Chapter 3

The Helmholtz problem in a smooth domain

In the preceding chapter, we built the theory of the Laplace transform of vector-valued distributions. We shall apply this theory to the heat equation in the next chapter. This will lead to the Helmholtz problem that will be considered in this chapter.

In section 3.1, we establish the well-posedness of the Helmholtz problem. In section 3.2, we examine the regularity of the solution of the Helmholtz problem in a smooth domain.

3.1 Well-posedness of the problem

We consider the following Dirichlet problem for the Helmholtz operator: given a complex number $p = \xi + i\eta$ and a complex-valued function g on Ω, find $w : \Omega \mapsto \mathbb{C}$, solution of

$$-\Delta w + p\, w = g \;\text{ in }\; \Omega \tag{3.1.1}$$

and

$$w = 0 \;\text{ on }\; \partial\Omega. \tag{3.1.2}$$

Here $\Omega \subset \mathbb{R}^2$ is a bounded domain. Despite the title of the chapter, we assume in this specific section that the boundary $\partial\Omega \equiv \Gamma$ is Lipschitz in the sense of Definition 2.1.1 because the results apply to the non-smooth case which is considered in the next chapter. Actual smoothness requirements on Γ will be made in the next section.

It is convenient to study problem (3.1.1)-(3.1.2) in the abstract setting of the following theorem ([40], [38]).

Theorem 3.1.1. *Let X be a Hilbert space with inner product and associated norm denoted by $(\cdot,\cdot)_X$ and $\|\cdot\|_X$ respectively. The conjugate dual of X is denoted by X' and its norm is $\|\cdot\|_{X'}$. Let $a(\cdot,\cdot)$ be a sesquilinear form, $l(\cdot)$ be a (conjugate) linear form on X. We make the following assumptions:*

1. *The linear form $l(\cdot)$ is continuous i.e. there exists a $M > 0$ such that,*

$$|l(v)| \leq M\|v\|_X, \quad \forall \ v \in X. \tag{3.1.3}$$

2. *The sesquilinear form $a(\cdot,\cdot)$ is continuous i.e there exists a constant $K > 0$*

$$|a(s,v)| \leq K||s\|_X \ \|v||_X \quad \forall \ \ s\, v \in X. \tag{3.1.4}$$

3. *The sesquilinear form $a(\cdot,\cdot)$ is X-elliptic or X-coercive i.e there exists a constant $\alpha > 0$ such that*

$$Re \ a(v,v) \geq \alpha\|v\|_X^2, \quad \forall \ v \in X. \tag{3.1.5}$$

Then the abstract variational problem of finding

$$s \in X \quad such\ that \ \ a(s,v) = l(v) \ \ \forall \ \ v \in X \tag{3.1.6}$$

is well-posed. In other words, there exists a unique $s \in X$, solution of (3.1.6) such that,

$$\|s\|_X \leq C\|l\|_{X'} \tag{3.1.7}$$

for some constant $C > 0$.

Proof. With the sesquilinear form $a(\cdot,\cdot)$, we associate the operator

$$A : X \longrightarrow X',$$

defined by

$$\langle Aw, v\rangle_{X'\times X} = a(w,v). \tag{3.1.8}$$

The variational problem (3.1.6) is then equivalent to the functional equation: find

$$s \in X \quad \text{such that} \quad As = l \quad \text{in} \quad X'. \tag{3.1.9}$$

It is clear from the sesquilinearity of $a(\cdot,\cdot)$ that A is linear. Likewise, A is bounded since the continuity in (3.1.4) of $a(\cdot,\cdot)$ yields

$$\|Aw\|_{X'} := \sup_{v \neq 0} \frac{|a(w,v)|}{\|v\|_X} \leq K\|w\|_X. \tag{3.1.10}$$

On the other hand, for $w \in X$, (3.1.5) and the boundedness of the form $Aw \in X'$ lead to

$$\begin{aligned} \alpha\|w\|_X^2 \leq Re\ a(w,w) &= Re\,\langle Aw, w\rangle \\ &\leq |\,\langle Aw, w\rangle\,| \\ &\leq \|Aw\|_{X'}\|w\|_X. \end{aligned}$$

Thus

$$\|Aw\|_{X'} \geq \alpha\|w\|_X \quad \forall \;\; w \in X. \tag{3.1.11}$$

Let $A^\star \in B(X, X')$ be the adjoint operator of A. In the present context, it should be noted that,

$$\langle A^\star w, v\rangle_{X' \times X} = \overline{a(v,w)}. \tag{3.1.12}$$

Therefore, following the above argument that lead to (3.1.11), we obtain

$$\|A^\star w\|_{X'} \geq \alpha\|w\|_X \quad \forall \;\; w \in X. \tag{3.1.13}$$

To prove the theorem, it is equivalent to show that the mapping $A : X \mapsto X'$ in the operator equation (3.1.9) is an isomorphism. We claim that the range $R(A)$ of A is dense in X'. Indeed, let φ in the bi-dual space X'' of X be such that

$$\varphi(Aw) = 0 \quad \forall \;\; w \in X.$$

We show that $\varphi = 0$. The space X is reflexive, being a Hilbert space. Thus, there exists

$v \in X$ such that $\varphi = \mathbf{C}(v)$ where

$$\mathbf{C} : X \longrightarrow X''$$

is the canonical mapping of X to X''. Now for $w \in X$,

$$\begin{aligned} 0 &= \varphi(Aw) \quad \text{(by assumption)} \\ &= \mathbf{C}(v)(Aw) \\ &= \langle Aw, v \rangle \quad \text{(by definition of } \mathbf{C}) \\ &= < \overline{A^\star v, w} > \quad \text{by (3.1.8) and (3.1.12).} \end{aligned}$$

Hence $A^\star v = 0$. By (3.1.13), it follows that $v = 0$. Thus $\varphi = \mathbf{C}(v) = \mathbf{C}(0) = 0$.

We also claim that $R(A)$ is closed in X'. In fact, let (Aw_n) be a sequence in $R(A)$ such that

$$Aw_n \longrightarrow h \quad \text{in} \quad X' \quad \text{as} \quad n \longrightarrow \infty.$$

Then (Aw_n) is a Cauchy sequence in X'. By (3.1.11) and the linearity of A, we have

$$\alpha \|w_n - w_m\|_X \leq \|Aw_n - Aw_m\|_{X'},$$

which implies that (w_n) is a Cauchy sequence in X. Since X is complete, the sequence (w_n) converges to some $w \in X$. Continuity of the operator A leads to

$$Aw_n \longrightarrow Aw \quad \text{in} \quad X' \quad \text{as} \quad n \rightarrow \infty.$$

By uniqueness of limits, we have

$$h = Aw.$$

Hence $R(A)$ is closed. The density and the closedness of $R(A)$ in X' mean that the operator A is surjective. Since A is injective by (3.1.11), the operator A is bijective. The Banach open mapping theorem guarantees that A is an isomorphism. $\square$

Remark 3.1.2. *1. Theorem 3.1.1 can be proved by the Banach contraction mapping theorem. It is indeed possible to choose $\rho > 0$ such that the map*

$$v \longrightarrow v - \rho\tau(Aw - l);$$

is a contraction from X into X where

$$\tau : X' \longrightarrow X;$$

is the Riesz-representation operator (see [16]).

2. In the case when the sesquilinear form $a(\cdot,\cdot)$ is hermitian i.e.

$$a(w, v) = \overline{a(v, w)} \text{ so that } A = A^\star,$$

Theorem 3.1.1 is the so-called Lax-Milgram lemma. Its proof is then a direct consequence of Riesz-representation theorem. In this case $a(\cdot,\cdot)$ defines an inner product on X the associated norm of which is equivalent to the norm $\|\cdot\|_X$. Note also that in this case, the variational problem (3.1.6) is equivalent to the minimization problem: find

$$s \in X \quad \text{such that} \quad J(s) = min_{v\in X} J(v) \tag{3.1.14}$$

where $J(v) := \frac{1}{2}a(v, v) - l(v)$ represents the total energy of the system under consideration. (See [16] for more details).

We want to put problem (3.1.1)-(3.1.2) in the general variational setting discussed in Theorem 3.1.1. The standard procedure to achieve this consists of four main steps described in [40]. To this end, we assume once and for all that, $g \in L^2(\Omega)$. We take $X = H_0^1(\Omega)$ and we define $a(\cdot,\cdot)$ and $l(\cdot)$ as follows:

$$a(w, v) := \int_\Omega \nabla w \nabla \bar{v} dx + \int_\Omega p w \bar{v} dx, \tag{3.1.15}$$

and

$$l(v) := \int_\Omega g \ \bar{v} dx. \tag{3.1.16}$$

We are therefore led to the following variational problem: find

$$w \in H_0^1(\Omega) \ \text{ such that } \ a(w,v) = l(v) \ \ \forall \ \ v \in H_0^1(\Omega). \tag{3.1.17}$$

Clearly, $a(\cdot,\cdot)$ is a sesquilinear form and $l(\cdot)$ is a conjugate or antilinear form. By the Cauchy-Schwarz inequality, the conjugate linear form in (3.1.16) is continuous on $H_0^1(\Omega)$ since

$$\begin{aligned} |l(v)| &\leq \left(\int_\Omega |g|^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega |v|^2 dx\right)^{\frac{1}{2}} \\ &\leq \|g\|_{0,\Omega}\|v\|_{1,\Omega}. \end{aligned} \tag{3.1.18}$$

Similarly, for $w, v \in H_0^1(\Omega)$, we have

$$\begin{aligned} |a(w,v)| &\leq \left(\int_\Omega |\nabla w|^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega |\nabla v|^2 dx\right)^{\frac{1}{2}} + |p| \left(\int_\Omega |w|^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega |v|^2 dx\right)^{\frac{1}{2}} \\ &\leq \|\nabla w\|_{0,\Omega}\|\nabla v\|_{0,\Omega} + |p|\|w\|_{0,\Omega}\|v\|_{0,\Omega} \\ &\leq (1+|p|)\|w\|_{1,\Omega}\|v\|_{1,\Omega} \end{aligned} \tag{3.1.19}$$

which show the continuity of the sesquilinear form. Regarding the H_0^1-ellipticity or H_0^1-coercivity of $a(\cdot,\cdot)$, we assume that

$$Re(p) = \xi \geq 0. \tag{3.1.20}$$

Under this assumption, we have for $w \in H_0^1(\Omega)$ and $Rep > 0$

$$\begin{aligned} Re\ a(w,w) &= \int_\Omega |\nabla w|^2 dx + Re(p) \int_\Omega |w|^2 dx \\ &\geq \min\{1, Re(p)\}\|w\|_{1,\Omega}. \end{aligned} \tag{3.1.21}$$

For $Re(p) = 0$, we have

$$Re\ a(w,w) \geq C\|w\|_{1,\Omega}^2, \tag{3.1.22}$$

by Poincaré Friedrichs inequality in Theorem 2.4.3. In summary, we have proved the following theorem:

Theorem 3.1.3. *Under the condition (3.1.20), the problem (3.1.17) is well-posed in $H_0^1(\Omega)$. More precisely, there exists a unique solution $w \in H_0^1(\Omega)$ of (3.1.17) and a constant K depending on p (except for $Re(p) = 0$) such that*

$$\|w\|_{1,\Omega} \le K\|g\|_{0,\Omega}. \tag{3.1.23}$$

Notice that the constant K in (3.1.23) does indeed depend on p for $Re(p) > 0$ since, from (3.1.17) and (3.1.21) we have

$$\begin{aligned} \min\{1, Re(p)\}\|w\|_{1,\Omega}^2 &\le \int_\Omega |\nabla w|^2 dx + Re(p)\int_\Omega |w|^2 dx \\ &= Rea(w,w) \\ &= Re\int_\Omega g\overline{w}dx \\ &\le \left(\int_\Omega |g|^2 dx\right)^{\frac{1}{2}}\left(\int_\Omega |w|^2 dx\right)^{\frac{1}{2}} \quad \text{by Cauchy-Schwarz's inequality} \\ &\le \|g\|_{0,\Omega}\|w\|_{1,\Omega}. \end{aligned} \tag{3.1.24}$$

Thus

$$\|w\|_{1,\Omega} \le \frac{1}{\min\{1, Re(p)\}}\|g\|_{0,\Omega} \text{ for } Re(p) > 0.$$

In the case when the unique solution w of (3.1.17) satisfies an estimate of the type (3.1.23) where the constant K does not depend on p, we will say that the problem (3.1.17) is uniformly well-posed. In order to achieve this, we work with weighted Sobolev spaces defined as follows:

Definition 3.1.4. *Given $\rho > 0$ and an integer $m \ge 0$, we denote by $H^m(\Omega, \rho)$, the Sobolev space $H^m(\Omega)$ equipped with the weighted norm*

$$\|s\|_{m,\,\Omega,\,\rho} := \sqrt{\int_\Omega \sum_{|\alpha|\le m} \rho^{2(m-|\alpha|)}|D^\alpha s(x)|^2 dx}. \tag{3.1.25}$$

