On zeros of hypergeometric polynomials

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by

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Declaration

I, the undersigned, hereby declare that this dissertation submitted herewith for the degree Magister Scientiae to University of Pretoria contains my own, independent work and has not been submitted for any degree at any other university.

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Abstract

Our focus, in this thesis, is on zeros of hypergeometric polynomials. Several problems in various areas of science can be seen in terms of the search of zeros of functions; and this search can be reduced to finding the zeros of approximating polynomials, since under some conditions, functions can be approximated by polynomials. In this thesis, we consider the zeros of a specific polynomial, namely the hypergeometric polynomial.
We review some work done on the zero location and the asymptotic zero distribution of Gauss hypergeometric polynomials with real parameters.
We extend some contiguous relations of ${}_2F_1$ functions, and then we deduce the zero location for some classes of Gauss polynomials with non-real parameters.
We study the asymptotic zero distribution of some classes of ${}_3F_2$ polynomials that extend results in the literature.

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Chapter 1

Hypergeometric functions

1.1 Definition and Remarks (cf. [26], p.45)

1.1.1 Definition

Let a, b and c be complex numbers such that $c \notin \{0, -1, -2, -3, \ldots\}$. The geometric series

$$\sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} z^k \tag{1.1}$$

is called the Gauss hypergeometric function, where $(\alpha)_k$ is the Pochhammer symbol or shifted factorial and is defined by:

$$(\alpha)_k = \begin{cases} 1 & \text{if } k = 0 \text{ and } \alpha \neq 0 \\ \alpha(\alpha+1)\ldots(\alpha+k-1) & \text{if } k \in \mathbb{N} \end{cases} .$$

1.1.2 Notation and Remarks

- The hypergeometric function in (1.1) is usually denoted by ${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right)$, to exhibit the numerator and denominator parameters.

- The series (1.1) converges:

$$\begin{cases} \text{if } |z| < 1 \text{ and } a, b \neq 0, -1, -2, \ldots \\ \text{if } |z| = 1 \text{ and } Re(c-a-b) > 0 \end{cases}$$

- For $n, k \in \mathbb{N}$, we have

$$(-n)_k = \begin{cases} (-1)^k \frac{n!}{(n-k)!} & \text{for } 0 \leq k \leq n \\ 0 & \text{for } k \geq n+1 \end{cases}$$

  Hence, if either a or b, or both, is zero or a negative integer, the series (1.1) terminates, and then convergence does not enter the discussion. For instance, if $n \in \mathbb{N}$,

$${}_2F_1\left(\begin{matrix} -n, b \\ c \end{matrix}; z\right) = 1 + \sum_{k=1}^{\infty} \frac{(-n)_k (b)_k}{(c)_k k!} z^k,$$

  which is a polynomial of degree n in z; and is called the hypergeometric polynomial.

1.2 Euler integral (cf. [26], p.47)

If $Re(c) > Re(b) > 0$ and $|z| < 1$, then the series (1.1) can be represented in the integral form

$${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} dt$$

where Γ is the gamma function.

1.3 The contiguous function relations (cf. [26], p.50)

In [19], Gauss defined each of the six functions obtained by increasing or decreasing one of the parameters a, b and c by unity as being contiguous to

${}_2F_1\left(\begin{array}{c} a,b \\ c \end{array};z\right)$. Clearly, these are:

$$
\begin{array}{ccc}
{}_2F_1\left(\begin{array}{c} a+1,b \\ c \end{array};z\right); & {}_2F_1\left(\begin{array}{c} a,b+1 \\ c \end{array};z\right); & {}_2F_1\left(\begin{array}{c} a,b \\ c+1 \end{array};z\right); \\
{}_2F_1\left(\begin{array}{c} a-1,b \\ c \end{array};z\right); & {}_2F_1\left(\begin{array}{c} a,b-1 \\ c \end{array};z\right); & {}_2F_1\left(\begin{array}{c} a,b \\ c-1 \end{array};z\right).
\end{array}
$$

We denote ${}_2F_1\left(\begin{array}{c} a+1,b \\ c \end{array};z\right)$ by $F(a^+)$, ${}_2F_1\left(\begin{array}{c} a-1,b \\ c \end{array};z\right)$ by $F(a^-)$ and so on; and we shall also denote ${}_2F_1\left(\begin{array}{c} a,b \\ c \end{array};z\right)$ simply by F that means, we shall omit the subscripts.

Gauss proved that between F and any two of its contiguous functions, there exits a linear relation with coefficients at most linear in z.

Some of these contiguous relations are

$$(a-b)F = aF(a^+) - bF(b^+) \tag{1.2}$$

$$(a-c+1)F = aF(a^+) - (c-1)F(c^-) \tag{1.3}$$

$$(1-z)F = F(a^-) - c^{-1}(c-b)zF(c^+) \tag{1.4}$$

$$(b-a)(1-z)F = (c-a)F(a^-) - (c-b)F(b^-) \tag{1.5}$$

1.4 Some transformations (cf. [26], p.58; [28], p.110)

Hypergeometric functions satisfy both linear identities and quadratic transformations, the latter only under certain necessary and sufficient constraints on the parameters a, b and c.

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Using the relation (cf.[26], eqn.1, p. 58)

$$(1-y)^{-a} = \sum_{k=0}^{\infty} \frac{(a)_k}{k!} y^k$$

and standard manipulations of Pochhammer symbol, one obtains

Theorem 1.4.1 (cf. [26], Theorem 20, p.60) *If* $|z| < 1$ *and* $|z/(1-z)| < 1$, $c > b$,

$${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right) = (1-z)^{-a} {}_2F_1\left(\begin{matrix} a, c-b \\ c \end{matrix}; \frac{-z}{1-z}\right)$$

If one observes that the numerator parameters in a hypergeometric series are always interchangeable, and applies the above transformation to the right hand side of the above identity, one has

Theorem 1.4.2 (cf. [26], Theorem 21, p.60) *For* $|z| < 1$,

$${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right) = (1-z)^{c-a-b} {}_2F_1\left(\begin{matrix} c-a, c-b \\ c \end{matrix}; z\right)$$

Theorem 1.4.3 (cf. [15]) *For* $n \in \mathbb{N}$ *and* $c, 1-b-n \notin \{0, -1, -2, \ldots -n+1\}$, *one has*

$${}_2F_1\left(\begin{matrix} -n, b \\ c \end{matrix}; z\right) = \frac{(b)_n}{(c)_n} (-z)^n {}_2F_1\left(\begin{matrix} -n, 1-c-n \\ 1-b-n \end{matrix}; \frac{1}{z}\right)$$

1.5 Relation between hypergeometric functions of z and $1-z$ (cf. [26], p.61)

The hypergeometric differential equation

$$z(1-z)w'' + [c - (a+b+1)z]w' - abw = 0 \tag{1.6}$$

has the linearly independent solutions

$$
\begin{aligned}
w_1 &= {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} ; z\right) \\
w_2 &= z^{1-c}{}_2F_1\left(\begin{matrix} a+1-c, b+1-c \\ 2-c \end{matrix} ; z\right)
\end{aligned}
$$

on $|z| < 1$.

On the other hand, let us introduce the new variable $t = 1 - z$ in (1.6). The hypergeometric differential equation becomes

$$
t(1-t)\omega'' + [(a+b+1-c) - (a+b+1)t]\omega' - ab\omega = 0
$$

which admits ${}_2F_1\left(\begin{matrix} a, b \\ a+b-c+1 \end{matrix} ; 1-z\right)$ as a solution. Therefore, there must exist constants A and B, such that

$$
\begin{aligned}
&{}_2F_1\left(\begin{matrix} a, b \\ a+b-c+1 \end{matrix} ; 1-z\right) \\
&= A{}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} ; z\right) + Bz^{1-c}{}_2F_1\left(\begin{matrix} a+1-c, b+1-c \\ 2-c \end{matrix} ; z\right)
\end{aligned}
\tag{1.7}
$$

for all $|z| < 1$ and $|1-z| < 1$. After calculations, one obtains that

$$
\begin{aligned}
A &= \frac{\Gamma(a+b+1-c)\Gamma(1-c)}{\Gamma(a+1-c)\Gamma(b+1-c)} \\
B &= \frac{\Gamma(a+b+1-c)\Gamma(c-1)}{\Gamma(a)\Gamma(b)};
\end{aligned}
$$

substituting A, and B in (1.7), we have the following theorem.

Theorem 1.5.1 (cf. [26], Theorem 22, p.62) *If $|z| < 1$ and $|1-z| < 1$, if $Re(c) < 1$ and $Re(c-a-b) > 0$, and if none of a, b, c, $c-a$, $c-b$,*

$c-a-b$ is an integer,

$$
\begin{aligned}
&{}_2F_1\left(\begin{matrix} a, b \\ a+b+1-c \end{matrix} ; 1-z\right) \\
&\quad = \frac{\Gamma(a+b+1-c)\Gamma(1-c)}{\Gamma(a+1-c)\Gamma(b+1-c)} {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} ; z\right) \\
&\qquad + \frac{\Gamma(a+b+1-c)\Gamma(c-1)}{\Gamma(a)\Gamma(b)} . z^{1-c} {}_2F_1\left(\begin{matrix} a+1-c, b+1-c \\ 2-c \end{matrix} ; z\right)
\end{aligned}
$$

In the next chapter we shall need the following special case of Theorem 1.5.1 for the case $a=-n$.

Theorem 1.5.2 (cf.[1], p.79, eqn. (2.3.14)) *For $n \in \mathbb{N}$, and $c,\ b+1-c-n \notin \{0, -1, \ldots, -n+1\}$,*

$$
{}_2F_1\left(\begin{matrix} -n, b \\ b+1-c-n \end{matrix} ; 1-z\right) = \frac{\Gamma(1-c)\Gamma(b+1-c-n)}{\Gamma(-n+1-c)\Gamma(b+1-c)} {}_2F_1\left(\begin{matrix} -n, b \\ c \end{matrix} ; z\right)
$$

Or, more elegantly:

$$
{}_2F_1\left(\begin{matrix} -n, b \\ b+1-c-n \end{matrix} ; 1-z\right) = \frac{(c)_n}{(c-b)_n} {}_2F_1\left(\begin{matrix} -n, b \\ c \end{matrix} ; z\right) \tag{1.8}
$$

This is called Pfaff's identity.

1.6 Rodrigues' formula for ${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} ; z\right)$

This formula is actually due to Jacobi, but formulas of this type which are derived usually from the underlying differential equation are called Rodrigues' formulas.

To prove this formula without using the differential equation, let us recall Leibnitz' formula for the n^{th} derivative of the product of two functions, given by $\displaystyle \frac{d^n}{dx^n} f(x)g(x) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x) g^{(n-k)}(x)$.

Theorem 1.6.1 (Rodrigues formula cf. [26], p. 161) *For $n \in \mathbb{N}$, b and c complex numbers such that $c \neq 0, -1, \ldots -n+1$,*

$$ {}_2F_1\left(\begin{matrix} -n, b \\ c \end{matrix}; x\right) = \frac{x^{1-c}(1-x)^{c-b+n}}{(c)_n} \frac{d^n}{dx^n}[x^{c+n-1}(1-x)^{b-c}] $$

Proof Let $f(x) = (1-x)^{b-c}$ and $g(x) = x^{c+n-1}$. Then, for $0 \leq k \leq n$,

$$ \begin{aligned} f^{(k)}(x) &= (c-b)_k (1-x)^{b-c-k} \\ g^{(n-k)}(x) &= \frac{(c)_n}{(c)_k} x^{c+k-1} \end{aligned} $$

So $\displaystyle \frac{x^{1-c}(1-x)^{c+n-b}}{(c)_n} \frac{d^n}{dx^n}[x^{c+n-1}(1-x)^{b-c}]$

$$ \begin{aligned} &= \frac{x^{1-c}(1-x)^{c+n-b}}{(c)_n} \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x) g^{(n-k)}(x) \\ &= \frac{x^{1-c}(1-x)^{c+n-b}}{(c)_n} \sum_{k=0}^{n} \binom{n}{k} (c-b)_k (1-x)^{b-c-k} \frac{(c)_n}{(c)_k} x^{n+k-1} \\ &= \sum_{k=0}^{n} \binom{n}{k} \frac{(c-b)_k}{(c)_k} (1-x)^n \left(\frac{x}{1-x}\right)^k \\ &= (1-x)^n \sum_{k=0}^{n} \frac{(-n)_k (c-b)_k}{(c)_k k!} \left(\frac{-x}{1-x}\right)^k \\ &\quad \text{since} \quad (-n)_k = \begin{cases} 0 & \text{if } k \geq n+1 \\ \frac{(-1)^k n!}{(n-k)!} & \text{for } 0 \leq k \leq n, \end{cases} . \\ &= (1-x)^n {}_2F_1\left(\begin{matrix} -n, c-b \\ c \end{matrix}; \frac{-x}{1-x}\right) \\ &= {}_2F_1\left(\begin{matrix} -n, b \\ c \end{matrix}; x\right) \text{ by Theorem 1.4.1} \end{aligned} $$

1.7 The function ${}_pF_q$ (cf. [26], p. 73)

In the last chapter of this thesis we shall deal with more general hypergeometric polynomials. So we give here the general definition of hypergeometric functions.

Definition 1.7.1 *Let $p,\ q \in \mathbb{N}$ and $a_1,\ a_2,\ \ldots,\ a_p,\ b_1,\ b_2,\ \ldots,\ b_q$ be complex numbers such that $b_1,\ b_2,\ \ldots,\ b_q \notin \{0, -1, -2, \ldots\}$. The function*

$$ {}_pF_q\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}; z\right) := 1 + \sum_{k=1}^{\infty} \frac{(a_1)_k \ldots (a_p)_k}{(b_1)_k \ldots (b_q)_k k!} z^k \tag{1.9}$$

where $(a)_k$ is the Pochhammer symbol, is called the generalized hypergeometric function.

1.7.1 Remark

$$\text{Let us set} \quad c_n := \frac{(a_1)_n \ldots (a_p)_n}{(b_1)_n \ldots (b_q)_n n!}.$$

Then

$$\frac{c_n}{c_{n+1}} = \frac{(b_1 + n) \ldots (b_q + n)(n + 1)}{(a_1 + n) \ldots (a_p + n)}.$$

So,

$$\lim_{n\to\infty} \frac{c_n}{c_{n+1}} = \begin{cases} \infty & \text{if} \quad p \le q \\ 1 & \text{if} \quad p = q + 1 \\ 0 & \text{if} \quad p > q + 1 \end{cases}$$

Hence, the series (1.9)

- converges if $p \le q$ for all $z \in \mathbb{C}$;
- converges for $|z| < 1$ if $p = q + 1$ and diverges for $|z| > 1$;
- diverges for $z \neq 0$ if $p > q + 1$.

If one of the a_i, $1 \leq i \leq p$, is a negative integer, the series (1.9) terminates.

If $p = 1$ and $q = 2$, we have the Gauss hypergeometric function and if $p = 3$ and $q = 2$, one obtains the hypergeometric function ${}_3F_2$, that will be considered in chapter 5.

1.8 Orthogonal polynomials

Definition 1.8.1 (cf. [26], p. 147) *A set of polynomials $\{\varphi_n(x)\}$, with $n \in \mathbb{N}$, is called a simple set if $\varphi_n(x)$ is a polynomial of degree precisely n in x and the set contains one polynomial of each degree.*

We have the following result

Theorem 1.8.1 (cf.[26], p.147, Theorem 53) *If $\{\varphi_n(x)\}$ is a simple set of polynomials and if $P(x)$ is a polynomial of degree m, there exist constants c_k such that*

$$P(x) = \sum_{k=0}^{m} c_k \varphi_k(x).$$

The c_k are functions of k and of any parameters involved in $P(x)$.

