Chapter 5

The heat equation

The material collected in the previous chapters enable us now to study the heat equation

$$\frac{\partial u}{\partial t} - \Delta u = f \ \text{ in } \ \Omega \tag{5.0.1}$$

appended with the initial condition

$$u(x, 0) = 0, \quad x \in \Omega \tag{5.0.2}$$

and the boundary condition

$$u(x, t) = 0 \ \text{ on } \ \partial\Omega \times (0, +\infty). \tag{5.0.3}$$

The investigation deals with two main steps. The first step is the quantitative and some particular qualitative analysis (section 5.1). The second step deals exclusively with the qualitative analysis regarding the regularity and the corner singularity of the solution (section 5.2).

5.1 Well-posedness and tangential regularity

In the heat equation and generally in parabolic problems, the time variable "t" plays a special role compared to the space variable "x". We will reflect the different roles of these variables by separating them as follows in the Sobolev spaces where the solution lives. For a function $v : (x, t) \in \Omega \times (0, +\infty) \to v(x, t) \in \mathbb{R}$, we write $v(t) \equiv v(\cdot, t) : \Omega \to \mathbb{R}, \quad v(t)(x) = v(x, t)$ and $v(x) \equiv v(x, \cdot) : (0, +\infty) \to \mathbb{R}, \quad v(x)(t) = v(x, t)$ when the variables t and x are fixed, respectively. The definitions and the comments below can be found in Lions and Magenes [39] though Ω is a polygon in our case.

Definition 5.1.1. *Given two integers $r \geq 0$ and $s \geq 0$, we denote by $H^{r,s}(\Omega \times (0, +\infty))$ the anisotropic Sobolev space defined by*

$$H^{r,s}(\Omega \times (0, +\infty)) := L^2 \left((0, +\infty), H^r(\Omega)\right) \cap H^s \left((0, +\infty), L^2(\Omega)\right)$$

and equipped with the Hilbert structure via the norm

$$\|v\|_{H^{r,s}(\Omega\times(0,+\infty))} := \left[\int_0^{+\infty} \left(\|v(\cdot,t)\|^2_{r,\Omega} + \sum_{j=0}^{s} \|\frac{\partial^j v(\cdot,t)}{\partial t^j}\|^2_{0,\Omega}\right) dt\right]^{\frac{1}{2}}.$$

Remark 5.1.2. *Notice that $H^{0,0}(\Omega \times (0, +\infty)) = L^2(\Omega \times (0, +\infty)) = L^2((0, +\infty),\ L^2(\Omega))$. The following subspaces of $H^{r,s}(\Omega \times (0, +\infty))$ will be used from time to time:*

- $$H^{r,s}_{0,}(\Omega \times (0, +\infty)) := L^2 \left((0, +\infty), H^r_0(\Omega)\right) \cap H^s \left((0, +\infty), L^2(\Omega)\right);$$
  This is characterized as the closure in $H^{r,s}(\Omega \times (0, +\infty))$ of the subspace of functions which are equal to zero in a neighborhood of the set $\Gamma \times (0, +\infty)$;

- $$H^{r,s}_{,0}(\Omega \times (0, +\infty)) := L^2 \left((0, +\infty), H^r(\Omega)\right) \cap H^s_0 \left((0, +\infty), L^2(\Omega)\right),$$
  which is also the closure in $H^{r,s}(\Omega \times (0, +\infty))$ of the subspace of functions that are equal to zero near $t = 0$ and $t = \infty$;

- $$H^{r,s}_{0,0}(\Omega \times (0, +\infty)) := H^{r,s}_{0,}(\Omega \times (0, +\infty)) \cap H^{r,s}_{,0}(\Omega \times (0, +\infty)),$$
  which is the closure in $H^{r,s}(\Omega \times (0, +\infty))$ of the space $\mathcal{D}(\Omega \times (0, +\infty))$ of test functions;

- $$\widetilde{H}^{r,s}(\Omega \times (0, +\infty) := L^2 \left((0, +\infty), H^r(\Omega)\right) \cap \widetilde{H}^s \left((0, +\infty), L^2(\Omega)\right),$$
  where $\widetilde{H}^s\left((0, +\infty), L^2(\Omega)\right)$ is the space of functions $v \in H^s\left((0, +\infty), L^2(\Omega)\right)$ such that their extension $\widetilde{v}$ by zero outside $(0, +\infty)$ belong to $H^s(\mathbb{R}, L^2(\Omega))$. Notice that

  $$H^{r,s}_{,0}(\Omega \times (0, +\infty)) \subset \widetilde{H}^{r,s}(\Omega \times (0, +\infty)).$$

In what follows in this section, we consider the well-posedness of the boundary value problem associated with the heat operator and the tangential regularity of its solution. Though these results are classical, we give the proofs in detail for convenience. (See Lions and Magenes [38]).

Theorem 5.1.3. *Under the assumption $f \in L^2\left[(0,+\infty), L^2(\Omega)\right] \equiv L^2((0,+\infty)\times\Omega)$, there exists a unique variational solution*

$$u \in \widetilde{H}^{1,1}_{0,}\left(\Omega \times (0,+\infty)\right) \tag{5.1.1}$$

of the heat equation (5.0.1)-(5.0.3) such that

$$\|u\|_{\widetilde{H}^{1,1}_{0,}(\Omega\times(0,+\infty))} \leq C\|f\|_{0,\Omega\times(0,+\infty)}. \tag{5.1.2}$$

In other words, the solution

$$u \in L^2\left[(0,+\infty), H^1_0(\Omega)\right] \tag{5.1.3}$$

satisfying

$$\|u\|_{L^2\left[(0,+\infty),H^1_0(\Omega)\right]} \leq C\|f\|_{0,\Omega\times(0,+\infty)}. \tag{5.1.4}$$

is also tangentially regular in the sense that

$$u \in \widetilde{H}^1\left[(0,+\infty), L^2(\Omega)\right]$$

which is the optimal differentiability smoothness in the time variable "t", such that

$$\|u\|_{\widetilde{H}^1(0,+\infty),L^2(\Omega)} \leq C\|f\|_{0,\Omega\times(0,+\infty)}. \tag{5.1.5}$$

Proof. The fact that $f \in L^2\left[(0,+\infty), L^2(\Omega)\right]$ implies that f is a vector-valued distribution, $f \in \mathcal{D}'(L^2(\Omega))$, such that, for $\xi \geq 0$, $e^{-\xi t}\widetilde{f} \in \mathcal{S}'(L^2(\Omega))$ is a vector-valued tempered distribution. (see Definition 2.5.14 and 2.5.23). Therefore, it is natural to look for a solution u of (5.0.1)-(5.0.3) which is a vector-valued distributions $u \in \mathcal{D}'(L^2(\Omega))$. We proceed by necessary conditions and assume that a solution $u \in \mathcal{D}'(L^2(\Omega))$ exists such that for $p = \xi + i\eta \quad \xi \geq 0$, we have $e^{-\xi t}u \in \mathcal{S}'(L^2(\Omega))$.

Since $f \in L^2\left[\Omega \times (0,+\infty)\right] = L^2\left[(0,+\infty), L^2(\Omega)\right]$, Proposition 2.5.36 implies that its

Laplace transform $\widehat{f}(\cdot,p)$ exists for $Re(p) \geq 0$, with more precisely $\widehat{f}(\cdot,p)$ belonging to the Hardy-Lebesgue space: $\widehat{f}(p) \in H^2\left[0; L^2(\Omega)\right]$.

For the class of solutions we are interested in, Definition 2.5.32 guarantees the existence of the Laplace transform $\widehat{u}(p)$. Therefore, taking the Laplace transform of the distributional equation (5.0.1)-(5.0.3) leads to the Helmholtz problem

$$-\Delta\widehat{u} + p\,\widehat{u} = \widehat{f} \text{ in } \Omega \tag{5.1.6}$$

$$\widehat{u} = 0 \text{ on } \partial\Omega. \tag{5.1.7}$$

We now make use of the results established in chapters 3 and 4 about the Helmholtz problem.

Firstly, since $\widehat{f}(p) \in L^2(\Omega)$ for $Re(p) \geq 0$, Theorem 3.1.7 guarantees that there exists a unique variational solution

$$\widehat{u} \in H_0^1(\Omega, 1+|p|)$$

of (5.1.6)-(5.1.7) satisfying the relation

$$\|\widehat{u}(p)\|_{1,\Omega,1+|p|} \leq C\|\widehat{f}(p)\|_{0,\Omega}^2 \tag{5.1.8}$$

From (5.1.8) and the fact that $\widehat{f}(p) \in H^2\left[0; L^2(\Omega)\right]$, we deduce

$$\begin{aligned} \sup_{\xi>0}\left(\int_{-\infty}^{\infty}\|\widehat{u}(\cdot,\xi+i\eta)\|_{1,\Omega,1+|p|}^2 d\eta\right) &\leq C\sup_{\xi>0}\left(\int_{-\infty}^{\infty}\|\widehat{f}(\cdot,\xi+i\eta)\|_{0,\Omega}^2 d\eta\right) \\ &< +\infty. \end{aligned} \tag{5.1.9}$$

Secondly, the function $p \rightsquigarrow \widehat{u}(p)$ is holomorphic in the complex region $Re(p) \geq 0$ since $\widehat{f}(p)$ enjoys this property and the operator $-\Delta+p$, is analytic hypo-elliptic (Theorem 2.5.38). Thus

$$\widehat{u}(p) \in H^2\left[0; H_0^1(\Omega)\right].$$

In the third step, we use the conclusion of the second step, which enables us to apply the Paley-Wiener theorem (Theorem 2.5.37): there exists a function

$$v \in L^2\left[(-\infty,+\infty); H_0^1(\Omega)\right], \text{ with } v(\cdot,t) = 0 \text{ for } t<0$$

and

$$\widehat{v}(p) = \widehat{u}(p) \quad \text{for} \quad \xi = Re(p) \geq 0.$$

By injectivity of the Laplace transform, we have $u = v$. This proves (5.1.3).

In the fourth step, we take $p = i\eta$ in (5.1.8) and integrate both sides, to obtain

$$\int_{-\infty}^{\infty} \left[\|\nabla\widehat{u}(i\eta)\|_{0,\Omega}^2 + (1+|\eta|)^2\|\widehat{u}(i\eta)\|_{0,\Omega}^2\right] d\eta \leq C\int_{-\infty}^{\infty} \|\hat{f}(i\eta)\|_{0,\Omega}^2 d\eta. \tag{5.1.10}$$

Using the Plancherel-Parseval theorem, the relation (5.1.10) leads to

$$\int_{-\infty}^{\infty} \left[\|\nabla u(t)\|_{0,\Omega}^2 + \|u(t)\|_{0,\Omega}^2 + \|\frac{\partial u(t)}{\partial t}\|_{0,\Omega}^2\right] dt \leq C\int_{-\infty}^{\infty} \|f(t)\|_{0,\Omega}^2 dt. \tag{5.1.11}$$

The relation (5.1.11) implies in particular that

$$u \in H^1\left[(-\infty,+\infty); L^2(\Omega)\right].$$

By the Sobolev embedding theorem, valid for vector-valued Sobolev spaces, the space

$$H^1\left[(-\infty,+\infty); L^2(\Omega)\right]$$

is continuously embedding in $C^0\left[(-\infty,+\infty); L^2(\Omega)\right]$. Therefore $u(0) = 0$ because $u(t) = 0$ for $t < 0$. Consequently u satisfies the inclusion (5.1.1) and the relation (5.1.11) leads to (5.1.2), (5.1.4), (5.1.5).

Conversely, if we do not start from a solution $u \in \mathcal{D}'(L^2(\Omega))$ such that $e^{-\xi t}u \in \mathcal{S}'(L^2(\Omega))$ for $\xi \geq 0$, we consider the Helmholtz problem in (5.1.6)-(5.1.7) where $\widehat{u}$ is unknown. All the arguments following (5.1.7) remain valid and lead to the existence of a unique solution satisfying (5.1.1)-(5.1.5). The theorem is proved. $\square$

Remark 5.1.4. *It can be shown that $u \in \widetilde{H}_{0,}^{1,1}(\Omega\times(0,+\infty))$ obtained in Theorem 5.1.3 is the only function of this class such that, for $t > 0$,*

$$\int_\Omega \left[\frac{\partial u}{\partial t}(x,t)v(x) + \nabla_x u(x,t)\nabla_x v(x)\right] dx = \int_\Omega f(x,t)v(x)dx, \quad \forall v \in H_0^1(\Omega). \tag{5.1.12}$$

Equation (5.1.12) is the variational formulation of the heat problem (5.0.1)-(5.0.3). For the *study of the variational problem (5.1.12), we refer the reader to [37].*

5.2 Regularity and singularities of the solution

In section 5.2, we assumed that the domain Ω has a boundary $\partial\Omega \equiv \Gamma$ of class C^2 in the sense of Definition 2.1.1. For the problem (5.1.6)-(5.1.7), the relation (5.1.8) combined with Theorem 3.2.2 regarding the regularity of the solution of this problem implies that we have the estimate

$$\sum_{|\alpha|=2} \|D^\alpha \widehat{u}(i\eta)\|_{0,\Omega}^2 + \|\nabla \widehat{u}(i\eta)\|_{0,\Omega}^2 + \|\widehat{u}(i\eta)\|_{0,\Omega}^2 \leq C\|\widehat{f}(i\eta)\|_{0,\Omega}^2. \tag{5.2.1}$$

By the Plancherel-Parseval theorem and the fact that $u(t) = 0$ for $t \leq 0$, we have

$$\int_0^{+\infty} \left[\sum_{|\alpha|=2} \|D^\alpha u(t)\|_{0,\Omega}^2 + \|\nabla u(t)\|_{0,\Omega}^2 + \|u(t)\|_{0,\Omega}^2 \right] dt \leq C \int_0^{+\infty} \|f(t)\|_{0,\Omega}^2 dt. \tag{5.2.2}$$

Consequently, we have proved the following regularity result:

Theorem 5.2.1. *Under the assumption that the domain Ω has a boundary of class C^2, the solution u of the heat equation obtained in Theorem 5.1.3 is regular in the sense that*

$$u \in \widetilde{H}^{2,1}\left(\Omega \times (0, +\infty)\right)$$

such that

$$\|u\|_{\widetilde{H}^{2,1}(\Omega\times(0,+\infty))} \leq C\|f\|_{0,\Omega\times(0,+\infty)}.$$

The non-smooth case addresses the study of the regularity and singularity of the solution of the heat equation specifically in the polygonal domain. The result reads as follows:

