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A general discrete-time arbitrage theorem

by

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DECLARATION

I, the undersigned, hereby declare that the dissertation submitted herewith for the degree Magister Scientiae to the University of Pretoria contains my own, independent work and has not been submitted for any degree at any other university.

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Abstract

Title: *A general discrete-time arbitrage theorem*
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We study discrete-time stochastic processes as if they represent financial prices. The thesis comprises the development of one major result, the discrete-time arbitrage theorem.

It states that a given discrete time stochastic process (on a general probability space) has the no-arbitrage property if and only if there is an equivalent measure under which the process is a martingale. The sufficiency part of this statement is easy to show, but proving the necessity part requires that a certain set $K - L^0_+$, which represents the possible final values of the portfolio less a random nonnegative amount, be shown to be closed in probability.

The space $L^0(\Omega, \mathcal{F}, P; \mathbb{R}^d)$ of all $\mathbb{R}^d$-valued measurable functions is decomposed into two subspaces. These spaces are analysed by Hilbert space orthogonality techniques (applied to subspaces of $L^2(\Omega, \mathcal{F}, P; \mathbb{R}^d)$), to get conclusions which can be extended to L^0. Finally, to show that the above-mentioned set is closed, a substitute for the "sequential compactness of the unit ball" (which is unavailable here since L^0 is not reflexive) is given.

The proof presented here follows that of W. Schachermayer [Sch].

Samevatting

Titel: *A general discrete-time arbitrage theorem*
Kandidaat: *Augustinus Johannes van Zyl*
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Ons bestudeer diskrete-tyd stogastiese prosesse onder 'n finansiële interpretasie. Die skripsie behels die ontwikkeling van een belangrike resultaat, die diskrete-tyd *arbritage* stelling.

Hierdie stelling sê dat 'n gegewe stogastiese proses (op 'n algemene maatruimte) die geen-arbitrage eienskap het as en slegs as daar 'n ekwivalente maat bestaan waaronder die proses 'n *martingale* is. Dat die bestaan van 'n ekwivalente martingalemaat voldoende is, is maklik om aan te toon. Om die noodwendigheid van die bestaan van hierdie maat te bewys verg egter dat 'n sekere versameling $K - L^0_+$, wat 'n voorstelling is van die moontlike finale waardes van die portefeulje minus 'n ewekansige nie-negatiewe bedrag, aangetoon moet word as geslote in waarskynlikheid.

Die ruimte $L^0(\Omega, \mathcal{F}, P; \mathbb{R}^d)$ van alle $\mathbb{R}^d$-waardige meetbare funksies word ontbind as die direkte som van twee deelruimtes. Hierdie ruimtes word ontleed deur Hilbert-ruimtetegnieke (ortogonaliteitstegnieke toegepas op deelruimtes van $L^2(\Omega, \mathcal{F}, P; \mathbb{R}^d)$), om gevolgtrekkings te kry wat uitgebrei kan word na L^0. Laastens, om aan te toon dat die versameling hierbo vermeld geslote is, word 'n substituut vir die "rykompakheid van die eenheidsbal" (wat ongeldig is in hierdie nie-refleksiewe ruimte) gegee.

Die bewys wat hier gevolg word is dié van W. Schachermayer [Sch].

Contents

Chapter 1

The model and some basic facts

Historical remarks

An "arbitrage" opportunity is a situation where it is possible to construct a risk-less profitable investment strategy. That is, it is a case where there is a portfolio which does not suffer any losses under any contingency, and has a strictly positive probability that the strategy will yield something. A simple example of this [DybRo] is the opportunity to lend *and* borrow at two different interest rates without any further costs. In this case the "strategy" to borrow money at the lower interest rate and simultaneously lend the same amount at the higher rate will always yield something positive - the difference of the two interest rates times the amount borrowed.

A formal definition of the concept will be given later. Let it be noted that the no-arbitrage condition (the requirement that there be no arbitrage opportunities) is a property of the stochastic process which represents the prices of the assets under different states and in different time periods. For example, the stochastic process of Brownian motion satisfies the no-arbitrage condition while the discrete-time process $(1, 2, 3, \dots)$ does not, as will be clear later on. The arbitrage theorem, also called the fundamental theorem of asset pricing [DybRo], essentially states the following: A given stochastic process satisfies the no-arbitrage condition if and only if there exists an equivalent measure which makes the process a martingale. This basic result has many different forms, depending on whether the probability space is finite or not, and on whether the time is discrete or continuous. This dissertation concerns itself with the case of a general probability space but discrete (and finite) time.

Even in the more simple setting, that of a stochastic process (S_0, S_1) and a finite-state probability space, this result is not totally trivial: If $(x_i)_{i=1}^n$ is a vector, define "$x >> 0$" to mean $x_i \geq 0$ for all i and there exists an i such that $x_i > 0$. Let s_{ij} denote the payoff of the i^{th} asset $(1 \leq i \leq d)$ in state j $(1 \leq j \leq m)$. Let $S = (s_{ij})_{1 \leq i \leq d,\ 1 \leq j \leq m}$. The form of the no-arbitrage condition in this setting is that there exists no $y \in \mathbb{R}^d$ such that $\sum_{i=1}^d y_i s_{ij} >> 0$. The arbitrage theorem for one time step and a finite probability space says that S satisfies the no-arbitrage condition if and only if there exists a $q \in \mathbb{R}^m$, every $q_j > 0$, such that $\sum_{j=1}^m q_j s_{ij} = 0$ for every i. Now it is clear that this q can be chosen such that $\sum_j q_j = 1$, i.e. it can be interpreted as a probability measure. And in discrete probability spaces the equivalence of q to the original probability measure - that they have the same null sets - reduces to the statement that for every $j \in \{1, 2, \ldots, m\}$, $q_j > 0$. The arbitrage theorem given here is proved in [DybRo] who uses notation similar to what we will use in the text (this reference should also be consulted by anyone interested in more background on the concept of "arbitrage" and connections with the ideas of equilibrium, linear pricing rules and the "law of one price").

In the case where the state space is infinite (the situation which will be treated in the text), instead of $\{1, 2, \ldots, m\}$, the situation becomes more difficult. However, R.C. Dalang, A. Morton and W. Willinger [DMW] showed that in this case the no-arbitrage theorem can still be given a clear-cut formulation: A stochastic process satisfies the no-arbitrage condition if and only if there exists an equivalent martingale measure. This thesis follows another proof of that result, that of W. Schachermayer [Sch]. Another discussion of his paper, as well as other proofs of the same theorem, is by F. Delbaen [Delb]. The latter author has also, together with Schachermayer, proved what must be termed the no-arbitrage theorem in the most general case, for continuous-time stochastic processes (in general probability spaces). Here it is not exactly the no-arbitrage condition which is equivalent to the existence of an equivalent martingale measure, but a closely related concept, "no free lunch with vanishing risk".

But a discussion of the continuous-time arbitrage concepts is beyond the scope of the present text; the interested reader is referred to [DeSc].

Mathematics

1.1. Unless otherwise stated, we work in the probability space $(\Omega, \mathcal{F}, P)$.

We use the standard notation and abbreviations of measure theory: If A is an event, i.e. $A \in \mathcal{F}$, the set $\{\omega \in \Omega | \; \omega \in A\}$ will sometimes be written just $\{\omega \in A\}$. For example, the notation $\{f(\omega) > 0\}$ or even just $\{f > 0\}$ will stand for the set $\{\omega \in \Omega | \; f(\omega) > 0\}$. And with $P(\omega \in A)$, or sometimes with square brackets like $P[\omega \in A]$, we mean $P(\{\omega \in \Omega | \; \omega \in A\})$.

1.2. The mathematical description of the price process is as follows. *Time* is indexed by the set $T = \{1, 2, \ldots, N\}$, i.e. there are N time steps, where $N \in \mathbb{N}$. There are *d financial assets* (or "stocks"). The i^{th} financial asset $(1 \leq i \leq d)$ has, at a given time $t \in T$ and state of the world $\omega \in \Omega$, one price denoted $S_t^i(\omega)$, at which it can be bought and sold. So the price process $S := (S_t)_{t=1}^N$ is a d-dimensional discrete time stochastic process defined in the probability space $(\Omega, \mathcal{F}, P)$.

This probability space possesses a filtration of σ-fields $(\mathcal{F}_t)_{t=1}^N$, and we assume that S is adapted to that filtration, i.e. S_t is $\mathcal{F}_t$-measurable for every $t \in T$.

Remark. A decisive aspect of the mathematical modeling of price processes is as indicated above the representation of the flow of information by the sequence of σ-fields. (This is standard practice in the theory of stochastic processes.) Some basic comments on this: Suppose that $X : \Omega \to \mathbb{R}^d$ is a random variable and $\mathcal{F}$ is the collection of sets $\{X^{-1}(B) | \text{ B a Borel set in } \mathbb{R}^d\}$ (i.e. $\mathcal{F}$ is the smallest σ-algebra such that X is $\mathcal{F}$-measurable), then $\mathcal{F}$ represents all the events we can deduce from $X(\omega)$. This is because given $A \in \mathcal{F}$ with $A = X^{-1}(B)$ we can decide the question "$\omega \in A$" by inspecting whether "$X(\omega) \in B$" holds (since $\omega \in X^{-1}(B) \Leftrightarrow X(\omega) \in B$). So $\mathcal{F}$ can be interpreted as the information we have about ω. If we have an adapted stochastic process instead of a single random variable, then under mild conditions of measurability, we can interpret the element $\mathcal{F}_t$ of the filtration $(\mathcal{F}_t)_{t \in T}$ to be the information we have at our disposal at time t, obtained by observing X_s for $s \leq t$. $\mathcal{F}_t$ is therefore often called the "history of the process up to time t".

It is usually also assumed that $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_N$ (which means that we do not forget information), but we do not need this assumption for the results in the sequel. It is also not assumed that $\mathcal{F}$ is complete (That is, we do not assume that it contains all the sets of measure zero).

