Contributions to the theory of tensor norms and their relationship with vector-valued function spaces

by

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Abstract

Let (Ω, Σ, μ) be a non-atomic probability space, $1 \leq p < \infty$ and X be an infinite dimensional Banach space. We shall prove that each of the inclusions

$$L^p(\mu) \overset{\wedge}{\otimes} X \hookrightarrow L^p_{\text{strong}}(\mu, X) \hookrightarrow L^p(\mu) \overset{\vee}{\otimes} X \hookrightarrow L^p_{\text{weak}}(\mu, X) \hookrightarrow \mathcal{L}(X^*, L^p(\mu))$$

is injective with norm ≤ 1, and also that the inclusions

$$L^p(\mu) \overset{\wedge}{\otimes} X \hookrightarrow L^p_{\text{strong}}(\mu, X) \hookrightarrow L^p(\mu) \overset{\vee}{\otimes} X$$

are strict.

The Radon-Nikodym property and the Lewis-Radon-Nikodym property for tensor norms will be introduced and discussed. In particular, it will be shown that the Hilbertian tensor norm h introduced in (Grothendieck (1956a), §3) has the Lewis-Radon-Nikodym property but does not have the Radon-Nikodym property. However, we shall single out another of Grothendieck's natural tensor norms, namely the projective tensor norm $\wedge$, that does have the Radon-Nikodym property. Furthermore, it will be shown that if α is a tensor norm with the Radon-Nikodym property, then $\alpha/$, $\backslash\alpha$ and $/\alpha$ have the property as well, but that in general $\alpha\backslash$ need not have the property. However, the tensor norms γ_p and $\gamma_p\backslash$ will both be shown to have the Lewis-Radon-Nikodym property.

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DECLARATION

I, the undersigned, hereby declare that the thesis submitted herewith for the degree Philosophiae Doctor to the University of Pretoria contains my own, independent work and has not been submitted for any degree at any other university.

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[1]Family praisename derived from the word '*crocodile*'

Contents

Chapter 1

Tensor products and preliminaries

In this chapter we lay the foundations for the ensuing developments that will prove the main results. After the historical introduction of tensor products, we clarify the notation and introduce the preliminary results. In the process, we describe the historical evolution of our central problems.

1.1 Historical introduction

Tensor products apparently appeared in Functional Analysis for the first time during the late thirties in the work of Murray and John von Neumann on Hilbert spaces. The systematic study of tensor products of Banach spaces took off the ground in the 1940's with the works of Dunford, Schatten, and von Neumann.

The first systematic study of classes of norms on tensor products of Banach spaces is due to Schatten who in 1943 continued his work in a series of papers, some co-authored with von Neumann. Schatten organised and extended the basic results of the early works in his influential monograph "*A Theory of Cross-Spaces*" (Schatten (1950)). The most fascinating applications of the theory dealt with operator ideals on Hilbert spaces, the Hilbert-Schmidt operators and the Schatten-von Neumann classes. Also, the greatest and least reasonable crossnorms made their debut appearance in this monograph. The testimony borne by these early works is that tensor products were developed as a tool for studying spaces of operators, the very objective that impeded the very progress in the study of tensor products. Interpretations of tensor products other than as spaces of operators was given little attention.

The real breakthrough in the study of tensor products of Banach spaces came with the genius of Grothendieck in his *"Résumé de la théorie métrique des produits tensoriels topologiques"* submitted in 1954 and published in 1956 in the Bulletin of the Mathematical Society of São Paulo (Grothendieck (1956a)). Grothendieck wrote the Résumé (as this paper later came to be called) independently of Schatten's work. This paper deeply influenced the course of Functional Analysis in that it demonstrated enormous possibilities for the using of tensor products in Banach space theory and anticipated the study of Banach spaces in terms of their finite dimensional subspaces - the so-called the 'local' theory - which has substantially enriched our understanding of Banach spaces. A great deal of the more elementary aspects of Grothendieck's theory were known to Schatten, but he was not aware of the important role of the finite dimensional behaviour of tensor norms. He, therefore, did not succeed in developing a useful duality theory for tensor products of general Banach spaces. However, the idea of ideals of operators was always apparent in the study of tensor products in both Schatten's and Grothendieck's works.

More attention was given to the Résumé in 1968 through the famous paper *"Absolutely summing operators in $\mathcal{L}_p$-spaces and their applications"* (Lindenstrauss and Pelczynski (1968)) which presented Grothendieck's fundamental theorem, the main result of the Résumé, as an inequality concerning $n \times n$ matrices and Hilbert spaces. Various applications were given, mainly on the class of absolutely p-summing operators, without using tensor products. This approach incredibly revived the Banach space theory which was deemed almost complete by certain people in the middle of the sixties. Most important results today trace their roots back to the Résumé, including many classical results which "... are already contained in Grothendieck's paper, although sometimes in a hidden way" (Defant and Floret (1993), Introduction, page 1, 3rd paragraph, lines 11 and 12).

Going along with Grothendieck in 1956, call a norm on a tensor product $X \otimes Y$ a *tensor norm* if it arises as a special case of a method for norming $X \otimes Y$ for every pair of Banach spaces X and Y. The excellent examples of such tensor norms are the greatest and least reasonable crossnorms. The norm on $L^1_{\text{strong}}(\mu, X)$ clearly comes from a tensor norm on $L^1(\mu) \otimes X$. Kwapien showed in 1972 that for $1 < p < \infty$ there is no tensor norm such that for all μ and X the norm on $L^p_{\text{strong}}(\mu, X)$ comes from a tensor norm. In particular, the $L^p_{\text{strong}}(\mu, X)$ norm is not the completion of $L^p(\mu) \otimes X$ $(1 < p < \infty)$ under either the greatest or the least reasonable crossnorms. The question arising out of this setup is:

- *For $1 < p < \infty$ and for a Banach space X, how is $L^p_{strong}(\mu, X)$ related to the completion of $L^p(\mu) \otimes X$ under the Grothendieck's natural tensor norms $\wedge$ and $\vee$?*

This question is central to our later investigation.

In the case where $\Omega = \mathbb{N}$ and μ is a counting measure on $\mathbb{N}$, $L^p(\mu)$ is a classical sequence space ℓ^p, $1 \leq p < \infty$, and the above question has been addressed by Grothendieck under the greatest (called the *projective*) and the least (called the *injective*) tensor norms. The presentation appears in the article, "*Sur certaines classes de suites dans les espaces de Banach, et le théorème de Dvoretzky-Rogers*" contained in the same volume of *Boletin de Sociedade de Matemética de São Paulo* in which the Résumé appeared (Grothendieck (1956b)), namely:

- *Let X be a Banach space and $1 \leq p < \infty$. Then*

$$\ell^p \overset{\wedge}{\otimes} X \hookrightarrow \ell^p_{strong}(X) \hookrightarrow \ell^p \overset{\vee}{\otimes} X,$$

  where '$\hookrightarrow$' denotes a 'canonical inclusion map'. Both inclusions have norm ≤ 1, and are injective.

As discovered by D.R.Lewis and C.Stegall (Diestel and Uhl Jr (1977), III.1.8) the space ℓ^1 plays an important role in the study of the Radon-Nikodym property. The central role of this space in the Radon-Nikodym theory was exploited further by D.R. Lewis by embarking on an intensive study of $\otimes$ norms with the Radon-Nikodym property (Lewis (1976)). This property, as observed in (Defant and Floret (1993), 1st paragraph on page 430), paved the way towards the appreciation of the full duality theory of the projective and injective tensor norms, namely the description of the conditions under which $X^* \overset{\wedge}{\otimes} Y^* = (X \overset{\vee}{\otimes} Y)^*$ holds, where $\wedge$ and $\vee$ are the notations for the projective and injective $\otimes$ norms, respectively. While our hat is off to these developments and their role in the Radon-Nikodym theory, a question that is our central concern is:

- *Can we abstract the properties enjoyed by the space ℓ^1, namely the Radon-Nikodym property and the approximation property, and consider any spaces that enjoy the same properties to study the Radon-Nikodym property of $\otimes$ norms?*

We shall address this question affirmatively in the sequel.

1.2 Notation and tensor products

Let X and Y be infinite dimensional real or complex Banach spaces. Then X^* denotes the Banach space dual of X and B_X is the closed unit ball of X. The space of all continuous (or bounded) linear operators from X to Y is denoted by $\mathcal{L}(X,Y)$. If Z is another infinite dimensional Banach space, then $\mathcal{B}(X,Y;Z)$ denotes the space of all continuous (or bounded) bilinear mappings from $X \times Y$ to Z.

Let $T \in \mathcal{L}(X,Y)$ and $\phi \in \mathcal{B}(X,Y;Z)$. Then

$$\|T\| := \sup\{\|Tx\| : x \in B_X\},$$

the *operator norm*, and

$$\|\phi\| := \sup\{\|\phi(x,y)\| : x \in B_X, y \in B_Y\},$$

where ':=' is used for a definition.

The topology $\sigma(X^*,X)$ on X^* is denoted by w^*. Hence we denote by $\mathcal{K}_{w^*}(X^*,Y)$ the subspace of $\mathcal{L}(X^*,Y)$ consisting of all the compact operators from X^* to Y that are w^*-to-weakly continuous and equipped with the operator norm. All the spaces defined above *are* Banach spaces.

We shall not give any formal definition of the algebraic tensor product $X \otimes Y$ of X and Y: for the formal definition see, for instance, (Defant and Floret (1993), Chapter I, Greub (1978), Chapter I, Jarchow (1981), page 22).

A norm α on $X \otimes Y$ is called a *reasonable crossnorm* if it satisfies the following conditions:

(a) $\alpha(x \otimes y) \leq \|x\|\|y\|$ for all $x \in X$ and $y \in Y$,

(b) for $x^* \in X^*$ and $y^* \in Y^*$, $x^* \otimes y^* \in (X \otimes Y, \alpha)^*$ and

$$\|x^* \otimes y^*\|_{(X\otimes Y,\alpha)^*} \leq \|x^*\|\|y^*\|.$$

If α is a reasonable crossnorm on $X \otimes Y$, then α satisfies the following conditions:

(á) $\alpha(x \otimes y) = \|x\|\|y\|$,

(b́) for $x^* \in X^*$ and $y^* \in Y^*$,

$$\|x^* \otimes y^*\|_{(X\otimes Y,\alpha)^*} = \|x^*\|\|y^*\|.$$

Two reasonable crossnorms are of particular interest: the *least* reasonable crossnorm and the *greatest* reasonable crossnorm. Let us describe them.

Let $u \in X \otimes Y$. Define $|u|_\vee$ by

$$|u|_\vee := \sup\{|(x^* \otimes y^*)(u)| : x^* \in B_{X^*}, y^* \in B_{Y^*}\}.$$

Then $|.|_\vee$ *is* a norm. In addition, $|.|_\vee$ is a reasonable crossnorm and is the least reasonable crossnorm, in the sense that, if α is any other reasonable crossnorm on $X \otimes Y$, then $|u|_\vee \leq \alpha(u)$ for all $u \in X \otimes Y$. Denote by $X \overset{\vee}{\otimes} Y$ the completion of the normed space $(X \otimes Y, |.|_\vee)$, called the *injective* tensor product of X and Y.

Alternatively, $|.|_\vee$ can be described as follows: with $\mathcal{B}(X,Y) := \mathcal{B}(X,Y;\mathbb{C})$, $\mathbb{C}$ the Banach space of complex numbers, each member of $X \otimes Y$ acts naturally as a continuous bilinear functional on $X^* \otimes Y^*$. Moreover, if $u \in X \otimes Y$ is viewed as a member of $\mathcal{B}(X^*, Y^*)$, then $|u|_\vee = \|u\|$. Thus there is a natural isometry of $X \overset{\vee}{\otimes} Y$ into $\mathcal{B}(X^*, Y^*)$.

Define $|.|_\wedge$ on $X \otimes Y$ by

$$|u|_\wedge := \sup\{|\phi(u)| : \phi \in \mathcal{B}(X,Y), \|\phi\| \leq 1\}$$

for $u \in X \otimes Y$. Then $|.|_\wedge$ *is* a norm. It is a reasonable crossnorm and is the greatest reasonable crossnorm, in the sense that, if α is any reasonable crossnorm on $X \otimes Y$, then $\alpha(u) \leq |u|_\wedge$ for all $u \in X \otimes Y$. In particular, for the following result see (Diestel and Uhl Jr (1977), VIII.1.8):

Proposition 1.2.1. *Let $u \in X \otimes Y$. Then $|u|_\vee \leq |u|_\wedge$.*

Proof. $X^* \overset{\vee}{\otimes} Y^*$ is isometric to a closed subspace of $\mathcal{B}(X^{**}, Y^{**})$, so we may write

$$\|x^* \otimes y^*\|_{\mathcal{B}(X^{**},Y^{**})} = \|x^*\|\|y^*\|.$$

Hence the restriction $x^* \otimes y^*|_{\mathcal{B}(X,Y)}$ to $X \otimes Y$ satisfies

$$\|x^* \otimes y^*|_{X \otimes Y}\|_{\mathcal{B}(X,Y)} \leq \|x^*\|\|y^*\|.$$

Therefore, if $u \in X \otimes Y$, then

$$\begin{aligned} |u|_\vee &= \sup\{|x^* \otimes y^*(u)| : x^* \in B_{X^*}, y^* \in B_{Y^*}\} \\ &\leq \sup\{|\phi(u)| : \phi \in \mathcal{B}(X,Y), \|\phi\| \leq 1\} \\ &= |u|_\wedge. \end{aligned}$$

□

Denote by $X \overset{\wedge}{\otimes} Y$ the completion of $X \otimes Y$ under $|.|_\wedge$ and call it the *projective* tensor product of X and Y. Next, we describe a more convenient alternative version of $|.|_\wedge$.

Let $u \in X \otimes Y$. Then

$$|u|_\wedge = \inf\{\sum_{i \leq n} \|x_i\|\|y_i\| : x_i \in X, y_i \in Y, u = \sum_{i \leq n} x_i \otimes y_i\}.$$

The proof of this description emphasizes once more the fact that $|.|_\wedge$ is greater than all crossnorms on $X \otimes Y$ (Diestel and Uhl Jr (1977), page 227, Diestel et al. (2002a), Diestel et al. (2002b)).

Furthermore, for $u \in X \overset{\wedge}{\otimes} Y$ and $\epsilon > 0$, there exist sequences (x_n) in X and (y_n) in Y such that $\lim_{n\to\infty} \|x_n\| = 0 = \lim_{n\to\infty} \|y_n\|, u = \sum_{n=1}^{\infty} x_n \otimes y_n$ in $|.|_\wedge$ norm and $|u|_\wedge \leq \sum_{n=1}^{\infty} \|x_n\|\|y_n\| \leq |u|_\wedge + \epsilon$. See (Diestel and Uhl Jr (1977), page 227) for the proof.

In fact, if we allow representations of 'infinite length' for $u \in X \overset{\wedge}{\otimes} Y$, i.e. $u = \sum_{n=1}^{\infty} x_n \otimes y_n$, and define

$$\alpha_f(u) := \inf\{\sum_{i\leq n} \|x_i\|\|y_i\| : u = \sum_{i\leq n} x_i \otimes y_i\},$$

where the infimum is taken over all finite representations $u = \sum_{i\leq n} x_i \otimes y_i$ of $u \in X \overset{\wedge}{\otimes} Y$, and

$$\alpha_\infty(u) := \inf\{\sum_{n=1}^{\infty} \|x_n\|\|y_n\| : u = \sum_{n=1}^{\infty} x_n \otimes y_n \in X \overset{\wedge}{\otimes} Y\},$$

then we have $|u|_\wedge = \alpha_\infty(u)$, since it can be shown that

$$|u|_\wedge \leq \alpha_\infty(u) \leq \alpha_f(u),$$

for all $u \in X \overset{\wedge}{\otimes} Y$. In fact, it is enough if u is taken from $(X \otimes Y, |.|_\wedge)$. The infinite representation can be consulted in (Diestel et al. (2002a), Diestel et al. (2002b)). We shall need the following theorem:

Theorem 1.2.2. *(The Universal Mapping Property of $\overset{\wedge}{\otimes}$) For any Banach spaces X, Y and Z, the space $\mathcal{L}(X \overset{\wedge}{\otimes} Y, Z)$ of all bounded linear operators from $X \overset{\wedge}{\otimes} Y$ to Z is isometrically isomorphic to the space $\mathcal{B}(X, Y; Z)$ of all bounded bilinear mappings taking $X \times Y$ to Z. The natural correspondence establishing this isometric isomorphism is given by*

$$v \in \mathcal{L}(X \overset{\wedge}{\otimes} Y, Z) \leftrightarrow \phi \in \mathcal{B}(X, Y; Z)$$

via

$$v(x \otimes y) := \phi(x, y).$$

In particular, $(X \overset{\wedge}{\otimes} Y)^*$ is the Banach space dual $\mathcal{B}(X, Y)$ of continuous bilinear functionals on $X \times Y$.

The proof can be found in (Diestel and Uhl Jr (1977), VIII.2.1, Diestel et al. (1997)).

A *tensor norm* ($\otimes$ norm, for short) α is a method of ascribing to any pair (E, F) of finite dimensional Banach spaces (over the scalar field $\mathbb{R}$ or $\mathbb{C}$) a reasonable crossnorm α for $E \otimes F$ in such a way that should E, F, G, H be finite dimensional Banach spaces and $u : E \longrightarrow F$ and $v : G \longrightarrow H$ be bounded linear operators, then $u \otimes v : E \otimes G \longrightarrow F \otimes H$ has bound

$$\|u \otimes v\|_{\mathcal{L}(E \overset{\alpha}{\otimes} G, F \overset{\alpha}{\otimes} H)} \leq \|u\| \|v\|.$$

This inequality is referred to as the *uniform crossnorm property* of α.

This definition extends to infinite dimensional Banach spaces. In particular, $|.|_{\vee}$ and $|.|_{\wedge}$ *are* tensor norms.

Given vector spaces G and H, the transposition map $t : G \otimes H \longrightarrow H \otimes G$ is the isomorphism generated by $t(g \otimes h) = h \otimes g$. If $u \in G \otimes H$, denote by ${}^t u$ the image $t(u) \in H \otimes G$.

Let α be a tensor norm. We define the *transpose* ${}^t\alpha$ of α as follows: If E, F are any finite dimensional Banach spaces, then for $u \in E \otimes F$

$${}^t\alpha(u) = \alpha({}^t u).$$

It can be proved that, if α is a tensor norm, then so, too, is ${}^t\alpha$ and ${}^t({}^t\alpha) = \alpha$ (Diestel et al. (1997), Chapter I, §2). We shall write ${}^{tt}\alpha$ for ${}^t({}^t\alpha)$.