Proposition 3.1.5. *Let $\rho > 0$ be such that $\frac{x}{\rho} \in \Omega$ whenever $x \in \Omega$. Then on $H^m(\Omega)$, $m \ge 1$, integer, the weighted norm $\|\cdot\|_{m,\,\Omega,\,\rho}$ in Definition 3.1.4 is equivalent (with constants not* *depending on ρ) to the more economical weighted norm $|||\cdot|||_{m,\,\Omega,\,\rho}$ given by*

$$|||s|||^2_{m,\,\Omega,\,\rho} := \int_\Omega \left[\sum_{|\alpha|=m} |D^\alpha s(y)|^2 + \rho^{2m}|s(y)|^2 \right] dy. \tag{3.1.26}$$

Proof. Let us consider the change of variable

$$y = \frac{x}{\rho}, \text{ so that } dy = \rho^{-2}dx.$$

Given $s \in H^m(\Omega)$, we introduce the function $s_{\frac{1}{\rho}}$ given by

$$s_{\frac{1}{\rho}}(x) = s(\frac{x}{\rho}).$$

By the chain rule, we readily get

$$D_x^\alpha s_{\frac{1}{\rho}}(x) = \rho^{-|\alpha|} D_y s(y), \text{ for } |\alpha| \le m.$$

This implies that we have

$$\rho^{1-m}\|s\|_{m,\,\Omega,\,\rho} = \|s_{\frac{1}{\rho}}\|_{m,\,\Omega} \text{ and } \rho^{1-m}|||s|||_{m,\,\Omega,\,\rho} = |||s_{\frac{1}{\rho}}|||_{m,\,\Omega} \tag{3.1.27}$$

where the economical norm $|||\cdot|||_{m,\,\Omega}$ is defined by

$$|||v|||^2_{m,\,\Omega} = \int_\Omega \left(\sum_{|\alpha|=m} |D^\alpha v(y)|^2 + |v(y)|^2 \right) dy. \tag{3.1.28}$$

But for Ω bounded (as in our case), the usual norm $\|\cdot\|_{m,\,\Omega}$ on $H^m(\Omega)$ is equivalent to $|||\cdot|||_{m,\Omega}$. (see Theorem 1.8 in [54]). This combined with (3.1.27) proves the proposition. □

Remark 3.1.6. *From Proposition 3.1.5, it follows that one can either work with the norm (3.1.25) or (3.1.26). The latter weighted norm is the one adopted in [19] and [46]. Note that the equivalence of norms stated in Proposition 3.1.5 holds for bounded domains. That is why in the case of G an infinite sector we will work with (3.1.25).*

Theorem 3.1.7. *Under the condition (3.1.20), the problem (3.1.17) is uniformly well-posed in the sense that its unique solution w obtained in Theorem 3.1.3 is such that*

$$\|w\|_{1,\Omega,1+|p|} \leq C\|g\|_{0,\Omega} \tag{3.1.29}$$

where $C > 0$ represents here and after in the thesis various constants that depend neither on p nor on other parameters such as the space step size $h = \Delta x$ and the time step size $k = \Delta t$ in the numerical part of the work.

Proof. We know from (3.1.15), (3.1.16) and (3.1.17) where v is replaced by the solution w that

$$\int_\Omega \left(|\nabla w|^2 + p|w|^2\right) dx = \int_\Omega g\bar{w}dx,$$

or

$$\int_\Omega |\nabla w|^2 dx + \xi \int_\Omega |w|^2 dx + i\eta \int_\Omega |w|^2 dx = \int_\Omega g\bar{w}dx. \tag{3.1.30}$$

Taking the real parts of each side of (3.1.30), we have in view of (3.1.20)

$$\int_\Omega \xi^2 |w|^2 dx \leq \int_\Omega |g| \; |\xi w| dx. \tag{3.1.31}$$

By Cauchy-Schwarz inequality, (3.1.31) leads to

$$\int_\Omega \xi^2 |w|^2 dx \leq \left(\int_\Omega |g|^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega \xi^2 |w|^2 dx\right)^{\frac{1}{2}},$$

which implies that

$$\int_\Omega \xi^2 |w|^2 dx \leq \int_\Omega |g|^2 dx. \tag{3.1.32}$$

Similarly, considering the imaginary parts of both sides of (3.1.30) yields

$$\int_\Omega |\eta|^2 |w|^2 dx \leq \int_\Omega |g|^2 dx. \tag{3.1.33}$$

Finally from the real part of (3.1.30) using Cauchy-Schwarz inequality, we have

$$\int_\Omega |\nabla w|^2 dx \le \left(\int_\Omega |g|^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega |w|^2 dx\right)^{\frac{1}{2}}$$

from where we have, in view of Poincaré Friedrichs inquality in Theorem 2.4.3

$$\int_\Omega \left(|\nabla w|^2 + |w|^2\right) dx \le C \int_\Omega |\nabla w|^2 dx \le C \left(\int_\Omega |g|^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega |\nabla w|^2 + |w|^2 dx\right)^{\frac{1}{2}}.$$

Thus

$$\int_\Omega \left(|\nabla w|^2 + |w|^2\right) dx \le C \int_\Omega |g|^2 dx \tag{3.1.34}$$

Adding (3.1.32), (3.1.33) and (3.1.34), we have

$$\int_\Omega |\nabla w|^2 dx + (1+|p|)^2 \int_\Omega |w|^2 dx \le 2(2+C) \int_\Omega |g|^2 dx, \tag{3.1.35}$$

in view of the identity

$$(1+|p|^2) \le (1+|p|)^2 \le 2(1+|p|^2). \tag{3.1.36}$$

Hence the theorem follows from (3.1.35). □

Remark 3.1.8. *The variational problem (3.1.17) solved in Theorem 3.1.3 is the distributional formulation of the Helmholtz problem (3.1.1)-(3.1.2) as explained below. Since the two sides of (3.1.17) are continuous on $H_0^1(\Omega)$ and $\mathcal{D}(\Omega)$ is dense in $H_0^1(\Omega)$, then the variational equation (3.1.17) is equivalent to the one obtained by replacing $v \in H_0^1(\Omega)$ with $v \in \mathcal{D}(\Omega)$. Furthermore, by the definition of the differentiation of distributions (Definition 2.3.8), (3.1.17) is equivalent to*

$$\langle -\Delta w + p\, w, \bar{v}\rangle_{\mathcal{D}'\times\mathcal{D}} = \langle g, \bar{v}\rangle \quad \textit{for all} \ \ v \in \mathcal{D}(\Omega). \tag{3.1.37}$$

Thus w is the solution of the distributional partial differential equation,

$$w \in H_0^1(\Omega), \ -\Delta w + p\, w = g \ \ \textit{in} \ \ \mathcal{D}'(\Omega). \tag{3.1.38}$$

Remembering that $H_0^1(\Omega) = \{w \in H^1(\Omega), \gamma w = 0\}$ where γ is the trace operator and that

$g \in L^2(\Omega)$ with $L^2(\Omega)$ contained in $L^1_{loc}(\Omega)$, which is continuously embedded in $\mathcal{D}'(\Omega)$, we deduce from (3.1.38) that $w \in H^1_0(\Omega)$ is the solution of the problem

$$-\Delta w + p\, w = g \;\; a.e \; in \;\; \Omega, \quad \gamma w = 0.$$

Remark 3.1.9. *We consider the Helmholtz problem (3.1.1)-(3.1.2) when the condition (3.1.20) is not satisfied. Consider the linear operator $-\Delta$ acting from the subspace $E = \{v \in H^1_0(\Omega); -\Delta v \in L^2(\Omega)\}$ equipped with the topology of $L^2(\Omega)$ into $L^2(\Omega)$:*

$$-\Delta : E \subset L^2(\Omega) \to L^2(\Omega).$$

By Green formula, the operator $-\Delta$ is self-adjoint and positive. Furthermore, Theorem 3.1.3 and Rellich-Kondrachov Theorem 2.4.5 guarantee that the operator $-\Delta$ has a bounded compact inverse operator

$$(-\Delta)^{-1} : L^2(\Omega) \to E \hookrightarrow_c L^2(\Omega).$$

Consequently, Fredholm theory [67] guarantee that there exists a sequence (λ_j) of positive eigenvalues of $(-\Delta)^{-1}$ with associated eigenvectors w_j in $H^1_0(\Omega)$ such that $\lambda_j \to +\infty$ as $j \to +\infty$. Transposed to the operator $-\Delta$, we have $-\Delta w_j + \xi_j w_j = 0$ where $\xi_j = \frac{-1}{\lambda_j}$. Now if in (3.1.1) $p \neq \xi_j < 0$ for every j, then Fredholm theory guarantees that the Helmholtz equation (3.1.1)-(3.1.2) has a unique solution in $E \subset H^1_0(\Omega)$. However if $p = \xi_j < 0$ for some j, then Fredholm theory states that (3.1.1)-(3.1.2) has a solution (not unique) if and only if the right-hand side g is orthogonal in $L^2(\Omega)$ to any solution $z \in H^1_0(\Omega)$ of the homogeneous equation

$$-\Delta z + \xi_j z = 0.$$

Notice that for the Helmholtz problem considered on unbounded domains, the unique solutions can be achieved by imposing the so called Sommerfeld's radiation condition at infinity (see [20]).

3.2 Regularity of the solution in a smooth domain

After the study of the variational solution of the Helmholtz problem in section 3.1, we study in this section, the regularity of the solution of the said problem. We begin the section with the definition of the regularity of the solution.

Definition 3.2.1. *Let w be the variational solution of (3.1.17) given by Theorem 3.1.3. Then the solution w is said to be regular, if $w \in H^2(\Omega)$ with*

$$\|w\|_{2,\Omega} \leq K\|g\|_{0,\Omega}, \tag{3.2.1}$$

for some constant $K > 0$ which depends on p and is independent of w. In other words, the linear operator $g \rightsquigarrow w$ is bounded from $L^2(\Omega)$ into $H^2(\Omega)$.The solution is uniformly regular if K does not depend on p.

Theorem 3.2.2. *We assume that the domain Ω has a boundary Γ of class C^2. Then the variational solution w of (3.1.17) is uniformly regular. More precisely, there exists a constant $C > 0$ independent of p such that*

$$\|w\|_{2,\ \Omega,\sqrt{1+|p|}} \leq C\|g\|_{0,\Omega}$$

The proof of Theorem 3.2.2 is presented in several auxiliary results stated below. Our presentation is based on [12].

Lemma 3.2.3. *We assume that $\Omega = \mathbb{R}^2$, $\;g \in L^2(\mathbb{R}^2)$ and $p \in \mathbb{C}$ with condition (3.1.20) satisfied.*
Then any variational solution, $w \in H^1(\mathbb{R}^2)$ of the problem

$$-\Delta w + p\, w = g \;\text{ in }\; \mathbb{R}^2 \tag{3.2.2}$$

is such that $w \in H^2(\mathbb{R}^2)$ and

$$\|w\|_{2,\mathbb{R}^2,\sqrt{|p|}} \leq 3\|g\|^2_{0,\mathbb{R}^2}. \tag{3.2.3}$$

Proof. First of all the variational solution $w \in H^1(\mathbb{R}^2)$ of the Helmholtz problem (3.2.2) satisfies the equation

$$\int_{\mathbb{R}^2} (\nabla w \nabla \overline{v} + p\, w\overline{v})dx = \int_{\mathbb{R}^2} g\, \overline{v}\, dx \quad \forall \;\; v \in H^1(\mathbb{R}^2). \tag{3.2.4}$$

Take $v = w$ in (3.2.4) to obtain

$$\int_{\mathbb{R}^2} \left(|\nabla w|^2 + p|w|^2\right) dx = \int_{\mathbb{R}^2} g\bar{w}dx$$

Taking separately the real and imaginal parts in this relation, we have for $p \neq 0$

$$\begin{aligned}
\int_{\mathbb{R}^2} \left(|\nabla w|^2 + |p||w|^2\right) dx &\leq \int_{\mathbb{R}^2} |g||w|dx \\
&\leq (\int_{\mathbb{R}^2} |g|^2 dx)^{\frac{1}{2}} (\int_{\mathbb{R}^2} |w|^2 dx)^{\frac{1}{2}} \text{ by Cauchy-Schwartz inequality} \\
&\leq \frac{1}{\sqrt{|p|}} (\int_{\mathbb{R}^2} |g|^2 dx)^{\frac{1}{2}} (\int_{\mathbb{R}^2} (|\nabla w|^2 + |p||w|^2 dx))^{\frac{1}{2}}
\end{aligned}$$

which implies that

$$\left(\int_{\mathbb{R}^2} \left(|\nabla w|^2 + |p||w|^2\right) dx\right)^{\frac{1}{2}} \leq \frac{1}{\sqrt{|p|}} (\int_{\mathbb{R}^2} |g|^2 dx)^{\frac{1}{2}}.$$

Thus

$$\left(\int_{\mathbb{R}^2} \left(|p||\nabla w|^2 + |p|^2|w|^2\right) dx\right)^{\frac{1}{2}} \leq (\int_{\mathbb{R}^2} |g|^2 dx)^{\frac{1}{2}}. \tag{3.2.5}$$

We next use the technique of the difference quotient or the translation method due to Agmon, Douglis and Nirenberg [2]. Given a real-valued function v defined almost every where on $\mathbb{R}^2$ and given a vector $h \neq 0$ in $\mathbb{R}^2$, the difference quotient of v by h is denoted and defined by

$$(D_h v)(x) = \frac{(\tau_h v)(x) - v(x)}{|h|},$$

where $(\tau_h v)(x) = v(x + h)$ is the translation of v in the direction of h. Fix $h \neq 0$ in $\mathbb{R}^2$. Replacing v by $D_{-h}(D_h w)$ in (3.2.4) we have

$$\int_{\mathbb{R}^2} \left[\nabla w \nabla D_{-h}(D_h \overline{w}) + p\ w\ D_{-h}(D_h \overline{w})\right] dx = \int_{\mathbb{R}^2} g D_{-h}(D_h \overline{w}) dx. \tag{3.2.6}$$

