APPENDIX A

SELECTED PROPERTIES OF THE Q-FUNCTION

This appendix contains a number of properties of the Q function that are used in the text of this dissertation.

1.
$$Q(x) = \int_x^\infty \frac{1}{2\pi} e^{-t^2/2} dt. \tag{A.1}$$

2.
$$Q(x) = P[X > x], \tag{A.2}$$

where X is a zero-mean, unit-variance Gaussian random variable.

3. $Q(x)$ is monotonically decreasing.

4.
$$\lim_{\sigma \to 0} \frac{Q\left(\frac{\alpha}{\sigma}\right)}{Q\left(\frac{\beta}{\sigma}\right)} = \begin{cases} +\infty, & [\alpha]^+ < \beta; \\ 2, & \alpha < \beta = 0; \\ 1, & \alpha = \beta \text{ or } \max\{\alpha, \beta\} < 0; \\ 1/2, & \beta < \alpha = 0; \\ 0, & [\beta]^+ < \alpha; \end{cases} \tag{A.3}$$

where

$$[z]^+ = \max\{0, z\}. \tag{A.4}$$

5.
$$2 \lim_{\sigma \to 0} \sigma^2 \log Q\left(\frac{x}{\sigma}\right) = -\left([x]^+\right)^2. \tag{A.5}$$

6. If X is a zero-mean, unit variance, normal random variable, then

$$E\left[Q\left(\mu+\lambda X\right)\right]=Q\left(\frac{\mu}{\sqrt{1+\lambda^2}}\right). \tag{A.6}$$

7.

$$\int_0^\infty x\exp\left(-\frac{x^2}{2}\right)Q\left(\frac{x}{\sigma}\right)dx=\frac{1}{2}\left(1-\frac{1}{\sqrt{1+\sigma^2}}\right). \tag{A.7}$$

APPENDIX B

SIMULATION OF MOBILE CHANNEL

This appendix contains the mobile channel model used to obtain the simulation results. In the first section, the Doppler filter is described, and how it is used in a baseband Rayleigh fading simulator. In the second section, the Rayleigh fading simulator in the first section is used to implement several independent fading paths in a mobile fading simulator.

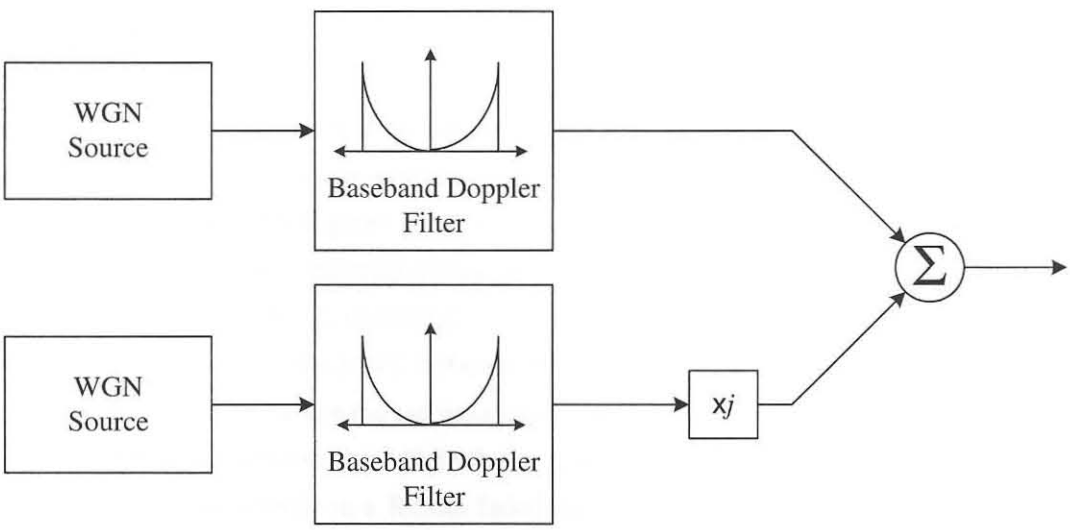

Figure B.2: Baseband complex Rayleigh fading coefficient simulator.