Due to the finite dimensional nature of the spaces E and F, $E \otimes F$ is algebraically identical to $(E^* \overset{\alpha}{\otimes} F^*)^*$. If α is a tensor norm and for $u \in E \otimes F$ we define $\alpha^*(u)$ by

$$\alpha^*(u) := \|u\|_{(E^* \overset{\alpha}{\otimes} F^*)^*},$$

then it can be shown that α^* is also a tensor norm and $(\alpha^*)^* = \alpha$. We shall write α^{**} for $(\alpha^*)^*$. Furthermore, $({}^t\alpha)^* = {}^t(\alpha^*)$ (Diestel et al. (1997), Chapter I, §2).

The *contragradient* of a tensor norm α is defined by

$$\overset{\vee}{\alpha} := {}^t(\alpha^*) = ({}^t\alpha)^*,$$

and this is also a tensor norm.

Given tensor norms α and β, we define $\alpha \leq \beta$ to mean that $\alpha(u) \leq \beta(u)$ for any $u \in E \otimes F$ and for any finite dimensional Banach spaces E and F.

It is easily established that for any tensor norms α and β, $\alpha \leq \beta$ if and only if ${}^t\alpha \leq {}^t\beta$ (respectively, $\beta^* \leq \alpha^*$) (Diestel et al. (1997), Chapter I, §2).

Since for any tensor norms α and β, $\alpha \leq \beta$ if and only if $\beta^* \leq \alpha^*$ and $\alpha^{**} = \alpha$, it follows that the tensor norms constitute a complete lattice. So we can naturally build new tensor norms from a given tensor norm α. To do this, we need the following proposition (Diestel et al. (1997), Chapter II, §3): $\mathcal{F}$ denotes the class of all finite dimensional Banach spaces over the scalar field $\mathbb{K}$.

Proposition 1.2.3. *Let α be a tensor norm. Then the following statements regarding α are equivalent:*

1. *Given $E, F, G \in \mathcal{F}$ with $F \subseteq E$, the natural inclusion $F \overset{\alpha}{\otimes} G \hookrightarrow E \overset{\alpha}{\otimes} G$ is an isometry.*
2. *Given $E, F, G \in \mathcal{F}$ with $F \subseteq E$, the canonical map $E \overset{\alpha^*}{\otimes} G \longrightarrow (E/F) \overset{\alpha^*}{\otimes} G$ is a quotient map.*
3. *Given $E, F, G \in \mathcal{F}$ with $F \subseteq E$, every α^* form on $F \times G$ extends to an α^* form on $E \times G$ having the same α^* norm.*
4. *Given $E, F, G \in \mathcal{F}$ with $F \subseteq E$, every α form on $E \times G$ that vanishes on $F \times G$ induces an α form on $(E/F) \times G$ having the same α norm.*

   Should α enjoy any (and, hence, all) of 1 through 4, then α also enjoys the following properties:
5. *Given Banach spaces X, Y, Z with Y a closed linear subspace of X, the natural inclusion $Y \overset{\alpha}{\otimes} Z \hookrightarrow X \overset{\alpha}{\otimes} Z$ is an isometry.*
6. *Given Banach spaces X, Y, Z with Y a closed linear subspace of X, the canonical map $X \overset{\alpha^*}{\otimes} Z \longrightarrow (X/Y) \overset{\alpha^*}{\otimes} Z$ is a quotient map.*
7. *Given Banach spaces X, Y, Z with Y a closed linear subspace of X, every α^* form on $Y \times Z$ extends to an α^* form on $X \times Z$ having the same α^* norm.*
8. *Given Banach spaces X, Y, Z with Y a closed linear subspace of X, every α form on $X \times Z$ that vanishes on $Y \times Z$ induces an α form on $(X/Y) \times Z$ of the same α norm.*

Let α be a tensor norm. We say α is *left injective* if the conditions of Proposition 1.2.3 hold for α; α is said to be *right injective* if ${}^t\alpha$ is left injective; α is *injective* if it is both left and right injective. We say α is *left projective* if α^* is left injective; α is *right projective* if α^* is right injective, that is , if $\overset{\vee}{\alpha}$ is left injective; α is *projective* if it is both left and right projective.

Define

$$/\alpha := \sup\{\beta : \beta \text{ is a } \otimes \text{ norm } \leq \alpha,\ \beta \text{ left injective}\}$$

$$\alpha\backslash := \sup\{\beta : \beta \text{ is a} \otimes \text{ norm } \leq \alpha,\ \beta \text{ right injective}\}$$

$$\backslash\alpha := \inf\{\beta : \beta \text{ is a } \otimes \text{ norm } \geq \alpha, \beta \text{ left projective}\}$$

$$\alpha/ := \inf\{\beta : \beta \text{ is a } \otimes \text{ norm } \geq \alpha,\ \beta \text{ right projective}\}.$$

For a tensor norm α, the notations $/\alpha$, $\alpha\backslash$, $\backslash\alpha$ and $\alpha/$ are called, respectively, the left injective, right injective, left projective and right projective hulls of α.

We define

$$/\alpha\backslash := (/\alpha)\backslash = /(\alpha\backslash),$$

called the *injective hull* of the tensor norm α, and

$$\backslash\alpha/ := (\backslash\alpha)/ = \backslash(\alpha/),$$

called the *projective hull* of the tensor norm α.

1.3 Other preliminary results

We shall also need some standard results of Banach space theory and vector measures. Firstly, we put on record the following definitions for our later use.

A Banach space X is said to be *metrically accessible* (or has the *metric approximation property*) if for any finite dimensional subspace E of X and any $\epsilon > 0$ there is a finite rank operator $u : X \longrightarrow X$ with norm $\leq 1 + \epsilon$ and such that for any $x \in E$, $\|x - ux\| \leq \epsilon$.

A Banach space is said to be *accessible* (or has the *approximation property*) if given a compact set K in X and an $\epsilon > 0$, there is a finite rank bounded linear operator $u : X \longrightarrow X$ such that for any $x \in K$, $\|x - ux\| \leq \epsilon$.

The second definition is weaker than the first one and will be indispensable in the formulation of one of our main results.

Let (Ω, Σ, μ) be a complete positive finite measure space. A function $f : \Omega \longrightarrow X$ is called *simple* if there are disjoint members $E_1, \ldots, E_n$ of Σ and vectors $x_1, \ldots, x_n$ in X for which $f(w) = \sum_{i=1}^{n} \chi_{E_i}(w)x_i$ holds for all $w \in \Omega$, where χ_E denotes the characteristic function of the set $E \subseteq \Omega$. Such functions *are* measurable. A function $f : \Omega \longrightarrow X$ is said to be *μ-measurable* (or *strongly measurable*) if there exists a sequence (f_n) of simple functions such that $\lim_{n\to\infty} f_n = f$ μ-almost everywhere. It *must* be noted that measurable functions are stable under sums, scalar multiples and pointwise almost everywhere convergence. A function $f : \Omega \longrightarrow X$ is said to be *scalarly μ-measurable* (or *weakly measurable*) if x^*f is μ-measurable for each $x^* \in X^*$.

The proof of the next theorem can be found in (Diestel (1984), page 25, Diestel and Uhl Jr (1977), page 42).

Theorem 1.3.1. (*Pettis Measurability Theorem*) *A function $f : \Omega \longrightarrow X$ is μ-measurable if and only if f is scalarly μ-measurable and there exists an $E \in \Sigma$ with $\mu(E) = 0$ such that $f(\Omega \backslash E)$ is a norm-separable subset of X.*

Even stronger results than the above theorem appear in its proof and these are crystallized in the next two corollaries.

Corollary 1.3.2. *A function $f : \Omega \longrightarrow X$ is μ-measurable if and only if f is the μ-almost everywhere uniform limit of a sequence of countably valued μ-measurable functions.*

A set $\Gamma \subseteq X^*$ is said to be *norming* if

$$\|x\| = \sup\{\frac{|x^*x|}{\|x^*\|} : x^* \in \Gamma\},$$

for all $x \in X$.

Corollary 1.3.3. *A μ-essentially separably valued function $f : \Omega \longrightarrow X$ is μ-measurable if there exists a norming set $\Gamma \subseteq X^*$ such that the numerical function x^*f is μ-measurable for each $x^* \in \Gamma$.*

Let $f : \Omega \longrightarrow X$ be a simple function, say, $f(w) = \sum_{i=1}^{n} \chi_{E_i}(w)x_i$. Then

$$\int_E f(\omega)d\mu(\omega) := \sum_{i=1}^{n} \mu(E \cap E_i)x_i,$$

for all $E \in \Sigma$. A μ - measurable function $f : \Omega \longrightarrow X$ is called *Bochner integrable* if there exists a sequence of simple functions (f_n) such that

$$\lim_{n\to\infty} \int_\Omega \|f_n(\omega) - f(\omega)\|d\mu(\omega) = 0.$$

In this case, $\int_E f(\omega)d\mu(\omega)$ is defined for each $E \in \Sigma$ by

$$\int_E f(\omega)d\mu(\omega) := \lim_{n\to\infty} \int_E f_n(\omega)d\mu(\omega).$$

Theorem 1.3.4. (*Bochner Characterization of Integrable Functions*) *A μ-measurable function $f : \Omega \longrightarrow X$ is Bochner integrable if and only if*

$$\int_\Omega \|f(\omega)\|d\mu(\omega) < \infty.$$

The proof can be consulted in (Diestel and Uhl Jr (1977), page 45, Diestel (1984), page 26). Although our interest is in the case $1 < p < \infty$, our next consideration holds for $1 \leq p \leq \infty$. We define

$$L^p_{\text{strong}}(\mu, X) := \{f : \Omega \longrightarrow X\ \mu\text{-measurable} : \int_\Omega \|f(\omega)\|^p d\mu(\omega) < \infty\},$$

in the case $1 \leq p < \infty$, and

$$L^\infty_{\text{strong}}(\mu, X) := \{f : \Omega \longrightarrow X\ \mu\text{-measurable} : f \text{ is } \mu\text{-essentially bounded}\},$$

normed, respectively, by

$$\|f\|_{L^p_{\text{strong}}(\mu,X)} := \left(\int_\Omega \|f(\omega)\|^p d\mu(\omega)\right)^{\frac{1}{p}} < \infty$$

and

$$\|f\|_{L^\infty_{\text{strong}}(\mu,X)} := \operatorname{ess\,sup}\{\|f(\omega)\| : \omega \in \Omega\} < \infty.$$

When X is the scalar field, we denote the above *Banach spaces*, respectively, by $L^p(\mu)$ and $L^\infty(\mu)$. We write

$$L^p_{\text{weak}}(\mu, X) := \{f : \Omega \longrightarrow X\ \mu\text{-measurable} : x^*f \in L^p(\mu) \text{ for all } x^* \in X^*\},$$

for $1 \leq p < \infty$, and

$$L^\infty_{\text{weak}}(\mu, X) := \{f : \Omega \longrightarrow X\ \mu\text{-measurable} : x^*f \in L^\infty(\mu) \text{ for all } x^* \in X^*\},$$

normed, respectively, by

$$\|f\|_{L^p_{\text{weak}}(\mu,X)} := \sup\{\|x^*f\|_{L^p(\mu)} : x^* \in B_{X^*}\} < \infty,$$

and

$$\|f\|_{L^\infty_{\text{weak}}(\mu,X)} := \sup\{\|x^*f\|_{L^\infty(\mu)} : x^* \in B_{X^*}\} < \infty.$$

The finiteness of $\|.\|_{L^p_{\text{weak}}(\mu,X)}$ and $\|.\|_{L^\infty_{\text{weak}}(\mu,X)}$ follow from the Closed Graph Theorem: we will justify this claim for our case of interest, namely $1 < p < \infty$, later on in the proof of Theorem 2.2.1.

Let K be a compact Hausdorff space. We define $C(K, X)$ to be the Banach space of all X-valued continuous functions on K equipped with the supremum norm; namely, if $f \in C(K, X)$ then we define

$$\|f\| := \sup\{\|f(k)\| : k \in K\}.$$

If $X = \mathbb{K}$, the scalar field $\mathbb{R}$ or $\mathbb{C}$, then $C(K, X) = C(K)$. The proof of the next result appears in (Diestel et al. (1997), Diestel et al. (2002a), Diestel et al. (2002b)).

Theorem 1.3.5. *Let X be a Banach space, K a compact Hausdorff space and (Ω, Σ, μ) a finite measure space. Then*

1. *$C(K) \overset{\vee}{\otimes} X = C(K, X)$ isometrically isomorphically.*
2. *$L^1(\mu) \overset{\wedge}{\otimes} X = L^1_{strong}(\mu, X)$ isometrically isomorphically.*

The next result is a consequence of the first part of the preceding theorem.

Corollary 1.3.6. *If K and S are compact Hausdorff spaces, then*

$$C(K) \overset{\vee}{\otimes} C(S) = C(K \times S).$$

One naturally asks whether we can interchange the tensor norms in Theorem 1.3.5 above. This can be done with *some* care.

Theorem 1.3.7. *Let X be a Banach space, K and S compact Hausdorff spaces and (Ω, Σ, μ) a finite measure space. Then*

1. *$L^1(\mu) \overset{\vee}{\otimes} X$ is isometrically isomorphic to the completion of $P^1(\mu, X)$, the space of all Pettis integrable functions $f : \Omega \longrightarrow X$ equipped with the norm*
$$\|f\|_{P_1} = \sup_{\|x^*\| \leq 1} \int_\Omega |x^* f| d\mu.$$
2. *A continuous function $f : K \times S \longrightarrow \mathbb{K}$ belongs to $C(K) \overset{\wedge}{\otimes} C(S)$ if and only if there are continuous functions $g : K \longrightarrow \ell^2$ and $h : S \longrightarrow \ell^2$ such that $f(x, y) =< g(x), h(y) >$, $(x, y) \in K \times S$, where $<,>$ denotes the inner product of ℓ^2.*

The proof of the first part appears in (Diestel and Uhl Jr (1977), VIII.1.5) while the proof of the second part appears in (Defant and Floret (1993), II.14.5. Corollary 2).

We hasten to mention that by the Uniform Boundedness Theorem the next result is seen to hold. The notation '$\hookrightarrow$' will be used for a '*canonical inclusion map*'.

Theorem 1.3.8. *Let (Ω, Σ, μ) be a complete finite measure space. Then*

$$L^\infty_{weak}(\mu, X) = L^\infty_{strong}(\mu, X)$$

isometrically isomorphically.

Proof. Indeed, if $f \in L^\infty_{\text{weak}}(\mu, X)$, then f is μ-measurable, and so Pettis Measurability Theorem 1.3.1 assures us that there exists an $E \in \Sigma$ with $\mu(E) = 0$ such that $f(\Omega \setminus E)$ is a norm-separable subset of X. Set

$$A_{f,E} := f(\Omega \setminus E).$$

Then

$$\sup\{|x^* f(\omega)| : f(\omega) \in A_{f,E}\} < \infty$$

for any $x^* \in X^*$. It follows from the Uniform Boundedness Theorem that

$$\sup\{\|f(w)\| : f(w) \in A_{f,E}\} < \infty.$$

Therefore,

$$\text{ess-}\sup\{\|f(\omega)\| : \omega \in \Omega\} < \infty,$$

and so, $f \in L^\infty_{\text{strong}}(\mu, X)$. We have shown that

$$L^\infty_{\text{weak}}(\mu, X) \hookrightarrow L^\infty_{\text{strong}}(\mu, X)$$

is a bounded inclusion of norm ≤ 1.

On the other hand, if $f \in L^\infty_{\text{strong}}(\mu, X)$, then there is an $M \geq 0$ such that $\|f(\omega)\| \leq M$ μ-a.e.. That is,

$$\sup\{|x^* f(\omega)| : x^* \in B_{X^*}\} \leq M$$

μ-a.e.. Hence $|x^* f(\omega)| \leq M$ μ-a.e. for each $x^* \in B_{X^*}$, and so there is a $K \geq 0$ such that $|x^* f(\omega)| \leq K$ for all $x^* \in X^*$ μ-a.e.. It follows that

$$\text{ess-}\sup\{|x^* f(\omega)| : \omega \in \Omega\} < \infty.$$

That is, $x^* f(.) \in L^\infty(\mu)$ for each $x^* \in X^*$ so that $f \in L^\infty_{\text{weak}}(\mu, X)$. Therefore

$$L^\infty_{\text{strong}}(\mu, X) \hookrightarrow L^\infty_{\text{weak}}(\mu, X)$$

is an inclusion of norm ≤ 1. This shows that

$$L^\infty_{\text{weak}}(\mu, X) = L^\infty_{\text{strong}}(\mu, X)$$

isomorphically. Furthermore,

$$\begin{aligned}
\|f\|_{L^\infty_{\text{strong}}(\mu,X)} &= \text{ess-}\sup\{\|f(\omega)\| : \omega \in \Omega\} \\
&= \text{ess-}\sup_{\omega\in\Omega} \sup_{\|x^*\|\leq 1} |x^* f(\omega)| \\
&= \sup_{\|x^*\|\leq 1} \text{ess-}\sup_{\omega\in\Omega} |x^* f(\omega)| \\
&= \sup_{\|x^*\|\leq 1} \|x^* f(.)\|_{L^\infty(\mu)} \\
&= \|f\|_{L^\infty_{\text{weak}}(\mu,X)}.
\end{aligned}$$

Hence, the equality of the sets holds isometrically isomorphically as claimed. □

In particular, the above theorem holds for a discrete measure μ on the set $\Omega = \mathbb{N}$.

Chapter 2

Vector-valued sequence and function spaces

2.1 Tensoring with sequence spaces

Let $\Omega = \mathbb{N}$, the set of all positive integers. Then a sequence $(x_i)_i$ of members of X is said to be p-th *power summable* (or *strongly p-summable*) if $\sum_i \|x_i\|^p < \infty$. Then we write $(x_i)_i \in \ell^p_{\text{strong}}(X)$, in which case we define the $\ell^p_{\text{strong}}(X)$-norm of $(x_i)_i$ by

$$\|(x_i)_i\|_{\ell^p_{\text{strong}}(X)} := (\sum_i \|x_i\|^p)^{\frac{1}{p}}.$$

This norm makes $\ell^p_{\text{strong}}(X)$ a Banach space. A sequence $(x_i)_i$ of members of X is said to be *scalarly* p-th *power summable* or (*weakly p-summable* if for each $x^* \in X^*$, the scalar sequence $(x^*(x_i))_i$ belongs to ℓ^p. By the Closed Graph Theorem, the linear map

$$x^* \mapsto (x^*(x_i))_i$$

from X^* to ℓ^p is continuous. Consequently the quantity

$$\|(x_i)_i\|_{\ell^p_{\text{weak}}(X)} := \sup\{(\sum_i |x^*(x_i)|^p)^{\frac{1}{p}} : \|x^*\| \leq 1\} < \infty,$$

and so it defines a norm on the space $\ell^p_{\text{weak}}(X)$ of all weakly p-summable sequences from X. This norm makes $\ell^p_{\text{weak}}(X)$ a Banach space.

