CHAPTER 3

A SINGLE CHANNEL QUEUEING MODEL WITH OPTIONAL SERVICE AND SERVICE INTERRUPTION

3.1 Introduction

One of the important characteristics of a queueing system is the service process. Entities in the system may be served individually or in batches. An arriving entity may not get satisfactory service rendered by the server. An intelligent entity may think of better service from the same server or may seek some other server (i.e. leaving the system unsatisfied). If the service given to an intelligent entity is not satisfactory and if it needs a second service, it has to join the end of the queue and wait for its turn of service. Either it may join the queue for the second service, or balk since it has waited too long. To satisfy such intelligent entities, the server can offer two kinds of service. Either an arriving entity can choose one of two servers before service starts. For example, a patient decides to undergo ordinary surgery or laparoscopic surgery; or a vehicle uses the existing road or the by-pass road. Or if an arriving entity is not satisfied by the first essential service, it can opt for the second optional service immediately. Else it can opt for the second optional service immediately. The former kind of service has been studied by Madan [16] and later by Madan [17] where there is no waiting capacity. However, in queueing systems where the server offers two services, one essential and the other optional, and interruption may take place; the system implies queueing models with service interruption which have been extensively studied in the past by Takagi [46]. Service interruption models with more than one service offered by a single server have not been considered so far. To fill the gap this chapter presents a Markov queueing model where the server offers two services, one essential and the other optional.

3.2 Model description

The server offers two services, one essential and the other optional. The essential service follows an exponential distribution and the optional service follows an arbitrary distribution. The server offers only one service at a time. The first service is essential for all entities while the second service is optional. In addition the service is interrupted for a random period whenever the system becomes empty. It is assumed that the duration of interruptions is independent and, identically randomly distributed and is independent of the arrival process and the service time. Therefore the system has three states, namely

(i) The operating state providing the first essential service.
(ii) The operating state providing the second optional service.
(iii) The state of interruption.

3.3 Assumptions and notation

λ : average arrival rate of entities.

μ : the average service rate of the server when offering essential service.

W_n (t): the joint probability that at time t, there are n>0 entities in the system and the server is providing essential service for the entities.

S_n (x, t): the joint probability that at time t, there are n>0 entities in the system with elapsed service time between R and R+dx and the server is offering optional service to the entities.

V_n (x, t): the joint probability that at time t there are n>0 entities in the system with elapsed interruption time lying between R and R+dx and the server is interrupted.

On completion of the regular (essential) service, an entity leaves the system with probability p and desires to have the second optional service with probability q; p+q=1.

$\mu_1(x)dx$: the first order probability that the optional service will be completed in the time interval x and x+dx given that the same was not completed before time x and is related to the density function B1(x) by the hazard function relation.

$$B_1(\mu) = \mu_1(x)e^{-\int_0^x \mu_1(x)dx}$$

$\alpha(x)dx$: The first order probability that the elapsed interruption will take place in time x and x+dx given that the same was not complete until time x and is related to the density function V (x) by the relation

$$V(x) = \alpha(x)e^{-\int_0^x \alpha(t)dt}$$

Equations governing the system are as follows:

$$\frac{d}{dt}W_n(t) = -(\lambda+\mu)W_n(t) + \lambda W_{n-1}(t) + p\mu W_{n+1}(t) + \int_0^\infty V_n(x,t)\alpha(x)dx + \int_0^\infty S_{n+1}(x,t)\mu(x)dx \quad ,n > 1 \tag{3.4.1}$$

$$\frac{d}{dt}W_1(t) = -(\lambda+\mu)W_1(t) + \lambda W_2(t) + \int_0^\infty V_1(x,t)\alpha(x)dx + \int_0^\infty S_2(x,t)\mu(x)dx \tag{3.4.2}$$