Definition 1.8.2 *Consider a simple set $\{\varphi_n(x)\}$ of real polynomials. If there exits an interval $a < x < b$ and a function $\omega(x) > 0$ on that interval, and if*

$$\int_a^b \omega(x)\varphi_n(x)\varphi_m(x)dx = 0 \text{ for } m \neq n,$$

we say that the polynomials $\varphi_n(x)$ are orthogonal with respect to the weight function $\omega(x)$ over the interval $a < x < b$.

$$\text{Now, } \int_a^b \omega(x)\varphi_n^2(x)dx \neq 0, \text{for all } n, \text{ since } \omega(x) > 0 \text{ and } \varphi_n(x) \text{ real.}$$

An equivalent condition for orthogonality is given by

Theorem 1.8.2 (cf. [26], p. 149, Theorem 54) *Let $\{\varphi_n(x)\}$ be a simple set of real polynomials and $\omega(x) > 0$ on the interval $a < x < b$. $\varphi_n(x)$ are orthogonal with respect to $\omega(x)$ over the interval $a < x < b$ if and only if*

$$\int_a^b \omega(x)x^k\varphi_n(x)dx = 0 \quad \textit{for} \quad k = 0, 1, 2, \ldots, n-1.$$

With regard to zeros of orthogonal polynomials, we can consider the following result.

Theorem 1.8.3 (cf. [28], p.137, Theorem 6.3) *Let $\{\varphi_n(x)\}$ be a simple set of real orthogonal polynomials on the interval $a < x < b$. Then for $n \geq 1$, $\varphi_n(x)$ has n real simple zeros x_k satisfying $a < x_k < b$, for $1 \leq k \leq n$.*

Proof. Let $\omega(x)$ be the weight function of $\{\varphi_n(x)\}_n$ over $[a, b]$; and let n be a naturel number. According to Theorem 1.8.2, considering $n > 0$ and $k = 0$, we have

$$\int_a^b \omega(x)\varphi_n(x)dx = 0$$

Then $\omega(x)\varphi(x)$ must change sign at least once in $a < x < b$. Since $\omega(x) > 0$ on the interval, $\varphi(x)$ has to have at least one zero of odd multiplicity in $a < x < b$.

Let $x_1, x_2, \ldots, x_l$ be the distinct zeros of $\varphi_n(x)$ of odd multiplicity in $a < x < b$. With the above considering, $l \geq 1$. Now let us define

$$q(x) = \prod_{j=1}^{l}(x - x_j)$$

Then $q(x)$ is a real polynomial of degree exactly l, having simple zeros at the zeros of $\varphi_n(x)$ which have odd multiplicity.

Suppose $l \leq n-1$. Then from Theorem 1.8.1, there are constants $c_0, c_1, c_2, \ldots, c_l$ such that

$$q(x) = \sum_{k=0}^{l} c_k\varphi_k(x).$$

So,

$$\begin{aligned}\int_a^b \varphi_n(x)q(x)\omega(x)dx &= \sum_{k=0}^{l} c_k \int_a^b \varphi_k(x)\varphi_n(x)dx \\ &= 0 \quad \text{by virtue of Theorem 1.8.2}\end{aligned}$$

This situation is impossible because by construction of $q(x)$, $\varphi_n(x)q(x)$ does not change sign in $a < x < b$, and $\omega(x) > 0$ on the interval.

We deduce that $l \geq n$. But $q(x)$ has zeros exactly at the zeros of $\varphi_n(x)$ in $a < x < b$ of odd multiplicity, and $\varphi_n(x)$ can not have more than n zeros. Therefore $l = n$ and $\varphi_n(x)$ has n simple zeros in $a < x < b$.

Using the Rodrigues' formula (cf. Theorem 1.6.1), one obtains the following result.

Theorem 1.8.4 ([26], pp. 257-261) *Let* $F = {}_2F_1\left(\begin{array}{c} -n, b \\ c \end{array}; z\right)$ *where* $n \in \mathbb{N}$, *b and c are real such that* $c \notin \{0, -1, -2, \ldots, -n+1\}$; *then*

(1) for $c > 0$ and $b > c+n-1$, F is orthogonal to polynomials of lower degree on the interval $0 < z < 1$ with respect to the weight function $z^{c-1}(1-z)^{b-c-n}$

(2) for $b < 1-n$ and $c < b-n+1$, F is orthogonal to polynomials of lower degree to polynomials of lower degree on $1 \leq z < \infty$ with respect to the weight function $z^{c-1}(1-z)^{b-c-n}$

(3) For $b < 1-n$ and $c > 0$, F is orthogonal to polynomials of lower degree on $-\infty < z \leq 0$ with respect to the weight function $z^{c-1}(1-z)^{b-c-n}$

1.9 Additional result

Since we shall use Euler constant, we need to define it and state an inequality which will be used later.

Definition 1.9.1 (cf. [26], p.8) *The Euler constant γ is defined by*

$$\gamma = \lim_{n\to\infty} (H_n - Logn)$$

in which $H_n = \sum_{k=1}^{n} \frac{1}{k}$

Obviously γ exists and $0 \le \gamma < 1$. Actually, $\gamma \simeq 0.5772$.

Lemma 1.9.1 *For $n \ge 1$,*

$$\sum_{k=1}^{n} \frac{1}{k} - Log(n+1) \le \gamma$$

Proof. For $n \ge 1$ let us set

$$a_n = \sum_{k=1}^{n} \frac{1}{k} - \log(n+1) \quad \text{and} \quad b_n = \sum_{k=1}^{n} \frac{1}{k} - \log(n).$$

The sequences (a_n) and (b_n) above satisfy the following properties:

(i) $a_n < b_n$ for all n, since $\log(n) < \log(n+1)$;

(ii) $\gamma = \lim_{n\to\infty} b_n$, by definition of γ;

(iii) The sequence (a_n) is monotone increasing. In fact, for $n > 1$, we have

$$\begin{aligned} a_n - a_{n-1} &= \frac{1}{n} - \log(n+1) + \log(n) \\ &= \frac{1}{n} - \log\frac{n+1}{n} \\ &\ge \frac{1}{n} - \frac{1}{n} = 0 \end{aligned}$$

Then for n, one gets

$$\begin{aligned} a_n &\le \lim_{n\to\infty} a_n, \quad \text{from} \quad \text{(iii)} \\ &\le \lim_{n\to\infty} b_n, \quad \text{from} \quad \text{(i)} \\ &= \gamma \quad \text{from} \quad \text{(ii)} \end{aligned}$$

Thus, for all $n \ge 1$,

$$\sum_{k=1}^{n} \frac{1}{k} - Log(n+1) \le \gamma.$$

1.10 Brief overview

Our study is on zeros of hypergeometric polynomials. The structure of this thesis is as follows.

In Chapter 2, we shall consider the zero location of ${}_2F_1$ polynomials. We start by considering the more general cases, that is to say, cases for which the parameters are not restricted. The more general result is by F. Klein (cf. [21]), and is a result for zeros of hypergeometric functions (not necessarily polynomials); but his approach is geometric and difficult to penetrate. A more simple approach for polynomials is given in [11]; it is based on the Rodrigues' formula and uses the orthogonality. We then discuss recent results for classes of ${}_2F_1$ polynomials with restrictions on the parameters (cf. [5], [6], [14], [15]). In addition we consider recent results for Jacobi and ultraspherical polynomials and the implication of these results for hypergeometric polynomials.

The goal in Chapter 3 is to study the zero location of some ${}_2F_1$ polynomials with non-real parameters. In this respect, we shall first generalize some contiguous relations of ${}_2F_1$ functions; and then, we shall use these contiguous relations to find the zero location of some classes of ${}_2F_1$ polynomials with non-real parameters.

In Chapter 4, we shall review some results (cf. [5], [8], [10], [14]) on the asymptotic zero distribution of ${}_2F_1$ polynomials.

Chapter 5 is devoted to asymptotic zero distribution of certain ${}_3F_2$ polynomials. We shall start this chapter by giving an alternative proof of the result stated in [13] for the asymptotic zero distribution of

$$ {}_3F_2\left(\begin{array}{c} -n, n+1, \frac{1}{2} \\ b+n+1, 1-b-n \end{array}; z\right). $$

This result will be generalized and we shall deal with the asymptotic zero distribution of some other classes of ${}_3F_2$ polynomials.

Chapter 2

Zero location of ${}_2F_1$ polynomials for real parameters

2.1 Introduction

The problem of finding the zeros of functions is an important one with far-reaching applications in almost all areas of science.

Some important functions can be written in form of hypergeometric functions. For instance (cf. [28], p.109), if $|z| < 1$, one has

$$(1-z)^{-a} = {}_2F_1\left(\begin{matrix} a, b \\ b \end{matrix}; z\right)$$

$$-\frac{\ln(1-z)}{z} = {}_2F_1\left(\begin{matrix} 1, 1 \\ 2 \end{matrix}; z\right)$$

$$\frac{1}{2z}\ln\left(\frac{1+z}{1-z}\right) = {}_2F_1\left(\begin{matrix} \frac{1}{2}, 1 \\ \frac{3}{2} \end{matrix}; z^2\right)$$

$$\frac{\arctan z}{z} = {}_2F_1\left(\begin{matrix} \frac{1}{2}, 1 \\ \frac{3}{2} \end{matrix}; -z^2\right)$$

$$\frac{\arcsin z}{z} = {}_2F_1\left(\begin{matrix}\frac{1}{2},\frac{1}{2}\\ \frac{3}{2}\end{matrix};z^2\right)$$

$$\frac{\ln\left(z+\sqrt{1+z^2}\right)}{z} = {}_2F_1\left(\begin{matrix}\frac{1}{2},\frac{1}{2}\\ \frac{3}{2}\end{matrix};-z^2\right)$$

Also (cf. [1], p.116)

$$\cos(mz) = {}_2F_1\left(\begin{matrix}\frac{m}{2},\frac{-m}{2}\\ \frac{1}{2}\end{matrix};\sin^2(z)\right)$$

$$\sin(mz) = m\sin(z){}_2F_1\left(\begin{matrix}\frac{1+m}{2},\frac{1-m}{2}\\ \frac{3}{2}\end{matrix};\sin^2(z)\right)$$

The problem of finding zeros of functions can often be reduced to finding the zeros of the approximating polynomials, since some functions can be approximated by polynomials (cf. [24]).

To find the zeros of polynomials poses problems. Indeed, if the degree of a polynomial is $n \geq 5$, one does not have a solution by radicals (cf. [17]).

For our work, we are interested in one type of polynomial: the hypergeometric polynomials.

In fact, several classical polynomials can be written in form of hypergeometric polynomials. For instance,

$$P_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha)_n}{n!}{}_2F_1\left(\begin{matrix}-n,1+\alpha+\beta+n\\ 1+\alpha\end{matrix};\frac{1-x}{2}\right),\ \text{for Jacobi's polynomials}$$

$$\frac{n!}{(2\lambda)_n}C_n^{\lambda}(1-2z) = {}_2F_1\left(\begin{matrix}-n,n+2\lambda\\ \lambda+\frac{1}{2}\end{matrix};z\right)\ \text{for ultraspherical polynomials.}$$

In this chapter, our aim is to review work done on the location of zeros of some classes of ${}_2F_1$ polynomials with real parameters.

The chapter is organized as follows.

In Section 2.2, we consider the zero location of ${}_2F_1\left(\begin{matrix}-n,b\\ c\end{matrix};z\right)$ when b and c are not restricted.

In Section 2.3, we shall discuss zero location of ${}_2F_1\left(\begin{matrix}-n,b\\ c\end{matrix};z\right)$ with some restrictions on parameters b and c.

2.2 More general cases

Felix Klein (cf. [21], p.585-590) proved results on zero location for ${}_2F_1$ functions (not necessarily polynomials). Klein's result is stated in the polynomial case as follows.

Let us define the Klein's symbol

$$E(u) = \begin{cases} 0 & \text{if } u \leq 0 \\ [u] & \text{if } u > 0,\ u \notin \mathbb{Z} \\ u-1 & \text{if } u = 1,2,3,\ldots \end{cases}.$$

Theorem 2.2.1 (cf. [21]) *Let* $F = {}_2F_1\left(\begin{matrix}-n,b\\ c\end{matrix};z\right)$ *and*

$$X = E\left\{\frac{1}{2}\left(-|1-c| + |n+b| - |b-c-n| + 1\right)\right\}$$

where E is the Klein's symbol.

(i) If $c \geq 1$, then F has exactly X zeros in $(0,1)$.

(ii) If $c < 1$, then F has either X or $X+1$ zeros in $(0,1)$.

(iii) If $c < 1$ and $c+n-b > 0$, and either $c+n$ or $c-b = 0,-1,-2,\ldots$, then F has X zeros in $(0,1)$.

(iv) *If* $c < 1$ *and* $c + n - b > 0$*, but neither* $c + n$ *nor* $c - b = 0, -1, -2, \ldots$*, then* F *has an even number of zeros in* $(0, 1)$*, if*

$$\Gamma(c)\Gamma(c+n)\Gamma(c-b) > 0 \tag{2.1}$$

and an odd number of zeros in $(0, 1)$*, if the product is negative.*

(v) *If* $c < 1$ *and* $c + n - b < 0$*, then* F *has* X *zeros in* $(0, 1)$*.*

Klein's proof is geometric and difficult to penetrate. A more transparent approach in the polynomial case can be obtained by using the connection between Gauss hypergeometric polynomials and the Jacobi polynomials which are orthogonal, and Hilbert-Klein formulas.

Szegö (cf. [27]) used the representation of Jacobi polynomials in terms of hypergeometric polynomials and he has got the zero location, for ${}_2F_1$ polynomials, which is equivalent to Klein's result.

Indeed, ${}_2F_1$ polynomials and Jacobi polynomials are closely connected by the equation (cf. [25], p.464, eq. 142)

$${}_2F_1\left(\begin{matrix} -n, b \\ c \end{matrix}; z\right) = \frac{n! z^n}{(c)_n} P_n^{(\alpha,\beta)}(1 - \frac{2}{z})$$

where $\alpha = -n - b$ and $\beta = b - c - n$. $P_n^{(\alpha,\beta)}(1 - \frac{2}{z})$ being orthogonal, the number of its real zeros in the intervals $(-\infty, -1)$, $(-1, 1)$ and $(1, \infty)$ are given by Hilbert-Klein formulas (cf. [27], p. 145, Theorem 6.72).

Remark that with the transformation $\omega = 1 - \frac{2}{z}$, the intervals $-\infty < \omega < -1$, $-1 < \omega < 1$ and $1 < \omega < \infty$ become respectively $0 < z < 1$, $1 < z < \infty$ and $-\infty < z < 0$. Thus, the Hilbert-Klein formulas for hypergeometric polynomials can be stated as follows.

Theorem 2.2.2 ([12], Theorem 3.1) *Let* $b,\ c \in \mathbb{R}$ *with* $b, c, c-b \neq 0, -1, \ldots, -n+1$. *Let*

$$\begin{aligned} X &= E\left\{\frac{1}{2}\left(|1-c|-|n+b|-|b-c-n|+1\right)\right\} \\ Y &= E\left\{\frac{1}{2}\left(-|1-c|+|n+b|-|b-c-n|+1\right)\right\} \\ Z &= E\left\{\frac{1}{2}\left(-|1-c|-|n+b|+|b-c-n|+1\right)\right\}. \end{aligned}$$

Then the numbers of zeros of ${}_2F_1\left(\begin{matrix}-n, b \\ c\end{matrix}; z\right)$ *in the intervals* $(1,\infty)$, $(0,1)$ *and* $(-\infty,0)$ *respectively are*

$$N_1 = \begin{cases} 2[(X+1)/2] & \text{if } (-1)^n\binom{-b}{n}\binom{b-c}{n} > 0 \\ 2[X/2]+1 & \text{if } (-1)^n\binom{-b}{n}\binom{b-c}{n} < 0 \end{cases}$$

$$N_2 = \begin{cases} 2[(Y+1)/2] & \text{if } \binom{-c}{n}\binom{b-c}{n} > 0 \\ 2[Y/2]+1 & \text{if } \binom{-c}{n}\binom{b-c}{n} < 0 \end{cases}$$

$$N_3 = \begin{cases} 2[(Z+1)/2] & \text{if } \binom{-c}{n}\binom{-b}{n} > 0 \\ 2[Z/2]+1 & \text{if } \binom{-c}{n}\binom{-b}{n} < 0. \end{cases}$$

Then, one obtains following theorems on the zero location of ${}_2F_1$ polynomials with real parameters b and c.