B.1 DOPPLER SPREAD RAYLEIGH FADING

To be able to simulate Clarke's model, we will have to analyze the spectrum of received electric field. This is derived in [32] and is shown to be

$$S_{E_z}(f) = \frac{1.5}{\pi f_m \sqrt{1 - \left(\frac{f-f_c}{f_m}\right)}} \tag{B.1}$$

with a vertical $\lambda/4$ antenna ($G(\alpha) = 1.5$), and a uniform distribution of incoming power over 0 to 2π. In this equation, f_m is maximum Doppler shift and f_c the carrier frequency. The baseband power spectral density is given by equation (B.2),

$$S_{bbE_z}(f) = \frac{1}{8\pi f_m} K \left[\sqrt{1 - \left(\frac{f}{2f_m}\right)^2} \right], \tag{B.2}$$

where $K[\cdot]$ is the complete elliptical integral of the first kind. When we wish to simulate the Rayleigh fading channel, we can do this by sending both the in-phase and quadrature baseband independent Gaussian noise samples through baseband filters with the transfer function given in equation (B.2). The resulting complex signal can then be utilized as a complex Rayleigh fading coefficient in a baseband simulation environment.

In this dissertation, a infinite impulse response (IIR) third order approximation of a 50Hz Doppler filter is used for simulation purposes. The power spectral density of this filter is shown in Figure B.3.

B.2 FREQUENCY SELECTIVE MULTIPATH RAYLEIGH FADING

In the mobile channel, the receiver antenna picks up the sum of independent Rayleigh (Doppler spread) faded multiple paths. In the previous section, the simulator that implements the fading coefficients of each of these paths is given. Here we present the simulation model to implement multiple independent fading paths with different delays. This introduces the harsh frequency selectivity that is frequently encountered in mobile channel environments. The frequency selective (multipath) mobile channel model for simulation purposes is shown in Figure B.4. This is basically a finite impulse response (FIR) filter structure, where the tap weights are the path strengths corresponding to each of the time delay- or multipath components. Note that to get a line-of-sight component, an unfaded signal may be added. This will result in a Ricean faded signal. It is possible to vary the strength of the specular (line-of-sight) path by a constant weighting of this component.

