpha_2)zW_0^*(s) = \int_0^{\infty} W^*(x, s, z)\mu_1(x)dx + \beta_2 zS^*(s, z) + \beta_1 zR^*(s, z) + \lambda^2 zW_0^*(s) + z \quad (5.5.17)$$

$\sum_{n=2}^{\infty} z^{n+1}$ *(5.5.11) + z^2 *(5.5.12)+ z^* (5.5.6) and using (5.5.13), it results in

$$zS^*(0, s, z) + (s + \lambda + \alpha_1 + \beta_2)zS_0^*(s) = \int_0^{\infty} S^*(x, s, z)\mu_2(x)dx + \alpha_2 zW^*(s, z) + \lambda z^2 S_0^*(s) \quad (5.5.18)$$

Denoting $s + \lambda - \lambda z$ by η and using (5.5.14) in (5.5.17), and then integrating from 0 to ∞ , it is clear that

$$\frac{(\eta + \alpha_2)[z - \beta_1^*(\eta + \alpha_2)]}{1 - \beta_1^*(\eta + \alpha_2)} W^*(s, z) = \frac{(\eta + \alpha_2)(z - 1)\beta_1^*(\eta + \alpha_2)}{1 - \beta_1^*(\eta + \alpha_2)} W_0^*(s) + \beta_2 zS^*(s, z) + \beta_1 zR^*(s, z) + z \quad (5.5.19)$$

Using (5.5.15) in (5.5.18), and then integrating from 0 to ∞ , it follows that

$$\frac{(\eta + \alpha_1 + \beta_2)[z - \beta_2^*(\eta + \alpha_2 + \beta_2)]}{1 - \beta_2^*(\eta + \alpha_2 + \beta_2)} S^*(s, z) = \frac{(\eta + \alpha_1 + \beta_2)(z - 1)\beta_2^*(\eta + \alpha_2 + \beta_2)}{1 - \beta_2^*(\eta + \alpha_2 + \beta_2)} S_0^*(s) + \alpha_2 z W^*(s, z) \quad (5.5.20)$$

$$(\eta + \beta_1)R^*(s, z) = \alpha_1 S^*(s, z) \quad (5.5.21)$$

Using (5.4.20) and (5.5.21) and simplifying

$$\begin{aligned} K(s, z)W^*(s, z) &= \frac{(\eta + \alpha_2)(z - 1)\beta_1^*(\eta + \alpha_2)}{1 - \beta_1^*(\eta + \alpha_2)} * \frac{(\eta + \alpha_1 + \beta_2)[z - \beta_2^*(\eta + \alpha_1 + \beta_2)]}{1 - \beta_2^*(\eta + \alpha_2 + \beta_2)} * (\eta + \beta_1)W_0^*(s) \\ &+ \frac{z(z - 1)(\eta + \alpha_1 + \beta_2)}{1 - \beta_2^*(\eta + \alpha_1 + \beta_2)} * [\beta_2(\eta + \beta_1) + \alpha_1\beta_1][\beta_2^*(\eta + \alpha_1 + \beta_2)]S_0^*(s) \\ &+ \frac{z(\eta + \beta_1)(\eta + \alpha_1 + \beta_2)[z - \beta_2^*(\eta + \alpha_1 + \beta_2)]}{1 - \beta_2^*(\eta + \alpha_1 + \beta_2)} \end{aligned} \quad (5.5.22)$$

$$\begin{aligned} K(s, z)S^*(s, z) &= \frac{(\eta + \alpha_2)[z - \beta_2^*(\eta + \alpha_2)]}{1 - \beta_1^*(\eta + \alpha_2)} * \frac{(\eta + \beta_1)(\eta + \alpha_1 + \beta_2)(z - 1)\beta_2^*(\eta + \alpha_1 + \beta_2)}{1 - \beta_2^*(\eta + \alpha_1 + \beta_2)} * S_0^*(s) \\ &+ \alpha_2 z (\eta + \beta_1) \left[z + \frac{(\eta + \alpha_2)(z - 1)\beta_2^*(\eta + \alpha_1 + \beta_2)}{1 - \beta_1^*(\eta + \alpha_2)} * W_0^*(s) \right] \end{aligned} \quad (5.5.23)$$

$$\begin{aligned} K(s, z)R^*(s, z) &= \frac{\alpha_1(\eta + \alpha_2)[z - \beta_1^*(\eta + \alpha_2)]}{1 - \beta_1^*(\eta + \alpha_2)} * \frac{(\eta + \alpha_1 + \beta_2)(z - 1)\beta_2^*(\eta + \alpha_1 + \beta_2)}{1 - \beta_2^*(\eta + \alpha_1 + \beta_2)} * S_0^*(s) \\ &+ \alpha_1 \alpha_2 z \left[z + \frac{(\eta + \alpha_2)(z - 1)\beta_1^*(\eta + \alpha_2)}{1 - \beta_1^*(\eta + \alpha_2)} * W_0^*(s) \right] \end{aligned} \quad (5.5.24)$$

where

$$K(s, z) = (\eta + \beta_1) \left[\frac{(\eta + \alpha_2)(z - \beta_1^*(\eta + \alpha_2))}{1 - \beta_1^*(\eta + \alpha_2)} * \frac{(\eta + \alpha_1 + \beta_2)(z - \beta_2^*(\eta + \alpha_1 + \beta_2))}{1 - \beta_2^*(\eta + \alpha_1 + \beta_2)} - \alpha_2 \beta_2 z^2 \right] - \alpha_1 \alpha_2 \beta_1 z^2$$