1.3. Let $\mathcal{B}$ be a Banach space. The space of all $\mathcal{F}$-measurable functions from Ω to $\mathcal{B}$, with probability measure P, will be denoted $L^0(\Omega, \mathcal{F}, P;\ \mathcal{B})$. (It will also be denoted L^0 or $L^0(\mathcal{F})$ or any other way in which the arguments which are clear from the context can be omitted). For the concept of measurability, and other versions of this condition, the reader may consult [DS]. In the dissertation, $\mathcal{B}$ will always be either $\mathbb{R}$ or $\mathbb{R}^d$. In the case $\mathcal{B} = \mathbb{R}$, L^0_+ will refer to the set $\{x \in L^0 |\ x \geq 0\ P - a.s.\}$, and $L^0_- := \{x \in L^0 |\ x \leq 0\ P - a.s.\}$. The space L^0 is given a topology, namely that induced by convergence *in probability* (also called convergence in *measure*). That is, a set $A \subseteq L^0$ is closed (in probability) if and only if A possesses all its probability convergence limits, a concept which will be defined next. A sequence $(f_n)_{n=1}^{\infty} \subseteq L^0(\Omega, \mathcal{F}, P; \mathbb{R}^d)$ converges to f *in probability*, or is a probability convergence limit of the sequence, if

$$(\forall \epsilon > 0)(P[\|f_n(\omega) - f(\omega)\|_{\mathbb{R}^d} \geq \epsilon] \to 0 \text{ as } n \to \infty)$$

holds.

Regarding the norm on $\mathbb{R}^d$, it is a consequence of the fact that all the norms on this finite-dimensional space generate the Euclidean topology, that we can use any such norm in the above definition. Let us fix it to be the ℓ^2 norm, $\|\cdot\|_2 : \mathbb{R}^d \to \mathbb{R}_+,\ x \mapsto \sqrt{\sum_{i=1}^d |x_i|^2}$. We will use this convention in the rest of the manuscript.

Importantly, all statements regarding measurable functions f should be taken as statements about their equivalence classes (those functions which differ from f only on a set of measure zero). For example, unless stated otherwise, $\{0\}$ should be read as $\{f | f = 0\ P - a.s.\}$.

Note that the space L^0 is metrizable, by the metric $d(X, Y) = \inf\{\alpha > 0 |\ P(\|X(\cdot) - Y(\cdot)\|_2 \geq \alpha) < \alpha\}$. (Many other metrics exist.)

L^0 is a complete metric space (for the completeness, see e.g. [DS III.6.5]); however, it is not a locally convex space. In fact, its continuous dual is $\{0\}$.

1.4. For $1 \leq p < \infty$, we define $L^p(\Omega, \mathcal{F}, P; \mathbb{R}^d)$ (or just L^p if the context is clear), as the set of all $\mathcal{F}$-measurable functions $f : \Omega \to \mathbb{R}^d$ such that $\int_\Omega \|(f(\omega)\|_2)^p dP(\omega) < \infty$.

Let $\|f\|_p := [\int_\Omega(\|f(\cdot)\|_2)^p dP]^{\frac{1}{p}}$ (This makes L^p a Banach space).

We set $L^\infty(\Omega, \mathcal{F}, P; \mathbb{R}^d)$ to be the set of all $\mathcal{F}$-measurable functions which are essentially bounded. In this space, we use the norm

$\|f\|_\infty = \inf\{\alpha \in \mathbb{R}_+ |\ P(\|f(\omega)\|_2 \geq \alpha) = 0\} = \inf\{\alpha > 0 |\ f \leq \alpha\ a.s.\}$.

When d=1, we will define $L^p_+ := L^0_+ \cap L^p$ and $L^p_- := L^0_- \cap L^p$.

The relationships between the different modes of convergence (e.g. that of L^p-convergence, convergence in probability and convergence almost surely) will be frequently used. The main results are the following. They will not be proved here, but we will provide references. For the next three results see

[DS, Theorems III.6.13(b), III.6.13(a) and III.3.6].

1.5. **Theorem.** *Let $(f_n)_{n=1}^\infty$ be a sequence in $L^0(\Omega, \mathcal{F}, P; \mathbb{R}^d)$. If $f_n \to f$ a.s., then $f_n \to f$ in probability.*

1.6. **Theorem.** *Let $(f_n)_{n=1}^\infty$ be a sequence in $L^0(\Omega, \mathcal{F}, P; \mathbb{R}^d)$. If $f_n \to f$ in probability, then there is a subsequence $(f_{n_r})_{r=1}^\infty$ converging to f almost surely.*

1.7. **Theorem.** *Convergence in mean [i.e. in L^1] implies convergence in probability.*

1.8. From the last two results it is trivial that ...

Theorem. *Suppose $(f_n)_{n=1}^\infty$ converges to f in the L^1-norm. Then there exists a subsequence $(f_{n_r})_{r=1}^\infty$ which converges to f almost surely.*

1.9. Under certain special conditions we can do more, due to the Vitali convergence theorem [DS, III.3.6]. (Note that condition (iii) in [DS] is superfluous in the context of finite measure spaces.)

Theorem. *We have $f \in L^1$ and $\|f_n - f\|_{L^1} \to 0$ if and only if $f_n \to f$ in measure and $(f_n)_{n=1}^\infty$ is uniformly integrable.*

It is also important to realise that, if the spaces are defined on the same probability space, we have $L^\infty \subseteq L^2 \subseteq L^1$ (and also $\subseteq L^0$). It is actually easy to show that if $p > q$ then $L^p(P) \subseteq L^q(P)$. Keeping this in mind, it is not difficult to see that (in probability spaces) the following result holds.

1.10. **Theorem.** *Convergence in L^2 implies convergence in L^1.*

1.11. Suppose that the economic agent has at each time step t and state ω the d-tuple of asset holdings (called a *portfolio*) $h_t(\omega)$. h is a d-dimensional random variable: $h_t^i(\omega)$ is the investment (amount of stock) in the i^{th} asset, under these circumstances. This amount is real-valued, and the case if $h_t^i(\omega) < 0$ can be interpreted as the result of "short selling" of the i^{th} asset.

The *value* of the portfolio is defined to be $(h(\omega), S(\omega)) := \sum_{i=1}^d h^i(\omega)S^i(\omega)$. (The notation $(\cdot,\cdot)$ will be the only one used for this inner product). The sequence of portfolios, $(h_t)_{t=1}^N$, is called a *strategy* if every h_t is $\mathcal{F}_{t-1}$-measurable (i.e. h is a so-called *predictable* process). The rationale of this definition is that we want to allow portfolio selections that do not look into the future, but only use information which can be deduced from the history of S up till now, i.e. S_u for $u \leq t$.

To state the no-arbitrage condition, we define the subspace of random variables

$$K := \left\{ \sum_{k=1}^N (h_k, S_k - S_{k-1}) | h_i \in L^0(\Omega, \mathcal{F}_{k-1}, P; \mathbb{R}^d) \text{ for all } k \in \{1, 2, \ldots, N\} \right\}.$$

K is a set of random variables which represents the possible final portfolio values under "self-financing" strategies: At time k the trader can invest h_k, using the information $\mathcal{F}_{k-1}$; this costs him (h_k, S_{k-1}). After time step k his portfolio is worth (h_k, S_k), so his gain from this time step is $(h_k, S_k - S_{k-1})$. Now we say that the process $(S_t)_{t=0}^N$ satisfies the *no-arbitrage* condition if the total gain, which is an element of K, cannot be strictly positive with strictly positive probability without risking a negative outcome with strictly positive probability: (the no-arbitrage condition) $K \cap L_+^0 = \{0\}$. (For continuous-time stochastic processes, which are not needed in the text, the no-arbitrage condition is defined similarly, by replacing, under suitable assumptions, the sums in K by stochastic integrals.)

Most of modern finance is based on either the intuitive or the (The weak topology can be defined for more general spaces, but here we only need the Hilbert space case) mathematical theory of the absence of arbitrage, cf. [DybRo]. There are good economical arguments behind this assumption, here we mention two. Firstly in a "rational" and liquid market the arbitrage opportunities will tend to vanish. Secondly it is natural to require that a pricing model assign prices to commodities in a way that does not depend on the intermediary transactions which are made to obtain that commodity, shortly that the model is *consistent*.

Recall that two probability measures P and Q are called *equivalent* if for every A in the associated σ-field we have "$P(A) = 0 \Leftrightarrow Q(A) = 0$". The significance of this is that then P and Q have mutual Radon-Nikodým derivatives. If Q is a measure which is equivalent to P, then if the no-arbitrage condition holds for P, it holds for Q as well.

1.12. As an example of a process that does not have an equivalent martingale measure, take S defined by $S_k = k,\ k = \{0, 1, \ldots, T\}$. Because $E_P[S_k] = k$, which is not constant w.r.t. k for any probability measure P, S cannot be a martingale under any measure. And the strategy $h_k = 1$ for all k, is enough to verify that in this case the no-arbitrage condition is not satisfied either.

1.13. We use the symbol $\mathbb{E}$ for the conditional expectation operator (which is defined in most introductory texts on stochastic calculus or mathematical finance). As an exercise we can prove the sufficiency part of our main theorem.

Theorem. *Let $S = (S_t)_{t=0}^N$ be a stochastic process which is adapted to the filtration $(\mathcal{F}_k)_{k=0}^N$. If S is a martingale, then S satisfies the no-arbitrage condition.*

Proof: Suppose S is a martingale, and let $h \in L^0(\Omega, \mathcal{F}_{t-1}, P; \mathbb{R}^d)$. Then (we refer to the i^{th} coordinate of h by h^i; h^i is also $\mathcal{F}_{t-1}$-measurable)

$$\begin{aligned}
\mathbb{E}[(h, S_t - S_{t-1})|\mathcal{F}_{t-1}] =& \mathbb{E}[\sum_{i=1}^{d} h^i(S_t^i - S_{t-1}^i)|\mathcal{F}_{t-1}] \\
=& \sum_{i=1}^{d} h^i \mathbb{E}[S_t^i - S_{t-1}^i|\mathcal{F}_{t-1}] \qquad \text{(since } h \text{ is } \mathcal{F}_{t-1} \text{ -measurable)} \\
=& 0 \qquad \text{(because } S \text{ is a martingale).}
\end{aligned}$$

So $E[(h(\cdot),(S_t - S_{t-1})(\cdot)] = 0$. Now the condition $(h(\cdot), S_t(\cdot) - S_{t-1}(\cdot)) \geq 0$ a.s. implies that, because its expectation is zero, $(h(\cdot), S_t(\cdot) - S_{t-1}(\cdot)) = 0$ a.s. We have shown that the no-arbitrage condition is satisfied. $\square$

Chapter 2

Some techniques in weak topology

2.1. It is useful to have the Bolzano-Weierstrass property, namely that every bounded sequence has a convergent subsequence. However, this principle does not hold in infinite dimensional space, e.g. an orthonormal sequence in infinite dimensional Hilbert space cannot have a convergent subsequence (it doesn't satisfy the Cauchý criterion), although all the elements of that sequence have norm one, so that the sequence is bounded. This is one of the important reasons to introduce the *weak* topology, which is the smallest topology on the Hilbert space H under which all the bounded linear functionals are continuous.. It is not difficult to show that a sequence $(x_n)_{n=1}^{\infty} \subseteq H$ converges *weakly* to x iff.

$$\langle y, x_n \rangle \to \langle y, x \rangle \qquad \text{for all } y \in H. \tag{1}$$

The weak topology is clearly coarser than the norm topology usually defined on a Hilbert space, because in the norm topology the bounded linear functionals are already continuous. (The weak topology can be defined for more general spaces, but here we only need the Hilbert space case.)

