CHAPTER 4

AN M/M/1 QUEUEING SYSTEM WITH BATCH ARRIVALS OF VARYING SIZE, SERVICE OF FIXED BATCH SIZE AND TWO MODES OF FAILURE OF SERVICE FACILITY

4.1 Introduction

In many industrial processes, the service is interrupted because of the occurrence of breakdown in the facility that provides the service. The entities will not be serviced unless the facility is repaired. The server if human, may be in need of rest from time to time (Yadavalli *et al* [44]) or if non-human may be subject to two modes of failure, partial or total. That is, when the service facility is in partial failure mode, it gives service with a lower rate than in normal operating conditions. Various authors have analysed queueing systems where the service facility is subject to two modes of failure (Madan [49], Jain and Sharma [50], Reddy [51], and Sridharan and Jayashree [52]). Queueing systems with two modes of failure and arrivals and services in batches have not been considered so far. Such types of service interruptions are common in industry, factories, telephone booths and in operation of mechanical devices such as electronic computers, etc. In this chapter an M/M/1 queueing system is considered where the service facility is subject to two modes of failure, arrivals are in batches of varying size and service is rendered for batches of fixed size.

4.2 Model description

In this model units arrive at the system in batches of varying size and batches are pre-ordered for service purposes. The service of units is rendered in batches of fixed size and the service times of successive batches are distributed exponentially by a single server with rate μ_1 in normal working condition and at a slower rate μ_2 ,$(\mu_2 < \mu_1)$ in case of partial failure of the service channel.

One of the underlying assumptions about the repair process is that it starts instantaneously. If the service channel repair in the partial failure mode is complete, the unit enters the normal working mode; otherwise it goes to the failure mode.

After the repair of the service channel in total failure is complete, the unit goes directly to the normal working mode without passing through the partial failure mode. The repair times of failure modes, and the failure times are exponentially distributed with different derivatives.

4.3   Assumptions and notation

The system may be described as follows:

1.   Entities arrive in batches in varying size. Let $\lambda c_i dt (i = 1,2,3...,k)$ denote the probability that a batch of *i* entities arrives in a small interval of time *dt*, where $0 \leq c_i < 1$ and $\sum_{i=1}^{k} c_i = 1$ and $\lambda > 0$ is the mean arrival rate of batches which are pre-ordered for service purposes.

2.   The service of entities is rendered in batches of fixed size b, (b$\geq$1) and the service times of successive batches are distributed exponentially with mean service time $\dfrac{1}{\mu_1}$, (μ_1>0) and $\dfrac{1}{\mu_2}$, (μ_2>0) when the service channel is in the normal and partial failure mode respectively.

3. $\alpha_1(\alpha_2)dt$ is the first order probability that a total (partial) failure occurs during a short interval of time *dt*.

4. $\beta_1(\beta_2)dt$ is the first order probability of completion of a repair of total (partial) failure during a short interval of time *dt*.

5. $W_n(t)$ is the joint probability that at time *t*, there are (n$\geq$ 0) entities in the queue when the service channel is in the normal working mode (i.e. excluding the batch of entities in service if any).

6. $S_n(t)$ is the joint probability that at time t. there are n entities (n$\geq$ 0) in the queue when the service channel is in partial failure mode(i.e. excluding the batch of entities in service if any).

7. $R_n(t)$ is the probability that at time t, there are (n$\geq$ 0) entities in the queue when the service channel is in total failure mode (i.e. excluding the batch of entities in service if any)

8.   Q(t) is the probability that at time t, there are no entities either in service or in the queue and the service channel is in normal working mode, and though operative, is idle

9.   F(t) is the probability that at time t, there are no entities either in service or in the queue, and the service channel is in partial failure mode, and though partially operative, is idle.

10. $P_n(t)$ is the probability that there are $(n \geq 0)$ entities in the queue irrespective of the state of the service channel and that $P_n(t)=W_n(t)+S_n(t)+R_n(t)$.

11.   If repair in the partial failure mode is in the process of being completed, the system will not enter the total failure mode.