Since $W^*(s, z)$ is a regular function and denominator of (5.5.22) i.e. $K(s, z)$ vanishes for some z in $|z| < 1$, the numerator also vanishes for the same value of z . Applying Rouché's theorem the unknown $W_0^*(s)$ and $S_0^*(s)$ can be determined. Hence $W^*(s, z), S^*(s, z)$ and $R^*(s, z)$ can be completely determined.

Special cases

If it is assumed that the service is in the normal working condition and that partial failure of the service channel is exponential, then

$$\beta_1^*(s + \lambda - \lambda z + \alpha_2) = \frac{\mu_1}{s + \lambda - \lambda z + \alpha_2 + \mu_1}$$

$$\beta_2^*(s + \lambda - \lambda z + \alpha_1 + \beta_2) = \frac{\mu_2}{s + \lambda - \lambda z + \alpha_1 + \beta_2 + \mu_2}$$

Consequently equations (5.5.22) to (5.5.24) would become

$$\begin{aligned} K_1(s, z)W^*(s, z) &= [(\eta + \alpha_1 + \beta_1 + \mu_1)z - \mu_2]^*(\eta + \beta_1)^*[z + (z - 1)\mu_1 W_0^*(s)] \\ &\quad + z(z - 1)^*[\beta_2(\eta + \beta_1) + \alpha_1 \beta_1]^* \mu_2 S_0^*(s) \end{aligned} \quad (5.5.25)$$

$$\begin{aligned} K_1(s, z)S^*(s, z) &= \alpha_2 z(\eta + \beta_1)^*[z + \mu_1(z - 1)W_0^*(s)] \\ &\quad + [(\eta + \alpha_2 + \mu_1)z - \mu_1]^*(\eta + \beta_1)\mu_2(z - 1)S_0^*(s) \end{aligned} \quad (5.5.26)$$

$$K(s, z)R^*(s, z) = \alpha_1 \alpha_2 z[z + \mu_1(z - 1)W_0^*(s)] + \alpha_2 [(\eta + \alpha_2 + \mu_1)z - \mu_1]\mu_2(z - 1)S_0^*(s) \quad (5.5.27)$$

where

$$K_1(s, z) = (s + \lambda - \lambda z + \beta_1) * [(s + \lambda - \lambda z + \alpha_2) - \mu_1] *$$

$$[(s + \lambda - \lambda z + \alpha_1 + \beta_2 + \mu_2)z - \mu_2] - \alpha_2 \beta_2 z^2 - \alpha_1 \alpha_2 \beta_1 z^2$$

Since $W^*(s, z)$ is a regular function $W_0^*(s)$ and $S_0^*(s)$ can be determined as before.

5.6 Steady state solution

In taking the steady state probability corresponding to $W_n(t)$, $S_n(t)$ $R_n(t)$ as W_n , S_n , R_n , and the corresponding probability generations as $W(z)$, $S(z)$ and $R(z)$ by using the Tauberian property (see Widder [47]).

$$\lim_{s \rightarrow 0} s f^*(s) = \lim_{t \rightarrow \infty} f(t)$$

if the limit on the right exists.

The steady state solution can be obtained from (5.5.22), (5.5.23) and (5.5.24) as

$$K(z)W(z) = \frac{(\eta_1 + \alpha_2)(z-1)\beta_1(\eta_1 + \alpha_2)}{1 - \beta_1(\eta_1 + \alpha_2)} * \frac{(\eta_1 + \alpha_1 + \beta_2) * [z - \beta_2(\eta_1 + \alpha_1 + \beta_2)]}{1 - \beta_2(\eta_1 + \alpha_1 + \beta_2)} * (\eta_1 + \beta_1)W_0$$

$$+ \frac{z(z-1) * (\eta_1 + \alpha_1 + \beta_2)}{1 - \beta_1(\eta_1 + \alpha_1 + \beta_2)} * [\beta_2(\eta_1 + \beta_1) + \alpha_1 \beta_1] * S_0$$