Theorem 5.2.2. *Let Ω be a bounded open polygonal subset of $\mathbb{R}^2$ with only one non-convex vertex of interior angle $\omega > \pi$, $f \in L^2\left(\Omega \times (0, +\infty)\right)$ and u the solution of the heat equation given in Theorem 5.1.3. Then there holds the singular decomposition*

$$u = u_R + \left[K *_t \phi(r,t)\right] r^{\frac{\pi}{\omega}} \sin\frac{\pi}{\omega}\theta \tag{5.2.3}$$

where:

- *The function*
$$u_R \in \widetilde{H}^{1,1}_{0,} \left(\Omega \times (0, +\infty)\right) \cap \widetilde{H}^{2,1} \left(\Omega \times (0, +\infty)\right)$$
is the regular part;

- *The function*
$$K \in \widetilde{H}^{\frac{1}{2}-\frac{\pi}{2\omega}}(0, +\infty)$$
is the "coefficient" of singularity;

- *The function $\phi(r, t)$ is a regularizing kernel family to be specified shortly in the proof;*

- *The symbol $*_t$ represents the convolution in the variable t.*

Moreover, we have the estimate

$$\|u_R\|_{\widetilde{H}^{2,1}(\Omega\times(0,+\infty))} + \|K\|_{\widetilde{H}^{\frac{1}{2}-\frac{\pi}{2\omega}}(0,+\infty)} \leq C\|f\|_{0,\Omega\times(0,+\infty)}. \tag{5.2.4}$$

Proof. By performing the Laplace transform of vector-valued distributions, (5.0.1)-(5.0.3) becomes the Helmholtz problem (4.1.1) or (5.1.6)-(5.1.7) where $w(p) = \widehat{u}(p)$ and $g(p) = \widehat{f}(p)$. We use extensively the notation in Theorem 4.3.4. Let $\delta_0 > 0$ be as in Theorem 4.3.4 where we take $\xi = 0$ i.e. $p = i\eta$. From this theorem we define

$$w_R(i\eta) := \begin{cases} w_R^1(i\eta) & \text{if} \quad |\eta| \leq \delta_0 \\ \\ w_R^2(i\eta) & \text{if} \quad |\eta| > \delta_0, \end{cases}$$

$$B(i\eta) := \begin{cases} B_1(i\eta) & \text{if} \quad |\eta| \leq \delta_0 \\ \\ B_2(i\eta) & \text{if} \quad |\eta| > \delta_0, \end{cases}$$

and

$$M(r, i\eta) := \begin{cases} \psi(r) & \text{if} \quad |\eta| \leq \delta_0 \\ \\ \psi(r\sqrt{|\eta|}) & \text{if} \quad |\eta| > \delta_0. \end{cases}$$

Here and after, for the purpose of Remark 5.2.3 below, the cut-off function ψ is considered to be slightly different from the previous one in (4.1.3) in the sense that $\psi \equiv \psi(r) \in C_0^\infty(\mathbb{R})$ is an even function satisfying $\psi(r) = 1$ if $r \leq \delta_0$ and $\int_{\mathbb{R}_t} \mathcal{F}^{-1}\{\psi(\sqrt{|\eta|})\}(t)dt = 1$, where $\mathcal{F}^{-1}$ is the inverse Fourier transform.

Since the function $\eta \leftrightsquigarrow \sqrt{1+|\eta|}$ is equivalent to the function $\eta \leftrightsquigarrow \sqrt{|\eta|}$ for $|\eta| > \delta_0$ and to the constant function $\eta \to 1$ for $|\eta| \leq \delta_0$, then the two parts of Theorem 4.3.4 can be combined as

$$w(i\eta) = w_R(i\eta) + B(i\eta)M(r, i\eta)r^{\frac{\pi}{\omega}} \sin\frac{\pi}{\omega}\theta \tag{5.2.5}$$

with

$$\|w_R(i\eta)\|_{2,\,\Omega,\,\sqrt{1+|\eta|}} + |B(i\eta)|(1+|\eta|)^{\frac{1}{2}-\frac{\pi}{2\omega}} \leq C\|g(i\eta)\|_{0,\Omega}. \tag{5.2.6}$$

Notice that the estimate (5.2.6) is valid if $i\eta$ is replaced by $p = \xi + i\eta$ with $\xi \geq 0$ in the reasoning above. This shows that in terms of the Hardy-Lebesgue space $w_R(p) \in H^2(0, L^2(\Omega))$ and $B(p) \in H^2(0)$. Denote by $u_R(t), K(t)$ and $\phi(r,t)$ the inverse Fourier transform of $w_R(i\eta), B(\eta)$ and $M(r, i\eta)$, respectively. From (5.2.6), the Plancherel-Parseval theorem and Paley-Wiener Theorem 2.5.37 yield

$$u_R \in \widetilde{H}^{2,1}\left((\Omega \times (0+\infty)\right) \quad \text{and} \quad K \in \widetilde{H}^{\frac{1}{2}-\frac{\pi}{2\omega}}(0,+\infty)$$

with the decomposition (5.2.3) as well as the estimate (5.2.4). □

Remark 5.2.3. *The function*

$$\phi(r,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{it\eta}M(r, i\eta)d\eta$$

is a regularizing kernel family in the following sense ([46] Lemma 2.20, [19] and [54]). If $K(t) \in H^s(\mathbb{R})$, then:

- *$\phi(r,t) *_t K(t) \in C^\infty(\mathbb{R})$ such that $\phi(r,t) *_t K(t) \in C_0^\infty(\mathbb{R})$ if $K(t)$ has a compact support,*
- *$\phi(r,t) *_t K(t)$ converges to $K(t)$ in $H^s(\mathbb{R})$ as $r \to 0$.*

An alternative proof of Theorem 5.2.2 can be found in [30] and [31] where the kernel $\phi(r,t)$ is replaced by $\phi(r,t) = \frac{1}{\sqrt{t}}e^{\frac{-r^2}{4t}}$ $t > 0$.

For the function $w(p) \equiv \widehat{u}(p)$, which is the solution of the problem (5.1.6)-(5.1.7) and which admits the singular decomposition (5.2.5) and (5.2.6), Theorem 4.4.4 applies. Thus $\widehat{u}(p) \in H^{2,\beta}(\Omega)$, for $0 < \beta < 1 - \frac{\pi}{\omega}$, such that

$$\|\widehat{u}(i\eta)\|_{H^{2,\beta}(\Omega)} \leq C\|\widehat{f}(i\eta)\|_{0,\Omega}. \tag{5.2.7}$$

Applying the Plancherel-Parseval Theorem, combined with the tangential regularity in Theorem 5.1.3, we obtain the following global regularity result for the heat equation, which as mentioned earlier, is one of our main contributions in which the numerical approach is based. The result was announced in [14] and [13].

Theorem 5.2.4. *Let Ω and f be as in Theorem 5.2.2. For $0 < \beta < 1 - \frac{\pi}{\omega}$ and the solution u in Theorem 5.1.3, we have the inclusion*

$$u \in \widetilde{H}^{2(\beta),1}\left(\Omega \times (0, +\infty)\right) \cap \widetilde{H}_{0,}^{1,1}\left(\Omega \times (0, +\infty)\right)$$

such that

$$\|u\|_{\widetilde{H}^{2(\beta),1}(\Omega \times (0,+\infty))} \leq C\|f\|_{0,\Omega \times (0,+\infty)},$$

where

$$\widetilde{H}^{2(\beta),1}\left(\Omega \times (0, +\infty)\right) := L^2\left((0, +\infty), H^{2,\beta}(\Omega)\right) \cap \widetilde{H}^1\left((0, +\infty), L^2(\Omega)\right).$$

Remark 5.2.5. *More tangential regularity can be achieved on the solution u by assuming such regularity on the datum f. More precisely, if $f \in \widetilde{H}^s\left[(0, +\infty); L^2(\Omega)\right]$, $s \geq 0$ an integer, then*

$$u \in \widetilde{H}^{s+1}\left[(0, +\infty); L^2(\Omega)\right] \cap L^2\left[(0, +\infty); H^{2,\beta}(\Omega) \cap H_0^1(\Omega)\right].$$

In [39] and [46] the datum is taken such that $f \in \widetilde{H}^{s-1, \frac{s-1}{2}}(\Omega \times (0, +\infty))$ in order to have $u \in \widetilde{H}^{1, \frac{s+1}{2}}(\Omega \times (0, +\infty))$ and $u_R \in \widetilde{H}^{s+1, \frac{s+1}{2}}(\Omega \times (0, +\infty))$ for the regular part in Theorem 5.2.2. In this case, we need to consider a weighted Sobolev space $H^{s+1,\beta}(\Omega)$ of higher order like in [42] for the global regularity.

Remark 5.2.6. *If Ω is convex i.e. $\omega < \pi$ in Theorem 5.2.2, then we take $\beta = 0$ in Theorem 5.2.4, which means that u has the classical optimal smoothness property.*

Chapter 6

Some numerical approximations

In the previous chapters, we obtained the solution $u \in L^2[(0,+\infty); H_0^1(\Omega)]$ of the heat equation (5.0.1)-(5.0.3) as the inverse Fourier transform of the variational solution $\widehat{u} \equiv \widehat{u}(p)$ of the Helmholtz problem (5.1.6)-(5.1.7), which satisfies: $\widehat{u} \in H_0^1(\Omega)$

$$\int_\Omega [\nabla \widehat{u}\ \nabla \bar{v} + p\ \widehat{u}\ \bar{v}]\, dx = \int_\Omega \widehat{f}\ \bar{v}\ dx,\ \ p = \xi + i\eta,\ \xi \geq 0,\ \ \forall v \in H_0^1(\Omega). \tag{6.0.1}$$

In this chapter, we consider the discrete counterpart of this procedure. More precisely, to the discrete solution of (6.0.1), we apply the inverse Fourier transform to generate an approximate solution of the heat equation. This is done in three steps each of which deals specifically with two cases: smooth and non-smooth solutions. The first step (section 6.1) is a semi-discrete method where the finite element method is used in the space variable, while the time variable remains continuous. The second step (section 6.2) is a fully discrete method with Fourier discretization in time and finite element approximation in space. For the next step (section 6.3), the finite element approximation in space is maintained while the standard and non-standard finite difference methods are used in the time variable. The last part, (section 6.4) provides numerical experiments.

6.1 Semi-discrete finite element method

We assume that Ω in (6.0.1) is a polygon. Throughout this section, we assume further that the polygon Ω is convex. Let $(\mathcal{T}_h)_{h>0}$ be a regular family of triangulations of $\bar{\Omega}$ consisting of compatible triangles T with exterior diameter $h_T \leq h$ and interior diameter ρ_T. Thus there exists a constant $\sigma > 0$ such that

$$\frac{h_T}{\rho_T} \leq \sigma, \quad \forall\, T \in \cup_{h>0}\, \mathcal{T}_h \tag{6.1.1}$$

or equivalently, there exists $\theta_0 > 0$, such that

$$\theta_T \ \geq \theta_0, \ \forall\, T \ \in \cup_{h>0}\, \mathcal{T}_h \tag{6.1.2}$$

where θ is the smallest angle of the triangle T. With each $\mathcal{T}_h$, we associate the finite element space V_h of continuous piecewise linear functions that are zero on the boundary:

$$V_h := \left\{ v_h \in C^0(\bar{\Omega});\ v_h|_{\partial\Omega} = 0,\ v_h|_T \in P_1,\ \forall T \in \mathcal{T}_h \right\} \tag{6.1.3}$$

where P_1 is the space of polynomials of degree less than or equal to 1. It is well-known that V_h is a finite-dimensional subspace of the Sobolev space $H_0^1(\Omega)$.

The finite element method (FEM) for the problem (6.0.1) reads as follows: find $\widehat{u}_h \equiv \widehat{u}_h(p) \in V_h$, solution of

$$\int_\Omega [\nabla \widehat{u}_h\ \nabla \bar{v}_h + p\ \widehat{u}_h\ \bar{v}_h]\, dx = \int_\Omega \widehat{f}\ \bar{v}_h\ dx,\ \forall v_h \in V_h. \tag{6.1.4}$$

Our standard references for all concepts concerning the classical finite element method are [16], [57].

By the generalized Lax-Milgram lemma (Theorem 3.1.1), there exists a unique solution $\widehat{u}_h \in V_h$ to (6.1.4). As in the continuous case (Theorem 3.1.7), this discrete solution satisfies the estimate

$$\|\widehat{u}_h\|^2_{1,\Omega,1+\sqrt{|p|}} := \|\nabla \widehat{u}_h\|^2_{0,\Omega} + (1+\sqrt{|p|})^2 \|\widehat{u}_h\|^2_{0,\Omega} \leq C \|\widehat{f}\|^2_{0,\Omega}, \tag{6.1.5}$$

where we recall that $C > 0$ represent, here and after, various constants that are independent of the involved arguments and parameters (e.g Fourier arguments, step sizes, etc).