In view of the property

$$\int_{\mathbb{R}^2} v D_{-h} \bar{S} dx = \int_{\mathbb{R}^2} (D_h v) \bar{S} dx, \quad \text{for} s \in H^1(\mathbb{R}^2)$$

we have from (3.2.6) that

$$\int_{\mathbb{R}^2} [|\nabla D_h w|^2 + p|D_h w|^2]dx = \int_{\mathbb{R}^2} g D_{-h}(D_h \bar{w})dx.$$

Taking separately the real and imaginary parts in this identity, we obtain

$$\int_{\mathbb{R}^2} \left[|\nabla D_h w|^2 + p|D_h w|^2\right] dx \le 2|\int_{\mathbb{R}^2} g D_{-h}(D_h \overline{w})dx|, \tag{3.2.7}$$

in view of the relation

$$(1/2)(|\xi| + |\eta|) \le |p| \le |\xi| + |\eta|. \tag{3.2.8}$$

Application of Cauchy-Schwarz inequality in (3.2.7) yields

$$\begin{aligned} \int_{\mathbb{R}^2} \left[|\nabla D_h w|^2 + |p||D_h w|^2\right] dx &\le 2\left(\int_{\mathbb{R}^2} |g|^2 dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} |D_{-h}(D_h \overline{w})|^2 dx\right)^{\frac{1}{2}} \\ &= 2\|g\|_{0,\mathbb{R}^2} \|D_{-h}(D_h w)\|_{0,\mathbb{R}^2}. \end{aligned} \tag{3.2.9}$$

At this stage, we use the following well-known property of $H^1(\mathbb{R}^2)$:

$$\|D_{-h} v\|_{0,\mathbb{R}^2} \le \|\nabla v\|_{0,\mathbb{R}^2}, \ \ \forall \ \ v \in H^1(\mathbb{R}^2). \tag{3.2.10}$$

Moreover a function $v \in L^2(\mathbb{R}^2)$ is of class $H^1(\mathbb{R}^2)$ if and only if there exists a constant $C > 0$ such that

$$\|D_h v\|_{0,\mathbb{R}^2} \le C, \ \ \forall \ \ 0 \neq h \in \mathbb{R}^2. \tag{3.2.11}$$

In this case we have

$$\|\nabla v\|_{0,\mathbb{R}^2} \le C. \tag{3.2.12}$$

Taking $v := D_h w \in H^1(\mathbb{R}^2)$ in (3.2.10), the relation (3.2.9) yields

$$\int_{\mathbb{R}^2} \left[|\nabla D_h w|^2 + |p||D_h w|^2\right] dx \ \ \le \ \ 2\|g\|_{0,\mathbb{R}^2} \|\nabla D_h w\|_{0,\mathbb{R}^2}. \tag{3.2.13}$$

Thus

$$\|\nabla D_h w\|_{0,\mathbb{R}^2} \leq 2\|g\|_{0,\mathbb{R}^2} \quad \text{or} \quad \|D_h \frac{\partial w}{\partial x_j}\|_{0,\mathbb{R}^2} \leq 2\|g\|_{0,\mathbb{R}^2} \ \text{ for } j = 1, 2. \tag{3.2.14}$$

In view of (3.2.11) and (3.2.12), we have from (3.2.14) that $\frac{\partial w}{\partial x_j} \in H^1(\mathbb{R}^2)$ $\forall j$, with

$$\left\|\nabla \frac{\partial w}{\partial x_j}\right\|_{0,\mathbb{R}^2} \leq 2\|g\|_{0,\mathbb{R}^2} \ \ \forall j.$$

Therefore $\frac{\partial^2 w}{\partial x_i \partial x_j} \in L^2(\mathbb{R}^2)$ for $1 \leq i, j \leq 2$ and thus $w \in H^2(\mathbb{R}^2)$ such that

$$\left(\sum_{|\alpha|=2} \|D^\alpha w\|^2_{0,\mathbb{R}^2}\right)^{1/2} \leq 2\|g\|_{0,\mathbb{R}^2}. \tag{3.2.15}$$

Combining (3.2.5) with (3.2.15), we obtain (3.2.3). $\square$

Lemma 3.2.4. *Let $g \in L^2(\mathbb{R}^2_+)$ and $p \in \mathbb{C}$ such that condition (3.1.20) is satisfied. Then any variational solution $w \in H^1_0(\mathbb{R}^2_+)$ of the problem*

$$-\Delta w + pw = g \ \ in \ \ \mathbb{R}^2_+ \tag{3.2.16}$$

is such that $w \in H^2(\mathbb{R}^2_+)$ and

$$\|w\|_{2,\mathbb{R}^2_+,\sqrt{|p|}} \leq 6\|g\|_{0,\mathbb{R}^2_+} \tag{3.2.17}$$

Proof. The method as presented in the proof of Lemma 3.2.3 is still valid, but this time only in the tangential direction. In other words, we choose $0 \neq h \in \mathbb{R} \times \{0\}$, which means that h is parallel to the boundary $\partial\mathbb{R}^2_+$. We proceed by considering $w \in H^1_0(\mathbb{R}^2_+)$, the variational solution of (3.2.16). Thus

$$\int_{\mathbb{R}^2_+} (\nabla w \nabla \overline{v} + p\, w\overline{v})dx = \int_{\mathbb{R}^2_+} g, \ \overline{v}\ dx \ \forall v \in H^1_0(\mathbb{R}^2_+). \tag{3.2.18}$$

Arguing as in the proof of Lemma 3.2.3, we obtain the analogue of the inequality (3.2.9), which is

$$\int_{\mathbb{R}^2_+} \left[|\nabla D_h w|^2 + |p||D_h w|^2 \right] dx \leq 2\|g\|_{0,\mathbb{R}^2_+} \|D_{-h}(D_h w)\|_{0,\mathbb{R}^2_+}. \tag{3.2.19}$$

Since $w \in H_0^1(\mathbb{R}^2_+)$, its extension $\widetilde{w}$ by zero outside $\mathbb{R}^2_+$ is such that $\widetilde{w} \in H^1(\mathbb{R}^2)$. Moreover, we have

$$D_h \widetilde{w} = \widetilde{D_h w} \;\text{ and }\; \nabla \widetilde{w} = \widetilde{\nabla w}.$$

This then leads to

$$\begin{aligned} \|D_{-h}(D_h w)\|_{0,\mathbb{R}^2_+} &= \|D_{-h}(D_h \widetilde{w})\|_{0,\mathbb{R}^2} \\ &\leq \|\nabla D_h \widetilde{w}\|_{0,\mathbb{R}^2} \;\text{ by }\; (3.2.10) since \;\; D_h \widetilde{w} \in H^1(\mathbb{R}^2) \\ &= \|\nabla D_h w\|_{0,\mathbb{R}^2_+}. \end{aligned}$$

Using (3.2.19), we have

$$\int_{\mathbb{R}^2_+} \left[|\nabla D_h w|^2 + |p||D_h w|^2 \right] dx \leq 2\|g\|_{0,\mathbb{R}^2_+} \|\nabla D_h w\|_{0,\mathbb{R}^2_+}.$$

from where we in turn have

$$\left(\int_{\mathbb{R}^2_+} \left[|\nabla D_h w|^2 + |p||D_h w|^2 \right] dx \right)^{\frac{1}{2}} \leq 2\|g\|_{0,\mathbb{R}^2_+}, \tag{3.2.20}$$

and thus

$$\left(\int_{\mathbb{R}^2_+} |\frac{\partial}{\partial x_j} D_h w|^2 dx \right)^{\frac{1}{2}} \leq 2\|g\|_{0,\mathbb{R}^2_+} \;\; \forall \;\; 1 \leq j \leq 2.$$

Letting h tend to zero, we obtain

$$\left(\int_{\mathbb{R}^2_+} \left| \frac{\partial^2 w}{\partial x_j \partial x_1} \right|^2 dx \right)^{\frac{1}{2}} \leq 2\|g\|_{0,\mathbb{R}^2_+}, \;\; \text{for} \;\; 1 \leq j \leq 2. \tag{3.2.21}$$

In order to show that $\frac{\partial^2 w}{\partial x_2^2} \in L^2(\mathbb{R}^2_+)$ we go back to (3.2.16), which yields

$$-\frac{\partial^2 w}{\partial x_1^2} - \frac{\partial^2 w}{\partial x_2^2} + pw = g \text{ in } \mathbb{R}^2_+.$$

We then have by (3.2.21), the triangular inequality and by considering the variational formulation of (3.2.16) with $w \in H_0^1(\mathbb{R}^2_+)$ as test function

$$\begin{aligned} \|\frac{\partial^2 w}{\partial x_2^2}\|_{0,\mathbb{R}^2_+} &\leq \|g\|_{0,\mathbb{R}^2_+} + |p| \|w\|_{0,\mathbb{R}^2_+} + \|\frac{\partial^2 w}{\partial x_1^2}\|_{0,\mathbb{R}^2_+}; && (3.2.22) \\ &\leq 5\|g\|_{0,\mathbb{R}^2_+}. && (3.2.23) \end{aligned}$$

Combining (3.2.22) and (3.2.21) with the analogue of (3.2.5) for $\mathbb{R}^2_+$, which is valid by the same arguments we obtain Lemma 3.2.4. □

To come back to the set $\bar{\Omega}$ itself, we make use of its open covering $\{V_j\}_{j=0}^k$ constructed in chapter 2 (section 2.1) as well as of the C^∞-partition of unity $\{\theta_j\}_{j=0}^k$ given in formula (2.1.5). According to this formula, the solution $w \in H_0^1(\Omega)$ of (3.1.17) can be represented as

$$w = \sum_{j=0}^{k} \theta_j w \equiv \sum_{j=0}^{k} w_j. \tag{3.2.24}$$

We deal with the cases $j = 0$ and $1 \leq j \leq k$ differently in the next two results.

Lemma 3.2.5. *The variational solution $w \in H_0^1(\Omega)$ of the problem (3.1.1)-(3.1.2) is regular in the interior of Ω in the more precise sense that $\theta_0 w \in H^2(\Omega)$ and*

$$\|\theta_0 w\|_{2,\Omega,\sqrt{|p|}} \leq K\|g\|_{0,\Omega}, \tag{3.2.25}$$

where $K > 0$ is independent of p.

Proof. The function $\theta_0 w \in H_0^1(\Omega)$ because $\theta_0 \in \mathcal{D}(V_0)$ where $\bar{V}_0 \subset \Omega$. Thus $\widetilde{\theta_0 w} \in H^1(\mathbb{R}^2)$ such that

$$\begin{aligned} -\Delta(\widetilde{\theta_0 w}) + p\widetilde{\theta_0 w} &= \widetilde{\theta_0 g} - 2\nabla\widetilde{\theta}_0 \nabla\widetilde{w} - (\Delta\widetilde{\theta}_0)\widetilde{w} \\ &=: g_0 \in L^2(\mathbb{R}^2). \end{aligned}$$

By Lemma 3.2.3, we have

$$\|\widetilde{\theta_0 w}\|_{2,\mathbb{R}^2,\sqrt{|p|}} \leq 3\|g_0\|_{0,\mathbb{R}^2}.$$

Thus

$$\|\theta_0 w\|_{2,\Omega,\sqrt{|p|}} \leq K(\|w\|_{1,\Omega} + \|g\|_{0,\Omega})$$

and

$$\|\theta_0 w\|_{2,\Omega,\sqrt{|p|}} \leq K\|g\|_{0,\Omega} \tag{3.2.26}$$

since $\|w\|_{1,\Omega} \leq K\|g\|_{0,\Omega}$ by Theorem 3.1.7 with K depending on p. □

Regarding the case when $1 \leq j \leq k$ in (3.2.24), we have the following result:

Lemma 3.2.6. *The variational solution $w \in H_0^1(\Omega)$ of (3.1.1)-(3.1.2) is regular near the boundary of Ω in the sense that $\theta_j w \in H^2(V_j^+)$, $\quad V_j^+ = V_j \cap \Omega$, and*

$$\|\theta_j w\|_{2,\Omega,\sqrt{|p|}} \leq K\|g\|_{0,\Omega},$$

where $K > 0$ is independent of p.