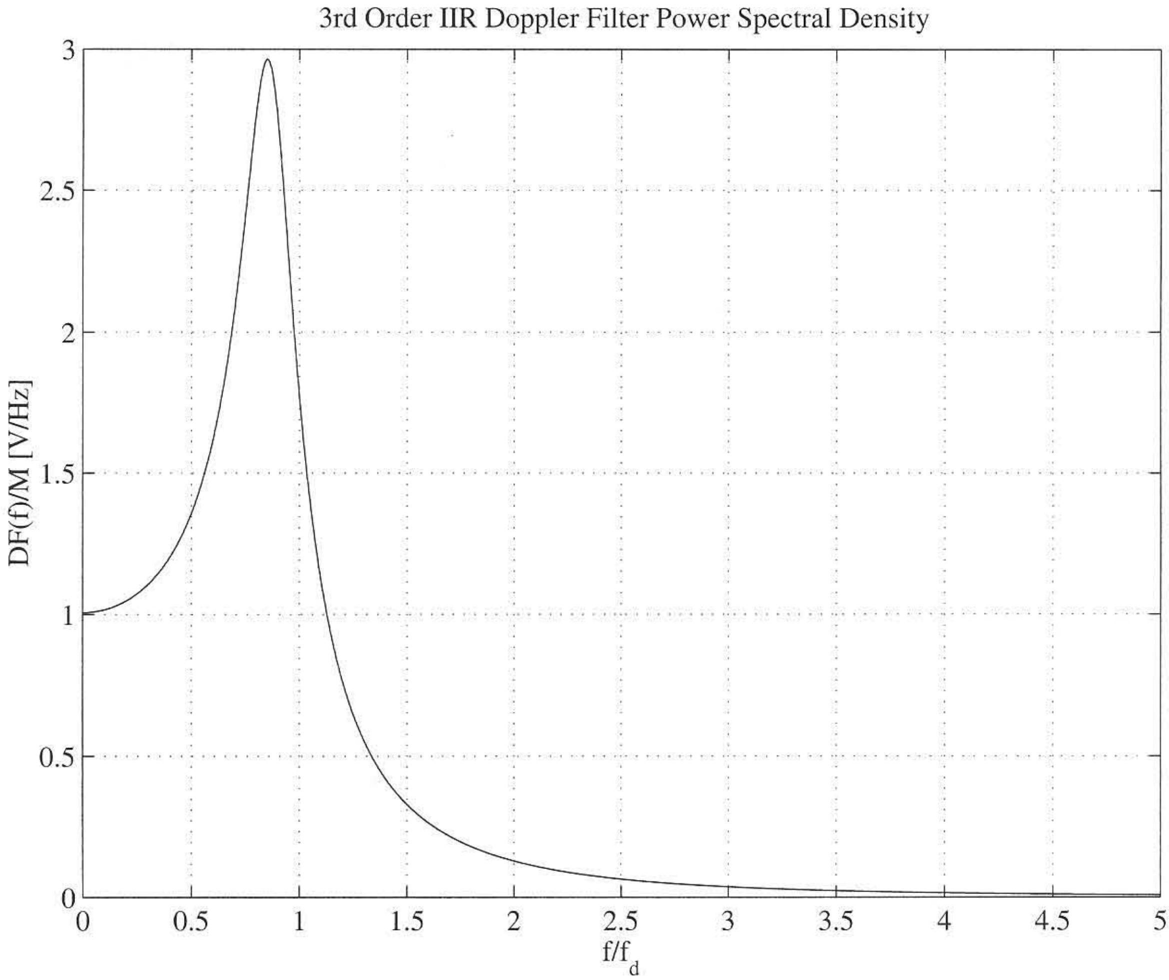

Figure B.3: Frequency spectrum of 3rd order approximation of a Doppler filter with a Doppler frequency of 50Hz.

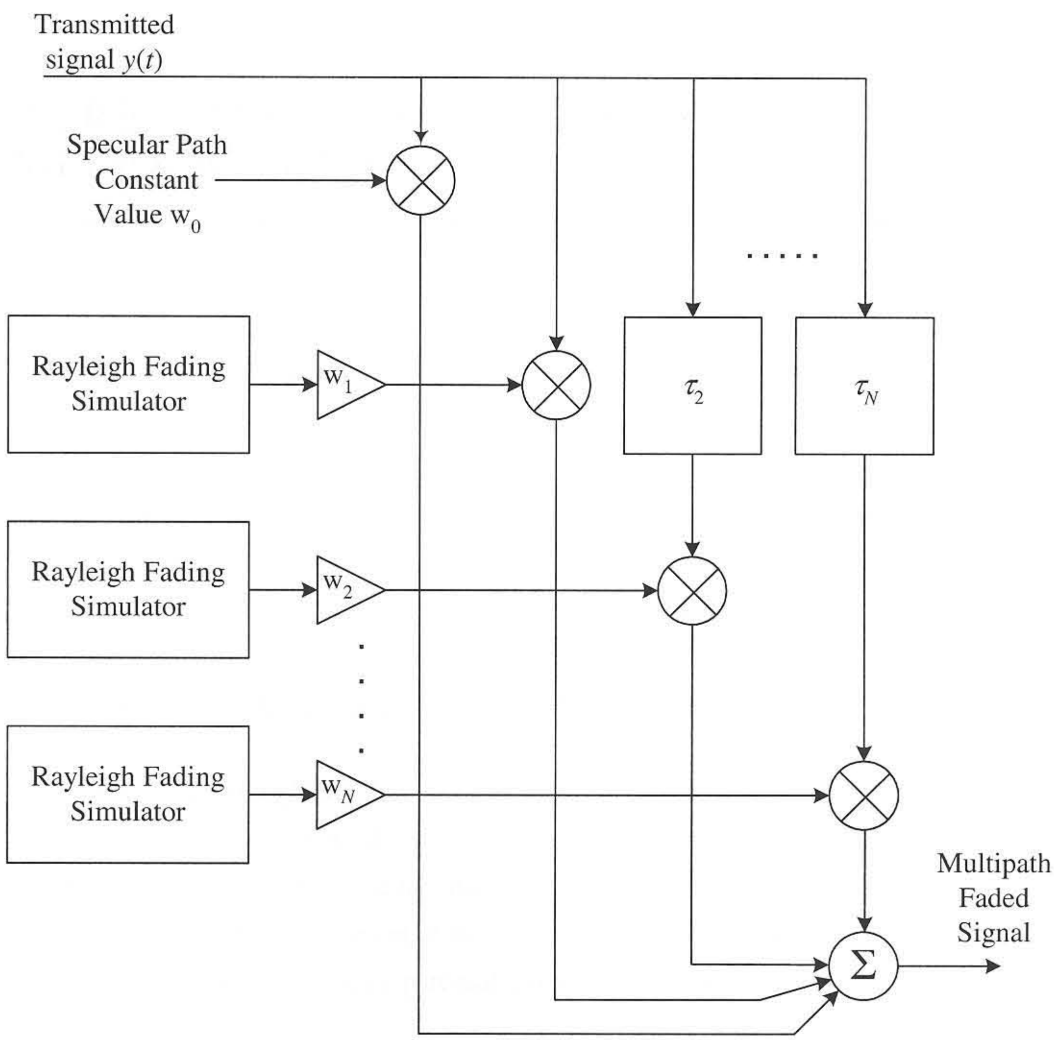

Figure B.4: Model of a frequency selective (multipath) fading channel.