It should be noted that each finite element (T, P_T, Σ_T), where $P_T = P_1(T)$ and $\Sigma_T = \{$ vertices of $T\}$, is affine-equivalent to the reference finite element $(\widetilde{T}, \widetilde{P}, \widetilde{\Sigma})$ where $\widetilde{T}$ is the unit triangle with vertices $\widetilde{\Sigma} = \{\widetilde{a}_1 = (0,0),\ \widetilde{a}_2 = (1,0),\ \widetilde{a}_3 = (0,1)\}$, $\widetilde{P} = P_1(\widetilde{T})$. This means that for any $T \in \mathcal{T}_h$, there exists an invertible affine mapping

$$F_T : \widetilde{x} \in \mathbb{R}^2 \rightsquigarrow x = F_T(\widetilde{x}) = B_T \widetilde{x} + b_T \in \mathbb{R}^2 \tag{6.1.6}$$

such that

$$T = F_T(\widetilde{T}), \;\; \Sigma_T = F_T(\widetilde{\Sigma}) \;\text{ and }\; P_T = \{p = \widetilde{p} \circ F_T^{-1}, \;\; \widetilde{p} \in \widetilde{P}\}.$$

We shall constantly use the notation

$$\widetilde{v} = v \circ F_T \;\text{ and }\; v = \widetilde{v} \circ F_T^{-1} \tag{6.1.7}$$

relating a function $v : x \in T \rightsquigarrow v(x) \in \mathbb{R}$ and the associated function $\widetilde{v} : \widetilde{x} \in \widetilde{T} \rightsquigarrow \widetilde{v}(\widetilde{x}) \in \mathbb{R}$ when considering the affine equivalent finite elements (T, P_T, Σ_T) and $(\widetilde{T}, \widetilde{P}, \widetilde{\Sigma})$. For such functions, we have $v \in H^m(T)$ if and only if $\widetilde{v} \in H^m(\widetilde{T})$ and there hold the estimates

$$|v|_{m,T} \leq C\|B_T^{-1}\|^m |detB_T|^{\frac{1}{2}} |\widetilde{v}|_{m,\widetilde{T}}, \tag{6.1.8}$$

and

$$|\widetilde{v}|_{m,\widetilde{T}} \leq C\|B_T\|^m |detB_T|^{-\frac{1}{2}} |v|_{m,T}, \tag{6.1.9}$$

where the Euclidean norms of the involved matrices are bounded as follows:

$$\|B_T^{-1}\| \leq \frac{\sqrt{2}}{\rho_T} \quad \text{and} \quad \|B_T\| \leq \sqrt{2} h_T. \tag{6.1.10}$$

By Céa's Lemma (Theorem 2.4.1 in Ciarlet [16]) we have the a priori estimate

$$\|\widehat{u} - \widehat{u}_h\|^2_{1,\,\Omega,\,1+\sqrt{|p|}} \leq C \inf_{v_h \in V_h} \|\widehat{u} - v_h\|^2_{1,\,\Omega,\,1+\sqrt{|p|}}. \tag{6.1.11}$$

In what follows, Π_h and Π_T denote suitable global and local interpolation operators that satisfy the relation

$$(\Pi_h v)|_T = \Pi_T v_T \quad \forall T \in \mathcal{T}_h. \tag{6.1.12}$$

Typically, we consider these to be the Lagrange interpolation operator when the argument v is of class $C^0(\bar{\Omega})$. When the domain of the operator consists of non-smooth functions such as those in the space $H^1(\Omega)$, we work with Π_h and Π_T as Clément's regularization operator ([16], [17] [28]). Using the latter operator and Theorem A4 in [28] or Exercise 3.2.3 in [16], we have

$$\inf_{v_h \in V_h} \|\widehat{u} - v_h\|_{0,\Omega}^2 \leq \|\widehat{u} - \Pi_h \widehat{u}\|_{0,\Omega}^2 \leq Ch^2 |\widehat{u}|_{1,\Omega}^2. \tag{6.1.13}$$

Since Ω is convex, the solution $\widehat{u}$ is of class $H^2(\Omega)$, which by Sobolev embedding theorem (Theorem 2.4.5) is embedded in $C^0(\bar{\Omega})$ and so the Lagrange interpolation operator is used. Therefore estimating $\inf_{v_h \in V_h} \|\nabla\widehat{u} - \nabla v_h\|_{0,\Omega}^2$ is reduced to estimating the local interpolation errors $\|\nabla\widehat{u} - \Pi_T \widehat{u}\|_{0,T}^2$ because

$$\|\nabla\widehat{u} - \nabla\Pi_h \widehat{u}\|_{0,\Omega}^2 = \sum_{T \in \tau_h} \|\nabla\widehat{u} - \nabla\Pi_T \widehat{u}\|_{0,T}^2. \tag{6.1.14}$$

We have the following result:

Lemma 6.1.1.

$$|\widehat{u} - \Pi_T \widehat{u}|_{m,T}^2 \leq Ch_T^{4-2m} |\widehat{u}|_{2,T}^2, \quad 0 \leq m \leq 1.$$

Proof. The proof of this classical result is reproduced here because the argument will help us to adjust the non-smooth case. We have

$$|\widehat{u} - \Pi_T \widehat{u}|_{m,T}^2 \leq C\|B_T^{-1}\|^{2m} |det B_T| |\widetilde{\widehat{u} - \Pi_T \widehat{u}}|_{m,\widetilde{T}}^2 \text{ by (6.1.8).} \tag{6.1.15}$$

Now for any polynomial $\widetilde{p} \in P_1(\widetilde{T})$, we have $\Pi_{\widetilde{T}} \widetilde{p} = \widetilde{p}$. Thus, we have

$$\begin{aligned}
|\widetilde{\widehat{u} - \Pi_T \widehat{u}}|_{m,\widetilde{T}}^2 &= |\widetilde{\widehat{u}} - \Pi_T \widetilde{\widehat{u}}|_{m,\widetilde{T}}^2 \\
&= |(I - \Pi_{\widetilde{T}})(\widetilde{\widehat{u}} + \widetilde{p})|_{m,\widetilde{T}}^2 \\
&\leq \|I - \Pi_{\widetilde{T}}\|_{\mathcal{L}(H^2(\widetilde{T}), H^m(\widetilde{T}))}^2 \|\widetilde{\widehat{u}} + \widetilde{p}\|_{H^2(\widetilde{T})}^2
\end{aligned}$$

The last inequality is true because the linear operator $\Pi_{\widetilde{T}} : H^2(\widetilde{T}) \to H^m(\widetilde{T})$ is bounded for $0 \leq m \leq 1$. This implies that

$$|\widetilde{\widehat{u} - \Pi_T \widehat{u}}|_{m,\widetilde{T}}^2 \leq C \inf_{\widetilde{p} \in P_1(\widetilde{T})} \|\widetilde{\widehat{u}} + \widetilde{p}\|_{2,\widetilde{T}}^2. \tag{6.1.16}$$

But the norm of the quotient space $\frac{H^2(\widetilde{T})}{P_1(\widetilde{T})}$ is equivalent to the associated semi-norm. This yields

$$
\begin{aligned}
|\widetilde{\widehat{u} - \Pi_T \widehat{u}}|^2_{m,\widetilde{T}} &\leq C|\widetilde{\widehat{u}}|^2_{2,\widetilde{T}} \\
&\leq C\|B_T\|^4 |detB_T|^{-1} |\widehat{u}|^2_{2,T},
\end{aligned} \tag{6.1.17}
$$

owing to (6.1.9). Due to (6.1.15) and (6.1.17), we obtain

$$
|\widehat{u} - \Pi_T \widehat{u}|^2_{m,T} \leq C\|B_T^{-1}\|^{2m} \|B_T\|^4 |\widehat{u}|^2_{2,T}. \tag{6.1.18}
$$

Now making use of (6.1.10) and the regularity (6.1.1) of the triangulation, we obtain from (6.1.18) the desired estimate in the Lemma. □

As a consequence of Lemma 6.1.1 as well as of the inequalities (6.1.5), (6.1.11) and (6.1.13) we have proved the estimate

$$
\begin{aligned}
\|\widehat{u} - \widehat{u}_h\|^2_{1,\Omega,1+\sqrt{|p|}} &\leq Ch^2\{|\widehat{u}|^2_{2,\Omega} + (1+\sqrt{|p|})^2 |\widehat{u}|^2_{1,\Omega}\} \\
&\leq Ch^2 \|\widehat{u}\|^2_{2,\Omega,1+\sqrt{|p|}} \\
&\leq Ch^2 \|\widehat{f}\|^2_{0,\Omega},
\end{aligned} \tag{6.1.19}
$$

the latter inequality being obtained similarly to Theorem 3.2.2. Notice that the Aubin-Nitsche duality argument (cf. Theorem 3.2.4 in [16]) yields the estimate

$$
\|\widehat{u} - \widehat{u}_h\|^2_{0,\Omega} \leq Ch^4 \|\widehat{f}\|^2_{0,\Omega}. \tag{6.1.20}
$$

Using Plancherel-Parseval theorem and the inverse Fourier transform (which works because the various constants C are independent of the Fourier argument), we have the following result:

Theorem 6.1.2. *Assume that the polygon Ω is convex. Then the semi-discrete solution*

$$
u_h(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{it\eta} \widehat{u}_h(i\eta) d\eta
$$

of the heat equation (5.0.1)-(5.0.3) is convergent, with optimal error estimate

$$
\|u - u_h\|^2_{0,(\Omega\times(0,+\infty))} + h^2 \|u - u_h\|^2_{L^2[(0,+\infty),\, H^1(\Omega)]} \leq C\, h^4 \|f\|^2_{0,(\Omega\times(0,+\infty))}.
$$

To deal with the case $\beta > 0$, we need the analogue of the arguments used in the classical case $\beta = 0$. The first result is related to that in [23] and reads as follows:

Lemma 6.1.3. *On the quotient space* $\frac{H^{2,\beta}(\widetilde{T})}{P_1(\widetilde{T})}$*, the semi-norm,*

$$|\dot{v}|_{H^{2,\beta}(\widetilde{T})} = \sqrt{\sum_{|\alpha|=2} \|r^\beta D^\alpha v\|^2_{0,\widetilde{T}}} \qquad v \in \dot{v},$$

is a norm equivalent to the usual norm

$$\|\dot{v}\|_{\frac{H^{2,\beta}(\widetilde{T})}{P_1(\widetilde{T})}} = \inf_{p \in P_1(\widetilde{T})} \|v + p\|_{H^{2,\beta}(\widetilde{T})}. \tag{6.1.21}$$

Proof. Let $\{p_1, p_2, p_3\}$ be an orthonormal basis of the space $P_1(\widetilde{T})$ with respect to the inner product of $H^{2,\beta}(\widetilde{T})$. For any $p \in P_1(\widetilde{T})$, we have

$$p = \sum_{i=1}^{3} (p;\ p_i)_{H^{2,\beta}(\widetilde{T})}\, p_i. \tag{6.1.22}$$

Firstly, we prove that there exists a constant $C > 0$ such that for any $v \in H^{2,\beta}(\widetilde{T})$, we have

$$\|v\|^2_{H^{2,\beta}(\widetilde{T})} \leq C \left[\sum_{|\alpha|=2} \|r^\beta D^\alpha v\|^2_{0,\widetilde{T}} + \sum_{i=1}^{3} |(v;\ p_i)_{H^{2,\beta}(\widetilde{T})}|^2 \right]. \tag{6.1.23}$$

Assume by contradiction that (6.1.23) is not true. Then for any integer n, there exists $v_n \in H^{2,\beta}(\widetilde{T})$ such that

$$\|v_n\|_{H^{2,\beta}(\widetilde{T})} = 1 \tag{6.1.24}$$

and

$$\sum_{|\alpha|=2} \|r^\beta D^\alpha v_n\|^2_{0,\widetilde{T}} + \sum_{i=1}^{3} |(v_n;\ p_i)_{H^{2,\beta}(\widetilde{T})}|^2 < \frac{1}{n}. \tag{6.1.25}$$

By the compactness of the embedding $H^{2,\beta}(\widetilde{T}) \hookrightarrow H^1(\widetilde{T})$ (Theorem 4.4.3) and (6.1.24), there exists a subsequence (v_{nj}) of (v_n) such that (v_{nj}) is convergent in $H^1(\widetilde{T})$, while (6.1.25) implies that the sequence $\left(r^\beta D^\alpha v_{nj}\right)$, $|\alpha| = 2$, converges to zero in $L^2(\widetilde{T})$.

These two facts imply that (v_{nj}) is a Cauchy-sequence in $H^{2,\beta}(\widetilde{T})$ and it converges therefore to some $v \in H^{2,\beta}(\widetilde{T})$, which in view of (6.1.24) and (6.1.25) satisfies

$$\|v\|_{H^{2,\beta}(\widetilde{T})} = 1, \tag{6.1.26}$$

$$(v,\ p_i)_{H^{2,\beta}(\widetilde{T})} = 0, \tag{6.1.27}$$

$$\|r^\beta D^\alpha v\|_{0,\widetilde{T}} = 0 \text{ for } |\alpha| = 2 \text{ and } v \in P_1(\widetilde{T}). \tag{6.1.28}$$

Using (6.1.22) and (6.1.27), we have $v = 0$ which is a contradiction to (6.1.26). The estimate (6.1.23) is therefore proved.

On the other hand, for $v \in H^{2,\beta}(\widetilde{T})$, let $q \in P_1(\widetilde{T})$ be such that

$$(v + q,\ p_i)_{H^{2,\beta}(\widetilde{T})} = 0 \ \text{ for } i = 1, 2, 3.$$

The inequality (6.1.23) applied to $v + q$ yields

$$\inf_{p \in P_1(\widetilde{T})} \|v + p\|_{H^{2,\beta}(\widetilde{T})} \le \|v + q\|_{H^{2,\beta}(\widetilde{T})} \le C\sqrt{\sum_{|\alpha|=2} \|r^\beta D^\alpha v\|_{L^2(\widetilde{T})}}.$$

This proves the equivalence of the norms. □

The second result reads as follows:

Lemma 6.1.4. *$\widetilde{v} \in H^{2,\beta}(\widetilde{T})$ if and only if $v \in H^{2,\beta}(T)$ with*

$$|\widetilde{v}|^2_{H^{2,\beta}(\widetilde{T})} \le C\|B_T\|^4 \|B_T^{-1}\|^{2\beta} |detB_T|^{-1} |v|^2_{H^{2,\beta}(T)}$$

and

$$|v|^2_{H^{2,\beta}(T)} \le C\|B_T^{-1}\|^4 \|B_T\|^{2\beta} |detB_T| |\widetilde{v}|^2_{H^{2,\beta}(\widetilde{T})}.$$

Proof. If $\widetilde{v} \in H^{2,\beta}(\widetilde{T})$, then by Definition 4.4.1 and setting from (6.1.7)

$v(x) = (\widetilde{v} \circ F_T^{-1})(x)$, $\widetilde{v}(\widetilde{x}) = v(F_T(\widetilde{x}))$, we have

$$\begin{aligned}
|\widetilde{v}|^2_{H^{2,\beta}(\widetilde{T})} &= \sum_{|\alpha|=2} \int_{\widetilde{T}} |r^\beta(\widetilde{x}) D^\alpha(\widetilde{v}(\widetilde{x}))|^2 d\widetilde{x} \\
&= \sum_{|\alpha|=2} \int_T |r^\beta(F_T^{-1}(x)) D^\alpha v(F_T(\widetilde{x}))|^2 |det B_T|^{-1} dx \\
&\leq \|B_T\|^4 |det B_T|^{-1} \|B_T^{-1}\|^{2\beta} \sum_{|\alpha|=2} \int_T |r^\beta(x) D^\alpha v(x)|^2 dx.
\end{aligned}$$

The last inequality is due to the fact that $\dfrac{\partial^2 v(F_T(\widetilde{x}))}{\partial \widetilde{x}_i \partial \widetilde{x}_j} = \displaystyle\sum_{p=1,\, l=1}^{2} \dfrac{\partial^2 v(x)}{\partial x_p \partial x_l} B_T^{l,i} B_T^{p,j}$ by the chain rule and for any vertex a of T, and $x \in \mathbb{R}^2$ we have from (6.1.6)

$$\|F_T^{-1}(x) - F_T^{-1}(a)\| = \|B_T^{-1}(x-a)\| \leq \|B_T^{-1}\| \|x-a\|$$

and

$$r(F_T^{-1}(x)) \leq \|B_T^{-1}\| r(x).$$

Thus

$$|\widetilde{v}|^2_{H^{2,\beta}(\widetilde{T})} \leq \|B_T\|^4 |det B_T|^{-1} \|B_T^{-1}\|^{2\beta} |v|^2_{H^{2,\beta}(T)}. \tag{6.1.29}$$

If on the other hand, $v \in H^{2,\beta}(T)$ then in a similar way, setting $\widetilde{v}(\widetilde{x}) = v(F_T(\widetilde{x}))$ we have

$$\begin{aligned}
|v|^2_{H^{2,\beta}(T)} &= \sum_{|\alpha|=2} \int_T |r^\beta(x) D^\alpha v(x)|^2 dx \\
&= \sum_{|\alpha|=2} \int_T |r^\beta(F_T(\widetilde{x})) D^\alpha \widetilde{v}(F_T^{-1}(x))|^2 |det B_T| d\widetilde{x} \\
&\leq C \|B_T^{-1}\|^4 |det B_T| \sum_{|\alpha|=2} \int_{\widetilde{T}} |r^\beta(F_T(\widetilde{x})) D^\alpha \widetilde{v}(\widetilde{x})|^2 d\widetilde{x}
\end{aligned}$$

because $\dfrac{\partial^2 \widetilde{v}(F_T^{-1}(x))}{\partial x_i\, \partial x_j} = \displaystyle\sum_{p=1,l=1}^{2} \dfrac{\partial^2 \widetilde{v}(\widetilde{x})}{\partial \widetilde{x}_p\, \partial \widetilde{x}_l} (B_T^{-1})^{l,i} (B_T^{-1})^{p,j}$ by chain rule.