2.2. Note that we have the following well-known fundamental property (cf. [DS, IV.7])

Theorem. (The weak sequential compactness property) *Every bounded sequence of elements in a Hilbert space contains a weakly convergent subsequence.*

2.3. If A is a set, we denote by *convA* the convex envelope of A, i.e. the smallest convex set which contains A. (We have $convA = \{\sum_{i=1}^{n} \alpha_i x_i | \, n \in \mathbb{N}, \ \sum_{i=1}^{n} \alpha_i = 1; \ 0 \leq \alpha_i \leq 1$ and $x_i \in A$ for $i = 1, 2, \ldots, n\}$. So *convA* is just the set of *convex combinations* of A. The idea of a *sequence* (y_n) *of convex combinations*, $y_n \in conv\{x_n, x_{n+1}, \ldots\}$ will also be encountered frequently. It simply means that the sequence satisfies the requirement that for every $n \in \mathbb{N}$, it holds that $y_n \in conv\{x_m | \, m \geq n\}$. Sometimes it will be convenient to just say $(y_n)_{n=1}^{\infty}$ is a *sequence of convex combinations from* $(x_n)_{n=1}^{\infty}$.

It should be noted that if (y_n) is a sequence of convex combinations from (x_n), any subsequence $(y_{n_r})_{r=1}^{\infty}$ will also be a sequence of convex combinations from (x_n).

Theorem. *Let $(f_n)_{n=1}^{\infty}$ be a sequence in the Hilbert space $L^2(\Omega, \mathcal{F}, P; \mathbb{R}^d)$. If f_n converges weakly to f, then there exists a sequence $(c_n)_{n=1}^{\infty}$ of convex combinations from $(f_n)_{n=1}^{\infty}$ which converges to f (in norm).*

Proof: In fact, we can choose it to be a sequence of arithmetic averages. Note that it follows directly from the uniform boundedness principle that there is a $B > 0$ such that $E[(f_n, f_n)] = \|f_n\|_{L^2} \leq B$ for all n. We can also control terms of the form $E[(g, h)]$ where g and h appear in the subsequence: for all $n \in \mathbb{N}$ set $m_1^{(n)} := n$. Since

$$E[(f_{m_1^{(n)}}, f_k)] \to 0 \text{ as } k \to \infty,$$

we can choose $m_2^{(n)} > n$ such that

$$E[(f_{m_1^{(n)}}, f_{m_2^{(n)}})] \leq 1.$$

Similarly we can set $m_3^{(n)} > m_2^{(n)}$ to satisfy

$$E[(f_{m_1^{(n)}}, f_{m_3^{(n)}})] \leq \frac{1}{2}, \ E[(f_{m_2^{(n)}}, f_{m_3^{(n)}})] \leq 1.$$

Continuing in the same fashion, it is clear that given $j \in \{1, 2, \ldots, n-1\}$, we can find an integer $m_{j+1}^{(n)}$ so that

$$E[(f_{m_i^{(n)}}, f_{m_{j+1}^{(n)}})] \leq \frac{1}{j} \qquad \text{for all } i < j.$$

Thus

$$\begin{aligned}
&\left\|\frac{f_{m_1^{(n)}} + f_{m_2^{(n)}} + \cdots + f_{m_n^{(n)}}}{n}\right\|_{L^2} \\
=&\frac{1}{n^2}\cdot\Big(\sum_{i=1}^{n} E[(f_{m_i^{(n)}}, f_{m_i^{(n)}})] + 2\sum_{i<2} E[(f_{m_i^{(n)}}, f_{m_2^{(n)}})] + \\
&\qquad + 2\sum_{i<3} E[(f_{m_i^{(n)}}, f_{m_3^{(n)}})] + \cdots + 2\sum_{i<n} E[(f_{m_i^{(n)}}, f_{m_n^{(n)}})]\Big) \\
\leq&\frac{1}{n^2}(nB + 2\cdot 1 + 2\cdot 2\cdot\frac{1}{2} + \cdots + 2\cdot n\cdot\frac{1}{n}) \\
=&\frac{nB + n(2)}{n^2} \to 0.
\end{aligned}$$

□

2.4. **Theorem.** *Let $(g_n)_{n=1}^{\infty}$ be a sequence of convex combinations from $(f_n)_{n=1}^{\infty}$. Then*

(i) $E[f_n] \to \alpha$ *as* $n \to \infty \Rightarrow E[g_n] \to \alpha$ *as* $n \to \infty$.

(ii) *If h is a random variable and the sequence (f_n) satisfies $f_n(\omega) \leq h(\omega)$ almost surely for all $n \in \mathbb{N}$, then every convex combination g_n satisfies $g_n(\omega) \leq h(\omega)$ a.s.*

Proof: (i) Suppose $N \in \mathbb{N}$ is such that if $n \geq N$ then $|E[f_n] - \alpha| < \epsilon$. Denote each convex combination g_n by $\sum_{i=1}^{k(n)} \alpha_{ni} f_{n_i}$ ($\sum_{i=1}^{k(n)} \alpha_{ni} = 1$, each $0 \leq \alpha_{ni} \leq 1$).Then, if $n \geq N$, we get the following. $|E[g_n] - \alpha| = |E[\sum_{i=1}^{k(n)} \alpha_{ni} f_{ni}] - \alpha| = |\sum_{i=1}^{k(n)} \alpha_{ni} E[f_{ni} - \alpha]| \leq \sum_{i=1}^{k(n)} \alpha_{ni}|E[f_{ni}] - \alpha| \leq (\sum_{i=1}^{k(n)} \alpha_{ni})\epsilon = \epsilon$. (We used the triangle inequality for absolute values and the supposition successively, in the inequalities). So if $E[f_n] \to \alpha$ then $E[g_n] \to \alpha$.

(ii)

$$g_n(\omega) = \sum_{i=1}^{m(n)} \alpha_{n,i} f_i(\omega) \leq \sum_{i=1}^{m(n)} \alpha_{i,n} h(\omega) = h(\omega) \qquad a.s.$$

since $f_i(\omega) \leq h(\omega)$ a.s. for any i. □

2.5. The above results will only be used in the following chapters via the next two lemmas.

Lemma. *Let* $(f_n)_{n=1}^{\infty}$ *be a sequence in* $L^0(\Omega, \mathcal{F}, P; \mathbb{R}^d)$ *with* $\sup_{n\in\mathbb{N}}\{\|f_n(\cdot)\|_2\}$ *almost surely finite. Then there exists a sequence* (g_n) *with the property that for every* $n \in \mathbb{N}$ *we have that* $g_n \in$ *conv*$\{f_n,\ f_{n+1}, \dots\}$*, and which converges almost surely.*

Proof: *Step 1:* First assume that (f_n) is a sequence bounded in $L^2(P;\ \mathbb{R}^d)$. Apply theorem 2 and corollary 4 to obtain a sequence of convex combinations $(g_n)_{n=1}^{\infty}$, $g_n \in$ *conv*$\{f_n, f_{n+1}, \dots\}$ converging in L^2. Take an a.s. converging subsequence, which is as mentioned in note 4 still a sequence of convex combinations from $(f_n)_{n=1}^{\infty}$.

Step 2: Now we relax the assumption of boundedness. Define a "weight" function,

$$w(\omega) := \sup_{n\in\mathbb{N}}\{\|f_n(\omega)\|_{\mathbb{R}^d}\}.$$

Apply step 1 to the sequence (a_n) which is defined to satisfy

$$a_n(\omega) = \begin{cases} (f_n(\omega)/w(\omega)) & \text{if } w(\omega) \neq 0 \\ 0 & \text{otherwise} \end{cases}.$$

(Step 1 applies because every a_n satisfies $\|a_n(\omega)\| \leq 1$ a.s., so $\int_\Omega \|a_n(\cdot)\|^2 dP \leq 1$.

Let's say (c_n) is the sequence of convex combinations from (a_n) converging almost surely to c. Then $(g_n)_{n=1}^{\infty} := (c_n \cdot w)_{n=1}^{\infty}$ is a sequence of convex combinations from $(f_n)_{n=1}^{\infty} = (a_n(\omega) \cdot w(\omega))_{n=1}^{\infty}$ which converges almost surely to $c \cdot w$ □

Remark. This result cannot be strengthened to get convergence of the sequence itself (i.e. not needing convex combinations): let $\Omega = [0, 1]$, P the Lebesgue measure and (r_n) be the Rademacher sequence. That is,

$$r_n(\omega) = \begin{cases} +1 & \text{if the integer part of } \omega \cdot 2^n \text{ is odd} \\ -1 & \text{otherwise} \end{cases}.$$

This sequence cannot have an a.s. convergent subsequence since for any $m \neq n$, we have that $P(r_m \neq r_n) = \frac{1}{2}$, so the subsequence cannot be almost surely Cauchý.

The Rademacher sequence $(r_n)_{n=1}^{\infty}$ also gives an example of a weakly convergent sequence which fails to be convergent. Because it is orthonormal, it does not converge in L^2, as noted above. And by the Bessel inequality if given $f \in L^2$, we have for $(r_n)_{n=1}^{\infty}$ that $\sum_{k=1}^{\infty} |\langle f, r_k\rangle|^2 \leq \|f\|^2 < \infty$. The left hand side being a convergent series, $|\langle f, r_k\rangle| \to 0$ as $k \to 0$. $f \in L^2$ is arbitrary, so (r_n) is weakly convergent in $L^2([0,1], \mathcal{F}, \mu_L)$.