4.4   Equations describing the system

Using probability arguments, the following difference-differential equations are obtained:

$$\frac{d}{dt}W_n(t)+(\lambda+\mu_1+\alpha_2)W_n(t)=\sum_{i=1}^{n}\lambda c_i W_{n-i}(t)+\mu_1 W_{n+b}(t)+\beta_1 R_n(t)+\beta_2 S_n(t); \quad n>0,$$

(4.4.1)

$$\frac{d}{dt}W_0(t)+(\lambda+\mu_1+\alpha_2)W_0(t)=\lambda Q(t)+\mu_1\sum_{k=1}^{b}W_k(t)+\beta_1 R_0(t)+\beta_2 S_0(t),$$

(4.4.2)

$$\frac{d}{dt}S_n(t)+(\lambda+\mu_2+\alpha_1+\beta_2)S_n(t)=\sum_{i=1}^{n}\lambda c_i S_{n-i}(t)+\mu_2 S_{n+b}(t)+\alpha_2 W_n(t); \quad n>0,$$

(4.4.3)

$$\frac{d}{dt}S_0(t)+(\lambda+\mu_2+\alpha_1+\beta_2)S_0(t)=\mu_2\sum_{k=1}^{b}S_k(t)+\alpha_2 W_0(t)+\lambda F(t),$$

(4.4.4)

$$\frac{d}{dt}R_n(t)+(\lambda+\beta_1)R_n(t)=\sum_{i=1}^{n}\lambda c_i R_{n-i}(t)+\alpha_1 S_n(t)$$

(4.4.5)

$$\frac{d}{dt}R_0(t)+(\lambda+\beta_1)R_0(t)=\alpha_1 S_0(t)$$

(4.4.6)

$$\frac{d}{dt}Q(t) + \lambda Q(t) = \mu_1 W_0(t)$$

(4.4.7)

$$\frac{d}{dt}F(t) + \lambda F(t) = \mu_2 S_0(t)$$

(4.4.8)

It is assumed that the system initially starts when there are m entities in the queue and the service channel is in the normal working condition so that the initial conditions are

$W_n(0) = \delta_{n,m}$ where

$$\delta_{n,m} = \left\{ \begin{array}{l} 1 \text{ if } n = m \\ 0 \text{ if } n \neq m \end{array} \right\}$$

$$S_n(0) = 0, \forall n \geq 0$$

$$R_n(0) = 0, \forall n \geq 0$$

(4.4.9)

4.5   Time dependant solution

Let $f^*(s)$ be the Laplace transform of *f(t)*. Taking the Laplace transform of equations (4.4.1)-(4.4.8) and using (4.4.9), it follows that:

$$(s + \lambda + \mu_1 + \alpha_2)W_n^*(s) = \delta_{n,m} + \sum_{i=1}^{n} \lambda c_i W_{n-i}^*(s) + \mu_1 W_{n+b}^*(s) + \beta_1 R_n^*(s) + \beta_2 S_n^*(s) \text{, n>0}$$

(4.5.1)

$$(s + \lambda + \mu_1 + \alpha_2)W_0^*(s) = \delta_{0,m} + \lambda Q^*(s) + \mu_1 \sum_{k=1}^{b} W_k^*(s) + \beta_1 R_0^*(s) + \beta_2 S_0^*(s)$$

(4.5.2)

$$(s + \lambda + \mu_2 + \alpha_2 + \beta_2)S_n^*(s) = \sum_{i=1}^{n} \lambda c_i S_{n-i}^*(s) + \mu_2 S_{n+b}^*(s) + \alpha_2 W_n^*(s)$$

(4.5.3)

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$$(s + \lambda + \mu_2 + \alpha_1 + \beta_2)S_0^*(s) = \mu_2 \sum_{k=1}^{b} S_k^*(s) + \alpha_2 W_0^*(s) + \lambda F^*(s)$$

(4.5.4)

$$(s + \lambda + \beta_1)R_n^*(s) = \sum_{i=1}^{n} \lambda c_i R_{n-i}^*(s) + \alpha_1 S_n^*(s)$$

(4.5.5)

$$(s + \lambda + \beta_1)R_0^*(s) = \alpha_1 S_0^*(s)$$

(4.5.6)

$$(s + \lambda)Q^*(s) = \mu_1 W_0^*(s)$$

(4.5.7)

$$(s + \lambda)F^*(s) = \mu_2 S_0^*(s)$$

(4.5.8)

