CHAPTER THREE

THE MATCHED FILTER RECEIVER AND MULTIUSER DETECTION PERFORMANCE MEASURES

This chapter contains the analysis of the single user matched filter. The single user matched filter is the simplest method to demodulate CDMA signals. Several criteria are given by which to measure the performance of multiuser detection schemes. The analysis contained here is largely based on the approach followed by Verdu in [31].

The analyses done in this chapter are done in the real domain. This is to facilitate a geometric understanding of the CDMA multiuser detection problem. In the following chapters, the complex valued CMDA model will be utilized.

3.1 OPTIMAL DECISION RULES AND SUFFICIENT STATISTIC

3.1.1 DECISION RULES AND DECISION REGIONS

To obtain a comprehensive understanding of CDMA detection, we will have to explore the subject of *hypothesis testing* [34]. A certain observed random quantity has a distribution known to belong to a finite set of distributions, each of which is associated with a possible outcome. By sampling and observing the random quantity, we must make a decision as to which distribution (or possible outcome) the sample belongs to. The set of possible outcomes or distributions is often referred to as *hypotheses* in statistical terms. The analysis of the observation or sample is mapped to a decision by means of a *decision rule*. Data demodulation is a hypothesis testing experiment in which the observed quantity is a noise corrupted version of the transmitted signal. There are as many decisions as different values for the transmitted data. For example, in the basic synchronous K user CDMA channel model (2.1), there are 2^K possible decisions, and the observed quantity is a waveform on the interval $[0, T]$.

To make a decision as to what data was transmitted, we need to partition the observation space into *decision regions*, each of which corresponds to a possible transmitted data symbol or hypothesis. Knowledge concerning the distribution of the information source is called *a-priori* knowledge. Let us first assume an equiprobable information source at the transmitter. Assume that within the whole observation space m optimum or non-optimum decision regions $R_i \quad i = 1, \ldots, m$ exist corresponding to m hypotheses. Each hypothesis is distributed according to a probability density function of a random variable Z:

$$\begin{aligned} H_1 &: Z \sim f_{Z|1} \\ &\vdots \\ H_m &: Z \sim f_{Z|m} \end{aligned}$$

When referring to optimum regions, the regions are so chosen that they minimize the error probability. We write the probability of error P_e for arbitrary decision regions, as

$$\begin{aligned} P_e &= 1 - \frac{1}{m} \sum_{i=1}^{m} P[Z \in R_i | i] \\ &= 1 - \frac{1}{m} \sum_{i=1}^{m} \int_{R_i} f_{Z|i}(z) dz \\ &\geq 1 - \frac{1}{m} \int \max_{j=1,\ldots,m} f_{Z|j}(z) dz \end{aligned} \tag{3.1}$$

where the last integral is over the whole observation space. Inequality (3.1) is a lower bound which corresponds to the optimum error probability. There may exist several optimum solutions for the choice of decision boundaries. This non-uniqueness of optimum decision regions arises because there may exist points in the observation space at which the maximum density is achieved by several densities simultaneously. If these elements are arbitrarily assigned to the maximizing hypotheses with the lowest index, we obtain the following optimal decision rule for equiprobable hypotheses.

Proposition 3.1 *(Optimal decision rule - Equiprobable hypotheses) Consider m equiprobable hypotheses under which an observed random vector Z has the following probability density functions*[1]

$$\begin{aligned} H_1 &: Z \sim f_{Z|1} \\ &\vdots \\ H_m &: Z \sim f_{Z|m}, \end{aligned} \tag{3.2}$$

then the following decision regions minimize the error probability

$$R_i = \{z : f_{Z|i}(z) = \max_{j=1,\ldots,m} f_{Z|j}(z)\} - \bigcup_{j=1}^{i-1} R_j, \quad i = 1, \ldots, m. \tag{3.3}$$

[1]The symbol $\sim$ denotes "is distributed according to"

For the case of non-equiprobable transmitted symbols or hypotheses, the a-priori probabilities are denoted as $P[H_i]$. The *a posteriori* probabilities can be computed using Bayes' rule. Conditioned on a particular realization of z of the observation, the conditional (a-posteriori) probabilities for hypothesis H_i is given by

$$P[H_i|z] = \frac{f_{Z|i}(z)P[H_i]}{\sum_{j=1}^{m} f_{Z|j}(z)P[H_j]}. \tag{3.4}$$

In general, the minimum error probability decision rule is termed the *Maximum a posteriori* (MAP) rule, which selects the hypothesis with the highest $P[H_i|z]$. In the case of unknown a-priori probabilities or equiprobable hypotheses (as in (3.3)), the decisions are known as *Maximum Likelihood* (ML) decisions.

Consider the case of a m-hypothesis testing problem where the observation is a Gaussian vector with dimension L, with independent components, and variance equal to σ^2. The distributions under each of the hypotheses are distinguished by their means. For example, the mean of the jth vector component under hypothesis H_i is denoted by a_{ij}. The probability density function corresponding to H_i is given by

$$f_{Z|i}(z_1, \ldots, z_L) = \frac{1}{(2\pi)^{L/2}\sigma^L} \exp\left(-\frac{1}{2\sigma^2}\sum_{j=1}^{L}(z_j - a_{ij})^2\right), \tag{3.5}$$

and the optimum decision regions for equiprobable hypotheses are

$$R_i = \{(z_1, \ldots, z_L) : \sum_{j=1}^{L}(z_j - a_{ij})^2 = \min_{k=1,\ldots,m} \sum_{j=1}^{L}(z_j - a_{kj})^2\} - \bigcup_{j=1}^{i-1} R_j, \tag{3.6}$$

which means that we select the hypothesis whose mean vector is closest to the observed vector in Euclidian distance.

3.1.2 CONTINUOUS-TIME GAUSSIAN SIGNALS

In many hypothesis testing problems, the observed quantity is not a vector as in equations (3.5) and (3.6), but a real valued function over a finite time interval. This is the case with both the synchronous and asynchronous CDMA receivers. Sometimes a structure can be placed at the receiver input so that the decisions are based on functions of the received waveforms (called *observables* or *decision statistics*) which can be either scalars or vectors. In the case of a real valued observed quantity, we invoke the following counterpart to proposition 3.1.

Proposition 3.2 *(Optimal decision rule - Equiprobable hypotheses and real valued functions) Let $x_1, \ldots, x_m$ be finite energy deterministic functions defined on an interval $\mathcal{R}$ of the real line. Let $n(t)$ be white Gaussian noise with unit power spectral density. Consider m equiprobable hypotheses.*

$$\begin{aligned} H_1 &: y(t) = x_1(t) + \sigma n(t),\ t \in \mathcal{R} \\ &\vdots \\ H_m &: y(t) = x_m(t) + \sigma n(t),\ t \in \mathcal{R}, \end{aligned} \tag{3.7}$$

then the following decision regions minimize the error probability

$$R_i = \{y = \{y(t),\ t \in \mathcal{R}\} : f[y|x_i] = \max_{j=1,\ldots,m} f[y|x_j]\} - \bigcup_{j=1}^{i-1} R_j, \tag{3.8}$$

where

$$f[y|x_i] = \exp\left(-\frac{1}{2\sigma^2}\int_{\mathcal{R}} [y(t) - x_i(t)]^2 dt\right). \tag{3.9}$$

The function $f[y|x_i]$ in (3.9) is termed the *likelihood function*, and corresponds to the unnormalized conditional probability density function $f_{Z|j}(z)$ in proposition 3.1. As with (3.6), minimizing $[y(t) - x_i(t)]^2$, maximizes (3.6), giving us the *minimum distance* decision region

$$R_i = \{y = \{y(t),\ t \in \mathcal{R}\} : [y(t) - x_i(t)]^2 = \min_{k=1,\ldots,m} [y(t) - x_k(t)]^2\} - \bigcup_{j=1}^{i-1} R_j. \tag{3.10}$$

This means that the decision regions that minimize the error probability are minimum distance regions. The waveform $x_i(t)$ that is closest to y(t) in mean-square distance is inferred.

3.1.3 SUFFICIENT STATISTIC

A function of an observable random variable $Y = g(y)$, which does not depend on any unknown parameters, is called a *statistic*. A *sufficient statistic* can formally and generally be defined as follows [35]. In a statistical inference problem where a parameter Θ is to be inferred[2] on the basis of observations y, we say that a function of the observation $Y = g(y)$ is a sufficient statistic for Θ if the conditional distribution of y given $g(y)$, denoted as $f_{y|Y}$, does not depend on Θ. This means that if Y is observed, then additional information cannot be obtained from y if the conditional distribution of y given Y is free of Θ. We will later see that in the case of a single user receiver, the decision statistic Y is given by

$$Y = \langle y, x_i \rangle = \int_{\mathcal{R}} y(t) x_i(t) dt; \quad i = 1, \ldots, m. \tag{3.11}$$

[2] In hypothesis testing Θ takes a finite or countably infinite number of values, whereas in *estimation* problems, it takes an uncountable number of values

To prove that (3.11) is a sufficient statistic for $\Theta = \{H_1, \ldots, H_m\} \equiv \{x_1, \ldots, x_m\}$, we will need another definition of sufficient statistic termed the *factorization criterion* [35]. If y has a probability density function $f[y; x_i]$, then Y is a sufficient statistic for Θ if and only if

$$f[y; x_i] = g(Y; x_i)h(y), \tag{3.12}$$

where $g(Y; x_i)$ does not depend on y, except through Y, and $h(y)$ does not involve x_i. The proof that Y as defined in (3.11) is a sufficient statistic for Θ, is given by

$$\begin{aligned} f[y; x_i] = f[y|x_i] &= \exp\left(\frac{-1}{2\sigma^2}\int_{\mathcal{R}}[y(t) - x_i(t)]^2 dt\right) \\ &= \exp\left(\frac{-1}{2\sigma^2}\left[\int_{\mathcal{R}} y(t)^2 dt - 2\int_{\mathcal{R}} y(t)x_i(t)dt + \int_{\mathcal{R}} x_i(t)^2 dt\right]\right) \\ &= \exp\left(\frac{-1}{2\sigma^2}\left[\int_{\mathcal{R}} y(t)^2 dt - 2Y + \int_{\mathcal{R}} x_i(t)^2 dt\right]\right) \\ &= \exp\left(\frac{-1}{2\sigma^2}\left[2Y + \int_{\mathcal{R}} x_i(t)^2 dt\right]\right)\exp\left(\frac{-1}{2\sigma^2}\int_{\mathcal{R}} y(t)^2 dt\right) \\ &= g(Y; x_i)h(y), \end{aligned} \tag{3.13}$$

where we have split the function $f[y; x_i]$ into the factors $g(Y; x_i)h(y)$. This satisfies the factorization criterion, and proves that (3.11) contains all the information in the original observations to make an optimal decision.

3.2 THE OPTIMAL RECEIVER - SINGLE USER

In this section we will study the optimal receiver for the single user CDMA channel. For a single user, the channel simplifies to

$$y(t) = Abs(t) + \sigma n(t), \quad t \in [0, T] \tag{3.14}$$

where $s(t)$ is deterministic and has unit energy, the noise term $n(t)$ is white and Gaussian and bit $b \in \{\pm 1\}$. The amplitude of the single user is denoted by A.