Theorem 2.2.3 (cf. [21], [12], Theorem 3.2) *Let* $F = {}_2F_1\left(\begin{matrix}-n, b \\ c\end{matrix}; z\right)$ *where* $b,\ c \in \mathbb{R}$ *and* $c > 0$.

(i) For $b > c+n$, *all zeros of* F *are real and lie in the interval* $(0,1)$.

(ii) *For $c < b < c+n$, $c+j-1 < b < c+j$, $j = 1, 2, \dots, n$; F has j real zeros in $(0,1)$. The remaining $(n-j)$ zeros of F are all non-real if $(n-j)$ is even, while if $(n-j)$ is odd, F has $(n-j-1)$ non-real zeros and one additional real zero in $(1,\infty)$.*

(iii) *For $0 < b < c$, all the zeros of F are non-real if n is even, while if n is odd, F has one real zero in $(1,\infty)$ and the other $(n-1)$ zeros are non-real.*

(iv) *For $-n < b < 0$, $-j < b < -j+1$, $j = 1, 2, \dots, n$, F has j real negative zeros. The remaining $(n-j)$ zeros of F are all non-real if $(n-j)$ is even, while if $(n-j)$ is odd, F has $(n-j-1)$ non-real zeros and one additional real zero in $(1,\infty)$.*

(v) *For $b < -n$, all zeros of F are real and negative.*

Theorem 2.2.4 (cf.[12], Theorem 3.3) *Let $F = {}_2F_1\left(\begin{matrix} -n, b \\ c \end{matrix}; z\right)$. Suppose that $c < 0$, $b > 0$, $c - b > 1 - n$. Then*

(i) *$1 - n < c - b < 0$, $0 < b < n-1$ and $1 - n < c < 0$.*

(ii) *If $-k < c < -k+1$, $k = 1, \dots, n-1$ and*

$$-j < c - b < -j+1, \quad j = 1, \dots, n-1,$$

then ${}_2F_1\left(\begin{matrix} -n, b \\ c \end{matrix}; z\right)$ has $(j-k) \geq 0$ real zeros in $(0,1)$. For the remaining $(n-j+k)$ zeros of F

(a) *$(n-j+k)$ are non-real if $(n-j)$ and k are even.*

(b) *$(n-j+k-1)$ are non-real and one real zero lies in $(1,\infty)$ if $(n-j)$ is odd and k is even.*

(c) *$(n-j+k-1)$ are non-real if $(n-j)$ is even, k odd and one zero is real and negative.*

(d) *$(n-j+k-2)$ are non-real if $(n-j)$ is odd and k is odd with one real negative zero and one real zero in $(1,\infty)$.*

Theorem 2.2.5 (cf.[12], Theorem3.4) *Let $F = {}_2F_1\left(\begin{matrix}-n, b\\ c\end{matrix}; z\right)$ where b and c are such that*

$$1-n < c-b < 0, \quad 1-n < b < 0, \quad 1-n < c < 0.$$

If $-j < b < -j+1$, $j = 1, \ldots, n-1$; $-k < c < -k+1$, $k = 1, \ldots, n-1$ and $-\ell < c-b < -\ell+1$, $\ell = 1, \ldots, n-1$, then F has no real zeros if $n+j+\ell$, $k+\ell$, $j+k$ are even, one real zero in $(1,\infty)$ if $n+j+\ell$ is odd, one real zero in $(0,1)$ if $k+\ell$ is odd and one real negative zero if $j+k$ is odd.

Another approach for locating the zeros of ${}_2F_1$ polynomials is one which uses Rodrigues' formula and the orthogonality of ${}_2F_1\left(\begin{matrix}-n, b\\ c\end{matrix}; z\right)$. As we saw (Theorem 1.6.1) Rodrigues' formula for ${}_2F_1$ (cf. [26], p.162) is given by

$${}_2F_1\left(\begin{matrix}-n, b\\ c\end{matrix}; z\right) = \frac{z^{1-c}(1-z)^{c+n-b}}{(c)_n}\frac{d^n}{dz^n}[z^{c+n-1}(1-z)^{b-c}]$$

By virtue of Theorem1.8.3 and Theorem 1.8.4, one gets

Theorem 2.2.6 (cf. [13]) *Let $F(x) = {}_2F_1\left(\begin{matrix}-n, b\\ c\end{matrix}; x\right)$, where $n \in \mathbb{N}$, $b, c \in \mathbb{R}$ and $c \notin \{0, -1, -2, \ldots, -n+1\}$.*

(i) *For $c > 0$ and $b > c+n-1$, all n zeros of F are real and simple and lie in $(0,1)$.*

(ii) *For $b < 1 - n$ and $c < b + 1 - n$, all n zeros of F are real and simple and lie in $(1, \infty)$.*

(iii) *For $b < 1 - n$ and $c > 0$, all zeros of F are real and simple and lie in $(-\infty, 0)$.*

Recently, in (cf. [22]), A. Martínez et al. used Riemann-Hilbert analysis to show that the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ are orthogonal on

$$\Delta = \begin{cases} [-1, 1] & \text{if } \alpha, \beta > -1 \\ (-\infty, -1] & \text{if } 2n + \alpha + \beta < 0 \text{ and } \beta > -1 \\ [1, \infty) & \text{if } 2n + \alpha + \beta < 0 \text{ and } \alpha > -1 \end{cases} .$$

with respect to the weight function $w(z; \alpha, \beta) := (1 - z)^{\alpha}(1 + z)^{\beta}$.

Since the hypergeometric polynomials ${}_2F_1$ are closely connected to the classical Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, by the formula (cf. [26], p.254)

$$P_n^{(\alpha,\beta)}(z) = \frac{(1+\alpha)_n}{n!} {}_2F_1\left(\begin{matrix} -n, 1+\alpha+\beta+n \\ 1+\alpha \end{matrix} ; \frac{1-z}{2} \right), \tag{2.2}$$

these results may be used to locate the zeros of some ${}_2F_1$ polynomials.

Theorem 2.2.7 *Let $n \in \mathbb{N}$. If $c > 0$ and $b < -n + 1$, then the polynomial ${}_2F_1\left(\begin{matrix} -n, b \\ c \end{matrix} ; z \right)$ has only real negative and simple zeros.*

Proof. Indeed, the corresponding Jacobi polynomials are $P_n^{(\alpha,\beta)}(z)$, with $\alpha = c - 1$ and $\beta = b - c - n$. Since $c > 0$, $\alpha > -1$; also, $b < -n + 1$ implies $2n + \alpha + \beta < 0$. So, according to Corollary 5.2 in (cf. [22]), $P_n^{(\alpha,\beta)}(z)$ are orthogonal on $[1, +\infty)$. Then, the zeros of $P_n^{(\alpha,\beta)}(z)$ are simple real and lie in

$[1,+\infty)$. Thus, using (2.2), we conclude that the zeros of ${}_2F_1\left(\begin{array}{c}-n,b\\ c\end{array};z\right)$ lie in $(-\infty,-1]$ if one has the above stated conditions.

Note that this is an alternative approach to obtain information already given by Theorem 2.2.3.

2.3 Some special classes of ${}_2F_1$ polynomials with restrictions imposed on the parameters

Ultraspherical polynomials are Jacobi polynomials $P_n^{(\alpha,\beta)}(z)$ in which

$$\alpha=\beta=\gamma-\frac{1}{2}$$

(cf. [28], p.144; [26], p.276). A connection between $C_n^{\lambda}(z)$ and ${}_2F_1$ polynomials is given by

$$\frac{n!}{(2\lambda)_n}C_n^{\lambda}(1-2z)={}_2F_1\left(\begin{array}{c}-n,n+2\lambda\\ \lambda+\frac{1}{2}\end{array};z\right)\tag{2.3}$$

and another relationship is given by

$${}_2F_1\left(\begin{array}{c}-n,b\\ 2b\end{array};z\right)=\frac{n!2^{-2n}z^n}{(b+\frac{1}{2})_n}C_n^{\lambda}(1-\frac{2}{z})\tag{2.4}$$

where $\lambda=\frac{1}{2}-b-n$.

Using (2.3), (2.4) and the fact that $C_n^{\lambda}(z)$ are orthogonal over the interval $(-1,1)$ when $\lambda>-\frac{1}{2}$, K.Driver and P.Duren, gave the following results

Theorem 2.3.1 (cf. [5] and [6]) *Let* $F={}_2F_1\left(\begin{array}{c}-n,b\\ 2b\end{array};z\right)$, *where* $b\in\mathbb{R}$ *and* $n\in\mathbb{N}$.

(i) *For* $b > -\frac{1}{2}$, *all zeros of* F *are real and simple and lie on the circle* $|z-1| = 1$.

(ii) *For* $b > 0$, *all zeros of* F *satisfy the inequality* $Re\{z_k\} \geq \frac{2b^2}{(b+n)^2}$

(iii) *For* $b < 1-n$, *all zeros of* F *are real and greater than* 1.

From (cf. [28], p.123), we know that the hypergeometric function

$$F = {}_2F_1\left(\begin{array}{c} a, b \\ c \end{array} ; z\right),$$

admits the quadratic transformation if and only if the numbers

$$1-c, \quad a-b, \quad a+b-c$$

satisfy one of the following properties:

- One of them is equal to $\pm 1/2$.
- One of then is equal to another one or equal to the opposite of another one.

On the other hand, ${}_2F_1\left(\begin{array}{c} a, b \\ c \end{array} ; z\right)$ admits a cubic transformation (cf. [2], p.67) if either two of the numbers :

$$\pm(1-c) \quad \pm(-n-b) \quad \pm(c+n-b)$$

are equal to $\frac{1}{3}$, or

$$1-c = \pm(-n-b) = \pm(c+n-b)$$

We note that ${}_2F_1\left(\begin{matrix}-n, b\\ \frac{1}{2}\end{matrix}; z\right)$ and ${}_2F_1\left(\begin{matrix}-n, b\\ \frac{3}{2}\end{matrix}; z\right)$ can be expressed in terms of ultraspherical polynomials, namely

$$\begin{aligned}&{}_2F_1\left(\begin{matrix}-n, b\\ \frac{1}{2}\end{matrix}; z\right)\\ &= \frac{n!}{(1-b)_n} C_{2n}^{b-n}(\sqrt{z}) \quad \text{for} \quad b \notin \{1, 2, \ldots, n\}\end{aligned} \tag{2.5}$$

$$\begin{aligned}&{}_2F_1\left(\begin{matrix}-n, b\\ \frac{3}{2}\end{matrix}; z\right)\\ &= -\frac{n!}{2(1-b)_{n+1}} \frac{1}{\sqrt{z}} C_{2n+1}^{b-n-1}(\sqrt{z}) \quad \text{for} \quad b \notin \{1, 2, \ldots, n+1\}\end{aligned} \tag{2.6}$$

K. Driver and M. Möller, using the quadratic and cubic transformations, as well as (2.5) and (2.6) proved the two following special cases:

Theorem 2.3.2 (cf. [15], Theorem 2.2) *Let* $F = {}_2F_1\left(\begin{matrix}-n, b\\ \frac{1}{2}\end{matrix}; z\right)$ *with* b *real and* $n \in \mathbb{N}$.

(i) *For* $b > n - \frac{1}{2}$, *all* n *zeros of* F *are real and simple and lie in* $(0, 1)$.

(ii) *For* $n - \frac{1}{2} - j < b < n + \frac{1}{2} - j$, $j = 1, 2, \ldots, n-1$, $(n-j)$ *zeros of* F *lie in* $(0, 1)$ *and the remaining* j *zeros of* F *form* $\left[\frac{j}{2}\right]$ *non-real complex pairs of zeros and one real zero lying in* $(1, \infty)$ *when* j *is odd.*

(iii) *For* $0 < b < \frac{1}{2}$, F *has* $\left[\frac{n}{2}\right]$ *non-real complex conjugate pairs of zeros with one real zero in* $(1, \infty)$ *when* n *is odd.*

(iv) *For* $-j < b < -j + 1$, $j = 1, 2, \ldots, n-1$, F *has exactly* j *real negative zeros. There is exactly one further real zero greater than 1 only when* $(n-j)$ *is odd and all the remaining zeros of* F *are non–real.*

(v) For $b < 1 - n$, all zeros of F are real and negative.

Theorem 2.3.3 (cf. [15], Theorem 2.3) *Let $F = {}_2F_1\left(\begin{array}{c} -n, b \\ 3/2 \end{array}; z\right)$ with $b \in \mathbb{R}$ and $n \in \mathbb{N}$.*

(i) For $b > n + \frac{1}{2}$, all n zeros of F are real and simple and lie in $(0, 1)$.

(ii) For $n + \frac{1}{2} - j < b < n + \frac{3}{2} - j$, $j = 1, 2, \ldots, n-1$, $(n - j)$ zeros of F lie in $(0, 1)$. F has $\left[\frac{j}{2}\right]$ pairs of non-real complex conjugate zeros with one additional real zero in $(1, \infty)$ when j is odd.

(iii) For $0 < b < \frac{3}{2}$, F has $\left[\frac{n}{2}\right]$ non-real complex conjugate pairs of zeros with one real zero in $(1, \infty)$ when n is odd.

(iv) For $-j < b < -j + 1$, $j = 1, 2, \ldots, n-1$, F has exactly j real negative zeros. There is exactly one further real zero greater than 1 only when $(n - j)$ is odd and all the remaining zeros of F are non–real.

(v) For $b < 1 - n$, all zeros of F are real and negative.

Let us come back to Theorem 2.2.2. If in Theorem 2.2.2, $\beta = -\alpha$, then the numbers of the zeros of Jacobi polynomials $P_n^{(\alpha,-\alpha)}(\omega)$ in the intervals $(-1, 1)$, $(-\infty, -1)$, and $(1, \infty)$ are respectively

$$\begin{aligned} N_1 &= \begin{cases} 2[(X+1)/2] & \text{if } (-1)^n \binom{n+\alpha}{n} > 0 \\ 2[X/2] + 1 & \text{if } (-1)^n \binom{n+\alpha}{n} < 0 \end{cases} \\ N_2 &= 0 \\ N_3 &= \begin{cases} 0 & \text{if } \binom{n+\alpha}{n} > 0 \\ 1 & \text{if } \binom{n+\alpha}{n} < 0 \end{cases} \end{aligned} \tag{2.7}$$

In addition, $${}_2F_1\left(\begin{array}{c}-n, b\\ -2n\end{array}; z\right) = \frac{n! z^n}{(-2n)_n} P_n^{(\alpha,-\alpha)}(1-\frac{2}{z}) \qquad (2.8)$$

where $\alpha = -b - n$.

Using Hilbert's result (2.7) and (2.8), K. Driver and M. Möller stated and proved

Theorem 2.3.4 (cf. [14], Theorem 3.1 and Corollary 3.2) *Let* $F = {}_2F_1\left(\begin{array}{c}-n, b\\ -2n\end{array}; z\right)$ *with b real and $n \in \mathbb{N}$.*

(i) For $b > 0$, F has n non-real zeros if n is even whereas if n is odd, F has exactly one real negative zero and the remaining $(n-1)$ zeros of F are all non-real.

