The general structure and ergodic properties of quantum and classical mechanics: A unified C*-algebraic approach

by

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DECLARATION

I, the undersigned, hereby declare that the thesis submitted herewith for the degree Philosophiae Doctor to the University of Pretoria contains my own independent work, and has not been submitted for any degree at any other university.

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Samevatting

Titel: Die algemene struktuur en ergodiese eienskappe van kwantum en klassieke meganika: 'n Verenigde C*-algebraïese benadering
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Kwantum meganika is in wese 'n statistiese teorie. Klassieke meganika word egter normaalweg nie as inherent statisties beskou nie. Tog kan laasgeneoemde ook statisties geformuleer word. Wat meer is, 'n statistiese formulering van albei kan in terme van unitale C*-algebras uitgedruk word, in welke geval dit duidelik word dat hulle dieselfde algemene struktuur het, met kwantum meganika 'n nie-kommutatiewe veralgemening van klassieke meganika. Suiwer wiskundig beteken dit bloot dat kwantum meganika 'n nie-kommutatiewe veralgemening van waarskynlikheidsteorie is. Die belangrikste insig in dié verband is dat die projeksie postulaat van kwantum meganika 'n nie-kommutatiewe voorwaardelike waarskynlikheid is. Dit is die onderwerp van Hoofstuk 1.

Soos ergodiese teorie (die teorie van langtermyn gedrag in 'n dinamiese stelsel) in klassieke waarskynlikheidsteorie gedoen word, word ergodiese teorie dan in Hoofstuk 2 ook in nie-kommutatiewe waarskynlikheidsteorie gedoen. In besonder word veralgemenings van Khintchine se rekurrensie stelling en 'n variasie daarvan vir ergodiese stelsels bewys, asook verskeie karakteriserings van nie-kommutatiewe ergodisiteit.

Laastens, in Hoofstuk 3, word rekurrensie en ergodisiteit dan vanuit 'n fisiese oogpunt in kwantum en klassieke meganika ondersoek, deur middel van 'n kwantum meganiese analogie van Liouville se Stelling in klassieke meganika wat in Hoofstuk 1 voorgestel is.

Summary

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Quantum mechanics is essentially a statistical theory. Classical mechanics, however, is usually not viewed as being inherently statistical. Nevertheless, the latter can also be formulated statistically. Furthermore, a statistical formulation of both can be expressed in terms of unital C*-algebras, in which case it becomes clear that they have the same general structure, with quantum mechanics a noncommutative generalization of classical mechanics. Purely mathematically this merely means that quantum mechanics is a noncommutative generalization of probability theory. The most important insight in this respect is that the projection postulate of quantum mechanics is a noncommutative conditional probability. This is the subject of Chapter 1.

As ergodic theory (the theory of long term behaviour of a dynamical system) is done in classical probability theory, ergodic theory is then done in Chapter 2 also in noncommutative probability theory. In particular generalizations of Khintchine's recurrence theorem and a variation thereof for ergodic systems is proved, as well as various characterizations of noncommutative ergodicity.

Lastly, in Chapter 3, recurrence and ergodicity is then investigated from a physical perspective in quantum and classical mechanics, by means of a quantum mechanical analogue of Liouville's Theorem in classical mechanics which was suggested in Chapter 1.

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0.1 List of symbols and terms

General symbols

$\varnothing$ is the empty set.
$A \subset B$ means that the set A is contained in the set B, with $A = B$ allowed.
$\overline{S}$ is the closure of a set S in a topological space.
$\mathbb{C}$ is the set of complex numbers.
$\mathbb{N} = \{1, 2, 3, ...\}$.
$\mathbb{R}$ is the set of real numbers.

$f^* = \overline{f}$ is the complex-conjugate of a complex-valued function f.
λ is the Lebesgue measure on $\mathbb{R}^{2n}$.
$\mathfrak{L}(V)$ is the space of bounded linear operators $V \to V$ on a normed space V.
Tr denotes the trace of a bounded linear operator on a Hilbert space (see **[Mu]**).