3.2.1 LINEAR DETECTORS

Before deriving the optimum demodulator for the single user channel, it is insightful to consider the class of detectors termed *linear detectors*. A detector that outputs the sign of the correlation of the received signal with a deterministic signal $\varphi(t)$ of duration T is given by

$$\hat{b} = \text{sgn}\left(\langle y, \varphi\rangle\right) = \text{sgn}\left(\int_0^T y(t)\varphi(t)dt\right) \tag{3.15}$$

The detector extracts the information contained in the observed waveform $y(t)$ by means of the scalar decision statistic $\langle y, \varphi \rangle$. The decision statistic is given by

$$Y = \langle y, \varphi \rangle = Ab \langle s, \varphi \rangle + \sigma \langle n, \varphi \rangle \tag{3.16}$$

The linearity of the decision statistic makes it easy to discern the respective contributions of signal and noise, whereby the choice of φ can be optimized. Having the signal and noise terms separated, we will attempt to determine an optimum value for φ. A sensible way to do this, is to maximize the signal-to-noise ratio (SNR) γ of the decision statistic Y. The signal variance is simply $A^2(\langle s, \varphi \rangle)^2$. A property of white Gaussian noise is that $E[\langle n, \varphi \rangle^2] = ||\varphi||^2$. The noise variance is thus equal to $\sigma^2||\varphi||^2$. The SNR of the decision statistic Y maximized with respect to φ is given by

$$\gamma_{\max} = \max_{\varphi} \frac{A^2 \left(\langle s, \varphi \rangle\right)^2}{\sigma^2 \left\|\varphi\right\|^2}. \tag{3.17}$$

Equation (3.17) can readily be solved by means of the Cauchy-Schwarz inequality $(\langle s, \varphi \rangle)^2 \leq ||\varphi||^2||s||^2$, where the equality is only satisfied if and only if φ is a nonzero multiple α of s. Thus we have the maximized SNR given by

$$\gamma_{\max} = \frac{A^2 \left\|\alpha s\right\|^2 \left\|s\right\|^2}{\sigma^2 \left\|\alpha s\right\|^2} = \frac{A^2 \left\|s\right\|^2}{\sigma^2} \tag{3.18}$$

We conclude that any nonzero multiple α of the signal s will maximize the SNR of the decision statistic Y. This excludes the negative multiples of s, as they will yield erroneous decisions in the absence of noise. The value of the constant will have no effect on the maximum SNR, as well as the decisions

$$\hat{b} = \operatorname{sgn}\left(\langle y, \alpha s \rangle\right) = \operatorname{sgn}\left(\int_0^T y(t)s(t)dt\right). \tag{3.19}$$

The detector in (3.19) is known as the *matched filter* or *conventional* detector. We have seen that the matched filter detector is optimal, in that it maximizes the SNR of the decision statistic Y. A linear filter with an impulse response $s(T-t)$ sampled at multiples of time T is equivalent to the decision statistic $\langle y, \varphi \rangle$ in (3.16).

3.2.2 ERROR PROBABILITY - OPTIMAL SINGLE USER LINEAR DETECTOR

Let us investigate the conditional distributions of the decision statistic Y for a DS-CDMA system with binary antipodal modulation. We assume that the noise term $n(t)$ is a Gaussian process. A property of a Gaussian process $n(t)$, is that the inner product $\langle n, \varphi \rangle$ is a Gaussian random variable. Therefore from (3.16), the decision statistic conditioned on $\{-1, +1\}$ is Gaussian with mean $\{-A\langle y, \varphi \rangle, +A\langle y, \varphi \rangle\}$ respectively. The variance for both distributions is equal to $\sigma^2||\varphi||^2$. The Gaussian conditional distributions of Y is abbreviated by $\mathcal{N}(-A\langle y, \varphi \rangle, \sigma^2||\varphi||^2)$ for a minus one and

$\mathcal{N}(+A\langle y,\varphi\rangle,\sigma^2||\varphi||^2)$ for a one sent, respectively. Figure 3.1 shows the conditional distributions of Y conditioned on the transmission of $b=-1$ and $b=1$.

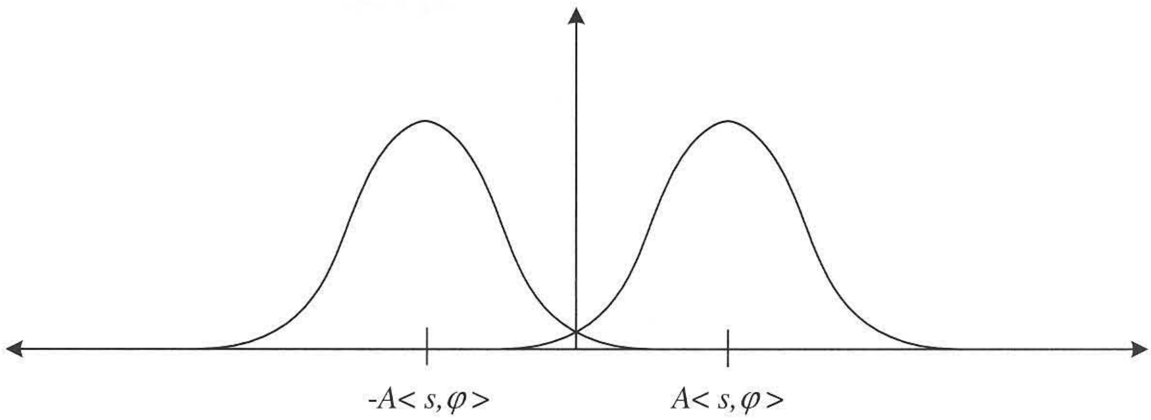

Figure 3.1: Conditional distributions of Y given $b=-1$ and $b=+1$

In the single user binary antipodal case, we have the hypothesis testing problem:

$$\begin{aligned} H_{-1} &: \quad f_{Y|-1} = \mathcal{N}\left(-A\left\langle s,\varphi\right\rangle, \sigma^2||\varphi||^2\right) \\ H_{+1} &: \quad f_{Y|+1} = \mathcal{N}\left(+A\left\langle s,\varphi\right\rangle, \sigma^2||\varphi||^2\right) \end{aligned} \tag{3.20}$$

which is a special case of the vector Gaussian problem in (3.6) with $L=1$ and $m=2$. The corresponding decision regions are

$$R_{-1} = \{y \in (-\infty,\infty) : f_{Y|-1}(y) > f_{Y|+1}(y)\} = (-\infty, 0) \tag{3.21}$$

$$R_{+1} = \{y \in (-\infty,\infty) : f_{Y|+1}(y) > f_{Y|-1}(y)\} = [0, \infty), \tag{3.22}$$

which means that the boundary or threshold between the two regions is at $x=0$. Using the decision regions in (3.21) and (3.22) the probability of error is given by

$$\begin{aligned} P_e &= \frac{1}{2}\int_0^\infty f_{Y|-1}(v)dv + \frac{1}{2}\int_{-\infty}^0 f_{Y|+1}(v)dv \\ &= \frac{1}{2}\int_{A\langle s,\varphi\rangle}^\infty \frac{1}{\sqrt{2\pi}\sigma\left\|\varphi\right\|^2}\exp\left(-\frac{v^2}{2\sigma^2\left\|\varphi\right\|^2}\right)dv \\ &\quad + \frac{1}{2}\int_{-\infty}^{-A\langle s,\varphi\rangle} \frac{1}{\sqrt{2\pi}\sigma\left\|\varphi\right\|^2}\exp\left(-\frac{v^2}{2\sigma^2\left\|\varphi\right\|^2}\right)dv \\ &= \int_{\frac{A\langle s,\varphi\rangle}{\sigma\|\varphi\|}}^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{v^2}{2}}dv \qquad (3.23) \\ &= Q\left(\frac{A\left\langle s,\varphi\right\rangle}{\sigma\left\|\varphi\right\|}\right), \qquad (3.24) \end{aligned}$$

where (3.23) follows by symmetry and a change of integration variable, and (3.24) follows from the notation of the complementary cumulative distribution function of the unit normal random variable or Q function. Assuming a matched filter receiver, the error probability simplifies to

$$P_e = Q\left(\frac{A}{\sigma}\right) = Q\left(\sqrt{\frac{A^2}{\sigma^2}}\right) = Q\left(\sqrt{\gamma}\right), \tag{3.25}$$

where γ denotes the SNR.

In much of the literature on digital communication systems, bit error probability (BEP) is given in terms of bit energy E_b and N_0 where N_0 is related to noise variance by $\sigma^2 = N_0/2$. The bit energy E_b is simply equal to A^2, since the signature waveform is assumed to have unit energy. Thus the matched filter probability of error can also be written as

$$P_e = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) \tag{3.26}$$

which is equal to the BEP of a BPSK system [27].

3.2.3 ERROR PROBABILITY - OPTIMAL SINGLE USER NON-LINEAR DETECTOR

Let us now search for the detector that achieves the minimum error probability among all detectors, by dropping the linearity constraint as imposed in (3.16). This means that we can no longer assume that the observable is $\langle s, \varphi \rangle$ and we have to work with the received process $\{y(t), t \in [0, T]\}$ itself. This is a special case of the problem solved in Proposition 3.2 with $m = 2$, $\mathcal{R} = [0, T]$ and $x_1(t) = As(t)$, $x_2(t) = -As(t)$. Because the energies of x_1 and x_2 are identical, the minimum error probability detector decides $\hat{b} = 1$ if

$$\int_{\mathcal{R}} y(t)x_1(t)dt \geq \int_{\mathcal{R}} y(t)x_2(t)dt, \tag{3.27}$$

and

$$\int_{\mathcal{R}} y(t)x_1(t)dt = -\int_{\mathcal{R}} y(t)x_2(t)dt = A\int_0^T y(t)s(t)dt, \tag{3.28}$$

which means the matched filter output statistic $\langle s, \varphi \rangle$ is a sufficient statistic, and the detector in (3.19) is optimal among all detectors. The shape of the transmitted signal does not affect the minimum bit-error-rate, because of the inherent symmetry of white Gaussian noise, i.e., its projections along every direction has the same distribution.

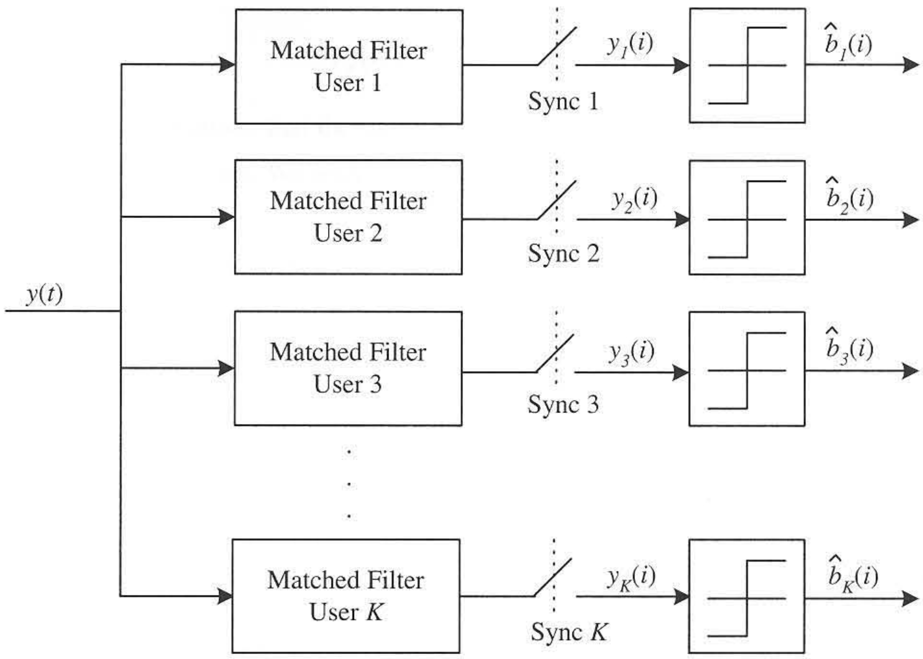

Figure 3.2: Block diagram depicting the bank of matched filters for multiple CDMA users

3.3 MATCHED FILTER ERROR PROBABILITY - SYNCHRONOUS USERS

In this section we will analyze the performance of the single user matched filter in a multiuser CDMA environment. In the multiuser case, demodulation is achieved by a bank of matched filters (Figure 3.2), each matched to a specific user's signature waveform. In the synchronous case we need only to concern ourselves with the timing of a single synchronizer to sample the matched filter outputs of all the users. The output of the kth matched filter in a K user channel is given by

$$y_k = \int_0^T y(t)s_k(t)dt = A_k b_k + \sum_{j \neq k} A_j b_j \rho_{jk} + n_k \tag{3.29}$$

as in equation (2.10), with

$$n_k = \sigma \int_0^T n(t)s_k(t)dt \tag{3.30}$$

a Gaussian random variable with zero mean and variance equal to σ^2. Consider the case of orthogonal signature waveforms, then $\rho_{jk} = 0$ for $j \neq k$, and the problem reduces to the single user case with $y_k = A_k b_k + n_k$. The error probability with orthogonal signature waveforms also reduces to the single user case with

$$P_e(\sigma, k) = Q\left(\frac{A_k}{\sigma}\right), \tag{3.31}$$

which leads us to the conclusion that the matched filter is optimal in a K user CDMA channel with orthogonal signature waveforms. We return to the non-orthogonal CDMA channel.