(ii) For $-n < b < 0$, if $-k < b < -k+1$, $k = 1, \ldots, n$, F has k real zeros in the interval $(1, \infty)$. In addition, if $(n-k)$ is even, F has $(n-k)$ non-real zeros whereas if $(n-k)$ is odd, F has one real negative zero and $(n-k-1)$ non-real zeros.

(iii) For $-n > b > -2n$, if $-n-k > b > -n-k-1$, $k = 0, 1, \ldots, n-1$, F has $(n-k)$ real zeros in the interval $(1, \infty)$. In addition, if k is even F has k non-real zeros while if k is odd, F has one real zero in $(0, 1)$ and $(k-1)$ non-real zeros.

(iv) For $b < -2n$, all n zeros of F are non–real for n even whereas for n odd, F has exactly one real zero in the interval $(0, 1)$.

The last result gives place to the following proposition.

Proposition. 2.3.5 *For $b > 0$ and $n \in \mathbb{N}$, the zeros of* ${}_2F_1\left(\begin{array}{c}-n, n+1\\ 1-n-b\end{array}; z\right)$ *and* ${}_2F_1\left(\begin{array}{c}-n, n+1\\ 1+n+b\end{array}; z\right)$ *approach the lemniscate $|z(1-z)| = \frac{1}{4}$ as $n \to \infty$.*

Proof The identity (1.4.3) yields

$$ {}_2F_1\left(\begin{array}{c} -n, n+1 \\ 1-n-b \end{array}; z\right) = \frac{(n+1)_n}{(1-b-n)_n}(-z)^n {}_2F_1\left(\begin{array}{c} -n, b \\ -2n \end{array}; \frac{1}{z}\right). $$

Then it follows from Theorem 2.3.5 that for $b > 0$, $n \in \mathbb{N}$, the zeros of ${}_2F_1\left(\begin{array}{c} -n, n+1 \\ 1-n-b \end{array}; z\right)$ approach the Cassini curve $|(2z-1)^2 - 1| = 1$ i.e. $|z(z-1)| = \frac{1}{4}$, as $n \to \infty$.

Furthermore, identity (1.8) yields ${}_2F_1\left(\begin{array}{c} -n, n+1 \\ 1+n+b \end{array}; 1-z\right) = \frac{(b)_n}{(1+b+n)_n} {}_2F_1\left(\begin{array}{c} -n, n+1 \\ 1-n-b \end{array}; z\right)$. Since the Cassini curve $|z(z-1)| = \frac{1}{4}$ is invariant under the transformation $z \to 1-z$, the stated result follows immediately.

2.4 Conclusion

In this chapter, we reviewed some results on the location of the zeros of ${}_2F_1$ polynomials with real parameters. The cases and techniques mentioned are not exhaustive; but they show that different methods may be used to locate the zeros of ${}_2F_1$ polynomials with real parameters.

In the next chapter, we shall consider the zero location of ${}_2F_1$ polynomials with non-real parameters.

Chapter 3

Zero location of ${}_2F_1$ polynomials for some non-real parameters

3.1 Introduction

In the previous chapter, we considered the location of the real and some non real zeros of ${}_2F_1\left(\begin{array}{c}-n, b\\ c\end{array}; z\right)$ when b and c are real numbers.

In this chapter, we present the location of the real zeros of ${}_2F_1$ polynomials with non-real parameters, with restriction on the parameters. We base our method on the contiguous relations of ${}_2F_1$ functions.

The chapter is divided into two sections.

In Section 3.2, we generalize some contiguous relations; and in Section 3.3, we apply these generalized contiguous relations to study the zeros of some special cases of ${}_2F_1$ polynomial with non-real parameters.

3.2 Generalization of some contiguous relations for ${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right)$

R. Vidũnas in [30] dealt with generalizations of contiguous relations and obtained several properties for the coefficients of these generalized three term contiguous relations.

In this paragraph, we obtain a different generalization of some contiguous relations involving more than three terms, that will be applied to investigate the zero location of some families of Gauss hypergeometric polynomials with non real parameters.

Let $F = {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right)$ be the hypergeometric function where a, b and c are real or complex numbers with $c \notin \{0, -1, -2, -3, \ldots\}$. Let us consider $c \neq 1$, $b \neq c$, and $b \neq 0$. Rewriting the contiguous relations (1.2), (1.3), (1.4), (1.5), in the first chapter, one gets:

$$F(c^-) = \frac{1}{c-1}[aF(a^+) + (c-a-1)F] \tag{3.1}$$

$$F(c^+) = \frac{c}{(c-b)z}[F(a^-) + (z-1)F] \tag{3.2}$$

$$F(b^+) = \frac{1}{b}[aF(a^+) + (b-a)F] \tag{3.3}$$

$$F(b^-) = \frac{1}{c-b}[(c-a)F(a^-) - (b-a)(1-z)F] \tag{3.4}$$

Our goal is to generalize the above relations.

For k, l, $t \in \mathbb{N}$, we write:

$$\begin{aligned} F(a^{k\pm}) &= {}_2F_1(a \pm k, b; c; z) \\ F(b^{l\pm}) &= {}_2F_1(a, b \pm l; c; z) \\ F(c^{t\pm}) &= {}_2F_1(a, b; c \pm t; z) \end{aligned}$$

where $a^{j+} = a + j$ and $a^{j-} = a - j$.

Hence, with these notations, we have:
$F(c^-) = F(c^{1-})$, $F(c^+) = F(c^{1+})$, $F(a^-) = F(a^{1-})$, $F(a^+) = F(a^{1+})$, $F(b^-) = F(b^{1-})$, $F(b^+) = F(b^{1+})$, and $F(a^{0+}) = F(a^{0-}) = F$.

Lemma 3.2.1 *For $k \in \mathbb{N}$ and $c - k \notin \{0, -1, -2, \ldots, -k+1\}$, we have*

$$F(c^{k-}) = \frac{1}{(c-k)_k} \sum_{j=0}^{k} \binom{k}{j} (a)_j (c-a-k)_{k-j} F(a^{j+}). \tag{3.5}$$

Proof From (3.1) we have:

$$\begin{aligned} F(c^{1-}) &= \frac{aF(a^{1+}) + (c-a-1)F}{c-1} \\ &= \frac{1}{(c-1)} \sum_{j=0}^{1} \binom{1}{j} (a)_j (c-a-1)_{1-j} F(a^{j+}) \end{aligned} \tag{3.6}$$

Suppose the identity is true for k; that means:

$$F(c^{k-}) = \frac{1}{(c-k)_k} \sum_{j=0}^{k} \binom{k}{j} (a)_j (c-a-k)_{k-j} F(a^{j+})$$

and replacing c by $c-1$, one obtains

$$\begin{aligned} &F(c^{(k+1)-}) \\ &= \frac{1}{(c-k-1)_k} \sum_{j=0}^{k} \binom{k}{j} (a)_j (c-a-k-1)_{k-j} F(a^{j+}, c^{1-}), \end{aligned}$$

It follows from (3.6) that

$$F(a^{j+}, c^{1-}) = \frac{(a+j)F(a^{(j+1)+}) + (c-a-j-1)F(a^{j+})}{c-1},$$

hence

$$F(c^{(k+1)-}) = \frac{1}{(c-1)(c-k-1)_k}[\sum_{j=0}^{k} \binom{k}{j} (a)_j (a+j)(c-a-k-1)_{k-j} F(a^{(j+1)+})$$

$$
\begin{aligned}
& + \sum_{j=0}^{k} \binom{k}{j} (a)_j (c-a-k-1)_{k-j} (c-a-j-1) F(a^{j+})] \\
= \quad & \frac{1}{(c-(k+1))_{k+1}} \Big[\sum_{j=0}^{k} \binom{k}{j} (a)_{j+1} (c-a-k-1)_{k-j} F(a^{(j+1)+}) \\
& + \sum_{j=0}^{k} \binom{k}{j} (a)_j (c-a-k-1)_{k-j+1} F(a^{j+}) \Big] \\
= \quad & \frac{1}{(c-(k+1))_{k+1}} \Big[\binom{k}{k} (a)_{k+1} F(a^{(k+1)+}) \\
& + \sum_{j=0}^{k-1} \binom{k}{j} (a)_{j+1} (c-a-k-1)_{k-j} F(a^{(j+1)+}) \\
& + \sum_{j=1}^{k} \binom{k}{j} (a)_j (c-a-k-1)_{k+1-j} F(a^{j+}) + \binom{k}{0} (c-a-k-1)_{k+1} F \Big] \\
= \quad & \frac{1}{(c-(k+1))_{k+1}} \Big[\binom{k}{k} (a)_{k+1} F(a^{(k+1)+}) \\
& + \sum_{j=1}^{k} \binom{k}{j-1} (a)_j (c-a-k-1)_{k-j+1} F(a^{j+}) \\
& + \sum_{j=1}^{k} \binom{k}{j} (a)_j (c-a-k-1)_{k+1-j} F(a^{j+}) + \binom{k}{0} (c-a-k-1)_{k+1} F \Big] \\
= \quad & \frac{1}{(c-(k+1))_{k+1}} \Big[\binom{k+1}{k+1} (a)_{k+1} F(a^{(k+1)+}) \\
& + \sum_{j=1}^{k} \Big[\binom{k}{j-1} + \binom{k}{j} \Big] (a)_j (c-a-k-1)_{k+1-j} F(a^{j+}) \\
& + \binom{k+1}{0} (c-a-k-1)_{k+1} F \Big] \\
= \quad & \frac{1}{(c-(k+1))_{k+1}} \sum_{j=0}^{k+1} \binom{k+1}{j} (a)_j (c-a-(k+1))_{k+1-j} F(a^{j+})
\end{aligned}
$$

which proves the result.

Lemma 3.2.2 *For $c-b \notin \{0,-1,-2,\dots,-k+1\}$, we have:*

$$F(c^{k+}) = \frac{(c)_k}{(c-b)_k z^k} \sum_{j=0}^{k} \binom{k}{j} F(a^{j-})(z-1)^{k-j}$$

Proof Considering the relation (3.2), we have

$$F(c^{1+}) = \frac{c}{(c-b)z}(F(a^{1-}) + (z-1)F)$$

Let us suppose that for $k \in \mathbb{N}$, we have

$$F(c^{k+}) \quad = \quad \frac{(c)_k}{(c-b)_k z^k} \sum_{j=0}^{k} \binom{k}{j} F(a^{j-})(z-1)^{k-j}. \tag{3.7}$$

Then, replacing c by $c+1$ in (3.7), one has:

$$F(c^{(k+1)+}) = \frac{(c+1)_k}{(c-b+1)_k z^k} \sum_{j=0}^{k} \binom{k}{j} F(a^{j-}, c^{1+})(z-1)^{k-j}.$$

According to (3.2), we have

$$F(a^{j-}, c^{1+}) = \frac{c}{(c-b)z}[F(a^{(j+1)-}) + (z-1)F(a^{j-})]$$

Then, one obtains

$$\begin{aligned}
&F(c^{(k+1)+}) \\
&= \frac{(c+1)_k}{(c-b+1)_k z^k} \sum_{j=0}^{k} \binom{k}{j} \frac{c}{(c-b)z}[F(a^{(j+1)-}) + (z-1)F(a^{j-})](z-1)^{k-j} \\
&= \frac{c(c+1)_k}{(c-b)(c-b+1)_k z^{k+1}} \sum_{j=0}^{k} \binom{k}{j} [(z-1)^{k-j} F(a^{(j+1)-}) + (z-1)^{k-j+1} F(a^{j-})] \\
&= \frac{(c)_{k+1}}{(c-b)_{k+1} z^{k+1}} \Big[\sum_{j=0}^{k} \binom{k}{j} (z-1)^{k-j} F(a^{(j+1)-}) + \sum_{j=0}^{k} \binom{k}{j} (z-1)^{k-j+1} F(a^{j-}) \Big] \\
&= \frac{(c)_{k+1}}{(c-b)_{k+1} z^{k+1}} \Big[\binom{k}{k} F(a^{(k+1)-}) + \sum_{j=1}^{k} \binom{k}{j-1} (z-1)^{k-j+1} F(a^{j-})
\end{aligned}$$

$$
\begin{aligned}
&+ \binom{k}{0}(z-1)^{k+1}F + \sum_{j=1}^{k}\binom{k}{j}(z-1)^{k-j+1}F(a^{j-})\Big] \\
= &\ \frac{(c)_{k+1}}{(c-b)_{k+1}z^{k+1}}\Big\{\binom{k+1}{k+1}F(a^{(k+1)-}) + \sum_{j=1}^{k}\Big[\binom{k}{j-1}+\binom{k}{j}\Big](z-1)^{k-j+1}F(a^{j-}) \\
&+ \binom{k+1}{0}(z-1)^{k+1}F\Big\} \\
= &\ \frac{(c)_{k+1}}{(c-b)_{k+1}z^{k+1}}\sum_{j=0}^{k+1}\binom{k+1}{j}F(a^{j-})(z-1)^{k+1-j}
\end{aligned}
$$

$$\text{Thus} \quad \forall k \in \mathbb{N}, F(c^{k+}) = \frac{(c)_k}{(c-b)_k z^k}\sum_{j=0}^{k}\binom{k}{j}F(a^{j-})(z-1)^{k-j}$$

Lemma 3.2.3 *If* $\quad b \notin \{0,-1,-2,\ldots,-k+1\}, \quad$ *then*

$$F(b^{k+}) = \frac{1}{(b)_k}\sum_{j=0}^{k}\binom{k}{j}(a)_j F(a^{j+})(b-a)_{k-j}$$

Proof Using the relation (3.3),we have

$$
\begin{aligned}
F(b^{1+}) &= \frac{aF(a^{1+}) + (b-a)F}{b} \\
&= \frac{1}{(b)_1}\sum_{j=0}^{1}\binom{1}{j}(a)_j F(a^{j+})(b-a)_{1-j}
\end{aligned} \tag{3.8}
$$

Suppose that the relation is true for $k \in \mathbb{N}$; that means

$$F(b^{k+}) = \frac{1}{(b)_k}\sum_{j=0}^{k}\binom{k}{j}(a)_j F(a^{j+})(b-a)_{k-j}.$$

Then, replacing b by $b+1$, we obtain:

$$
\begin{aligned}
&F(b^{(k+1)+}) \\
&= \frac{1}{(b+1)_k}\sum_{j=0}^{k}\binom{k}{j}(a)_j F(a^{j+}, b^{1+})(b+1-a)_{k-j}
\end{aligned} \tag{3.9}
$$

In (3.8), replacing a by a^{j+}, one obtains:

$$\begin{aligned}
&F(a^{j+},b^{1+})\\
&\quad= \frac{a^{j+}F(a^{(j+1)+})+(b-a^{j+})F(a^{j+})}{b}\\
&\quad= \frac{(a+j)F(a^{(j+1)+})+(b-a-j)F(a^{j+})}{b}
\end{aligned} \tag{3.10}$$