The space $\ell^p_{\text{strong}}(X)$ is a linear subspace of $\ell^p_{\text{weak}}(X)$ with a continuous inclusion of norm one. These spaces are equal precisely when X is finite dimensional, by a Weak Dvoretzky-Rogers Theorem. Otherwise the inclusion is strict.

Let $1 \leq p < \infty$ and define p' by $\frac{1}{p} + \frac{1}{p'} := 1$. Then

$$\ell^p_{\text{strong}}(X)^* = \ell^{p'}_{\text{strong}}(X^*) \quad \text{(isometrically)},$$

where for a member $(x_i^*)_i$ of $\ell^{p'}_{\text{strong}}(X^*)$ the evaluation at a member $(x_i)_i$ of $\ell^p_{\text{strong}}(X)$ is given by

$$\sum_i x_i^*(x_i),$$

a series that is clearly absolutely convergent.

As an example of a strict inclusion of $\ell^p_{\text{strong}}(X)$ into $\ell^p_{\text{weak}}(X)$, let X be $\ell^{p'}$ (respectively c_0 if $p = 1$). Then the standard unit vector basis can be seen to be a weakly p-summable sequence in $\ell^{p'}$ (respectively c_0) that is not strongly p-summable.

In this section we shall prove the following *setwise* inequality which forms an important part of the developments in (Diestel et al. (1997), Grothendieck (1956b)) and whose vector-valued function spaces analogue is the subject of the next section:

$$\ell^p \overset{\wedge}{\otimes} X \hookrightarrow \ell^p_{\text{strong}}(X) \hookrightarrow \ell^p \overset{\vee}{\otimes} X \hookrightarrow \ell^p_{\text{weak}}(X) \hookrightarrow \mathcal{L}(X^*, \ell^p),$$

for $1 \leq p < \infty$, where '$\hookrightarrow$' denotes a '*canonical inclusion map*'. Each inclusion is injective and has norm ≤ 1 with the exception of the last one which *will* be shown to be an isometric isomorphism. The inclusions

$$\ell^p \overset{\wedge}{\otimes} X \hookrightarrow \ell^p_{\text{strong}}(X) \hookrightarrow \ell^p \overset{\vee}{\otimes} X$$

are strict.

Proposition 2.1.1. *Let $1 < p \leq \infty$ and X be Banach space. Define p' by $\frac{1}{p} + \frac{1}{p'} = 1$. Then the space $\mathcal{L}(\ell^{p'}, X)$ of all bounded linear operators from $\ell^{p'}$ to X is isometrically isomorphic to the space $\ell^p_{weak}(X)$ of all scalarly p-th power summable sequences of vectors in X, equipped with the norm $\|.\|_{\ell^p_{weak}(X)}$.*

Proof. Let $1 < p < \infty$ and let $(x_i)_i \in \ell^p_{\text{weak}}(X)$. Define the operator

$$u : X^* \longrightarrow \ell^p$$
$$u(x^*) := (x^*(x_i))_i.$$

Then the adjoint u^* of u defines a bounded linear operator from $\ell^{p'}$ to X^{**}. However, for e_j denoting the j-th unit coordinate vector in $\ell^{p'}$, we have that $u^* e_j = x_j$. For, if $x^* \in X^*$, then

$$x^*(x_j) = u(x^*)_j = u^*(e_j)(x^*),$$

and so $u^*(e_j) = x_j$. Since $u^*(e_j)$ lies in X for each j and the linear span of the e_j's is dense in $\ell^{p'}$, we have that $u^*(\ell^{p'}) \subseteq X$. Hence, *each weakly p-summable sequence $(x_i)_i$ of vectors in X determines an operator $u^* : \ell^{p'} \longrightarrow X$ by $u^* e_j = x_j$.*

Conversely, let $v : \ell^{p'} \longrightarrow X$ be a bounded linear operator with $v(e_j) := x_j$. Suppose that $(\lambda_i)_i \in \ell^{p'}$ and $(\mu_i)_i \in B_{\ell^\infty}$. Then

$$\begin{aligned} \|\sum_{i=m+1}^{n} \mu_i \lambda_i x_i\| &= \|v(\sum_{i=m+1}^{n} \mu_i \lambda_i e_i)\| \\ &\leq \|v\| \|\sum_{i=m+1}^{n} \lambda_i e_i\|_{\ell^{p'}} \to 0, \end{aligned}$$

as $m, n \to \infty$. From this and the Bounded Multiplier Test, it follows that $\sum_i \lambda_i x_i$ is an unconditionally convergent series in X. Moreover, for $x^* \in X^*$, the unconditional summability of the sequence $(\lambda_i x_i)_i$ for each $(\lambda_i)_i \in \ell^{p'}$ ensures that $(\lambda_i x^*(x_i))_i$ is absolutely summable for each $(\lambda_i)_i \in \ell^{p'}$, and in this way, the membership of $(x^*(x_i))_i$ in ℓ^p is defined for each $x^* \in X^*$. That is, *each $v : \ell^{p'} \longrightarrow X$ determines a weakly p-summable sequence $(x_i)_i$ of vectors in X by $v(e_i) = x_i$.* This completes the proof for $1 < p < \infty$.

Next we address the case $p = \infty$. Precisely, we shall show that *$\mathcal{L}(\ell^1, X)$ is isometrically isomorphic to $\ell^\infty_{weak}(X)$.* We have already shown in Theorem 1.3.8 that, by the Uniform Boundedness Theorem, $\ell^\infty_{\text{weak}}(X) = \ell^\infty_{\text{strong}}(X)$ isometrically isomorphically. Now let $(x_i)_i \in \ell^\infty_{\text{weak}}(X)$. Consider the map

$$u : X^* \longrightarrow \ell^\infty$$

defined by

$$u(x^*) := (< x^*, x_i >)_i, \quad \forall x^* \in X^*.$$

Then u is well-defined. The adjoint u^* of u is a bounded linear operator from $\ell^{\infty *}$ $(\frac{1}{\infty} + \frac{1}{1} := 1)$ with values in X^{**}. Let $v := u^*|_{\ell^1}$. Let us show that $v(e_j) = x_j$. Take $x^* \in X^*$. Then

$$\begin{aligned} x^*(x_j) &= u(x^*)_j \\ &= u^*(e_j)(x^*) \\ &= v(e_j)(x^*). \end{aligned}$$

This is the case for all $x^* \in X^*$, so $v(e_j) = x_j$. Since the linear span of the e_j's is dense in ℓ^1 and $v(e_j) \in X$ for each j, it follows that $v(\ell^1) \subseteq X$. Therefore, *each weak ℓ^∞-sequence $(x_i)_i$ of vectors in X; that is, a sequence $(x_i) \subset X$ such that $(x^*(x_i))_i \in \ell^\infty$ for each $x^* \in X^*$, determines an operator $v : \ell^1 \longrightarrow X$ by $v(e_j) = x_j$.*

Conversely, let $w : \ell^1 \longrightarrow X$ be a bounded linear operator defined by $w(e_j) := x_j$ for each j. Then proceeding similarly to the case of a finite p, let $(\lambda_i) \in \ell^1$ and $(\mu_i) \in B_{\ell^\infty}$. Then

$$\begin{aligned}\|\sum_{i=m+1}^{n} \mu_i\lambda_i x_i\| &= \|w(\sum_{i=m+1}^{n} \mu_i\lambda_i e_i)\| \\ &\leq \|w\|\|\sum_{i=m+1}^{n} \mu_i\lambda_i e_i\|_{\ell^1} \\ &= \|w\|(\sum_{i=m+1}^{n} |\mu_i||\lambda_i|) \leq \|w\|(\sum_{i=m+1}^{n} |\lambda_i|) \\ &= \|w\|\|\sum_{i=m+1}^{n} \lambda_i e_i\|_{\ell^1} \to 0,\end{aligned}$$

as $m, n \to \infty$. By the Bounded Multiplier Test, $\sum_i \lambda_i x_i$ converges unconditionally in X, and so for $x^* \in X^*$, the sequence $(\lambda_i x^*(x_i))_i$ is absolutely summable for each $(\lambda_i) \in \ell^1$. This determines, by duality, the membership of $(x^*(x_i))_i$ in ℓ^∞ for every x^* in X^*. Therefore, *each operator* $w : \ell^1 \longrightarrow X$ *determines a weak* ℓ^∞*-sequence* $(x_i)_i$ *of vectors in* X *by* $w(e_i) = x_i$. □

For some examples of strict inclusions, let $1 < p \leq \infty$. Then we have the isometric relations $\ell^p \otimes X = \mathcal{F}(\ell^{p'}, X) \hookrightarrow \mathcal{L}(\ell^{p'}, X) = \ell^p_{\text{weak}}(X)$ so that

$$\ell^p \overset{\vee}{\otimes} X \hookrightarrow \ell^p_{\text{weak}}(X),$$

isometrically. This inclusion is generally strict as can be seen, for instance, by considering $1 < p < \infty$. Then the metric accessibility of ℓ^p ensures that

$$\ell^p \overset{\vee}{\otimes} \ell^{p'} = \mathcal{K}(\ell^{p'}, \ell^{p'}),$$

the compact operators on $\ell^{p'}$, while $id_{\ell^{p'}}$ is doubtlessly a non-compact member of

$$\mathcal{L}(\ell^{p'}, \ell^{p'}) = \ell^p_{\text{weak}}(\ell^{p'}),$$

by Proposition 2.1.1. For $p = \infty$, note that sequences that belong to $\ell^\infty \overset{\vee}{\otimes} X$ all have relatively compact ranges, while there are sequences in $\ell^\infty_{\text{weak}}(X) = \ell^\infty_{\text{strong}}(X)$ that do not have relatively compact ranges for dim $X = \infty$.

Write

$$c_{0\,\text{weak}}(X)$$

for the closed subspace of $\ell^{\infty}_{\text{weak}}(X) = \ell^{\infty}_{\text{strong}}(X)$ consisting of all sequences (x_n) in X with $\lim_{n\to\infty} < x^*, x_n >= 0$ for all $x^* \in X^*$. Such sequences are said to be *weakly null sequences* in X. Also, $c_{0\,\text{weak}}(X)$ has a closed subspace

$$c_{0\,\text{strong}}(X)$$

of all sequences (x_n) in X with $\lim_{n\to\infty} \|x_n\| = 0$. Such sequences are said to be *strongly null sequences* in X.

By the Schur's ℓ^1 Theorem (Diestel et al. (1995), page 6), it can happen that $c_{0\,\text{weak}}(X) = c_{0\,\text{strong}}(X)$ for infinite dimensional spaces X, in particular when $X = \ell^1$. This phenomenon is, however, not typical since the unit vector basis in ℓ^p $(1 < p < \infty)$ is weakly null but not strongly null. However, we have the following fact:

Proposition 2.1.2. *Let X be a Banach space. Then*

$$c_0 \overset{\vee}{\otimes} X = c_{0\,strong}(X)$$

isometrically.

Proof. Let $(\lambda_i^1)_i, \ldots, (\lambda_i^n)_i \in c_0$ and $x_1, \ldots, x_n \in X$, then the natural inclusion

$$c_0 \otimes X \longrightarrow \ell^{\infty}_{\text{strong}}(X) : \sum_{j\leq n} (\lambda_i^j)_i \otimes x_j \mapsto (\sum_{j\leq n} \lambda_i^j x_j)_i$$

clearly takes values in $c_{0\,\text{strong}}(X)$ and satisfies

$$\begin{aligned}
\|\sum_{j\leq n} (\lambda_i^j)_i \otimes x_j\|_{\vee} &= \sup_{\gamma\in B_{\ell^1}, x^*\in B_{X^*}} |\sum_{j\leq n} \gamma((\lambda_i^j)_i) x^*(x_j)| \\
&= \sup_{\gamma\in B_{\ell^1}, x^*\in B_{X^*}} |\gamma(\sum_{j\leq n} x^*(x_j)(\lambda_i^j)_i)| \\
&= \sup_{x^*\in B_{X^*}} \|(\sum_{j\leq n} x^*(x_j)\lambda_i^j)_i\|_{c_0} \\
&= \|(\sum_{j\leq n} \lambda_i^j x_j)_i\|_{\ell^{\infty}_{\text{weak}}(X)} \\
&= \|(\sum_{j\leq n} \lambda_i^j x_j)_i\|_{\ell^{\infty}_{\text{strong}}(X)} \\
&= \|(\sum_{j\leq n} \lambda_i^j x_j)_i\|_{c_{0\,\text{strong}}(X)}.
\end{aligned}$$

□

A scalarly summable sequence in a Banach space need not be strongly summable. For instance, consider the sequence of unit coordinate vectors in c_0. For any $x^* \in (c_0)^* = \ell^1$ with $x^* = (\lambda_i)_i$, say, we have

$$\sum_j |x^*(e_j)| = \sum_j |\lambda_j| < \infty,$$

so that $(e_i)_i \in \ell^1_{\text{weak}}(c_0)$. But for each j, $\|e_j\| = 1$ and so $(e_i)_i \notin \ell^1_{\text{strong}}(c_0)$.

If $p' = \infty$, the proof given above for a finite p' does not apply because the linear span of the e_j's is not dense in ℓ^∞. However, we can restrict attention to the subspace c_0 of ℓ^∞ and obtain the following:

Proposition 2.1.3. *Let $p = 1$. Then the space $\mathcal{L}(c_0, X)$ of bounded linear operators from c_0 to X is isometrically isomorphic to the space of scalarly summable sequences of vectors in X, where an operator $u \in \mathcal{L}(c_0, X)$ corresponds to the sequence $(x_i)_i \in \ell^1_{weak}(X)$ by means of the formula $u(e_i) = x_i$.*

Proof. Let $u : c_0 \longrightarrow X$ be a bounded linear operator and $x^* \in X^*$. Let $ue_i := x_i$ and consider

$$\begin{aligned}
\sum_{i\leq n} |x^*(x_i)| &= \sum_{i\leq n} |x^*u(e_i)| \\
&= \sum_{i\leq n} (\operatorname{sign} x^*u(e_i)) \,.\, x^*u(e_i) \\
&= x^*(u(\sum_{i\leq n} \operatorname{sign} x^*u(e_i) \,.\, e_i)) \\
&\leq \|x^*\|\|u\|\|\sum_{i\leq n} (\operatorname{sign} x^*u(e_i))\, e_i\|_\infty \\
&\leq \|x^*\|\|u\|.
\end{aligned}$$

Hence

$$\|(x_i)_i\|_{\ell^1_{\text{weak}}(X)} \leq \|u\|.$$

Conversely, let $(x_i)_i \in \ell^1_{\text{weak}}(X)$ and define $u : c_0 \longrightarrow X$ by $ue_i := x_i$. If $x^* \in B_{X^*}$, then for any $(\lambda_i)_i \in c_0$ and any $n \in \mathbb{N}$ we have

$$\begin{aligned}
|x^*u(\sum_{i=1}^n \lambda_i e_i)| &\leq \sum_{i=1}^n |\lambda_i x^*(u(e_i))| \\
&\leq (\sup_{1\leq i\leq n} |\lambda_i|) \sum_{i=1}^n |x^*(u(e_i))| \\
&\leq \|(\lambda_i)_i\|_\infty \|(x_i)_i\|_{\ell^1_{\text{weak}}(X)},
\end{aligned}$$

whence

$$\|u(\sum_{i=1}^{n} \lambda_i e_i)\| \leq \|(\lambda_i)_i\|_\infty \|(x_i)_i\|_{\ell^1_{\text{weak}}(X)}$$

for each n, and so

$$\|u(\sum_{i} \lambda_i e_i)\| \leq \|(\lambda_i)_i\|_\infty \|(x_i)_i\|_{\ell^1_{\text{weak}}(X)}.$$

It follows that

$$\|u\| \leq \|(x_i)_i\|_{\ell^1_{\text{weak}}(X)}.$$

□

Corollary 2.1.4. *$\ell^1 \overset{\vee}{\otimes} X$ identifies with the space of unconditionally summable sequences in X. Furthermore, $\ell^1 \overset{\vee}{\otimes} X$ can be identified with the space of weak*-weak continuous compact linear operators from $\ell^\infty (= (\ell^1)^*)$ to X or with the space of compact linear operators from c_0 into X.*

Proof. We start by indentifying $\ell^1 \overset{\vee}{\otimes} X$ with the space of unconditionally summable sequences in X. Let $(x_n)_n \in \ell^1 \overset{\vee}{\otimes} X$. Then

$$(x_n)_n = |.|_\vee - \lim_n \sum_{i \leq n} e_i \otimes x_i.$$

It follows that

$$\lim_n |(0, \ldots, 0, x_n, x_{n+1}, \ldots)|_\vee = 0.$$

Choose $1 = n_0 < n_1 < \ldots$ so that

$$\sup_{\|x^*\| \leq 1} \sum_{i \geq n_j} |x^*(x_i)| < (j+1)^{-3}$$

for $j = 1, 2, \ldots$. Let $s_i = j^{-1}$ for $n_{j-1} \leq i < n_j$ and let $y_i = s_i^{-1} x_i$ for all $i \in \mathbb{N}$. Then $(s_i)_i \in c_0$ and if $n_{j-1} \leq m < n < n_k$

$$\sup_{x^* \in B_{X^*}} \sum_{i=m}^{n} |x^*(y_i)| \leq \sum_{l=j}^{k} l^{-2}.$$

which ensures that $(x^*(y_i))_i \in \ell^1$ for each $x^* \in X^*$ and $(y_i) \in \ell^1_{\text{weak}}(X)$ so that $\sum_n t_n y_n$ converges in X for each $(t_n)_n \in c_0$ by Proposition 2.1.3. Hence $\sum_n \lambda_n x_n = \sum_n \lambda_n s_n y_n$ converges in X for each $(\lambda_n)_n \in \ell^\infty$. By the Bounded Multiplier Test $\sum_n x_n$ is unconditionally convergent in X.