3.3.1 THE TWO USER CASE

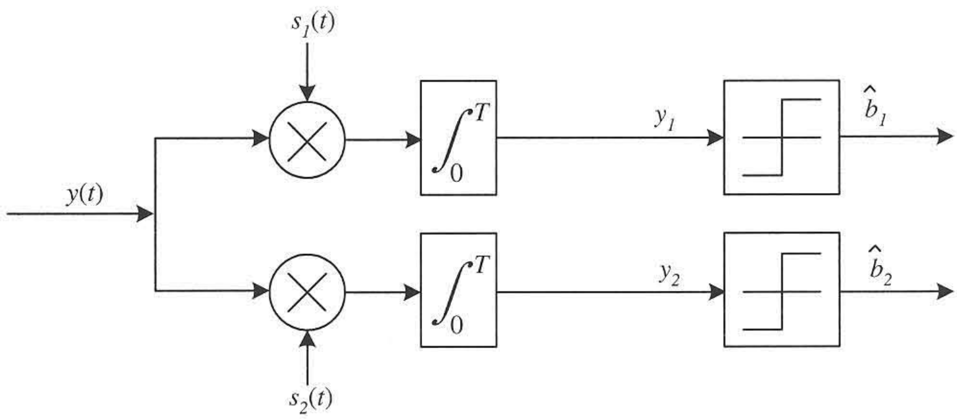

Figure 3.3: Block diagram depicting the special case of the two user CDMA matched filter receiver structure

The two user CDMA channel (Figure 3.3) is instrumental to developing a thorough intuitive and visual understanding of the multiuser interference problem. We start by determining the error probability of user 1 as given by

$$\begin{aligned} P_e(\sigma, 1) &= P\left[\hat{b}_1 \neq b_1\right] \\ &= P\left[b_1 = +1\right] P\left[y_1 < 0 | b_1 = +1\right] \\ &\quad + P\left[b_1 = -1\right] P\left[y_1 \geq 0 | b_1 = -1\right], \end{aligned} \tag{3.32}$$

but y_1 conditioned on b_1 is not Gaussian, so we will have to condition on b_2 as well, with

$$\begin{aligned} P\left[y_1 \geq 0 | b_1 = -1\right] &= P\left[y_1 \geq 0 | b_1 = -1, b_2 = +1\right] P\left[b_2 = +1\right] \\ &\quad + P\left[y_1 \geq 0 | b_1 = -1, b_2 = -1\right] P\left[b_2 = -1\right]. \end{aligned} \tag{3.33}$$

Substitute (3.29) into (3.33) for $\{b_1 = -1, b_2 = +1\}$ and $\{b_1 = -1, b_2 = -1\}$, to obtain

$$\begin{aligned} P\left[y_1 \geq 0 | b_1 = -1\right] &= P\left[n_1 \geq A_1 - A_2\rho\right] P\left[b_2 = +1\right] \\ &\quad + P\left[n_1 \geq A_1 + A_2\rho\right] P\left[b_2 = -1\right] \\ &= \frac{1}{2} Q\left(\frac{A_1 - A_2\rho}{\sigma}\right) + \frac{1}{2} Q\left(\frac{A_1 + A_2\rho}{\sigma}\right) \end{aligned} \tag{3.34}$$

where in the two user case, $\rho_{12} = \rho$. Due to the fact that we assumed equiprobable bitstreams b_1 and b_2, and due to symmetry, we get exactly the same expression for $P\left[y_1 < 0 | b_1 = +1\right]$. The bit error probability BEP of the conventional receiver with one interfering user is given by

$$\begin{aligned} P_e(\sigma, 1) = P_e(\sigma, 2) &= \frac{1}{2} Q\left(\frac{A_1 - A_2\rho}{\sigma}\right) + \frac{1}{2} Q\left(\frac{A_1 + A_2\rho}{\sigma}\right) \\ &= \frac{1}{2} Q\left(\frac{A_1 - A_2\left|\rho\right|}{\sigma}\right) + \frac{1}{2} Q\left(\frac{A_1 + A_2\left|\rho\right|}{\sigma}\right) \end{aligned} \tag{3.35}$$

due to the fact that user 1 is arbitrary. Since the Q function is monotonically decreasing, we readily obtain the upper bound

$$P_e(\sigma, 1) \leq Q\left(\frac{A_1 - A_2\left|\rho\right|}{\sigma}\right). \tag{3.36}$$

This bound is smaller than $1/2$, provided that the interferer is not dominant, i.e.

$$\frac{A_2}{A_1} < \frac{1}{\left|\rho\right|}. \tag{3.37}$$

In this case, because of the asymptotic behavior ($\sigma \to 0$) of the Q function, equation (3.35) is dominated by the term with the smallest argument. Thus, the upper bound (3.36) is an excellent approximation (modulo a factor two) to $P_e(\sigma, 1)$ for all but low SNRs. This implies that the BEP of the conventional receiver behaves like the BEP of a single user system with a reduced SNR, i.e.

$$\gamma_{\text{equiv}} = \left(\frac{A_1 - A_2\left|\rho\right|}{\sigma}\right)^2 \tag{3.38}$$

On the other hand, if the relative amplitude of the interferer is such that

$$\frac{A_2}{A_1} > \frac{1}{\left|\rho\right|}, \tag{3.39}$$

then the conventional receiver exhibits a highly anomalous behavior called the *near-far problem*. For example, the error probability is not monotonic with σ. When we consider the limit $\sigma \to \infty$, we obtain the error probability from (3.35) as

$$\lim_{\sigma \to \infty} P_e(\sigma, 1) = \frac{1}{2}, \tag{3.40}$$

which is what we would expect from any detector. At the other extreme for $\sigma \rightarrow 0$, we get

$$\lim_{\sigma \rightarrow 0} P_e(\sigma, 1) = \frac{1}{2}, \tag{3.41}$$

because due to (3.39), as $\sigma \rightarrow 0$, the polarity of the output of the matched filter for user 1 tends to be governed by the bitstream of user 2, rather than that of user 1. In this case, a little Gaussian noise is better than no noise. With zero noise, it can be seen that the interference shifts the matched filter output to the wrong side of the threshold, as in Figure 3.4. The addition of noise can have one of three effects on the decision,

1. no effect,
2. to prevent an error and
3. to induce an error.

The noise sample amplitude needed for 3. is at least $|\rho|A_2 + A_1$, whereas the noise excursion for 2. is only $|\rho|A_2 - A_1$.

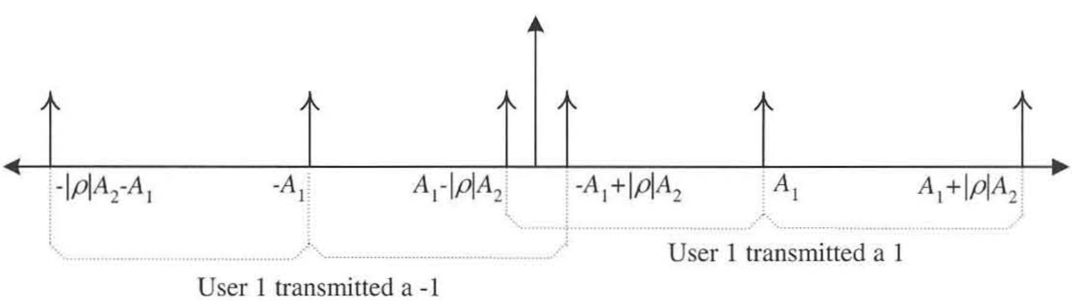

Figure 3.4: Output of the matched filter with one interfering user and $A_2/A_1 > 1/|\rho|$

The noise level that minimizes the BEP under (3.39) is (from [31]):

$$\sigma^2 = \frac{A_1 A_2 \rho}{\operatorname{arctanh}\left(\frac{A_1}{A_2 \rho}\right)} \tag{3.42}$$

Finally we consider the case of equality with

$$\frac{A_2}{A_1} = \frac{1}{\rho}. \tag{3.43}$$

Then the error probability of the single user matched filter reduces to

$$P_e(\sigma, 1) = \frac{1}{4} + \frac{1}{2} Q\left(\frac{2A_1}{\sigma}\right) \tag{3.44}$$

which means that the signal of user 2 exactly cancels the signal of user 1 with a probability of $\frac{1}{2}$ at the matched filter output. It becomes a zero mean Gaussian random variable; with probability $\frac{1}{2}$, the signal of user 2 doubles the contribution of the desired signal to the matched filter output. With respect to the two user case, we will now consider methods of using the BEP as a performance measure. This will give us insight and intuition when considering the K user scenario.

3.3.1.1 BEP AS PERFORMANCE MEASURE - THE TWO USER CASE

When evaluating the performance of digital communication systems, the BEP with respect to the SNR, or alternatively E_b/N_0, is commonly used in the literature. Figure 3.5 shows the BEP for the two user matched filter detector with $\rho = 0.2$ and different relative amplitude values for A_1 and A_2. It can be seen that the BEP degrades rapidly as the relative amplitude of the interferer increases. The top curve is an example of the near-far problem under the condition (3.39).

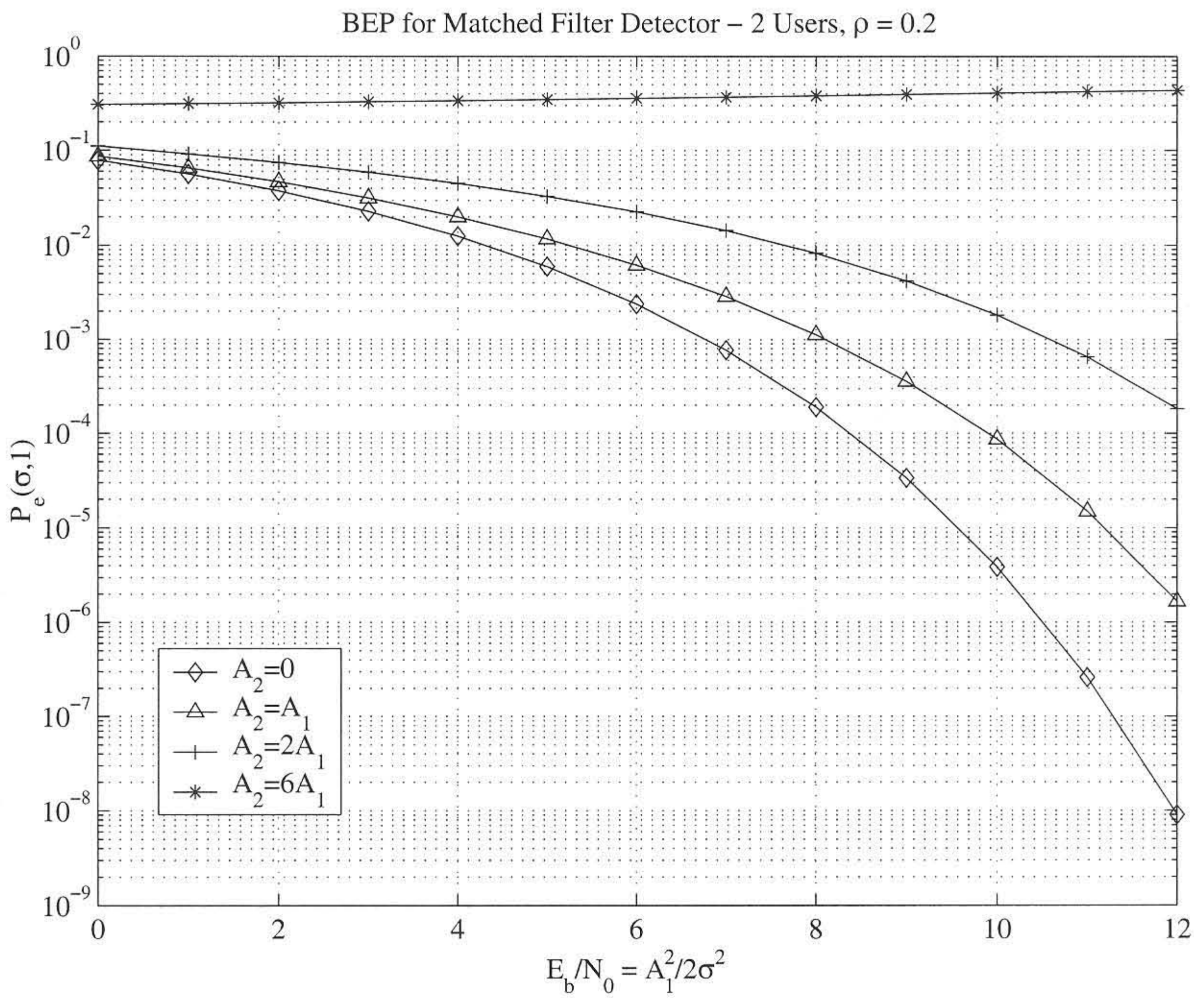

Figure 3.5: BEP of the matched filter detector for different relative amplitudes and $\rho = 0.2$

It is often the case that a digital communication system needs to be designed with a maximum tolerable BEP in mind. The necessary bit energies then need to be found to satisfy that BEP. Figure 3.6 represents the *power-tradeoff* regions so that both users have a BEP of 1×10^{-5}, with the cross-correlation between the two users characterized on the z-axis. In the case of orthogonal users, the objective will be reached for both users if their SNRs are greater than $Q^{-1}(3 \times 10^{-5}) = 12\text{dB}$. From Figure 3.6 it can be seen that as the cross correlation increases:

- even at equal amplitudes the necessary signal energy increases rapidly;
- the sensitivity to imbalances in the received signal grows, making power control necessary.