Proof. For a fixed $1 \leq j \leq k$, we have

$$-\Delta(\theta_j w) + p\theta_j w = \theta_j g - 2\nabla\theta_j \nabla w - (\Delta\theta_j)w := g_j \ \in L^2(V_j^+). \tag{3.2.27}$$

For simplicity, we use the notation $w_j = \theta_j w \in H_0^1(V_j^+)$. From (2.1.3), we use the C^2-diffeomorphism T_j that transforms $x \in V_j^+$ into $y = T_j(x) \in Q_+$ and we set

$$v_j(y) = w_j \circ T_j^{-1}(y) \in H_0^1(Q_+)$$

where T_j^{-1} is defined in (2.1.4). In short the idea of the rest of the proof is as follows: The equation (3.2.27) is transformed to the analogue in Q_+ of the form

$$L_j v_j + p\, v_j = f_j \ \in L^2(Q_+) \tag{3.2.28}$$

where L_j is a strongly elliptic operator of order 2. We then apply the analogue of Lemma 3.2.4 to problem (3.2.28) to obtain an estimate similar to (3.2.17). We come back to the

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desired estimate on V_j^+ by using the transformation

$$T_j : V_j^+ \longrightarrow Q_+.$$

The details are provided below. By the chain rule, we have

$$\begin{aligned}\frac{\partial v_j(y)}{\partial y_1} &= \sum_{k=1}^{2} \frac{\partial w_j(T_j^{-1}(y))}{\partial x_k}\frac{\partial x_k}{\partial y_1} = \frac{\partial w_j}{\partial x_1}\frac{\partial x_1}{\partial y_1} + \frac{\partial w_j}{\partial x_2}\frac{\partial x_2}{\partial y_1} \\ &= \alpha\frac{\partial w_j}{\partial x_1} + \alpha\varphi'(x_1)\frac{\partial w_j}{\partial x_2}\end{aligned}$$

because

$$T_j^{-1}(y) = (\alpha y_1,\ \varphi(\alpha y_1) - \beta y_2) \equiv (x_1,\ x_2) \text{ and } \frac{\partial x_1}{\partial y_1} = \alpha \text{ while } \frac{\partial x_2}{\partial y_1} = \alpha\varphi'(x_1).$$

Similarly

$$\frac{\partial v_j(y)}{\partial y_2} = \frac{\partial w_j}{\partial x_1}\frac{\partial x_1}{\partial y_2} + \frac{\partial w_j}{\partial x_2}\frac{\partial x_2}{\partial y_2} = -\beta\frac{\partial w_j}{\partial x_2}$$

since

$$\frac{\partial x_1}{\partial y_2} = 0 \text{ and } \frac{\partial x_2}{\partial y_2} = -\beta.$$

In the variational formulation of (3.2.27), the contribution of $-\Delta w_j$ is the following integral, which is transformed on Q_+ by change of variable: For ψ a test function, we have

$$\int_{V_j^+} \nabla w_j \nabla\psi dx = \int_{Q_+} \left[\begin{pmatrix} \frac{1}{\alpha} & \varphi'(x_1) \\ 0 & \frac{-1}{\beta}\end{pmatrix}\begin{pmatrix}\frac{\partial v_j}{\partial y_1} \\ \frac{\partial v_j}{\partial y_2}\end{pmatrix}\begin{pmatrix} \frac{1}{\alpha} & \varphi'(x_1) \\ 0 & \frac{-1}{\beta}\end{pmatrix}\begin{pmatrix}\frac{\partial \psi}{\partial y_1} \\ \frac{\partial \psi}{\partial y_2}\end{pmatrix}\right]\alpha\beta dy \qquad (3.2.29)$$

Evaluating equation (3.2.29) leads to the following relation

$$\begin{aligned}&\int_{V_j^+} \nabla w_j \nabla\psi dx \\ &= \int_{Q_+}\left[\frac{1}{\alpha^2}\frac{\partial v_j}{\partial y_1}\frac{\partial \psi}{\partial y_1} + \frac{1}{\alpha\beta}\frac{\partial v_j}{\partial y_1}\frac{\partial \psi}{\partial y_2}\varphi'(x_1) + \frac{1}{\alpha\beta}\varphi'(x_1)\frac{\partial v_j}{\partial y_2}\frac{\partial \psi}{\partial y_1} + (\frac{1}{\beta^2}\varphi'(x_1)^2+1)\frac{\partial v_j}{\partial y_2}\frac{\partial \psi}{\partial y_2}\right]\alpha\beta dy.\end{aligned}$$

By Green formula the operator L_j is explicitly given by the following relation:

$$L_j := -\frac{\partial^2}{\partial y_1^2} - \frac{\partial}{\partial y_2}\left(\varphi'(x_1)\frac{\partial}{\partial y_1}\right) - \frac{\partial}{\partial y_1}\left(\varphi'(x_1)\frac{\partial}{\partial y_2}\right) - \frac{\partial}{\partial y_2}\left(1 + (\varphi'(x_1))^2\frac{\partial}{\partial y_2}\right).$$

Lemma 3.2.7. *The operator*

$$L_j := -\frac{\partial^2}{\partial y_1^2} - \frac{\partial}{\partial y_2}\left(\varphi'(x_1)\frac{\partial}{\partial y_1}\right) - \frac{\partial}{\partial y_1}\left(\varphi'(x_1)\frac{\partial}{\partial y_2}\right) - \frac{\partial}{\partial y_2}\left(1 + (\varphi'(x_1))^2\frac{\partial}{\partial y_2}\right)$$

is strongly uniformly elliptic in Q_+. That is, there exists a real number $\alpha > 0$ and a complex number γ such that

$$Re\left[-\gamma\left(\xi_1^2 + 2\xi_1\xi_2\varphi'(x_1) + (1 + (\varphi'(x_1)))^2\xi_2^2\right)\right] \geq \alpha|\xi|^2, \ \ \forall \xi \in \mathbb{R}^2, \ \ y \in Q_+.$$

Proof. We take $\gamma = -1$ and $0 < \alpha < 1/2$. Then we have consecutively

$$\begin{aligned} Re\left[-\gamma\left(\xi_1^2 + 2\xi_1\xi_2\varphi'(x_1) + (1 + (\varphi'(x_1)))^2\xi_2^2\right)\right] - \alpha|\xi|^2 &= \xi_1^2(1-\alpha) + \xi_2^2(1 - \alpha + (\varphi'(x_1))^2) \\ &+ 2\rho'(x_1)\xi_1\xi_2 \\ &\geq 1/2\xi_1^2 + 1/2(\varphi'(x_1))^2\xi_2^2 + \varphi'(x_1)\xi_1\xi_2 \\ &= (\frac{\sqrt{2}}{2}\xi_1 + \frac{\sqrt{2}}{2}\varphi'(x_1)\xi_2)^2 \\ &\geq 0. \end{aligned}$$

Hence the proof of the Lemma. □

Applying the analogue of the Lemma 3.2.4 to (3.2.29) we obtain

$$\left(\sum_{|\alpha|=2} \|D^\alpha v_j\|_{0,Q_+}^2\right)^{\frac{1}{2}} \leq K_j\|f_j\|_{0,Q_+} \qquad (3.2.30)$$

which is the analogue of (3.2.17) in Q_+. Making the change of variables $y = T_j(x)$ and $\theta_j w = v_j \circ T_j, \ \ g_j = f_j \circ T_j$ in (3.2.30) we obtain

$$\left(\sum_{|\alpha|=2} \|D^\alpha \theta_j w\|_{0,V_j^+}^2\right)^{\frac{1}{2}} \leq K_j\|g\|_{0,V_j^+} \qquad (3.2.31)$$

together with

$$\|\theta_j w\|^2_{1,V_j^+} \leq K_j \|g\|_{0,V_j^+} \text{ (By Theorem 3.1.3)} \tag{3.2.32}$$

Adding (3.2.31) and (3.2.32) proves Lemma 3.2.6. □

Remark 3.2.8. *The underlying point in the proofs of Lemma 3.2.6 and 3.2.7 is that the ellipticity property is preserved by translation.*

Proof. of Theorem 3.2.2
We prove Theorem 3.2.2 by adding (3.2.26), (3.2.31) and (3.2.32) with (3.1.24) through $j = 0$ to $j = k$. □

Remark 3.2.9. *The inequality in the Theorem 3.2.2 is the particular case of some more general inequalities established in Agronovitch and Vishik [3].*

Chapter 4

The Helmholtz problem in a non-smooth domain

In the preceding chapter we study the regularity of the solution of Helmholtz problem in a smooth domain. In this chapter we study the same problem in the non-smooth domain specifically the polygonal domain. We begin the chapter with section 4.1 where we study the regularity of the solution of the Helmholtz problem far away from the corner. In section 4.2 and 4.3, we study the regularity of the solution of the problem at the corner for $p = 0$ and for $p \neq 0$ respectively. Finally, we show in section 4.4, that the solution of the Helmholtz problem attains its global regularity in a weighted Sobolev space $H^{2,\beta}(\Omega)$ to be defined.

4.1 Regularity far away from corners and reduction to a sector

The results of section 3.2 show that the solution of the Helmholtz problem is regular far away from the vertices (corners) of the polygonal domain. More precisely, we have the following result:

Theorem 4.1.1. *Let E be an open subset of the polygonal domain Ω such that the distance from E to the vertices of Γ is strictly positive. Then, the variational solution of the Helmholtz problem*

$$w \in H_0^1(\Omega), \quad -\Delta w + p\, w = g, \tag{4.1.1}$$

corresponding to $g \in L^2(\Omega)$, $Re(p) \geq 0$ *is such that*

$$w \in H^2(E).$$

Proof. We proceed by partition of unity as in section 3.2, observing that either $\bar{E} \cap \Gamma = \phi$ or $\bar{E} \cap \Gamma \neq \phi$. The first case corresponds to the interior regularity stated in Lemmas 3.2.3 and 3.2.5. The second case include the situation where the arc-length of $\bar{E} \cap \Gamma$ is positive, in which case $\bar{E} \cap \Gamma$ is locally represented as the graph of C^∞ functions. This then corresponds to the regularity near the boundary stated in Lemmas 3.2.4 and 3.2.6. □

In view of Theorem 4.1.1, the singular behavior of the solution of (4.1.1) is a local problem which is related to each corner. Thus we focus on one corner of Ω and assume for convenience that this corner is at the origin of $\mathbb{R}^2$. In the neighborhood of this corner, we assume that Ω coincides with the sector G defined by

$$G = \{(r\cos\theta, r\sin\theta); r > 0, 0 < \theta < \omega\}, \tag{4.1.2}$$

in the usual polar co-ordinate (r, θ) where ω is the size of the interior angle at the corner. It is further assumed that this is the only non-convex corner i.e $\omega > \pi$ of Ω as seen in Figure 4.1.

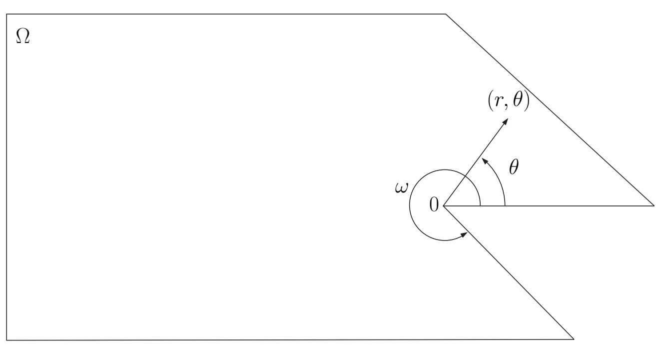

Figure 4.1: Model Polygonal domain

To be more specific on the local nature of the problem, we consider once and for all a cut-off function $\psi \equiv \psi(r) \in \mathcal{D}(\mathbb{R}^2)$ such that

$$\psi(r) = \begin{cases} 1 & \text{for} \quad 0 \leq r \leq \frac{r_0}{2} \\ \\ 0 & \text{for} \quad r \geq r_0, \end{cases} \tag{4.1.3}$$

where the number $r_0 > 0$ is so small that no other corner point of Ω lies in the disk $|x| < r_0$.

With $\widetilde{w} \in H_0^1(\mathbb{R}^2)$ being the extension of w by zero outside Ω, the solution of the local problem we will deal with is $\widetilde{w}\psi$. The right hand side is $\psi\widetilde{g} - \widetilde{w}\Delta\psi - 2\nabla\psi\nabla\widetilde{w}$. For simplicity, we write $\widetilde{w}\psi$ as w. Equally $\psi\widetilde{g} - \widetilde{w}\Delta\psi - 2\nabla\psi\nabla\widetilde{w}$ will be written as g. In summary, the local problem we deal with reads as follows: $w \in H_0^1(G)$ is solution of

$$-\Delta w + p\, w = g \in L^2(G) \tag{4.1.4}$$

where the involved functions have bounded supports in the following specific way:

$$w(r, \theta) = 0 \quad \text{for} \;\; r \geq r_0, \tag{4.1.5}$$

$$g(r, \theta) = 0 \quad \text{for} \;\; r \geq r_0. \tag{4.1.6}$$

Remark 4.1.2. *When there is no risk of confusion, a real-valued function v on the sector G will be written indistinctly by $v(x)$, $v(x_1, x_2)$, $v(r\sin\theta, r\cos\theta)$ or $v(r, \theta)$.*

By Hardy inequality [29], it follows that the local solution $w \in H_0^1(G)$ satisfies the inclusion

$$r^{|\alpha|-1} D^\alpha w \in L^2(G) \;\; \text{for all} \;\; |\alpha| \leq 1. \tag{4.1.7}$$

This leads us to consider the so-called weighted Sobolev spaces introduced first by Kondratiev [36].

Definition 4.1.3. ([*29*], [*36*])
We denote by $P_2^k(G)$ *the space of all distributions* v *0n* G *such that,*

$$r^{|\alpha|-k} D^\alpha v \in L^2(G) \ \textit{for all} \ |\alpha| \le k,$$

where k *is a non-negative integer. We equip* $P_2^k(G)$ *with the natural norm defined by*

$$\|v\|^2_{P_2^k(G)} := \sum_{|\alpha|\le k} \|r^{|\alpha|-k} D^\alpha v\|^2_{0,G}. \tag{4.1.8}$$

By using the chain rule and the change of variables in integrals via the Euler transformation

$$r = e^t, \tag{4.1.9}$$

the weighted Sobolev space on the sector G is linked to the usual Sobolev space on the strip $B = \mathbb{R} \times (0, \omega)$ as specified in the next Lemma.