This fact has a bearing on the next theorem, where convergence to a non-zero limit is obtained, because of the following observation. No matter which convex combinations from (r_n) are chosen, the new sequence cannot have an a.s. limit $c \in L^2$ other than $c = 0$ a.s. This can be seen by the following argument. Let $c_n \in conv\{r_n, r_{n+1}, \ldots\}$ for every n, with $c_n \to c$ a.s., and A be an arbitrary member of $\mathcal{F}$. Then we get $\int_A c\, dP = \int_A \lim c_n dP = \lim \int_A c_n dP$, due to the boundedness of the sequence. And the last term must be zero as from the weak convergence of $(r_n)_{n=1}^{\infty}$ we get that $\langle \chi_A, r_n\rangle \to 0$, so (properties, 2.4) $\int_A c_n dP = \langle \chi_A, c_n\rangle \to 0$. That is, for every $A \in \mathcal{F}$ we have that $\int_A c\, dP = 0$, therefore the limit is zero almost surely.

This will be the case, for instance, if we take $c_n = \frac{1}{n}(r_{n+1} + r_{n+2} + \cdots + r_{2n})$ for all n. We know by the strong law of large numbers that if we set $S_k = \sum_{1 \leq j \leq k} R_j$, then $\frac{1}{k} S_k \to 0$ a.s. Now

$$c_n = \frac{1}{n}(S_{2n} - S_n) = \left(2\frac{S_{2n}}{2n} - \frac{S_n}{n}\right) \to 2(0) - 0 = 0.$$

2.6. Under some restrictions a non-zero limit can be found.

Lemma. *Let $(h_n)_{n=1}^{\infty}$ be a bounded sequence in $L^2(\Omega, \mathcal{F}, P; \mathbb{R}^d)$ which stays bounded away from zero in probability, i.e., there is a constant $\alpha > 0$ such that $P(\|h_n(\cdot)\|_2 \geq \alpha) \geq \alpha$, for all $n \in \mathbb{N}$.*

Then there is a bounded sequence $(g_n)_{n=1}^{\infty} \subseteq L^{\infty}(\Omega,\ \mathcal{F},\ P)$ and a sequence of convex combinations $f_n \in conv\{g_n h_n,\ g_{n+1}h_{n+1}, \ldots\}$ such that f_n converges almost surely to some $f_0 \in L^2(\Omega,\ \mathcal{F},\ P;\ \mathbb{R}^d)$ with $f_0 \neq 0$.

Proof: Chebychev's inequality says that for any real-valued random variable X and constant α, we have that

$$E[|X|\,] \geq \alpha P(|X| \geq \alpha).$$

Applying this to $X : \omega \mapsto \|h_n(\omega)\|_2$ we get

$$\begin{aligned} E[\|h_n(\cdot)\|_2] &\geq \alpha P(\|h_n(\cdot)\|_2 \geq \alpha) \\ &\geq \alpha \cdot \alpha = \alpha^2. \end{aligned}$$

So

$$\begin{aligned} \max_i E[|h_n^i(\cdot)|] &\geq \frac{1}{d}\sum_{i=1}^{d} E[|h_n^i(\cdot)|] = \frac{1}{d}E[\sum_{i=1}^{d}|h_n^i(\cdot)|] \\ &= \frac{1}{d}E[\|h_n(\cdot)\|_1] \geq \frac{1}{d}E[k\|h_n(\cdot)\|_2] \geq \frac{k}{d}\alpha^2, \end{aligned}$$

where $k > 0$ is a constant such that $\|(\cdot)\|_1 \geq k\|(\cdot)\|_2$ (which exists by the topological "equivalence" of all norms in finite-dimensional space $\mathbb{R}^d$). There will be such an i for every $n \in \mathbb{N}$. Let i be such that $E[|h_n^i|] \geq 0$ for infinitely many n's (for every $n \in \mathbb{N}$ there will be an $i_n \in \{1, 2, \ldots, d\}$ satisfying (2); $(i_n)_{n=1}^{\infty}$ must have a constant subsequence), and $(h_{nr})_{r=1}^{\infty}$ be the subsequence such that (2) holds for every $r \in \mathbb{N}$. We can assume without loss of generality that the subsequence is also $(h_n)_{n=1}^{\infty}$, since the goal of the argument is to find a sequence of convex combinations from $(k_n)_{n=1}^{\infty} = (g_n h_n)_{n=1}^{\infty}$; and a sequence of convex combinations from a subsequence $(k_{nr})_{r=1}^{\infty}$ is also a sequence of convex combinations from $(k_n)_{n=1}^{\infty}$. Set $g_n(\omega) := sign(h_n^i(\omega))$. Then we have $E[g_n h_n^i] = E[|h_n^i|] > \frac{\alpha^2}{d}$. Define $k = (k_n)_{n=1}^{\infty}$ by $k_n := g_n \cdot h_n$ for every n. The sequence k is bounded in L^2 since $(g_n)_{n=1}^{\infty}$ is, so we can now apply theorem 2, the weak compactness property, to get a weakly convergent subsequence. To that subsequence we apply corollary 4, to find a sequence of convex combinations $(f_n)_{n=1}^{\infty}$ from $(g_n \cdot h_n)_{n=1}^{\infty}$, converging almost surely to say f_0.

Now $(f_n)_{n=1}^{\infty}$ is uniformly integrable as the following shows:

$$\begin{aligned} \int_{\{\|f_n(\cdot)\|_2 > c\}} \|f_n(\cdot)\|_2 dP &= c^{-1}\int_{\{\|f_n(\cdot)\|_2 > c\}} c\|f_n(\cdot)\|_2 \\ &\leq c^{-1}\int_{\{\|f_n(\cdot)\|_2 > c\}} (\|f_n(\cdot)\|_2)^2 dP \\ &\leq c^{-1}\int_{\Omega} (\|f_n(\cdot)\|_2)^2 dP \\ &\leq \frac{M}{c}, \end{aligned}$$

where $M \in \mathbb{R}$ is the L^2 upper bound of the sequence. That is, it holds for every $n \in \mathbb{N}$ that $\int_{\{\|f_n(\cdot)\|_2 > c\}} \|f_n(\cdot)\|_2 dP \leq \frac{M}{c}$, a constant independent of n which converges to 0 if $c \to \infty$. Therefore $(f_n)_{n=1}^{\infty}$ is uniformly integrable.

We have also shown above that $f_n \to f_0$ a.s. and therefore $f_n \to f_0$ in measure. By the Vitali convergence theorem 1.9, we can conclude that $\|f_k - f_0\|_{L^1} \to 0$, therefore $E[|(f_k^i - f_0^i)(\cdot)|] \to 0$ which implies $E[f_k^i] \to E[f_0^i]$. But by construction f_k^i, for any $k \in \mathbb{N}$, is a convex combination of elements with expected value $> \frac{\alpha^2}{d}$, so $E[f_0^i] \neq 0$ and we conclude that $f_0 \neq 0$. □

Remark. It is in general possible for $E[|f|] > 0$ and $E[f] = 0$ to be true simultaneously. That is why multiplication by g_n is needed to ensure that $E[h_n] \neq 0$. In our Rademacher example of above, $g_n = r_n$ will do.

Remark. The usefulness of these two results lies in the fact that convex combinations of elements of a (vector) subspace remain in the subspace, and that convex combinations preserve inequalities in the manner of (2.4).

Chapter 3

Reduction to a "topological problem"

In this chapter we prove our main theorem (3.4), which is a corollary of (3.3). We assume in the proof of (3.3) that under the no-arbitrage assumption we have that $K - L^0_+$ is closed in probability. This fact, that $K - L^0_+$ is closed when assuming no-arbitrage, is shown independently in lemma 6.1. Therefore this chapter reduces our main theorem to lemma 6.1.

3.1. We begin by giving another form of the the no-arbitrage condition, which will be used often. We say that there is a *single-step arbitrage* opportunity if there is a $k \in \mathbb{N}$, $1 \leq k \leq N$, and $h \in L^0(\mathcal{F}_{k-1}; \mathbb{R}^d)$ such that $(h, S_k - S_{k-1}) \in K \cap L^0_+ \setminus \{0\}$.

Theorem. *The process $(S_t)_{t=0}^N$ satisfies the no-arbitrage condition, i.e. $K \cap L^0_+(\Omega, \mathcal{F}, P) = \{0\}$, if and only if there does not exist a single-step arbitrage opportunity.*

Proof: Note that in general the condition "$x \geq 0$ a.s. $\Rightarrow x = 0$ a.s." is equivalent to "$x \notin L^0_+ \backslash \{0\}$".

Suppose $K \cap L^0_+ = \{0\}$. Let $t \in \{1, \ldots, N\}$, and $h \in L^0(\mathcal{F}_{t-1}; \mathbb{R}^d)$ be arbitrary. Let $h_i = 0$ if $i \neq t$, and $h_t = h$. Then for $i \in \{1, \ldots, N\}$ we have that h_i is $\mathcal{F}_{i-1}$-measurable, so $(h, S_t - S_{t-1}) = \sum_{i=1}^N (h_{i-1}, S_i - S_{i-1})$ belongs to K. By supposition,

$(h, S_t - S_{t-1}) \in L^0_+$ would then imply $h = 0$ (almost surely). So there is no single-step arbitrage opportunity.

Now we prove the sufficiency part of the statement by induction. For $n \in \mathbb{N}$, let $P(n)$ be the statement that for all stochastic processes of the type $(S_k)_{k=0}^n$, the absence of single-step arbitrage opportunities implies that $K \cap L^0_+ = \{0\}$. When $k = 1$, $P(k)$ can be shown by observing that the no single-step arbitrage condition is simply that for all $\mathbb{R}^d$-valued $\mathcal{F}_0$-measurable h it holds that $(h, S_1 - S_0) \notin L^0_+ \backslash \{0\}$. And $K \cap L^0_+ = \{0\}$ means that for all $\mathcal{F}_0$-measurable h_0 it is true that $\sum_{i=1}^1 (h_{i-1}, S_i - S_{i-1}) \notin L^0_+ \backslash \{0\}$. So $P(1)$ is true.

Suppose $P(N)$ holds, and we have the no single-step arbitrage condition for the process $(S_k)_{k=0}^{N+1}$. We need to show that $P(N+1)$ holds, i.e. that under this assumption it can be stated that $K \cap L^0_+ = \{0\}$, where

$$K = \Big\{ \sum_{i=1}^{N+1} (h_{i-1}, S_i - S_{i-1}) | \; h_i \in L^0(\Omega, \mathcal{F}_{i-1}, P; \mathbb{R}^d) \text{ for all } i \in \{1, \ldots, N\} \Big\}.$$

Let $x = \sum_{i=1}^{N+1} (h_{i-1}, S_i - S_{i-1}) \in K \cap L^0_+$, where every h_i satisfies the conditions needed for membership of K.