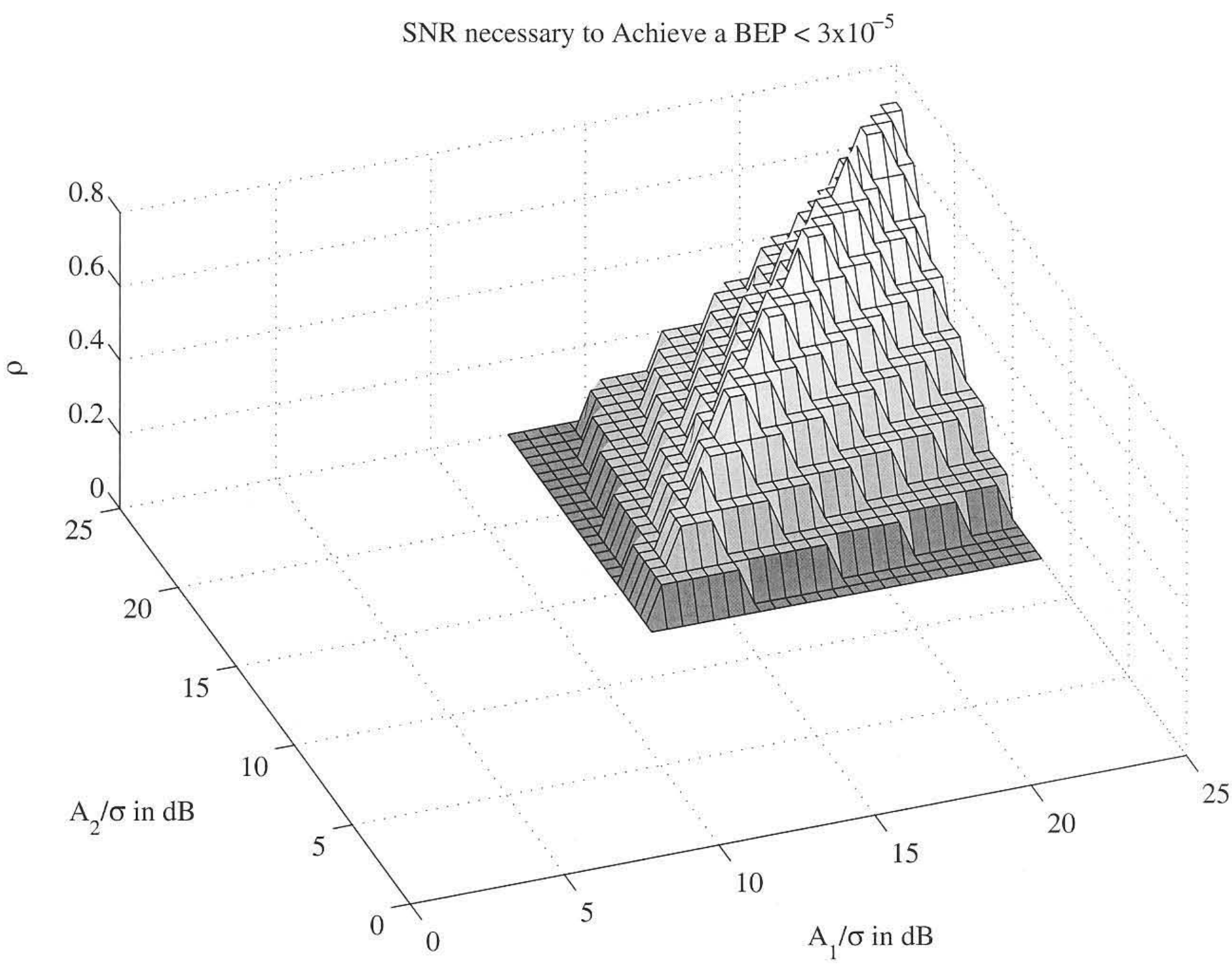

Figure 3.6: Regions of signal-to-noise ratios to attain a BEP of 3×10^{-5} for both users

3.3.1.2 THE TWO USER SIGNAL SPACE REPRESENTATION

Verdu [31] mentions another useful visualization of CDMA detector operation that involves decision regions on a signal space diagram. The signal space representation of detector operation was conceived by Shannon [36] and popularized in the textbook of Wozencraft and Jacobs [37].

For a K user synchronous channel, there are 2^K hypotheses within the observation space on $[0, T]$. This space has infinite dimensions, but the conventional K user demodulator has decision vector space of K dimensions

$$(y_1, \ldots, y_K) = \left(\int_0^T y(t)s_1(t)dt, \ldots, \int_0^T y(t)s_K(t)dt \right). \tag{3.45}$$

To represent the decision regions on a signal space diagram, we will need K dimensions or axes. It is obvious that the two user (two dimensional) case will yield a practical visualization of the signal space. In this case, (y_1, y_2) conditioned on (b_1, b_2) is a Gaussian vector (3.29), (3.30) with mean

$$(A_1b_1 + A_2b_2\rho, A_2b_2 + A_1b_1\rho) \tag{3.46}$$

and covariance matrix

$$\text{cov}(y_1, y_2) = \sigma^2 \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}. \tag{3.47}$$

In the (y_1, y_2) signal space (Figure 3.7), we can depict each of the mean vectors for each of the four hypotheses where $A_1 = A_2 = 1$ and $\rho = 0.2$.
The received vector can be viewed as the sum of the transmitted vector (3.46) and a zero mean Gaussian vector (n_1, n_2). The two user received joint Gaussian vector density functions for all four hypotheses is depicted in Figure 3.8 with $\rho = 0.2$ and $\sigma = 1$.
In the absence of noise, as depicted in Figure 3.7, the detector will make correct decisions, since the signal points lie in the correct regions. The probability of error found in (3.34) is the average of the probabilities that the received vector satisfies $y_1 < 0$ given that $(+,+)$ and $(+,-)$ has been transmitted. There is a shortcoming in the (y_1, y_2) signal space diagram in Figure 3.7 in that the noise components (n_1, n_2) are correlated, i.e.

$$E[n_1n_2] = \sigma^2\rho \tag{3.48}$$

This has the consequence that the noise vector is not symmetric, nor does the norm of the noise vector determine the likelihood of that realization. This can be seen in Figure 3.9 in the 'overhead' view of Figure 3.8.

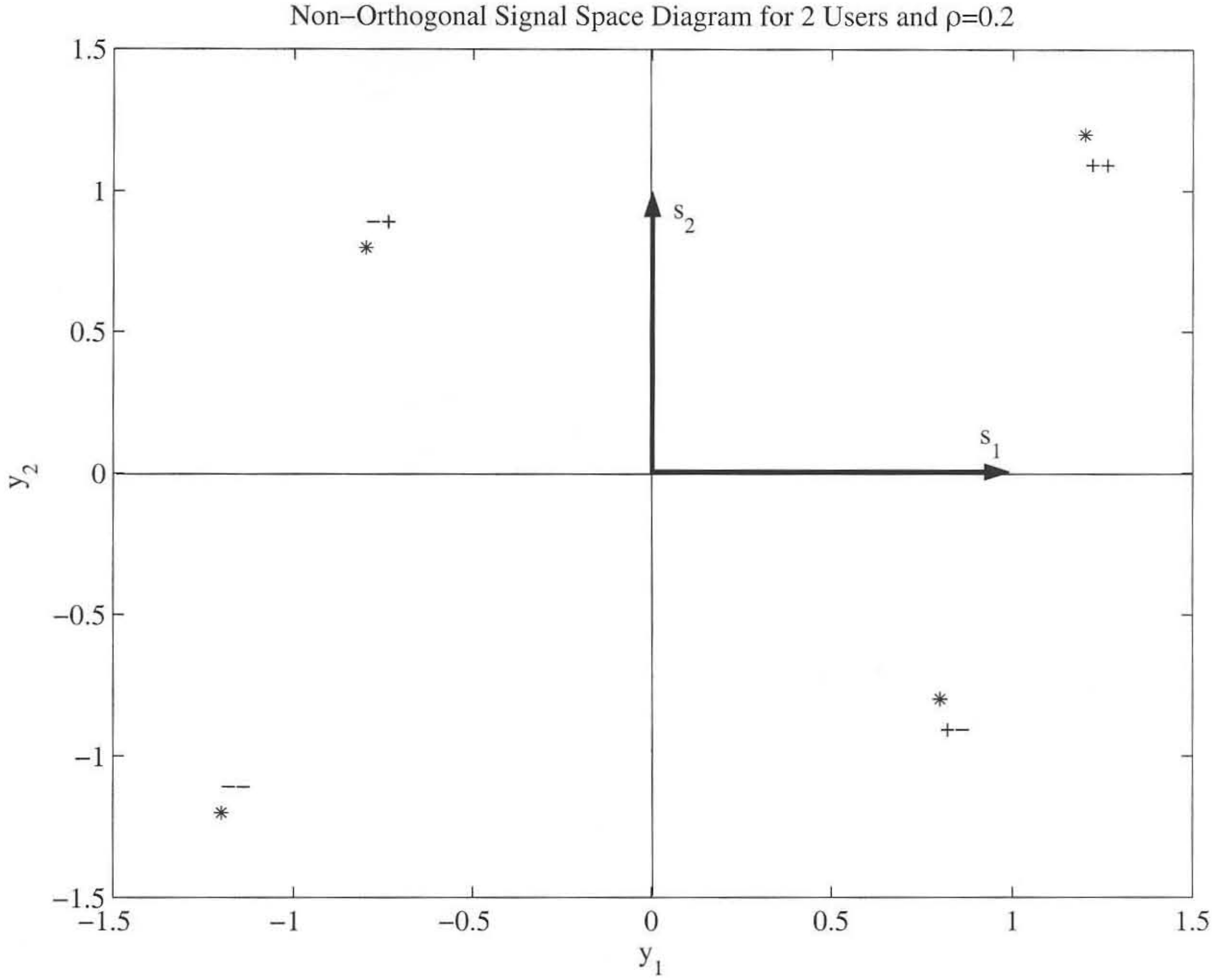

Figure 3.7: Signal space diagram in the (y_1, y_2) space for equal amplitudes and $\rho = 0.2$

A more suitable diagram than the (y_1, y_2) signal space diagram is the $(\tilde{y}_1, \tilde{y}_2)$ signal space diagram whose axes are equal to the correlations of the received waveform with an arbitrary orthonormal basis (ψ_1, ψ_2) that spans the linear space generated by the signals (s_1, s_2). For example, a choice for that orthonormal basis by means of the Gram-Schmidt procedure is

$$\psi_1 = s_1 \tag{3.49}$$

$$\psi_2 = \frac{1}{\sqrt{1-\rho^2}}\, s_2 - \frac{\rho}{\sqrt{1-\rho^2}}\, s_1. \tag{3.50}$$