Lemma 4.1.4. ([*29*])
Assume that $u \in P_2^k(G)$ *with* k *a positive integer and define* v *by,*

$$v(t, \theta) = u(e^t \cos\theta, e^t \sin\theta) e^{(-k+1)t}. \tag{4.1.10}$$

Then, $v(t, \theta) \in H^k(B)$.

4.2 Regularity and singularities when $p = 0$

We consider (4.1.4) in the particular case when $p = 0$. We are then dealing with the Dirichlet problem for the Laplace operator:

$$w \in H_0^1(G), -\Delta w = g \ \in L^2(G), \tag{4.2.1}$$

where w and g satisfy (4.1.5)-(4.1.6).

Theorem 4.2.1. *For the solution* $w \in H_0^1(G)$ *of the problem (4.2.1), we have the following singular decomposition : there exists a scalar* A *such that*

$$w_R := w - A r^{\frac{\pi}{\omega}} \sin \frac{\pi}{\omega}\theta \ \ \in P_2^2(G) \cap H_0^1(G),$$

$$w_R^1 := w - A\psi(r)r^{\frac{\pi}{\omega}} \sin\frac{\pi}{\omega}\theta \quad \in H^2(G) \cap H_0^1(G),$$

and

$$\|w_R\|_{P_2^2(G)} + \|w_R^1\|_{2,G} + |A| \leq C\|g\|_{0,G}, \tag{4.2.2}$$

where $\psi \equiv \psi(r)$ is the cut-off function in (4.1.3), w_R or w_R^1 is the regular part, $r^{\frac{\pi}{\omega}} \sin\frac{\pi}{\omega}\theta$ or $\psi(r)r^{\frac{\pi}{\omega}} \sin\frac{\pi}{\omega}\theta$ is the singular function and A is the coefficient of the singular function.

The method used in proving Theorem 4.2.1 was developed by Kondratiev [36] and it demands a lot of theoretical knowledge. We shall essentially quote the important steps. For more details see for instance [29]. In polar co-ordinate, equation (4.2.1) takes the form

$$-\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right) = g(r,\theta) \ \text{ in } \ G. \tag{4.2.3}$$

Now, we use the Euler transformation (4.1.9) and make a change of dependent variable

$$s(t,\theta) = w(e^t,\theta) = w(r,\theta). \tag{4.2.4}$$

Since

$$\frac{\partial w}{\partial r} = e^{-t}\frac{\partial s}{\partial t} \ \text{ and } \ \frac{\partial^2 w}{\partial r^2} = e^{-2t}\frac{\partial^2 s}{\partial t^2} - e^{-2t}\frac{\partial s}{\partial t},$$

(4.2.3) becomes

$$-\left(\frac{\partial^2 s}{\partial t^2} + \frac{\partial^2 s}{\partial \theta^2}\right) = e^{2t}g(t,\theta) \ \text{ in } \ B \tag{4.2.5}$$

with boundary conditions

$$s(t,\omega) = s(t,0) = 0, \tag{4.2.6}$$

where $s \in H_0^1(B)$ and $g(e^t\cos\theta, e^t\sin\theta)e^t \in L^2(B)$ in view of Lemma 4.1.4.

Taking the Fourier transform, the problem (4.2.5)-(4.2.6) becomes the following family of ordinary differential equation that depend on the parameter λ:

$$-\frac{d^2\widehat{s}(i\lambda,\theta)}{d\theta^2} + \lambda^2\widehat{s}(i\lambda,\theta) = \widehat{e^t g}(-\lambda_2 - 1 + i\lambda_1,\theta) \equiv \widehat{e^t g}(i\lambda - 1,\theta) \quad 0 < \theta < \omega \tag{4.2.7}$$

$$\widehat{s}(i\lambda, 0) = \widehat{s}(i\lambda, \omega) = 0. \tag{4.2.8}$$

Remark 4.2.2. *For a function $h : r \to h(r)$, the composition of the Euler transformation (4.1.9) and the Fourier transform is called the Mellin transform of h see [29]. Formally we have:*

$$(Mh)(\lambda) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} r^{-i\lambda-1} h(r) dr.$$

We apply Proposition 2.5.36 (corresponding to the scalar Theorem 2.5.1) to the L^2 vector-valued functions

$$t \in (-\infty, +\infty) \rightsquigarrow \frac{\partial^\beta s(t,\theta)}{\partial^{\beta_1} t \partial^{\beta_2} \theta}, \quad |\beta| \leq 1 \text{ and } t \in (-\infty, +\infty) \rightsquigarrow e^t g(t,\theta) \in L^2(-\infty, +\infty),$$

observing that the support of all these functions are contained in $I_\alpha = (-\infty, \alpha)$ where $\alpha = \ln r_0$.

We obtain that $\widehat{s}(i\lambda, \theta)$ is holomorphic in the region $\lambda_2 > 0$ and $\widehat{e^t g}(i\lambda - 1, \theta)$ is holomorphic in the region $\lambda_2 > -1$ such that the following estimates hold:

$$\sum_{j=0}^{1} \int_{-\infty}^{+\infty} \int_0^\omega |\lambda_1 + i\lambda_2|^{2j} |\frac{\partial^{1-j}\widehat{s}}{\partial\theta^{1-j}}(i\lambda, \theta)|^2 d\lambda_1 d\theta \leq r_0^{2\lambda_2} \sum_{|\beta|\leq 1} \int_{-\infty}^{+\infty} \int_0^\omega |\frac{\partial^\beta s(t,\theta)}{\partial t^{\beta_1} \partial\theta^{\beta_2}}|^2 dt d\theta$$

$$\int_{-\infty}^{+\infty} \int_0^\omega |\widehat{e^t g}(i\lambda - 1, \theta)|^2 d\lambda_1 d\theta \leq r_0^{2(\lambda_2+1)} \int_{-\infty}^{+\infty} \int_0^\omega |e^t g(t,\theta)|^2 dt d\theta.$$

In view of the above holomorphic property of $\widehat{s}(i\lambda, \theta)$ and $\widehat{e^t g}(i\lambda + 1, \theta)$, Theorem 2.5.38 implies that the solution $\widehat{s}(i\lambda, \theta)$ of (4.2.7)-(4.2.8) admits a meromorphic extension (which we denote in the same way) to the complex strip

$$-\infty < \lambda_1 < +\infty, \quad -1 < \lambda_2 < 0.$$

We want to say a bit more about this meromorphic extension. Firstly, the considerations in Remark 3.1.9 can be made more precise in this one-dimensional case. Indeed, it is well-known that the operator $u \rightsquigarrow -u''$ with boundary conditions $u(0) = u(\omega) = 0$ has the eigenvalues

$$\lambda_k^2 = (\frac{k\pi}{\omega})^2, \quad k \in \mathbb{N}, \quad k \neq 0,$$

with, for each k, the associated eigenvector

$$v_k = \sin \frac{k\pi}{\omega}\theta.$$

Now in the extension $\widehat{s}(i\lambda, \theta)$ of the solution, if we take

$$i\lambda = \sqrt{\lambda_k} = \frac{k\pi}{\omega}, \quad i.e.\ \lambda = \frac{-ik\pi}{\omega},$$

then it is clear that the only possible pole of the meromorphic function $\widehat{s}(i\lambda, \theta)$ in the strip $-\infty < \lambda_1 < +\infty, \quad -1 < \lambda_2 < 0$ is $\lambda = \frac{k\pi}{\omega}$. We distinguish two cases: if $\omega < \pi$, there is no pole in the said strip. However, there is indeed a unique pole in the non convex case $\omega > \pi$.

Secondly, we introduce the Green function $N \equiv N(i\lambda, \theta, \gamma)$ of the operator

$$v \in C^2(0, \omega) \rightsquigarrow -\frac{d^2 v}{d\theta^2} + \lambda^2 v, \quad \lambda = \frac{-ik\pi}{\omega}, \quad -1 < \lambda_2 < 0$$

with homogeneous Dirichlet boundary conditions $v(0) = v(\omega) = 0$.
By definition [66], the Green function satisfies the following properties:

1. The function $(\theta, \gamma) \rightsquigarrow N \equiv N(i\lambda, \theta, \gamma)$ is continuous on the square $(0, \omega) \times (0, \omega)$;

2. The partial derivatives $\frac{\partial N}{\partial \theta}$, $\frac{\partial^2 N}{\partial \theta^2}$ exist and are continuous on the triangles $0 \leq \theta \leq \gamma \leq \omega$ and $0 \leq \gamma \leq \theta \leq \omega$;

3. For each fixed $\gamma \in [0, \omega]$, $\frac{d^2 N}{\partial \theta^2} + \lambda^2 N = 0$ for $0 \leq \theta \leq \omega, \quad \theta \neq \gamma$;

4. On the diagonal $\theta = \gamma$, the first derivative makes a jump such that

$$\frac{\partial N(0^+, \theta)}{\partial \theta} - \frac{\partial N(0^-, \theta)}{\partial \theta} = -1 \ \text{ for } \ 0 < \theta < \omega;$$

5. $N(i\lambda, 0, \gamma) = N(i\lambda, \omega, \gamma) = 0$ for each $\gamma \in (0, \omega)$.

Following the classical procedure (see [66]), it can be shown that the Green function is given by the formula

$$N(i\lambda, \theta, \gamma) = \frac{-1}{\omega\lambda} \begin{cases} \gamma \ \sinh \lambda(\theta - \omega) & , \ \text{if} \quad 0 \leq \gamma \leq \theta \leq \omega \\ \\ \theta \ \sinh \lambda(\theta - \omega) & , \ \text{if} \quad 0 \leq \theta \leq \gamma \leq \omega. \end{cases} \tag{4.2.9}$$

Notice that

$$N(0,\ \theta,\ \gamma) = \begin{cases} \gamma\,(\theta - 1) & , \ \text{for} \quad 0 \leq \gamma \leq \theta \leq \omega \\ \\ \theta\,(\gamma - 1) & , \ \text{for} \quad 0 \leq \theta \leq \gamma \leq \omega \end{cases}$$

which is in agreement with the Green function given in Walter [66] and Gustafson [33]. In view of the expression of $N(i\lambda, \theta, \gamma)$, the solution of (4.2.7)-(4.2.8) admits the representation

$$\widehat{s}(i\lambda, \theta) = \int_0^\omega N(i\lambda, \theta, \gamma)\widehat{e^t g}(i\lambda - 1, \gamma)d\gamma; \ \text{ when } \ \lambda \neq \frac{-ik\pi}{\omega}, \ \ \lambda_2 > -1. \tag{4.2.10}$$

The regularity of this extended solution of (4.2.7)-(4.2.8) is described in the next result.

Lemma 4.2.3. *There exist constants $C > 0$ and $K > 0$, such that*

$$\sum_{j=0}^{2} |\lambda_1|^{2-j} \|\widehat{s}(i\lambda, \theta)\|_{j,(0,\omega)} \leq C \|\widehat{e^t g}(i\lambda - 1, \cdot)\|_{0,(0,\omega)}, \ \ \textit{for} \ \ |\lambda_1| \geq K, \ \ -1 \leq \lambda_2 \leq 0.$$

Proof. For general problems, the proof of Lemma 4.2.3 is given in Grisvard [29] and Kondratiev [36]. For the case under consideration, the proof can be obtained explicitly either by using the Green function $N(i\lambda, \theta, \gamma)$ in (4.2.9) and the representation (4.2.10), which is valid or by simple arguments. We prefer the latter approach.