The following probability generating functions are defined as follows:

$$W^*(s, z) = \sum_{n=0}^{\infty} W_n^*(s)z^n$$

$$S^*(s, z) = \sum_{n=0}^{\infty} S_n^*(s)z^n$$

$$R^*(s, z) = \sum_{n=0}^{\infty} R_n^*(s)z^n$$

$$C(z) = \sum_{i=1}^{\infty} c_i z^i$$

(4.5.9)

$\sum_{z=1}^{\infty} z^{n+b} * (4.5.1) + z^b * (4.5.2)$ and using (4.5.9), it follows that

$$\left[(\eta + \mu_1 + \alpha_2)z^b - \mu_1\right]W^*(s, z) = z^{m+b} + \mu_1 \sum_{i=1}^{b}(z^b - z^i)W_i^*(s) - \mu_1 W_0^*(s)$$
$$+ \beta_1 z^b R^*(s, z) + \beta_2 z^b S^*(s, z) + \lambda z^b Q^*(s)$$

(4.5.10)

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$\sum_{z=1}^{\infty} z^{n+b} * (4.5.3) + z^b * (4.5.4)$ and using (4.5.9), the result is

$$\left[(\eta + \alpha_1 + \beta_2 + \mu_2) z^b - \mu_2 \right] S^*(s,z) = \mu_2 \sum_{i=1}^{b} (z^b - z^i) S_i^*(s) - \mu_2 S_0^*(s)$$
$$+ \alpha_2 z^b W^*(s,z) + \lambda z^b F^*(s)$$

(4.5.11)

$\sum_{z=1}^{\infty} z^n * (4.5.5) + (4.5.6)$ and using (4.5.9), it follows that

$$(\eta + \beta_1) R^*(s,z) = \alpha_1 S^*(s,z)$$

(4.5.12)

where $\eta = s + \lambda - \lambda C(z)$

Simplification of (4.5.10),(4.5.11) and (4.5.12) yields

$$K(s,z) W^*(s,z) = (\eta + \beta_1) \left[(\eta + \mu_2 + \alpha_1 + \beta_2) z^b - \mu_2 \right] *$$
$$\left[z^{m+b} + \mu_1 \sum_{i=1}^{b} (z^b - z^i) W_i^*(s) + \lambda Q^*(s) z^b - \mu_1 W_0^*(s) \right] +$$
$$z^b \left[(n + \beta_1) \beta_2 + \alpha_1 p_1 \right] *$$
$$\left[\mu_2 \sum_{i=1}^{b} (z^b - z^i) S_i^*(s) + \lambda F^*(s) z^b - \mu_2 S_0^*(s) \right]$$

(4.5.13)

$$K(s,z) S^*(s,z) = (\eta + \beta_1) \left[\begin{array}{l} \alpha_2 z^b \left(z^{m+b} + \mu_1 \sum_{i=1}^{b} (z^b - z^i) W_i^*(s) + \lambda Q^*(s) z^b - \mu_1 W_0^*(s) \right) \\ + ((\eta + \mu_1 + \alpha_2) z^b - \mu_1) * \left(\mu_2 \sum_{i=1}^{b} (z^b - z^i) S_i^*(s) \right) + \lambda F^*(s) z^b - \mu_2 S_0^*(s) \end{array} \right]$$

(4.5.14)

$$K(s,z) R^*(s,z) = \alpha_1 \left[\alpha_2 z^b \left(z^{m+b} + \mu_1 \sum_{i=1}^{b} (z^b - z^i) W_i^*(s) + \lambda Q^*(s) z^b - \mu_1 W_0^*(s) \right) \right]$$
$$+ \left[((\eta + \mu_1 + \alpha_2) z^b - \mu_1) \left(\mu_2 \sum_{i=1}^{b} (z^b - z^i) \right) S_i^*(s) + \lambda F^*(s) z^b - \mu_2 S_0^*(s) \right]$$

(4.5.15)

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$$(s + \lambda)Q^*(s) = \mu_1 W_0^*(s)$$