$$\frac{\partial}{\partial x}S_n(x,t) + \frac{\partial}{\partial t}S_n(x,t) = -(\lambda+\mu_1(x))S_n(x,t) + \lambda S_{n-1}(x,t) \quad ,n > 1 \tag{3.4.3}$$

$$\frac{\partial}{\partial x}S_1(x,t) + \frac{\partial}{\partial t}S_1(x,t) = -(\lambda+\mu_1(x))S_1(x,t) \tag{3.4.4}$$

$$\frac{\partial}{\partial x}V_n(x,t) + \frac{\partial}{\partial t}V_n(x,t) = -(\lambda+\alpha(x))V_n(x,t) + \lambda V_{n-1}(x,t) \quad ,n > 0 \tag{3.4.5}$$

$$\frac{\partial}{\partial x}V_0(x,t) + \frac{\partial}{\partial t}V_0(x,t) = -(\lambda+\alpha(x))V_0(x,t)$$

(3.4.6)

Subject to boundary conditions

$S_n(0,t) = v\mu W_n(t);$ n >0

(3.4.7)

$V_n(0,t) = 0;$ n >0

(3.4.8)

$$V_n(0,t) = \rho\mu W_1(t) + \int_0^\infty S_1(x,t)\mu(x)dx$$

(3.4.9)

Further, it is assumed that the system starts initially when there are k units in the system so that the initial conditions are:

$$W_n(0) = \delta_{nk} = \begin{cases} 1 \text{ if } n = k \\ 0 \text{ if } n \neq k \end{cases}$$

where δ_{nk} is Kronecker's delta function

$S_n(0) = 0$ for all n>0

$V_n = 0$ for all n$\geq$0

(3.4.10)

3.4 Time dependant solution

The Laplace transform of a function *f(t)* is defined as

$$f^*(s) = L\{f(t)\} = \int_0^\infty e^{-st} f(t) st, \qquad \text{Re(s)>0}$$

(3.5.1)

The Laplace transform of $\frac{d}{dt} f(t)$ is

$$L\left[\frac{d}{dt} f(t)\right] - sf^*(s) \quad f(0)$$

(3.5.2)

Using the Laplace transform of equations (3.4.1) to (3.4.9) and equations (3.4.10), (3.5.1) and (3.5.2) result in:

$$(s+\lambda+\mu)W_n^*(s) = \delta_{nk} + \lambda W_{n-1}^*(s) + p\mu W_{n+1}^*(s) + \int_0^\infty V_n^*(x,s)\alpha(x)dx + \int_0^\infty S_{n+1}^*(x,s)\mu_1(x)dx$$

n >1

(3.5.3)

$$(s+\lambda+\mu)W_1^*(s) = \delta_{1k} + p\mu W_2^*(s) + \int_0^\infty V_1^*(x,s)\alpha(x)dx + \int_0^\infty S_2^*(x,s)\mu_1(x)dx$$

(3.5.4)

$$\frac{\partial}{\partial x}S_n^*(x,s) + (s+\lambda+\mu_1(x))S_n^*(x,s) = \lambda S_{n-1}^*(x,s) \qquad ,n>1$$

(3.5.5)

$$\frac{\partial}{\partial x}S_1^*(x,s) + (s+\lambda+\mu_1(x))S_1^*(x,s) = 0$$

(3.5.6)

$$\frac{\partial}{\partial x}V_n^*(x,s) + (s+\lambda+\alpha(x))V_n^*(x,s) = \lambda V_{n-1}^*(x,s) \qquad ,n>0$$

(3.5.7)

$$\frac{\partial}{\partial x}V_0^*(x,s) + (s+\lambda+\alpha(x))V_0^*(x,s) = 0$$

(3.5.8)

Subject to boundary conditions:

$S_n^*(0,s) = q\mu W_n^*(s)$; ,n >0

(3.5.9)

$V_n^*(0,s) = 0$; ,n >0

(3.5.10)

$$V_0^*(0,s) = p\mu W_1^*(s) + \int_0^x S_1^*(x,s)\mu_1(x)dx$$

(3.5.11)