We assume that $\lambda_1 \geq K > 0$ for a constant to be determined shortly and we assume that $-1 \leq \lambda_2 \leq 0$. Then $\lambda \neq \frac{-ik\pi}{\omega}$. The arguments used below are similar to those that led to the proof of the inequality (3.1.29). Multiply both sides of (4.2.7) by $\lambda_1^2 \bar{\widehat{s}}(i\lambda, \theta)$ and integrate by parts to obtain the following after using (4.2.8):

$$\int_0^\omega \left[\lambda_1^2 |\frac{d\widehat{s}(i\lambda, \theta)}{d\theta}|^2 + (\lambda_1^2 - \lambda_2^2 + 2i\lambda_1\lambda_2)|\widehat{s}(i\lambda, \theta)|^2 \right] d\theta = \int_0^\omega \widehat{e^t g}(i\lambda - 1, \theta)\lambda_1^2 \bar{\widehat{s}}(i\lambda, \theta)d\theta.$$

Using the real part of this identity and Cauchy-Schwarz inequality, we obtain:

$$\int_0^\omega \left[\lambda_1^2 |\frac{d\widehat{s}(i\lambda, \theta)}{d\theta}|^2 + \lambda_1^4 (1 - \frac{\lambda_2^2}{\lambda_1^2})|\widehat{s}(i\lambda, \theta)|^2 \right] d\theta$$
$$\leq \left(\int_0^\omega |\widehat{e^t g}(i\lambda - 1, \theta)|^2 d\theta \right)^{\frac{1}{2}} \left(\int_0^\omega \lambda_1^4 |\widehat{s}(i\lambda, \theta)|^2 d\theta \right)^{\frac{1}{2}}. \tag{4.2.11}$$

Notice that $0 \leq \lambda_2^2 \leq 1$. We assume at this point in time that

$$K \geq 2, \text{ and } \frac{\lambda_2^2}{\lambda_1^2} < \frac{1}{2} \text{ so that } \frac{1}{2} < (1 - \frac{\lambda_2^2}{\lambda_1^2}).$$

Then, with $|\lambda_1| \geq K$ and so $|\lambda_1| \geq 2$, we obtain from (4.2.11)

$$\frac{1}{2}\int_0^\omega \left[\lambda_1^2 |\frac{d\widehat{s}(i\lambda,\theta)}{d\theta}|^2 + \lambda_1^4 |\widehat{s}(i\lambda,\theta)|^2\right] d\theta \leq$$
$$\left(\int_0^\omega |\widehat{e^t g}(i\lambda - 1,\theta)|^2 d\theta\right)^{\frac{1}{2}} \left(\int_0^\omega \left[\lambda_1^2 |\frac{d\widehat{s}(i\lambda,\theta)}{d\theta}|^2 + \lambda_1^4 |\widehat{s}(i\lambda,\theta)|^2\right] d\theta\right)^{\frac{1}{2}}.$$

Thus

$$\left(\int_0^\omega \left[\lambda_1^2 |\frac{d\widehat{s}(i\lambda,\theta)}{d\theta}|^2 + \lambda_1^4 |\widehat{s}(i\lambda,\theta)|^2\right] d\theta\right)^{\frac{1}{2}} \leq 2\left(\int_0^\omega |\widehat{e^t g}(i\lambda - 1,\theta)|^2 d\theta\right)^{\frac{1}{2}}. \tag{4.2.12}$$

On the other hand, from (4.2.7) we have

$$\begin{aligned}\left(\int_0^\omega |\frac{d^2\widehat{s}(i\lambda,\theta)}{d\theta^2}|^2 d\theta\right)^{\frac{1}{2}} \leq \quad & |\lambda|^2 \left(\int_0^\omega |\widehat{s}(i\lambda,\theta)|^2 d\theta\right)^{\frac{1}{2}} + \left(\int_0^\omega |\widehat{e^t g}(i\lambda - 1,\theta)|^2 d\theta\right)^{\frac{1}{2}} \\ \leq \quad & 2\lambda_1^2 \left(\int_0^\omega |\widehat{s}(i\lambda,\theta)|^2 d\theta\right)^{\frac{1}{2}} + \left(\int_0^\omega |\widehat{e^t g}(i\lambda - 1,\theta)|^2 d\theta\right)^{\frac{1}{2}}\end{aligned}$$

because

$$|\lambda|^2 = \lambda_1^2 + \lambda_2^2 \leq \lambda_1^2 + 4 \leq 2\lambda_1^2 \text{ and } |\lambda_1| \geq 2.$$

Using then (4.2.12), we have

$$\left(\int_0^\omega |\frac{d^2\widehat{s}(i\lambda,\theta)}{d\theta^2}|^2 d\theta\right)^{\frac{1}{2}} \leq 5\left(\int_0^\omega |\widehat{e^t g}(i\lambda - 1,\theta)|^2 d\theta\right)^{\frac{1}{2}} \tag{4.2.13}$$

Since $|\lambda_1| \geq 2$, it follows from (4.2.12) that

$$\left(\int_0^\omega \left[|\frac{d\widehat{s}(i\lambda,\theta)}{d\theta}|^2 + |\widehat{s}(i\lambda,\theta)|^2\right] d\theta\right)^{\frac{1}{2}} \leq 2\left(\int_0^\omega |\widehat{e^t g}(i\lambda - 1,\theta)|^2 d\theta\right)^{\frac{1}{2}}. \tag{4.2.14}$$

Taking the squares of (4.2.12), (4.2.13), (4.2.14) and adding these inequalities, we obtain the

Lemma 4.2.3 for the specific choice $K \geq 2$. □

Remark 4.2.4. *In terms of the weighted Sobolev space $H^m((0,\omega),\rho)$ introduced in Definition 3.1.4, the proof of Lemma 4.2.3 shows that*

$$\|\widehat{s}(i\lambda,\cdot)\|_{2,(0,\omega),|\lambda_1|} \leq C\|\widehat{e^t g}(i\lambda-1,\cdot)\|_{0,(0,\omega)} \;\; \textit{for} \;\; |\lambda_1| \geq 2, \; -1 \leq \lambda_2 \leq 0.$$

Once again, this inequality is as mentioned in the proof of Theorem 3.2.2, a particular case of the results of Agranovitch and Vishik [3].

Corollary 4.2.5. *There exists a sequence (N_m) of integers such that*

$$N_m \geq K, \;\; \forall m \quad \lim_{m\to+\infty}\int_{-1}^{0} |\widehat{s}(\pm iN_m - \lambda_2,\theta)|d\lambda_2 = 0$$

for almost every $0 < \theta < \omega$.

Proof. From Lemma 4.2.3, we have

$$\int_0^{\omega}\int_{-1}^{0} |\widehat{s}(i\lambda_1 - \lambda_2,\theta)|d\lambda_2 d\theta \leq \frac{C}{|\lambda_1|^2} \;\; \text{for} \;\; |\lambda_1| \geq K.$$

This implies that

$$\lim_{K\leq|N|\to+\infty}\int_0^{\omega}\int_{-1}^{0} |\widehat{s}(\pm iN - \lambda_2,\theta)|d\lambda_2 d\theta = 0.$$

By the fact that a Cauchy sequence in $L^P(0,\omega)$ admits a point-wise convergent subsequence (see Adams [1], Corollary 2.11), we can find a sequence $(N_m)_{m\geq 1}$ of integers which have the desired property. □

Proof. (Theorem 4.2.1)
At this point, we make use of the fact that the polygonal domain is non-convex, i.e $\omega > \pi$. This implies as observed earlier that no pole of the meromorphic function $\widehat{s}(i\lambda,\theta)$ or eigenvalue $\lambda = \frac{-i\pi}{\omega}$ of the problem (4.2.7)-(4.2.8) belongs to the line

$$\lambda_2 = -1.$$

Under this condition, the Plancherel-Parseval Theorem, implies that the function

$$\lambda_1 \rightsquigarrow \widehat{s}(i\lambda_1 + 1,\theta),$$

has the inverse Fourier transform

$$s_R(t,\theta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{i\lambda_1 t} \widehat{s}(i\lambda_1 + 1, \theta) d\lambda_1,$$

that belongs to the Sobolev space $H^2(B)$ such that

$$\|s_R\|_{2,B} \leq C\|e^t g\|_{0,B}. \tag{4.2.15}$$

Notice that the inverse Fourier transform of the function $\lambda_1 \rightsquigarrow \widehat{s}(i\lambda_1, \theta)$ i.e $\lambda_2 = 0$, given by

$$s(t,\theta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{i\lambda_1 t} \widehat{s}(i\lambda_1, \theta) d\lambda_1 \tag{4.2.16}$$

is of class $L^2(B)$ (in fact of class $H_0^1(B)$).

In order to link $s_R(t,\theta)$ to $s(t,\theta)$, we use the sequence (N_m) in the Corollary 4.2.5, observing that

$$\begin{aligned} s(t,\theta) &= \lim_{N_m \to \infty} \frac{1}{\sqrt{2\pi}} \int_{-N_m}^{N_m} e^{i\lambda t} \widehat{s}(i\lambda, \theta) d\lambda \\ &= \lim_{N_m \to \infty} \frac{1}{\sqrt{2\pi}} \left[\int_{-N_m+0i}^{-N_m-i} + \int_{-N_m-i}^{N_m-i} + \int_{N_m-i}^{N_m+0i} + \int_{Q_m} \right] e^{i\lambda t} \widehat{s}(i\lambda, \theta) d\lambda \end{aligned} \tag{4.2.17}$$

where Q_m is the rectangle with vertices $-N_m + 0i, -N_m - i, N_m - i$ and $N_m + 0i$ illustrated in Figure 4.2.

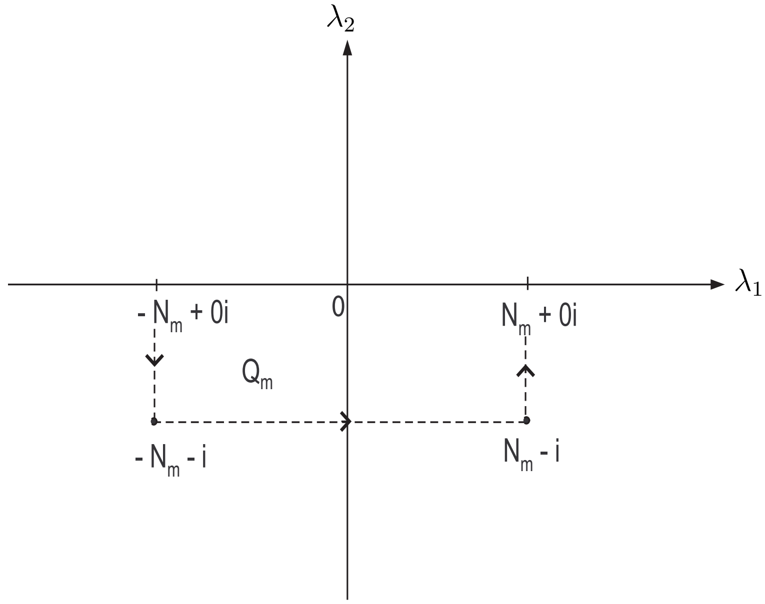

Figure 4.2: Application of the Residue theorem

By Corollary 4.2.5, we know that the limits corresponding to the first and the third integrals are zero. Recall that we are in the non-convex case for the sector G i.e. $\omega > \pi$. The only pole of $\widehat{s}(i\lambda, \theta)$ in the region Q_m being then $\frac{-i\pi}{\omega}$, the Laurent expansion of this function has the form

$$\widehat{s}(i\lambda, \theta) = \frac{P_1(\theta)}{\lambda + \frac{i\pi}{\omega}} + \alpha(\lambda, \theta), \tag{4.2.18}$$

with $\alpha(\lambda, \theta)$ being analytic. Applying to (4.2.18) the operator $u \rightsquigarrow u'' + \lambda^2 u$ with boundary conditions $u(0) = u(\omega) = 0$, it is easy to show in terms of the eigenvalues and associated eigenvectors of this operator that

$$P_1(\theta) = A_1 \sin \frac{\pi}{\omega}\theta, \text{ for some scalar } A_1. \tag{4.2.19}$$

By the Residue Theorem, the fourth integral in (4.2.17) is given by

$$\frac{1}{2\pi} \int_{Q_m} i\sqrt{2\pi}\, e^{i\lambda t} \widehat{s}(i\lambda, \theta) d\lambda = Res \left(i\sqrt{2\pi}\, e^{i\lambda t} \widehat{s}(i\lambda, \theta) \right)_{\lambda = \frac{-i\pi}{\omega}}.$$

Now considering the Taylor's expansion of $e^{i\lambda t}$ about $\lambda = \frac{-i\pi}{\omega}$ and the expression of $P_1(\theta)$ in

(4.2.19), we obtain

$$Res\left(i\sqrt{2\pi}\, e^{i\lambda t}\widehat{s}(i\lambda,\theta)\right)_{\lambda=\frac{-i\pi}{\omega}} = Ae^{\frac{\pi}{\omega}t}\sin\frac{\pi}{\omega}\theta.$$

Therefore (4.2.17) leads to

$$s(t,\theta) = s_R + A\, e^{\frac{\pi}{\omega}t}\sin\frac{\pi}{\omega}\theta. \tag{4.2.20}$$

where s_R satisfies (4.2.15).

In terms of the Euler transformation (4.1.9), the decomposition (4.2.20) becomes

$$w(r,\theta) = w_R(r,\theta) + A\, r^{\frac{\pi}{\omega}}\sin\frac{\pi}{\omega}\theta \tag{4.2.21}$$

where in view of (4.2.4), we have

$$w(r,\theta) \equiv w(e^t,\theta) = s(t,\theta) \ \text{ and } \ w_R(r,\theta) \equiv w_R(e^t,\theta) = s_R(t,\theta).$$

Furthermore, by a simple change of variables, we have (see Lemma 4.1.4) $w_R \in P_2^2(G) \cap H_0^1(G)$, with the inequality (4.2.15) becoming

$$\|w_R\|_{P_2^2(G)} \leq C\|g\|_{0,G}. \tag{4.2.22}$$

Finally, we use the cut-off function $\psi \equiv \psi(r)$ in (4.1.3) to rewrite (4.2.21) in the form

$$w(r,\theta) = w_R^1(r,\theta) + A\psi(r)r^{\frac{\pi}{\omega}}\sin\frac{\pi}{\omega}\theta \tag{4.2.23}$$

where

$$w_R^1(r,\theta) := (1-\psi(r))w(r,\theta) + \psi(r)w_R(r,\theta) \in H^2(G)\cap H_0^1(G)$$

such that

$$\|w_R^1\|_{2,G} \leq C\|g\|_{0,G} \tag{4.2.24}$$

because w is regular far away from the corner $(0,0)$ (see Theorem 4.1.1). Thus (4.2.23) and

(4.2.24) yield

$$|A|\|\psi(r)r^{\frac{\pi}{\omega}}\sin\frac{\pi}{\omega}\theta\|_{1,G} \leq \|w_R^1\|_{2,G} + \|w\|_{1,G} \leq \|g\|_{0,G}$$

from where we have

$$|A| \leq C\|g\|_{0,G}. \tag{4.2.25}$$

This completes the proof of Theorem 4.2.1. □

4.3 Regularity and singularities when $p \neq 0$

In the case $p \neq 0$, we proceed by first drawing a consequence of Theorem 4.2.1.