(5.6.1)

$$K(z)S(z) = \alpha_2 z (\eta_1 + \beta_1) * \frac{(\eta_1 + \alpha_2)(z-1)\beta_1(\eta_1 + \alpha_2)}{1 - \beta_1(\eta_1 + \alpha_2)} * W_0 + \frac{(\eta_1 + \alpha_2)[z - \beta_1(\eta_1 + \alpha_2)]}{1 - \beta_1(\eta_1 + \alpha_2)}$$

$$* \frac{(\eta_1 + \alpha_1 + \beta_2)(z-1)\beta_2(\eta_1 + \alpha_1 + \beta_2)}{1 - \beta_2(\eta_1 + \alpha_1 + \beta_2)} * (\eta_1 + \beta_1)S_0$$

(5.6.2)

$$\begin{aligned}
 K(z)R(z) = & \alpha_1 * \frac{(\eta_1 + \alpha_2)[z - \beta_1(\eta_1 + \alpha_2)]}{1 - \beta_1(\eta_1 + \alpha_2)} * \frac{(\eta_1 + \alpha_1 + \beta_2)(z-1)\beta_2(\eta_1 + \alpha_1 + \beta_2)}{1 - \beta_2(\eta_1 + \alpha_1 + \beta_2)} * S_0 \\
 & + \alpha_1 \alpha_2 z * \frac{(\eta_1 + \alpha_2)(z-1)\beta_1(\eta_1 + \alpha_2)}{1 - \beta_1(\eta_1 + \alpha_2)} * W_0
 \end{aligned}
 \tag{5.6.3}$$

where $\eta_1 = \lambda - \lambda z$ and

$$K(z) = (\eta_1 + \beta_1) * \left[\frac{(\eta_1 + \alpha_2)[z - \beta_1(\eta_1 + \alpha_2)]}{1 - \beta_1(\eta_1 + \alpha_2)} * (\eta_1 + \alpha_1 + \beta_2) * \right. \\
 \left. \frac{[z - \beta_2(\eta_1 + \alpha_1 + \beta_2)]}{1 - \beta_2(\eta_1 + \alpha_1 + \beta_2)} - \alpha_2 \beta_2 z^2 \right] - \alpha_1 \alpha_2 \beta_1 z^2$$

As earlier, applying Rouché's theorem, the two unknowns W_0 and S_0 may be determined.

5.7 Some special cases

Case 1

If the service times in the normal working mode and the partial failure mode are exponential, then

$$\beta_1(\lambda - \lambda z + \alpha_2) = \frac{\mu_1}{\lambda - \lambda z + \alpha_2 + \mu_1}$$

$$\beta_2(\lambda - \lambda z + \alpha_1 + \beta_2) = \frac{\mu_2}{\lambda - \lambda z + \alpha_1 + \beta_2 + \mu_2}$$

Therefore equations (5.6.1), (5.6.2) and (5.6.3) become

$$\begin{aligned}
 K_1(z)W(z) = & (z-1) * [(\eta_1 + \alpha_1 + \beta_2 + \mu_2)z - \mu_2] * (\eta_1 + \beta_1)\mu_1 W_0 \\
 & + z(z-1) * [\beta_2(\eta_1 + \beta_1) + \alpha_1 \beta_1] * \mu_2 S_0
 \end{aligned}
 \tag{5.6.4}$$

$$K_1(z)S(z) = \alpha_2 z(\eta_1 + \beta_1)^* (z-1)\mu_1 W_0 + (\eta_1 + \beta_1)^* [(\eta_1 + \alpha_2 + \mu_1)z - \mu_1]^* (z-1)\mu_2 S_0$$

(5.6.5)

$$K_1(z)R(z) = \alpha_1 \alpha_2 z(z-1)\mu_1 W_0 + \alpha_1 [(\eta_1 + \alpha_2 + \mu_1)z - \mu_1]^* (z-1)\mu_2 S_0$$

(5.6.6)

where

$$K_1(z) = (\eta_1 + \beta_2)^* [(\eta_1 + \alpha_2 + \mu_1)z - \mu_1]^* ((\eta_1 + \alpha_1 + \beta_2 + \mu_2)z - \mu_2) - \alpha_2 \beta_2 z^2 - \alpha_1 \alpha_2 \beta_1 z^2$$

The two unknowns W_0 and S_0 can be determined as before

Case 2

If there is no failure at all, then $\alpha_1 = \alpha_2 = 0$. Using this in (5.6.1) to (5.6.3), it follows that

$$W(z) = \frac{(z-1)\beta_1(\lambda - \lambda z)}{z - \beta_1(\lambda - \lambda z)} W_0$$

$$S(z) = \frac{(z-1)\beta_2(\lambda - \lambda z)}{z - \beta_2(\lambda - \lambda z)} S_0$$

$$R(z) = 0$$

The identical forms of $W(z)$ and $S(z)$ confirm that the service time distributions in the normal working condition and the partial failure mode are the same when there is no failure at all.

5.8 Concluding remarks

At this juncture of the modelling process one may admire the modelling elegance achieved despite the attendant intricacy of applications in practice.