Conversely, suppose $(x_n)_n$ is unconditionally summable in X. Then $\sum_n \lambda_n x_n$ converges for each $\lambda = (\lambda_n)_n \in \ell^\infty$. It follows that $(x_n)_n \in \ell^1_{\text{weak}}(X)$. By Proposition 2.1.3 the operator

$$v : c_0 \longrightarrow X$$

defined by

$$v(\lambda) := \sum_n \lambda_n x_n,$$

is a bounded linear operator. Suppose that $m \leq n$. Then

$$\begin{aligned} |\sum_{i=m}^{n} e_i \otimes x_i|_\vee &= \sup_{\lambda \in B_{\ell^\infty}, x^* \in B_{X^*}} |\sum_{i=m}^{n} \lambda_i x^*(x_i)| \\ &= \sup_{\lambda \in B_{c_0}, x^* \in B_{X^*}} |\sum_{i=m}^{n} \lambda_i x^*(x_i)| \quad \text{(by Goldstine's Theorem)} \\ &= \sup_{\lambda \in B_{c_0}} \|\sum_{i=m}^{n} \lambda_i x_i\| \\ &= \sup_{\lambda \in B_{c_0}} \|v(\sum_{i=m}^{n} \lambda_i e_i)\| \\ &\leq \|v\| \|\sum_{i=m}^{n} \lambda_i e_i\|_\infty . \end{aligned}$$

Since $\lambda \in c_0$, $\lim_{m,n} \|\sum_{i=m}^{n} \lambda_i e_i\|_\infty = 0$; whence the unconditional summability of $(x_n)_n$ in $\ell^1 \overset{\vee}{\otimes} X$ follows.

Since c_0 is metrically accessible,

$$\ell^1 \overset{\vee}{\otimes} X = \overline{\mathcal{F}(c_0, X)}^{\|\cdot\|} = \mathcal{K}(c_0, X).$$

A weak* - weakly continuous linear operator $u : \ell^\infty \longrightarrow X$ is automatically compact since $u = v^*$ where $v : X^* \longrightarrow \ell^1$ is a weakly compact linear operator into ℓ^1, a Banach space in which relatively weakly compact sets are relatively norm compact by the Schur's ℓ^1 Theorem.

We identify $\mathcal{K}(c_0, X)$ with $\mathcal{K}_{w^*}(\ell^\infty, X)$ by realizing that u belongs to $\mathcal{K}_{w^*}(\ell^\infty, X)$ precisely when there is a v in $\mathcal{K}(c_0, X)$ such that $v^{**} = u$. In this case $v = u|_{c_0}$. □

Proposition 2.1.5. *Let $1 \leq p < \infty$. Then for every Banach space X, each of the inclusions*

$$\ell^p \overset{\wedge}{\otimes} X \hookrightarrow \ell^p_{strong}(X) \hookrightarrow \ell^p \overset{\vee}{\otimes} X,$$

is injective and has norm ≤ 1.

Proof. Define $\Phi : \ell^p \times X \longrightarrow \ell^p_{\text{strong}}(X)$ by

$$\Phi(\lambda, x) \mapsto \lambda.x := (\lambda_i x)_i,$$

where $\lambda = (\lambda_i)_i \in \ell^p$ and $x \in X$. Then $\|\lambda.x\|_p \leq \|\lambda\|_p\|x\|$ so that Φ is a bilinear operator from $\ell^p \times X$ into $\ell^p_{\text{strong}}(X)$ of norm ≤ 1. By the Universal Mapping Property Φ induces a linear operator, still denoted by Φ, of norm ≤ 1 from $\ell^p \overset{\wedge}{\otimes} X$ into $\ell^p_{\text{strong}}(X)$ which is the inclusion $\ell^p \overset{\wedge}{\otimes} X \hookrightarrow \ell^p_{\text{strong}}(X)$.

Let $(x_i)_i \in \ell^p_{\text{strong}}(X)$. Then $(x_i)_i \in \ell^p_{\text{weak}}(X)$ since

$$\begin{aligned} \sum_i |x^*(x_i)|^p &\leq \sum_i \|x^*\|^p\|x_i\|^p \\ &= \|x^*\|\|(x_i)_i\|^p_{\ell^p_{\text{strong}}(X)} \end{aligned}$$

for each $x^* \in X^*$; whence

$$\|(x_i)_i\|_{\ell^p_{\text{weak}}(X)} \leq \|(x_i)_i\|_{\ell^p_{\text{strong}}(X)}.$$

Moreover, on $\ell^p \otimes X$, a dense linear subspace of $\ell^p_{\text{strong}}(X)$, the natural inclusion of $\ell^p_{\text{strong}}(X)$ into $\ell^p_{\text{weak}}(X)$ is the natural identity taking $\overline{\ell^p \otimes X}^{\ell^p_{\text{strong}}(X)}$ into $\overline{\ell^p \otimes X}^{\ell^p_{\text{weak}}(X)} = \ell^p \overset{\vee}{\otimes} X$ since for each $u \in \ell^p \otimes X$ such that $u = \sum_{i \leq n} f_i \otimes x_i$ we have

$$\begin{aligned} \|u\|_{\ell^p_{\text{weak}}(X)} &= \|\sum_{i\leq n} f_i \otimes x_i\|_{\ell^p_{\text{weak}}(X)} \\ &= \|\sum_{i\leq n}(\sum_k < f_i, e_k > e_k) \otimes x_i\|_{\ell^p_{\text{weak}}(X)} \\ &= \|\sum_k e_k \otimes (\sum_{i\leq n} < f_i, e_k > x_i)\|_{\ell^p_{\text{weak}}(X)} \\ &= \lim_m \|\sum_{k\leq m} e_k \otimes x'_k\|_{\ell^p_{\text{weak}}(X)} \quad \text{where} \quad x'_k = \sum_{i\leq n} < f_i, e_k > x_i \\ &= \lim_m \|(x'_k)_{k\leq m}\|_{\ell^p_{\text{weak}}(X)} \\ &= \lim_m \sup_{x^* \in B_{X^*}} \|(x^*(x'_k))_{k\leq m}\|_{\ell^p} \\ &= \lim_m \sup_{x^* \in B_{X^*}} \sup_{\varphi \in B_{\ell^{p'}}} | < \varphi, (x^*(x'_k))_{k\leq m} > | \\ &= \lim_m \sup_{x^* \in B_{X^*}, \varphi \in B_{\ell^{p'}}} |\sum_{k\leq m} \varphi_k x^*(x'_k)| \\ &= \lim_m \sup_{x^* \in B_{X^*}, \varphi \in B_{\ell^{p'}}} |\sum_{k\leq m} \varphi(e_k) x^*(x'_k)| \\ &= |u|_\vee. \end{aligned}$$

Since ℓ^p is metrically accessible, the inclusion

$$\ell^p \overset{\wedge}{\otimes} X \hookrightarrow \ell^p \overset{\vee}{\otimes} X$$

is injective. Hence the set theoretical considerations force the inclusion

$$\ell^p \overset{\wedge}{\otimes} X \hookrightarrow \ell^p_{\text{strong}}(X)$$

to be injective as well. That the inclusion

$$\ell^p_{\text{strong}}(X) \hookrightarrow \ell^p \overset{\vee}{\otimes} X$$

is also injective is shown as follows: let $(x_n) \in \ell^p_{\text{strong}}(X)$ with $\sum_n e_n \otimes x_n = 0$ in $\ell^p \overset{\vee}{\otimes} X$. Then for all $\lambda \in \ell^{p'}, x^* \in X^*$ it follows that

$$(\lambda \otimes x^*)(\sum_n e_n \otimes x_n) = 0.$$

That is,

$$\sum_n \lambda_n x^*(x_n) = 0.$$

In particular, if $\lambda = e_k$ for a fixed $k \in \mathbb{N}$, then

$$\sum_n e_k(n) x^*(x_n) = 0$$

for every $x^* \in X^*$. Hence $x^*(x_k) = 0$ for every $x^* \in X^*$; whence $x_k = 0$. Since k is arbitray, it follows that $(x_n) = 0$. Therefore, the inclusion

$$\ell^p_{\text{strong}}(X) \hookrightarrow \ell^p \overset{\vee}{\otimes} X$$

is injective. This completes the proof of the proposition. □

Grothendieck's version of the famous Dvoretzsky-Rogers lemma is stated below and its proof can be consulted in (Grothendieck (1956b), Lemme, page 97).

Proposition 2.1.6. *Let E be an n-dimensional Banach space. There exist $x_1, \ldots, x_n \in S_E$ such that if $1 \leq r \leq n$, then for any real $\lambda_1, \ldots, \lambda_r$*

$$\|\sum_{i=1}^{r} \lambda_i x_i\| \leq M_r \|(\lambda_i)\|_{\ell^2_r},$$

where

$$M_r = 1 + \frac{1}{n}(1^2 + 2^2 + \ldots + (r-1)^2)^{1/2} \leq 1 + \frac{r\sqrt{r}}{n\sqrt{3}}.$$

This lemma gives rise to the following keystone result (Grothendieck (1956b), Théorèm 2, page 99):

Proposition 2.1.7. *If X is an infinite dimensional Banach space, $(a_i)_i$ is a sequence of non-negative real numbers with each $a_i < 1$ and $\lim_i a_i = 0$, then there is a sequence $(x_i)_i \in X$ such that $(x_i)_i \in \ell^2 \overset{\vee}{\otimes} X$ yet $\|x_i\| = a_i$ for each i.*

Proposition 2.1.7 has consequences that underly our claim of strict inclusions in Proposition 2.1.5. The first major consequence is Théorèm 3 in (Grothendieck (1956b), page 100):

Proposition 2.1.8. *If X is an infinite dimensional Banach space, $1 \le p \le 2$ and q satisfies $\frac{1}{q} = \frac{1}{p} - \frac{1}{2}$ (so that $2 \le q \le \infty$), then for any sequence (a_i) of non-negative real numbers with $(a_i) \in \ell^q$ (respectively c_0 if $q = +\infty$), there exists $(x_i)_i \in \ell^p \overset{\vee}{\otimes} X$ such that $\|x_i\| = a_i$. Moreover, given $\epsilon > 0$, $(a_i)_i \in \ell^p \overset{\vee}{\otimes} X$ can be chosen so that $\|(x_i)_i\|_{\ell^p_{weak}(X)} \le \|(a_i)_i\|_{\ell^q} + \epsilon$.*

The following corollary (Grothendieck (1956b), Corollaire 2, page 101) shows that the second inclusion in Proposition 2.1.5 is strict.

Corollary 2.1.9. *If X is an infinite dimensional Banach space and $1 \le p < \infty$, then $\ell^p \overset{\vee}{\otimes} X$ and $\ell^p_{strong}(X)$ are not the same. Hence, there is a scalarly p-th power summable sequence of members of X which is not p-th power summable.*

Proof. Let $1 \le p \le 2$. Since $\frac{1}{q} = \frac{1}{p} - \frac{1}{2}$ implies that $p < q$, we need only take $(a_i) \in \ell^q \setminus \ell^p$ and apply Proposition 2.1.8 above.

Suppose $2 \le p < \infty$. Let $(a_i)_i \in c_0 \setminus \ell^p$. By Proposition 2.1.7 we choose $(x_i)_i$ so that $(x_i)_i \in \ell^2 \overset{\vee}{\otimes} X$ and $\|x_i\| = |a_i|$. Then $(x_i)_i \in \ell^p \overset{\vee}{\otimes} X$ but $\sum \|x_i\|^p = +\infty$. □

The second major consequence of Proposition 2.1.7 is Théorèm 4 in (Grothendieck (1956b), page 101):

Proposition 2.1.10. *Let X be an infinite dimensional Banach space, $2 \le p' \le \infty$ and q' satisfy $\frac{1}{q'} = \frac{1}{p'} + \frac{1}{2}$ (so that $1 \le q' \le 2$). If $(a_i)_i \notin \ell^{q'}$ is a sequence of non-negative real numbers, then there is a sequence $(z_i)_i$ of members of X so that $\|z_i\| = a_i$ and $(z_i) \notin \ell^{p'} \overset{\wedge}{\otimes} X$.*

Corollary 2.1.11. *Let X be an infinite dimensional Banach space.*

1. *If (a_i) is a sequence of non-negative real numbers that is not summable, then there is a sequence $(x_i)_i$ of members of X with $\|x_i\| = a_i$ yet $(x_i) \notin \ell^2 \overset{\wedge}{\otimes} X$.*

2. *If $(a_i)_i$ is a sequence of non-negative real numbers that is not square summable, then there is a sequence $(x_i)_i$ of members of X with $\|x_i\| = a_i$ yet $(x_i)_i \notin \ell^\infty \overset{\wedge}{\otimes} X$.*

The next corollary (Grothendieck (1956b), Corollaire 2, page 103) shows that the first inclusion in Proposition 2.1.5 is strict.

Corollary 2.1.12. *If X is an infinite dimensional Banach space and $1 < p < \infty$, then $\ell^p_{strong}(X) \neq \ell^p \overset{\wedge}{\otimes} X$.*

Proof. Suppose that $2 < p$. Take $(a_i)_i \in \ell^p \setminus \ell^q$ where $\frac{1}{q} = \frac{1}{p} + \frac{1}{2}$. By Proposition 2.1.10 there is a sequence $(x_i)_i$ of members of X so that $\|x_i\| = |a_i|$ yet $(x_i)_i \notin \ell^p \overset{\wedge}{\otimes} X$.

Suppose that $1 < p \leq 2$. Take $(a_i)_i \in \ell^p \setminus \ell^1$. By Corollary 2.1.11 there is a sequence $(x_i)_i$ of members of X so that $\|x_i\| = |a_i|$ yet $(x_i)_i \notin \ell^2 \overset{\wedge}{\otimes} X$. On the one hand, $\|x_i\| = |a_i|$ so $(x_i)_i \in \ell^p_{\text{strong}}(X)$ while, on the other hand, $p < 2$ puts $\ell^p \overset{\wedge}{\otimes} X$ setwise inside $\ell^2 \overset{\wedge}{\otimes} X$ and so $(x_i)_i \notin \ell^p \overset{\wedge}{\otimes} X$. □

2.2 Tensoring with function spaces

We shall prove the following result:

Theorem 2.2.1. (*The Main Theorem*) *Let (Ω, Σ, μ) be a non-atomic probability measure space, $1 < p < \infty$ and X be an infinite dimensional Banach space. Then each of the inclusions*

$$L^p(\mu) \overset{\wedge}{\otimes} X \hookrightarrow L^p_{strong}(\mu, X) \hookrightarrow L^p(\mu) \overset{\vee}{\otimes} X \hookrightarrow L^p_{weak}(\mu, X) \hookrightarrow \mathcal{L}(X^*, L^p(\mu))$$

is injective and has norm ≤ 1. The inclusions

$$L^p(\mu) \overset{\wedge}{\otimes} X \hookrightarrow L^p_{strong}(\mu, X) \hookrightarrow L^p(\mu) \overset{\vee}{\otimes} X$$

are strict.

Lemma 2.2.2. *Let $1 < p < \infty$ and X be an infinite dimensional Banach space. Then*

$$L^p_{strong}(\mu, X) \hookrightarrow \mathcal{L}(X^*, L^p(\mu))$$

is an inclusion of norm ≤ 1.

Proof. Define

$$\Phi : L^p_{\text{strong}}(\mu, X) \longrightarrow \mathcal{L}(X^*, L^p(\mu))$$

by

$$\Phi : f \mapsto u_f,$$

where

$$u_f(x^*) := x^*f(.), \quad x^* \in X^*.$$

Take an $f \in L^p_{\text{strong}}(\mu, X)$. Then u_f *is* well-defined, linear and continuous.

To see that u_f is continuous we'll show it has a closed graph. So let $(x_n^*) \subseteq X^*$ and $x_n^* \to x_0^*$ and $u_f(x_n^*) \to g$ in $L^p(\mu)$. Since $x_n^* \to x_0^*$,

$$x_n^* f(\omega) \to x_0^* f(\omega)$$

for each $\omega \in \Omega$. On the other hand,

$$x_n^*(f(.)) = u_f(x_n^*)(.) \to g(.)$$

in $L^p(\mu)$; so there is a subsequence $(x_{n_k}^*)$ of (x_n^*) so that

$$x_{n_k}^* f(.) \to g(.)$$

μ-almost everywhere. Of course,

$$x_{n_k}^* f(\omega) \to x_0^* f(\omega) \quad \omega \in \Omega.$$

It follows that g and $x_0^* f = u_f(x_0^*)$ agree μ-almost everywhere and so as members of $L^p(\mu)$. Therefore u_f has a closed graph. □

Furthermore, u_f takes B_{X^*} into the *lattice bounded* set: for each $x^* \in B_{X^*}$,

$$\begin{aligned} |u_f(x^*)| &= |x^* f(.)| \\ &\leq \|x^*\| \|f(.)\| \\ &\leq \|f(.)\| \in L^p(\mu). \end{aligned}$$

Lemma 2.2.3. *Let $1 < p < \infty$. Then*

$$L^p_{strong}(\mu, X) \hookrightarrow L^p_{weak}(\mu, X)$$

boundedly with norm at most 1.

Proof. Take $f \in L^p_{\text{strong}}(\mu, X)$ and $x^* \in X^*$. Then

$$\int_\Omega |x^* f(\omega)|^p d\mu \leq \|x^*\|^p \int_\Omega \|f(\omega)\|^p d\mu < \infty,$$

and so $f \in L^p_{\text{weak}}(\mu, X)$. Therefore,

$$\|f\|_{L^p_{\text{weak}}(\mu,X)} \leq \|f\|_{L^p_{\text{strong}}(\mu,X)}.$$

□

Remark 2.2.4. *In general, the inclusion in Lemma 2.2.3 is strict.*

Lemma 2.2.5. *Let* $1 < p < \infty$. *Then*

$$L^p(\mu) \overset{\vee}{\otimes} X \hookrightarrow \mathcal{L}(X^*, L^p(\mu))$$

isometrically.