Now $x = \sum^N + f_{N+1}$, where we define $\sum^N := \sum_{i=1}^N (h_{i-1}, S_i - S_{i-1})$ and $f_{N+1} := (h_N, S_{N+1} - S_N)$.

Now if, case 1, $\sum_N < 0$ holds with positive probability; and if $x \in L^0_+ \setminus \{0\}$ then $\sum_N(\omega) < 0 \Rightarrow f_{N+1}(\omega) > 0$. Let $h^* = \chi_{\{\sum_N < 0\}} h_N$ (where h_N is such that $f_{N+1} = (h_N, S_{N+1} - S_N)$). Then clearly $(h^*, S_{N+1} - S_N) \in L^0_+ \setminus \{0\}$, contradicting the inductive assumption that that there is no single-step arbitrage.

In case 2, $\sum_N \geq 0$ a.s., we have by $P(N)$ that $\sum_N = 0$ a.s. So $x = f_{N+1}$, but $f_{N+1} \notin L^0_+ \backslash \{0\}$, due to absence of single-step arbitrage opportunities; so we conclude (since $x \in K$ is arbitrary) that $K \cap L^0_+ = \{0\}$. □

3.2. The martingale measure Q satisfying $dQ = g \cdot dP$ for some $g \in L^\infty$, will be found by applying a separation theorem. Now if $g \in L^\infty$, it can be seen as a functional both on Ω and on L^1. The former use will be denoted (as usual) $g(\omega)$, and the latter use by $\langle f, g \rangle$ $(= E[fg])$. Roughly the motivation why we need a result like the next theorem is as follows: in order to get $\int_\Omega (S_1(\omega) - S_0(\omega)) dQ = 0$, we need a g which makes all

$\mathcal{F}_0$-measurable h satisfy $\int_\Omega (h(\omega), S_1(\omega) - S_0(\omega)) \cdot g\ dP = 0$. It follows that $g|_K = 0$ must hold. But for Q to be an *equivalent* measure, which can also be scaled to a *probability* measure, we also need $g(\omega) > 0$ a.s.

Theorem (Yan). *Let $C \subseteq L^1$ be a closed convex cone (with basis 0, i.e. if $x \in C$, $\lambda > 0$, then $\lambda x \in C$) containing L^1_-. Suppose $C \cap L^1_+ = \{0\}$. Then there exists a $g \in (L^1)^* = L^\infty$ such that $g|_C \leq 0$ and $g(\omega) > 0$ $P - a.s.$*

Proof: For the fact that the continuous dual of $L^1(P)$ is $L^\infty(P)$, or rather that the map $g \mapsto (f \mapsto E[fg])$ defines an isomorphism between $(L^1)^*$ and L^∞, consult standard texts such as [DS, p.381]. (This fact extends to σ-finite spaces)

Observe that if $g \in L^\infty$ separates C from L^1_+, let's say $\sup\{\langle g, x\rangle | \ x \in C\} = \alpha$, then $\alpha = 0$. This is because C is a cone and g is linear, so if $\langle g, x\rangle = \alpha$ then $\langle g, 2x\rangle = 2\alpha$, thus α cannot be strictly positive. Similarly α cannot be strictly negative, since $\frac{1}{2}x \in C$. Also note that if $g|_C \geq 0$ then $g' := -g$ satisfies $g'|_C \leq 0$, so we can assume $g|_C \leq 0$ if we already have that g separates C from L^1_+. Therefore we only need to show that there is a $g \in L^\infty$ which separates (as a linear functional on L^1) the subsets L^1_+ and C; and that $g(\omega) > 0$ a.s. a.s.

Step 1: We show that every $h \in L^1_+$ can be strictly separated from from C, and the separating functional is an element of L^∞_+. Suppose non-zero $h \in L^1_+$ is given. Then a well known corollary of the Hahn-Banach separation theorem [Robertson & Robertson, p.30], which says that in locally convex spaces a given point can always be strictly separated from a given disjoint closed convex set, implies that there is a $g \in L^\infty$ such that $h \notin closure(g(C))$. By the observations above, therefore, $g|_C \leq 0$ and $\langle g, h\rangle > 0$. And because $L^1_- \subseteq C$, we have that $g|_{L^1_-} \leq 0$. Therefore $g \in L^\infty_+$, for the supposition that on the contrary for $\omega \in A \subseteq \Omega$, $P(A) > 0$ it holds that $g(\omega) < 0$, leads to the contradiction that $\langle g, -\chi_A\rangle > 0$.

Step 2: Now we use step 1 to separate L^0_+ from C by an almost surely positive $g \in L^\infty$. Denote by $\mathcal{G}$ the set $\{g \in L^\infty_+ | \ g|_C \leq 0\}$, and $s := \sup\{P[g(\omega) > 0] | \ g \in \mathcal{G}\}$. Let $(g_n)_{n=1}^\infty \subset \mathcal{G}$ be a sequence such that $P[g_n(\omega) > 0] \to s$. Let

$$g := \sum_{n=1}^{\infty} \frac{g_n}{2^n \|g_n\|_\infty}.$$

Then clearly $g \in \mathcal{G}$, and $P[g(\omega) > 0] \geq P[g_n(\omega) > 0]$ for all n. That is, $P[g > 0] = s$, since s is the supremum. Now let g' be the functional separating $\chi_{\{g(\omega)=0\}}$ from C, and $g'' := g + g'$. Suppose $s < 1$. Then $P[g'' > 0] = P[g > 0] + P[g' > 0] > s$, a contradiction showing $s = 1$. So $g > 0$ a.s., and $g|_C \leq 0$. □

3.3. Here we show a preliminary to the main result of these notes, assuming (as mentioned in the beginning of this chapter), lemma 6.1, which will then be shown independently in the sequel.

Theorem. *Let $S = (S_0, S_1)$ be a stochastic process which is adapted to the filtration $(\mathcal{F}_0, \mathcal{F}_1)$, and suppose it satisfies the no-arbitrage condition. Then there is an equivalent measure Q, under which S is a martingale.*

Proof: Let $Y(\omega) := (S_1(\omega) - S_0(\omega))/w(\omega)$, where

$$w(\omega) := max(\|S_0(\omega)\|_1,\ \|S_1(\omega)\|_1, 1).$$

(Then $Y \in L^\infty$). Theorem 3.1 tells us that that the no-arbitrage assumption simply means that $K \cap L^0_+(\Omega, \mathcal{F}_1, P; \mathbb{R}^d) = \{0\}$. Adding lemma 6.1, this means that $K - L^0_+$ (Algebraic difference: $K - L^0_+ := \{k - l :\ k \in K,\ l \in L^0(\Omega, \mathcal{F}_0, P; \mathbb{R})\}$) is closed in $L^0(\Omega,\ \mathcal{F}_1,\ P)$.

Now look at the subspace topology induced by $L^1(\Omega,\ \mathcal{F}_1,\ P)$, as a subset of $L^0(\Omega,\ \mathcal{F}_1,\ P)$ under its topology of convergence in probability. Because $K - L^0_+$ is closed in L^0, $C := ((K - L^0_+) \cap L^1)$ will be closed in the subspace topology. And the norm-topology on L_1 is finer than the topology of convergence in probability, so C is closed in L^1 (w.r.t. the norm-topology). Furthermore, it will be disjoint from $L^1_+(\Omega, \mathcal{F}_1, P)\backslash\{0\}$, because, as noted in theorem 3.1, the no-arbitrage condition implies that $K \cap L^0_+ = \{0\}$.

In addition, C is convex: K and L_1 are both convex, since they are vector spaces. To show that L^0_+ is a convex set, suppose $x, y \in L^0_+(\mathcal{F}_1)$, so $x(\omega) \geq 0$ a.s. and $y(\omega) \geq 0$ a.s. Then if $z = tx + (1-t)y$, $0 \leq t \leq 1$, then $z \geq 0$ holds trivially. z is also a linear combination of $\mathcal{F}_1$-measurable functions, therefore $z \in L^0_+(\mathcal{F}_1)$, and L^0_+ is convex. Now the algebraic difference of two convex sets is convex, and the intersection of convex sets is convex, so $C = (K - L^0_+) \cap L_1$ is convex.

Finally, since $0 \in K$ it is a fact that $L^1_- = -L^1_+ \subseteq -L^0_+ \subseteq C$.

This allows us to apply Yan's theorem (3.2) to C to get a $g \in L^\infty(\Omega, \mathcal{F}_1, P)$, $g(\omega) > 0$ a.s., such that $g|_C \leq 0$ (g is again treated as a functional on $L^1(\Omega, \mathcal{F}_1, P)$). Now let $h \in L^\infty(\Omega, \mathcal{F}_0, P; \mathbb{R}^d)$. Then $(h, Y) \in L^\infty \subseteq L^1$, so $f := (h, Y) = ((h, Y) - 0) \in ((K - L^0_+) \cap L^1)$. Applying g to f, we get

$$\int_\Omega (h(\omega,\ S_1(\omega) - S_0(\omega)) \cdot g(\omega)/w(\omega))dP = \int (h, Y) \cdot g\ dP$$
$$= \int fg\ dP \geq 0.$$

By repeating the argument for $-h$ instead of h (K is a vector space), we deduce that $\int(h(\omega,\ S_1(\omega) - S_0(\omega)) \cdot g(\omega)/w(\omega))dP = 0$ for all $h \in K$.