Since as above $r(F_T(\widetilde{x})) \leq \|B_T\| r(\widetilde{x})$, we then have

$$|v|^2_{H^{2,\beta}(T)} \leq C\|B_T^{-1}\|^4 \|B_T\|^{2\beta} |detB_T| |\widetilde{v}|^2_{H^{2,\beta}(\widetilde{T})}, \tag{6.1.30}$$

which completes the proof of the Lemma. □

We are now in a position to deal with the case when $\beta > 0$. Indeed, following the argument of the classical case, that led to (6.1.16), we have

$$|\widetilde{\widehat{u} - \Pi_T \widehat{u}}|^2_{m,\widetilde{T}} \leq C \inf_{p \in P_1(\widetilde{T})} \|\widetilde{\widehat{u}} + p\|^2_{H^{2.\beta}(\widetilde{T})}.$$

Then Lemma 6.1.3 implies that

$$|\widetilde{\widehat{u} - \Pi_T \widehat{u}}|^2_{m,\widetilde{T}} \leq C |\widetilde{\widehat{u}}|^2_{H^{2,\beta}(\widetilde{T})}. \tag{6.1.31}$$

Although we are in the non-smooth case, we still use the Lagrange interpolation operator because the solution belongs to the space $H^{2,\beta}(\Omega)$ which is embedded in $C^0(\bar{\Omega})$ (cf. Theorem 4.4.3). The right hand side of (6.1.31) is dealt with by using Lemma 6.1.4, which yields

$$|\widetilde{\widehat{u}}|^2_{H^{2,\beta}(\widetilde{T})} \leq C\|B_T\|^4 \|B_T^{-1}\|^{2\beta} |detB_T|^{-1} |\widehat{u}|^2_{H^{2,\beta}(T)}. \tag{6.1.32}$$

Combining (6.1.15), (6.1.31), (6.1.32) and (6.1.10) yield

$$\begin{aligned} |\widehat{u} - \Pi_T \widehat{u}|^2_{m,T} &\leq C\|B_T^{-1}\|^{2m+2\beta} \|B_T\|^4 |\widehat{u}|^2_{H^{2,\beta}(T)} \\ &\leq C\rho_T^{-2m-2\beta} h_T^4 |\widehat{u}|^2_{H^{2,\beta}(T)}. \end{aligned}$$

Using the regularity of the triangulation (6.1.1), we end up with

$$|\widehat{u} - \Pi_T \widehat{u}|^2_{m,T} \leq C h_T^{4-2m-2\beta} |\widehat{u}|^2_{H^{2,\beta}(T)}. \tag{6.1.33}$$

The analysis covered so far is valid for the case when the critical vertex $(0,0)$ which is responsible for the singularity belongs to T. In the case when $(0,0) \notin T$, we have $\widehat{u} \in H^2(T)$.

Therefore for $0 \le m \le 1$ we have

$$\begin{aligned}
|\widehat{u} - \Pi_T \widehat{u}|^2_{m,T} &\le Ch_T^{4-2m} \sum_{|\alpha|=2} \int_T |D^\alpha \widehat{u}|^2 dx \quad \text{by Lemma 6.1.1} \\
&= Ch_T^{4-2m} \sum_{|\alpha|=2} \int_T |r^\beta(x) D^\alpha \widehat{u}|^2 r^{-2\beta}(x) dx \\
&\le Ch_T^{4-2m} \sup_{x\in T} r^{-2\beta}(x) |\widehat{u}|^2_{H^{2,\beta}(T)}. \qquad (6.1.34)
\end{aligned}$$

At this stage, we require the triangulation $(\mathcal{T}_h)$ to satisfy the mesh requirement conditions:

$$h_T \le \begin{cases} Ch^{\frac{1}{1-\beta}}, & \text{if } (0,0) \in T \\ Ch \inf_{x\in T} r^\beta(x), & \text{if } (0,0) \notin T, \end{cases} \qquad (6.1.35)$$

In view of (6.1.11), (6.1.13) and (6.1.14) which are valid for the non-smooth case, the relations (6.1.33), (6.1.34) and (6.1.35) imply that

$$\begin{aligned}
\|\widehat{u} - \widehat{u}_h\|^2_{1,\,\Omega,\,1+\sqrt{|p|}} &\le Ch^2 \left\{ |\widehat{u}|^2_{H^{2,\beta}(\Omega)} + (1+\sqrt{|p|})^2 |\widehat{u}|^2_{1,\Omega} \right\} \\
&\le Ch^2 \|\widehat{u}\|^2_{H^{2,\beta}(\Omega,\,1+\sqrt{|p|})} \\
&\le Ch^2 \|\widehat{f}\|^2_{0,\Omega}
\end{aligned}$$

where the norm of the weighted Sobolev space $H^{2,\beta}(\Omega, \rho)$ is defined in (4.4.2). It should be noted that the inequality

$$\|\widehat{u}\|_{H^{2,\beta}(\Omega,\,1+\sqrt{|p|})} \le C \|\widehat{f}\|_{0,\Omega}$$

used here can be deduced from the proof of Theorem 4.4.4 where this weighted Sobolev space appeared for the first time. Using the Plancherel-Parseval Theorem and the inverse Fourier transform together with the Aubin-Nitsche duality argument yield the following result.

Theorem 6.1.5. *Assume that the triangulations are refined according to (6.1.35). Then the semi-discrete solution*

$$u_h(t) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{it\eta} \widehat{u}_h(i\eta) d\eta$$

is such that

$$\|u-u_h\|^2_{L^2[\Omega\times(0,+\infty)]}+h^2\|u-u_h\|^2_{L^2[(0,+\infty),\,H^1(\Omega)]\cap H^{\frac{1}{2}}[(0,+\infty),L^2(\Omega)]}\leq C\,h^4\|f\|^2_{L^2[\Omega\times(0,+\infty)]}.$$

Remark 6.1.6. *The Mesh Refinement Method (MRM) (6.1.3), (6.1.4) and (6.1.35) was introduced by Babuska [8]. An alternative approach to it is the so-called Singular Function Method (SFM) introduced initially by Strang and Fix [63]. The SFM consists in replacing V_h in (6.1.3) by the family of augemented finite element spaces $V_h^+(p)$, $p=i\eta$, defined by*

$$V_h^+(p)=V_h\oplus\quad span\left\{M(r,p)r^{\frac{\pi}{\omega}}\sin\frac{\pi}{\omega}\theta\right\}$$

where $M(r,p)r^{\frac{\pi}{\omega}}\sin\frac{\pi}{\omega}\theta$ is the singular function given in (5.2.5) for the Helmholtz equation. The SFM for problems with edge singularities is investigated in [43] and [44]. Further contributions on the MRM and SFM can be found in [10].

6.2 Fourier finite element method

From the practical point of view, the semi-discrete finite element method in the previous section must be coupled with some discretization in the time-variable t, so that we have a fully discrete method. In this section, we use for the time variable t, the Fourier series method, which is the backbone of many modern techniques such as the spectral method and the wavelets method; see for instance [9, 11, 41, 51].

The Fourier-Finite Element method presented here is along the lines of [34] and has been extensively used in the literature for elliptic problems. (See for instance [35, 44, 41, 50]). Here, we implement this method for the heat equation, which is a parabolic problem.

The starting point is to consider the Fourier series of the solution $u(x,t)$ and of the datum $f(x,t)$ for the heat equation (5.0.1)-(5.0.3). More precisely, for $x\in\Omega$ and $t\in(0,2\pi)$, we have the expansions

$$u(x,t)=\sum_{k\in\mathbb{Z}}e^{ikt}u_k(x)\;\text{ and }\;f(x,t)=\sum_{k\in\mathbb{Z}}e^{ikt}f_k(x)\tag{6.2.1}$$

which mean that

$$\lim_{N\to+\infty}\|u-u^N\|_{L^2[(0,2\pi),H^1(\Omega)]}=0=\lim_{N\to+\infty}\|f-f^N\|_{L^2[(0,2\pi),L^2(\Omega)]}.\tag{6.2.2}$$

Here, for $N \in \mathbb{N}$,

$$u^N(x,t) := \sum_{|k|\leq N} e^{ikt} u_k(x) \;\text{ and }\; f^N(x,t) := \sum_{|k|\leq N} e^{ikt} f_k(x) \tag{6.2.3}$$

are truncated Fourier series, whereas each $u_k,\ k \in \mathbb{Z}$, is the unique variational solution (see Theorems 3.1.3 and 3.1.7) of the Helmholtz problem (3.1.1)-(3.1.2) with right side f_k and $p = ik$.

The following estimate between Fourier series and truncated Fourier series is useful (the beneath properties are available in [11]).

Lemma 6.2.1.

$$\|u - u^N\|_{0,(\Omega\times(0,2\pi))} \leq C\ N^{-1}\|u\|_{H^1[(0,2\pi),L^2(\Omega)]} \leq C\ N^{-1}\|f\|_{0,(\Omega\times(0,2\pi))}.$$

Furthermore, we have

$$\|u - u^N\|_{L^2((0,2\pi),H^1(\Omega))} \leq C\ N^{-1}\|u\|_{H^1((0,2\pi),\ H^1(\Omega))}$$

whenever $u \in H^1\left((0,2\pi),\ H^1(\Omega)\right)$.

Proof. We have

$$\begin{aligned}
\|u - u^N\|^2_{L^2[(0,2\pi),L^2(\Omega)]} &= \|\sum_{|k|\geq N} e^{ikt} u_k(x)\|^2_{L^2[(0,2\pi),L^2(\Omega)]} \text{ by } (6.2.1) - (6.2.3) \\
&\leq \frac{1}{N^2}\|\sum_{|k|\geq N} e^{ikt} iku_k(x)\|^2_{L^2[(0,2\pi),L^2(\Omega)]} \\
&\leq \frac{C}{N^2}\sum_{|k|\geq N}\|iku_k\|^2_{0,\Omega} \text{ by Plancherel-Parseval theorem} \\
&\leq \frac{C}{N^2}\sum_{k\in\mathbb{Z}}\|iku_k\|^2_{0,\Omega} \\
&\leq \frac{C}{N^2}\|u\|^2_{H^1[(0,2\pi),L^2(\Omega)]} \\
&\leq \frac{C}{N^2}\|f\|_{L^2((0,2\pi),\ L^2(\Omega))} \text{ by } (5.1.4).
\end{aligned}$$

The second but the last inequality is due to Plancherel-Parseval Theorem and the fact that the solution has the tangential regularity $H^1\left[(0,2\pi), L^2(\Omega)\right]$. $\square$

Fix $N \in \mathbb{N}$. For each $k \in \mathbb{Z}$ with $|k| \leq N$, let $u_{k,h} \in V_h$ be the unique solution of the finite element method (6.1.3)-(6.1.4) in which $p = ik$. Notice that $u_{k,h}$ is an approximation of the solution u_k of the Helmholtz problem (3.1.1)-(3.1.2) with $\widehat{f}$ inline of g.

The fully discrete solution of interest to us is $u_h^N(x,t)$ defined as follows:

$$u_h^N(x,t) := \sum_{|k| \leq N} e^{ikt} u_{k,h}(x), \quad x \in \Omega, \quad t \in (0, 2\pi). \tag{6.2.4}$$

The quality of the discrete solutions u_h^N is described in the following result.

Theorem 6.2.2. *The discrete solution u_h^N converges to the exact solution u in $L^2\left[(0,2\pi), H^1(\Omega)\right]$ as $N \to +\infty$ and $h \to 0$.*

Proof. The theorem is proved without making use of any smoothness property in the x-variable of the exact solution u. Let $\epsilon > 0$ be given. By the convergence of the Fourier expansion (6.2.2), there exists $N_0 \in \mathbb{N}$ such that for $N \geq N_0$ we have

$$\|u - u^N\|^2_{L^2[(0,2\pi),H^1(\Omega)]} < \frac{\epsilon^2}{2}. \tag{6.2.5}$$

On the other hand, we have for each $N \in \mathbb{N}$

$$\begin{aligned}
\|u^N - u_h^N\|^2_{L^2[(0,2\pi),H^1(\Omega)]} &= \sum_{|k| \leq N} \|u_k - u_{k.h}\|^2_{1,\Omega} \text{ by Plancherel-Parseval Theorem} \\
&\leq C \sum_{|k| \leq N} \inf_{v_h \in V_h} \|u_k - v_h\|^2_{1,\Omega,\sqrt{|k|}} \text{ by Cea's Lemma} \\
&\leq C \sum_{|k| \leq N} \|u_k - \Pi_h v_k\|_{1,\Omega,\sqrt{|k|}}
\end{aligned}$$

for any $v \in \mathcal{D}\left(\Omega \times (0, 2\pi)\right)$ such that v_k are Fourier coefficients of v. Thus, using triangular inequality, interpolation theory in Sobolev spaces and Plancherel-Parseval Theorem, we have

$$\begin{aligned}
\|u^N - u_h^N\|^2_{L^2[(0,2\pi),H^1(\Omega)]} &\leq C \sum_{|k| \leq N} \left\{ \|u_k - v_k\|^2_{1,\Omega,\sqrt{|k|}} + \|v_k - \Pi_h v_k\|^2_{1,\Omega,\sqrt{|k|}} \right\} \\
&\leq C \left\{ \|u - v\|^2_{L^2[(0,2\pi),H^1(\Omega)]} + h^2 \|v\|^2_{L^2[(0,2\pi),H^2(\Omega)]} \right\}.
\end{aligned}$$

Since $\mathcal{D}\left(\Omega \times (0,2\pi)\right)$ is dense in $L^2\left[(0,2\pi), H_0^1(\Omega)\right]$, we can choose

$$v \in \mathcal{D}\left(\Omega \times (0, 2\pi)\right) \;\text{ such that }\; \|v - u\|_{L^2[(0,2\pi),H^1(\Omega)]} < \frac{\epsilon}{2\sqrt{C}}.$$

This implies that for every N

$$\|u^N - u_h^N\|^2_{L^2[(0,2\pi),H^1(\Omega)]} \leq C\left(\frac{\epsilon^2}{4C} + h^2\|v\|^2_{L^2[(0,2\pi),H^2(\Omega)]}\right).$$

Furthermore, there exists $h_0 > 0$ such that for $h \leq h_0$ we have

$$h^2\|v\|^2_{L^2[(0,2\pi),H^2(\Omega)]} < \frac{\epsilon^2}{4C}$$

and thus

$$\|u^N - u_h^N\|^2_{L^2[(0,2\pi),H^1(\Omega)]} < \frac{\epsilon^2}{2} \quad \text{for} \;\; h \leq h_0 \;\; \text{and for any} \;\; N.$$

Combining this with (6.2.5) and the triangle inequality, we have

$$\|u - u_h^N\|^2_{L^2[(0,2\pi),H^1(\Omega)]} < \epsilon^2 \;\; \text{for} \;\; N \geq N_0 \;\; \text{and} \;\; h \leq h_0.$$

□

Further qualities of the discrete solution u_h^N are specified in the next result.