Substituting (3.10) into (3.9) gives

$$\begin{aligned}
&F(b^{(k+1)+})\\
&= \frac{1}{b(b+1)_k}\sum_{j=0}^{k}\binom{k}{j}(a)_j\Big[(a+j)F(a^{(j+1)+})\\
&\quad + (b-a-j)F(a^{j+})\Big](b-a+1)_{k-j}\\
&= \frac{1}{(b)_{k+1}}\Big[\sum_{j=0}^{k}\binom{k}{j}(a)_{j+1}F(a^{(j+1)+})(b-a+1)_{k-j}+\\
&\quad +\sum_{j=0}^{k}\binom{k}{j}(a)_jF(a^{j+})(b-a-j)(b-a+1)_{k-j}\Big]\\
&= \frac{1}{(b)_{k+1}}\Big[\sum_{j=1}^{k+1}\binom{k}{j-1}(a)_jF(a^{j+})(b-a+1)_{k-j+1}\\
&\quad +\sum_{j=0}^{k}\binom{k}{j}(a)_jF(a^{j+})(b-a-j)(b-a+1)_{k-j}\Big]\\
&= \frac{1}{(b)_{k+1}}\Big[\binom{k}{0}(b-a)(b-a+1)_kF+\sum_{j=1}^{k}\Big[\binom{k}{j-1}(b-a+1)_{k-j+1}\\
&\quad +\binom{k}{j}(b-a-j)(b-a+1)_{k-j}\Big](a)_jF(a^{j+})+\binom{k}{k}(a)_{k+1}F(a^{(k+1)+})\Big]\\
&= \frac{1}{(b)_{k+1}}\Big[\binom{k}{0}(b-a)_{k+1}F+\sum_{j=1}^{k}\Big[\binom{k}{j-1}(b-a+1+k-j)\\
&\quad +\binom{k}{j}(b-a-j)\Big](b-a+1)_{k-j}(a)_jF(a^{j+})
\end{aligned}$$

$$
\begin{aligned}
&\quad + \binom{k}{k}(a)_{k+1}F(a^{(k+1)+})\Big] \\
&= \frac{1}{(b)_{k+1}}\Big[\binom{k}{0}(b-a)_{k+1}F + \sum_{j=1}^{k}\Big[\Big[\binom{k}{j-1} + \binom{k}{j}\Big](b-a) \\
&\quad + \binom{k}{j-1}(k-j+1) - j\binom{k}{j}\Big](b-a+1)_{k-j}(a)_jF(a^{j+}) \\
&\quad + \binom{k}{k}(a)_{k+1}F(a^{(k+1)+})\Big] \\
&= \frac{1}{(b)_{k+1}}\Big[\binom{k}{0}(b-a)_{k+1}F \\
&\quad + \sum_{j=1}^{k}\binom{k+1}{j}(b-a)(b-a+1)_{k-j}(a)_jF(a^{j+}) \\
&\quad + \binom{k}{k}(a)_{k+1}F(a^{(k+1)+})\Big] \\
&= \frac{1}{(b)_{k+1}}\Big[\binom{k+1}{0}(b-a)_{k+1}F + \sum_{j=1}^{k}\binom{k+1}{j}(b-a)_{k+1-j}(a)_jF(a^{j+}) \\
&\quad + \binom{k+1}{k+1}(a)_{k+1}F(a^{(k+1)+})\Big] \\
&= \frac{1}{(b)_{k+1}}\sum_{j=0}^{k+1}\binom{k+1}{j}(b-a)_{k+1-j}(a)_jF(a^{j+}) \qquad (3.11)
\end{aligned}
$$

We have then the announced result.

Lemma 3.2.4 *If* $c-b \notin \{0,-1,-2,\ldots,-k+1\}$, *then*

$$F(b^{k-}) = \frac{1}{(c-b)_k}\sum_{j=0}^{k}\binom{k}{j}(a-b)_j(1-z)^j(c-a)_{k-j}F(a^{(k-j)-})$$

Proof From the relation (3.4) we have

$$F(b^{1-})$$

$$= \frac{1}{c-b}\Big[(c-a)F(a^{1-}) + (a-b)(1-z)F\Big] \tag{3.12}$$
$$= \frac{1}{(c-b)_1}\sum_{j=0}^{1}\binom{1}{j}(a-b)_j(1-z)^j(c-a)_{1-j}F(a^{(1-j)-})$$

Let us suppose true the relation for k; that means:

$$F(b^{k-})$$
$$= \frac{1}{(c-b)_k}\sum_{j=0}^{k}\binom{k}{j}(a-b)_j(1-z)^j(c-a)_{k-j}F(a^{(k-j)-})$$

Then

$$F(b^{(k+1)-})$$
$$= \frac{1}{(c-b+1)_k}\sum_{j=0}^{k}\binom{k}{j}(a-b+1)_j(1-z)^j(c-a)_{k-j}F(a^{(k-j)-}, b^{-})$$

In (3.12), replacing a by $a^{(k-j)-}$, we get

$$\begin{aligned} &F(a^{(k-j)-}, b^{-}) \\ &\quad = \frac{1}{(c-b)}[(c-a+k-j)F(a^{(k-j+1)-}) \\ &\qquad + (a-k+j-b)(1-z)F(a^{(k-j)-})]; \end{aligned} \tag{3.13}$$

Hence

$$\begin{aligned} &F(b^{(k+1)-}) \\ &= \frac{1}{(c-b+1)_k}\Big[\sum_{j=0}^{k}\binom{k}{j}(a-b+1)_j(1-z)^j(c-a)_{k-j}\Big[\frac{1}{c-b}\Big[(c-a \\ &\quad + k-j)F(a^{(k-j+1)-}) + (a-k+j-b)(1-z)F(a^{(k-j)-})\Big]\Big]\Big] \\ &= \frac{1}{(c-b)_{k+1}}\Big[\sum_{j=0}^{k}\binom{k}{j}(a-b+1)_j(1-z)^j(c-a)_{k-j+1}F(a^{(k-j+1)-}) \\ &\quad + \sum_{j=0}^{k}\binom{k}{j}(a-b+1)_j(a-b-k+j)(c-a)_{k-j}(1-z)^{j+1}F(a^{(k-j)-})\Big] \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{(c-b)_{k+1}}\Big[\binom{k}{0}(c-a)_{k+1}F(a^{(k+1)-})\\
&\quad+\sum_{j=0}^{k-1}\Big[\binom{k}{j+1}(a-b+1)_{j+1}(1-z)^{j+1}(c-a)_{k-j}\\
&\quad+\binom{k}{j}(a-b-k+j)(a-b+1)_j(c-a)_{k-j}(1-z)^{j+1}\Big]F(a^{(k-j)-})\\
&\quad+\binom{k}{k}(a-b+1)_k(a-b)(1-z)^{k+1}F\Big]\\
&= \frac{1}{(c-b)_{k+1}}\Big[\binom{k}{0}(c-a)_{k+1}F(a^{(k+1)-})\\
&\quad+\sum_{j=0}^{k-1}\Big\{\binom{k}{j+1}(a-b+j+1)\\
&\quad+\binom{k}{j}(a-b-k+j)\Big\}(a-b+1)_j(c-a)_{k-j}(1-z)^{j+1}F(a^{(k-j)-})+\\
&\quad+\binom{k}{k}(a-b)_{k+1}(1-z)^{k+1}F\Big]\\
&= \frac{1}{(c-b)_{k+1}}\Big[\binom{k}{0}(c-a)_{k+1}F(a^{(k+1)-})\\
&\quad+\sum_{j=0}^{k-1}\Big\{\Big(\binom{k}{j+1}+\binom{k}{j}\Big)(a-b)+\binom{k}{j+1}(j+1)\\
&\quad-(k-j)\binom{k}{j}\Big\}(a-b+1)_j(c-a)_{k-j}(1-z)^{j+1}F(a^{(k-j)-})\\
&\quad+\binom{k}{k}(a-b)_{k+1}(1-z)^{k+1}F\Big]\\
&= \frac{1}{(c-b)_{k+1}}\Big[\binom{k}{0}(c-a)_{k+1}F(a^{(k+1)-})\\
&\quad+\sum_{j=0}^{k-1}\binom{k+1}{j+1}(a-b)(a-b+1)_j(c-a)_{k-j}(1-z)^{j+1}F(a^{(k-j)-})\\
&\quad+\binom{k}{k}(a-b)_{k+1}(1-z)^{k+1}F\Big]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{(c-b)_{k+1}}\Big[\binom{k}{0}(c-a)_{k+1}F(a^{(k+1)-}) \\
&\quad + \sum_{j=0}^{k-1}\binom{k+1}{j+1}(a-b)_{j+1}(c-a)_{k-j}(1-z)^{j+1}F(a^{(k-j)}) \\
&\quad + \binom{k}{k}(a-b)_{k+1}(1-z)^{k+1}F\Big] \\
&= \frac{1}{(c-b)_{k+1}}\Big[\binom{k}{0}(c-a)_{k+1}F(a^{(k+1)-}) \\
&\quad + \sum_{l=1}^{k}\binom{k+1}{l}(a-b)_{l}(c-a)_{k+1-l}(1-z)^{l}F(a^{(k+1-l)-}) \\
&\quad + \binom{k}{k}(a-b)_{k+1}(1-z)^{k+1}F\Big] \\
&= \frac{1}{(c-b)_{k+1}}\Big[\binom{k+1}{0}(c-a)_{k+1}F(a^{(k+1)-}) \\
&\quad + \sum_{j=1}^{k}\binom{k+1}{j}(a-b)_{j}(c-a)_{k+1-j}(1-z)^{j}F(a^{(k+1-j)-}) \\
&\quad + \binom{k+1}{k+1}(a-b)_{k+1}(1-z)^{k+1}F\Big]
\end{aligned}
$$

Thus $F(b^{k-}) = \frac{1}{(c-b)_k}\sum_{j=0}^{k}\binom{k}{j}(a-b)_j(1-z)^j(c-a)_{k-j}F(a^{(k-j)-})$ for all $k \in \mathbb{N}$.

3.3 Zeros of some classes of ${}_2F_1$ polynomials with complex parameters

Let $F = {}_2F_1\left(\begin{matrix}-n, ib+1 \\ ib\end{matrix}; z\right)$ be a hypergeometric polynomial with $n \in \mathbb{N}$ and b a real number such that $b \neq 0$. In the following theorems, we will consider the zeros of F and of some other polynomials close to F, using the results obtained in the previous section.

Theorem 3.3.1 *Let* $n \in \mathbb{N}$, $b \in \mathbb{R}$ *such that* $b \neq 0$, *and let* $F = {}_2F_1\left(\begin{array}{c} -n, ib+1 \\ ib \end{array}; x\right)$. *The* $n-1$ *real zeros of* F *are at 1 .*

Proof From the definition of ${}_2F_1$, we have

$$F = \sum_{k=0}^{n} \frac{(-n)_k (ib+1)_k}{(ib)_k k!} z^k.$$

Since $(ib+1)_k = \frac{(ib)_k(ib+k)}{ib}$, this becomes

$$\begin{aligned} F &= \sum_{k=0}^{n} \frac{(-n)_k (ib+k)}{ibk!} z^k \\ &= \sum_{k=0}^{n} \frac{(-n)_k (b-ik)}{bk!} z^k \\ &= \sum_{k=0}^{n} \frac{(-n)_k}{k!} z^k - i \sum_{k=0}^{n} \frac{(-n)_k k}{bk!} z^k \\ &= \sum_{k=0}^{n} \frac{(-n)_k}{k!} z^k - \frac{i}{b} \sum_{k=1}^{n} \frac{(-n)_k}{(k-1)!} z^k \\ &= \sum_{k=0}^{n} \frac{(-n)_k}{k!} z^k - \frac{i}{b} \sum_{k=0}^{n-1} \frac{(-n)_{k+1}}{k!} z^{k+1} \\ &= \sum_{k=0}^{n} \frac{(-n)_k}{k!} z^k + \frac{i}{b} nz \sum_{k=0}^{n-1} \frac{(-n+1)_k}{k!} z^k. \end{aligned}$$

$$\text{Because} \quad (1-y)^{-a} = \sum_{n=0}^{\infty} \frac{(a)_n}{n!} y^n \quad (cf.[26], p.47),$$

$$\begin{aligned} (3.14) \quad \text{becomes} \quad F &= (1-z)^n + i\frac{nz}{b}(1-z)^{n-1} \\ &= (1-z)^{n-1}[1 - z + i\frac{nz}{b}]. \end{aligned}$$

Then 1 is a zero of F with multiplicity $n-1$.

Theorem 3.3.2 *Let $n \in \mathbb{N}$, b a real number such that $b \neq 0$ and $n \geq 2$, and let*

$$F = {}_2F_1\left(\begin{matrix} -n, ib+1 \\ ib-1 \end{matrix}; z\right).$$

Then F has $n-2$ real zeros at 1.

Proof Using the contiguous relation (3.5), we have

$$\begin{aligned}
&{}_2F_1\left(\begin{matrix} -n, ib+1 \\ ib-1 \end{matrix}; z\right) \\
= &\ \frac{1}{ib-1}\Big[(ib+n-1){}_2F_1\left(\begin{matrix} -n, ib+1 \\ ib \end{matrix}; z\right) - n{}_2F_1\left(\begin{matrix} -n+1, ib+1 \\ ib \end{matrix}; z\right)\Big] \\
= &\ \frac{1}{ib-1}\Big[(ib+n-1)\frac{(1-z)^{n-1}}{b}[b-(b+in)z] \\
&\ -n\frac{(1-z)^{n-2}}{b}[b-(b+i(n-1))z]\Big] \\
= &\ \frac{(1-z)^{n-2}}{b(ib-1)}\Big[(ib+n-1)(1-z)[b-(b-in)z]-n[b-(b-i(n-1))z]\Big].
\end{aligned}$$

From the above relation, we observe that 1 is a zero with multiplicity $n-2$.

Now, consider

$$\begin{aligned}
p(z) &:= (ib+n-1)(1-z)[b-(b-in)z]-n[b-(b-i(n-1))z] \\
&= ip_1(z)+p_2(z);
\end{aligned}$$

where $p_1(z) = (n-1)z(1-z-n)+b^2(1-z)^2$, and $p_2(z) = b(z-1)(1-z(1-2n))$.Then

$${}_2F_1\left(\begin{matrix} -n, ib+1 \\ ib-1 \end{matrix}; z\right) = \frac{(1-z)^{n-2}}{b(ib-1)}\Big[ip_1(z)+p_2(z)\Big]$$

Note that a real number z_0 is a zero of $p(z)$ if and only $p_1(z_0) = 0 = p_2(z_0)$. However, $p_2(z)$ vanishes at 1, but 1 is not a zero of $p_1(z)$.

By similar arguments as used above, we obtain the following theorems.

Theorem 3.3.3 *Let $n \in \mathbb{N}$, $b \in \mathbb{R}$ such that $b \neq 0$ and $n \geq 3$, and* $F = {}_2F_1\left(\begin{matrix} -n, ib+1 \\ ib-2 \end{matrix}; z\right)$. *The $n-3$ zeros of F are at 1.*

Theorem 3.3.4 *Let $n \in \mathbb{N}, b \in \mathbb{R}$ such that $n \geq 2$. $b \neq 0$ and $F = {}_2F_1\left(\begin{matrix} -n, ib+2 \\ ib \end{matrix}; z\right)$ Then, $n-2$ real zeros of F are at 1.*

3.4 Remark

The fact that $z = 1$ turns out to be a multiple zero of the polynomials suggests that one can make use of a transformation mapping $1-z$ onto z and then apply the Theorems of Section 1.5.

However, the assumptions of Theorem 1.5.1 are not applicable. But, in Theorem 1.4.1, the restriction $c > b$ is not essential as both sides of the formula in this theorem are meromorphic in a, b, c, z and even rational functions in b, c, z for a a negative integer.

So, this formula holds for all parameters where both sides are well-defined. Then it immediately follows that, for $j \in \{1, 2, \ldots, n\}$,

$${}_2F_1\left(\begin{matrix} -n, c+j \\ c \end{matrix}; z\right) = (1-z)^n {}_2F_1\left(\begin{matrix} -n, -j \\ c \end{matrix}; \frac{-z}{1-z}\right)$$

has $n-j$ zeros at 1, and the remaining j zeros can be easily calculated as the zeros of the polynomial of degree j:

$${}_2F_1\left(\begin{matrix} -n, -j \\ c \end{matrix}; \frac{-z}{1-z}\right)$$

Then considering c as a pure imaginary number is a particular case.

3.5 Conclusion

As we can see, with some generalization of contiguous relations of ${}_2F_1$ functions, it is possible to locate real zeros of some classes of ${}_2F_1$ polynomials with non-real parameters.