Let Q be the measure with density function $c \cdot g(\omega)/w(\omega)$, where $c > 0$ is chosen to satisfy $Q(\Omega) = 1$. Then Q is an equivalent probability measure. And we have seen above that for each $h \in L^\infty(\Omega, \mathcal{F}_0, P; \mathbb{R}^d)$ we get

$$\int_\Omega (h(\omega), S_1(\omega) - S_0(\omega))dQ = 0. \tag{1}$$

This implies that (S_0, S_1) is a martingale under Q and the filtration $(\mathcal{F}_0, \mathcal{F}_1)$ (Reasoning: Suppose $A \in \mathcal{F}_0$. Let $1 \leq i \leq d$ and $h = \chi_A \cdot e_i \in L^\infty(\Omega, \mathcal{F}_0, P; \mathbb{R}^d)$, where e_i is the i^{th} unit vector in $\mathbb{R}^d$. We have $\int(h, Y)dQ = 0$ by the above argument (1), so $(\int_A(S_1 - S_0)dQ)_i = \int_A(S_1 - S_0)_i dQ = \int_\Omega(h, S_1 - S_0)dQ = 0$. This implies that for any $A \in \mathcal{F}_0$, it holds that $(\int_A(S_1 - S_0)dQ)_i = 0$. i is arbitrary, so $\int_A(S_1 - S_0)dQ = 0$; and $A \in \mathcal{F}_0$ is arbitrary, therefore (S_0, S_1) is a martingale) □

3.4. This is the main result of the dissertation.

The Dalang-Morton-Willinger theorem. *Let $S = (S_k)_{k=0}^N$ be a an $\mathbb{R}^d$-valued stochastic process in $(\Omega, \mathcal{F}, P)$ which is adapted to the filtration $(\mathcal{F}_k)_{k=0}^N$. Suppose S satisfies the no-arbitrage condition. Then there exists an equivalent measure Q, under which S is a martingale*

Proof: The main work, except for the "topological" part, was done in the previous lemma. Only induction from the $N = 2$-process to general N remains. Let $P(i)$ be the statement that this theorem is true for all processes with length i (i.e. i time steps, the process consists of $i+1$ random variables). By the previous result, $P(1)$ holds. Suppose $P(M-1)$ is true. Let S be a process of length M satisfying the suppositions of this theorem, say $S = (S_k)_{k=0}^{M}$, and $S' = (S_k)_{k=1}^{M}$. Now since $(S_0, \ldots, S_M)$ satisfies the no-arbitrage condition, $(S_1, \ldots, S_M)$ will also satisfy it. So there is an equivalent measure Q' making S' a martingale.

Observe the following. If $g \in L^0$ is $\mathcal{F}_1$-measurable, then the measure $Q = gdQ'$ preserves the martingale property for $(S_k)_{k=1}^{M}$. This is because then, if $1 \leq s \leq t$, we have $E_Q[(S_t - S_0)|\mathcal{F}_s] = E_{Q'}[(S_t - S_s)g|\mathcal{F}_s] = gE_{Q'}[S_t - S_s|\mathcal{F}_s] = 0$.

P and Q' are equivalent, so $S'' = (S_0, S_1)$, which as an $(\mathcal{F}_0, \mathcal{F}_1)$-adapted process in $(\Omega, \mathcal{F}_1, P)$ (strictly speaking P should be restricted in its domain to $\mathcal{F}_1$) doesn't allow arbitrage, will satisfy the no-arbitrage condition in the probability space $(\Omega, \mathcal{F}_1, Q')$ as well. There is therefore a $g \in L^0(\mathcal{F}_1)$ making Q, with $dQ = g\, dQ'$, an equivalent martingale measure for S''. By the previous paragraph, Q is still an equivalent martingale measure for S', so S is a martingale. (For if $0 \leq s < t$, either $s = 0$ or $s \geq 1$. In the first case, $E[S_t|\mathcal{F}_s] = E[E[S_t|\mathcal{F}_1]|\mathcal{F}_0] = E[S_1|\mathcal{F}_0] = S_0$. In the second case the fact that S' is already a martingale under Q implies that $\mathbb{E}[S_t|\mathcal{F}_s] = S_s$. Therefore Q is a martingale under Q and $P(M)$ is true. This finishes the induction proof. $\square$

Chapter 4

Further results independent from the no-arbitrage assumption

For the rest of these notes, we will fix an $Y \in L^0(\Omega, \mathcal{F}_1, P; \mathbb{R}^d)$. We define the following subspaces of $L^0(\Omega, \mathcal{F}_0, P; \mathbb{R}^d)$:

$$N := \{k \in L^0(\Omega, \mathcal{F}_0, P; \mathbb{R}^d) : \ (k(\omega),\ Y(\omega)) = 0 \qquad P - a.s.\},$$

$$N^{(\perp)} := \{h \in L^0(\Omega, \mathcal{F}_0, P; \mathbb{R}^d) : \ (k(\omega),\ h(\omega)) = 0 \ P - a.s. \text{ for each } k \in N\}.$$

Note both these subspaces are closed under multiplication by $\mathcal{F}_0$-measurable scalar-valued random variables.

4.1. **Lemma.** *$N \cap L^2(\Omega, \mathcal{F}_0, P; \mathbb{R})$ and $N^{(\perp)} \cap L^2(\Omega, \mathcal{F}_0, P; \mathbb{R})$ are orthogonal complements in the Hilbert space $H := L^2(\Omega, \mathcal{F}_0, P; \mathbb{R})$.*

Proof. Define $M := N \cap H$ and $M^{(\perp)} := N^{(\perp)} \cap H$. Let $x \in M$, $y \in M^{(\perp)}$. Then $\langle x, y \rangle = E[(x, y)] = 0$ since $(x, y) = 0$ a.s. by definition. So M and $M^{(\perp)}$ are orthogonal. Next we show that they are orthogonal complements. If not, then there will be an $h \in L^2(\Omega, \mathcal{F}_0, P; \mathbb{R})$ such that $h \perp M^{(\perp)}$, $h \notin M$ and $h \neq 0$. Let A be the set $\{(h(\omega), Y(\omega)) > 0\}$. h not being a member of M, $\{(h, Y) \neq 0\}$ is an event with strictly positive probability. So either "$(h, Y) > 0$" or "$(-h, Y) > 0$" happens with strictly positive probability. Therefore, by choosing $-h$ instead of h if necessary, we can assume $P(A) > 0$. Let $g := \chi_A \cdot h$, then $g \notin M$ since "$(\chi_A h, Y) > 0$" is a non-zero event.

Now, on the one hand, $\langle g, Y \rangle = E[(\chi_A h, Y)] = E[(h, \chi_A Y)]$, because χ_A is scalar-valued. And this is zero, because $\chi_A Y \in M^{(\perp)}$ since $Y \in M^{(\perp)}$, and $M^{(\perp)}$ is closed under $\mathcal{F}_0$-measurable scalar multiplication. So $\langle g, Y \rangle = 0$. On the other hand, for $\omega \in A$ we have $(g(\omega), Y(\omega)) = (\chi_A(\omega)h(\omega), y(\omega)) = (h(\omega), Y(\omega)) > 0$ by definition of A; and for $\omega \notin A$ we get $(g(\omega), y(\omega)) = (\chi_A(\omega)h(\omega), Y(\omega)) = 0$. A being of strictly positive probability, this shows that $\langle g, Y \rangle > 0$.

This is a contradiction, showing that M and $M^{(\perp)}$ are orthogonal complements, i.e. $M^\perp = M^{(\perp)}$. □

4.2. M and $M^{(\perp)}$ (as defined in the previous paragraph) are also closed (as Hilbert space sets) subspaces: Let $(x_n)_{n=1}^\infty \subset M$, $x_n \to x$. Then there's a subsequence $(x_{n_r})_{r=1}^\infty$ converging a.s. to x. So $(x_{n_r} - x, Y) \to 0$ a.s. But each (x_{n_r}, Y) equals zero almost surely, a fact which implies that $(x, Y) = 0$ a.s. We have that $x \in M$; in conclusion M is closed. The proof that $M^{(\perp)}$ is closed is the same, just substituting an arbitrary member of N for Y.

4.3. **Theorem.** *(i) N is closed in probability. (ii) $N \cap N^{(\perp)} = \{0\}$.*

Proof: (i) As the topology of convergence in probability is a metric topology, we can work with sequences. Suppose $x_n \to x$ in probability, for $(x_n)_{n=1}^\infty \subseteq N$. Then there is a subsequence, still called $(x_n)_{n=1}^\infty$, converging to x almost surely. Now as in the previous paragraph it must hold that $(x(\omega), Y(\omega)) = 0$ a.s., so $x \in N$.

(ii) If $x \in N \cap N^{(\perp)}$, then by definition $(x(\omega), x(\omega)) = 0$ with probability one, so x can be identified with 0. □

4.4. With the aim of obtaining a suitable projection from L^0 to $N^{(\perp)}$, we define $p : H \to M$ as the orthogonal projection of H onto $M^{(\perp)}$. Then $ker(p) = M$, and p is a bounded linear operator on the Hilbert space. Of course p is continuous (as an operator from the Hilbert space H), but we need to show that p is also continuous if H is given the topology inherited from L^0.

4.5. **Theorem.** *p is a continuous operator from $H \subseteq L^0$ to $N^{(\perp)}$, i.e. p is continuous with respect to convergence in measure.*

Proof: It is first shown that $\|p(x)(\omega)\|_2 \leq \|x(\omega)\|_2$ almost surely for any $x \in H$. Now $x = n + p(x)$ where $n \in N$, $p(x) \in N^{(\perp)}$. By the definition of the subspaces, $(n(\omega), p(x)(\omega)) = 0$ a.s., so $\|x(\omega)\|_2 = \|n(\omega)\|_2 + \|p(x)(\omega)\|_2$ which implies the statement.

Now to show continuity, we assume that $x_n \to x$ in measure and apply this result to $x - x_n$. Recall that a sequence $(x_n)_{n=1}^{\infty}$ converges to x in measure if for every $\epsilon > 0$ it holds that $P[\|(x-x_n)(\cdot)\|_2 \geq \epsilon] \to 0$. Now if, for a given $\omega \in \Omega$, $\|(x-x_n)(\omega)\|_2 \to 0$ then $\|p(x - x_n)(\omega)\|_2 \to 0$, so it is clear that $p(x - x_n) = (p(x) - p(x_n))$ converges to zero in measure, i.e. $p(x_n) \to x$ in measure. □

Fortunately $L^2(\Omega, \mathcal{F}_1, P; \mathbb{R}^d)$ is dense in $L^0(\Omega, \mathcal{F}_1, P; \mathbb{R}^d$, because it contains the simple functions, and $L^0(\mathcal{F}_1)$ is the (convergence in probability) closure of the set of $\mathcal{F}_1$-simple functions). So the projection $p : L^2 \to N^{(\perp)}$ can now be extended to $\pi : L^0 \to N^{(\perp)}$ using the fact that a continuous linear operator in a topological vector space is uniformly continuous, and by using a basic extension result [DS I.6.17], "the principle of extension by continuity". This construction π can then be shown to possess a few important properties.

4.6. **Lemma** *Let $h \in L^0(\mathcal{F}_0)$. Then $(h(\omega), Y(\omega)) = (\pi(h)(\omega), Y(\omega))$ with probability one.*

Proof: In fact π is linear: Let $x, y \in L^0$, then we can write (see the previous paragraph) $x = \lim_n x_n$, $y = \lim_n y_n$ with $(x_n)_{n=1}^{\infty}$, $(y_n)_{n=1}^{\infty} \subseteq L^2$. x_n and y_n being members of L^2, $\pi(x_n + y_n) = p(x_n + y_n) = p(x_n) + p(y_n) = \pi(x_n) + \pi(y_n)$. Taking limits, we see that $\pi(x + y) = \pi(x) + \pi(y)$. Similarly $\pi(\lambda x) = \lambda\pi(x)$ for $\lambda \in \mathbb{R}$.