Corollary 4.3.1. *Let $K \subset \mathbb{C}$ be a compact set and let the complex parameter p with $Re(p) \geq 0$ vary in the set K. Then there exist a complex valued function $p \rightsquigarrow B_1(p)$ and a constant C not depending on p such that the solution of (4.1.4), (4.1.5) and (4.1.6) admits the singular representation*

$$w(x,p) = w_R^1(x,p) + B_1(p)\ \psi(r)r^{\frac{\pi}{\omega}}\sin\frac{\pi}{\omega}\theta \tag{4.3.1}$$

with regular part $w_R^1 \in H^2(G)\cap H_0^1(G)$ and coefficient $B_1(p)$ of the singular function satisfying the estimate

$$\|w_R^1\|_{2,G} + |B_1(p)| \leq C\|g\|_{0,G}. \tag{4.3.2}$$

Proof. The decomposition into regular part and singular function stated in Theorem 4.2.1 above means that the bounded linear map $-\Delta+p$ operating from $H^2 \cap H_0^1$ into L^2 has closed range with finite co-dimension 1 or that $-\Delta + p$ has index -1 (See [25], [38]). Notice that (4.3.1) is valid from Theorem 4.2.1 if we re-write (4.1.4) as $-\Delta w = g - p\,w$.

Applying $-\Delta + p$ to both sides of equation (4.3.1), we have

$$g = (-\Delta + p)w_R^1(\cdot,p) + B_1(p)(-\Delta + p)\psi(r)r^{\frac{\pi}{\omega}}\sin\frac{\pi}{\omega}\theta.$$

Now, letting $(-\Delta+p)w_R^1(\cdot,p) =: g_R$ and denoting by $\|(-\Delta+p)^{-1}\|$ the norm of the operator $(-\Delta + p)^{-1}$ from $L^2(G)$ into $H^2(G)$ with domain $D = \{(-\Delta + p)u :\ u \in H^2(G) \cap H^1(G)\}$, we have

$$\begin{aligned}\|w_R^1(\cdot,p)\|_{2,G} &= \|(-\Delta+p)^{-1}\ g_R\|_{2,G}\\ &\leq \|((-\Delta+p))^{-1}\|\|g_R\|_{0,G}\\ &\leq C\|(-\Delta+p)^{-1}\|\|g\|_{0,G}\\ &\leq C\sup_{p\in K}\|((-\Delta+p))^{-1}\|\|g\|_{0,G}\\ &\leq C\|g\|_{0,G} \qquad (4.3.3)\end{aligned}$$

because the coefficients of the operator $-\Delta+p$ are continuous and K is compact.

Furthermore, by (4.3.1), (4.3.3) and the analogue of (3.1.29), we have

$$|B_1(p)|\|\psi(r)r^{\frac{\pi}{\omega}}\sin\frac{\pi}{\omega}\theta\|_{1,G}\leq\|w\|_{1,G}+\|w_R^1\|_{2,G}\leq C\|g\|_{0,G},$$

which yields

$$|B_1(p)|\leq C\|g\|_{0,G}.$$

$\square$

Theorem 4.3.2. *For $|p|$ large enough, there exist a regular function $w_R(x,p)\in H^2(G,\sqrt{|p|})$ and a complex valued-function $p\rightsquigarrow B_2(p)$ such that the solution of the problem (4.1.4), (4.1.5) and (4.1.6) admits the singular decomposition*

$$w(x,p)=w_R(x,p)+B_2(p)\ \psi(\sqrt{|p|}r)r^{\frac{\pi}{\omega}}\sin\frac{\pi}{\omega}\theta.$$

Furthermore, we have the estimate

$$\|w_R\|_{2,\,G,\,\sqrt{|p|}}+|B_2(p)||p|^{1-\frac{\pi}{\omega}}\leq C\|g\|_{0,G},$$

where we recall that here and after $C>0$ denotes various constants independent on p and the weighted norm $\|\cdot\|_{2,\,G,\,\sqrt{|p|}}$ is given in Definition 3.1.4.

Proof. We perform the change of variable $x \in G \to \rho x \in G$ where $\rho := \frac{1}{\sqrt{|p|}}$ and $\omega = \rho^2 p = \frac{p}{|p|}$. Problem (4.1.4) becomes

$$-\frac{1}{\rho^2}\left(\frac{\partial^2 w(\rho x)}{\partial x_1^2} + \frac{\partial^2 w(\rho x)}{\partial x_2^2}\right) + p\, w(\rho x) = g(\rho x)$$

or equivalently

$$(-\Delta + \omega)\, w_\rho(x) = h_\rho(x) \tag{4.3.4}$$

where $h_\rho(x) = \rho^2 g(\rho x)$ and $w_\rho(x) = w(\rho x)$. Since the complex parameter ω satisfies $|\omega| = 1$, Corollary 4.3.1 applies to (4.3.4). Thus w_ρ admits the singular decomposition

$$w_\rho = w^{R,\rho}(x,\omega) + \psi(r) B_1(\omega,\rho)\rho^{\frac{\pi}{\omega}} r^{\frac{\pi}{\omega}} \sin\frac{\pi}{\omega}\theta \tag{4.3.5}$$

or

$$w_\rho = w^{R,\rho}(x,\omega) + \psi(r) B_2(p) r^{\frac{\pi}{\omega}} \sin\frac{\pi}{\omega}\theta \tag{4.3.6}$$

where $B_1(\omega,\rho) = B_2(p)\rho^{\frac{\pi}{\omega}}$ with uniform estimate:

$$\|w^{R,\rho}\|_{2,G} + |B_2(p)|\rho^{\frac{\pi}{\omega}} \le C\|h_\rho\|_{0,G}. \tag{4.3.7}$$

Now from (4.3.7), we go back from the variable ρx in (4.3.4) to the initial variable x in (4.1.4) as follows: Put

$$w_R(x) = w^{R,\rho}(\frac{x}{\rho}) \text{ and } z = \frac{x}{\rho} \text{ so that } dz = \rho^{-2}dx,$$

$$\frac{\partial w_R}{\partial x_1}(x) = \frac{\partial w^{R,\rho}}{\partial z_1}(z)\frac{1}{\rho} \text{ and } \frac{\partial^2 w_R}{\partial x_1^2}(x) = \frac{\partial^2 w^{R,\rho}}{\partial z_1^2}(z)\frac{1}{\rho^2}$$

Thus we have

$$
\begin{aligned}
\|w^{R,\rho}\|_{2,G}^2 &= \int_G (|w^{R,\rho}(z)|^2 + |\nabla w^{R,\rho}(z)|^2 + \sum_{|\alpha|=2} |D^\alpha w^{R,\rho}(z)|^2) dz \\
&= \int_G (|w_R(x)|^2 + \rho^2 |\nabla_x w_R(x)|^2 + \rho^4 \sum_{|\alpha|=2} |D_x^\alpha w_R(x)|^2) \rho^{-2} dx \\
&= \int_G (\rho^{-2}|w_R(x)|^2 + |\nabla_x w_R(x)|^2 + \rho^2 \sum_{|\alpha|=2} |D_x^\alpha w_R(x)|^2) dx \\
&= \rho^2 \int_G (\rho^{-4}|w_R(x)|^2 + \rho^{-2}|\nabla w_R(x)|^2 + \sum_{|\alpha|=2} |D^\alpha w_R(x)|^2) dx \\
&= |p|^{-1} \int_G (\sum_{|\alpha|=2} |D^\alpha w_R(x)|^2 + |p||\nabla w_R(x)|^2 + |p|^2 |w_R(x)|^2) dx \\
&= |p|^{-1} \|w_R\|_{2,G,\sqrt{|p|}}^2,
\end{aligned}
$$

which implies that

$$
\frac{1}{\rho^2} \|w^{R,\rho}\|_{2,G}^2 = \|w_R\|_{2,G,\frac{1}{\rho}}^2. \tag{4.3.8}
$$

Similarly, the right hand side of (4.3.7) yields

$$
\|h_\rho(\frac{x}{\rho})\|_{0,G}^2 = \int_G |h_\rho(z)|^2 dz = \int_G |\rho^2 g(\rho z)|^2 dz = \rho^2 \|g\|_{0,G}^2. \tag{4.3.9}
$$

Using (4.3.7), (4.3.8) and (4.3.9) we have the desired estimate

$$
\|w_R\|_{2,G,\sqrt{|p|}} + |B_2(p)||p|^{\frac{1}{2}-\frac{\pi}{2\omega}} \leq C\|g\|_{0,G},
$$

together with the singular decomposition

$$
w(x) = w_R(x) + B_2(p)\psi(\sqrt{|p|}r) r^{\frac{\pi}{\omega}} \sin \frac{\pi}{\omega}\theta.
$$

□

Remark 4.3.3. *The second part of Theorem 4.3.2 (case $|p|$ large) and its proof constitute a particular case of the deep results stated and proved in [19], [46] and [47] for general elliptic and parabolic problems with edge corners. An alternative approach is presented in [30]. The nature of the Helmholtz operator $-\Delta + pI$ makes the above proof simple and explicit in the* *following manner compared to a general operator of the form* $p + L(x, D_x)$ *investigated in the above mentioned references with* $L(x, D_x)$ *being a proper elliptic operator of order 2 with principal part frozen at the origin denoted by* $L_0(D_x)$. *In making the change of variable* $x \to \rho x$, *the analogue of (4.3.4) has the form*

$$M_\rho(x, D_x)w_\rho = h_\rho \tag{4.3.10}$$

where the operator M_ρ *tends to* $\omega + L_0$ *as* $\rho \to 0$.

The analogue of (4.3.6) is neither explicit nor does it give a uniform estimate of the form (4.3.7). Such an estimate is achieved provided that a perturbation argument together with the convergence of M_ρ *to* $\omega + L_0$ *is used. On the contrary, for the Helmholtz operator,* M_ρ *is reduced to the constant operator* $-\Delta + \omega$.

So far, the analysis of the regularity and the singularity of the solution of problem (4.1.1) was done in two local steps: far away from vertices (Theorem 4.1.1) and near each vertex (Theorem 4.2.1, Corollary 4.3.1 and Theorem 4.3.2). We now combine these steps to obtain the following global result on Ω.

Theorem 4.3.4. *There exists a positive number* $\delta_0 > 0$ *such that the solution of the problem (4.1.1) admits the singular decomposition*

$$w(x, p) = w_R^1(x, p) + B_1(p)\psi(r)r^{\frac{\pi}{\omega}} \sin \frac{\pi}{\omega}\theta$$

with regular part $w_R^1(x, p) \in H^2(\Omega)$ *and coefficients of singularity* $B_1(p) \in \mathbb{C}$ *satisfying the estimate*

$$\|w_R^1\|_{2,\Omega} + |B_1(p)| \leq C\|g\|_{0,\Omega}$$

for $|p| \leq \delta_0$. *Furthermore, the singular decomposition becomes*

$$w(x, p) = w_R^2(x, p) + B_2(p)\psi(r\sqrt{|p|})r^{\frac{\pi}{\omega}} \sin \frac{\pi}{\omega}\theta$$

where $w_R^2(x, p) \in H^2(\Omega, \sqrt{|p|})$ *and*

$$\|w_R^2\|_{2,\Omega,\sqrt{|p|}} + |B_2(p)||p|^{\frac{1}{2}-\frac{\pi}{2\omega}} \leq C\|g\|_{0,\Omega}$$

for $|p| > \delta_0$.

Proof. Notice that Ω was assumed to have only one non-convex vertex, which is localized through the cut-off function $\psi = \psi(r)$ used before.

The solution w of (4.1.1) can then be written as

$$w(x,p) = (1-\psi)w(x,p) + \psi\, w(x,p) \ \text{ on } \ \Omega.$$

Corollary 4.3.1 and Theorem 4.3.2 guarantee the existence of $\delta_0 > 0$ such that the singular decompositions and the estimates in these two results apply to the local solution ψw of (4.1.4) with right-hand side

$$\psi\, g - w\Delta\psi - 2\nabla w \nabla\psi.$$

More precisely, for $|p| \leq \delta_0$, we have

$$w(x,p) = (1-\psi)w(x,p) + w_R^1(x,p) + B_1(p)\psi(r)r^{\frac{\pi}{\omega}}\sin\frac{\pi}{\omega}\theta \tag{4.3.11}$$

with

$$\|w_R^1\|_{2,\Omega} + |B_1(p)| \leq C\|\psi\, g - w\Delta\psi - 2\nabla w\nabla\psi\|_{0,\Omega}.$$

The desired regular part for w is

$$w_R^{1,1} := (1-\psi)w + w_R^1$$

which is indeed of class $H^2(\Omega)$ due to the regularity far away from the vertex that guarantees that $(1-\psi)w \in H^2(\Omega)$. Then with

$$\Omega_{r_0} := \{x \in \Omega; r_0/2 < |x| = r \leq r_0\}$$

we have

$$\begin{aligned}
\|w_R^{1,1}\|_{2,\Omega} + |B_1(p)| &\leq \|(1-\psi)w\|_{2,\Omega} + \|w_R^1\|_{2,\Omega} + |B_1(p)| \\
&\leq C\|w\|_{2,\Omega_{r_0}} + C\|\psi g - w\Delta\psi - 2\nabla w\nabla\psi\|_{0,\Omega} \\
&\leq C\|w\|_{2,\Omega_{r_0}} + C\|g\|_{0,\Omega} + C\|w\|_{1,\Omega_{r_0}} \\
&\leq C\|g\|_{0,\Omega},
\end{aligned}$$

by the regularity of the solution far away from the origin and specifically on Ω_{r_0}. Notice that the various constants C above do not depend on p because p moves in the the compact set $\bar{B}(0,\delta_0)$.