APPENDIX C

DIFFERENTIATION WITH RESPECT TO A COMPLEX MATRIX

An issue commonly encountered in the study of optimization theory, is that of differentiating a cost function with respect to a parameter vector or matrix. In the text, the normal gradient operator is used. The differentiation of a cost function with respect to a complex vector or matrix is more involved. This appendix will expand on the case of differentiation with respect to a complex vector in Appendix B of [57], to the case of differentiation with respect to a complex matrix. The relationship between the concepts of a gradient and a derivative for complex matrices is discussed here.

C.1 BASIC DEFINITIONS

Consider a complex function $f(\mathbf{M})$ that is dependent on a parameter matrix $\mathbf{M}$. When the entries of $\mathbf{M}$ are complex valued, there are two different mathematical concepts that require individual attention: (1) the matrix nature of $\mathbf{M}$, and (2) the fact that each entry of $\mathbf{M}$ is complex valued.

Let us start with the fact that the element of the of the lth row and the mth column of the matrix $\mathbf{M}$ is the sum of a real part and an imaginary part multiplied by $j = \sqrt{-1}$, i.e.

$$m_{vw} = x_{vw} + jy_{vw}. \tag{C.1}$$

The real and imaginary parts of equation (C.1) can alternatively be written in terms of the pair of complex conjugates m_{vw} and m^*_{vw} with

$$x_{vw} = \frac{1}{2}\left(m_{vw} + m^*_{vw}\right) \tag{C.2}$$

and

$$y_{vw} = \frac{1}{2j}\left(m_{vw} - m^*_{vw}\right), \tag{C.3}$$

where $(\cdot)^*$ denotes complex conjugation. It is evident that the real quantities x_{vw} and y_{vw} are functions of both m_{vw} and m^*_{vw}. Only when we deal with an analytic function f, may we abandon the complex-conjugated term m^*_{vw} by virtue of the Cauchy-Riemann equations. However, it is rare that one encounters analytic functions in physical sciences and engineering.

When considering a derivative, a connection should be made with the concept of a differential. In particular, the chain rule of changes of variables must be satisfied. Considering these points, the mathematician Schwartz [70] defined certain complex derivatives in terms of real derivatives:

$$\frac{\partial}{\partial m_{vw}} = \frac{1}{2}\left(\frac{\partial}{\partial x_{vw}} - j\frac{\partial}{\partial y_{vw}}\right) \tag{C.4}$$

and

$$\frac{\partial}{\partial m^*_{vw}} = \frac{1}{2}\left(\frac{\partial}{\partial x_{vw}} + j\frac{\partial}{\partial y_{vw}}\right). \tag{C.5}$$

The above derivatives satisfy the following basic requirements with respect to a differential:

$$\frac{\partial m_{vw}}{\partial m_{vw}} = 1 \tag{C.6}$$

and

$$\frac{\partial m_{vw}}{\partial m^*_{vw}} = \frac{\partial m^*_{vw}}{\partial m_{vw}} = 0. \tag{C.7}$$

(An analytic function f satisfies $\frac{\partial f}{\partial z^*} = 0$ everywhere.) Equations (C.4) and (C.5) are referred to as the *derivative* and the *conjugate derivative* respectively, both with respect to m_{vw}.