In the next chapter, we will deal with the asymptotic behavior of zeros of ${}_2F_1$ polynomials.

Chapter 4

Asymptotic zero distribution of ${}_2F_1$ polynomials

4.1 Introduction

Throughout this chapter, we consider ${}_2F_1$ polynomials with real parameters. Our purpose is to review some results on the asymptotic behavior of some classes of ${}_2F_1$. This chapter is organized as follows.

In the following section, we deal with some asymptotic results on zeros, results which are described in [5] and [8] and review a recent by K. Driver and S. Johnston in [10]

4.2 Some results on the asymptotic zero distribution of ${}_2F_1$ polynomials

Just as for the zero location, the connections between ${}_2F_1$ polynomials and some classical orthogonal polynomials play an important rule in the study of the asymptotic behavior of zeros of ${}_2F_1$ polynomials.

Using the connection

$$ {}_2F_1\left(\begin{array}{c}-n, b\\ 2b\end{array}; z\right) = \frac{n!2^{-2n}z^n}{(b+\frac{1}{2})_n} C_n^{\lambda}(1-\frac{2}{z}) $$

as well as the orthogonality of $C_n^{\lambda}(1-\frac{2}{z})$, K. Driver and P. Duren have proved

Theorem 4.2.1 ([5],Theorem 1) *For* $b < \frac{1}{2}-n$, *the zeros* z_k *of* ${}_2F_1\left(\begin{array}{c}-n, b\\ 2b\end{array}; z\right)$

$$ \textit{satisfy} \quad \frac{2}{(1+\lambda)} \le z_k \le \frac{2}{(1-\lambda)}, \quad \textit{where} \quad \lambda = 2\frac{\sqrt{n(1-2b-n)}}{(1-2b)}. $$

In particular, for each fixed n *the zeros all tend to 2 as* $b \to -\infty$, *and* $|z_k - 2| \le 2\sqrt{2n}|b|^{-\frac{1}{2}} + O(|b|^{-\frac{3}{2}})$, $b \to -\infty$.

Elbert and Laforgia (cf. [18]) proved that the zeros of an ultraspherical polynomial $C_n^b(z)$ satisfy the following inequality

$$ |z_k| \le \frac{\sqrt{n^2+2b}}{b+n} \quad \text{for} \quad b > 0. \tag{4.1} $$

Knowledge of the connection

$$ \frac{n!}{(2\lambda)_n} C_n^{\lambda}(1-2z) = {}_2F_1\left(\begin{array}{c}-n, n+2\lambda\\ \lambda+\frac{1}{2}\end{array}; z\right) \quad \text{and} \quad (4.1) $$

together with the orthogonality of C_n^{λ} are used by K. Driver and P. Duren to establish

Theorem 4.2.2 ([7],Theorem 2, Corollary) *For* $b > 0$, *all zeros of the hypergeometric polynomial* ${}_2F_1\left(\begin{array}{c}-n, b\\ 2b\end{array}; z\right)$ *satisfy the inequality*

$$ Re\{z_k\} \ge \frac{2b^2}{(b+n)^2} $$

As b *increases, the zeros move monotonically along the unit circle towards the point* $z = 2$. *Furthermore, all zeros tend to 2 as* $b \to \infty$.

For $b \ge n$, *all zeros* z_k *satisfy* $Re\{z_k\} \ge \frac{1}{2}$

A formula due to Clausen (cf. [25]) states:

$$\left[{}_2F_1\left(\begin{matrix} c,b \\ c+b+\frac{1}{2}\end{matrix};z\right)\right]^2 = {}_3F_2\left(\begin{matrix} 2c,2b,c+b \\ 2c+2b,c+b+\frac{1}{2}\end{matrix};z\right)$$

From the quadratic transformation

$${}_2F_1\left(\begin{matrix} a,b \\ 2b\end{matrix};z\right) = (1-z)^{-\frac{a}{2}}{}_2F_1\left(\begin{matrix} \frac{a}{2},b-\frac{a}{2} \\ b+\frac{1}{2}\end{matrix};\frac{z^2}{4z-4}\right),$$

with $b=-n$ and $u=\frac{4}{z}-\frac{4}{z^2}$, one obtains:

$${}_2F_1\left(\begin{matrix} -n,b \\ -2n\end{matrix};z\right) = (1-z)^{-\frac{b}{2}}{}_2F_1\left(\begin{matrix} \frac{b}{2},-n-\frac{b}{2} \\ -n+\frac{1}{2}\end{matrix};\frac{1}{u}\right)$$

Hence $\left[{}_2F_1\left(\begin{matrix} -n,b \\ -2n\end{matrix};z\right)\right]^2 = (1-z)^{-b}{}_3F_2\left(\begin{matrix} b,-2n-b,-n \\ -2n,-n+\frac{1}{2}\end{matrix};\frac{1}{u}\right)$

Using the above relations, K. Driver and M. Möller proved:

Theorem 4.2.3 (cf. [14], Theorem 4.1) *Let $b>0$ and $n\in\mathbb{N}$. For the zeros z of the hypergeometric polynomial ${}_2F_1\left(\begin{matrix} -n,b \\ -2n\end{matrix};z\right)$, we have that $w=\frac{1}{z}$ approaches the Cassini curve*

$$|(2w-1)^2-1|=1 \quad as \quad n\to\infty;$$

more precisely, if Z_n is the set of zeros of ${}_2F_1\left(\begin{matrix} -n,b \\ -2n\end{matrix};z\right)$, then

$$\max_{z\in Z_n}\min\{|\frac{1}{z}-\omega| : \omega\in\mathbb{C}, |(2\omega-1)^2-1|=1\}\to 0 \quad as \quad n\to\infty.$$

P. Borwein and W. Chen (cf. [4], Theorem 5.1) established that if

$$q_n(z)=\int_0^1 [t^\alpha(t(1+z)-1)]^n dt,$$

$$\text{and} \quad e_{\alpha n}(z) = \frac{(1+z)^{\alpha n+1}}{z^{n+1}} \int_{\frac{1}{1+z}}^{1} [t^{\alpha}(t(1+z)-1)]^n dt$$

then $q_n(z)$ and $e_{\alpha n}(z)$ have the same critical curve

$$\{z : |R_z(t^*)| = |R_z(1)|\}$$

where $R_z(t) = t^{\alpha}(t(1+z)-1)$, $t^* = \frac{\alpha}{(1+\alpha)(1+z)}$. That means, the limit points of the zeros of $q_n(z)$ or $e_{\alpha n}(z)$ can only cluster on the curve

$$\{z : |z(1+z)^{\alpha}| = \frac{\alpha^{\alpha}}{(1+\alpha)^{1+\alpha}}\}$$

Using the above result and the Euler integral representation of ${}_2F_1$, K. Driver and P. Duren proved that

Theorem 4.2.4 (cf. [8], Theorem 1) *For each number $k \in \mathbb{N}$, the zeros of the hypergeometric polynomials ${}_2F_1\left(\begin{matrix} -n, kn+1 \\ kn+2 \end{matrix}; z\right)$ approach the lemniscate*

$$\{z : |z^k(1-z)| = \frac{k^k}{(k+1)^{k+1}}\}$$

as $n \to \infty$. Furthermore, every point on this curve is a cluster point of zeros.

Now, considering the Euler integral representation of ${}_2F_1\left(\begin{matrix} -n, kn+1 \\ kn+2 \end{matrix}; z\right)$, that is to say,

$${}_2F_1\left(\begin{matrix} -n, kn+1 \\ kn+2 \end{matrix}; z\right) = (kn+1)\int_0^1 [t^k(1-zt)]^n dt,$$

Duren and Guillou, in [16], used the analysis of the saddle-point to extend Driver-Duren result to real number $k > 0$.

With the same technique of [16], Boggs and Duren (cf. [3]) extended the above result to ${}_2F_1\left(\begin{matrix} -n, kn+l+1 \\ kn+l+2 \end{matrix}; z\right)$ for $k > 0$ and $l \geq 0$.

Let us consider again the connection

$$P_n^{(\alpha,\beta)}(z) = \frac{(1+\alpha)_n}{n!}\,{}_2F_1\left(\begin{array}{c} -n, 1+\alpha+\beta+n \\ 1+\alpha \end{array}; \frac{1-z}{2}\right).$$

Martinez et al. in [23], studied the asymptotic zero distribution of $P_n^{(\alpha_n,\beta_n)}(z)$ when the limits $A = \lim_{n\to\infty} \frac{\alpha_n}{n}$ and $B = \lim_{n\to\infty} \frac{\beta_n}{n}$ exist. They considered several cases of values of A and B, described the zero distribution of $P_n^{(\alpha_n,\beta_n)}(z)$ and deduced several cases which have been studied always in the context of the corresponding hypergeometric polynomials.

The hypergeometric polynomial $F = {}_2F_1\left(\begin{array}{c} -n, \frac{n+1}{2} \\ \frac{n+3}{2} \end{array}; z\right)$ is not one of the classes of ${}_2F_1$ polynomials studied in [3] or in [16].

The Euler integral representation of ${}_2F_1\left(\begin{array}{c} -n, \frac{n+1}{2} \\ \frac{n+3}{2} \end{array}; z\right)$ is given by

$${}_2F_1\left(\begin{array}{c} -n, \frac{n+1}{2} \\ \frac{n+3}{2} \end{array}; z\right) = \frac{n+1}{2}\int_0^1 t^{\frac{n-1}{2}}(1-zt)^n dt$$

which is not a good form for employing the technique of the analysis of saddle-point used by Boggs and Duren in [3].

K.Driver and S.Johnston have first proved, in [9] that if $Re(c) > Re(b) > 0$, then

$${}_3F_2\left(\begin{array}{c} -n, \frac{b}{2}, \frac{b+1}{2} \\ \frac{c}{2}, \frac{c+1}{2} \end{array}; z\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt^2)^n dt$$

Taking $b = n+1$ and $c = n+2$, one obtains

$$\begin{aligned} {}_2F_1\left(\begin{array}{c} -n, \frac{n+1}{2} \\ \frac{n+3}{2} \end{array}; z\right) &= {}_3F_2\left(\begin{array}{c} -n, \frac{b}{2}, \frac{b+1}{2} \\ \frac{c}{2}, \frac{c+1}{2} \end{array}; z\right) \\ &= (n+1)\int_0^1 [t(1-zt^2)]^n dt. \end{aligned} \tag{4.2}$$

This last integral representation of F is useful when applying the method of the analysis of saddle-point.

Using (4.2) in conjunction with the analysis of saddle-point, K. Driver and S. Johnston proved the following results

Theorem 4.2.5 (cf. [10], Theorem 3.1) *For sufficiently large n, the polynomial* ${}_2F_1\left(\begin{array}{c} -n, \frac{n+1}{2} \\ \frac{n+3}{2} \end{array}; z\right)$ *has no zeros in the region* $Re(z) < \frac{1}{3}$

Theorem 4.2.6 (cf. [10], Theorem 4.1) *The zeros of the hypergeometric polynomial*

$$ {}_2F_1\left(\begin{array}{c} -n, \frac{n+1}{2} \\ \frac{n+3}{2} \end{array}; z\right) $$

approach the section of the lemniscate

$$ \{z : |z(1-z)^2| = \frac{4}{27}; Re(z) > \frac{1}{3}\}, $$

as $n \to \infty$.

Chapter 5

Asymptotic Zero Distribution of certain ${}_3F_2$ polynomials

5.1 Introduction

In the previous chapter, we showed the asymptotic behavior of zeros for some ${}_2F_1$ polynomials. In this chapter, we deal with the asymptotic zero distribution of some ${}_3F_2$ polynomials.

The chapter is divided as follows.

In Section 5.2, we shall give an alternative proof of ${}_3F_2\left(\begin{array}{c} -n, n+1, 1/2 \\ b+n+1, 1-b-n \end{array}; z\right)$ studied by K.Driver and K. Jordaan in [13].2

In Section 5.3, we shall study the asymptotic distribution of

$${}_3F_2\left(\begin{array}{c} -n, -pn-b, c \\ -pn-d, -n+t \end{array}; z\right), \text{where} \quad b>0, \quad c>0, \quad d>-1, \quad t<1, \quad \text{and} \quad p \geq 1$$

which is an extension of [14].

In section 5.4, we will consider the asymptotic zero distribution of

$${}_3F_2\left(\begin{array}{c}-n, pn-b, c\\ pn-d, -n+\lambda\end{array}; z\right) \text{ where } \quad p\geq 1, \quad c>0, \quad d<1, \quad \lambda<1, \quad \text{and} \quad -1<b<np.$$

In Section 5.5, we will deal with the class

$${}_3F_2\left(\begin{array}{c}-n, pn+b, c\\ pn+d, -n+l\end{array}; z\right) \text{ where } \quad b>0, \quad c>0, \quad d>-1, \quad l<1, \quad \text{and} \quad p\geq 1,$$

which is a generalization of one considered by K. Driver and K. Jordaan in [13].

We use the technique employed in [14] which utilizes classic analytical results such as Vitali's convergence theorem and Hurwitz' theorem, which we state here for the convenience of the reader.

Theorem 5.1.1 (cf. [29], 5.21, p.168) *Let $f_n(z)$ be a sequence of functions, each regular in a region D; let $|f_n(z)| \leq M$ for every $n \in \mathbb{N}$ and $z \in D$; and let $f_n(z)$ tend a limit, as $n \to \infty$, at a set of points having a limit-point inside D. Then $f_n(z)$ tends uniformly to a limit in any region bounded by a contour interior to D, the limit being, therefore, an analytic function of z.*

Theorem 5.1.2 (cf. [20], Theorem 14.3.4, p.205) *Let $f_n(z)$ be a sequence of functions, each analytic in a region bounded by a simple closed contour, and let $f_n(z) \to f(z)$ uniformly in D. Suppose that $f(z)$ is not identically zero. Let z_0 be an interior point of D. Then z_0 is a zero of $f(z)$ if , and only if, it is a limit-point of zeros of the functions $f_n(z)$, point which are zeros for an infinity of values of n being counted as limit-points.*

5.2 An alternative proof for the asymptotic zero behavior of ${}_3F_2\left(\begin{matrix}-n, n+1, 1/2\\ b+n+1, 1-b-n\end{matrix}; z\right)$

K.Driver and K. Jordaan proved (cf.[13], Theorem 2.2) that if $b > 0$, the zeros of ${}_3F_2\left(\begin{matrix}-n, n+1, 1/2\\ b+n+1, 1-b-n\end{matrix}; z\right)$ approach the unit circle when n tends to ∞. In this section, we propose an alternative proof of the result.

Theorem 5.2.1 (cf. [13], Theorem 2.2.) *For $b > 0$ fixed, the zeros of*

$$ {}_3F_2\left(\begin{matrix}-n, n+1, 1/2\\ b+n+1, 1-b-n\end{matrix}; z\right) $$

approach the unity circle as $n \to \infty$.

Proof Let us consider the hypergeometric function

$$ F = {}_3F_2\left(\begin{matrix}a, 1-a, 1/2\\ c, 2-c\end{matrix}; z\right), $$

where $a \in \mathbb{R}$, c and $2-c \notin \{0, -1, -2, \ldots\}$.