Regarding the case when $|p| > \delta_0$, the singular decomposition to be used in place of (4.3.11) is

$$w(x,p) = (1-\psi)w(x,p) + w_R^2(x,p) + B_2(p)\psi(r\sqrt{|p|})r^{\frac{\pi}{\omega}}\sin\frac{\pi}{\omega}\theta$$

with

$$\|w_R^2\|_{2,\Omega,\sqrt{|p|}} + |B_2(p)||p|^{\frac{1}{2}-\frac{\pi}{2\omega}} \leq C\|\psi g - w\Delta\psi - 2\nabla w\nabla\psi\|_{0,\Omega}.$$

Take $w_R^{2,2} := (1-\psi)w + w_R^2 \in H^2(\Omega)$ as the regular part. In view of the analogue of the Theorem 3.2.2 we have

$$\|(1-\psi)w\|_{2,\Omega,\sqrt{|p|}} \leq C\|g\|_{0,\Omega}.$$

Therefore we have as in the previous case

$$\|w_R^{2,2}\|_{2,\Omega,\sqrt{|p|}} + |B_2(p)||p|^{\frac{1}{2}-\frac{\pi}{2\omega}} \leq C\|g\|_{0,\Omega}.$$

□

4.4 Global regularity of the solution

We devote this section to show that the solution of the Helmholtz problem is regular in a weighted Sobolev space. This result is fundamental to our study as the constructive analysis to come is based on it. The weighted Sobolev space in question is defined as follows:

Definition 4.4.1. *For β a non-negative real number, we denote by $H^{2,\beta}(\Omega)$ the space of all distributions $v \in H^1(\Omega)$ such that*

$$r^\beta D^\alpha v \in L^2(\Omega) \;\; \forall\, \alpha \textit{ such that } |\alpha| = 2$$

where $r \equiv r(x) = d(x,\ vertices)$ is the distance to the vertices of the domain Ω.

The weighted Sobolev space $H^{2,\beta}(\Omega)$ is equipped with its natural Hilbert structure given by the inner product

$$(w,v)_{H^{2,\beta}(\Omega)} = (w,v)_{1,\Omega} + \sum_{|\alpha|=2} \int_\Omega r^\beta D^\alpha w . D^\alpha v dx.$$

The norm of the space $H^{2,\beta}(\Omega)$ is written $\|\cdot\|_{H^{2,\beta}(\Omega)}$ while the following is simply a semi-norm:

$$|v|_{H^{2,\beta}(\Omega)} := \left(\sum_{|\alpha|=2} \int_\Omega |r^\beta D^\alpha v|^2 dx \right)^{\frac{1}{2}}.$$

Remark 4.4.2. *The usual Sobolev space $H^2(\Omega)$ is continuously embedded in the weighted Sobolev space $H^{2,\beta}(\Omega)$:*

$$H^2(\Omega) \hookrightarrow H^{2,\beta}(\Omega).$$

Indeed, this is obvious for $\beta = 0$ since $H^2(\Omega) = H^{2,0}(\Omega)$.

For $\beta > 0$ and for $v \in H^2(\Omega)$, we have

$$\begin{aligned} \int_\Omega \left(|v|^2 + |\nabla v|^2 + \sum_{|\alpha|=2} |D^\alpha v|^2 \right) dx &= \int_\Omega \left(|v|^2 + |\nabla v|^2 + \sum_{|\alpha|=2} r^{2\beta} |D^\alpha v|^2 r^{-2\beta} \right) dx \\ &\geq C \int_\Omega \left(|v|^2 + |\nabla v|^2 + \sum_{|\alpha|=2} r^{2\beta} |D^\alpha v|^2 \right) dx \end{aligned}$$

where $C = \min \left\{ 1, \left(\frac{1}{diameter(\Omega)} \right)^{2\beta} \right\}$, observing that $\sup_{x \in \bar{\Omega}} d(x, vertices) \leq \ diameter(\Omega)$.

Theorem 4.4.3. *The space $H^{2,\beta}(\Omega)$ is continuously and compactly embedded in $C^0(\bar{\Omega})$ for $0 \leq \beta < 1$:*

$$H^{2,\beta}(\Omega) \hookrightarrow_c C^0(\bar{\Omega}).$$

Furthermore, the embedding of $H^{2,\beta}(\Omega)$ into $H^1(\Omega)$ is compact: $H^{2,\beta}(\Omega) \hookrightarrow_c H^1(\Omega)$

Proof. The case when $\beta = 0$ is well-known because $H^{2,0}(\Omega) = H^2(\Omega)$ (Sobolev and Rellich-Kondrachov embeddings, Theorem 2.4.5). So we assume that $\beta > 0$. Let v be in $H^{2,\beta}(\Omega)$ so that

$$v \in L^p(\Omega),\ \forall\, p \in [1, +\infty) \ \text{ and } \ D^\alpha v = (r^\beta D^\alpha v).r^{-\beta} \ \ \forall \ \ 1 \leq |\alpha| \leq 2. \tag{4.4.1}$$

The first inclusion in (4.4.1) is due to the fact that $v \in H^1(\Omega)$, which is embedded in $L^p(\Omega)\ \ \forall\, p \in [1, +\infty)$ by Theorem 2.4.5. We want to show that $D^\alpha v \in L^p(\Omega)\ \ 1 \leq |\alpha| \leq 2$ for some $p > 1$ and $p < 2$. Take $q_1 = \frac{2}{p}$ with conjugate $q_2 = \frac{2}{2-p}$ i.e. $\frac{1}{q_1} + \frac{1}{q_2} = 1$. Then $r^{-\beta p}$ is of class $L^{q_2}(\Omega)$ iff $1 \leq p < \frac{2}{1+\beta}$. By Hölder's inequality, we deduce from (4.4.1) and the choice of p, q_1 and q_2 that

$$\begin{aligned} \int_\Omega |D^\alpha v|^p dx &\leq \left(\int_\Omega (|r^\beta D^\alpha v|^p)^{q_1} dx \right)^{\frac{1}{q_1}} \left(\int_\Omega r^{-\beta p q_2} dx \right)^{\frac{1}{q_2}} \\ &= \left(\int_\Omega |r^\beta D^\alpha v|^2 dx \right)^{\frac{p}{2}} \left(\int_\Omega r^{\frac{-2\beta p}{2-p}} r dr \right)^{\frac{2-p}{2}} \\ &\leq C \|v\|^p_{H^{2,\beta}(\Omega)}. \end{aligned}$$

Notice that if $|\alpha| = 0$ and $\beta = 0$ in (4.4.1), we could show in a similar manner that $v \in L^p(\Omega)$ for the specific choice of p made above. Thus $H^{2,\beta}(\Omega) \hookrightarrow W^{2,p}(\Omega)$. But by the Sobolev and Rellich Kondrachov imbeddings, Theorem 2.4.5, the Sobolev space $W^{2,p}(\Omega)$ is continuously and compactly embedded into $C^0(\bar{\Omega})$ and $H^1(\Omega)$, respectively. This proves the first and the second claims and hence the proof of the Theorem is completed. $\square$

We are now in a position to state one of our main contributions that will have an impact on the heat equation and on its numerical approximation in the next chapter. This result is announced in [14] and [13].

Theorem 4.4.4. *Assume that $0 < \beta < 1 - \frac{\pi}{\omega}$. Then the solution w of the Helmholtz problem (4.1.1) is of class $H^{2,\beta}(\Omega)$ such that the following estimate holds for some constant $C > 0$ independent on p:*

$$\|w\|_{H^{2,\beta}(\Omega)} \leq C \|g\|_{0,\Omega}.$$

Proof. The existence of a number $\delta_0 > 0$ in Theorem 4.3.4 is the rephrasing of the requirement that $|p|$ is large enough in Theorem 4.3.2. From Remark 4.4.2, we have for the regular part in Theorem 4.3.4:

$$\|w_R^1\|_{H^{2,\beta}(\Omega)} \leq C \|w_R^1\|_{2,\Omega} \leq C \|g\|_{0,\Omega}$$

$$\|w_R^2\|_{H^{2,\beta}(\Omega,|p|)} \leq C\|w_R^2\|_{2,\Omega,|p|} \leq C\|g\|_{0,\Omega}$$

where the weighted norm on $H^{2,\beta}(\Omega, \sqrt{|p|})$ is defined in a similar manner as that of $H^2(\Omega, \sqrt{|p|})$ of Definition 3.1.4 by

$$\|v\|^2_{H^{2,\beta}(\Omega,\sqrt{|p|})} = \int_\Omega \left(|p|^2|v|^2 + |p||\nabla v|^2 + \sum_{|\alpha|=2} |r^\beta D^\alpha v|^2 \right) dx. \tag{4.4.2}$$

Regarding the singular part, we proceed as follows. Firstly, the function $\psi(r)r^{\frac{\pi}{\omega}} \sin\frac{\pi}{\omega}\theta$ belongs to $H^{2,\beta}(\Omega)$ because near the non-convex corner $(0,0)$, $r^\beta D^\alpha \psi(r)r^{\frac{\pi}{\omega}} \sin\frac{\pi}{\omega}\theta$ with $|\alpha| = 2$, behaves like $r^{\beta+\frac{\pi}{\omega}-2}$ which is of class $L^2(\Omega)$ in view of the condition $0 < \beta < 1 - \frac{\pi}{\omega}$.
Thus for $|p| \leq \delta_0$, the estimate for $|B_1(p)|$ in Theorem 4.3.4 yields

$$\|\psi(r)B_1(p)r^{\frac{\pi}{\omega}} \sin\frac{\pi}{\omega}\theta\|_{H^{2,\beta}(\Omega)} \leq C\|g\|_{0,\Omega}.$$

For $|p| > \delta_0$, the same argument as above shows that $\psi(r|p|)r^{\frac{\pi}{\omega}} \sin\frac{\pi}{\omega}\theta$ is of class $H^{2,\beta}(\Omega)$. Now the estimate for $B_2(p)$ in Theorem 4.3.4 leads to

$$\begin{aligned} \|\psi(r\sqrt{|p|})B_2(p)r^{\frac{\pi}{\omega}} \sin\frac{\pi}{\omega}\theta\|_{H^{2,\beta}(\Omega)} &\leq C|B_2(p)| \\ &\leq C|p|^{\frac{\pi}{2\omega}-\frac{1}{2}}\|g\|_{0,\Omega} \\ &\leq C(\delta_0)^{\frac{\pi}{2\omega}-\frac{1}{2}}\|g\|_{0,\Omega}. \end{aligned}$$

□

Remark 4.4.5. *The underlying point of our investigation is that the linear operator*

$$g(.,p) \in L^2(\Omega) \rightsquigarrow w(.,p) \in H^{2,\beta}(\Omega)$$

is bounded with norm independent on $p \in \mathbb{C}$ satisfying (3.1.20). Theorem 4.4.4 is proved in Grisvard [29] in the particular case when $p = 0$. This originates from the study by Raugel [58], [59] of the regularity in the general case where g is in the Sobolev space $H^m(\Omega)$, $m > 0$. In this case weighted Sobolev spaces $H^{m+2,\beta}(\Omega)$ of higher order are essential as demonstrated by Raugel.

Remark 4.4.6. *In this thesis we used three types of weighted Sobolev spaces which play completely different roles:*

- *The weighted Sobolev space $H^m(\Omega, \rho)$ (cf. Definition 3.1.4 and Proposition 3.1.5), which is exactly the usual Sobolev space $H^m(\Omega)$ equipped with a weighted norm. The space $H^m(\Omega, \rho)$ arises generally when the (partial) Fourier transform with respect to t is applied to functions $(t, x) \to v(x, t)$ in the usual Sobolev space $H^m(\Omega \times \mathbb{R})$. In fact the norm $\|v\|_{m,\Omega\times\mathbb{R}}$ is equivalent to $\left(\int_{\mathbb{R}} \|(\mathcal{F}v)(\eta)\|^2_{m,\Omega,1+|\eta|} d\eta\right)^{\frac{1}{2}}$ see Dauge [19].*
- *The Kondratiev weighted Sobolev space $P_2^k(G)$ (cf Definition 4.1.3) serves to investigate the regularity and the singularity for an elliptic problem localized in a sector G. The space $P_2^k(G)$ is not equal to the usual Sobolev space $H^k(G)$. However, it is related to H^k through Lemma 4.1.4 and we have $P_2^k(G) \subset H^k_{loc}(G)$ i.e. $v \in P_2^k(G) \Rightarrow \rho v \in H^k(G) \ \forall \rho \in \mathcal{D}(G)$.*
- *The weighted Sobolev space $H^{2,\beta}(\Omega)$ is a replacement for $H^2(\Omega)$ for the global regularity of the solutions.*