The following generating functions are defined;

$$W^*(s,z) = \sum_{n=1}^{\infty} W_n^*(s) z^n$$

$$S^*(x,s,z) = \sum_{n=1}^{\infty} S_n^*(x,s) z^n$$

$$S^*(0,s,z) = \sum_{n=1}^{\infty} S_n^*(0,s) z^n$$

$$S^*(s,z) = \int_0^{\infty} S^*(x,s,z)dx$$

$$V^*(x,s,z) = \sum_{n=0}^{\infty} V_n^*(x,s) z^n$$

$$V^*(0,s,z) = \sum_{n=1}^{\infty} V_n^*(0,s) z^n$$

$$V^*(s,z) = \int_0^{\infty} V^*(x,s,z)dx \qquad (3.5.12)$$

$\sum_{n-2}^{\infty} z^{n+1} *(3.5.3) + z^2 * (3.5.4)$ and using (3.5.12):

$$[(s+\lambda-\lambda z+\mu)z - p\mu]W^*(s,z) = z^{k+1} - z[p\mu W_1^*(s) + \int_0^{\infty} S_1^*(x,s)\mu_1(x)dx]$$

$$+ z\int_0^{\infty} V^*(x,s,z)\alpha(x)dx + \int_0^{\infty} S^*(x,s,z)\mu_1(x)dx$$

(3.5.13)

$\sum_{n=2}^{\infty} z^{n+1} *(3.5.5) + z * (3.5.6)$ and using (3.5.12):

$$\frac{\partial}{\partial x} S^*(x,s,z) - (s+\lambda-\lambda z+\mu_1(x))S^*(x,s,z) = 0$$

(3.5.14)

$\sum_{n=2}^{\infty} z^{n+1} *(3.5.7) + (3.5.8)$ and using (3.5.12):

$$\frac{\partial}{\partial} V^*(x,s,z) + (s+\lambda-\lambda z+\alpha(x))V^*(x,s,z) = 0$$

(3.5.15)

$\sum_{n=1}^{\infty} z^n$ * (3.5.9) gives

$$S^*(0,s,z) = q\mu W^*(s,z)$$

(3.5.16)

$\sum_{n=1}^{\infty} z^n$ * (3.5.10) + (3.5.11) gives

$$V^*(0,s,z) = p\mu W_1^*(s) + \int_0^{\infty} S_1^*(x,s)\mu_1(x)dx$$

(3.5.17)

Integrating the equations (3.5.14) and (3.5.15) from 0 to x, gives

$$S^*(x,s,z) = S^*(0,s,z)\exp[-(s+\lambda-\lambda z)x - \int_0^x \mu_1(t)]$$

(3.5.18)

$$V^*(x,s,z) = V^*(0,s,z)\exp[-(s+\lambda-\lambda z)x - \int_0^x \alpha(t)dt]$$

(3.5.19)

Integrating (3.5.8) from 0 to x, gives

$$V_0^*(x,s) = V_0^*(0,s)\exp[-(s+\lambda)x - \int_0^x \alpha(t)dt]$$

(3.5.20)

Integration of (3.5.20) from 0 to ∞ yields

$$V_0^*(s) = V_0^*(0,s)\frac{1-V^*(s+\lambda)}{s+\lambda}$$

(3.5.21)

From (3.5.11), (3.5.17) and (3.5.21):

$$V^*(0,s,z) = V_0^*(0,s) = \frac{(s+\lambda)V_0^*(s)}{1-V^*(s+\lambda)}$$

(3.5.22)

Using (3.5.16) in (3.5.18):

$$S^*(x,s,z) = q\mu W^*(s,z)\exp[-(s+\lambda-\lambda z)x - \int_0^x \mu_1(t)dt]$$