Theorem 6.2.3. *If the polygon Ω is convex, there holds the error estimate*

$$\|u_h^N - u\|_{0,\Omega\times(0,2\pi)} \leq C\left(h^2 + N^{-1}\right) \tag{6.2.6}$$

for the coupled Fourier series method (6.2.4) and classical FEM (6.1.3)-(6.1.4). When Ω is not convex, the same error estimate holds provided that the triangulations meet the mesh refinement conditions (6.1.35). Moreover, in the two cases, we have the error estimate

$$\|u_h^N - u\|_{L^2[(0,2\pi),H^1(\Omega)]} \leq C\left(h + N^{-1}\right) \tag{6.2.7}$$

whenever u has the tangential regularity $u \in H^1\left[(0,2\pi), H^1(\Omega)\right]$.

Proof. The proof is done in two parts: the convex and non-convex cases. We start with the first result by using the triangular inequality on the error as follows:

$$\|u - u_h^N\|^2_{0,\Omega\times(0,2\pi)} \leq \|u - u^N\|^2_{0,\Omega\times(0,2\pi)} + \|u^N - u_h^N\|^2_{0,\Omega\times(0,2\pi)}.$$

Using Lemma 6.2.1 on the first term and the Plancherel-Parseval theorem on the other term we have

$$\begin{aligned}\|u-u_h^N\|^2_{0,\Omega\times(0,2\pi)} \le & \quad C\left\{N^{-2}\|u\|^2_{H^1[(0,2\pi),\, L^2(\Omega)]}+\sum_{|k|\le N}\|u_k-u_{k,h}\|^2_{0,\Omega}\right\}\\ \le & \quad C\left\{N^{-2}\|f\|^2_{L^2[(0,2\pi),\, L^2(\Omega)]}+\sum_{|k|\le N}\|u_k-u_{k,h}\|^2_{0,\Omega}\right\}. \qquad (6.2.8)\end{aligned}$$

By Aubin-Nitsche duality argument, we have since $u_k \in H^2(\Omega, \sqrt{|k|})$, that

$$\begin{aligned}\sum_{|k|\le N}\|u_k-u_{k,h}\|^2_{0,\Omega} &\le Ch^4\sum_{|k|\le N}\|u_k\|^2_{2,\Omega,\sqrt{|k|}}\\ &\le Ch^4\sum_{|k|\le N}\|f_k\|^2_{0,\Omega}.\end{aligned}$$

This yields

$$\sum_{|k|\le N}\|u_k-u_{k,h}\|^2_{0,\Omega}\le Ch^4\|f\|^2_{0,\Omega\times(0,2\pi)}. \qquad (6.2.9)$$

The proof for the convex case is now followed from (6.2.8)-(6.2.9).

For the non-convex case, the same method works provided that after (6.2.8), we use the inclusion $u_k \in H^{2,\beta}(\Omega, \sqrt{|k|})$ and the mesh refinement conditions (6.1.35) instead of $u_k \in H^2(\Omega, \sqrt{|k|})$.

The proof of the second part is based on the second estimate in Lemma 6.2.1. Using this estimate, the method of proof is the same. Basically from,

$$\|u-u_h^N\|^2_{L^2[(0,2\pi),\, H^1(\Omega)]}\le\|u-u^N\|^2_{L^2[(0,2\pi),\, H^1(\Omega)]}+\|u^N-u_h^N\|^2_{L^2[(0,2\pi),\, H^1(\Omega)]},$$

we use the estimates

$$\|u-u^N\|^2_{L^2[(0,2\pi),\, H^1(\Omega)]}\le CN^{-2}\|u\|^2_{H^1[(0,2\pi),\, H^1(\Omega)]}.$$

and

$$
\begin{aligned}
\|u^N - u_h^N\|^2_{L^2[(0,2\pi),\, H^1(\Omega)]} &\leq \sum_{|k|\leq N} \|u_k - u_{k,h}\|^2_{1,\Omega,\sqrt{|k|}} \\
&\leq Ch^2 \sum_{|k|\leq N} \|u_k\|^2_{2,\Omega,\sqrt{|k|}} \\
&\leq Ch^2 \|f\|^2_{0,\Omega\times(0,2\pi)}.
\end{aligned}
$$

In view of the Plancherel-Parseval Theorem, we deduce that

$$
\|u - u_h^N\|_{L^2[(0,2\pi),\, H^1(\Omega)]} \;\; \leq \;\; CN^{-1}\|u\|_{H^1((0,2\pi),\, H^1(\Omega))} + Ch\|f\|_{0,\Omega\times(0,2\pi)}
$$

The case when Ω is non-convex is dealt with similarly, on replacing the inclusion $u_k \in H^2(\Omega, \sqrt{|k|})$ with $u_k \in H^{2,\beta}(\Omega, \sqrt{|k|})$ and using the mesh refinement conditions (6.1.35). $\square$

6.3 Coupled non-standard finite difference and finite element methods

Unlike section 6.2, where the time variable was discretized by Fourier series, we now discretize it using the non-standard Finite Difference (NSFD) method. The NSFD approach was initiated more than two decades ago by Mickens [52] as a powerful tool that replicates the dynamics of the differential system under consideration. Major contributions to the mathematics foundation of the NSFD method are due to Anguelov and Lubuma [5, 6, 7] (see [56] for and overview). Since then, the NSFD method has been extensively applied to many concrete problems in engineering and science (see for example [32], [53]).

To understand the relevance of the NSFD method in this thesis, we consider the heat equation in the following specific form:

$$
\frac{\partial u}{\partial t} - \Delta u + \lambda u = f, \quad \lambda > 0, \quad \text{on} \;\; \Omega \times (0, +\infty) \tag{6.3.1}
$$

$$
u = 0 \;\; \text{on} \;\; \partial\Omega \times (0, +\infty) \tag{6.3.2}
$$

$$
u(x, 0) = u^0(x), \quad \text{for} \;\; x \in \Omega. \tag{6.3.3}
$$

An appropriate trace theorem (see [39]) can reduce (6.3.1)-(6.3.3) to our standard model (5.0.1)-(5.0.3).

When $f = 0$, the space independent case of (6.3.1)-(6.3.3) is the decay ordinary differential equation

$$\frac{\partial u}{\partial t} = -\lambda u, \tag{6.3.4}$$

$$u(0) = u^0, \tag{6.3.5}$$

which has the unique solution

$$u(t) = u^0 e^{-\lambda t}. \tag{6.3.6}$$

Let $t_k := k\triangle t,\ k = 0, 1, 2, \cdots$ be the discrete time variable with $\triangle t$ representing the time step size. At the time $t = t_{k+1}$, the solution

$$u(t_{k+1}) = u^0 e^{-\lambda t_{k+1}}, \tag{6.3.7}$$

given by (6.3.6) can be written as $u(t_k)e^{-\lambda\triangle t}$ or

$$e^{\lambda\triangle t}u(t_{k+1}) = u(t_k) \tag{6.3.8}$$

in view of the semi-group property of solutions of ordinary differential equations. By adding and subtracting $u(t_{k+1})$ from (6.3.8), we obtain the following equivalent formulation of (6.3.7) where the notation $u^k := u(t_k)$ is used:

$$\frac{u^{k+1} - u^k}{\frac{e^{\lambda\triangle t} - 1}{\lambda}} + \lambda u^{k+1} = 0. \tag{6.3.9}$$

By definition, (6.3.9) is called the exact scheme for the decay equation (6.3.4) (see Mickens [52]). The terminology is self-explanatory: at the time $t = t_k$, the difference equation (6.3.9) has the same general solution as the differential equation (6.3.4).

Clearly, the exact scheme (6.3.9) is dynamically consistent with any property of the initial value problem (6.3.4)-(6.3.5) irrespective of the value of the step size $\triangle t$. In particular the discrete scheme (6.3.9) replicates the positivity and the decay to zero which are the main features of the solution (6.3.6) of (6.3.4)-(6.3.5).

Equation (6.3.9) is a typical non-standard finite difference scheme in the following sense (cf [4]):

Definition 6.3.1. *A difference equation*

$$u^{k+1} = g(u^k, u^{k+1})$$

for approximating a differential equation

$$\frac{du}{dt} = g(u)$$

is called a non-standard finite difference method, if at least one of the following conditions is met:

1. *In the first order discrete derivative*

$$\frac{u^{k+1} - u^k}{\Delta t}$$

the traditional denominator Δt is replaced by a positive function $\phi(\Delta t)$ satisfying the property

$$\phi(\Delta t) = \Delta t + O((\Delta t)^2) \quad \text{as} \;\; \Delta t \to 0. \tag{6.3.10}$$

2. *Non-local term in $g(u)$ are approximated in a non-local way, i.e. by a suitable function of several points of the mesh.*

Remark 6.3.2. *Condition 2 in the Definition 6.3.1 is not necessary in our case since we are dealing with a linear problem. However, the condition is very useful in non-linear problems.*

For more on non-standard finite difference schemes, we refer the reader to [45, 56] and edited volumes [32] and [53].

Our aim is to design for (6.3.1)-(6.3.3) a fully discrete method, which will preserve the properties in the limit case of space independent equation and $f = 0$. To this end, we approximate (6.3.1)-(6.3.3) by coupling the FEM in space and the NSFD scheme in time as follows: With the initial guess $u_h^0 := \Pi_h u^0 \in V_h$ via the interpolation operator Π_h, let $(u_h^k)_{k\geq 1}$

be the sequence in the finite element space V_h defined recursively as unique solution of

$$\int_\Omega \left[\frac{u_h^k - u_h^{k-1}}{\frac{e^{\lambda \Delta t} - 1}{\lambda}} \, v_h + \nabla u_h^k \, \nabla v_h + \lambda u_h^k \, v_h \right] dx = \int_\Omega f(t_k) \, v_h \, dx \;\; \forall \, v_h \in V_h, \tag{6.3.11}$$

$$u_h^0 = \Pi_h u^0, \;\; k = 1, 2, 3, ... \tag{6.3.12}$$

The idea of coupling the NSFD method with the FEM and their implementation presented here are new. The results are published in [14] and [13].

Let $\phi(\Delta t)$ denote the function $\frac{e^{\lambda \Delta t} - 1}{\lambda}$ that satisfies (6.3.10). It is clear that (6.3.11) can be written as follows for any $v_h \in V_h$:

$$\begin{aligned} \left(u_h^k, v_h\right)_{0,\Omega} + \phi(\Delta t) \left(\nabla u_h^k, \nabla v_h\right)_{0,\Omega} + \lambda \phi(\Delta t) \left(u_h^k, v_h\right)_{0,\Omega} = \; & \phi(\Delta t) \left(f(t_h), v_h\right)_{0,\Omega} \\ + \; & \left(u_h^{k-1}, v_h\right)_{0,\Omega}. \end{aligned} \tag{6.3.13}$$

Equation (6.3.13) will be considered in conjunction with the continuous relation below, which in view of (5.1.12) is the variational formulation of (6.3.1)-(6.3.3): $u \in H_{0,}^{1,1} (\Omega \times (0, +\infty))$ satisfying (6.3.3) is the unique solution of

$$\left(\frac{\partial u(t)}{\partial t}, v \right)_{0,\Omega} + \left(\nabla_x u(t), \nabla_x v\right)_{0,\Omega} + \lambda \left(u(t), v\right)_{0,\Omega} = \left(f(t), v\right)_{0,\Omega}, \;\; t > 0, \; \forall v \in H_0^1(\Omega). \tag{6.3.14}$$

We let p_h be the elliptic or Ritz projection onto V_h defined with respect to the energy inner product

$$(\nabla v, \; \nabla w)_{0,\Omega} + \lambda \, (v \; w)_{0,\Omega}$$

associated with the elliptic problem, which is the following stationary problem of (6.3.1)-(6.3.3):

$$-\Delta u + \lambda u = f \text{ in } \Omega \tag{6.3.15}$$

$$u = 0 \text{ on } \partial\Omega. \tag{6.3.16}$$

More precisely, for $u \in H_0^1(\Omega)$, its Ritz projection $p_h u \in V_h$ is the unique solution, for all $v_h \in V_h$, of the problem

$$(\nabla p_h u, \nabla v_h)_{0,\Omega} + \lambda (p_h u, v_h)_{0,\Omega} = (\nabla u, \nabla v_h)_{0,\Omega} + \lambda (u, v_h)_{0,\Omega}\,. \tag{6.3.17}$$

Thus $p_h u$ is the finite element approximation of the solution of the elliptic problem (6.3.15)-(6.3.16). This Ritz projection is used to rewrite the global error in the form below, which is convenient in what follows:

$$u_h^k - u(t_k) = \big(u_h^k - p_h u(t_k)\big) + (p_h u(t_k) - u(t_k)) \equiv \theta^k + \rho^k. \tag{6.3.18}$$

With these highlights, we have the following result:

Theorem 6.3.3. *Let the polygon Ω be convex. We assume that u^0 and u, are smoother to the extent that $u^0 \in H^2(\Omega)$ and $u \in H^2\left((0,+\infty), H^2(\Omega)\right)$. Fix a time $t^\star$ that can be written in several ways as $t^\star = k\Delta t$. Then, there exists a constant $C^\star \equiv C(t^\star)$, depending on $t^\star$ and there holds the error estimate*

$$\|u_h^k - u(t_k)\|_{0,\Omega} \le C^\star(\Delta t + h^2),$$

for the coupled NSFD method and classical FEM (6.3.11)-(6.3.12). When Ω is not convex, the same error estimate holds provided that $H^2(\Omega)$ is replaced with $H^{2,\beta}(\Omega)$, $0 < \beta < 1 - \frac{\pi}{\omega}$, in the regularity assumption of u with however $u^0 \in H^2(\Omega)$ and the triangulations, meeting the mesh refinement conditions (6.1.35).