Proof. Let $u \in L^p(\mu) \otimes X$, $u = \sum_{i \leq n} f_i \otimes x_i$. Define

$$\Phi : (L^p(\mu) \otimes \ X, |.|_\vee) \longrightarrow \mathcal{L}(X^*, L^p(\mu))$$

by

$$\Phi(u) := \tilde{u},$$

where $\tilde{u} \in \mathcal{L}(X^*, L^p(\mu))$ is defined by

$$\tilde{u}(x^*) := \sum_{i \leq n} x^*(x_i) f_i.$$

Then $\tilde{u}(x^*) \in L^p(\mu)$ and $\tilde{u}$ is well-defined. For, if $u = \sum_{j \leq m} g_j \otimes x'_j$ is another representation of u, then for all $\varphi \in (L^p(\mu))^*$

$$\begin{aligned}
\varphi(\sum_{j \leq m} x^*(x'_j) g_j) &= \sum_{j \leq m} x^*(x'_j) \varphi(g_j) \\
&= (\varphi \otimes x^*)(\sum_{j \leq m} g_j \otimes x'_j) \\
&= (\varphi \otimes x^*)(u) \\
&= (\varphi \otimes x^*)(\sum_{i \leq n} f_i \otimes x_i) \\
&= \sum_{i \leq n} x^*(x_i) \varphi(f_i) \\
&= \varphi(\sum_{i \leq n} x^*(x_i) f_i).
\end{aligned}$$

Since $\varphi \in L^{p'}(\mu)$ is arbitrary,

$$\sum_{j \leq m} x^*(x_j) g_j = \sum_{i \leq n} x^*(x_i) f_i,$$

as members of $L^p(\mu)$. Clearly, $\tilde{u}$ *is* linear. Furthermore, it is norm bounded:

$$\begin{aligned}
|u|_\vee &= \sup\{|\sum_{i \leq n} \varphi(f_i) x^*(x_i)| : \|\varphi\| \leq 1, \|x^*\| \leq 1\} \\
&= \sup\{|(\sum_{i \leq n} x^*(x_i) f_i)(\varphi)| : \|\varphi\| \leq 1, \|x^*\| \leq 1\} \\
&= \sup\{\|\sum_{i \leq n} x^*(x_i) f_i\|_{L^p(\mu)} : \|x^*\| \leq 1\} \\
&= \sup\{\|\tilde{u}(x^*)\|_{L^p(\mu)} : \|x^*\| \leq 1\} \\
&= \|\tilde{u}\|_{\mathcal{L}(X^*, L^p(\mu))}.
\end{aligned}$$

So Φ is an isometry. By density, it extends to an isometry, still denoted by Φ, from

$$L^p(\mu) \overset{\vee}{\otimes} X \longrightarrow \mathcal{L}(X^*, L^p(\mu)).$$

Therefore,

$$L^p(\mu) \overset{\vee}{\otimes} X \hookrightarrow \mathcal{L}(X^*, L^p(\mu))$$

isometrically. □

The proof of the next proposition can be consulted in (Diestel et al. (2002a), Diestel et al. (2002b)).

Proposition 2.2.6. *If either X^* or Y is accessible, then the natural inclusion $X^* \overset{\alpha}{\otimes} Y \hookrightarrow \mathcal{L}(X, Y)$ is one-to-one.*

Proof of Theorem 2.2.1. Recall that we assume that $1 < p < \infty$ and (Ω, Σ, μ) is a non-atomic probability measure space.

1. $L^p(\mu) \overset{\wedge}{\otimes} X \hookrightarrow L^p_{\text{strong}}(\mu, X)$, injectively.

   Define a bilinear map

   $$J : L^p(\mu) \times X \longrightarrow L^p_{\text{strong}}(\mu, X)$$

   by

   $$J(f, x) := f(.)x.$$

Then

$$\begin{aligned}(\int_\Omega \|f(\omega)x\|^p d\mu(\omega))^{\frac{1}{p}} &= (\int_\Omega |f(\omega)|^p \|x\|^p d\mu(\omega))^{\frac{1}{p}} \\ &= \|x\|(\int_\Omega |f(\omega)|^p d\mu(\omega))^{\frac{1}{p}} \\ &= \|x\|\|f\|_{L^p(\mu)}.\end{aligned}$$

So

$$\|J(f,x)\|_{L^p_{\text{strong}}(\mu,X)} = \|x\|\|f\|_{L^p(\mu)},$$

which implies that $\|J\| \leq 1$.

By the Universal Mapping Property of $\overset{\wedge}{\otimes}$, J induces a bounded linear operator, still denoted by J, from $L^p(\mu) \overset{\wedge}{\otimes} X$ to $L^p_{\text{strong}}(\mu, X)$, with $\|J\| \leq 1$. That is, for all $u \in L^p(\mu) \overset{\wedge}{\otimes} X$,

$$\|Ju\|_{L^p_{\text{strong}}(\mu,X)} \leq \|u\|_{L^p(\mu)\overset{\wedge}{\otimes}X},$$

and so

$$L^p(\mu) \overset{\wedge}{\otimes} X \hookrightarrow L^p_{\text{strong}}(\mu, X)$$

with $\|.\| \leq 1$.

Next, we address the question of the injectivity of J. First recall that $L^p(\mu)$ is metrically accessible (Diestel and Uhl Jr (1977), VIII.3.11) and (Diestel et al. (2002a), Diestel et al. (2002b)).

Of course, metric accessibility implies accessibility, and so we shall appeal to Proposition 2.2.6 with impunity.

Recall the definition of p' by $\frac{1}{p} + \frac{1}{p'} = 1$, and consider the configuration below:

$$L^p(\mu) \overset{\wedge}{\otimes} X = (L^{p'}(\mu))^* \overset{\wedge}{\otimes} X \hookrightarrow \mathcal{L}(L^{p'}(\mu), X) \longrightarrow \mathcal{L}(X^*, L^p(\mu)),$$

where the inclusion

$$L^p(\mu) \overset{\wedge}{\otimes} X = (L^{p'}(\mu))^* \overset{\wedge}{\otimes} X \hookrightarrow \mathcal{L}(L^{p'}(\mu), X)$$

is one-to-one by Proposition 2.2.6, and the map

$$\mathcal{L}(L^{p'}(\mu), X) \longrightarrow \mathcal{L}(X^*, L^p(\mu))$$

defined by

$$T \mapsto T^*$$

is an isometric isomorphism, and therefore, injective. It follows that the map

$$S : L^p(\mu) \overset{\wedge}{\otimes} X \longrightarrow \mathcal{L}(X^*, L^p(\mu))$$

is injective. On the other hand, the following configuration

$$L^p(\mu) \overset{\wedge}{\otimes} X \overset{J}{\hookrightarrow} L^p_{\text{strong}}(\mu, X) \overset{\Phi}{\hookrightarrow} \mathcal{L}(X^*, L^p(\mu)), \qquad S : L^p(\mu) \overset{\wedge}{\otimes} X \to \mathcal{L}(X^*, L^p(\mu))$$

tells us that

$$S = \Phi \circ J.$$

Since S is injective, it follows from set-theoretic arguments that J is injective. This inclusion, namely J, *is* strict. Indeed, Σ contains an infinite sequence (E_n) of pairwise disjoint sets with $0 < \mu(E_j) < \infty$, for $j = 1, 2, \ldots$, such that $\cup_{n=1}^{\infty} E_n = \Omega$. Let $(x_n)_{n=1}^{\infty} \in \ell^p_{strong}(X) \setminus \ell^p \overset{\wedge}{\otimes} X$ and consider $f := \sum_{n=1}^{\infty} \frac{\chi_{E_n}}{\mu(E_n)^{\frac{1}{p}}} x_n$. Then $f \in L^p_{\text{strong}}(\mu, X) \setminus L^p(\mu) \overset{\wedge}{\otimes} X$. Indeed,

$$\begin{aligned}
\int_\Omega \|f(\omega)\|^p d\mu(\omega) &= \int_\Omega \|\sum_{n=1}^{\infty} \frac{\chi_{E_n}(\omega)}{\mu(E_n)^{\frac{1}{p}}} x_n\|^p d\mu(\omega) \\
&= \sum_{m=1}^{\infty} \int_{E_m} \|\sum_{n=1}^{\infty} \frac{\chi_{E_n}(\omega)}{\mu(E_n)^{\frac{1}{p}}} x_n\|^p d\mu(\omega) \\
&= \sum_{m=1}^{\infty} \int_{E_m} \|\frac{x_m}{\mu(E_m)^{\frac{1}{p}}}\|^p d\mu(\omega) \\
&= \sum_{m=1}^{\infty} \frac{\|x_m\|^p}{\mu(E_m)} \mu(E_m) \\
&= \sum_{m=1}^{\infty} \|x_m\|^p < \infty.
\end{aligned}$$

As credit we obtain equality of norms: $\|f\|_{L^p_{\text{strong}}(\mu,X)} = \|(x_n)\|_{\ell^p_{\text{strong}}(X)}$. Furthermore, let us recall that for a measure space as above ℓ^p embeds isometrically into L^p under the mapping

$$\begin{aligned}
i : \ell^p &\longrightarrow L^p \\
(\xi_j) &\mapsto \sum_{j=1}^{\infty} \xi_j \mu(E_j)^{-\frac{1}{p}} \chi_{E_j}.
\end{aligned}$$

Then we have

$$\begin{aligned}
\|f\|_{L^p\hat{\otimes}X} &= |\sum_{n=1}^{\infty} \frac{\chi_{E_n}}{\mu(E_n)^{\frac{1}{p}}} \otimes x_n|_\wedge \\
&= \lim_N |\sum_{n=1}^{N} \frac{\chi_{E_n}}{\mu(E_n)^{\frac{1}{p}}} \otimes x_n|_\wedge \\
&= \lim_N |\sum_{n=1}^{N} i(e_n) \otimes x_n|_\wedge \\
&= \lim_N |(i \otimes id_X)(\sum_{n=1}^{N} e_n \otimes x_n)|_\wedge \\
&\leq |\sum_{n=1}^{\infty} e_n \otimes x_n|_\wedge \\
&= \|(x_n)\|_{\ell^p\hat{\otimes}X}.
\end{aligned}$$

Conversely,

$$\begin{aligned}
\|(x_n)\|_{\ell^p\hat{\otimes}X} &= |\sum_{n=1}^{\infty} e_n \otimes x_n|_\wedge \\
&\leq \lim_N \sum_{n=1}^{N} \|x_n\| \\
&= \lim_N \sum_{n=1}^{N} \|\frac{\chi_{E_n}}{\mu(E_n)^{\frac{1}{p}}}\| \|x_n\|.
\end{aligned}$$

Hence

$$\|(x_n)\|_{\ell^p\hat{\otimes}X} \leq \|f\|_{L^p\hat{\otimes}X},$$

so that

$$\|f\|_{L^p\hat{\otimes}X} \geq \|(x_n)\|_{\ell^p\hat{\otimes}X} = \infty.$$

So $f \notin L^p \hat{\otimes} X$.

2. $L^p_{\text{weak}}(\mu, X) \hookrightarrow \mathcal{L}(X^*, L^p(\mu))$ isometrically.

   We shall first show that $\sup\{\|x^*f\|_{L^p(\mu)} : \|x^*\| \leq 1\}$ is a finite quantity and defines a norm on $L^p_{\text{weak}}(\mu, X)$. Define

$$\begin{aligned}
\Phi : L^p_{\text{weak}}(\mu, X) &\longrightarrow \mathcal{L}(X^*, L^p(\mu)) \\
f &\mapsto u_f,
\end{aligned}$$

by

$$u_f(x^*) := x^* f(.),$$

for all $x^* \in X^*$. It is now routine to justify (as we did, for instance, in the proof of Lemma 2.2.2) the conclusion that u_f is a bounded linear operator: once again the Closed Graph Theorem comes to our rescue. Hence

$$\sup\{\|u_f(x^*)\|_{L^p(\mu)} : \|x^*\| \leq 1\} = \|u_f\| < \infty.$$

That is,

$$\sup\{\|x^* f(.)\|_{L^p(\mu)} : \|x^*\| \leq 1\} < \infty.$$

Putting

$$\|f\|_{L^p_{\text{weak}}(\mu,X)} := \sup\{\|x^* f(.)\|_{L^p(\mu)} : \|x^*\| \leq 1\},$$

it is routine to verify that this *is* a norm on $L^p_{\text{weak}}(\mu, X)$. In this way our first task is accomplished. To address the above inclusion we have

$$\begin{aligned} \|u_f\| &= \sup\{\|u_f(x^*)\|_{L^p(\mu)} : \|x^*\| \leq 1\} \\ &= \sup\{\|x^* f(.)\|_{L^p(\mu)} : \|x^*\| \leq 1\} \\ &= \|f(.)\|_{L^p_{\text{weak}}(\mu,X)}, \end{aligned}$$

so that

$$\|\Phi f\| = \|f(.)\|_{L^p_{\text{weak}}(\mu,X)},$$

as was to be shown.

3. $L^p(\mu) \overset{\vee}{\otimes} X \hookrightarrow L^p_{\text{weak}}(\mu, X)$ isometrically.

Define

$$\begin{aligned} \Psi : (L^p(\mu) \otimes X, |.|_{\vee}) &\longrightarrow L^p_{\text{weak}}(\mu, X) \\ f \otimes x &\mapsto f(.)x, \end{aligned}$$

and extend by linearity. Let $u = \sum_{i \leq n} f_i \otimes x_i \in (L^p(\mu) \otimes X, |.|_{\vee})$. Then for all $x^* \in X^*$, $\|x^*\| \leq 1$, we have

$$\begin{aligned}
\left(\int_\Omega |x^*(\sum_{i\leq n} f_i(\omega)x_i)|^p d\mu(\omega)\right)^{\frac{1}{p}} &= \left(\int_\Omega |\sum_{i\leq n} f_i(\omega)x^*(x_i)|^p d\mu(\omega)\right)^{\frac{1}{p}} \\
&= \|\sum_{i\leq n} x^*(x_i)f_i(.)\|_{L^p(\mu)} \\
&= \sup\{|\varphi(\sum_{i\leq n} x^*(x_i)f_i)| : \|\varphi\| \leq 1\} \\
&= \sup\{|\sum_{i\leq n} x^*(x_i)\varphi(f_i)| : \|\varphi\| \leq 1\} \\
&\leq \sup_{\|u^*\|\leq 1}\{|\sum_{i\leq n} u^*(x_i)\varphi(f_i)| : \|\varphi\| \leq 1\} \\
&= |\sum_{i\leq n} f_i \otimes x_i|_\vee .
\end{aligned}$$

Therefore,

$$\sup\{\left(\int_\Omega |x^*(\sum_{i\leq n} f_i(\omega)x_i)|^p d\mu(\omega)\right)^{\frac{1}{p}} : \|x^*\| \leq 1\} \leq |u|_\vee < \infty,$$

which implies that

$$\sum_{i\leq n} f_i(.)x_i \in L^p_{\text{weak}}(\mu, X)$$

and

$$\|\sum_{i\leq n} f_i(.)x_i\|_{L^p_{\text{weak}}(\mu,X)} \leq |u|_\vee .$$

That is,

$$\|\Psi u\|_{L^p_{\text{weak}}(\mu,X)} \leq |u|_\vee .$$

Extending by continuity and still denoting the extension by Ψ, we have

$$L^p(\mu) \overset{\vee}{\otimes} X \hookrightarrow L^p_{\text{weak}}(\mu, X)$$

with norm ≤ 1. Now we chase the following diagram:

$$L^p(\mu) \overset{\vee}{\otimes} X \overset{\Psi}{\hookrightarrow} L^p_{\text{weak}}(\mu, X) \overset{\Phi}{\hookrightarrow} \mathcal{L}(X^*, L^p(\mu))$$
$$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}_{\Theta}$$

Here Ψ is the bounded inclusion just proved in 3 above, Φ is the isometric embedding proved in 2 above and Θ is the isometric embedding in Lemma 2.2.5. We have that $\Theta = \Phi \circ \Psi$, and since Θ and Φ are isometric embeddings, it follows that Ψ is an isometric embedding too.

The above inclusion *can* be strict. Indeed, consider our earlier function $f = \sum_{n=1}^{\infty} \frac{\chi_{E_n}}{\mu(E_n)^{\frac{1}{p}}} x_n$ where, this time, $(x_i) \in \ell^p_{\text{weak}}(X)$ and $(E_n)_{n=1}^{\infty}$ is a pairwise disjoint sequence of sets in Σ with $0 < \mu(E_n) < \infty$ for all $n \in \mathbb{N}$ and $\cup_{n=1}^{\infty} E_n = \Omega$. Then

$$\|f\|_{L^p_{\text{weak}}(\mu,X)} = \sup\{\|x^* f\|_{L^p(\mu)} : \|x^*\| \leq 1\},$$

where

$$\begin{aligned}
\|x^* f\|_{L^p(\mu)} &= \left(\int_{\Omega} |x^* f(\omega)|^p d\mu(\omega)\right)^{\frac{1}{p}} \\
&= \left(\sum_{m=1}^{\infty} \int_{E_m} \left|\sum_{n=1}^{\infty} x^*(x_n) \frac{\chi_{E_n}(\omega)}{\mu(E_n)^{\frac{1}{p}}}\right|^p d\mu(\omega)\right)^{\frac{1}{p}} \\
&= \left(\sum_{m=1}^{\infty} \int_{E_m} \left|x^*(x_m) \frac{1}{\mu(E_m)^{\frac{1}{p}}}\right|^p d\mu(\omega)\right)^{\frac{1}{p}} \\
&= \left(\sum_{m=1}^{\infty} |x^*(x_m)|^p\right)^{\frac{1}{p}}.
\end{aligned}$$

Hence

$$\begin{aligned}
\|f\|_{L^p_{\text{weak}}(\mu,X)} &= \sup\{\|x^* f\|_{L^p(\mu)} : \|x^*\| \leq 1\} \\
&= \sup\{\left(\sum_{m=1}^{\infty} |x^*(x_m)|^p\right)^{\frac{1}{p}} : \|x^*\| \leq 1\} \\
&= \|(x_m)\|_{\ell^p_{\text{weak}}(X)} < \infty.
\end{aligned}$$

Since

$$\ell^p \overset{\vee}{\otimes} X \hookrightarrow \ell^p_{\text{weak}}(X)$$

is strict, it follows that by taking $(x_n) \in \ell^p_{\text{weak}}(X) \setminus \ell^p \overset{\vee}{\otimes} X$ and building

f as above, $f \notin L^p(\mu) \overset{\vee}{\otimes} X$. In fact,

$$\begin{aligned}
\|f\|_{L^p(\mu)\overset{\vee}{\otimes}X} &= |\sum_{n=1}^{\infty} \frac{\chi_{E_n}}{\mu(E_n)^{\frac{1}{p}}} \otimes x_n|_{\vee} \\
&= \lim_N |\sum_{n=1}^{N} \frac{\chi_{E_n}}{\mu(E_n)^{\frac{1}{p}}} \otimes x_n|_{\vee} \\
&= \lim_N \sup_{\|x^*\|\leq 1} \{|\sum_{n=1}^{N} \varphi(ie_n)x^*(x_n)| : \|\varphi\|_{L^{p'}(\mu)} \leq 1\} \\
&= \lim_N \sup_{\|x^*\|\leq 1} \{|\sum_{n=1}^{N} \psi(e_n)x^*(x_n)| : \|\psi\|_{\ell^{p'}} \leq 1\} \\
&= \lim_N |\sum_{n=1}^{N} e_n \otimes x_n|_{\vee} \\
&= \|(x_n)\|_{\ell^p\overset{\vee}{\otimes}X} = \infty.
\end{aligned}$$

4. $L^p_{\text{strong}}(\mu, X) \hookrightarrow L^p(\mu) \overset{\vee}{\otimes} X$ injectively.