(4.5.16)

$$(s + \lambda)F^*(s) = \mu_2 S_0^*(s)$$

(4.5.17)

where

$$K(s,z) = (\eta + \beta_1)\left[\left((\eta + \mu_1 + \alpha_2)z^b - \mu_1\right)\left((\eta + \mu_2 + \alpha_1 + \beta_2)z^b + \mu_2\right) - \alpha_2\beta_2 z^{2b}\right] - \alpha_1\alpha_2\beta_1 z^{2b}$$

$$f(z) = (s + \lambda - \lambda C(z) + \beta_1)\left[\begin{array}{l}\left((s + \lambda - \lambda C(z) + \mu_1 + \alpha_2)z^b - \mu_1\right)* \\ \left((s + \lambda - \lambda C(z) + \mu_2 + \alpha_1 + \beta_2)z^b - \mu_2\right) - \alpha_2\beta_2 z^{2b}\end{array}\right]$$

$$g(z) = \alpha_1\alpha_2\beta_1 z^{2b}$$

For $|z| = 1$

$$|f(z)| = \left|\begin{array}{l}(s + \lambda - \lambda C(z) + \beta_1)\left[\left((s + \lambda - \lambda C(z) + \mu_1 + \alpha_2)z^b - \mu_1\right)\right. \\ \left.\left((s + \lambda - \lambda C(z) + \mu_2 + \alpha_1 + \beta_2)z^b - \mu_2\right) - \alpha_2\beta_2 z^{2b}\right|\end{array}\right.$$

$$= \left|(s + \lambda - \lambda C(z) + \beta_1)\right|$$
$$\left|\left[\left((s + \lambda - \lambda C(z) + \mu_1 + \alpha_2)z^b - \mu_1\right)\left((s + \lambda - \lambda C(z) + \mu_2 + \alpha_1 + \beta_2)z^b - \mu_2\right) - \alpha_2\beta_2 z^{2b}\right|\right.$$

$$\geq \left|s + \lambda + \lambda C(z) + \beta_1\right|\left\|\begin{array}{l}\left((s + \lambda - \lambda C(z) + \mu_1 + \alpha_2)|z^b| - \mu_1\right) \\ \left((s + \lambda - \lambda C(z) + \mu_2 + \alpha_2 + \beta_2)|z^b| - \mu_2\right) - \alpha_2\beta_2|z^{2b}|\end{array}\right\|$$

$$= \left|s + \lambda - \lambda C(z) + \beta_1\right|\left|\begin{array}{l}(s + \lambda - \lambda C(z) + \mu_1 + \alpha_2) - \mu_1) \\ \left((s + \lambda - \lambda C(z) + \mu_2 + \alpha_2 + \beta_2) - \mu_2\right) - \alpha_2\beta_2\end{array}\right|$$

$$\geq \left|s + \lambda + \beta_1 - \lambda\|C(z)\|\right|\left|\begin{array}{l}\left((s + \lambda + \mu_1 + \alpha_2 - \lambda|C(z)|) - \mu_1\right) \\ \left((s + \lambda + \mu_2 + \alpha_1 + \beta_2 - \lambda|C(z)|) - \mu_2\right) - \alpha_2\beta_2\end{array}\right|$$

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$$\geq \left| s + \lambda + \beta_1 - \lambda \right| \left| \begin{matrix} (s + \lambda + \mu_1 + \alpha_2 - \lambda - \mu_1) \\ (s + \lambda + \mu_2 + \alpha_1 + \beta_2 - \lambda - \alpha_2) - \alpha_2 \beta_2 \end{matrix} \right|$$

$$= \left| s + \beta_1 \right| \left| (s + \alpha_2)(s + \alpha_1 + \beta_2) - \alpha_2 \beta_2 \right|$$

($\because$ *Re(s)>0*

$$\geq \left| \beta_1 \right| \left| \alpha_2 (\alpha_1 + \beta_2) - \alpha_2 \beta_2 \right|$$

$$\geq \left| \beta_1 \right| \left| \alpha_1 \alpha_2 \right|$$

$$= \alpha_1 \alpha_2 \beta_1$$

For $\left| z \right| = 1$

$$\left| g(z) \right| = \alpha_1 \alpha_2 \beta_1 \left| z^{2b} \right| = \alpha_1 \alpha_2 \beta_2$$

$$\therefore \left| f(z) \right| > \left| g(z) \right| \text{ on } \left| z \right| = 1$$