Conditioned on (b_1, b_2), (y_1, y_2) is Gaussian with mean

$$\begin{aligned}&\left(A_1 b_1 \langle s_1, \psi_1\rangle + A_2 b_2 \langle s_2, \psi_1\rangle, A_1 b_1 \langle s_1, \psi_2\rangle + A_2 b_2 \langle s_2, \psi_2\rangle\right)\\ &= \left(A_1 b_1 + A_2 b_2 \rho, A_2 b_2 \sqrt{1-\rho^2}\right)\end{aligned} \tag{3.51}$$

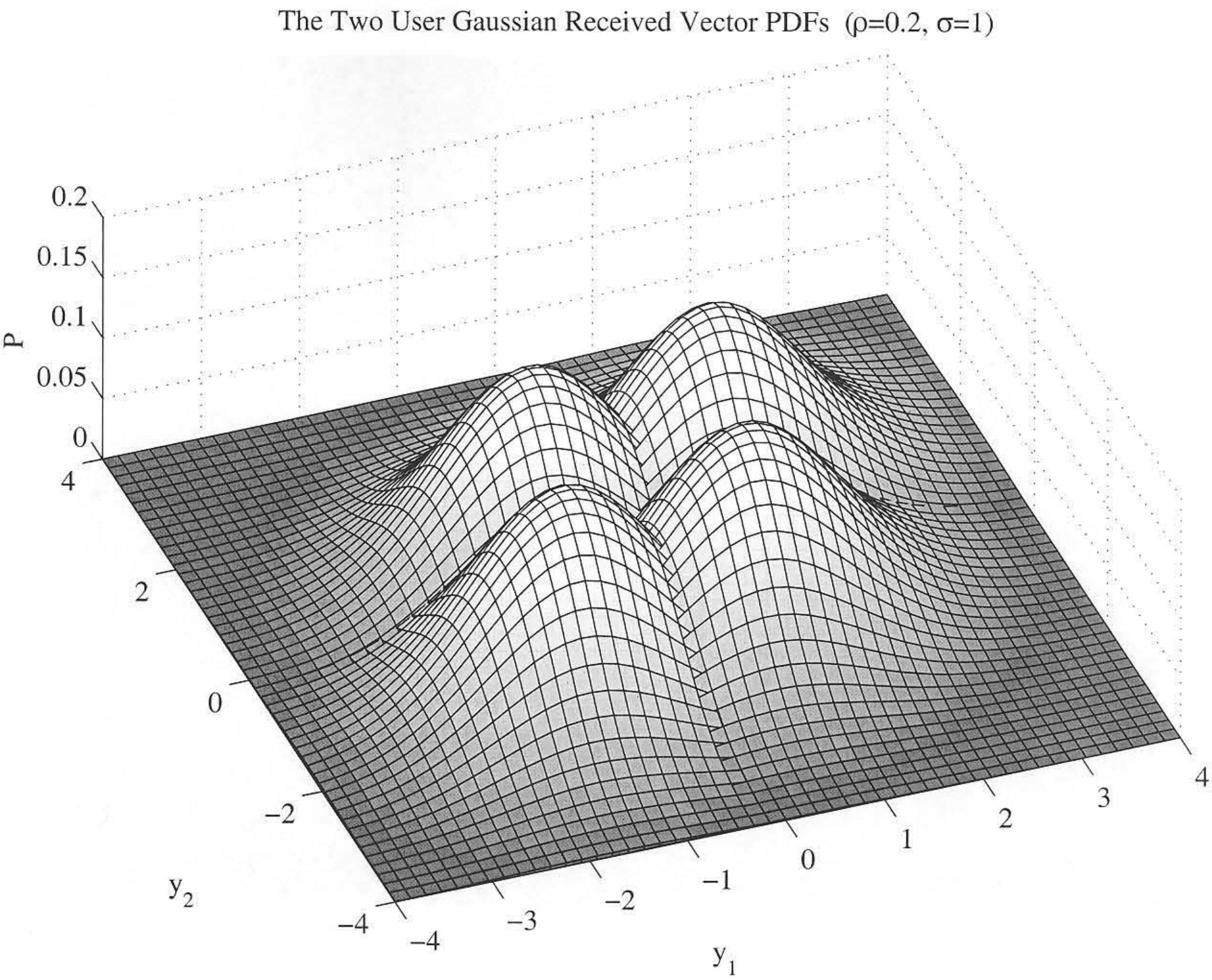

Figure 3.8: Joint probability density function in the (y_1, y_2) space for equal amplitudes, $\rho = 0.2$ and $\sigma = 1$

and covariance matrix equal to

$$\text{cov}(\tilde{y}_1, \tilde{y}_2) = \begin{bmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{bmatrix}. \tag{3.52}$$

The whitened counterpart to Figure 3.7 in alternative orthogonal representation $(\tilde{y}_1, \tilde{y}_2)$ is shown in Figure 3.10. Here, the decision regions are defined by the lines (or hyperplanes in K dimensional space) orthogonal to s_1 and s_2 respectively. With the alternative representation, the inner product between the vectors representing the signature waveforms s_1 and s_2 in Figure 3.10 are, in contrast with Figure 3.7, indeed equal to their cross-correlation.

Even though $(\tilde{y}_1, \tilde{y}_2)$ are not computed by the detector, it is useful to visualize the received vector as belonging to the alternative orthogonal two dimensional space. Indeed, the dimensions of the detector in Figure 3.10 are transparent to all the infinite components in $y(t)$ orthogonal to ψ_1 and ψ_2.

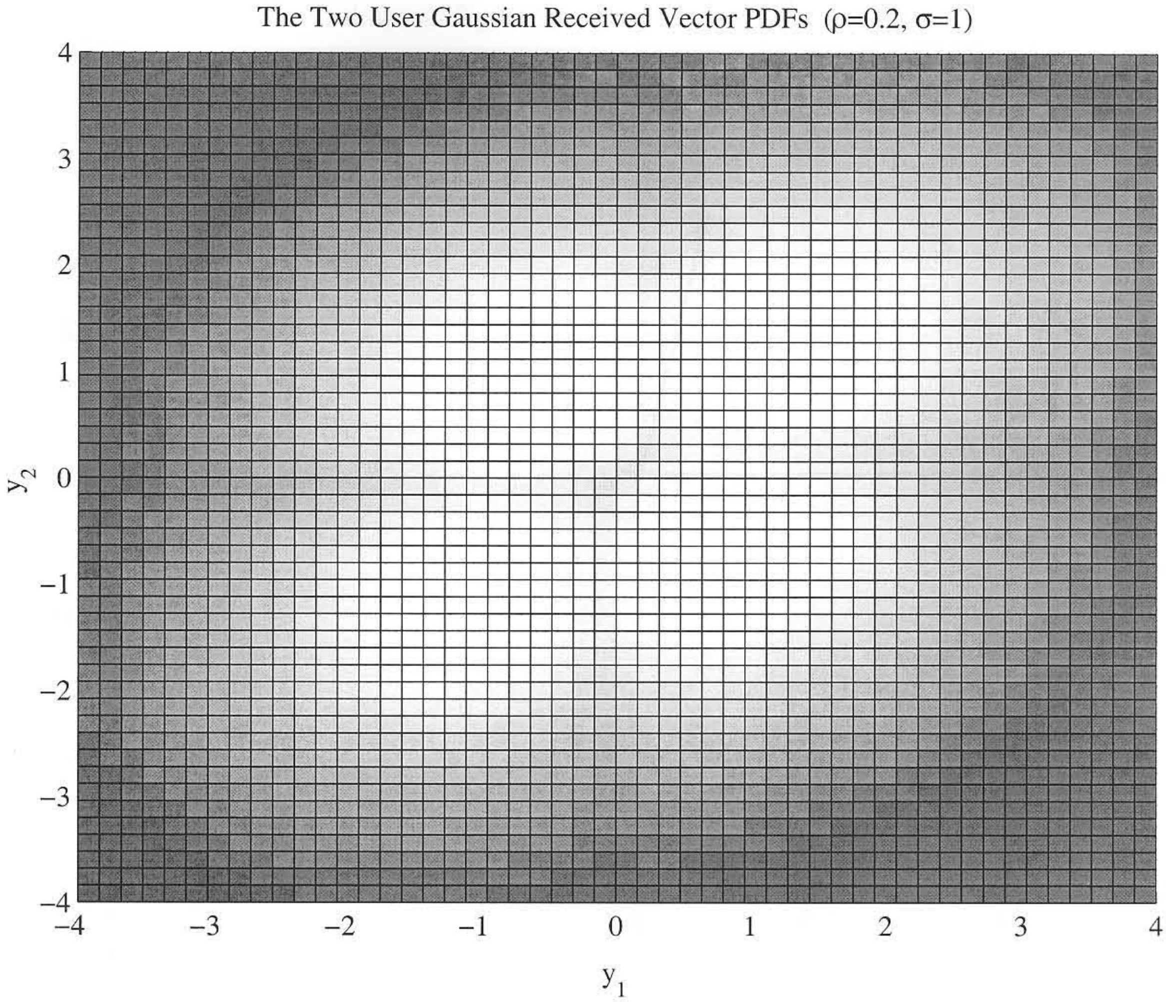

Figure 3.9: Overhead view of the joint probability density function in the (y_1, y_2) space for equal amplitudes, $\rho = 0.2$ and $\sigma = 1$

The anomalous behavior of the conventional matched filter detector in the near-far situation in (3.39) is illustrated in Figure 3.11 with $A_2 = 6A_1$. The decision regions stay exactly the same as in Figure 3.10. The transmitted vectors corresponding to $(+,-)$ and $(-,+)$ have now migrated outside the correct decision regions. This means that given $(+,-)$ or $(-,+)$ was transmitted, an error will occur unless the noise realization moves the vector back into the correct decision region. In a noiseless environment, the decisions of both users is equal to the data transmitted by user 1.

3.3.2 THE K-USER CASE

In the generalization of the BEP to the K user case, we will follow a similar approach a in the two user case. Following the same reasoning as before, the kth user BEP is given by

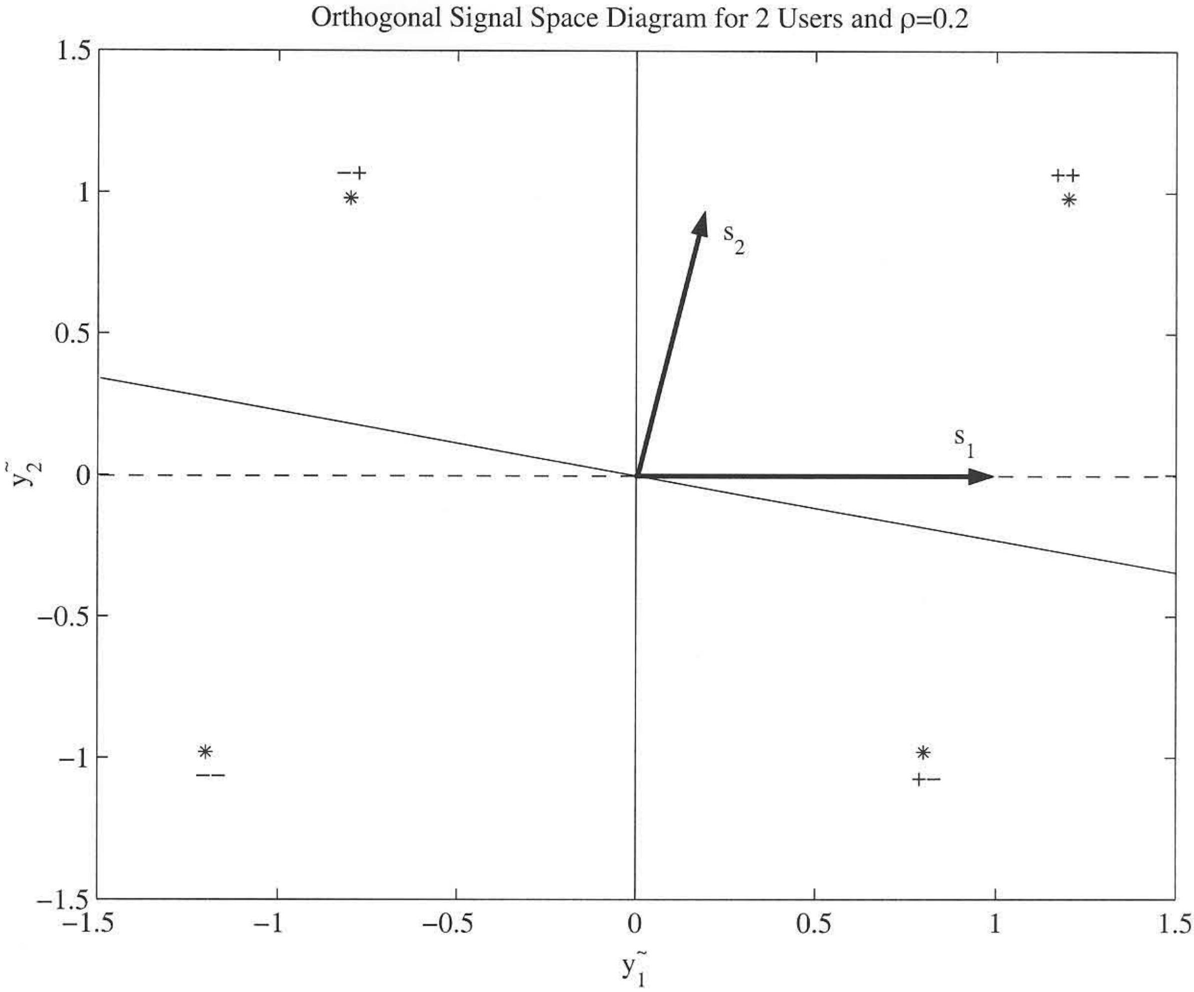

Figure 3.10: Signal space diagram in the alternative orthogonal $(\tilde{y}_1, \tilde{y}_2)$ space for equal amplitudes $A_1 = A_2$ and $\rho = 0.2$