   As Lemma 2.2.3 has shown,

$$f \in L^p_{\text{strong}}(\mu, X) \Longrightarrow f \in L^p_{\text{weak}}(\mu, X),$$

   and

$$\|f\|_{L^p_{weak}(\mu,X)} \leq \|f\|_{L^p_{\text{strong}}(\mu,X)}.$$

   The natural inclusion above is the natural identity on $L^p(\mu) \otimes X$. Since $L^p(\mu) \otimes X$ is a dense subspace of both $L^p_{\text{strong}}(\mu, X)$ and $L^p_{\text{weak}}(\mu, X)$, this inclusion takes $\overline{L^p(\mu) \otimes X}^{L^p_{\text{strong}}(\mu,X)}$ into $\overline{L^p(\mu) \otimes X}^{L^p_{\text{weak}}(\mu,X)}$.

   If we call this natural identity Γ, for a moment, then it holds that on $L^p(\mu) \otimes X$,

$$\Gamma(\sum_{i\leq n} f_i \otimes x_i) = \sum_{i\leq n} f_i \otimes x_i$$

   and

$$\|\Gamma(\sum_{i\leq n} f_i \otimes x_i)\|_{L^p_{\text{weak}}(\mu,X)} \leq \|\sum_{i\leq n} f_i \otimes x_i\|_{L^p_{\text{strong}}(\mu,X)}$$

   on $L^p(\mu) \otimes X$, and extending by continuity, using the same notation, Γ takes $\overline{L^p(\mu) \otimes X}^{L^p_{\text{strong}}(\mu,X)}$ into $\overline{L^p(\mu) \otimes X}^{L^p_{\text{weak}}(\mu,X)}$ boundedly with $\|\Gamma\| \leq 1$. But

$$\overline{L^p(\mu) \otimes X}^{L^p_{\text{weak}}(\mu,X)} = L^p(\mu) \overset{\vee}{\otimes} X$$

isometrically. Indeed, let $u = \sum_{i \leq n} f_i \otimes x_i$, and observe that

$$\begin{aligned}
\|u\|_{L^p_{\text{weak}}(\mu,X)} &= \|\sum_{i\leq n} f_i(.)x_i\|_{L^p_{\text{weak}}(\mu,X)} \\
&= \sup\{\|x^*(\sum_{i\leq n} f_i(.)x_i)\|_{L^p(\mu)} : \|x^*\| \leq 1\} \\
&= \sup\{|\sum_{i\leq n} x^*(x_i)\varphi(f_i)| : \|x^*\| \leq 1, \|\varphi\| \leq 1\} \\
&= |\sum_{i\leq n} f_i \otimes x_i|_{\vee} \\
&= |u|_{\vee}.
\end{aligned}$$

On extending by density we obtain

$$\overline{L^p(\mu) \otimes X}^{L^p_{\text{weak}}(\mu,X)} = L^p(\mu) \overset{\vee}{\otimes} X$$

isometrically. Thus we have shown that

$$L^p_{\text{strong}}(\mu, X) \hookrightarrow L^p(\mu) \overset{\vee}{\otimes} X$$

with norm ≤ 1. Since $L^p(\mu)$ is metrically accessible, it follows that

$$L^p(\mu) \overset{\wedge}{\otimes} X \hookrightarrow L^p(\mu) \overset{\vee}{\otimes} X$$

is an injection: one needs only apply Proposition 2.2.6 to the diagram below.

$$L^p(\mu) \overset{\wedge}{\otimes} X \overset{\Pi}{\hookrightarrow} L^p \overset{\vee}{\otimes} X \overset{\Theta}{\hookrightarrow} \mathcal{L}(X^*, L^p(\mu)), \qquad S : L^p(\mu) \overset{\wedge}{\otimes} X \to \mathcal{L}(X^*, L^p(\mu))$$

By Lemma 2.2.5, Θ is an isometric inclusion. From the size of its norm, $L^p(\mu) \overset{\wedge}{\otimes} X$ sits boundedly in $L^p(\mu) \overset{\vee}{\otimes} X$ so that Π is at least bounded, where

$$\Pi : L^p(\mu) \overset{\wedge}{\otimes} X \hookrightarrow L^p(\mu) \overset{\vee}{\otimes} X,$$

and S is the injection, by Proposition 2.2.6, established in the course of proving 1 above. It follows from

$$S = \Theta \circ \Pi$$

that Π is injective.

Next we lay to rest the fact that

$$L^p_{\text{strong}}(\mu, X) \hookrightarrow L^p(\mu) \overset{\vee}{\otimes} X$$

is injective. We shall show this directly by appealing to Pettis' Measurability Theorem. Let $f \in L^p_{\text{strong}}(\mu, X)$ and consider $u_f \in L^p(\mu) \overset{\vee}{\otimes} X$ with

$$u_f = 0,$$

where $u_f : X^* \longrightarrow L^p(\mu)$ is defined by $u_f(x^*) := x^* f(.)$. Then $\forall x^* \in X^*$, $\forall g \in L^{p'}(\mu)$,

$$\int_\Omega u_f(x^*)(\omega) g(\omega) d\mu(\omega) = 0,$$

which implies that

$$\int_\Omega x^* f(\omega) g(\omega) d\mu(\omega) = 0.$$

So $x^* f(.)$ (as a member of $L^p(\mu)$) $= 0$ μ-a.e. Since f is strongly μ-measurable, $\exists N_0$, a set of μ-measure zero so that outside N_0, f has a separable range; that is,

$$f(\Omega \setminus N_0) \subset S \subset X,$$

where S is a separable subset of X. There exists a countable set $\{x_n^*\} \subset X^*$ that norms S and so norms $f(\Omega \setminus N_0)$. Thus $\forall n$, $\exists N_n$ μ-null so that $x_n^* f(.) = 0$ outside N_n. Put

$$N = N_0 \cup N_1 \cup N_2 \cup \ldots.$$

Then N is a μ-null set and $x_n^* f(\omega) = 0$ whenever $\omega \notin N$. Since $\{x_n^*\}$ norms $f(\Omega \setminus N)$, we have

$$f(\omega) = 0$$

for all $\omega \notin N$. It follows that $f = 0$ μ-a.e. The injective inclusion

$$L^p_{\text{strong}}(\mu, X) \hookrightarrow L^p(\mu) \overset{\vee}{\otimes} X$$

can be strict. Indeed, take $(x_n) \in \ell^p \overset{\vee}{\otimes} X \setminus \ell^p_{\text{strong}}(X)$ and consider again the function $f = \sum_{n=1}^{\infty} \frac{\chi_{E_n}}{\mu(E_n)^{\frac{1}{p}}} x_n$. Then

$$\begin{aligned}
\|f\|_{L^p(\mu)\overset{\vee}{\otimes}X} &= \lim_N \sup_{\|x^*\|\leq 1} \{|\sum_{n=1}^{N} \varphi(\frac{\chi_{E_n}}{\mu(E_n)^{\frac{1}{p}}})x^*(x_n)| : \|\varphi\|_{L^{p'}(\mu)} \leq 1\} \\
&= \lim_N \sup_{\|x^*\|\leq 1} \{|\sum_{n=1}^{N} \varphi(ie_n)x^*(x_n)| : \|\varphi\|_{L^{p'}(\mu)} \leq 1\} \\
&= \lim_N \sup_{\|x^*\|\leq 1} \{|\sum_{n=1}^{N} \psi(e_n)x^*(x_n)| : \|\psi\|_{\ell^{p'}} \leq 1\} \\
&= \lim_N |\sum_{n=1}^{N} e_n \otimes x_n|_{\vee} \\
&= \|(x_n)\|_{\ell^p\overset{\vee}{\otimes}X},
\end{aligned}$$

so that $f \in L^p(\mu) \overset{\vee}{\otimes} X$. Since

$$\|f\|_{L^p_{\text{strong}}(\mu,X)} = \|(x_n)\|_{\ell^p_{\text{strong}}(X)} = \infty,$$

$f \notin L^p_{\text{strong}}(\mu, X)$.

□

Remark 2.2.7. *The injectivity of* Π *in the above proof forces the inclusion*

$$L^p(\mu) \overset{\wedge}{\otimes} X \hookrightarrow L^p_{strong}(\mu, X)$$

to be injective and so reproves 1. *One needs only take* Π *in conjunction with* 4 *above to appreciate this.*

Chapter 3

The Hilbertian tensor products

In the first section of this chapter we derive the basic facts about the tensor norms h and h^* as introduced in (Grothendieck (1956a), §3). We follow this with a discussion of the Radon-Nikodym property and the Lewis-Radon-Nykodym property for tensor norms. In particular we show that the Hilbertian tensor norm h *has* the Lewis-Radon-Nikodym property but *does not have* the Radon-Nikodym property. Along the way we show that if α is a tensor norm with the Radon-Nikodym property, then $\alpha/$, $\backslash\alpha$ and $/\alpha$ have the property as well, but that in general $\alpha\backslash$ doesn't need to also have this property.

3.1 The Hilbertian tensor norms h and h^*

Let α be a tensor norm and X and Y be Banach spaces. Since $\alpha(u) \leq |u|_\wedge$ for any $u \in X \otimes Y$, it follows that $(X \overset{\alpha}{\otimes} Y)^*$ consists of bilinear continuous forms on $X \times Y$ that are α-continuous.

A bilinear form $\varphi \in \mathcal{B}(X, Y)$ is said to be of *type* α or α-*integral* provided $\varphi \in (X \overset{\alpha^*}{\otimes} Y)^*$. The space of all these bilinear forms on $X \times Y$, the dual of $X \overset{\alpha^*}{\otimes} Y$ is denoted by $\mathcal{B}^\alpha(X, Y)$ and equipped with the norm $||.||_\alpha$, where

$$||\varphi||_\alpha := ||\varphi||_{(X \overset{\alpha^*}{\otimes} Y)^*}.$$

The space $\mathcal{B}^\alpha(X, Y)$ is a Banach space, a dual space, with this norm.

A bounded linear operator $u : X \longrightarrow Y$ is said to be of type α or α-*integral* if the bilinear map φ_u on $X \times Y^*$ given by

$$\varphi_u(x, y^*) := y^*(u(x))$$

is of type α. The space of α-integral linear operators from X to Y is denoted by $\mathcal{L}^\alpha(X,Y)$ and equipped with the norm $\|u\|_\alpha = \|\varphi_u\|_{\mathcal{B}^\alpha(X,Y^*)}$. The space $\mathcal{L}^\alpha(X,Y)$ is a Banach space, an isometric isomorph of $\mathcal{B}^\alpha(X,Y^*)$, in this norm.

The action $<,>$ of X^* on X induces the functional Tr on $X^* \overset{\wedge}{\otimes} X$. We can identify the space $\mathcal{L}^\alpha(X,Y^*)$ with the dual $(X \overset{\alpha^*}{\otimes} Y)^*$ as follows: if $u : X \longrightarrow Y^*$ is α-integral, the Kronecker product

$$u \otimes id_Y : X \otimes Y \longrightarrow Y^* \otimes Y$$

extends to a continuous map of norm $\leq \|u\|_\alpha$ from $X \overset{\alpha^*}{\otimes} Y$ into $Y^* \overset{\wedge}{\otimes} Y$. Then the action of u on $v \in X \overset{\alpha^*}{\otimes} Y$ is taken to be

$$< v, u >:=< (u \otimes id_Y)(v), Tr > .$$

There is always a canonical map

$$X^* \overset{\alpha}{\otimes} Y^* \hookrightarrow \mathcal{L}^\alpha(X,Y^*)$$

which send a simple tensor $x^* \otimes y^*$ to the operator defined by

$$u(x) :=< x, x^* > y^*.$$

This natural map has norm one and is injective if X^* or Y^* are accessible and is an into isometry if both X^* and Y^* are metrically accessible. We write

$$X^* \overset{\alpha}{\otimes} Y^* = \mathcal{L}^\alpha(X,Y^*)$$

isometrically if this canonical map is an onto isometry.

Let $1 < p < \infty$. An operator $u : X \longrightarrow Y$ is said to be γ_p-*integral* if $j_Y u$ has a factorization

$$X \xrightarrow{v} L^p(\nu) \xrightarrow{w} Y^{**}$$

for some measure ν, where $j_Y : Y \hookrightarrow Y^{**}$ is the canonical embedding. The γ_p-integral norm is

$$\gamma_p(u) := \inf \|v\|\|w\|,$$

with the infimum taken over all such factorizations.

We write $u \in \mathcal{L}^{\gamma_p}(X,Y)$ if u is γ_p-integral; the notation $\Gamma_p(X,Y)$ is also used in the literature (Diestel et al. (1995), Chapters 7 and 9). It is shown in (Kwapień (1972), §2, Proposition 1) that γ_p is a maximal Banach operator ideal norm. Then the isometric equality

$$(E \otimes F, \gamma_p) := \mathcal{L}^{\gamma_p}(E^*, F)$$

defines a tensor norm γ_p (using the same notation) on $E \otimes F$ for all finite dimensional normed spaces E and F associated with the maximal Banach operator ideal $\mathcal{L}^{\gamma_p}$ (Defant and Floret (1993), 17.3, Ryan (2002), Theorem 8.9); that is, if $u \in (E \otimes F, \gamma_p)$ and $\tilde{u} \in \mathcal{L}(E^*, F)$ is the associated operator, then

$$\gamma_p(u) := \gamma_p(\tilde{u})$$

defines γ_p to be a tensor norm: it *is* clearly a crossnorm. The uniform crossnorm property of γ_p is the same as the ideal property for the Banach operator ideal $\mathcal{L}^{\gamma_p}$ (for finite dimensional normed spaces) since for $u \in E \otimes F$,

$$\tilde{t} \circ \tilde{u} \circ \tilde{v}^* = \tilde{u}_{(\tilde{v} \otimes \tilde{t})(u)}$$

holds for all $\tilde{v} \in \mathcal{L}(E, E_1)$ and $\tilde{t} \in \mathcal{L}(F, F_1)$, where $\tilde{u}_{(\tilde{v} \otimes \tilde{t})(u)} \in \mathcal{L}(E_1^*, F_1)$ is the operator associated with $(\tilde{v} \otimes \tilde{t})(u)$. The extension of the definition of the tensor norm γ_p on $E \otimes F$ to infinite dimensional spaces is the following result the proof of which can be consulted in (Defant and Floret (1993), Chapter II, §17.5):

Proposition 3.1.1. (*Representation Theorem For Maximal Operator Ideals*) *Let $(\mathcal{U}, A)$ be a maximal normed ideal and α a finitely generated tensor norm which are associated with each other: $(\mathcal{U}, A) \sim \alpha$. Then for all Banach spaces X and Y the relations*

$$\mathcal{U}(X, Y^*) = (X \otimes Y, \alpha^*)^*$$

and

$$\mathcal{U}(X, Y) = (X \otimes Y^*, \alpha^*)^* \cap \mathcal{L}(X, Y)$$

hold isometrically.

A tensor norm α is said to be *metrically accessible* if the inclusion

$$X \overset{\alpha}{\otimes} Y \hookrightarrow B^{\alpha}(X^*, Y^*)$$

is an isometry provided that one of X, Y is finite dimensional.

The following result is proved in (Diestel et al. (1997)).

Proposition 3.1.2. *The canonical inclusion*

$$X \overset{\alpha}{\otimes} Y \hookrightarrow B^{\alpha}(X^*, Y^*)$$

is an isometric injection in case

1. *both X and Y are metrically accessible or*

2. *α is metrically accessible and one of X and Y is metrically accessible.*

In particular, the norms $\wedge$ and $\vee$ are metrically accessible (Diestel et al. (1997)).

A *semi-inner product* Φ on X is a continuous functional Φ on $X \times X$ such that:

(a) $\Phi(x,x) \geq 0$ for all $x \in X$,
(b) Φ is linear in its first variable,
(c) $\Phi(x,x') = \overline{\Phi(x',x)}$ for any $x, x' \in X$.

A proof analogous to the standard proof of the Cauchy-Schwarz Theorem establishes that if Φ is a semi-inner product on X, then

$$|\Phi(x,x')| \leq \Phi(x,x)^{\frac{1}{2}}\Phi(x',x')^{\frac{1}{2}}.$$

Proposition 3.1.3. *Let φ be a continuous bilinear functional on $X \times Y$. Then the following statements about φ are equivalent:*

1. *There exists a Hilbert space H and there are bounded linear operators $a : X \longrightarrow H$ and $b : Y \longrightarrow H^*$, each of norm ≤ 1, such that for any $x \in X$ and $y \in Y$,*
$$\varphi(x,y) = < ax, by > .$$

2. *There are Hilbert spaces H, K and bounded linear operators $c : X \longrightarrow H$ and $d : Y \longrightarrow K$, each of norm ≤ 1, such that for any $x \in X$ and $y \in Y$,*
$$|\varphi(x,y)| \leq \|cx\|\|dy\|.$$

3. *There are semi-inner products Φ and Θ on X and Y, respectively, such that $\Phi(x,x) \leq \|x\|^2$ for $x \in X$, $\Theta(y,y) \leq \|y\|^2$ for $y \in Y$ and*
$$|\varphi(x,y)| \leq \Phi(x,x)^{\frac{1}{2}}\Theta(y,y)^{\frac{1}{2}},$$
for $x \in X$ and $y \in Y$

Proof. $1 \Rightarrow 2$: This is easy and clear.