Now h can be written as $(\pi(h)) + (h - \pi(h))$. So from the linearity of π we get it only needs to be shown that $(h - \pi(h)) \in N$. Now let $(h_n)_{n=1}^{\infty} \subseteq L^2$ such that $h = \lim_n h_n$ in measure. Let $y_n = h_n - \pi(h_n) = h_n - p(h_n)$, then by the continuity of π we get that $(y_n)_{n=1}^{\infty}$ is a sequence in N (projection property of p) converging to $h - \pi(h)$. N is closed in probability, so $h - \pi(h) \in N$. □

4.7. Per definition the range of π is $N^{(\perp)}$, and the fact that the kernel of π is N can be readily shown: If $\pi(x) = 0$ then $(x(\omega), Y(\omega)) =$ (Lemma 4.6) $(\pi(x)(\omega), Y(\omega)) = 0\, a.s.$, so $x \in N$ per definition. Conversely, if $x \in N$ then again $(\pi(x)(\omega), Y(\omega)) = 0$ with probability one, therefore $\pi(x) \in N$. But from the first remark in this paragraph, $\pi(x)$ belongs to $N^{(\perp)}$ as well; this implies that $\pi(x) = 0$.

4.8. Only Lemma 4.6 and the fact that the range of π is $N^{(\perp)}$, will be needed from now on. But it is interesting that π can be explicitly determined.

Lemma. *If $f \in L^0(\mathcal{F}_0; \mathbb{R}^d)$, let $\tilde{p}(f) : \Omega \to \mathbb{R}^d$ be defined as follows. Let $\bar{f}$ be the normalised f; that is,*

$$\bar{f}(\omega) := \begin{cases} \frac{f(\omega)}{\|f(\omega)\|_2} & \text{if } f(\omega) \neq 0 \\ 0 & \text{otherwise} \end{cases}.$$

Let $\tilde{p}$ be the function

$$\omega \mapsto p(\bar{f})(\omega) \cdot \|f(\omega)\|_2.$$

Then $\tilde{p} = \pi$ almost surely.

Proof: We have the following steps: (i)$\tilde{p} = \pi = p$ on the integrable simple functions. (ii) $\tilde{p}$ is continuous w.r.t. convergence in measure.

Step (i): Suppose $y \in L^0(\mathcal{F}_0; \mathbb{R}^d)$ is of the form $y(\omega) = \sum_{i=1}^n c_i \chi_{A_i}(\omega)$, $c_i \in \mathbb{R}^d$, $A_i's$ disjoint. Then $\tilde{p}(f)(\omega) = p(\sum_{i=1}^n \frac{c_i}{\|c_i\|_2} \chi_{A_i}(\omega)) \cdot (\sum_{i=1}^n \|c_i\|_2 \chi_{A_i}(\omega))$ by definition ($\tilde{p}$ works as follows: divide by $\|y(\omega)\|_2$ pointwise, apply p, multiply by $\|y(\omega)\|_2$ pointwise). And this equals $\sum_{i=1}^n p(\frac{c_i}{\|c_i\|_2} \chi_{A_i})(\omega) \cdot (\sum_{i=1}^n \|c_i\|_2 \chi_{A_i}(\omega))$ because p is linear. Now the body of the proof of 4.5 shows that $\|p(f)(\omega)\|_2 \leq \|f(\omega)\|_2$ a.s. for any $f \in L^0$, so if $f(\omega) = 0$ then $p(f)(\omega) = 0$. This implies that $p(\frac{c_i}{\|c_i\|_2} \cdot \chi_{A_i}) =$

$\chi_{A_i} \cdot p(\frac{c_i}{\|c_i\|_2} \cdot \chi_{A_i})$. So

$$\begin{aligned}
\tilde{p}(f)(\omega) =& (\sum_{i=1}^{n} p(\frac{c_i}{\|c_i\|_2} \cdot \chi_{A_i})(\omega) \cdot \chi_{A_i})(\sum_{i=1}^{n} \|c_i\|_2 \chi_{A_i}(\omega)) \ a.s. \\
=& \sum_{i=1}^{n} p(\frac{c_i}{\|c_i\|_2} \cdot \chi_{A_i})(\omega) \cdot \chi_{A_i} \|c_i\|_2 \cdot \chi_{A_i}(\omega)) \qquad \chi_{A_i}\chi_{A_j} = \delta_{ij}\chi_{A_i} \\
=& \sum_{i=1}^{n} (p(c_i \chi_{A_i})(\omega)) \qquad \|c_i\|_2 \text{ a constant} \\
=& p(\sum_{i=1}^{n} c_i \chi_{A_i})(\omega) = p(f)(\omega).
\end{aligned}$$

So $\tilde{p}$ and p agree on the set of integrable simple functions. And since the simple functions belong to L^2 and π is the extension of p from L^2, $\pi = p = \tilde{p}$ on the integrable simple functions.

Step (ii): Given $f_n \to f$ in measure, we want $\tilde{p}(f_n) \to \tilde{p}(f)$ in measure, a condition which is equivalent to requiring that for every subsequence $(\tilde{p}(f_{n_r}))_{r=1}^{\infty}$ there is a further subsequence which converges almost surely. So given such a subsequence, we have that $f_{n_r} \to f$ (in measure) so we can choose a subsequence denoted $(g_k)_{k=1}^{\infty}$ such that $g_k \to f$ a.s. From this convergence we can get that $\bar{g}_k \to \bar{f}$ a.s. since if $g_k \to f$ a.s. then $\|g_k(\cdot)\|_2 \to \|f(\cdot)\|_2$ a.s., and the quotient of two sequences converging a.s. converges a.s. to the quotient of the limits provided the denominator does not equal zero (and we can assume $\|f(\omega)\|_2 \neq 0$ because for the ω for which it *is* zero $\bar{g}_k(\omega) \to \bar{f}(\omega)$ holds anyway). Now $(\bar{g}_k)_{k=1}^{\infty}$ is a bounded sequence, so an application of dominated convergence gives that $g_k \to f$ in L^2, so $p(g_k) \to p(f)$ in L^2. By a standard convergence theorem there is yet another subsequence, still called $(g_k)_{k=1}^{\infty}$ such that $p(\bar{g}_k) \to p(f)$ a.s. For the set, which has probability one, in which this convergence takes place pointwise, we have that $\tilde{p}(f)(\omega) = p(\bar{f})(\omega) \cdot \|f(\omega)\|_2 = \lim_{k\to\infty} p(\bar{g}_k(\omega) \cdot \|g_k(\omega)\|_2$.

Now that π and $\tilde{p}$ are both continuous extensions of p from a set (the integrable simple functions) which is dense in L^0, the fact that such extensions are unique ([DS I.6.17] again) implies that $\tilde{p} = \pi$ on L^0. □

Chapter 5

Two consequences of the no-arbitrage assumption

5.1. Let $(\Omega, \mathcal{F}_1, P)$ be a probability space, Y an $\mathcal{F}_1$-measurable $\mathbb{R}^d$-valued function, i.e., $Y \in L^0(\Omega, \mathcal{F}_1, P; \mathbb{R}^d)$ and $\mathcal{F}_0$ a sub-σ-field of $\mathcal{F}_1$. Let $K := \{(h, Y) : h \in L^0(\Omega, \mathcal{F}_0, P; \mathbb{R}^d)\}$. For the rest of this manuscript, we assume the no-arbitrage assumption, i.e. $K \cap L_0^+ = \{0\}$, with our specific choice of $Y(\omega) = (S_1(\omega) - S_0(\omega))/w(\omega)$, $w(\omega)$ as defined in 3.3. However, all the results of the remaining chapter hold for any $Y \in L^\infty(\Omega, \mathcal{F}_1, P; \mathbb{R}^d)$. In other words, we only use the fact that $K \cap L_+^0 = \{0\}$. It enters the argument through the following. We define the following complementary subsets of Ω.

$$\Omega_N := \{\omega \in \Omega | \ \mathbb{E}[\|Y(\cdot)\|_2 \ |\mathcal{F}_0](\omega) = 0\},$$

$$\Omega_N^c := \{\omega \in \Omega | \ \mathbb{E}[\|Y(\cdot)\|_2 \ |\mathcal{F}_0](\omega) > 0\}.$$

Lemma. *For $A \in \mathcal{F}_0$, $A \subseteq \Omega_N^c$ define*

$$\alpha(A) = \inf\{E[(h(\omega), Y(\omega))_+] : \ h \in N^{(\perp)}, \|h(\omega)\| = 1 \ for \ \omega \in A\}.$$

Then α is well defined and $P(A) > 0$ implies $\alpha(A) > 0$.

Proof: Suppose to the contrary that there exists a sequence $(h_n)_{n=1}^\infty \subseteq N^{(\perp)}$ such that $E[(h_n, Y)_+] \to 0$ as $n \to 0$. It can be assumed straight away that $\|h_n(\omega)\| = 0$

for all n and $\omega \in \Omega \setminus A$, because multiplication by χ_A will leave all the other properties intact.

Part 1: (From $E[(h_n, Y)_+] \to 0$, we get $E[(h_n, Y)_-] \to 0$)
Now $(h_n)_{n=1}^{\infty}$ is bounded so lemma 2.5 (page 12) can be applied to get a sequence $(g_n)_{n=1}^{\infty}$ of convex combinations from $(h_n)_{n=1}^{\infty}$, converging a.s. to say h.

And by theorem 2.4, and the fact that every $(g_n, Y)_+$ is a convex combination of $\{(h_m, Y)_+ : m \geq n\}$, it is the case that $E[(g_n, Y)_+] \to 0$. Now the fact that $g_n \to h$ a.s. implies that $E[(g_n, Y)_+] \to E[(h, Y)_+]$, so $E[(h, Y)_+] = 0$. *And here the assumption of no-arbitrage forces* $E[(h, Y)_-]$ *to be zero as well.* For otherwise, setting $f := -h$, we get $E[(f, Y)_+] > 0$ and $E[(f, Y)_-] = 0$, which means that $f \in K \cap L^0_+ \setminus \{0\}$, i.e. arbitrage.