Integrating this from 0 to ∞, gives

$$S^*(s,z) = q\mu W^*(s,z)\frac{1-B_1^*(s+\lambda-\lambda z)}{s+\lambda-\lambda z}$$

(3.5.23)

using (3.5.22) in (3.5.19) and integrating from 0 to ∞

$$V^*(s,z) = \frac{(s+\lambda)V_0^*(s)}{1-V^*(s+\lambda)} * \frac{1-V^*(s+\lambda-\lambda z)}{s+\lambda-\lambda z}$$

(3.5.24)

using (3.5.17), (3.5.18), (3.5.19) and (3.5.22) in (3.5.13) and simplifying yields

$$[(s+\lambda-\lambda z+\mu)z - \mu(p+qB_1^*(s+\lambda-\lambda z)]W^*(s,z)$$

$$= z^{k+1} + \frac{(s+\lambda)V_0^*(s)}{1\quad V^*(s\quad \lambda)}[V^*(s+\lambda-\lambda z)-1]$$

Thus

$$W^*(s,z) = \frac{z^{k+1} + \dfrac{z(s+\lambda)V_0^*(s)}{1-V^*(s+\lambda)}[V^*(s+\lambda-\lambda z)-1]}{[(s+\lambda-\lambda z+\mu)z - \mu(p+qB_1^*(s+\lambda-\lambda z))]}$$

(3.5.25)

Since $W^*(s,z)$ is a regular function and the denominator of the right hand side vanishes for some z in $|z|<1$, the number at 1 also vanishes for the same value of z. Applying Rouche's theorem the only unknown $V_0^*(s)$ can be determined. Hence $S^*(s,z), W^*(s,z)$ and $V^*(s,z)$ can be completely determined.

3.5 Some special Cases

Case 1:

If the optional service is exponential then:

$$B_1^*(s+\lambda-\lambda z) = \frac{\mu_1}{c+\lambda-\lambda z+\mu_1}$$

Therefore (3.5.23) to (3.5.25) become

$$W^*(s,z)=\frac{(s+\lambda-\lambda z+\mu_1)\left[z^{k+1}+\dfrac{z(s+\lambda)V_0^*(s)}{1-V^*(s+\lambda)}[V^*(s+\lambda-\lambda z)-1]\right]}{[(s+\lambda-\lambda z+\mu)z-p\mu](s+\lambda-\lambda z+\mu_1)-q\mu\mu_1}$$

(3.5.26)

$$S^*(s,z)=\frac{q\mu W^*(s,z)}{s+\lambda-\lambda z+\mu_1}$$

(3.5.27)

$$V^*(s,z)=\frac{(s+\lambda)V_0^*(s)}{1-V^*(s+\lambda)}*\frac{1-V^*(s+\lambda-\lambda z)}{s+\lambda-\lambda z}$$

(3.5.28)

Case 2:

In addition to the condition of case 1, if there is no optional service and the server is offering the essential service only, then $p=1, q=0$. (3.5.26) to (3.5.28) become

$$W^*(s,z)=\frac{\left[z^{k+1}+\dfrac{z(s+\lambda)V_0^*(s)}{1-V^*(s+\lambda)}[V^*(s+\lambda-\lambda z)-1]\right]}{[(s-\lambda-\lambda z+\mu)z-\mu}$$

$$S^*(s,z)=0$$

$$V^*(s,z)=\frac{(s+\lambda)V_0^*(s)}{1-V^*(s+\lambda)}*\frac{1-V^*(s+\lambda-\lambda z)}{s+\lambda-\lambda z}$$

3.6 The steady state solution

Taking W_n, S_n and V_n as the respective steady state probabilities corresponding to $W_n(t)$, $S_n(t)$ and $V_n(t)$ and correspondingly W(z), S(z) and V(z) as the probability generating functions, then the steady state solution can be obtained by using the Tauberian theorem Widder [47].