$2 \Rightarrow 1$: Suppose H and K are Hilbert spaces, $c : X \longrightarrow H$ and $d : Y \longrightarrow K$ are bounded linear operators, each of norm ≤ 1, $\varphi \in \mathcal{B}(X,Y)$ and φ satisfies
$$|\varphi(x,y)| \leq \|cx\|\|dy\|$$

for any $x \in X$ and $y \in Y$. Assume, without loss of generality, that $\overline{cX} = H$ and $\overline{dY} = K$. For $x \in X$ and $y \in Y$, define $\psi(cx, dy) := \varphi(x, y)$. Then ψ is well-defined since for $cx = cx'$ and $dy = dy'$ we have

$$\begin{aligned}|\psi(cx,dy) - \psi(cx',dy')| &= |\phi(x,y) - \phi(x',y')| \\ &= \frac{1}{2}|\varphi(x+x', y-y') + \varphi(x-x', y+y')| \\ &\leq \frac{1}{2}\{\|c(x+x')\|\|d(y-y')\| \\ &+ \|c(x-x')\|\|d(y+y')\|\}.\end{aligned}$$

By hypothesis,

$$|\psi(cx, dy)| = |\varphi(x,y)| \leq \|cx\|\|dy\|,$$

and so, ψ is continuous on $cX \times dY$. It is clear that ψ is bilinear and extends to a continuous bilinear form on $H \times K = \overline{cX} \times \overline{dY}$. Thus a bounded linear operator $u_\psi : K \longrightarrow H^*$ defined by $u_\psi(k)(h) := \psi(h,k)$ is born. Since $\|u_\psi\| = \|\psi\| \leq 1$, 1 follows from setting $a = c$ and $b = u_\psi \circ d$.

2 $\Rightarrow$ 3: If 2 holds, then the semi-inner products on X and Y defined, respectively, by

$$\Phi(x,x') := (cx, cx')_H, \quad \Theta(y,y') := (dy, dy')_K$$

satisfy $\Phi(x,x) \leq \|x\|^2$ for $x \in X$, $\Theta(y,y) \leq \|y\|^2$ for $y \in Y$ and

$$\begin{aligned}|\varphi(x,y)| &\leq \|cx\|\|dy\| \\ &= (cx,cx)_H^{\frac{1}{2}}(dy,dy)_K^{\frac{1}{2}} \\ &= \Phi(x,x)^{\frac{1}{2}}\Theta(y,y)^{\frac{1}{2}}\end{aligned}$$

for $x \in X$ and $y \in Y$ so that 3 holds, where $(,)_H$ and $(,)_K$ denote the inner products of H and K respectively.

3 $\Rightarrow$ 2: If 3 holds, we consider X equipped with the semi-inner product Φ. Define $|x|_\Phi$ on X by

$$|x|_\Phi := \Phi(x,x)^{\frac{1}{2}},$$

and $(x,x')_\Phi$ by

$$(x,x')_\Phi := \Phi(x,x').$$

Then $\{x \in X : |x|_\Phi = 0\}$ is a linear subspace of X. On lifting $|x|_\Phi$ and $(x,x')_\Phi$ to $X/\{x : |x|_\Phi = 0\}$, we obtain an inner product space whose completion H is a Hilbert space. The natural map of X onto $X/\{x : |x|_\Phi = 0\}$ is of norm ≤ 1 into H. It is a c in 2. Similarly, Y and Θ give rise to K and d for 2. $\square$

We shall write $\mathcal{H}(X,Y)$ for the collection of all continuous bilinear functionals on $X \times Y$ satisfying the conditions in Proposition 3.1.1. Then the following properties about $\mathcal{H}(X,Y)$ hold (Diestel et al. (1997)):

Proposition 3.1.4. *If X and Y are Banach spaces, then $\mathcal{H}(X,Y)$ satisfies the following properties:*

1. *$\mathcal{H}(X,Y)$ is a convex set.*
2. *$\mathcal{H}(X,Y)$ is compact in the topology of pointwise convergence on $X \times Y$.*
3. *$\mathcal{H}(X,Y)$ is a balanced set.*

It follows that $\mathcal{H}(X,Y)$ is an ideal candidate to be the closed unit ball of a dual space. We shall define the tensor norm h^* in such a way that the closed unit ball of $(X \overset{h^*}{\otimes} Y)^*$ is $\mathcal{H}(X,Y)$ (Diestel et al. (1997)).

Let $u \in X \otimes Y$. Then

$$|u|_{h^*} := \sup |(a \otimes b)(u)|_\wedge,$$

where the supremum is taken over all Hilbert spaces H, K and all bounded linear operators $a : X \longrightarrow H$, $b : Y \longrightarrow K$ of norm ≤ 1. Indeed, $(a \otimes b)(u) \in H \otimes K$ so that we can measure its size in $H \overset{\wedge}{\otimes} K$.

Proposition 3.1.5. *h^* is a norm on $X \otimes Y$. Moreover,*

1. *h^* is a reasonable crossnorm.*
2. *h^* is a uniform crossnorm.*
3. *$\mathcal{H}(X,Y) = B_{(X \otimes Y, |\cdot|_{h^*})^*}$.*
4. *h^* is a $\otimes$ norm.*

The next proposition introduces the tensor norm h (Diestel et al. (1997)).

Proposition 3.1.6. *Given any Banach spaces X and Y, there exists a unique $\otimes$-norm h such that a bilinear functional φ on $X \times Y$ is h-integral with $||\varphi||_h \leq 1$ if and only if a Hilbert space H and bounded linear operators $a : X \longrightarrow H$, $b : Y \longrightarrow H^*$, each of norm ≤ 1, exist such that*

$$\varphi(x,y) = < ax, by >$$

for any $x \in X$ and $y \in Y$.

More elegantly, we can describe the h-integral operators as follows:

Corollary 3.1.7. *A bounded linear operator $u : X \longrightarrow Y$ is h-integral with $\|u\|_h \leq 1$ if and only if there exist a Hilbert space H and bounded linear operators $v : X \longrightarrow H$, $w : H \longrightarrow Y$, each of norm ≤ 1, such that the following diagram*

$$\begin{array}{ccc} X & \xrightarrow{\quad u \quad} & Y \\ & {\scriptstyle v}\searrow \quad \nearrow{\scriptstyle w} & \\ & H & \end{array}$$

is commutative.

It follows that $\Gamma_2(X,Y) = \mathcal{L}^h(X,Y)$ isometrically. In fact, that

$$\mathcal{L}^h(X,Y) \hookrightarrow \Gamma_2(X,Y)$$

is an inclusion of norm ≤ 1 follows at once from the preceding corollary. On the other hand, $u \in \Gamma_2(X,Y)$ implies, by the same corollary that $j_Y u \in \mathcal{L}^h(X,Y^{**})$, where $j_Y : Y \hookrightarrow Y^{**}$ denotes the canonical isometric embedding. It follows from (Diestel et al. (1997), p67)[1] that $u \in \mathcal{L}^h(X,Y)$ with $\|u\|_h = \|j_Y u\|_h$. Therefore

$$\Gamma_2(X,Y) \hookrightarrow \mathcal{L}^h(X,Y)$$

is an inclusion of norm ≤ 1.

We shall denote by h the tensor norm whose existence is cited in Proposition 3.1.6 and call it the *Hilbertian* $\otimes$-norm. We shall also call the h-forms and h-integral linear operators Hilbertian. As a matter of fact, *every operator from or to a Hilbert space is h-integral with its h-integral norm precisely its operator norm.*

Proposition 3.1.8. *h is a symmetric injective metrically accessible $\otimes$-norm and h^* is a symmetric projective metrically accessible $\otimes$ norm.*

Hence, $h \leq / \wedge \backslash$ and $h^* \geq \backslash \vee /$. The next result follows from the former inequality.

Corollary 3.1.9. *A $/ \wedge \backslash$-integral bilinear functional φ is h-integral with $\|\varphi\|_h \leq \|\varphi\|_{/\wedge\backslash}$.*

Proposition 3.1.10. *A bilinear functional φ on $X \times Y$ is h^*-integral with $\|\varphi\|_{h^*} \leq 1$ if and only if for every Hilbert space H and for every bounded linear operator $u : H \longrightarrow X$, of norm ≤ 1, the form $\varphi \circ (u \otimes id_Y) \in \mathcal{B}^\wedge(H,Y)$ with $\|\varphi \circ (u \otimes id_Y)\|_\wedge \leq 1$.*

---

[1] If $u : X \longrightarrow Y$ is a bounded linear operator, then $u \in \mathcal{L}^\alpha(X,Y)$ if and only if $j_Y u \in \mathcal{L}^\alpha(X,Y^{**})$ with $\|u\|_\alpha = \|j_Y u\|_\alpha$ for any tensor norm α

3.2 Some generalities about the RNP and the Lewis-RNP for tensor norms

A tensor norm α has the *Radon-Nikodym property* (RNP) if whenever Y^* has the approximation property and the Radon-Nikodym property then

$$\mathcal{L}^\alpha(X, Y^*) = X^* \overset{\alpha}{\otimes} Y^*$$

for all Banach spaces X.

A classical result of Grothendieck says

$$(X \overset{\vee}{\otimes} Y)^* = X^* \overset{\wedge}{\otimes} Y^*$$

whenever Y^* has the approximation property and the Radon-Nikodym property (Diestel and Uhl Jr (1977), VIII §§ 3.8, 3.10 and 4.6).

In the proof of our next theorem, we shall apply the following result that exhibits the role of ℓ^1 in the study of the Radon-Nikodym property (Diestel and Uhl Jr (1977), Chapter III, Theorem 1.8):

Proposition 3.2.1. (*Lewis-Stegall*) *A Banach space X is said to have the Radon-Nikodym property with respect to (Ω, Σ, μ) if and only if every bounded linear operator $u : L^1(\mu) \longrightarrow X$ admits a factorization $u = wv$:*

$$L^1(\mu) \xrightarrow{v} \ell^1 \xrightarrow{w} X,$$

where $w : \ell^1 \longrightarrow X$ and $v : L^1(\mu) \longrightarrow \ell^1$ are continuous linear operators. In this case, for each $\epsilon > 0$, w, v can be chosen such that $\|v\| \leq \|u\| + \epsilon$ and $\|w\| \leq 1$.

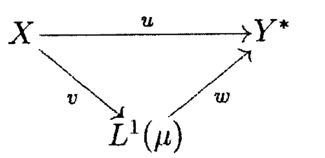
Theorem 3.2.2. *1. If α has the Radon-Nikodym property, then so does $\alpha/$.*

2. If α has the Radon-Nikodym property, then so does $\backslash\alpha$.

3. If α has the Radon-Nikodym property, then so does $/\alpha$.

Proof. Throughout this proof, Y^* has the approximation property and the Radon-Nikodym property (and hence the metric approximation property) (Diestel and Uhl Jr (1977), VIII.4.1).

1. Let $u : X \longrightarrow Y^*$ be $\alpha/$-integral and $\lambda > 1$. By Corollaire 1 (page 32) to Théorème 8 in §2.4 of (Grothendieck (1956a)), u admits a factorization:



where $v \in \mathcal{L}^{\alpha}(X, L^1(\mu))$, $w : L^1(\mu) \longrightarrow Y^*$ is bounded and linear, and

$$\|u\|_{\alpha/} \leq \|v\|_{\alpha}\|w\| \leq \lambda\|u\|_{\alpha/}.$$

Now Y^* has the Radon-Nikodym property so that Lewis-Stegall Theorem 3.2.1 says there is a set Γ and a pair of bounded linear operators s and l such that

$$s : L^1(\mu) \longrightarrow \ell^1(\Gamma), \quad l : \ell^1(\Gamma) \longrightarrow Y^*$$

so that

$$w = ls$$

and

$$\|w\| \leq \|l\|\|s\| \leq \lambda\|w\|.$$

Here is the picture we want:

$$\begin{array}{ccccc}
X^* \overset{\alpha}{\otimes} \ell^1(\Gamma) & \overset{\text{RNP}}{=\!=\!=} & \mathcal{L}^{\alpha}(X, \ell^1(\Gamma)) & & a \\
\Big\downarrow {\scriptstyle id_{X^*} \otimes l} & & \Big\downarrow & & \Big\downarrow \\
X^* \overset{\alpha/}{\otimes} Y^* & \longrightarrow & \mathcal{L}^{\alpha/}(X, Y^*) & & la
\end{array}$$

where the horizontal arrows are canonical inclusions. Here is what we see: sv is (the image of some) $b \in X^* \overset{\alpha}{\otimes} \ell^1(\Gamma)$ where

$$|b|_{\alpha} \leq \|s\|\|v\|_{\alpha}.$$

So

$$u = w \circ v = l \circ sv = (id_{X^*} \otimes l)(b)$$

and

$$\begin{aligned}
|u|_{\alpha/} &= |(id_{X^*} \otimes l)(b)|_{\alpha/} \\
&\leq \|l\|\|b\|_{\alpha} \\
&\leq \|l\|\|s\|\|v\|_{\alpha} \\
&\leq \lambda\|w\|\|v\|_{\alpha} \\
&\leq \lambda^2\|u\|_{\alpha/}.
\end{aligned}$$

Since $\lambda > 1$ is arbitrary, $u = (id_{X^*} \otimes l)(b)$ satisfies

$$|u|_{\alpha/} = |(id_{X^*} \otimes l)(b)|_{\alpha/} \leq \|u\|_{\alpha/}.$$

On the other hand $X^* \overset{\alpha/}{\otimes} Y^* \hookrightarrow \mathcal{L}^{\alpha/}(X, Y^*)$ is a norm-one inclusion, so the reverse inequality holds as well and so

$$X^* \overset{\alpha/}{\otimes} Y^* = \mathcal{L}^{\alpha/}(X, Y^*)$$

holds.

2. Let $u : X \longrightarrow Y^*$ be $\backslash\alpha$-integral and $\lambda > 1$. Again by Grothendieck's words on projective hulls (Corollaire 1 (page 32) to Théorème 8 in §2.4 of Grothendieck (1956a)), u admits a factorization

$$\begin{array}{ccc} X & \xrightarrow{\quad u \quad} & Y^* \\ & {\scriptstyle v}\searrow \quad \nearrow{\scriptstyle w} & \\ & C(K) & \end{array}$$

where $v : X \longrightarrow C(K)$ is a bounded linear operator, $w \in \mathcal{L}^{\alpha}(C(K), Y^*)$, and

$$\|w\|_{\backslash\alpha} \leq \|v\|\|w\|_{\alpha} \leq \lambda\|u\|_{\backslash\alpha}.$$

(We must keep in mind the fact that the operator $w : C(K) \longrightarrow Y^*$ is actually $\backslash\alpha$-integral, too, with $\|w\|_{\alpha} = \|w\|_{\backslash\alpha}$.) Here is the picture to be gazing at this time:

$$\begin{array}{ccccc} C(K)^* \overset{\alpha}{\otimes} Y^* & \overset{\text{RNP}}{=\!=\!=} & \mathcal{L}^{\alpha}(C(K), Y^*) & \qquad & a \\ {\scriptstyle v^* \otimes id_{Y^*}}\downarrow & & \downarrow & & \downarrow \\ X^* \overset{\backslash\alpha}{\otimes} Y^* & \longrightarrow & \mathcal{L}^{\backslash\alpha}(X, Y^*) & & av \end{array}$$

What we expect to see is this: $w \in \mathcal{L}^{\alpha}(C(K), Y^*)$ is the image of some $b \in C(K)^* \overset{\alpha}{\otimes} Y^*$ where $|b|_{\alpha} = \|w\|_{\alpha}$ and so $u = wv$ is the image of $(v^* \otimes id_{Y^*})(b) \in X^* \overset{\backslash\alpha}{\otimes} Y^*$ with

$$|(v^* \otimes id_{Y^*})(b)|_{\backslash\alpha} \leq \|v\|\|w\|_{\alpha} \leq \lambda\|u\|_{\backslash\alpha}.$$

Again $\lambda > 1$ is arbitrary so $|(v^* \otimes id_{Y^*})(b)|_{\backslash\alpha} \leq \|u\|_{\backslash\alpha}$. Since the inclusion $X^* \overset{\backslash\alpha}{\otimes} Y^* \hookrightarrow \mathcal{L}^{\backslash\alpha}(X, Y^*)$ is norm-one, the reverse inequality holds. Hence $\|u\|_{\backslash\alpha} = |(v^* \otimes id_{Y^*})(b)|_{\backslash\alpha}$ holds and with it

$$X^* \overset{\backslash\alpha}{\otimes} Y^* = \mathcal{L}^{\backslash\alpha}(X, Y^*).$$

3. Let $u : X \longrightarrow Y^*$ be $/\alpha$-integral and let L be an $L^1(\mu)$-space and $\phi : L \twoheadrightarrow X$ be a quotient operator. Again, a picture drives the proof:

$$\begin{array}{ccccc} X^* \overset{/\alpha}{\otimes} Y^* & \hookrightarrow & \mathcal{L}^{/\alpha}(X, Y^*) & & a \\ \Big\downarrow {\scriptstyle \phi^* \otimes id_{Y^*}} & & \Big\downarrow & & \Big\downarrow \\ L^* \overset{\alpha}{\otimes} Y^* & = & \mathcal{L}^{\alpha}(L, Y^*) & & a\phi \end{array}$$

- $\phi^* \otimes id_{Y^*}$ is an isometry thanks to Corollaire 3 to Théorème 6 of §2.4 of (Grothendieck (1956a)).

- $\phi : L \longrightarrow X$ induces a quotient $\phi \otimes id_Y$ of $L \overset{\alpha^*}{\otimes} Y$ onto $X \overset{\backslash(\alpha^*)}{\otimes} Y$ - this is what left projectivity of $\backslash(\alpha^*)$ is all about - by Corollaire 4 to Théorème 6 of §2.4 of (Grothendieck (1956a)).

- It follows that $(\phi \otimes id_Y)^*$ takes $\mathcal{L}^{/\alpha}(X, Y^*) = (X \overset{\backslash(\alpha^*)}{\otimes} Y)^*$ isometrically into $(L \overset{\alpha^*}{\otimes} Y)^* = \mathcal{L}^{\alpha}(L, Y^*)$. Checking what this adjoint looks like,

$$(\phi \otimes id_Y)^* : \mathcal{L}^{/\alpha}(X, Y^*) \longrightarrow \mathcal{L}^{\alpha}(L, Y^*)$$

  takes $a \in \mathcal{L}^{/\alpha}(X, Y^*)$ to $(\phi \otimes id_Y)^*(a) \in \mathcal{L}^{\alpha}(L, Y^*)$; take $l \in L$, any $y \in Y$ and compute

$$\begin{aligned} (\phi \otimes id_Y)^*(a)(l)(y) &= a((\phi \otimes id_Y)(l \otimes y)) \\ &= a(\phi(l))(y) \\ &= (a \circ \phi)(l)(y). \end{aligned}$$

- Equality on the bottom is just our RNP hypothesis for α at work.