Now $\lim E[(h_n, Y)_-] = \lim E[(g_n, Y)_-]$, because $((g_n, Y)_-)_{n=1}^{\infty}$ is a sequence of convex combinations from $((h_n, Y)_-)_{n=1}^{\infty}$. And $(g_n(\omega), Y(\omega)) \leq \|g_n(\omega)\|_{\mathbb{R}^d} \|Y(\omega)\|_{\mathbb{R}^d}$ by the Cauchy-Scwartz inequality, so the boundedness of $(h_n)_{n=1}^{\infty}$ (therefore also of $(g_n)_{n=1}^{\infty}$) and Y imply that we can use the Lebesgue dominated convergence theorem to get

$$\lim_{n\to\infty} E[(g_n, Y)_-] = E[(\lim_{n\to\infty} g_n, Y)_-] = E[(h, Y)_-] = 0.$$

Therefore $E[(h_n, Y)_-] \to 0$. So

$$E[|(h_n, Y)|] = E[(h_n, Y)_+] - E[(h_n, Y)_-] \to 0,$$

and by the boundedness we can extract a subsequence, also called $((h_n, Y))_{n=1}^{\infty}$, which converges to zero almost surely.

Part 2: (Application of a convergence theorem)
The fact that $\|h\|_n = 1$ on A, $P(A) > 0$, means $(h_n)_{n=1}^{\infty}$ is "bounded away from zero in probability", therefore the result of section 2.6 is applicable. Let $f_n \in conv\{g_n h_n, g_{n+1} h_{n+1}, \dots\}$ be the resulting sequence, $(g_n)_{n=1}^{\infty}$ the bounded sequence which is multiplied with $(h_n)_{n=1}^{\infty}$, and $f_n \to f_0$ ($\neq 0$) a.s.

On the one hand, in general, if $(b_n)_{n=1}^{\infty}$ is a bounded sequence of $\mathbb{R}^1$-valued random variables, and $(x_n)_{n=1}^{\infty} \to 0$ a.s., then $b_n x_n \to 0$ a.s. In this case, the fact that $(h_n, Y) \to 0$ a.s. implies that $(g_n h_n, Y) \to 0$ as well, and the same with (f_n, Y). So

$(f_0(\omega), Y(\omega)) = (\lim_{n\to\infty} f_n(\omega), Y(\omega)) = \lim(f_n(\omega), Y(\omega)) = 0$ a.s. This establishes that $f_0 \in N$.

On the other hand every $g_n h_n \in N^{(\perp)}$, and as noted above $N^{(\perp)}$ is a closed subspace, so $f_0 \in N^{(\perp)}$.

This is a contradiction: f_0 belongs to $N \cap N^{(\perp)} = \{0\}$ and was selected to be non-zero. □

5.2. **Lemma.** *Let $(h_n)_{n=1}^{\infty}$ be a sequence in $N^{\perp}$ such that*

$$\Psi(\omega) := \sup_{n\in\mathbb{N}}\{(h_n(\omega),\ Y(\omega))_+\}$$

is finite almost surely. Then

$$\Phi(\omega) := \sup_{n\in\mathbb{N}}\{\|h_n(\omega)\|\}$$

is finite almost surely too.

Proof: Observe that (under the hypotheses of the lemma) if $\|h_n(\omega)\|_2 \to \infty$ (as $n \to \infty$) for all $\omega \in A$, $P(A) > 0$, then $(\bar{h}_n)$, defined by $\bar{h}_n(\omega) := \frac{h_n(\omega)}{\|h_n(\omega)\|_2} \cdot \chi_A$, will satisfy the following: $(\bar{h}_n)_{n=1}^{\infty} \subseteq N^{\perp}$, $\|\bar{h}_n(\omega)\| = 1$ for $\omega \in A$ and $(\bar{h}_n(\omega), Y(\omega))_+ = \frac{1}{\|h_n(\omega)\|_{\mathbb{R}^d}}(h_n(\omega), Y(\omega))_+ \leq \frac{1}{\|h_n(\omega)\|_{\mathbb{R}^d}} \cdot \Psi(\omega) \to 0$ for $\omega \in A$. Then $E[(h_n, Y)_+] \to 0$. This contradicts the previous lemma. However, we have to be more sophisticated because to prove the theorem by assuming the converse, we do not have $\lim h_n(\omega) = \infty$ but $\limsup_{n\to\infty} h_n(\omega) = \infty$ for all $\omega \in A$.

So suppose $\Phi(\omega) = \infty$ for $\omega \in A$, and that $P(A) > 0$. Define for every $n \in \mathbb{N}$,

$$\bar{h}_n(\omega) = \frac{h_m(\omega)}{\|h_m(\omega)\|_2},$$

where m is a specific integer depending on ω and n, namely $m = \min\{k \geq n : \|h_n(\omega)\|_{\mathbb{R}^d} \geq n\}$. Or equivalently we can also define $\bar{h}_n$, to make its' measurability clear, as follows. For $m, n \in \mathbb{N}$ let $A_{m,n} := A \cap \{\|h_m \geq n\|_{\mathbb{R}^d}\}$. We know that "$(\forall n \in \mathbb{N})(\exists m \geq n)(\|h_m\| \geq n)$" holds, so $\cup_{m=n}^{\infty} A_{m,n} = A$. Define

$$\bar{h}_n(\omega) := \sum_{m=n}^{\infty} \frac{h_m(\omega)}{\|h_m(\omega)\|_2} \cdot \chi_{A_{m,n} \setminus \cup_{k=1}^{m} A_{k,n}}.$$

Then $(\bar{h}_n)$ is a sequence such that $(\bar{h}_n) \subseteq N^{\perp}$, $\|h_n(\omega)\|_2 = 1$ for all $\omega \in A, n \in \mathbb{N}$ and $(\bar{h}_n(\omega), Y(\omega))_+ \to 0$ a.s. But now an application of Lebesgue's dominated convergence theorem (Ψ being the dominating function) gives that $E[(h_n, Y)_+] \to 0$ a.s. This contradicts the previous lemma, so under the suppositions of the statement of this lemma we cannot have a set A with positive measure such that $\Phi(\omega) = \infty$ for $\omega \in A$.
□

Chapter 6

The key result

Finally we are in a position to show the lemma on which the arbitrage theorem stands. The task, to show that $K - L^0_+$ is closed in probability, can of course not be achieved by direct compactness arguments, since both K and L^0_+ are unbounded. The results of the previous two chapters, however, allow us to put an upper bound on the sequence of norms $(\|h_n(\omega)\|)_{n=1}^{\infty}$ if convergence in probability is given. Note that K is a (vector) subspace, and L^0_+ a cone, in the sense that if $x \in L^0_+, \lambda > 0$ then $\lambda x \in L^0_+$.

For purpose of comparison with the arbitrage theorem in *discrete* probability spaces, it should be stated that then we have that $K - L^0_+$ is a cone in the *finite*-dimensional space $\mathbb{R}^d$. Then it is an easy fact that $K - L^0_+$ is closed, since $K - L^0_+$ is an intersection of a finite-dimensional subspace (which is closed because all finite-dimensional spaces are) with a closed orthant obtained by restricting certain coordinates to $(-\infty, 0]$. In the *infinite*-dimensional case that is the most difficult part; it has consumed most of the pages of the thesis to show that $K - L^0_+$ is closed in probability.

6.1. **Lemma.** *If $K \cap L^0(\Omega, \mathcal{F}_1, P; \mathbb{R}^d) = \{0\}$, then $K - L^0_+(\Omega, \mathcal{F}_1, P)$ is closed in $L^0(\Omega, \mathcal{F}_1, P)$ with respect to the topology of convergence in measure.*

Proof: Given a sequence $(x_n)_{n=1}^{\infty} \subseteq K - L^0_+$ such that $x_n \to x$ we have to show that $x \in K - L^0_+$. Let for each n, $h_n \in L^0(\Omega, \mathcal{F}_0, P; \mathbb{R}^d)$ and $l_n \in L^0(\Omega, \mathcal{F}_0, P; \mathbb{R})$ be such that $x_n = (h_n, Y) - l_n$; we have to show that there exists $h \in L^0(\Omega, \mathcal{F}_0, P; \mathbb{R}^d), l \in L^0(\Omega, \mathcal{F}_0, P; \mathbb{R})$ such that $x = (h, Y) - l$. Now because the convergence is in probability, there is a subsequence, which can be denoted $x_n = (h_n, Y) - l_n$ as well without loss of generality, converging almost surely to x. And l being positive almost surely, it is enough to show that there exists an $h \in L^0(\Omega, \mathcal{F}_0, P; \mathbb{R}^d)$ such that $(h(\omega), Y(\omega)) \geq x(\omega)$ a.s.

We have $(-h_n(\omega), Y(\omega)) = -x_n(\omega) - l_n(\omega) \leq -x_n(\omega) < \infty$ a.s. By the previous lemma $\|h_n(\omega)\|_{\mathbb{R}^d} = \|-h_n(\omega)\|_{\mathbb{R}^d} < \infty$ a.s. So we can apply lemma 2.5 to get a sequence of convex combinations $(g_n)_{n=1}^{\infty}$ from $(f_n)_{n=1}^{\infty}$ converging to g almost surely. Now $(g_n(\omega), Y(\omega)) = (\sum_{i=1}^{k(n)} \alpha_{n,i} h_i(\omega), Y(\omega))$. Observe that for every $n \in \mathbb{N}$ we have that $(h_n(\omega), Y(\omega)) \geq x_n$ a.s., by definition. Thus $(\sum_{i=1}^{k(n)} \alpha_{n,i} h_i(\omega), Y(\omega)) \geq \sum_{i=1}^{k(n)} \alpha_{n,i} x_i$ a.s. Taking limits as $n \to \infty$,

$$\begin{aligned} (g(\omega), Y(\omega)) &= \lim_{n\to\infty} (g_n(\omega), Y(\omega))) \\ &\geq \lim_{n\to\infty} \sum_{i=1}^{k(n)} \alpha_{n,i} x_i \\ &= x \end{aligned}$$

as was to be shown. □

Notation

d	Number of assets
T	Final time period
$sign(x)$	$\begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$
x_+	$max\{x, 0\}, \qquad x \in \mathbb{R}$
x_-	$max\{-x, 0\}, \qquad x \in \mathbb{R}$
$N,\ N^{(\perp)}$	page 22
$L^0,\ L^p,\ etc.$	page 4
$S_t^i(\omega),\ 0 \leq t \leq T,\ 1 \leq i \leq d$	page 3
$A - B$	Algebraic difference, p. 19

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