Since *f(z)* and *g(z)* are differentiable inside and on the contour $\left| z \right| = 1$ and $\left| f(z) \right| > \left| g(z) \right|$ on $\left| z \right| = 1$, *f(z)-g(z)*, i.e. the denominator of equations (4.5.13),(4.5.14) and (4.5.15) have the same number of zeros inside $\left| z \right| = 1$ as that of *f(z)* by Rouche's theorem. The zeros are given by the equations

$$s + \lambda - \lambda C(z) + \beta_1 = 0 \text{ and}$$

$$\left[(s + \lambda - \lambda C(z) + \mu_1 + \alpha_2)z^b - \mu_1 \right] * \left[(s + \lambda - \lambda C(z) + \mu_2 + \alpha_1 + \beta_2)z^b - \mu_2 \right] - \alpha_2 \beta_2 z^{2b} = 0$$

The later equation has *2b* zeros inside $\left| z \right| = 1$. Thus the denominator *K(s,z)* has *2b* zeros inside $\left| z \right| = 1$. Since $\overline{W}(s, z)$ is regular inside the contour $\left| z \right| = 1$, the numerator must vanish from the zeros of the denominator, as such there are *2b* linear equations in *2b* unknowns, $\overline{W}_r^*(s), r = 1,2...b$, $\overline{S}_r^*(s), r = 1,2...b$. These together with (4.5.16) and (4.5.17) are sufficient to determine all the

2b+2 unknowns. $\overline{W}^*(s,z)$ and hence $\overline{S}^*(s,z)$ and $\overline{R}^*(s,z)$ can be completely determined.

4.6 The steady state solution

If W_n, S_n, R_n, Q and F represent the respective steady state probabilities corresponding to $W_n(t)$, $S_n(t)$ $R_n(t)$, Q(t) and F(t), and correspondingly W(z), S(z) and R(z) are the probability generating functions of W_n, S_n, R_n, then the steady state solution can be obtained by using the Tauberian property (see Widder [47]).

$\lim\limits_{s\to 0} sf^*(s) = \lim\limits_{t\to\infty} f(t)$ if the limit on the right hand side exists.

Thus equations (4.5.13),(4.5.14) and (4.5.15) yield

$$W(z) = \frac{N_1}{D_1} \qquad (4.6.1)$$

$$S(z) = \frac{N_2}{D_1} \qquad (4.6.2)$$

$$R(z) = \frac{N_3}{D_1} \qquad (4.6.3)$$

$$\lambda Q = \mu_1 W_0 \qquad (4.6.4)$$

$$\lambda F = \mu_2 S_0 \qquad (4.6.5)$$

where

$$N_1 = (\lambda - \lambda C(z) + \beta_1)((\lambda - \lambda C(z) + \mu_2 + \alpha_1 + \beta_2)z^b - \mu_2)\left[\mu_1 \sum_{i=1}^{b}(z^b - z^i)W_i + \lambda z^b Q - \mu_1 W_0\right]$$

$$+ z^b[(\lambda - \lambda C(z) + \beta_1)\beta_2 + \alpha_1 \beta_1]*\left[\mu_2 \sum_{i=1}^{b}(z^b - z^i)S_i + \lambda z^b F - \mu_2 S_0\right]$$

$$N_2 = (\lambda - \lambda C(z) + \beta_1)\left[\begin{array}{l} \alpha_2 z^b (\mu_1 \sum_{i=1}^{b}(z^b - z^i)W_i + \lambda z^b Q - \mu_1 W_0) + ((\lambda - \lambda C(z) + \mu_1 + \alpha_2)z^b - \mu_1)* \\ \left(\mu_2 \sum_{i=1}^{b}(z^b - z^i)S_i + \lambda F z^b - \mu_2 S_0\right) \end{array}\right]$$

$$N_3 = \alpha_1 \left[\begin{array}{l} \alpha_2 z^b (\mu_1 \sum_{i=1}^{b}(z^b - z^i)W_i + \lambda z^b Q - \mu_1 W_0) + ((\lambda - \lambda C(z) + \mu_1 + \alpha_2)z^b - \mu_1) * \\ \left(\mu_2 \sum_{i=1}^{b}(z^b - z^i)S_i + \lambda F z^b - \mu_2 S_0 \right) \end{array} \right]$$