Under the assumptions of the two cases above, we have the error estimate

$$\|u_h^k - u(t_k)\|_{1,\Omega} \le C^\star(\sqrt{\Delta t} + h),$$

whenever h is proportional to $\sqrt{\Delta t}$.

Proof. The proof in the case when Ω is convex follows from the arguments in Thomée [65], which work because $u(t_k) \in H^2(\Omega)$ in this case. In what follows, we adapt and give details to these arguments of [65] for the non-convex case. If Ω is not convex, then $u(t_k) \in H^{2,\beta}(\Omega)$ and from the interpolation theory discussed in section 5.1, we have, under the mesh refinement conditions (6.1.35),

$$\begin{aligned}\|\rho^k\|_{0,\Omega} &\leq Ch^2\|u(t_k)\|_{H^{2,\beta}(\Omega)} \\ &\leq Ch^2\left[\|u^0\|_{2,\Omega}+\int_0^{t_k}\|\frac{\partial u}{\partial s}\|_{H^{2,\beta}(\Omega)}ds\right] \end{aligned} \tag{6.3.19}$$

since $u(t_k)=u^0+\int_0^{t_k}\frac{\partial u}{\partial s}ds,\quad u^0\in H^2(\Omega)$ and $u\in H^1\left[(0,+\infty),\ H^{2,\beta}(\Omega)\right]$.

Given a sequence $(\gamma^k)_{k\geq 1}$ in $H_0^1(\Omega)$, we denote by $\frac{\bar{\partial}\gamma^k}{\partial t}$ the non-standard backward finite difference of γ^k defined by

$$\frac{\bar{\partial}\gamma^k}{\partial t}=\frac{\gamma^k-\gamma^{k-1}}{\phi(\Delta t)}. \tag{6.3.20}$$

Fix $v_h\in V_h$ and consider the sequence (θ^k) in $H_0^1(\Omega)$ defined in (6.3.18). Having the discrete and continuous variational problems (6.3.11) or (6.3.13) and (6.3.14) in mind, we have:

$$\begin{aligned}&\left(\frac{\bar{\partial}\theta^k}{\partial t},v_h\right)_{0,\Omega}+\left(\nabla\theta^k,\nabla v_h\right)_{0,\Omega}+\lambda\left(\theta^k,v_h\right)_{0,\Omega}\\
&\qquad=\left(\frac{\bar{\partial}(u_h^k-p_hu(t_k))}{\partial t},v_h\right)_{0,\Omega}+\left(\nabla(u_h^k-p_hu(t_k)),\nabla v_h\right)_{0,\Omega}\\
&\qquad+\lambda\left((u_h^k-p_hu(t_k)),v_h\right)_{0,\Omega},\text{by (6.3.18)}\\
&\qquad=-\left(p_h\frac{\bar{\partial}u(t_{t_k})}{\partial t},v_h\right)_{0,\Omega}+\left(f(t_k),v_k\right)_{0,\Omega}-\left(\nabla u(t_k),\nabla v_h\right)_{0,\Omega}\\
&\qquad-\lambda\left(u(t_k),v_h\right)_{0,\Omega},\text{by}\quad(6.3.17)\\
&\qquad=-\left(p_h\frac{\bar{\partial}u(t_k)}{\partial t},v_h\right)_{0,\Omega}+\left(\frac{\partial u(t_k)}{\partial t},\ v_h\right)_{0,\Omega},\quad\text{by}\quad(6.3.14)\\
&\qquad=\left((I-p_h)\frac{\bar{\partial}u(t_k)}{\partial t},v_h\right)_{0,\Omega}+\left(\frac{\partial u(t_k)}{\partial t}-\frac{\bar{\partial}u(t_k)}{\partial t},v_h\right)_{0,\Omega}\\
&\qquad\equiv\left(w^k,v_h\right)_{0,\Omega}\\
&\qquad\equiv\left(w_1^k,v_h\right)_{0,\Omega}+\left(w_2^k,v_h\right)_{0,\Omega}\end{aligned} \tag{6.3.21}$$

where $w_1^k=(I-p_h)\frac{\bar{\partial}u(t_k)}{\partial t}$ and $w_2^k=\frac{\partial u(t_k)}{\partial t}-\frac{\bar{\partial}u(t_k)}{\partial t}$.

If $v_h=\theta^k$ in (6.3.21), we have

$$\left(\frac{\bar{\partial}\theta^k}{\partial t},\theta^k\right)_{0,\Omega}+\left(\nabla\theta^k,\nabla\theta^k\right)_{0,\Omega}+\lambda\left(\theta^k,\theta^k\right)_{0,\Omega}=\left(w^k,\theta^k\right)_{0,\Omega}. \tag{6.3.22}$$

Using (6.3.20), we obtain

$$\begin{aligned}
\left(\frac{\bar{\partial}\theta^k}{\partial t}, \theta^k\right)_{0,\Omega} &= \phi^{-1}(\Delta t)\left(\theta^k - \theta^{k-1}, \theta^k\right)_{0,\Omega} \\
&= \phi^{-1}(\Delta t)\left(\theta^k, \theta^k\right)_{0,\Omega} - \phi^{-1}(\Delta t)\left(\theta^{k-1}, \theta^k\right)_{0,\Omega}, \\
&= \phi^{-1}(\Delta t)\|\theta^k\|_{0,\Omega}^2 - \phi^{-1}(\Delta t)\left(\theta^{k-1}, \theta^k\right)_{0,\Omega},
\end{aligned}$$

which combined with (6.3.22) yields

$$\phi^{-1}(\Delta t)\left[\|\theta^k\|_{0,\Omega}^2 - \left(\theta^{k-1}, \theta^k\right)_{0,\Omega}\right] \leq \left(w^k, \theta^k\right)_{0,\Omega}. \tag{6.3.23}$$

Using Cauchy-Schwarz inequality we have

$$\|\theta^k\|_{0,\Omega}^2 \leq \phi(\Delta t)\|w^k\|_{0,\Omega}\|\theta^k\|_{0,\Omega} + \|\theta^{k-1}\|_{0,\Omega}\|\theta^k\|_{0,\Omega}$$

and thus

$$\|\theta^k\|_{0,\Omega} \leq \phi(\Delta t)\|w^k\|_{0,\Omega} + \|\theta^{k-1}\|_{0,\Omega}. \tag{6.3.24}$$

By mathematical induction, (6.3.24) becomes

$$\|\theta^k\|_{0,\Omega} \leq \|\theta^0\|_{0,\Omega} + \phi(\Delta t)\sum_{j=1}^{k}\|w_1^j\|_{0,\Omega} + \phi(\Delta t)\sum_{j=1}^{k}\|w_2^j\|_{0,\Omega}. \tag{6.3.25}$$

Notice that

$$\begin{aligned}
\|\theta^0\|_{0,\Omega} &= \|u_h^0 - p_h u^0\|_{0,\Omega} \\
&= \|\Pi_h u^0 - p_h u^0\|_{0,\Omega} \quad \text{by (6.3.12)} \\
&\leq \|u^0 - \Pi_h u^0\|_{0,\Omega} + \|u^0 - p_h u^0\|_{0,\Omega} \\
&\leq Ch^2\|u^0\|_{2,\Omega} \text{ since } u^0 \in H^2(\Omega). \qquad (6.3.26)
\end{aligned}$$

A bound for $\phi(\Delta t)\sum_j^k \|w_1^j\|_{0,\Omega}$ is obtained by using (6.3.20) and (6.3.21) as follows:

$$\begin{aligned} w_1^j &= (I-p_h)\,\frac{\bar{\partial} u(t_j)}{\partial t} \\ &= (I-p_h)\,\phi^{-1}(\Delta t)\,(u(t_j - u(t_{j-1}))) \\ &= (I-p_h)\,\phi^{-1}(\Delta t)\int_{t_{j-1}}^{t_j} \frac{\partial u}{\partial s} ds. \end{aligned}$$

Thus we have

$$\begin{aligned} \phi(\Delta t)\sum_j^k \|w_1^j\|_{0,\Omega} &\le \sum_j^k \int_{t_{j-1}}^{t_j} \|(I-p_h)\frac{\partial u}{\partial s}\|_{0,\Omega} ds \\ &\le Ch^2 \int_{t_0}^{t_k} \|\frac{\partial u}{\partial s}\|_{H^{2,\beta}(\Omega)} ds \\ &\le Ch^2 \ \text{ since } \ u \in H^1\left((0,+\infty), H^{2,\beta}(\Omega)\right). \end{aligned} \tag{6.3.27}$$

On the other hand, a bound for $\phi(\Delta t)\sum_j^k \|w_2^j\|_{0,\Omega}$ is obtained using (6.3.21) as follows:

$$\begin{aligned} w_2^j &= \frac{\bar{\partial} u(t_j)}{\partial t} - \frac{\partial u(t_j)}{\partial t} \\ &= \phi^{-1}(\Delta t)(u(t_j) - u(t_{j-1})) - \frac{\partial u(t_j)}{\partial t}. \end{aligned}$$

This implies that

$$\begin{aligned} \phi(\Delta t)\sum_j^k w_2^j &= u(t_j) - u(t_{j-1}) - \Delta t\frac{\partial u(t_j)}{\partial t} + \Delta t\frac{\partial u(t_j)}{\partial t} - \phi(\Delta t)\frac{\partial u(t_j)}{\partial t} \\ &= (u(t_j) - u(t_{j-1})) - \Delta t\frac{\partial u(t_j)}{\partial t} + (\Delta t - \phi(\Delta t))\frac{\partial u(t_j)}{\partial t} \\ &= -\int_{t_{j-1}}^{t_j} (s - t_{j-1})\frac{\partial^2 u(s)}{\partial s^2} ds + (\Delta t - \phi(\Delta t))\frac{\partial u(t_j)}{\partial t}. \end{aligned}$$

by Taylor theorem with integral expression of the remainder term.

Taking the norm in $L^2(\Omega)$ and summing both sides of the equation, we have

$$\begin{aligned}
\phi(\Delta t)\sum_{j=1}^{k}\|w_2^j\|_{0,\Omega} \leq &\sum_{j=1}^{k}\|\int_{t_{j-1}}^{t_j}(s-t_{j-1})\frac{\partial^2 u(s)}{\partial s^2}ds\|_{0,\Omega} + C(\Delta t)^2\sum_{j=1}^{k}\|\frac{\partial u(t_j)}{\partial t}\|_{0,\Omega} \text{ by (6.3.10)}\\
\leq &\Delta t\int_0^{t_k}\|\frac{\partial^2 u(s)}{\partial s^2}\|_{0,\Omega}^2 ds + C(\Delta t)^2 k \sup_{1\leq j\leq k}\|\frac{\partial u(t)}{\partial t}\|_{0,\Omega}\\
\leq &\Delta t\left(\int_0^{t_k}\|\frac{\partial^2 u(s)}{\partial s^2}\|_{0,\Omega}^2 ds + Ct_k\|\frac{\partial u(s)}{\partial s}\|_{H^1((0,+\infty),L^2(\Omega))}\right) \text{ because}\\
&t^\star \equiv t_k = k\Delta t \text{ and } u \in H^2\left((0,+\infty),L^2(\Omega)\right) \text{ with } H^1\left((0,+\infty),L^2(\Omega)\right)\\
&\text{being continuously embedded in } C^0\left((0,+\infty),L^2(\Omega)\right),\\
\leq &C(t^\star)\Delta t. \qquad (6.3.28)
\end{aligned}$$

Combining (6.3.25), (6.3.26), (6.3.27) and (6.3.28) we have

$$\|\theta^k\|_{0,\Omega} \leq C(t^\star)\left(\Delta t + h^2\right). \qquad (6.3.29)$$

Hence, in view of (6.3.19) and (6.3.29), we obtain the required estimate

$$\|u_h^k - u(t_k)\|_{0,\Omega} \leq C(t^\star)\left(\Delta t + h^2\right). \qquad (6.3.30)$$

This proves the first part of the theorem.

The second part of the Theorem, is proved thanks to the relation (6.3.18) as follows:

$$\begin{aligned}
\|\nabla(u_h^k - u(t_k))\|_{0,\Omega} &\leq \|\nabla(u_h^k - p_h u(t_k))\|_{0,\Omega} + \|\nabla(p_h u(t_k) - u(t_k))\|_{0,\Omega}\\
&= \|\nabla\theta^k\|_{0,\Omega} + \|\nabla\rho^k\|_{0,\Omega}. \qquad (6.3.31)
\end{aligned}$$

Again, we give details for the non-convex case only, the convex case being mere classical due to the $H^2(\Omega)$ smoothness of the solution at every time $t > 0$. For Ω non-convex, we immediately bound $\nabla\rho^k$ by interpolation theory in section 5.1 as follows:

$$\|\nabla\rho^k\|_{0,\Omega} = \|\nabla(p_h u(t_k) - u(t_k))\|_{0,\Omega} \leq Ch\|u(t_k)\|_{H^{2,\beta}(\Omega)}. \qquad (6.3.32)$$

Letting $v_h = \theta^k$ in (6.3.21), we bound $\nabla\theta^k$ as follows:

$$\begin{aligned}
\|\nabla\theta^k\|^2_{0,\Omega} &\leq \left(w^k, \theta^k\right)_{0,\Omega} - \left(\frac{\bar{\partial}\theta^k}{\partial t}, \theta^k\right)_{0,\Omega} \\
&= \left(w^k - \frac{\bar{\partial}\theta^k}{\partial t}, \theta^k\right)_{0,\Omega} \\
&= \left(w^k, \theta^k\right)_{0,\Omega} - \frac{\left(\theta^k, \theta^k\right)_{0,\Omega}}{\phi(\Delta t)} + \frac{\left(\theta^{k-1}, \theta^k\right)_{0,\Omega}}{\phi(\Delta t)} \\
&\leq \left(w^k, \theta^k\right)_{0,\Omega} + \frac{\left(\theta^{k-1}, \theta^k\right)_{0,\Omega}}{\phi(\Delta t)}.
\end{aligned}$$

When Cauchy-Schwarz inequality is applied, we obtain

$$\begin{aligned}
\phi(\Delta t)\|\nabla\theta^k\|^2_{0,\Omega} &\leq \phi(\Delta t)\|w^k\|_{0,\Omega}\|\theta^k\|_{0,\Omega} + \|\theta^{k-1}\|_{0,\Omega}\|\theta^k\|_{0,\Omega} \\
&= \left(\phi(\Delta t)\|w^k\|_{0,\Omega} + \|\theta^{k-1}\|_{0,\Omega}\right)\|\theta^k\|_{0,\Omega} \\
&\leq \left(\phi(\Delta t)\|w^k\|_{0,\Omega} + \|\theta^{k-1}\|_{0,\Omega}\right)^2 \quad \text{by (6.3.24)}.
\end{aligned}$$

Using (6.3.29), (6.3.27) and (6.3.28), we have

$$\phi(\Delta t)\|\nabla\theta^k\|^2_{0,\Omega} \leq C^\star\left(h^2 + \Delta t\right)^2.$$

In the previous inequality, we let Δt be proportional to h^2 (i.e $h = C\sqrt{\Delta t}$), we divide both sides of the inequality by $\sqrt{\phi(\Delta t)}$ to obtain

$$\begin{aligned}
\|\nabla\theta^k\|_{0,\Omega} &\leq C^\star\left(h \cdot \frac{h}{\sqrt{\phi(\Delta t)}} + \sqrt{\Delta t}\sqrt{\frac{\Delta t}{\phi(\Delta t)}}\right) \\
&\leq C^\star(Ch + \sqrt{\Delta t})\sqrt{\frac{\Delta t}{\phi(\Delta t)}} \\
&\leq C^\star\left(h + \sqrt{\Delta t}\right) \quad \text{in view of (6.3.10)}. \qquad (6.3.33)
\end{aligned}$$

Combining (6.3.32), (6.3.33) together with Poincaré Friedrichs inequality (2.4.3), we have

$$\|u_h^k - u(t_k)\|_{1,\Omega} \leq C^\star\left(h + \sqrt{\Delta t}\right). \qquad (6.3.34)$$

This completes the proof of the theorem. □

By construction of the coupled NSFD scheme and FEM, we readily have the following qualitative stability result, which gives an indication on the relevance of this coupling.