Let us extend this notion to the general case of the derivative with respect to a matrix with complex elements. Extension of the derivative with respect to a complex vector was done in [71] and was also dealt with in Appendix B of [57]. We can extend equations (C.4) and (C.5) to the derivative with respect to a $K \times K$ matrix:

$$\frac{\partial}{\partial \mathrm{M}} = \frac{1}{2}\begin{bmatrix} \frac{\partial}{\partial x_{11}} - j\frac{\partial}{\partial y_{11}} & \frac{\partial}{\partial x_{12}} - j\frac{\partial}{\partial y_{12}} & \cdots & \frac{\partial}{\partial x_{1K}} - j\frac{\partial}{\partial y_{1K}} \\ \frac{\partial}{\partial x_{21}} - j\frac{\partial}{\partial y_{21}} & \frac{\partial}{\partial x_{22}} - j\frac{\partial}{\partial y_{22}} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial}{\partial x_{K1}} - j\frac{\partial}{\partial y_{K1}} & \cdots & \cdots & \frac{\partial}{\partial x_{KK}} - j\frac{\partial}{\partial y_{KK}} \end{bmatrix} \tag{C.8}$$

and

$$\frac{\partial}{\partial \mathrm{M}^*} = \frac{1}{2}\begin{bmatrix} \frac{\partial}{\partial x_{11}} + j\frac{\partial}{\partial y_{11}} & \frac{\partial}{\partial x_{12}} + j\frac{\partial}{\partial y_{12}} & \cdots & \frac{\partial}{\partial x_{1K}} + j\frac{\partial}{\partial y_{1K}} \\ \frac{\partial}{\partial x_{21}} + j\frac{\partial}{\partial y_{21}} & \frac{\partial}{\partial x_{22}} + j\frac{\partial}{\partial y_{22}} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial}{\partial x_{K1}} + j\frac{\partial}{\partial y_{K1}} & \cdots & \cdots & \frac{\partial}{\partial x_{KK}} + j\frac{\partial}{\partial y_{KK}} \end{bmatrix}. \tag{C.9}$$

Analogous to the scalar case, the above two derivatives obey the following relations:

$$\frac{\partial \mathrm{M}}{\partial \mathrm{M}} = \mathbf{I} \tag{C.10}$$

and

$$\frac{\partial \mathrm{M}}{\partial \mathrm{M}^*} = \frac{\partial \mathrm{M}^*}{\partial \mathrm{M}} = \mathbf{0} \tag{C.11}$$

where $\mathbf{I}$ and $\mathbf{0}$ are the $K^2 \times K^2$ identity and null matrices respectively.

For the purpose of differentiating with respect to complex matrices, equation (C.9) (the *conjugate derivative*) will be adopted as the *derivative with respect to a complex valued matrix.*

C.2 THE GRADIENT MATRIX IN TERMS OF THE DERIVATIVE WITH RESPECT TO A MATRIX

Consider a real cost function $J(\mathrm{M})$ that defines the $K \times K$ dimension error performance surface in terms of the $K \times K$ matrix M. The *complex gradient matrix* was defined in Chapter 4 as

$$\nabla_{\mathrm{M}}(J) = \begin{bmatrix} \frac{\partial J}{\partial x_{11}} + j\frac{\partial J}{\partial y_{11}} & \cdots & \frac{\partial J}{\partial x_{1K}} + j\frac{\partial J}{\partial y_{1K}} \\ \vdots & \ddots & \vdots \\ \frac{\partial J}{\partial x_{K1}} + j\frac{\partial J}{\partial y_{K1}} & \cdots & \frac{\partial J}{\partial x_{KK}} + j\frac{\partial J}{\partial y_{KK}} \end{bmatrix}, \tag{C.12}$$

where $x_{vw} + jy_{vw}$ is the element of the vth row and the wth column of the linear transformation M. The gradient matrix is *normal* to the $K \times K$ dimensional error surface. If we relate (C.12) to (C.9), we find that the gradient is related to the conjugate derivative by

$$\nabla_{\mathrm{M}}(J) = 2\frac{\partial J}{\partial \mathrm{M}^*}. \tag{C.13}$$

This means that the gradient is a scaled form of the conjugate derivative of (C.9).

C.3 DIFFERENTIATING THE COMPONENTS OF THE MMSE COST FUNCTION

With these preceding ideas in mind, let us attempt to differentiate the cost function in (4.29) to arrive at the result in (4.31). Let us start with the gradient of the cost function:

$$4\frac{\partial}{\partial \mathbf{M}^*}\left(\mathrm{tr}\left\{\mathbf{I}\right\} - \mathrm{tr}\left\{\mathbf{ARM}^H\right\} - \mathrm{tr}\left\{\mathbf{MAR}\right\} + \mathrm{tr}\left\{\mathbf{MRA}^2\mathbf{RM}^H\right\} + \mathrm{tr}\left\{\mathbf{M}\sigma^2\mathbf{RM}^H\right\}\right). \tag{C.14}$$