Symbols defined in the text

$A \leq B$ in a $*$-algebra, 2.7
$B_\infty(F)$, 1.7.2
$B_\infty(\mathbb{R}^{2n})$, 1.3
$B_\infty(\Sigma)$, 1.7, 2.1
$L(V)$, 2.2
tr, 1.6
χ_A, 1.3
$x \otimes y$, 2.3
$\|\cdot\|_\varphi$ where φ is a state on a unital $*$-algebra, 2.2
$[\cdot]$, 2.3

Terms

accurate, precise, 1.4
bounded quantum system, 1.7.3
Cauchy-Schwarz inequality for states, 2.2.2, 2.5.6 (in the proofs)
conditional probability, 1.3, 1.5, 1.6.1
constant energy surface, 3.2.8
density operator, 1.2
ergodic, 2.3.2
factor, finite factor, 1.8.2
finite von Neumann algebra, 1.7
faithful, 1.7, 1.7.4, 2.7.1
flow, 1.3
Hamiltonian flow, 1.3
ideal measurement, 1.5.1

information, 1.4(b), 1.6.1
measurement, 1.1.2, 1.6
noncommutative information, 1.6, 1.6.1
normal state, 1.7.4
phase space, phase point, 1.3
pure state, 1.3, 1.4(b), 1.7.4, 1.9.2
$*$-dynamical system, 2.3.1
spectral projection, 1.4.2, 1.2, 1.3
state, 1.2, 1.4, 2.2
unital $*$-algebra, 2.1

All Hilbert spaces are assumed complex.
We will use units in which $\hbar = h/2\pi = 1$, where h here denotes Planck's constant.

0.2 Introduction

The main motivation for this thesis is to gain a deeper understanding of the structure and nature of quantum mechanics. This will be achieved by a careful analysis of the relationship between quantum mechanics and classical mechanics. Quantum mechanics is inherently a statistical theory, while classical mechanics is not. The essential idea is therefore to study the general structure of statistical mechanics in a mathematical framework that applies to both quantum mechanics and classical mechanics. The language of abstract C*-algebras is ideally suited for this, since it provides a unified formulation of quantum mechanics and classical mechanics, with classical mechanics then viewed as a special case of quantum mechanics where we have commutativity. The concrete realizations of the C*-algebras in quantum mechanics consist of linear operators on Hilbert spaces, which are mathematical objects that differ very much from the measurable functions that make up the concrete realizations of the C*-algebras in classical mechanics. For this reason the abstract approach clarifies the general structure of mechanics (quantum and classical), enabling the above mentioned unified formulation of mechanics. This is discussed in detail in Sections 1.1 to 1.5 of Chapter 1.

From a mathematical point of view the general structure of classical mechanics to be presented is nothing more than probability theory (or, a probabilistic description of information) with dynamics, while the general structure of quantum mechanics is noncommutative probability theory (or, a probabilistic description of "noncommutative information") with dynamics. From a physical point of view the information referred to here is the information an observer has regarding the state of the physical system in question, while the dynamics describes the time-evolution of the system. The mathematics then suggests an interpretation of quantum mechanics in terms of the idea of noncommutative information, which clarifies several conceptual problems surrounding the measuring process. This interpretation is discussed in Section 1.6.