and

$$D_1 = (\lambda - \lambda C(z) + \beta_1)\left[\begin{array}{l} ((\lambda - \lambda C(z) + \mu_1 + \alpha_2)z^b - \mu_1)z^b - \mu_1) * \\ ((\lambda - \lambda C(z) + \mu_2 + \alpha_1 + \beta_2)z^b - \mu_2)) - \alpha_2 \beta_2 z^{2b} \end{array} \right] - \alpha_1 \alpha_2 \beta_1 z^{2b}$$

As earlier the unknowns can be determined by applying Rouche's theorem.

4.7   Some special cases

Case 1

If there are single arrivals and single departures, then C1=1, Ci=0 for i$\neq$1, C(z)=z and b=1. Substituting these values in equations (4.6.1),(4.6.2) and (4.6.3) and using (4.6.4), it follows that

$$W(z) = \frac{(z-1)[((\lambda - \lambda z + \mu_2 + \alpha_1 + \beta_2)z - \mu_2)(\lambda - \lambda z + \beta_1) * \mu_1 W_0 + \alpha_1 \beta_1 z \mu_2 S_0]}{D_2}$$

(4.6.6)

$$S(z) = \frac{(z-1)(\lambda - \lambda z + \beta_1)[\alpha_2 z \mu_1 W_0 + ((\lambda - \lambda z + \mu_1 + \alpha_2)z - \mu_1) * \mu_2 S_0]}{D_2}$$

(4.6.7)

$$R(z) = \frac{\alpha_1 (z-1)[\alpha_2 z \mu_1 W_0 + ((\lambda - \lambda z + \mu_1 + \alpha_2)z - \mu_1) * \mu_2 S_0]}{D_2}$$

(4.6.8)

where

$$D_2 = (\lambda - \lambda z + \beta_1)[((\lambda - \lambda z + \mu_1 + \alpha_2)z - \mu_1)((\lambda - \lambda z + \mu_2 + \alpha_1 + \beta_2)z - \mu_2) - \alpha_2 \beta_2 z^2] - \alpha_1 \alpha_2 \beta_1 z^2$$

As earlier, the two unknowns can be determined by applying Rouche's theorem.

To determine the unknown Q+F, one may use the normalising condition

W(1)+S(1)+R(1)+Q+F=1

For z=1, the equations (4.6.6),(4.6.7) and (4.5.8) are indeterminate.

Hence using L'Hospital rule one obtains

$$W(1) = \frac{\lambda \beta_2 (\alpha_1 + \beta_2)(Q + F)}{D} \tag{4.6.9}$$

$$S(1) = \frac{\lambda \alpha_2 \beta_1 (Q + F)}{D} \tag{4.6.10}$$

$$R(1) = \frac{\lambda \alpha_1 \alpha_2 (Q + F)}{D} \tag{4.6.11}$$

where

$$D = \beta_1(\mu_1(\alpha_1 + \beta_2) + \alpha_2 \mu_2) + \lambda(\alpha_2 \beta_1 + \alpha_2 \beta_1 + \beta_1 \beta_2 + \alpha_1 \alpha_2)$$

and using (4.6.9)-(4.6.11), and simplifying

$$Q + F = 1 - \frac{\lambda(\alpha_1 \beta_1 + \alpha_2 \beta_1 + \beta_1 \beta_2 + \alpha_1 \alpha_2)}{\beta_1(\mu_1(\alpha_1 + \beta_2) + \mu_2 \alpha_2)} \tag{4.6.12}$$

This is the steady state probability that the system is in a working condition, but idle. Therefore, the utilization factor is:

$$\rho = \frac{\lambda(\alpha_1 \beta_1 + \alpha_2 \beta_1 + \beta_1 \beta_2 + \alpha_1 \alpha_2)}{\beta_1(\mu_1(\alpha_1 + \beta_2) + \mu_2 \alpha_2)} \tag{4.6.13}$$

and the steady state condition is given by $\rho < 1$.