Now let $u : X \longrightarrow Y^*$ be $/\alpha$-integral. We know that

$$u\phi \in \mathcal{L}^{\alpha}(L, Y^*) = L^* \overset{\alpha}{\otimes} Y^*,$$

so we just need to show that $u\phi$ is in the image

$$(\phi^* \otimes id_{Y^*})(X^* \overset{/\alpha}{\otimes} Y^*)$$

of $X^* \overset{/\alpha}{\otimes} Y^*$ under $\phi^* \otimes id_{Y^*}$. To this end, take $v \in (L^* \overset{\alpha}{\otimes} Y^*)^*$ to be a functional that vanishes on the range of $\phi^* \otimes id_{Y^*}$. View v as a member of $\mathcal{L}^{\alpha^*}(L^*, Y^{**}) = (L^* \overset{\alpha}{\otimes} Y^*)^*$ and suppose that $v(\phi^*) = 0$ - remember that ϕ is a metric quotient map, so $\phi^* : X^* \hookrightarrow L^*$ is an isometric inclusion. What we have is that

$$v((\phi^* \otimes id_{Y^*})(x^* \otimes y^*)) = 0 \quad \text{for each} \quad x^* \in X^*, y^* \in Y^*.$$

Consider the operator $v \otimes id_{Y^*} : L^* \overset{\alpha}{\otimes} Y^* \longrightarrow Y^{**} \overset{\wedge}{\otimes} Y^*$ such that $(v \otimes id_{Y^*})^*(\mathrm{tr}) = v$. If $a \longrightarrow \tilde{a}$ denotes the natural map of $Y^{**} \overset{\wedge}{\otimes} Y^*$ into $\mathcal{L}(Y^*)$, then $a \longrightarrow \tilde{a}$ is one-to-one thanks to Y^* having the approximation property. If we let $a = (v \otimes id_{Y^*})(u\phi)$, we see that

$$(\tilde{a})^* = v\phi^* u^* = v\phi^*(u^*) = 0$$

and so

$$\mathrm{tr}\, a = 0.$$

But then

$$v(u\phi) = \mathrm{tr}\, a = 0.$$

But $v \in \mathcal{L}^{\alpha^*}(L^*, Y^{**}) = (L^* \overset{\alpha}{\otimes} Y^*)^*$ was arbitrary. So there is an $s \in X^* \overset{/\alpha}{\otimes} Y^*$ so

$$u\phi = (\phi^* \otimes id_{Y^*})(s)$$

where $|s|_{/\alpha} = |u\phi|_\alpha = \|u\|_{/\alpha}$ - keeping in mind the fact that $\phi^* \otimes id_{Y^*}$ is an isometry and

$$\mathcal{L}^\alpha(X, Y^*) = (X \overset{\alpha^*}{\otimes} Y)^*.$$

If $u : X \longrightarrow Y^*$ is α-integral then $u \otimes id_Y : X \otimes Y \longrightarrow Y^* \otimes Y$ acts on $v \in X \otimes Y$ via the formula

$$u(v) := \mathrm{tr}((u \otimes id_Y)(v)).$$

For elementary tensors $x \otimes y$ in $X \otimes Y$ we have $(u \otimes id_Y)(x \otimes y) = u(x) \otimes y$ and $u(x \otimes y) = u(x)(y) = tr((u \otimes id_Y)(x \otimes y))$. Further, if $\alpha^*(v) \leq 1$,

$$\begin{aligned} |u(v)| &= |\mathrm{tr}((u \otimes id_Y)(v))| \\ &= |u(v)| \\ &\leq \|u\|_\alpha \alpha^*(v) \\ &\leq \|u\|_\alpha. \end{aligned}$$

So extending u to $(X \overset{\alpha^*}{\otimes} Y)^*$ there is but one choice - and that's *the* functional

$$v \longrightarrow \mathrm{tr}((u \otimes id_Y)(v))$$

whose norm is $\leq \|u\|_\alpha$ too. But then we see u to be the original member of $(X \overset{\alpha^*}{\otimes} Y)^* = \mathcal{L}^\alpha(X, Y^*)$. $\square$

Since the tensor norm $\wedge$ has the Radon-Nikodym property, the following corollary holds too.

Corollary 3.2.3. *The tensor norms* $\wedge/$, $\backslash\wedge$ *and* $/\wedge$ *have the Radon-Nikodym property.*

Next is the question: does the right injective hull of a tensor norm with the Radon-Nikodym property have the Radon-Nikodym property? We give an example below that bears testimony to the answer that the right injective hull of a tensor norm with the Radon-Nikodym property *need not* have the Radon-Nikodym property. Firstly, we need a proposition.

Proposition 3.2.4. *Let X and Y be Banach spaces. Then* $\mathcal{L}^{\wedge\backslash}(X, Y) = AS(X, Y)$, *where $AS(X, Y)$ denotes the absolutely summing operators.*

Proof. By definition, $u \in \mathcal{L}^{\wedge\backslash}(X, Y)$ if and only if for every C-space C and operator $i : Y \longrightarrow C$, with $\|i\| \leq 1$ it holds that the operator $iu : X \longrightarrow C$ is integral, and hence 1-summing. In particular, if we choose $C = C(B_{Y^*})$ and $i = i_{Y^*}$ where $i_{Y^*} : Y \hookrightarrow C(B_{Y^*})$ is the canonical isometric embedding.

Since the 1-summing operators are, by definition, injective (i.e. the property of an operator being 1-summing does not depend on the range space) (Diestel et al. (1995), 2.5) it follows that $u : X \longrightarrow Y$ is already 1-summing.

For the converse, we shall first prove the following known *fact* (cf Stegall and Retherford (1972)):

- Absolutely summing operators into $C(K)$ are integral.

For, let $v : X \longrightarrow C(K)$ be absolutely summing. Then $v^{**} : X^{**} \longrightarrow C(K)^{**}$ is absolutely summing too (Diestel et al. (1995), Proposition 2.19). Since $C(K)^{**}$ is injective (as an L^∞-space (Diestel et al. (1995), Theorem 4.14)), v^{**} is integral (Diestel et al. (1995), Corollary 5.7), and so, v is integral too (Diestel et al. (1995), Theorem 5.14).

Now, let $u : X \longrightarrow Y$ be absolutely summing. Then for every C-space C and operator $i : Y \longrightarrow C$ such that $\|i\| \leq 1$, it holds that $iu : X \longrightarrow C$ is absolutely summing too by the ideal property, and so, integral by the afore-mentioned fact. Therefore $u : X \longrightarrow Y$ is $\wedge\backslash$-integral. $\square$

Example 3.2.5. *The tensor norm* $\wedge\backslash$ *does not have the Radon-Nikodym property.*

Proof. If we were to assume that $\wedge\backslash$ has the Radon-Nikodym property, then it would imply that $\mathcal{L}^{\wedge\backslash}(X,Y) = X^* \overset{\wedge\backslash}{\otimes} Y$ for every Banach spaces X and Y, where Y is reflexive and has the approximation property. In particular, any $u \in \mathcal{L}^{\wedge\backslash}(X,Y)$ would have to be compact.

By Grothendieck's Theorem, every operator $u : \ell^1 \longrightarrow \ell^2$ is summing (Diestel et al. (1995), 1.13, Wojtaszczyk (1991), III.F.7). So $\mathcal{L}(\ell^1,\ell^2) = \mathcal{L}^{\wedge\backslash}(\ell^1,\ell^2)$ by Proposition 3.2.4. Hence every $u \in \mathcal{L}(\ell^1,\ell^2)$ is compact. This is a contradiction since the natural inclusion $\ell^1 \hookrightarrow \ell^2$ is *not* compact. For, the unit vector basis has no convergent subsequence in ℓ^2. □

A tensor norm α has the *Lewis-Radon-Nikodym property* (Lewis-RNP) if $X^* \overset{\alpha}{\otimes} \ell^1 = \mathcal{L}^\alpha(X,\ell^1)$ for every Banach space X.

The following lemma, whose proof is presented in (Lewis (1976), Lemma 1, page 59), will be applied immediately in the proof of Proposition 3.2.7 below.

Lemma 3.2.6. *Let* X^* *be metrically accessible or let* α *be an accessible tensor norm. If* $u : X \longrightarrow Y$ *is* α*-integral and* $v : Y \longrightarrow Z$ *is in the norm closure of the finite rank operators, then* vu *is the image, under the canonical inclusion of* $X^* \overset{\alpha}{\otimes} Z$ *into* $\mathcal{L}^\alpha(X,Z)$*, of a* $t \in X^* \overset{\alpha}{\otimes} Z$ *satisfying* $|t|_\alpha \le \|u\|_\alpha \|v\|$.

Proposition 3.2.7. *If* $1 < p < \infty$*, then* γ_p *has the Lewis-Radon-Nikodym property.*

Proof. Let $\varphi : X \longrightarrow \ell^1$ be γ_p-integral and $\lambda > 1$. Write $\varphi = uv$ with $\|u\|\|v\| \le \lambda\gamma_p(\varphi)$, where $v : X \longrightarrow L^p(\nu)$, $u : L^p(\nu) \longrightarrow \ell^1$ for some measure ν. Let $(e_i)_i \subset \ell^\infty$ be the sequence of unit vectors of ℓ^∞. We shall first show that $u = u_2u_1$, where u_2 is a compact operator on ℓ^1 with $\|u_2\| \le 1$ and $u_1 : L^p(\nu) \longrightarrow \ell^1$ is γ_p-integral for some ν with $\gamma_p(u_1) \le \lambda\|u\|$.

Since $1 < p$, $u : L^p(\nu) \longrightarrow \ell^1$ is weakly compact and hence compact by the Schur's ℓ^1 Theorem. So

$$\lim_n \sup_{\|x\|\le 1} \sum_{i\ge n} |< u(x), e_i >| = 0.$$

Hence there is a positive sequence $(a_i) \in c_0$ of norm one which satisfies

$$\sup_{\|x\|\le 1} \sum_{i\ge 1} a_i^{-1} |< ux, e_i >| \le \lambda\|u\|.$$

Define $u_1 : L^p(\nu) \longrightarrow \ell^1$ by

$$u_1(x) := (a_i^{-1} < ux, e_i >)_{i \geq 1},$$

and $u_2 : \ell^1 \longrightarrow \ell^1$ by

$$u_2(z) := (a_i < z, e_i >)_{i \geq 1}.$$

Then u_1 is γ_p-integral with $\gamma_p(u_1) \leq \lambda \|u\|$ and u_2 is compact with $\|u_2\| \leq 1$.

Now consider the following commutative diagram:

$$\begin{array}{ccc} L^p(\nu)^* \overset{\gamma_p}{\otimes} \ell^1 & \longrightarrow & \mathcal{L}^{\gamma_p}(L^p(\nu), \ell^1) \\ {\scriptstyle v^* \otimes id_{\ell^1}} \downarrow & & \downarrow {\scriptstyle a \mapsto av} \\ X^* \overset{\gamma_p}{\otimes} \ell^1 & \longrightarrow & \mathcal{L}^{\gamma_p}(X, \ell^1) \end{array}$$

The unlabeled arrows are the canonical inclusions. Since $L^p(\nu)^*$ is metrically accessible, it follows from Lemma 3.2.6 that $u = u_2 u_1 \in \mathcal{L}^{\gamma_p}(L^p(\nu), \ell^1)$ is the image of a $t \in L^p(\nu)^* \overset{\gamma_p}{\otimes} \ell^1$ such that

$$|t|_{\gamma_p} \leq \|u_2\| \|u_1\|_{\gamma_p} \leq \lambda \|u\|.$$

Then $(v^* \otimes id_{\ell^1})(t)$ goes to $uv = \varphi$ under the natural map and

$$\begin{aligned} |(v^* \otimes id_{\ell^1})(t)|_{\gamma_p} &\leq \lambda \|v^*\| \|u\| \\ &= \lambda \|v\| \|u\| \\ &\leq \lambda^2 \gamma_p(\varphi). \end{aligned}$$

Since $\lambda > 1$ is arbitrary and the inclusion

$$X^* \overset{\gamma_p}{\otimes} \ell^1 \hookrightarrow \mathcal{L}^{\gamma_p}(X, \ell^1)$$

is one-to-one (by the accessibility of ℓ^1), it is also an onto isometry. □

Corollary 3.2.8. *If $1 < p < \infty$, then $\gamma_p \backslash$ has the Lewis-Radon-Nikodym property.*

This is, in fact, a corollary to the proof of Proposition 3.2.7; in fact, if we factor $\varphi \in \mathcal{L}^{\gamma_p \backslash}(X, \ell^1)$ as $\varphi = uv$, then v's codomain is a subspace of an $L^p(\mu)$-space which is, in turn, the domain of u and little else changes.

Corollary 3.2.9. *The tensor norm h has the Lewis-Radon-Nikodym property.*

However, the next example shows that the tensor norm h *does not* have the Radon-Nikodym property.

Example 3.2.10. *The tensor norm h does not have the Radon-Nikodym property.*

Proof. 1st *proof.* Suppose that h has the Radon-Nikodym property. Then for every Banach space Y^* that has the Radon-Nikodym property and the approximation property, $\mathcal{L}^h(X,Y^*) = \mathcal{L}_h(X,Y^*)$ for all Banach spaces X. Consider the operator $u : L^2[0,1] \longrightarrow H$, where H is a Hilbert space. Then u *is* Hilbertian, and so belongs to $\mathcal{L}_h(L^2[0,1],H) = \overline{\mathcal{F}(L^2[0,1],H)}^{\|\cdot\|_h} = \overline{\mathcal{F}(L^2[0,1],H)}^{\|\cdot\|_{op}}$. Therefore, u is compact. In particular,

$$id_{L^2[0,1]} : L^2[0,1] \longrightarrow L^2[0,1]$$

is compact and this is a contradiction.

2nd *proof.* Consider $u : L^2[0,1] \longrightarrow H$. Assume that u is representable:

$$u(g) = \int_0^1 fgd\mu,$$

where $f \in L^2([0,1],H)$. All these operators are bounded since for $g \in B_{L^2[0,1]}$,

$$\begin{aligned}\|u(g)\| &= \|\int_0^1 fgd\mu\|_H \\ &\leq (\int_0^1 \|f\|^2)^{\frac{1}{2}}\|g\|_{L^2[0,1]}.\end{aligned}$$

So $\|u\| \leq (\int_0^1 \|f\|^2)^{\frac{1}{2}} < \infty$. Every $f \in L^2([0,1],H)$ is the limit of simple functions by Pettis' Measurability Theorem 1.3.1. But simple functions define finite rank operators. So u is the limit of the finite rank operators. It follows that u is compact. Now look at

$$id_{L^2[0,1]} : L^2[0,1] \longrightarrow L^2[0,1].$$

If $id_{L^2[0,1]}$ were representable, then it would be compact. □

In the special case when $F = c_0$, our Radon-Nikodym property coincides with the Lewis-Radon-Nikodym property. Also note that Corollary 3.2.8 also follows from Proposition 3.2.7 and Theorem 3.2.2. Furthermore, the next corollary is not unexpected.

Corollary 3.2.11. *The tensor norms $h/$, $\backslash h$ and $/h$ have the Lewis-Radon-Nikodym property.*

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Summary

Title: Contributions to the theory of tensor norms and their relationship with vector-valued function spaces

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Chapter 1 presents the historical introduction of tensor products as well as the preliminary results on them and describes the evolution of our central problems, namely:

- For $1 < p < \infty$ and for a Banach space X, how is $L^p_{\text{strong}}(\mu, X)$ related to the completion of $L^p(\mu) \otimes X$ under the Grothendieck's natural tensor norms $\wedge$ and $\vee$?

- Can we abstract the properties enjoyed by the space ℓ^1, namely the Radon-Nikodym property and the approximation property, and consider any spaces that enjoy the same properties to study the Radon-Nikodym property of $\otimes$ norms?

Answers to these questions occupy much of the next two chapters.

In Chapter 2 we present the results on tensor products with sequence spaces. The most important of these is a result of Grothendieck, namely if $1 \leq p < \infty$ and X is an infinite dimensional Banach space, then

$$\ell^p \overset{\wedge}{\otimes} X \hookrightarrow \ell^p_{\text{strong}}(X) \hookrightarrow \ell^p \overset{\vee}{\otimes} X \hookrightarrow \ell^p_{\text{weak}}(X),$$

where '$\hookrightarrow$' denotes a norm ≤ 1 inclusion map. The inclusions

$$\ell^p \overset{\wedge}{\otimes} X \hookrightarrow \ell^p_{\text{strong}}(X) \hookrightarrow \ell^p \overset{\vee}{\otimes} X$$

are strict. Then we state and prove the main result, namely:

- Let (Ω, Σ, μ) be a non-atomic probability measure space, $1 < p < \infty$ and X be an infinite dimensional Banach space. Then each of the inclusions

$$L^p(\mu) \overset{\wedge}{\otimes} X \hookrightarrow L^p_{\text{strong}}(\mu, X) \hookrightarrow L^p(\mu) \overset{\vee}{\otimes} X \hookrightarrow L^p_{\text{weak}}(\mu, X) \hookrightarrow \mathcal{L}(X^*, L^p(\mu))$$

is injective and has norm ≤ 1. The inclusions

$$L^p(\mu) \overset{\wedge}{\otimes} X \hookrightarrow L^p_{\text{strong}}(\mu, X) \hookrightarrow L^p(\mu) \overset{\vee}{\otimes} X$$

are strict.

This result extends the inclusion diagrams mentioned above to the case where ℓ^p is replaced by $L^p(\mu)$ for the given measure space. Thus, the first question raised above is addressed.

In Chapter 3, the basic facts about the tensor norms h and h^* are derived. This is followed by a discussion of the Radon-Nikodym property and the Lewis-Radon-Nikodym property for tensor norms. We single out the projective norm $\wedge$ as the tensor norm that enjoys the Radon-Nikodym property. We show that if α is a tensor norm with the Radon-Nikodym property, then $\alpha/$, $\backslash\alpha$ and $/\alpha$ have the property as well, but that in general $\alpha\backslash$ need not also have this property. Both the γ_p and $\gamma_p\backslash$ tensor norms, $1 < p < \infty$, are shown to have the Lewis-Radon-Nikodym property. Furthermore, it is shown that the Hilbertian tensor norm h has the Lewis-Radon-Nikodym property but does not have the Radon-Nikodym property. These deliberations address the second question raised above.