$$
\begin{aligned}
P_e(\sigma, k) &= P\left[b_k = 1\right] P\left[y_k < 0 | b_k = 1\right] \\
&\quad + P\left[b_k = -1\right] P\left[y_k > 0 | b_k = -1\right] \\
&= \frac{1}{2} P\left[n_k < -A_k - \sum_{j \neq k} A_j b_j \rho_{jk}\right] \\
&\quad + \frac{1}{2} P\left[n_k > A_k - \sum_{j \neq k} A_j b_j \rho_{jk}\right].
\end{aligned}
\tag{3.53}
$$

Because of the symmetry of the two terms in (3.53), they are equal, and the BEP of the kth user becomes

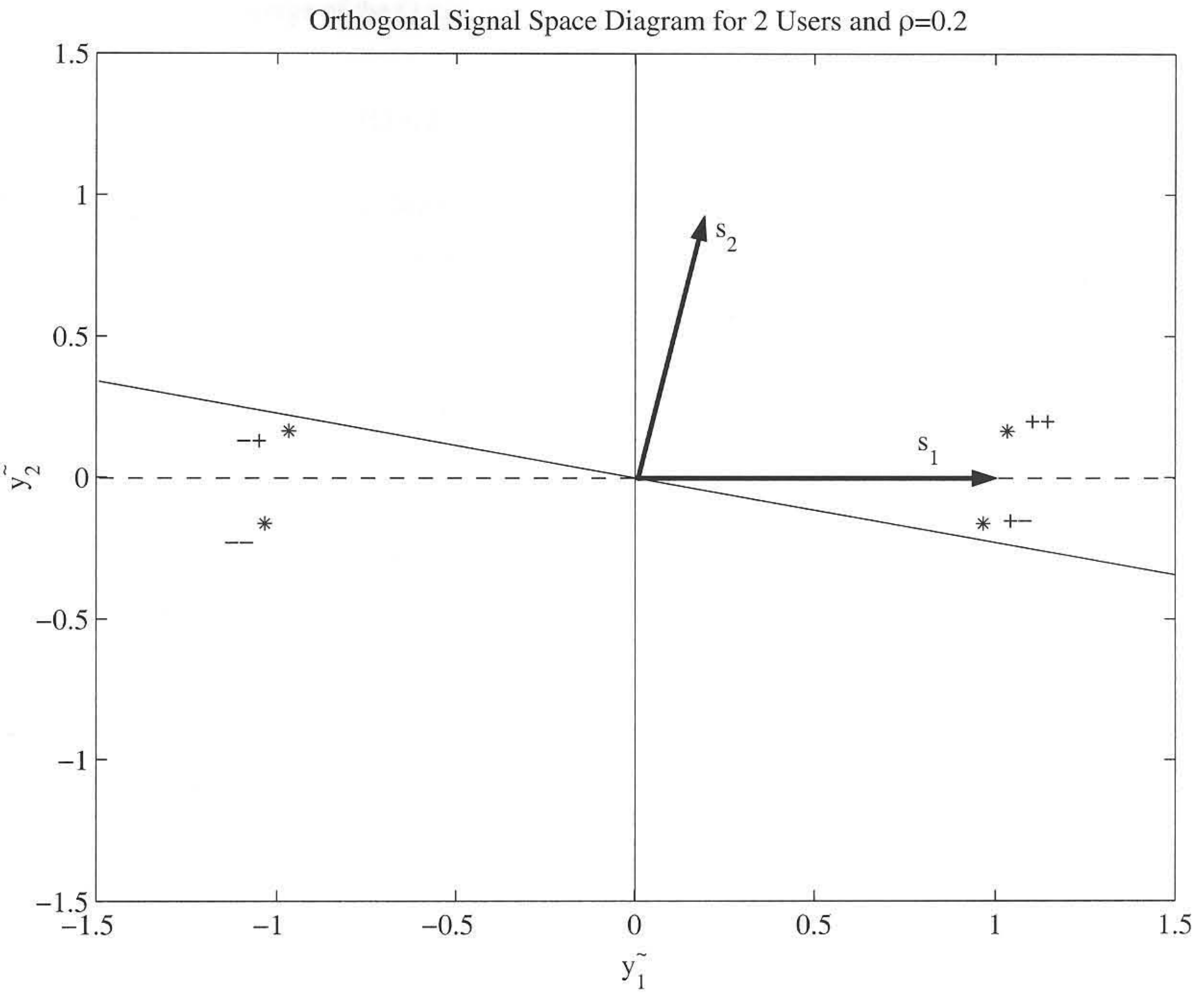

Figure 3.11: Signal space diagram in the alternative orthogonal $(\tilde{y}_1, \tilde{y}_2)$ space for $A_2 = 6A_1$ and $\rho = 0.2$

$$P_e(\sigma, k) = P\left[n_k > A_k - \sum_{j \neq k} A_j b_j \rho_{jk}\right] \tag{3.54}$$

$$= \frac{1}{2^{K-1}} \sum_{(b_1,\ldots,b_K)=(\{-1,1\},\ldots,\{-1,1\})} Q\left(\frac{A_k}{\sigma} + \sum_{j \neq k} b_j \frac{A_j}{\sigma} \rho_{jk}\right) \tag{3.55}$$

where (3.55) is conditioned on all the interfering bits. We see from equation (3.55) that the kth user error probability depends only on the shape of the signature waveforms through their cross-correlations over the interval $[0, T]$, as determined by the receiver. This is also due to the fact that the noise is white and Gaussian. The error probability, as in all digitally modulated systems, depend on the SNR $\frac{A_k}{\sigma}$ and in the CDMA case on the relative amplitudes of the interfering users. As in (3.36), error probability or average of the Q functions in (3.55) is upper bounded by

$$P_e(\sigma, k) \leq Q\left(\frac{A_k}{\sigma} - \sum_{j \neq k} \frac{A_j}{\sigma} |\rho_{jk}|\right). \tag{3.56}$$

When we look at the anomalous behavior of the condition (3.39) in the K user case, we note that (3.55) goes to zero as $\sigma \rightarrow 0$ if and only if the argument of each of the Q functions therein is positive, that is if

$$A_k > \sum_{j \neq k} A_j |\rho_{jk}| \tag{3.57}$$

The condition in (3.57) is commonly referred to as the *open eye* condition. Under this condition, the bound (3.56) becomes tight (modulo a factor independent of σ) as $\sigma \rightarrow 0$.

3.3.3 THE GAUSSIAN APPROXIMATION FOR BEP

Equation (3.55) is cumbersome in the sense that the number of operations required increases exponentially with the number of users. It is for this reason that a number of authors, including the classical papers of Pursley [38] and Yao [39], have approximated (3.55) by replacing the binomial random variable

$$\sum_{j \neq k} A_j b_j |\rho_{jk}| \tag{3.58}$$

with a Gaussian random variable with identical variance. The Gaussian approximated BEP becomes

$$\tilde{P}_e(\sigma, k) = Q\left(\frac{A_k}{\sqrt{\sigma^2 + \sum_{j \neq k} A_j^2 \left|\rho_{jk}^2\right|}}\right). \tag{3.59}$$

The approximation in (3.59) is fairly accurate at low SNRs, but for high SNRs it may become more unreliable. A comparison of the exact BEP versus the Gaussian approximation is shown in Figure 3.12 and Figure 3.13 for 10 and 14 equal energy users, respectively. The cross correlation ρ is set at 0.08. Figure 3.12 is representative of the open eye situation and Figure 3.13 of the closed eye situation. In the latter case we notice that the behavior of the BEP of the single user matched filter detector is non-monotonic. This was also observed in the two user case as the "anomalous" near far situation. In the limit as $\sigma \rightarrow 0$, equations (3.55) and (3.59) behave differently. Equation (3.59) has a nonzero limit, even if the open eye condition is satisfied. The reason for this is that when we approximate the binomial random variable with a Gaussian random variable, the error is greatest in the tails, which determine the BEP for high SNRs. When the performance is averaged with respect to random carrier phases, the multiuser interference is no longer binomially distributed, but remains amplitude limited. This is in contrast to a Gaussian random variable with the same variance.

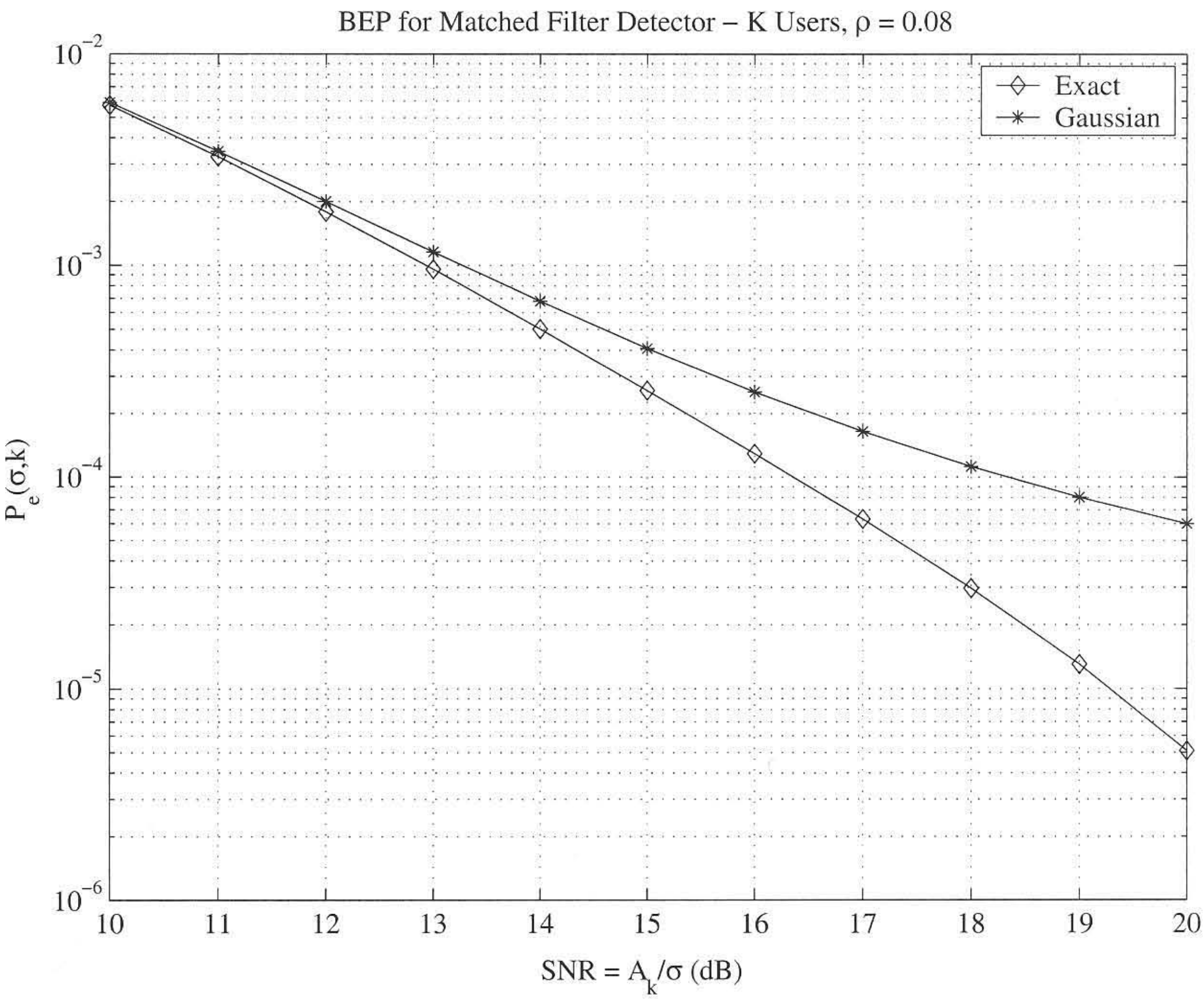

Figure 3.12: BEP as a function of SNR with $K = 10$ equal energy users and $\rho = 0.08$ (eye open)