Since differentiation is a linear operation, we can differentiate the terms of the cost function individually. It is trivial to see that the differentiation of the first term results in zero, i.e.

$$\frac{\partial}{\partial \mathbf{M}^*}\mathrm{tr}\left\{\mathbf{I}\right\} = 0. \tag{C.15}$$

To find the derivative of the second term of (C.14), we will exploit the definition of the trace of a matrix. We start with the ijth element of the product $\mathbf{ARM}^H$. We have

$$\left(\mathbf{ARM}^H\right)_{ij} = \sum_l a_{il}(\mathbf{RM})_{lj} = \sum_l\sum_k a_{il}r_{lk}m^*_{jk}, \tag{C.16}$$

where a_{il}, r_{lk} and m^*_{jk} are the complex elements of the matrices $\mathbf{A}$, $\mathbf{R}$ and $\mathbf{M}^H$ respectively. The diagonal elements of the product $\mathbf{ARM}^H$ are

$$\left(\mathbf{ARM}^H\right)_{ii} = \sum_l\sum_k a_{il}r_{lk}m^*_{ik}. \tag{C.17}$$

Since the trace of a matrix (or product of matrices) is the sum of the diagonal elements, we have

$$\mathrm{tr}\left\{\mathbf{ARM}^H\right\} = \sum_i\sum_l\sum_k a_{il}r_{lk}m^*_{ik}. \tag{C.18}$$

To apply $\frac{\partial}{\partial \mathbf{M}^*}$ to the trace of $\mathbf{ARM}^H$, we turn our attention to the derivative with respect to the individual elements $\frac{\partial}{\partial m^*_{ik}}$. Applying this to (C.18), we obtain

$$\begin{aligned}
\frac{\partial}{\partial m^*_{ik}}\left(\mathrm{tr}\left\{\mathbf{ARM}^H\right\}\right) &= \frac{\partial}{\partial m^*_{ik}}\left(\sum_i\sum_l\sum_k a_{il}r_{lk}m^*_{ik}\right) \\
&= \sum_l a_{il}r_{lk} \qquad \text{(C.19)} \\
&= (\mathbf{AR})_{ik}, \qquad \text{(C.20)}
\end{aligned}$$

and consequently

$$\frac{\partial}{\partial \mathbf{M}^*}\left(\mathrm{tr}\left\{\mathbf{ARM}^H\right\}\right) = \mathbf{AR}. \tag{C.21}$$

To determine the complex derivative of the third term, $\frac{\partial}{\partial \mathbf{M}^*}\left(\operatorname{tr}\{\mathbf{MAR}\}\right)$, we follow similar reasoning. In this case we have

$$\operatorname{tr}\{\mathbf{MAR}\} = \sum_i \sum_l \sum_k m_{il} a_{lk} r_{ki}. \tag{C.22}$$

Again, applying the derivative with respect to the individual elements $\frac{\partial}{\partial m_{ik}^*}$ we get

$$\begin{aligned} \frac{\partial}{\partial m_{il}^*}\left(\operatorname{tr}\{\mathbf{MAR}\}\right) &= \frac{\partial}{\partial m_{il}^*}\left(\sum_i \sum_l \sum_k m_{il} a_{lk} r_{ki}\right) \\ &= 0, \end{aligned} \tag{C.23}$$

since $\frac{\partial m_{il}}{\partial m_{il}^*} = 0$ from (C.7). Thus we have

$$\frac{\partial}{\partial \mathbf{M}^*}\left(\operatorname{tr}\{\mathbf{MAR}\}\right) = \mathbf{0}, \tag{C.24}$$

where $\mathbf{0}$ is the null matrix.

The derivatives of quadratic terms in (C.14) can be evaluated by means of the product rule and following the same reasoning as with the previous two terms. Let us first evaluate

$$\frac{\partial}{\partial \mathbf{M}^*}\operatorname{tr}\left\{\mathbf{MRA}^2\mathbf{RM}^H\right\}. \tag{C.25}$$

To simplify matters, we let $\mathbf{B} = \mathbf{RA}^2\mathbf{R}$. This means that

$$\operatorname{tr}\left\{\mathbf{MBM}^H\right\} = \sum_i \sum_l \sum_k m_{il} b_{lk} m_{ik}^*. \tag{C.26}$$