Theorem 6.3.4. *The discrete method (6.3.11)-(6.3.12) reduces to a numerical procedure for the space independent limit case of the boundary value problem (6.3.1)-(6.3.3). The latter method corresponds to the exact scheme (6.3.9) of the decay equation (6.3.4)-(6.3.5) when* $f \equiv 0$.

Remark 6.3.5. *In line with Theorem 6.3.4, the following comments are in order to understand the good performance of the NSFD method in the numerical experiment in the next section. The convergence (6.3.34) in* H^1 *norm implies that there exists a subsequence of* u_h^k *still denoted in the same way such that* u_h^k *converges point-wise to* u *as* $h \to 0$ *and* $k \to +\infty$ *(see [1], Corollary 2.11). Assume that* $\Delta u = 0$ *near a point* $a \in \Omega$*. Now if* v_h *in (6.3.11) is chosen in such a way that its support containing the point* a*, is very small and* $v_h = 1$ *near* a*, then we can use the approximation*

$$\int_\Omega g v_h dx = g(a)K \;\; \textit{where} \;\; K \;\; \textit{is the measure of the} \;\; \textit{supp}(v_h).$$

Using this approximation in (6.3.11), it follows that $u_h^k(a)$ *is a discrete solution of the ordinary differential equation associated with (6.3.1) and (6.3.3) when we fix* $x = a$*. Of course* $u_h^k(a)$ *is the solution of the exact scheme (6.3.9) if we also have* $f(a,t) = 0$.

More generally, the above reasoning could be used without considering a subsequence of u_h^k. Indeed, the practical implementation of the method (6.3.11) amounts to considering what Strang and Fix [63] call "variational crimes". That is using numerical integration in (6.3.11). In this regard, assume that a is the barycenter of a fixed triangle T of triangulation of $\bar{\Omega}$ and let us assume as above that $\Delta u = 0$ near the point a. We take v_h in (6.3.11) having its support in such that $v_h = 1$ near a and we use the approximation

$$\int_\Omega g v_h dx = g(a) \;\; \text{measure } (T).$$

We then proceed as before to conclude that $u_h^k(a)$ is a discrete solution of the associated ordinary differential equation.

6.4 Numerical experiment

This section is devoted to demonstrate computationally the optimal convergence of some of the numerical schemes presented in the preceding sections of this chapter.

Before we proceed with the numerical experiments that support the theory, we want to show that triangulations $(\mathcal{T}_h)$ of the polygonal domain $\bar{\Omega}$ that are refined according to the condition (6.1.35) exist in practice. To this end, we follow the procedure proposed by Raugel [59] and summarized in [30].

More precisely, observing that the vertex that is responsible for singularities is placed at the origin $(0,0)$, we consider the following steps:

1. Divide the polygon Ω into big triangles;

2. Divide each side of each of the big triangles that has no vertex at $(0,0)$ into $n = \frac{1}{h}$ subsegments of equal length and proceed, following the usual triangulation technique (See Ciarlet [16], Raviart and Thomas [57]);

3. Divide each of the big triangles that has a vertex at $(0,0)$, according to the ratios

$$\left(\frac{i}{n}\right)^{\frac{1}{1-\beta}}, \quad 1 \leq i \leq n,$$

   along the sides that ends at $(0,0)$; divide the third side in the usual way and proceed as usual.

Figure 6.1 illustrates this case, for $n = 4$, with one of the sides that ends at the vertex $(0,0)$ lying on the $0x_1$ axis.

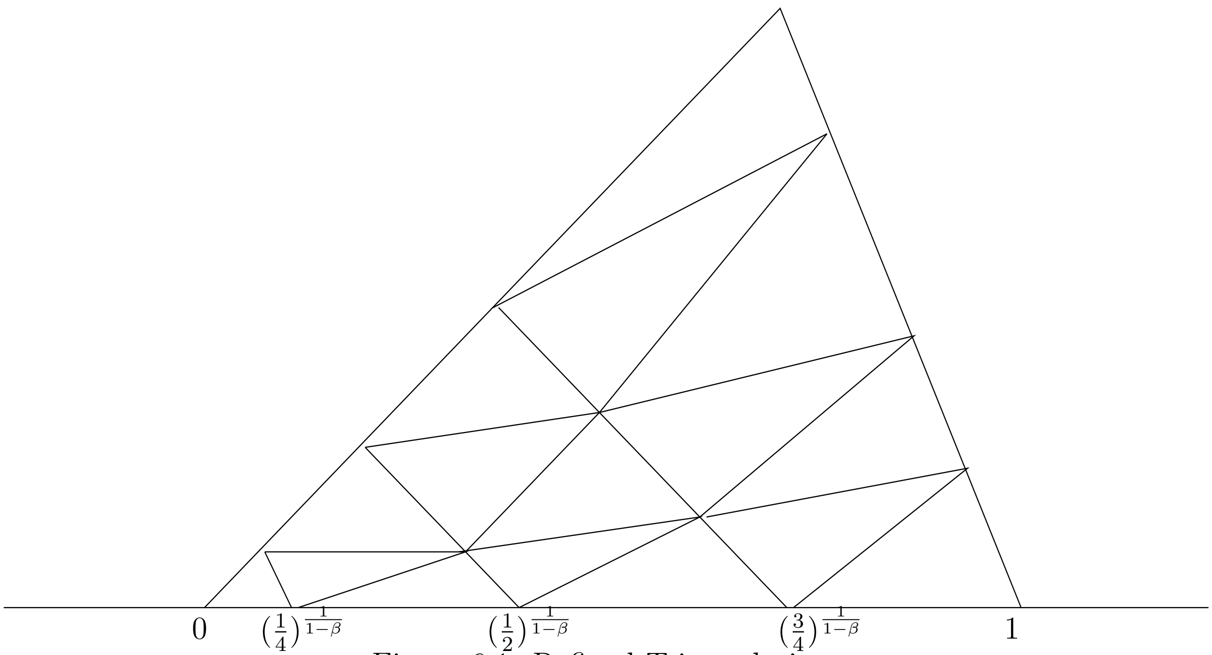

Figure 6.1: Refined Triangulation

The mesh refinement conditions (6.1.35) in this case reduce to

$$h_i \leq \begin{cases} C\left(\frac{1}{n}\right)^{\frac{1}{1-\beta}}, & \text{if} \quad i = 0 \\ \\ C\frac{1}{n}\inf_{T_i} r^{\beta}, & \text{if} \quad i \neq 0 \end{cases} \tag{6.4.1}$$

where

$$h_i = \left(\frac{i+1}{n}\right)^{\frac{1}{1-\beta}} - \left(\frac{i}{n}\right)^{\frac{1}{1-\beta}} \quad \text{and} \quad h = \frac{1}{n}$$

and $C > 0$ is a constant independent of n.

Let us prove (6.4.1). The proof for $i = 0$ is obvious by the definition of h_i.
In the case when $i \neq 0$, we have

$$\begin{aligned} h_i &= \frac{1}{1-\beta}(\xi)^{\frac{\beta}{1-\beta}}\frac{1}{n}, \quad \text{with} \quad \frac{i}{n} < \xi < \frac{i+1}{n}, \quad \text{by the Mean-Value Theorem.} \\ &\leq \frac{1}{1-\beta}\left(\frac{i+1}{n}\right)^{\frac{\beta}{1-\beta}}\frac{1}{n} \quad \text{since} \quad 0 < \beta < 1. \end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\frac{\left(\frac{i+1}{n}\right)^{\frac{\beta}{1-\beta}}}{\left(\frac{i}{n}\right)^{\frac{\beta}{1-\beta}}} &= \left(\frac{i+1}{i}\right)^{\frac{\beta}{1-\beta}} \\
&= \left(1+\frac{1}{i}\right)^{\frac{\beta}{1-\beta}} \\
&\leq (2)^{\frac{\beta}{1-\beta}} \text{ because } i \geq 1.
\end{aligned}$$

Therefore

$$h_i \leq C(\frac{i}{n})^{\frac{\beta}{1-\beta}}\frac{1}{n} \text{ and } h_i \leq C\frac{1}{n}\inf_{T_i} r^{\beta}.$$

This proves (6.4.1).

After this justification of the existence of the mesh refined triangulations, we proceed by considering Ω to be an L-shaped domain as shown in Figure 6.2. This consists of the re-entrant angle $\omega = \frac{3\pi}{2}$ that is responsible for singularities at the origin of the plane. The

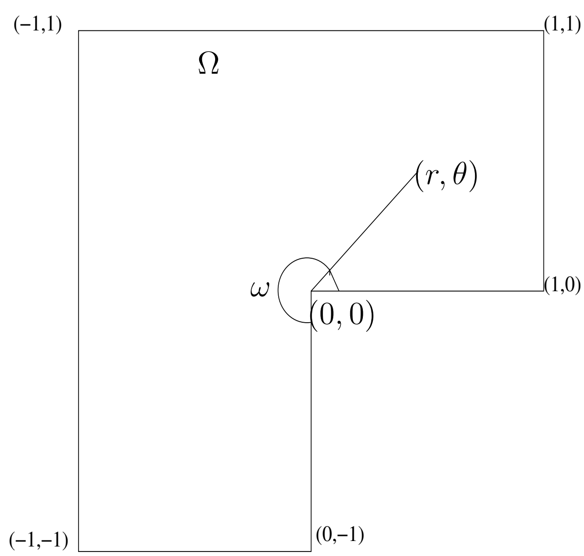

Figure 6.2: L-shaped domain

right-hand side f of equation (6.3.1) is taken in such a way that

$$u(x,t) = te^{-t}\psi(r)r^{\frac{2}{3}}\sin\frac{2}{3}\theta. \tag{6.4.2}$$

is the exact solution of the problem (6.3.1)-(6.3.3) where $\psi(r)$ is a smooth cut-off function such that $\psi = 1$ for $r \leq 1/4$ and $\psi = 0$ for $r \geq 1/2$. We use a uniform mesh for $\beta = 0$ and a refined mesh for $\beta = 1/3$ on the method (6.3.11)-(6.3.12). A similar construction is done when the denominator of the first term of (6.3.11) is replaced by Δt. The pictures resulting from these techniques are illustrated in Figures 6.3 and 6.4 for $n = 10$.

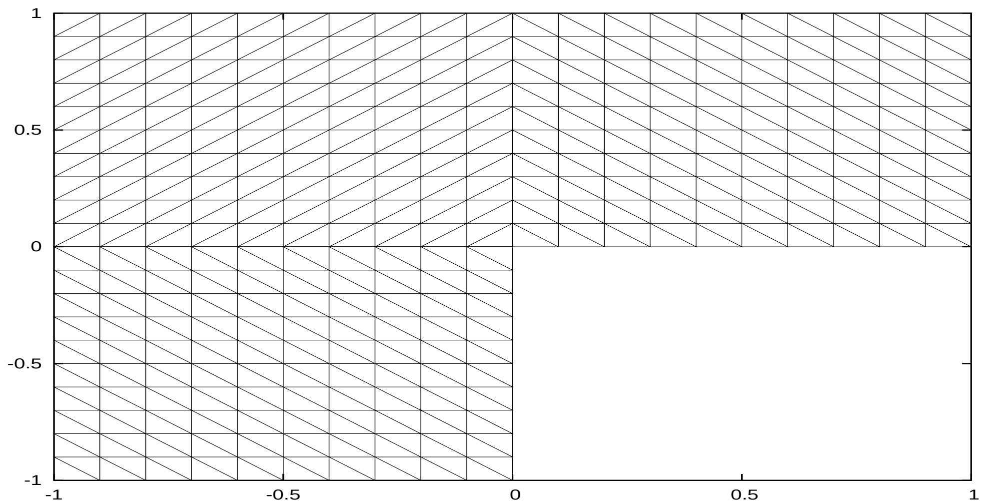

Figure 6.3: Uniform mesh for $n = 10$

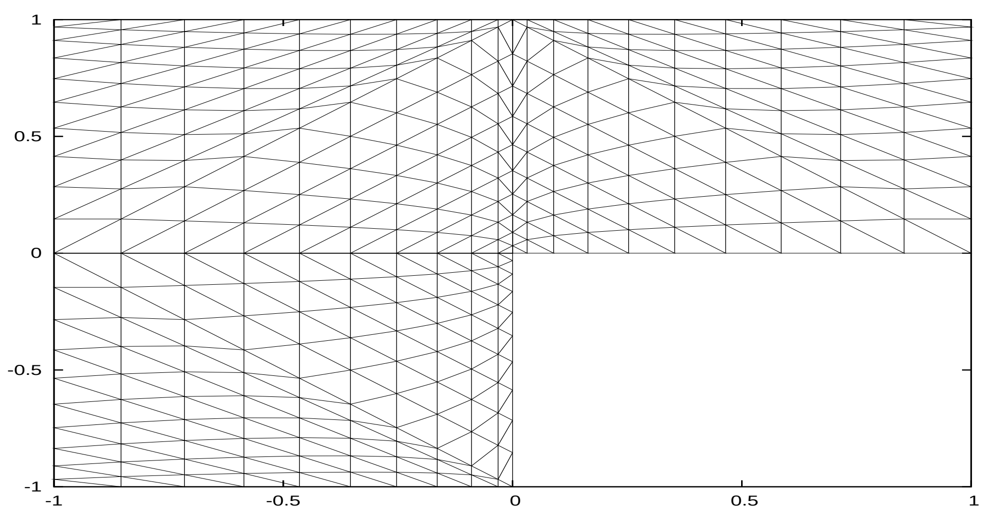

Figure 6.4: Refined mesh for $n = 10$

In Figure 6.3, the domain Ω is filled with a uniform mesh of identical triangles in the classical manner. This is followed by Figure 6.4, where the domain Ω is refined following the procedure of Raugel [59] presented above and illustrated in Figure 6.1.