Proposition 3.3 *Suppose that the random direct sequence model is used and BEP is averaged with respect to the choice of binary sequences with spreading gain N. If $K \to \infty$ and $N \to \infty$, but their ratio is kept constant*

$$\frac{K}{N} = \beta, \tag{3.60}$$

then the averaged BEP converges to

$$\lim_{K,N\to\infty} E\left[P_e(\sigma, 1)\right] = Q\left(\frac{A_1}{\sqrt{\sigma^2 + \beta \bar{A}^2}}\right) \tag{3.61}$$

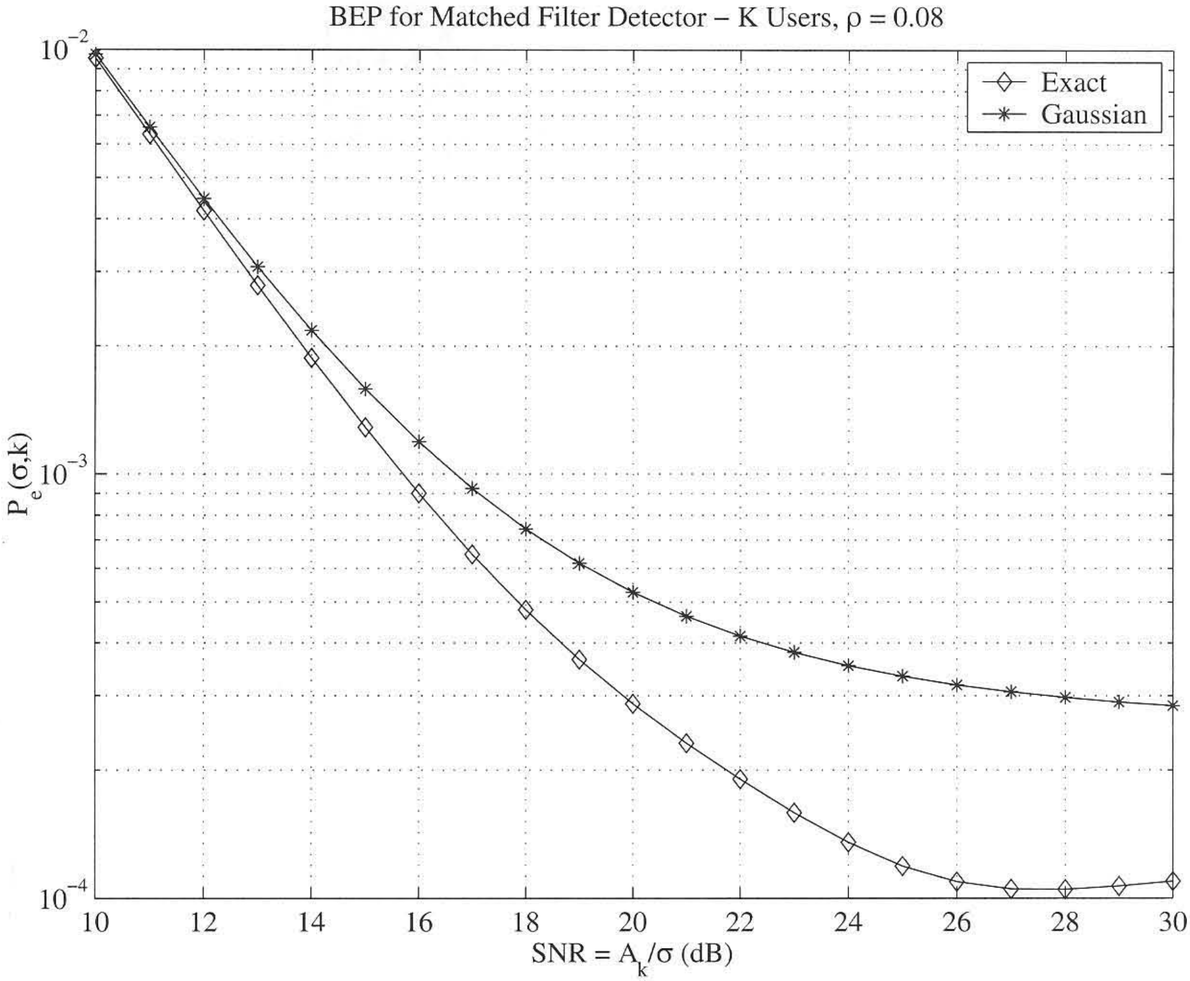

Figure 3.13: BEP as a function of SNR with $K = 14$ equal energy users and $\rho = 0.08$ (eye closed)

where

$$\bar{A}^2 \stackrel{\text{def}}{=} \lim_{K\to\infty} \frac{1}{K} \sum_{j=2}^{K} A_j^2 \tag{3.62}$$

A sufficient condition for the validity of (3.62) is that the amplitudes A_j be bounded.

Let us justify (3.62) under the condition that all energies are equal for all users, i.e. $A_k = A$. According to (3.54) we need to compute the limit of

$$\begin{aligned} &P\left[n_1 + A\sum_{j=2}^{K} b_j\rho_{1j} > A\right] \\ &= P\left[n_1 + A\sum_{j=2}^{K} b_j\left(\frac{1}{N}\sum_{n=1}^{N} d_{jn}\right) > A\right] \\ &= P\left[n_1 + A\sqrt{\frac{K-1}{N}} \cdot \frac{1}{\sqrt{(k-1)N}}\sum_{j=2}^{K}\sum_{n=1}^{N} d_{jn} > A\right] \end{aligned} \tag{3.63}$$

where the random variables d_{jn} in (3.63) are independent and equally likely to be $\{-1, +1\}$. The De Moivre-Laplace Central Limit Theorem dictates convergence in distribution as $K \to \infty$ of the random variable

$$\frac{1}{\sqrt{(k-1)N}}\sum_{j=2}^{K}\sum_{n=1}^{N} d_{jn} \tag{3.64}$$

to a zero mean, unit variance Gaussian random variable. The right side of (3.63) converges to

$$Q\left(\frac{A}{\sqrt{\sigma^2 + \beta A^2}}\right), \tag{3.65}$$

which is what we wanted to verify.

The limiting result in (3.61) can be strengthened to show that even if the BEP is not averaged with respect to random sequences, it converges as $K = \beta N \to \infty$ to the right side of (3.61) with probability one for any signal to noise ratio [40]. It must be said however, that convergence is very slow with K for high SNRs. An easily computable upper bound to $P_e(\sigma, k)$ can be found by partitioning the set of users into

$$\{1, \ldots, K\} = \{k\} \cup G \cup \bar{G} \tag{3.66}$$

where G is a subset of interferers that satisfies the partial eye open condition, i.e.

$$A_k > \sum_{j \in G} A_j \left|\rho_{jk}\right|. \tag{3.67}$$

Then the error probability of the single user matched filter is bounded by

$$P_e(\sigma, k) \leq \exp\left(-\frac{\left(A_k - \sum_{j \in G} A_j \left|\rho_{jk}\right|\right)^2}{2\left(\sigma^2 + \sum_{j \in \bar{G}} A_j^2 \rho_{jk}^2\right)}\right) \tag{3.68}$$

of which the justification is given in [31]. This bound is known as the *Chernoff bound* [41]. The freedom to choose G subject to (3.67) can be exploited to minimize the upper bound in (3.68). The conditions of $G = \emptyset$ and $\bar{G} = \emptyset$ deserve special attention.

First, if the fully open eye condition in (3.57) is satisfied and $\bar{G} = \emptyset$, then

$$P_e(\sigma, k) \leq \exp\left(-\frac{\left(A_k - \sum_{j \neq k} A_j \left|\rho_{jk}\right|\right)^2}{2\sigma^2}\right). \tag{3.69}$$

Second, we can set $G = \emptyset$, then (3.68) becomes (cf. (3.59))

$$P_e(\sigma, k) \leq \exp\left(-\frac{A_k^2}{2\left(\sigma^2 + \sum_{j \neq k} A_j^2 \rho_{jk}^2\right)}\right). \tag{3.70}$$

Equations (3.69) and (3.70) are the two extreme conditions for the upper bound of the single-user matched filter BEP.

There have been several other attempts to find better approximations for CDMA BEP bounds for random signature sequences. Some of these are presented in [42], [43] and [44].

3.4 MATCHED FILTER ERROR PROBABILITY - ASYNCHRONOUS USERS

In an asynchronous CDMA system where all users use the same basic chip waveform, the continuous-time to discrete-time conversion can be carried out by a single chip matched filter sampled at K times the chip rate, with the sampling instants determined by the synchronizers.

The analysis of the asynchronous case is identical, except for the fact that each bit is affected by $2K - 2$ interfering bits. This doubles the number of terms in (3.55)

$$P_e(\sigma, k) = \frac{1}{4^{K-1}} \sum_{((b_1,d_1),\ldots,(b_K,d_K))=(\{-1,1\}^2,\ldots,\{-1,1\}^2)} Q\left(\frac{A_k}{\sigma} + \sum_{j \neq k} \frac{A_j}{\sigma}(b_j \rho_{jk} + d_j \rho_{kj})\right). \tag{3.71}$$

The condition in (3.57) can be extended to the asynchronous case,

$$A_k > \sum_{j \neq k} A_j (\left|\rho_{jk}\right| + \left|\rho_{kj}\right|) \tag{3.72}$$

The asynchronous cross correlations in (3.71) depend on the relative timing offset between users. These parameters are time varying random variables. Given a set of signature waveforms, it is possible to compute the distribution (or simply expectation) of (3.71). This however is computationally intensive.

The infinite user limit as $K \to \infty$ can be extended to the asynchronous case by incorporating two fictitious interferers per actual interferer. Averaging over the received uniformly distributed delays and considering that the autocorrelation for rectangular chip waveforms is

$$R_p(\tau) = 1 - \tfrac{\tau}{T_c}, \quad 0 \leq \tau \leq T_c \tag{3.73}$$

we can obtain the second moment of the asynchronous cross correlations ρ_{jk} and ρ_{kj} [31]

$$\begin{aligned} \frac{1}{T}\int_0^T E\left[\rho_{jk}^2(\tau)\right] dt &= \frac{1}{NT_c}\int_0^{T_c} R_p^2(\tau)d\tau \\ &= \frac{1}{N}\int_0^1 (1-x)^2 dx \\ &= \frac{1}{3N}. \end{aligned} \tag{3.74}$$

The second moment of ρ_{kj} is equal to that of ρ_{jk} due to symmetry and a uniformly distributed delay. This implies that the BEP is equivalent to that of a synchronous system with $(2/3) \times (K-1)$ interferers.

3.5 ASYMPTOTIC MULTIUSER EFFICIENCY AND RELATED MEASURES

We already considered BEP as a performance measure for the multiuser CDMA environment. There are several other performance measures that can be derived from BEP that will be of value in the comprehension of CDMA detector operation. One such performance measure mentioned earlier, is the power tradeoff region of SNRs that results in a given guaranteed BEP level.