Differentiating (C.26) by means of the product rule, we obtain

$$\begin{aligned} \frac{\partial}{\partial m_{il}^*}\left(\operatorname{tr}\left\{\mathbf{MBM}^H\right\}\right) &= \frac{\partial}{\partial m_{il}^*}\left(\sum_i \sum_l \sum_k m_{il} b_{lk} m_{ki}^*\right) \\ &= 0 + \sum_l m_{il} b_{lk} \qquad \text{(C.27)} \\ &= (\mathbf{MB})_{ik} \qquad \text{(C.28)} \end{aligned}$$

where (C.27) follows from (C.6) and (C.7). Thus we have

$$\frac{\partial}{\partial \mathbf{M}^*}\left(\operatorname{tr}\left\{\mathbf{MBM}^H\right\}\right) = \mathbf{MB}. \tag{C.29}$$

If we apply this result to the two quadratic terms, we obtain

$$\frac{\partial}{\partial \mathbf{M}^*}\left(\operatorname{tr}\left\{\mathbf{MRA}^2\mathbf{RM}^H\right\}\right) = \mathbf{MRA}^2\mathbf{R} \tag{C.30}$$

and

$$\frac{\partial}{\partial \mathbf{M}^*}\left(\mathrm{tr}\left\{\mathbf{M}\sigma^2\mathbf{R}\mathbf{M}^H\right\}\right) = \mathbf{M}\sigma^2\mathbf{R}. \tag{C.31}$$

Using all of the above results, the gradient of the cost function J is given by

$$\nabla_{\mathbf{M}}(J) = -4\mathbf{AR} + 4\mathbf{MRA}^2\mathbf{R} + 4\mathbf{M}\sigma^2\mathbf{R} \tag{C.32}$$

Thus verifying eq. (4.31).

APPENDIX D

EVALUATING THE EXPECTED VALUE IN THE LCCM AND LCDCM COST FUNCTIONS

D.1 EXPECTED VALUE IN THE LCCM COST FUNCTION

From the first term in equation (5.3) we have

$$\begin{aligned} E\left[\mathbf{u}^H\mathbf{b}\mathbf{b}^H\mathbf{u}\mathbf{u}^H\mathbf{b}\mathbf{b}^H\mathbf{u}\right] &= E\left[\left(\sum_{i=1}^{K}\sum_{j=1}^{K} u_i^* b_i b_j^* u_j\right)\left(\sum_{k=1}^{K}\sum_{l=1}^{K} u_k^* b_k b_l^* u_l\right)\right] \\ &= E\left[\sum_{i=1}^{K}\sum_{j=1}^{K}\sum_{k=1}^{K}\sum_{l=1}^{K} u_i^* b_i b_j^* u_j u_k^* b_k b_l^* u_l\right] \\ &= E\left[\sum_{i=1}^{K}\sum_{j=1}^{K}\sum_{k=1}^{K}\sum_{l=1}^{K} u_i^* u_j u_k^* u_l b_i b_j^* b_k b_l^*\right] \\ &= \sum_{i=1}^{K}\sum_{j=1}^{K}\sum_{k=1}^{K}\sum_{l=1}^{K} \left(u_i^* u_j u_k^* u_l E\left[b_i b_j^* b_k b_l^*\right]\right). \end{aligned} \tag{D.1}$$

Consider equation (D.1). Since the different users' bits are independent, a zero result will be produced if $i \neq j \neq k \neq l$. If $i = k$ and $l = j$, and $b_k \in \{\pm 1 \pm j\}$ the argument of the expected value produces a result of either 4 or -4, each with a probability of $P(4) = P(-4) = 0.5$. It is easy to see that the expected value in this case will also be zero. There are three cases in which the expected value will produce a nonzero result. These are:

1. $i = j = k = l$;
2. $i = j$ and $k = l$, but $i = j \neq k = l$;
3. $j = k$ and $i = l$, but $j = k \neq i = l$.