For our numerical experiments, we take $n = 10, 50, 100, 125$. The refinement parameter β is taken to be $\beta = 0$ for a uniform mesh and $\beta = 1/3$ for a refined mesh. A similar approach to this choice of n values was done for the Laplace equation in a polygon in [24]. We approach the numerical solution to the problem using two techniques. The first technique is by coupling the standard finite difference method (SFDM) and finite element method. The second technique is by combining the non-standard finite difference and the finite element methods. In both cases we keep once and for all, the time fixed.

For the numerical solution obtained by coupling the SFDM and FEM, the error $\|u - u_h\|_{1,\Omega}$ was computed. Table 6.1 shows the rates of convergence for the uniform mesh ($\beta = 0$) and the refined mesh ($\beta = 1/3$). Figure 6.5 shows in logarithm scale the slope of the curves that correspond to the approximate rates of convergence, which are 0.27 (poor) for the uniform and 0.8123 for the refined mesh.

Similarly, for the numerical solution obtained by combining the NSFD method and FEM, the error $\|u - u_h\|_{0,\Omega}$ was computed. Table 6.2 shows the rates of convergence for the uniform mesh and the refined mesh whereas Figure 6.6 shows in logarithm scale that the approximate rates are 0.5 (poor) and 1.95 for the uniform mesh and the refined mesh, respectively.

We have therefore proved computationally that the refined mesh provides better (optimal) rates of convergence than the classical uniform mesh.

Table 6.1: Error in the H^1-norm for both uniform and refined meshes

n	**Uniform Mesh**	**Refined Mesh**
	$\|\|u-u_h\|\|_{1,\Omega}$	$\|\|u-u_h\|\|_{1,\Omega}$
10	3.0854E-3	2.9221E-3
50	1.2875E-3	5.8442E-4
100	9.0016E-4	2.9110E-4
125	8.1009E-4	2.3288E-4

Table 6.2: Error in the L^2-norm for both uniform and refined meshes

n	**Uniform Mesh**	**Refined Mesh**
	$\|\|u-u_h\|\|_{0,\Omega}$	$\|\|u-u_h\|\|_{0,\Omega}$
10	1.2469E-3	1.3411E-6
50	2.4939E-4	5.5372E-8
100	1.3027E-4	1.3860E-8
125	1.0457E-4	8.8717E-9

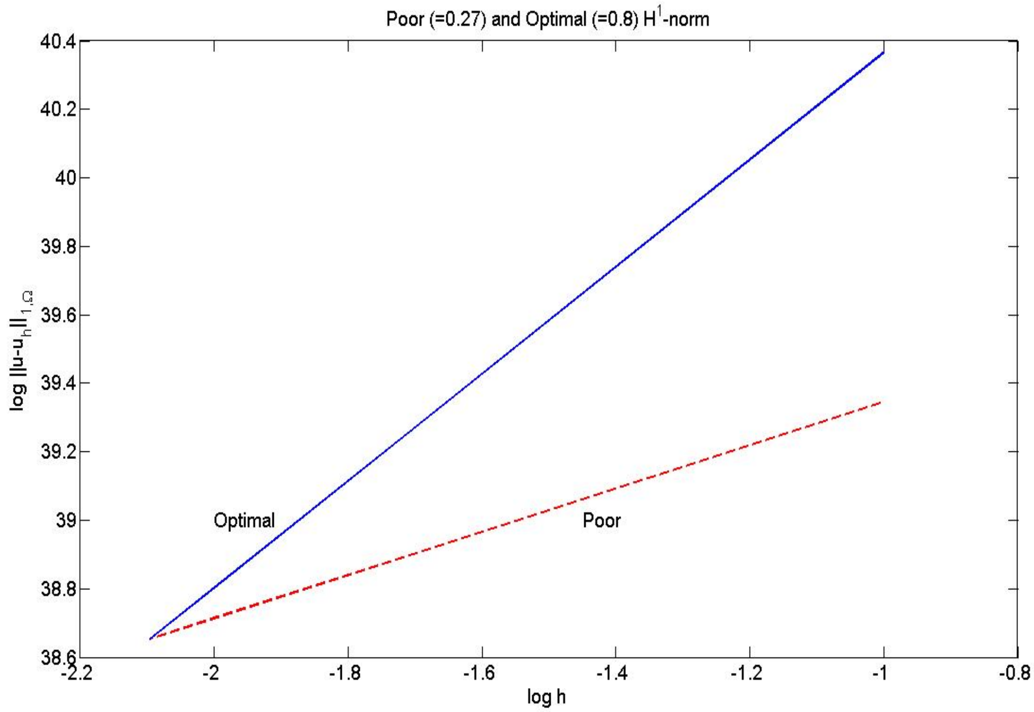

Figure 6.5: Rate of convergence for H^1-norm

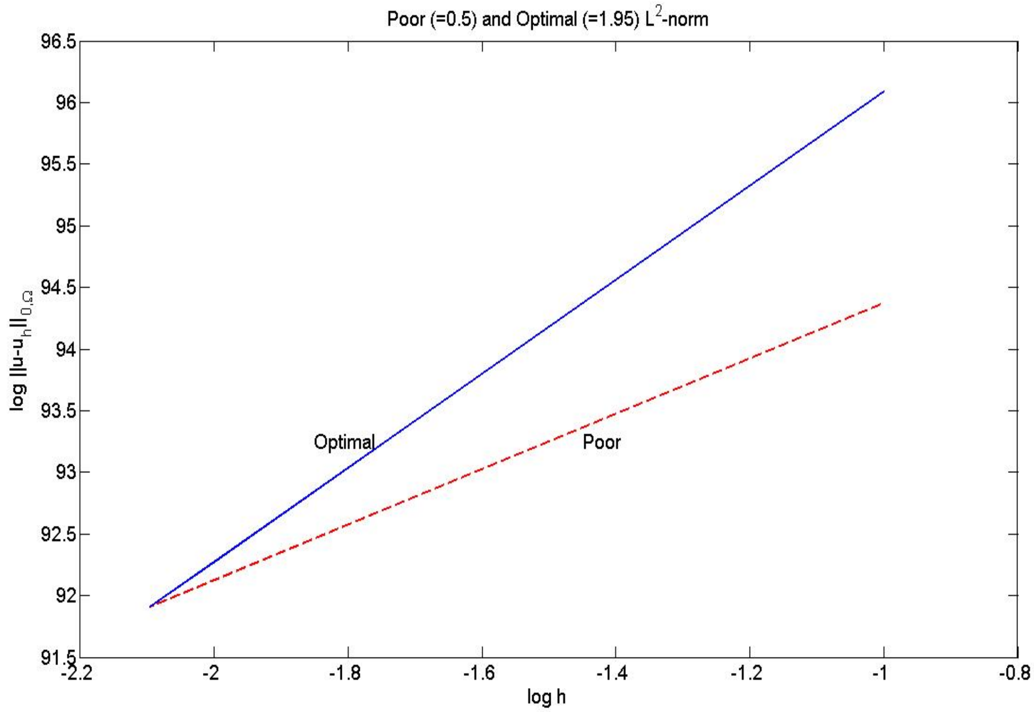

Figure 6.6: Rate of convergence for L^2-norm

We conclude this section by studying the impact and the power of the non-standard finite difference method. Let f in (6.3.1) be such that

$$u(x,t) = \alpha e^{-\lambda t}\psi(r)r^{2/3}\sin 2/3\theta \tag{6.4.3}$$

is the solution of (6.3.1)-(6.3.3) for the parameters λ and α. We fix once and for all, $x = (-0.0316, 0.0554)$, $\lambda = 3$ and $\alpha = \pm 0.5$. Since $|x| \leq \frac{1}{4}$, then $u(t) \equiv u(x,t)$ is a solution of the decay equation (6.3.4)-(6.3.6); $u(t)$ is plotted against the time on Fig 6.7(a) and (b). For the same fixed x, Fig 6.7(c) and (d) depict $u_h^k \equiv u_h^k(x)$ obtained from the NSFDM-FEM (6.3.1) as well as from the classical finite difference method with $\Delta t = 0.5$. For the latter method, there is no restriction on the value of Δt since it is implicit [65]. The figures speak for themselves.

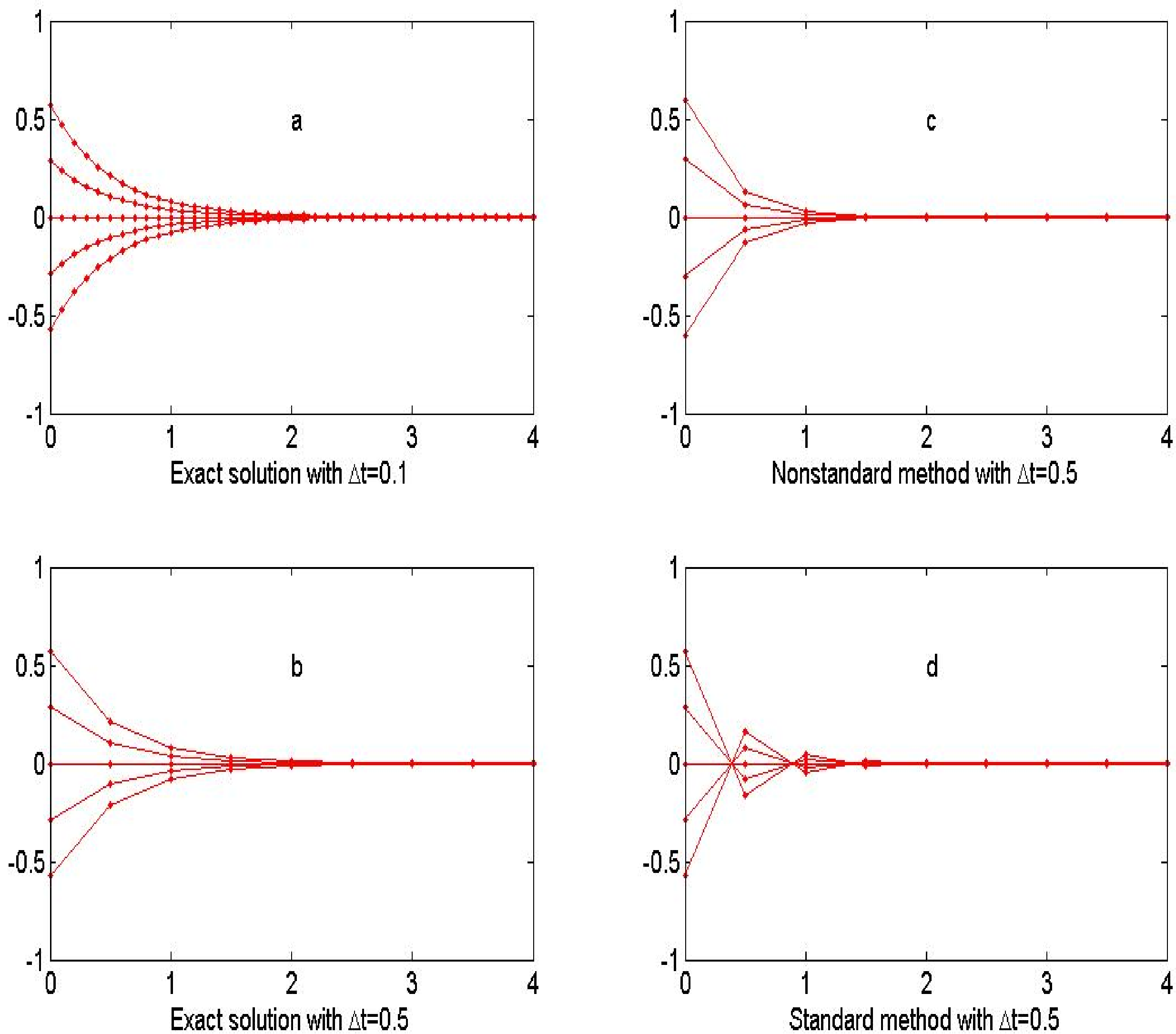

Figure 6.7: Impact of non-standard and Standard approaches

Chapter 7

Conclusion

This work was initially motivated by the Ph.D thesis of Maghnouji [46] where the singularities of a parabolic equation for a strongly elliptic operator on a polygonal domain are studied.

The initial aim was to provide the numerical analysis counterpart of [46]. However, given the complexity and level of generality of [46], we opted to work with the heat equation in order to better understand the difficulties and to obtain explicit results in which the geometry of the domain is clearly reflected. Some results for the heat equation are obtained in Grisvard [29] and [31]; but the approach used here is different as we are mostly concerned with the Laplace transform of vector-valued distributions.

The main results we obtained can be summarized as follows:

- We established the singular decomposition of the solution of the heat equation with an explicit representation of the singular part;
- We established the tangential regularity of the solution in the time variable;
- We showed that the solution is globally regular in a weighted Sobolev space in which the weight depends on the corners of the domain Ω;
- The mesh size being suitably refined in the triangulations of the space domain $\bar{\Omega}$, we implemented two optimally convergent numerical methods: the coupled Fourier-finite element method and the coupled Nonstandard finite difference method-finite element method. The latter method has the advantage of replicating some intrinsic properties of the exact solution.

Possible extensions of this thesis that we will consider in future include:

- The numerical study of parabolic problems in the general framework of [46]. For elliptic problems, this is done for instance in [43].
- The study of the heat equation with more regular right hand side. This would require the introduction and better understanding of anisotropic Sobolev spaces as in [46].
- The extension of the study to domains with both edge and vertex singularities such as polyhedrons. This is done in [42, 44] for elliptic problems as well as in [18, 19], [48, 49].
- Extension to nonlinear reaction diffusion equations and construction of suitable numerical methods. Reliable NSFD schemes in this case were considered in [4].