The function F has the simple representation (cf. [25], p.497, eqn.10)

$$ {}_3F_2\left(\begin{matrix}a, 1-a, 1/2\\ c, 2-c\end{matrix}; z\right) $$
$$ = {}_2F_1\left(\begin{matrix}a, 1-a\\ 2-c\end{matrix}; \frac{1-\sqrt{1-z}}{2}\right) {}_2F_1\left(\begin{matrix}a, 1-a\\ c\end{matrix}; \frac{1-\sqrt{1-z}}{2}\right) $$

Taking $a = -n$, and $c = b+n+1$, we get

$$ {}_3F_2\left(\begin{matrix}-n, n+1, 1/2\\ 1+b+n, 1-n-b\end{matrix}; z\right) $$

$$= \quad {}_2F_1\left(\begin{matrix}-n, 1+n\\ 1-n-b\end{matrix}; \frac{1-\sqrt{1-z}}{2}\right)\ {}_2F_1\left(\begin{matrix}-n, 1+n\\ 1+b+n\end{matrix}; \frac{1-\sqrt{1-z}}{2}\right)$$

By Proposition 2.3.5 for $b > 0$, the zeros $w = \frac{1-\sqrt{1-z}}{2}$ of ${}_2F_1\left(\begin{matrix}-n, 1+n\\ 1-n-b\end{matrix}; \frac{1-\sqrt{1-z}}{2}\right)$ and ${}_2F_1\left(\begin{matrix}-n, 1+n\\ 1+b+n\end{matrix}; \frac{1-\sqrt{1-z}}{2}\right)$ approach the curve $|w(1-w)| = \frac{1}{4}$. This means that the zeros of ${}_3F_2\left(\begin{matrix}-n, 1+n, 1/2\\ b, 2-b\end{matrix}; z\right)$ approach the curve $|(\frac{1-\sqrt{1-z}}{2})(1-\frac{1-\sqrt{1-z}}{2})| = \frac{1}{4}$ i.e. $|1-(1-z)| = 1$ as $n \to \infty$, which proves the result.

5.3 Asymptotic zero distribution of ${}_3F_2\left(\begin{matrix}-n, -pn-b, c\\ -pn-d, -n+t\end{matrix}; z\right)$

Let us consider $n \in \mathbb{N}$, $b > 0$, $c > 0$, $d > -1$, $p \geq 1$ and $t < 1$, and let set

$$\begin{aligned} F &= {}_3F_2\left(\begin{matrix}-n, -pn-b, c\\ -pn-d, -n+t\end{matrix}; z\right) \\ &= \sum_{k=0}^{n} r_{n,k} z^k \quad \text{where} \quad r_{n,k} = \frac{(-n)_k(-pn-b)_k(c)_k}{(-pn-d)_k(-n+t)_k k!} \end{aligned}$$

The main aim in this section is to establish that the zeros of F approach the unit circle when $n \to \infty$. Note that for $p = 2$, $c = b$, $d = 0$, and $t = \frac{1}{2}$, one gets the class of hypergeometric polynomials which was considered by K. Driver and M. Möller in [14]. The techniques used are the same that in [14].

Lemma 5.3.1 *Let n a natural number, $b > 0$, $c > 0$, $d > -1$, $p \geq 1$, γ the Euler constant, $t < 1$, and set*

$$r_{n,k} = \frac{(-n)_k(-pn-b)_k(c)_k}{(-pn-d)_k(-n+t)_k k!}, \quad \textit{with} \quad 0 \leq k \leq n$$

(i) For all integers $0 \leq k \leq n$,

$$r_{n,k} \leq (k+1)^{|b-d|+c+|t|} e^{\frac{|t|}{1-t}+\frac{|b-d|}{1+d}\gamma(|b-d|+c+|t|)}.$$

(ii) For all integers $1 \leq k \leq n$,

$$\frac{r_{n,n-k}}{r_{n,n}} \leq (k+1)^{|d-b|+|t|+1} e^{\frac{1}{c}+\gamma(|d-b|+|t|+1)}$$

Proof.

(i) For all integers $0 \leq k < n$, we have

$$\frac{r_{n,k+1}}{r_{n,k}} = \frac{(n-k)(pn-k+b)(k+c)}{(n-k-t)(pn-k+d)(k+1)} \tag{5.1}$$

This implies that for all integers $0 \leq k < n$,

$$\begin{aligned} \ln\left(\frac{r_{n,k+1}}{r_{n,k}}\right) &\leq \frac{t}{n-k-t} + \frac{b-d}{pn-k+d} + \frac{c-1}{k+1} \\ &\leq \frac{t}{n-k-t} + \frac{b-d}{pn-k+d} + \frac{c}{k+1} \\ &\leq \frac{|t|}{n-k-t} + \frac{|b-d|}{pn-k+d} + \frac{c}{k+1} \end{aligned}$$

Moreover, for all integers $\quad 1 \leq k \leq n, r_{n,k} = \prod_{j=0}^{k-1} \frac{r_{n,j+1}}{r_{n,j}}$.

Then, for $1 \leq k \leq n$,

$$\ln(r_{n,k}) = \ln\left(\prod_{j=0}^{k-1} \frac{r_{n,j+1}}{r_{n,j}}\right)$$

$$
\begin{aligned}
&\leq \sum_{j=0}^{k-1} \frac{|t|}{n-j-t} + \sum_{j=0}^{k-1} \frac{|b-d|}{pn-j+d} + \sum_{j=0}^{k-1} \frac{c}{j+1} \\
&= \sum_{j=0}^{k-1} \frac{|t|}{n-k+1-t+j} + \sum_{j=0}^{k-1} \frac{|b-d|}{pn-k+1+d+j} + \sum_{j=1}^{k} \frac{c}{j} \\
&\leq \frac{|t|}{1-t} + \frac{|b-d|}{1+d} + \sum_{j=1}^{k} \frac{|b-d|+c+|t|}{j} \\
&\leq \frac{|t|}{1-t} + \frac{|b-d|}{1+d} + (|b-d|+c+|t|)[\ln(k+1)+\gamma]
\end{aligned}
$$

by Lemma 1.9.1.

It follows that for all integers $1 \leq k \leq n, r_{n,k} \leq (k+1)^{|b-d|+c+|t|} e^{\frac{|t|}{1-t}+\frac{|b-d|}{1+d}} \gamma(|b-d|+c+|t|)$.

Since $r_{n,0} = 1$, then the relation holds for all integers $0 \leq k \leq n$.

(ii) From 5.1, for all integers $0 \leq k < n$, we have

$$
\frac{r_{n,k}}{r_{n,k+1}} = \frac{(n-k-t)(pn-k+d)(k+1)}{(n-k)(pn-k+b)(k+c)}.
$$

Then, for $1 \leq k \leq n$, $\dfrac{r_{n,n-k}}{r_{n,n-k+1}} = \dfrac{(k-t)((p-1)n+k+d)(n-k+1)}{k((p-1)n+k+b)(n-k+c)}$.

One obtains then

$$
\begin{aligned}
\ln\left(\frac{r_{n,n-k}}{r_{n,n-k+1}}\right) &\leq \frac{-t}{k} + \frac{d-b}{(p-1)n+k+b} + \frac{1-c}{n-k+c} \\
&\leq \frac{|t|}{k} + \frac{|d-b|}{k} + \frac{1}{n-k+c}
\end{aligned}
$$

In addition, for all integers $1 \leq k \leq n$, $\dfrac{r_{n,n-k}}{r_{n,n}} = \prod_{j=1}^{k} \dfrac{r_{n,n-j}}{r_{n,n-j+1}}$.

Thus, for all integers $1 \leq k \leq n$,

$$
\ln\left(\frac{r_{n,n-k}}{r_{n,n}}\right) = \sum_{j=1}^{k} \ln\left(\frac{r_{n,n-j}}{r_{n,n-j+1}}\right)
$$

$$
\begin{aligned}
&\le \sum_{j=1}^{k}\frac{|t|}{j}+\sum_{j=1}^{k}\frac{|d-b|}{(p-1)n+j+b}+\sum_{j=1}^{k}\frac{1}{n-j+c}\\
&= \sum_{j=1}^{k}\frac{|t|}{j}+\sum_{j=1}^{k}\frac{|d-b|}{(p-1)n+j+b}+\sum_{j=0}^{k-1}\frac{1}{n-k+c+j}\\
&\le \frac{1}{c}+\sum_{j=1}^{k}\frac{|d-b|+|t|+1}{j}\\
&\le \frac{1}{c}+(|d-b|+|t|+1)[\ln(k+1)+\gamma].
\end{aligned}
$$

Theorem 5.3.2 *Let $n\in\mathbb{N}$, $b>0$, $c>0$, $d>-1$, $p\ge 1$ and $t<1$, and let*

$$F={}_3F_2\left(\begin{matrix}-n,-pn-b,c\\ -pn-d,-n+t\end{matrix};z\right).$$

The zeros of F approach the unit circle as $n\to\infty$.

Proof.

$$\text{Let us set}\quad r_{n,k}=\frac{(-n)_k(-pn-b)_k(c)_k}{(-pn-d)_k(-n+t)_k k!}\quad\text{for}\quad 0\le k\le n.$$

Then $F=\sum_{k=0}^{n}r_{n,k}z^k$. According to Lemma5.3.1(i),

$$
\begin{aligned}
|\sum_{k=0}^{n}r_{n,k}z^k| &\le \sum_{k=0}^{n}r_{n,k}|z|^k\\
&\le e^{\frac{|t|}{1-t}+\frac{|b-d|}{1+d}+\gamma(|b-d|+c+|t|)}\sum_{k=0}^{n}(k+1)^{|b-d|+c+|t|}|z|^k
\end{aligned}
$$

Applying D'Alembert's ratio test, we observe that

$$
\begin{aligned}
\lim_{k\to\infty}\frac{(k+2)^{|b-d|+c+|t|}|z|^{k+1}}{(k+1)^{|b-d|+c+|t|}|z|^k} &= \lim_{k\to\infty}\left(\frac{1+\frac{2}{k}}{1+\frac{1}{k}}\right)^{|b-d|+c+|t|}|z|\\
&= |z|;
\end{aligned}
$$

so the radius of convergence of the series is 1 and the sequence of polynomials $\sum_{k=0}^{n}r_{n,k}z^k$ is uniformly bounded on $\Omega=\{z\in\mathbb{C}:|z|<\rho,\quad 0<\rho<1\}$.

Furthermore, since for fixed k,

$$\begin{aligned} r_{n,k} &= \frac{(-n)_k(-pn-b)_k(c)_k}{(-pn-d)_k(-n+t)_k k!} \\ &\to \frac{(c)_k}{(1)_k} \quad \text{as} \quad n \to \infty, \end{aligned}$$

the sequence $\sum_{k=0}^{n} r_{n,k} z^k$ converges pointwise, and therefore uniformly by Vitali's theorem, to

$$\sum_{k=0}^{n} \frac{(c)_k}{(1)_k} z^k = {}_1F_0(-c; -, z) = (1-z)^{-c}$$

Since the function $(1-z)^{-c}$ does not have any zeros inside the unit disc, by Hurwitz' theorem, there exists an index n_0 such that $\sum_{k=0}^{n} r_{n,k} z^k$ does not have zeros on Ω for $n > n_0$. Hence there exist numbers ρ_n, with $0 < \rho_n < 1$, so that $\rho_n \to 1$ and we can ensure that $\rho_n \geq \rho$ for $n > n_0$.

On the other hand, from Lemma 5.3.1(ii), the sequence of polynomials $\sum_{k=0}^{n} \frac{r_{n,n-k}}{r_{n,n}} z^k$ is uniformly bounded on Ω.

For fixed k, $1 \leq k \leq n$,

$$\begin{aligned} r_{n,n-k} &= \frac{(-n)_{n-k}(-pn-b)_{n-k}(c)_{n-k}}{(-pn-d)_{n-k}(-n+t)_{n-k}(n-k)!} \\ r_{n,n} &= \frac{(-n)_n(-pn-b)_n(c)_n}{(-pn-d)_n(-n+t)_n n!} \\ \text{This implies that} \quad \frac{r_{n,n-k}}{r_{n,n}} &= \frac{(-n)_k((p-1)n+d+1)_k(1-t)_k}{(-n-c+1)_k((p-1)n+b+1)_k(1)_k} \\ &\to \frac{(1-t)_k}{(1)_k} \text{ as n tends to } \infty. \end{aligned}$$

Hence $\displaystyle\sum_{k=0}^{n} \frac{r_{n,n-k}}{r_{n,n}}$ converges pointwise, and therefore uniformly by Vitali's theorem, to

$$\sum_{k=0}^{\infty} \frac{(1-t)_k}{(1)_k} z^k = (1-z)^{t-1}.$$

Since the latter function does not have any zeros inside the unit disc, by Hurwitz' theorem, there exists an index n_0 such that $\sum_{k=0}^{n} r_{n,k}z^k$ does not have zeros on Ω for $n > n_0$; this means that $\sum_{k=0}^{n} r_{n,k}z^k$ does not have zeros $|z| > \frac{1}{\rho}$.
Hence, there exists numbers π_n with $\pi_n > 1$, so that π_n tends to 1 and we can ensure that $\pi_n \le \frac{1}{\rho}$ for $n > n_0$.
Then all the zeros of $\sum_{k=0}^{n} r_{n,k}z^k$ lie in the annulus $\{z \in C : \rho_n \le |z| \le \pi_n\}$, which proves the result.

5.4 Asymptotic zero distribution of ${}_3F_2\left(\begin{matrix} -n, pn-b, c \\ pn-d, -n+\lambda \end{matrix}; z\right)$

We assume in this section that $n \in \mathbb{N}$, $c > 0$, $d < 1$, $\lambda < 1$, $p \ge 1$ and $-1 < b < pn$; and let

$$F = {}_3F_2\left(\begin{matrix} -n, pn-b, c \\ pn-d, -n+\lambda \end{matrix}; z\right)$$

Remark that this class of hypergeometric polynomials is different from the class considered in the previous section.

We want to study the asymptotic zero distribution of F. As for the last theorem, we shall use the following lemma.

Lemma 5.4.1 *Let $n \in \mathbb{N}$, $c > 0$, $d < 1$, $\lambda < 1$, $p \ge 1$ and*

$$-1 < b < pn; \quad \textit{and let set} \quad q_{n,k} = \frac{(-n)_k(pn-b)_k(c)_k}{(-n+\lambda)_k(pn-d)_k k!}$$

(1) For all integers $0 \le k \le n$,

$$q_{n,k} \le (k+1)^{c+|d-b|+|\lambda|} e^{\frac{|\lambda|}{1-\lambda}+\frac{|d-b|}{1-d}+\gamma(c+|d-b|+|\lambda|)}.$$

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(2) For all integers $1 \leq k \leq n$,

$$\frac{q_{n,n-k}}{q_{n,n}} \leq (k+1)^{|b-d|+1-\lambda} e^{\frac{|b-d|+1}{\delta}+\gamma(|b-d|+1-\lambda)}$$

where γ *is Euler's constant and* $\delta = \min\{pn-b, c\}$.

Proof.

(1) As for the last lemma, for all integers $0 \leq k < n$, we have

$$\frac{q_{n,k+1}}{q_{n,k}} = \frac{(n-k)(pn+k-b)(c+k)}{(n-k-\lambda)(pn+k-d)(k+1)} \tag{5.2}$$

So, for all integers $0 \leq k < n$,

$$\begin{aligned}\ln(\frac{q_{n,k+1}}{q_{n,k}}) &\leq \frac{\lambda}{n-k-\lambda} + \frac{d-b}{pn+k-d} + \frac{c-1}{k+1} \\ &\leq \frac{|\lambda|}{n-k-\lambda} + \frac{|d-b|}{pn+k-d} + \frac{c}{k+1}.\end{aligned}$$

And then for all integers $1 \leq k \leq n$,

$$\begin{aligned}\ln(q_{n,k}) &= \ln(\prod_{j=0}^{k-1} \frac{q_{n,j+1}}{q_{n,j}}) \\ &= \sum_{j=0}^{k-1} \ln(\frac{q_{n,j+1}}{q_{n,j}}) \\ &\leq \sum_{j=0}^{k-1} \frac{|\lambda|}{n-j-\lambda} + \sum_{j=0}^{k-1} \frac{|d-b|}{pn+j-d} + \sum_{j=0}^{k-1} \frac{c}{j+1} \\ &= \sum_{j=0}^{k-1} \frac{|\lambda|}{n-k+1-\lambda+j} + \sum_{j=0}^{k-1} \frac{|d-b|}{pn-d+j} + \sum_{j=1}^{k} \frac{c}{j} \\ &\leq \sum_{j=0}^{k-1} \frac{|\lambda|}{1-\lambda+j} + \sum_{j=0}^{k-1} \frac{|d-b|}{1-d+j} + \sum_{j=1}^{k} \frac{c}{j} \\ &\leq \frac{|\lambda|}{1-\lambda} + \frac{|d-b|}{1-d} + \sum_{j=1}^{k} \frac{|\lambda|+|d-b|+c}{j} \\ &\leq \frac{|\lambda|}{1-\lambda} + \frac{|d-b|}{1-d} + (c+|d-b|+|\lambda|)[\ln(k+1)+\gamma].\end{aligned}$$

Then the result is true for all integers $1 \leq k \leq n$. Since for $k = 0$, $q_{n,0} = 1$, then we obtain the claimed result.