Taking all of the above into account, we can write (D.1) as

$$\begin{aligned}
E\left[\mathbf{u}^H\mathbf{b}\mathbf{b}^H\mathbf{u}\mathbf{u}^H\mathbf{b}\mathbf{b}^H\mathbf{u}\right] &= 4\left(\sum_{k=1}^{K} u_k^* u_k u_k^* u_k + \sum_{i=1}^{K}\sum_{\substack{k=1\\ i\neq k}}^{K} u_i^* u_i u_k^* u_k + \sum_{j=1}^{K}\sum_{\substack{l=1\\ j\neq l}}^{K} u_i^* u_i u_k^* u_k\right) \\
&= 4\left(\sum_{i=1}^{K}\sum_{k=1}^{K} u_i^* u_i u_k^* u_k + \sum_{j=1}^{K}\sum_{\substack{l=1\\ j\neq l}}^{K} u_i^* u_i u_k^* u_k\right) \\
&= 4\left(\sum_{i=1}^{K}\sum_{k=1}^{K} u_i^* u_i u_k^* u_k + \sum_{j=1}^{K}\sum_{l=1}^{K} u_i^* u_i u_k^* u_k - \sum_{k=1}^{K} u_k^* u_k u_k^* u_k\right) \\
&= 8\left(\mathbf{u}^H\mathbf{u}\right)^2 - 4\sum_{k=1}^{K}|u_k|^4. \qquad \text{(D.2)}
\end{aligned}$$

In the same way we can write the expectation from the second term in equation (5.3) as

$$\begin{aligned}
E\left[\mathbf{u}^H\mathbf{b}\mathbf{b}^H\mathbf{u}\right] &= E\left[\sum_{i=1}^{K}\sum_{j=1}^{K} u_i^* b_i b_j^* u_j\right] \\
&= \sum_{i=1}^{K}\sum_{j=1}^{K} u_i^* u_j E\left[b_i b_j^*\right]. \qquad \text{(D.3)}
\end{aligned}$$

The expected value will produce a nonzero result only if $i = j$. In this case we have

$$\begin{aligned}
E\left[\mathbf{u}^H\mathbf{b}\mathbf{b}^H\mathbf{u}\right] &= 2\sum_{i=1}^{K}\sum_{j=1}^{K} u_i^* u_j \\
&= 2\mathbf{u}^H\mathbf{u}. \qquad \text{(D.4)}
\end{aligned}$$

D.2 EXPECTED VALUE IN THE LCDCM COST FUNCTION

Exactly in the same way as we have derived (D.2), we can show that the first and last terms of equation (5.26) are

$$E\left[\mathbf{u}^H\mathbf{b}[i]\mathbf{b}^H[i]\mathbf{u}\mathbf{u}^H\mathbf{b}[i]\mathbf{b}^H[i]\mathbf{u}\right] = 8\left(\mathbf{u}^H\mathbf{u}\right)^2 - 4\sum_{k=1}^{K}|u_k|^4, \qquad \text{(D.5)}$$

and

$$E\left[\mathbf{u}^H\mathbf{b}[i-D]\mathbf{b}^H[i-D]\mathbf{u}\mathbf{u}^H\mathbf{b}[i-D]\mathbf{b}^H[i-D]\mathbf{u}\right] = 8\left(\mathbf{u}^H\mathbf{u}\right)^2 - 4\sum_{k=1}^{K}|u_{k\cdot}|^4. \qquad \text{(D.6)}$$

Since D is large enough for bits separated by D seconds from the same user to be independent, we have from (5.26) the middle term as

$$E\left[\mathbf{u}^H\mathbf{b}[i]\mathbf{b}^H[i]\mathbf{u}\mathbf{u}^H\mathbf{b}[i-D]\mathbf{b}^H[i-D]\mathbf{u}\right] = \sum_{i=1}^{K}\sum_{j=1}^{K}\sum_{k=1}^{K}\sum_{l=1}^{K}\left(u_i^* u_j u_k^* u_l E\left[b_i[i]b_j^*[i]b_k[i-D]b_l^*[i-D]\right]\right). \tag{D.7}$$

In this case only two possibilities will produce a nonzero expected value result. These are:

1. $i = j = k = l$;
2. $i = j$ and $k = l$, but $i = j \neq k = l$.

Using this, we have equation (D.7) equal to

$$\begin{aligned} E\left[\mathbf{u}^H\mathbf{b}[i]\mathbf{b}^H[i]\mathbf{u}\mathbf{u}^H\mathbf{b}[i-D]\mathbf{b}^H[i-D]\mathbf{u}\right] &= 4\left(\sum_{k=1}^{K} u_k^* u_k u_k^* u_k + \sum_{i=1}^{K}\sum_{\substack{k=1\\ i\neq k}}^{K} u_i^* u_i u_k^* u_k\right) \\ &= 4\left(\sum_{i=1}^{K}\sum_{k=1}^{K} u_i^* u_i u_k^* u_k\right) \\ &= 4\left(\mathbf{u}^H\mathbf{u}\right)^2. \end{aligned} \tag{D.8}$$