Further, if there is no failure in the service channel, then setting $\alpha_1 = \alpha_2 = 0$ in (4.6.12) and (4.6.13), the result is

$$Q + F = 1 - \frac{\lambda}{\mu}$$

$$\rho = \frac{\lambda}{\mu_1} \text{ and } \rho < 1.$$

The system state probabilities are

$$W(1) = \frac{\lambda \beta_1 (\alpha_1 + \beta_2)}{\beta_1 (\mu_1 (\alpha_1 + \beta_2) + \mu_2 \alpha_2)}$$

$$S(1) = \frac{\lambda \alpha_2 \beta_1}{\beta_1 (\mu_1 (\alpha_1 + \beta_2) + \mu_2 \alpha_2)}$$

$$R(1) = \frac{\lambda \alpha_1 \alpha_2}{\beta_1 (\mu_1 (\alpha_1 + \beta_2) + \mu_2 \alpha_2)}$$

Case 2

In addition to the condition of Case 1, if the repair rates, service rates and failure rates are identical, then setting $\alpha_1 = \alpha_2 = \alpha$, $\beta_1 = \beta_2 = \beta$ and $\mu_1 = \mu_2 = \mu$ it follows that

$$\rho = \frac{\lambda (\alpha + \beta)^2}{\beta \mu (\beta + 2\alpha)}$$

and the steady state condition is

$$\frac{\lambda}{\mu} < \frac{\beta(\beta + 2\alpha)}{(\alpha + \beta)^2}$$

If P(z) denotes the queue length probability generating function irrespective of the state of system, then P(z)= W(z)+S(z)+R(z)

Further, if $E(N_q)$ denotes the expected number of entities in the queue, then

$$E(N_q) = \frac{d}{dz} P(z) \Big|_{z=1}$$

Denoting

$$\beta_1(\mu_1\alpha_1 + \mu_1\beta_2 + \mu_2\alpha_2) - \lambda(\alpha_1\alpha_2 + \alpha_1\beta_1 + \alpha_2\beta_1 + \beta_1\beta_2) \text{ by } \mathbf{X} \text{ and}$$

$$\lambda^2(\alpha_1 + \alpha_2 + \beta_1 + \beta_2) - \lambda(\mu_1\alpha_1 + \mu_2\alpha_2 + 2\alpha_1\alpha_2 + \mu_1\beta_2 + \mu_1\beta_1 + \mu_2\beta_1 + 2\alpha_1\beta_1 + 2\alpha_2\beta_1 + 2\beta_1\beta_2) + \beta_1\mu_1\mu_2 + \beta_1\mu_1\alpha_1 + \mu_1\beta_1\beta_2 + \beta_1\mu_2\alpha_2$$

by Y.

Then

$$E(N_q) = [X[Q(-\lambda^2(\alpha_1 + \alpha_2 + \beta_1 + \beta_2) + \lambda\alpha_1\alpha_2 + \lambda\mu_2\beta_1 + \lambda\alpha_1\beta_1 + \lambda\alpha_2\beta_1 + \lambda\beta_1\beta_2) +$$

$$F(-\lambda^2(\alpha_1 + \alpha_2 + \beta_1 + \beta_2) + \lambda\alpha_1\alpha_2 + \lambda\mu_1\beta_1 + \lambda\alpha_1\beta_1 + \lambda\alpha_2\beta_1 + \lambda\beta_1\beta_2 + \lambda\mu_1\alpha_1) -$$

$$(Q + F)(\lambda\alpha_1\alpha_2 + \lambda\alpha_1\beta_1 + \lambda\alpha_2\beta_1 + \lambda\beta_1\beta_2)Y]/X^2$$

(4.6.14)

Further, if there are no failures at all, setting $\alpha_1 = \alpha_2 = 0$ and $\mu_1 = \mu_2 = \mu$ in (4.6.14), the end result is

$$E(N_q) = \frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{\rho^2}{1 - \rho}$$

which is the expected number of entities in the queue for the M/M/1 queueing system (Saaty [53]).

4.8   Concluding remark

The model of the system considered in this chapter of the thesis emphasizes that the model is constructed mathematically in an advanced and elegant fashion. However it is suspected that its utility would be limited if it were to be used in practical applications as a result of complexity of the time dependant solution model of the system.