(2) For all integers $0 \leq k < n$, one has (cf.5.2)

$$\frac{q_{n,k}}{q_{n,k+1}} = \frac{(n-k-\lambda)(pn+k-d)(k+1)}{(n-k)(pn+k-b)(k+c)},$$

which implies that for all integers $1 \leq k \leq n$,

$$\frac{q_{n,n-k}}{q_{n,n-k+1}} = \frac{(k-\lambda)((p+1)n-k-d)(n-k+1)}{k((p+1)n-k-b)(n-k+c)}$$

$$\begin{aligned}
\Rightarrow \ln(\frac{q_{n,n-k}}{q_{n,n-k+1}}) &\leq \frac{-\lambda}{k} + \frac{b-d}{(p+1)n-k-b} + \frac{1-c}{n-k+c} \\
&\leq \frac{-\lambda}{k} + \frac{b-d}{(p+1)n-k-b} + \frac{1}{n-k+c} \\
&\leq \frac{-\lambda}{k} + \frac{|b-d|}{(p+1)n-k-b} + \frac{1}{n-k+c}.
\end{aligned}$$

Now, for all integers $1 \leq k \leq n$,

$$\begin{aligned}
\ln(\frac{q_{n,n-k}}{q_{n,n}}) &\leq \sum_{j=1}^{k} \frac{-\lambda}{j} + \sum_{j=1}^{k} \frac{|b-d|}{(p+1)n-j-b} + \sum_{j=1}^{k} \frac{1}{n-j+c} \\
&= \sum_{j=1}^{k} \frac{-\lambda}{j} + \sum_{j=0}^{k-1} \frac{|b-d|}{(p+1)n-k-b+j} + \sum_{j=0}^{k-1} \frac{1}{n-k+c+j} \\
&\leq \sum_{j=1}^{k} \frac{-\lambda}{j} + \sum_{j=0}^{k-1} \frac{|b-d|}{n-k+\delta+j} + \sum_{j=0}^{k-1} \frac{1}{n-k+\delta+j}
\end{aligned}$$

with $\delta = \min\{pn-b, c\}$. The inequality becomes

$$\begin{aligned}
\ln(\frac{q_{n,n-k}}{q_{n,n}}) &\leq \sum_{j=1}^{k} \frac{-\lambda}{j} + \frac{|b-d|+1}{\delta} + \sum_{j=1}^{k} \frac{|b-d|+1}{j} \\
&\leq \frac{|b-d|+1}{\delta} + (|b-d|+1-\lambda)[\ln(k+1)+\gamma]
\end{aligned}$$

then $\frac{q_{n,n-k}}{q_{n,n}} \leq (k+1)^{|b-d|+1-\lambda} e^{\frac{|b-d|+1}{\delta}+\gamma(|b-d|+1-\lambda)}$.

Theorem 5.4.2 *Let $n \in \mathbb{N}$, $c > 0$, $d < 1$, $\lambda < 1$, $p \geq 1$ and $-1 < b < pn$. The zeros of the hypergeometric polynomial*

$$F = {}_3F_2\left(\begin{array}{c} -n, pn-b, c \\ pn-d, -n+\lambda \end{array}; z\right)$$

approach the unit circle as $n \to \infty$.

Proof. Since the proof is fundamentally the same as for Theorem 5.3.2, we shall just give the main lines.

Let us consider the sequence of polynomials $(\phi_n(z))_n$ where $\phi_n(z) = \sum_{j=0}^{n} q_{n,k} z^k$ with $q_{n,k} = \frac{(-n)_k(pn-b)_k(c)_k}{(-n+\lambda)_k(pn-d)_k k!}$. According to Lemma 5.4.1, the sequence $(\phi_n(z))_n$ is uniformly bounded on
$\Omega = \{z \in \mathbb{C} : |z| < \rho, 0 < \rho < 1\}$. Moreover, for fixed k, $0 \leq k \leq n$,

$$\begin{aligned} q_{n,k} &= \frac{(-n)_k(pn-b)_k(c)_k}{(-n+\lambda)_k(pn-d)_k k!} \\ &\to \frac{(c)_k}{(1)_k} \quad \text{as } n \text{ tends to} \quad \infty. \end{aligned}$$

Then, by Vitali's theorem, $\phi_n(z) \to (1-z)^{-c}$, uniformly when $n \to \infty$. Remark that for $1 \leq k \leq n$,

$$\begin{aligned} \frac{q_{n,n-k}}{q_{n,n}} &= \frac{(1-(p+1)n+d)_k(-n)_k(1-\lambda)_k}{(1-(p+1)n+b)_k(1-c-n)_k(1)_k} \\ &\to \frac{(1-\lambda)_k}{(1)_k}, \quad \text{when} \quad n \to \infty. \end{aligned}$$

So, we conclude that $q_{n,n}^{-1}\phi_n(z) \to (1-z)^{\lambda-1}$.

5.5 Asymptotic zero distribution of ${}_3F_2\left(\begin{matrix} -n, pn+b, c \\ pn+d, -n+l \end{matrix}; z\right)$

Let $b > 0$, $c > 0$, $d > -1$, $l < 1$, $n \in \mathbb{N}$, and $p \geq 1$; and let

$$F = {}_3F_2\left(\begin{matrix} -n, pn+b, c \\ pn+d, -n+l \end{matrix}; z\right).$$

In this section, we will focus on the asymptotic zero distribution of F.

Remark that this class of hypergeometric polynomials is both an extension of one considered by K. Driver and K. Jordaan in [13], and it is different to one studied in the previous section. We shall use same techniques just as for the last section.

Lemma 5.5.1 *Let $b > 0$, $c > 0$, $d > -1$, $l < 1$, $n \in \mathbb{N}$, and $p \geq 1$, and let us denote $g_{n,k} = \frac{(-n)_k (pn+b)_k (c)_k}{(pn+d)_k (-n+l)_k k!}$*

(1) For all integers $0 \leq k \leq n$, $g_{n,k} \leq (k+1)^{b+c} e^{\frac{|l|}{1-l} + \frac{|b-d|}{1+d} + \gamma(|b-d|+|l|+c)}$.

(2) For integers $1 \leq k \leq n$, $\frac{g_{n,n-k}}{g_{n,n}} \leq (k+1)^{|d-b|+1-l} e^{\frac{|d-b|}{b} + \frac{1}{c} + \gamma(|d-b|+1-l)}$;

with γ the Euler constant.

Proof.

(1) For all integers $0 \leq k < n$,

$$\begin{aligned} \frac{g_{n,k+1}}{g_{n,k}} &= \frac{(-n+k)(pn+b+k)(c+k)}{(pn+d+k)(-n+l+k)(k+1)} \\ &= \frac{(n-k)(pn+k+b)(c+k)}{(n-k-l)(pn+k+d)(k+1)} \end{aligned} \tag{5.3}$$

Using the logarithm function, for $0 \leq k < n$, one gets

$$\begin{aligned}\ln(\frac{g_{n,k+1}}{g_{n,k}}) &\leq \frac{l}{n-k-l} + \frac{b-d}{pn+k+d} + \frac{c-1}{k+1} \\ &\leq \frac{|l|}{n-k-l} + \frac{|b-d|}{pn+k+d} + \frac{c}{k+1}.\end{aligned}$$

Then for $1 \leq k \leq n$,

$$\begin{aligned}\ln(g_{n,k}) &\leq \sum_{j=0}^{k-1} \frac{|l|}{n-j-l} + \sum_{j=0}^{k-1} \frac{|b-d|}{pn+j+d} + \sum_{j=0}^{k-1} \frac{c}{j+1} \\ &= \sum_{j=0}^{k-1} \frac{|l|}{n-k+1-l+j} + \sum_{j=0}^{k-1} \frac{|b-d|}{pn+j+d} + \sum_{j=1}^{k} \frac{c}{j} \\ &\leq \frac{|l|}{1-l} + \sum_{j=1}^{k} \frac{|l|}{j} + \frac{|b-d|}{1+d} + \sum_{j=1}^{k} \frac{|b-d|}{j} + \sum_{j=1}^{k} \frac{c}{j} \\ &\leq \frac{|l|}{1-l} + \frac{|b-d|}{1+d} + (|b-d|+c+|l|)[\ln(k+1)+\gamma].\end{aligned}$$

So, for all integers $1 \leq k \leq n$, one has $g_{n,k} \leq (k+1)^{|b-d|+c+|l|} e^{\frac{|l|}{1-l} + \frac{|b-d|}{1+d} + \gamma(|b-d|+c+|l|)}$. Since $g_{n,0} = 1$, then the conclusion follows.

(2) For all integers $0 \leq k < n$, from eqn 5.3, we have

$$\frac{g_{n,k}}{g_{n,k+1}} = \frac{(n-k-l)(pn+k+d)(k+1)}{(n-k)(pn+k+b)(c+k)}$$

This implies that for $1 \leq k \leq n$, $\frac{g_{n,n-k}}{g_{n,n-k+1}} = \frac{(k-l)((p+1)n-k+d)(n-k+1)}{k((p+1)n-k+b)(n-k+c)}$, which gives

$$\begin{aligned}\ln(\frac{g_{n,n-k}}{g_{n,n-k+1}}) &\leq \frac{-l}{k} + \frac{d-b}{((p+1)n-k+b)} + \frac{1-c}{n-k+c} \\ &\leq \frac{-l}{k} + \frac{|d-b|}{((p+1)n-k+b)} + \frac{1}{n-k+c}.\end{aligned}$$

So, for all integers $1 \leq k \leq n$, we get

$$\ln(\frac{g_{n,n-k}}{g_{n,n}}) \leq \sum_{j=1}^{k} \frac{-l}{j} + \sum_{j=1}^{k} \frac{|d-b|}{(p+1)n-j+b} + \sum_{j=1}^{k} \frac{1}{n-j+c}$$

$$\begin{aligned}
&= \sum_{j=1}^{k} \frac{-l}{j} + \sum_{j=0}^{k-1} \frac{|d-b|}{(p+1)n-k+b+j} \sum_{j=0}^{k-1} \frac{1}{n-k+c+j} \\
&\leq \sum_{j=1}^{k} \frac{-l}{j} + \frac{|d-b|}{b} + \frac{1}{c} + \sum_{j=1}^{k} \frac{|d-b|+1}{j} \\
&\leq \frac{|d-b|}{b} + \frac{1}{c} + (|d-b|+1-l)[\ln(k+1)+\gamma]
\end{aligned}$$

and the result follows.

Theorem 5.5.2 *Let $b > 0$, $c > 0$, $d > -1$, $l < 1$, $n \in \mathbb{N}$, and $p \geq 1$, and let set*

$$F = {}_3F_2\left(\begin{array}{c} -n, pn+b, c \\ pn+d, -n+l \end{array} ; z \right).$$

The zeros of F approach the unit circle as $n \to \infty$.

Proof. By applying a similar argument to that one for the proof of Theorem 5.4.2, we will give just the main lines.

The sequence $(\sum_{k=0}^{n} g_{n,k} z^k)_n$, where $g_{n,k} = \frac{(-n)_k (pn+b)_k (c)_k}{(pn+d)_k (-n+l)_k k!}$, is uniformly bounded on $\Omega = \{z \in \mathbb{C} : |z| < \rho, 0 < \rho < 1\}$ by Lemma 5.3(1).

Furthermore, for fixed k, $0 \leq k \leq n$,

$$\begin{aligned}
g_{n,k} &= \frac{(-n)_k (pn+b)_k (c)_k}{(pn+d)_k (-n+l)_k k!} \\
&\to \frac{(c)_k}{k!}, \quad \text{as} \quad n \to \infty
\end{aligned}$$

With Vitali's theorem, we conclude that $\lim_{n\to\infty} \sum_{k=0}^{n} g_{n,k} z^k = (1-z)^{-c}$.

Also, for $1 \leq k \leq n$,

$$\frac{g_{n,n-k}}{g_{n,n}} = \frac{(-n)_k (1-(p+1)n-d)_k (1-l)_k}{(-n-c+l)_k (1-(p+1)n-b)_k (1)_k},$$

then we have
$$\begin{aligned}
\lim_{n\to\infty} \frac{g_{n,n-k}}{g_{n,n}} &= \lim_{n\to\infty} \frac{(-n)_k (1-(p+1)n-d)_k (1-l)_k}{(-n-c+l)_k (1-(p+1)n-b)_k (1)_k} \\
&= \frac{(1-l)_k}{(1)_k}.
\end{aligned}$$

We come to conclude that $\displaystyle\lim_{n\to\infty}\sum_{k=0}^{n}\frac{g_{n,n-k}}{g_{n,n}}z^k = (1-z)^{l-1}.$

And the result follows.

5.6 Conclusion

In this thesis we focussed on zeros of hypergeometric polynomials.

In chapter 2 and in chapter 4, we reviewed some work done on the zero location and the asymptotic zero behavior of Gauss hypergeometric polynomials respectively.

We remarked that different techniques are used for the location of the zeros or for the analysis of the asymptotic zero distribution. Some techniques capitalized on some relations linking Gauss hypergeometric polynomials and other classical orthogonal polynomials; others used analytic tools such as the analysis of saddle-points.

In chapter 3, we extended some contiguous relations of ${}_2F_1$ functions. These extended relations gave enabled us to locate the zeros of some classes of Gauss hypergeometric polynomials with non-real parameters.

In chapter 5, we considered the asymptotic zero distribution of some classes of ${}_3F_2$ hypergeometric polynomials. In particular, we dealt with ${}_3F_2\left(\begin{array}{c}-n,-pn-b,c\\ -pn-d,-n+t\end{array};z\right)$, ${}_3F_2\left(\begin{array}{c}-n,pn-b,c\\ pn-d,-n+\lambda\end{array};z\right)$ and ${}_3F_2\left(\begin{array}{c}-n,pn+b,c\\ pn-d,-n+l\end{array};z\right)$ with some constraints on the parameters.

The techniques used in this chapter are the same as in [14].

The class ${}_3F_2\left(\begin{array}{c}-n,-pn-b,c\\ -pn-d,-n+t\end{array};z\right)$, with $b>0$, $c>0$, $d>-1$, $p\geq 1$ and $t<1$, is an extension of the class ${}_3F_2\left(\begin{array}{c}-n,-2n-b,b\\ -2n,-n+\frac{1}{2}\end{array};z\right)$

with $b > 0$ considered by K. Driver and M. Möller in [14]; while the class ${}_3F_2\left(\begin{matrix} -n, pn+b, c \\ pn+d, -n+l \end{matrix}; z\right)$ where $p \geq 1$, $b > 0$, $c > 0$, $d < 1$, $l < 1$ is a generalization of the class ${}_3F_2\left(\begin{matrix} -n, n+1, \frac{1}{2} \\ n+b+1, 1-b-n \end{matrix}; z\right)$ studied by K. Driver and K. Jordaan in [13].

As the next step, it may be interesting to consider classes of polynomials, where the zeros approach circles with different radii.

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