As is implied above, our view of statistical mechanics is as a description of situations where we have incomplete information about the state of a physical system (quantum or classical). In practice this is generally the case, since exact measurements are impossible, except for some simple quantum systems whose observables have discrete values which are separated enough to be distinguished by our measurements. If an observable has a continuous spectrum of values, then the best we can hope for when measuring the observable, is to obtain an interval of values containing the "actual" value of the observable (if we do not measure an observable of a quantum system exactly, then it does not really make sense to say that the observable has an actual precise value, unlike in classical mechanics where it is possible to think of an observable as having an exact value, even if we did not measure it exactly). For classical mechanics the most important observables (like energy, momentum and position) are not discrete but continuous, the major exception being the "number of particles" which is important in the statistical mechanics of large systems, but usually not exactly determinable, simply because in this case there is typically a huge number of very small particles involved. For this reason we view the statistical nature of physics as fundamental, even for classical mechanics. Mathematically, the case where we do have complete information is simply a special case of statistical mechanics, and hence is covered by our work. We will therefore usually refer simply to "mechanics" (quantum or classical), rather than "statistical mechanics". When we do use the term "statistical mechanics", it will be in the traditional sense, namely to refer to large systems where there are too many parts (usually small particles) for each to be measured individually (so we do not know the position, momentum and so on of each particle). In this case only a small number of parameters referring to the system as a whole (or to pieces of the system much larger than its individual parts) can in practice be measured, for example the temperature, volume, mass and pressure of a gas confined to some container.

Having set up a unified framework for quantum and classical mechanics, we proceed to consider recurrence and ergodicity. These concepts originated respectively in Poincaré's work on celestial mechanics and in Boltzmann's work on classical statistical mechanics, and now form part of what is known as ergodic theory. We want to study recurrence and ergodicity in our unified framework for mechanics to gain some insight into the properties of quantum mechanics as opposed to classical mechanics.

The notion of Poincaré recurrence in classical mechanics is quite well-known. Roughly it means that within experimental error a classical system confined to a finite volume in phase space will eventually return to its initial state. This happens because of Liouville's Theorem, which states that Lebesgue measure is invariant under the Hamiltonian flow in the phase space $\mathbb{R}^{2n}$. Ergodicity in classical mechanics refers to the situation where for every observable and (almost) every pure state of a system, the time mean of the observable (for that pure state) is equal to its average value on the constant energy surface containing the pure state, in which case the system is called ergodic. Again Liouville's Theorem is an implicit ingredient, since it induces a time-invariant measure on the constant energy surface (see Remark 3.2.8 for a brief discussion). It should be noted that ergodicity is of some importance in physics, since it forms the starting point of many developments of statistical mechanics (see for example [**Rue**, Section 1.1]). To study recurrence and ergodicity in quantum mechanics, we can expect from these remarks that we will need a quantum mechanical analogue of Liouville's Theorem. We propose such an analogue in Section 1.7 of Chapter 1, and in the process we are naturally led to consider finite von Neumann algebras.

Recurrence does in fact occur in quantum mechanics. One approach to recurrence in quantum mechanics has been through the theory of almost periodic functions (see for example [**BL**], [**HH**] and [**Perc**]). Another line of research, involving coherent states, along with possible applications of quantum recurrence, can be traced in

[**SLB**] and references therein. However, these methods differ considerably from the measure theoretic techniques employed to study recurrence in classical mechanics. In Section 3.1 of Chapter 3 we will see how recurrence (in a probabilistic sense) in quantum mechanics can be cast in a mathematical form that looks the same as the classical case, using our unified formulation of mechanics. More precisely, the quantum case is a noncommutative extension of the classical case.

The mathematical aspects of recurrence and ergodicity is the subject of Chapter 2, where one clearly sees that these concepts are not really measure theoretic in nature, as it might seem from numerous books (for example [**Pet**] and [**Wa**]), but rather $*$-algebraic, with the basic tools being some Hilbert space techniques. The idea is to study recurrence and ergodicity in the most general mathematical setting possible. This then includes our unified framework for mechanics as a special case. In Chapter 3 we look at a few physical aspects of recurrence and ergodicity, including some speculation on the relevance of these ideas in quantum mechanics.

The original inspiration for this thesis came from [**NSZ**], where recurrence is studied in a C*-algebraic framework from a purely mathematical point of view. The work presented here is for the most part based on [**D2**], [**D3**] and [**DS**].