$$\lim_{s \longrightarrow 0} sf^*(s) = \lim_{t \longrightarrow \infty} f(t)$$

If the limit on the right exists the equations (3.5.23) to (3.5.25) become

$$W(z) = \frac{\frac{\lambda z}{1-V(\lambda)}[V(\lambda-\lambda z)-1]V_0}{(\lambda-\lambda z+\mu)z-\mu(p+qB_1(\lambda-\lambda z))} \tag{3.6.1}$$

$$S(z) = q\mu W(z)\frac{1-B_1(\lambda-\lambda z)}{\lambda-\lambda z} \tag{3.6.2}$$

$$V(z) = \frac{\lambda V_0}{1-V(\lambda)} * \frac{1-V(\lambda-\lambda z)}{\lambda-\lambda z} \tag{3.6.3}$$

If P(z) is the probability generating function of the number of entities in the system irrespective of the state, then P(z)=W(z)+S(z)+V(z),

Thus :

$$P(z) = \frac{[1-V(\lambda-\lambda z)][p+qB_1(\lambda-\lambda z)]}{[p+qB_1(\lambda-\lambda z)]\mu - z(\lambda-\lambda z+\mu)} + \frac{V_0}{1-V(\lambda)} \tag{3.6.4}$$

Using the normalization condition P(1)=1,

$$V_0 = \frac{\mu-\lambda[1+q\mu E(B_1)]}{\lambda E(V)}[1-V(\lambda)]$$

where $E(B_1)$ is the expected optional service time and E(V) is the expected interruption time

Therefore (3.6.4) becomes

$$P(z) = \frac{[\mu - \lambda[1 + q\mu E(B_1)]}{\lambda E(V)} * \frac{[1 - V(\lambda - \lambda z)][p + qB_1(\lambda - \lambda z)]}{\mu(p + qB_1(\lambda - \lambda z) - z(\lambda - \lambda z + \mu))}$$

or

$$P(z) = \frac{\mu - \lambda[1 + q\mu E(B_1)][p + qB_1(\lambda - \lambda z)](1 - z)}{\mu(\rho + qB_1(\lambda - \lambda z) - z(\lambda - \lambda z + \mu)} * \frac{1 - V(\lambda - \lambda z)}{(\lambda - \lambda z)E(V)}$$

(3.6.5)

$$= \alpha\_(z) * P_{M/M/1}(z)$$

Where

$$\alpha\_(z) = \frac{1 - V(\lambda - \lambda z)}{(\lambda - \lambda z)E(V)}$$

$$P_{M/M/1}(z) = \frac{[\mu - \lambda[1 + q\mu E(B_1)][p + qB_1(\lambda - \lambda z)](1 - z)}{\mu(p + qB_1(\lambda - \lambda z) - z(\lambda - \lambda z + \mu)}$$

$\alpha\_(z)$ is the probability generating function of the number of entities which arrive before an arbitrary entity during an interruption period in which the arbitrary entity arrives (Fuhrmann [20]) and PM/M/1(z) is the probability generating function of the number of entities in the M/M/1 queueing system with additional optional service and

$$\delta = \frac{\lambda(1 + q\mu E(B_1))}{\mu}$$

3.7 Some special Cases

Case 1:

If there is no essential service (the server is offering only the optional service), then $p = 0, q = 1$ and $\frac{1}{\mu} \longrightarrow 0$ (3.6.5) becomes

$$P(z) = \frac{[1-\lambda E(B_1)]B_1(\lambda - \lambda z)(1-z)}{B_1(\lambda - \lambda z) - z} * \frac{1 - V(\lambda - \lambda z)}{(\lambda - \lambda z)E(V)} \tag{3.6.6}$$

$= \alpha\_(z) * P_{M/G/1}(z)$

which is the stochastic decomposition for the M/G/1 queueing system Takagi [46]

Case 2:

Suppose there is no additional optional service so that $\rho = 1,\ q = 0$ Then (3.6.5) becomes:

$$P(z) = \frac{\mu(1-\delta)(1-z)}{\mu - z(\lambda - \lambda z + \mu)} * \frac{1 - V(\lambda - \lambda z)}{(\lambda - \lambda z)E(V)} \tag{3.6.7}$$

where $\delta = \frac{1}{\mu}$

$P(z) = \alpha\_(z) * P_{M/M/1}(z)$

which is the stochastic decomposition for the M/M/1 queueing system by Takagi [46]

Case 3:

Suppose the additional service follows an exponential distribution. Then

$$B_1(\lambda - \lambda z) - \frac{\mu_1}{\lambda - \lambda z + \mu_1}$$

Thus (3.6.5) becomes

$$P(z) = \frac{[\mu - \lambda[1 + q\mu\mu_1)][\rho(\lambda - \lambda z + \mu_1) + q\mu_1]}{\mu[p(\lambda - \lambda z + \mu_1) + q\mu_1] - z(\lambda - \lambda z + \mu)(\lambda - \lambda z + \mu_1)} * \frac{1 - V(\lambda - \lambda z)}{\lambda E(V)}$$

(3.6.8)

Further if there is no additional optional service, then equation (3.6.8) leads to (3.6.7)

Case 4:

If E (N) denotes the expected number of entities in the system, then

$E(N) = \frac{d}{dz} P(z)$ at z=1

Thus

$$E(N) = \frac{\lambda E(V^2)}{qE(V)} + \frac{\lambda}{\mu - \lambda(1 + q\mu E(B_1))} + \frac{\lambda^2 \mu q E(B_1^2)}{2[\mu - \lambda(1 + q\mu E(B_1))]} + \lambda q E(B_1) \qquad (3.6.9)$$

where $E(B_1) = (-1)^i B_1^{*(i)}(0)$

$= (-1)^i \frac{d^i B_1^*(s)}{ds^i}$ at s=0

$E(V^i) = (-1)^i V^{*(i)}(0) = (-1)^i \frac{d^i V^*(s)}{ds^i}$ at s=0

By Little's [46] formula, the mean entity response time is given by

$$E(T) = \frac{E(N)}{\lambda}$$

Therefore,

$$E(T) = \frac{E(V^2)}{2E(V)} + \frac{1}{\mu - \lambda(1 + q\mu E(B_1))} + \frac{\lambda \mu q E(B_1^2)}{2[\mu - \lambda(1 + q\mu E(B_1))]} + qE(B_1)$$

(3.6.10)

Further if there is no additional optional service so that $\rho = 1, q = 0$ then equation (3.6.9) and (3.6.10) become:

$$E(N) = \frac{\lambda E(V^2)}{\lambda E(V)} + \frac{\lambda}{\mu - \lambda} \text{ and}$$

$$E(T) = \frac{E(V^2)}{2E(V)} + \frac{1}{\mu - \lambda}$$

which are the results given in (Takagi [46]) for the M/M/1 queueing system with interruption. On the other hand, if the server offers only the optional service so that $\rho = 0, q = 1$ and $\frac{1}{\mu} \longrightarrow 0$ then equations (3.6.9) and (3.6.10) become

$$E(N) = \frac{\lambda E(V^2)}{2E(V)} + \frac{\lambda^2 E(B_1^2)}{2[1 - \lambda E(B_1)]} + \lambda E(B_1)$$

$$E(T) = \frac{E(V^2)}{2E(V)} + \frac{\lambda E(B_1^2)}{2[1 - \lambda E(B_1)]} + E(B_1)$$

which are the results given in Takagi [46] where

$$\alpha(z) = V(\lambda - \lambda z)$$

3.8 Concluding remark

The description of the queueing system given in the introduction of this chapter leads the reader to believe that the low degree of system complexity would result in ease of mathematical modelling. The eventual mathematical manipulations required to create the model are far from insignificant, rather they are extensive and involved, and demand treatment by a highly proficient practioner.