When we consider a slowly time varying channel with respect to delays, phases, and most importantly, SNRs, averaging BEPs may be misleading. This is due to the fact that the channel may be dominated by particularly unfavorable, but rare channel conditions. It is common practice to design a digital communication system with *outage* as design parameter. Outage is defined as the percentage of time that the system performs below a certain level. When designing according to outage as design parameter, the cumulative distribution function of the BEP is more informative than its average.

In this section we will consider signal to interference ratio, multiuser efficiency, asymptotic multiuser efficiency and near-far resistance as CDMA multiuser detector performance measures.

interference caused by other users after detection. We can achieve this by letting $\sigma \to 0$ in (3.81). The *asymptotic multiuser efficiency* of user k is defined in [2] and [45] as

$$\eta_k = \lim_{\sigma \to 0} \frac{e_k(\sigma)}{A_k^2} \tag{3.82}$$

and is the log BEP of the kth user going to zero with the same slope as that of a single user with energy $\eta_k A_k^2$. That is,

$$\eta_k = \sup\left\{0 \le r \le 1 : \lim_{\sigma \to 0} P_e(\sigma, k)/Q\left(\frac{\sqrt{r}A_k}{\sigma}\right) = 0\right\} \tag{3.83}$$

where "sup" denotes the supremum of the argument and is formally defined as the smallest upper bound with respect to r for which the condition to the right of the semicolon is true. Let us prove the relation between (3.82) and (3.83). We start with the condition

$$\lim_{\sigma \to 0} P_e(\sigma, k)/Q\left(\frac{\sqrt{r}A_k}{\sigma}\right) = 0 \tag{3.84}$$

where $P_e(\sigma, k)$ is given by (3.76). From (A.4) in appendix A, we can determine the following

$$\begin{aligned} \left[\sqrt{r}A_k\right]^+ &< \sqrt{e_k(\sigma)} \\ \sqrt{r}A_k &< \sqrt{e_k(\sigma)} \\ r &< \frac{e_k(\sigma)}{A_k^2}, \end{aligned} \tag{3.85}$$

where the operation $[.]^+$ chooses either zero or the argument, depending on which is the larger of the two. From (3.78), and since e_k and A_k^2 can only be positive, r can take a value between zero and one. We can now make the right side of the inequality (3.85) a minimum upper bound of r by taking the limit $\sigma \to 0$:

$$r < \lim_{\sigma \to 0} \frac{e_k(\sigma)}{A_k^2}. \tag{3.86}$$

Since the right side of the inequality (3.86) is the minimum upper bound of r,

$$\sup\left\{0 \le r \le 1 : \lim_{\sigma \to 0} P_e(\sigma, k)/Q\left(\frac{\sqrt{r}A_k}{\sigma}\right) = 0\right\} = \eta_k = \lim_{\sigma \to 0} \frac{e_k(\sigma)}{A_k^2}. \tag{3.87}$$

An equivalent expression for η_k in [31] is

$$\eta_k = \frac{2}{A_k^2} \lim_{\sigma \to 0} \sigma^2 \log 1/P_e(\sigma, k). \tag{3.88}$$

From (3.88) it can be concluded that in the situations where the BEP does not approach zero as $\sigma \to 0$, such as the single user closed eye situation, the multiuser efficiency is 0. On the other hand, if the multiuser efficiency is a positive value, the bit error rate approaches zero exponentially as $\sigma \to 0$. The multiuser efficiency is very close to the asymptotic multiuser efficiency, unless the SNR is very low.

Verdu [31] defines the *worst asymptotic effective energy* ω as the minimum effective energy among all users as $\sigma \to 0$. That is

$$\omega(A_1, \ldots, A_K) \stackrel{\text{def}}{=} \min_{k=1,\ldots,K} \lim_{\sigma\to 0} e_k(\sigma) \tag{3.89}$$

$$= \min_{k=1,\ldots,K} \lim_{\sigma\to 0} A_k^2 \eta_k \tag{3.90}$$

$$= 2 \lim_{\sigma\to 0} \sigma^2 \log 1/P \left[\bigcup_{k=1}^{K} \{b_k \neq \hat{b}_k\} \right] \tag{3.91}$$

provided that $\omega(A_1, \ldots, A_K) > 0$. Equation (3.90) follows from (3.82), and (3.91) follows from taking the $\lim_{\sigma\to 0} \sigma^2 \log(\cdot)$ of both sides of

$$\max_{k=1,\ldots,K} P_e(\sigma, k) \leq P \left[\bigcup_{k=1}^{K} \{b_k \neq \hat{b}_k\} \right] \leq \sum_{k=1}^{K} P_e(\sigma, k). \tag{3.92}$$

The *near-far resistance* [46] is a figure of merit which defines the detector in terms of the near-far capture immunity and is defined as the minimum asymptotic efficiency over the received energies of all the other users, i.e.

$$\bar{\eta}_k = \inf_{\substack{A_j > 0 \\ j \neq k}} \eta_k, \tag{3.93}$$

where "inf" denotes the infimum, and is defined as the maximum lower bound of the argument. In the case where we have received energies which vary with time (such a the mobile channel), we have a more restrictive definition

$$\bar{\eta}_k = \inf_{\substack{A_j[i] > 0 \\ (i,j) \neq (0,k)}} \eta_k. \tag{3.94}$$

3.5.1 ASYMPTOTIC MULTIUSER EFFICIENCY OF THE TWO USER MATCHED FILTER

Let us consider asymptotic multiuser efficiency in terms of the two user matched filter case. For the matched filter receiver we have the case of the closed eye under the condition

$$A_1 \leq A_2 |\rho|, \tag{3.95}$$

where we have from equation (3.88) that if the BEP does not approach zero as $\sigma \to 0$ the asymptotic multiuser efficiency is

$$\eta_k = 0 \quad \text{if } A_1 \leq A_2 \,|\rho|. \tag{3.96}$$

Conversely, in the open eye condition if

$$A_1 > A_2|\rho|, \tag{3.97}$$

we have from (A.3) and (3.35)

$$\begin{aligned} \lim_{\sigma \to 0} \frac{P_e(\sigma,1)}{Q\left(\frac{\sqrt{r}A_1}{\sigma}\right)} &= \lim_{\sigma \to 0} \frac{\frac{1}{2}Q\left(\frac{A_1 - A_2\rho}{\sigma}\right) + \frac{1}{2}Q\left(\frac{A_1 + A_2\rho}{\sigma}\right)}{Q\left(\frac{\sqrt{r}A_1}{\sigma}\right)} \\ &= 0, \quad \sqrt{r}A_1 < A_1 - A_2|\rho|, \end{aligned} \tag{3.98}$$

from which we get

$$r < \left(1 - \frac{A_2}{A_1}|\rho|\right)^2. \tag{3.99}$$

Taking $\sup\{r\}$, we get

$$\eta_1 = \left(1 - \frac{A_2}{A_1}|\rho|\right)^2. \tag{3.100}$$

Combining the asymptotic multiuser efficiency for both regions of (3.95) and (3.97), we get

$$\eta_1 = \left[\max\left\{0, 1 - \frac{A_2}{A_1}|\rho|\right\}\right]^2, \tag{3.101}$$

which is the asymptotic efficiency for the two user matched filter receiver. A linear plot of the asymptotic multiuser efficiency for the two user matched filter detector is given in Figure 3.14.

3.5.2 ASYMPTOTIC MULTIUSER EFFICIENCY OF THE K USER MATCHED FILTER

It is trivial to expand the expression for matched filter asymptotic multiuser efficiency to the K user case. Using the same reasoning as before, we can combine (A.3) and (3.55) to obtain

$$\eta_k = \left[\max\left\{0, 1 - \sum_{j \neq k} \frac{A_j}{A_k}|\rho_{jk}|\right\}\right]^2 \tag{3.102}$$

for the synchronous case, and combine (A.3) and (3.71) to obtain

$$\eta_k = \left[\max\left\{0, 1 - \sum_{j \neq k} \frac{A_j}{A_k}\left(|\rho_{jk}| + |\rho_{kj}|\right)\right\}\right]^2 \tag{3.103}$$

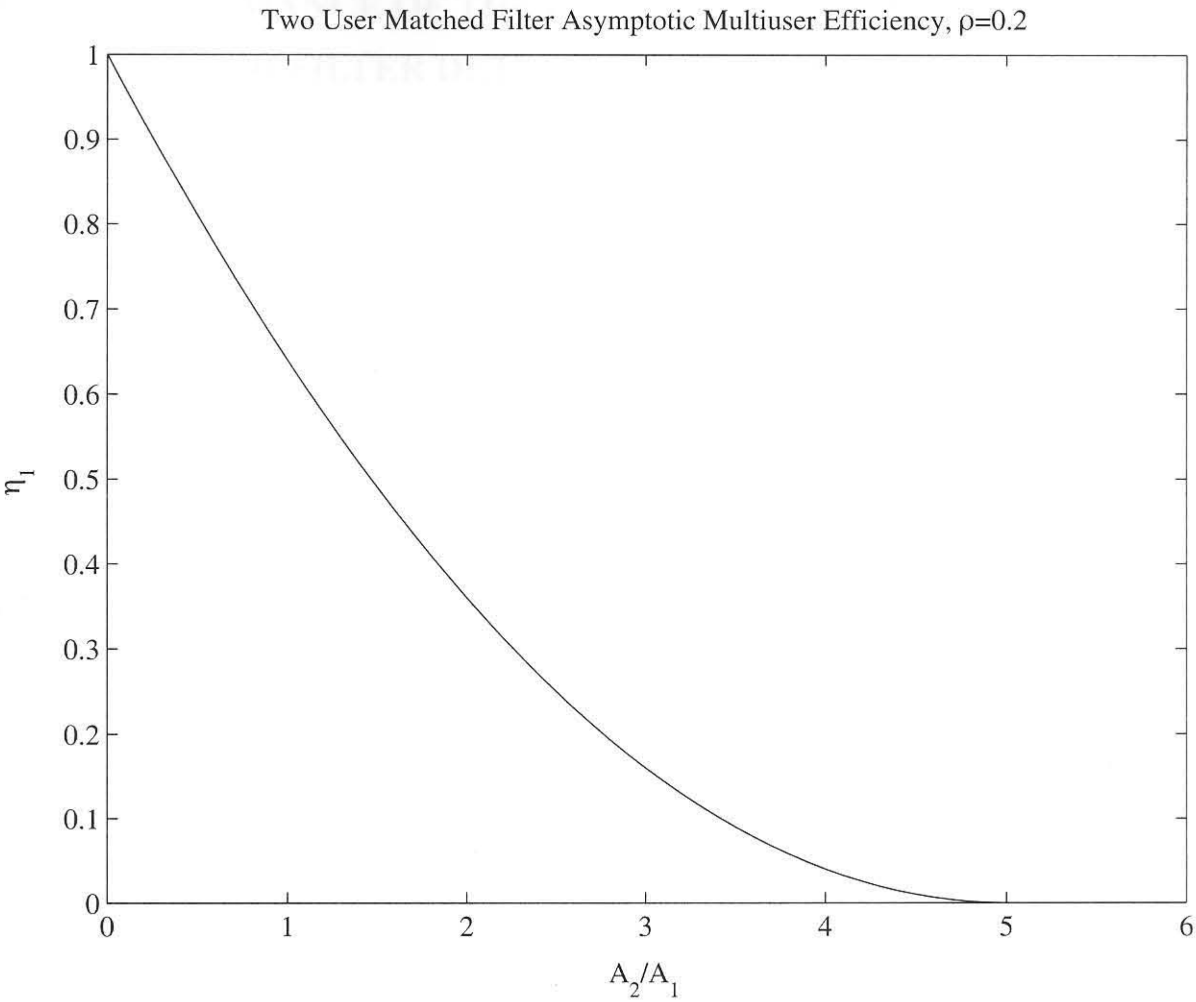

Figure 3.14: Asymptotic multiuser efficiency for a matched filter detector with two equal energy users and $\rho = 0.2$

for the asynchronous case. The asymptotic multiuser efficiency can be viewed as a normalized measure of the openness of the eye (refer to equations (3.57) and (3.72)).

Form (3.102) and (3.103) we can see that minimizing the asymptotic multiuser efficiency over all users, the near-far resistance of user k is equal to zero unless $\rho_{jk} = \rho_{kj} = 0 \; \forall \; j \neq k$ for all overlapping user bits for both the synchronous and asynchronous cases. This means that the single-user matched filter detector is not near-far resistant, since it is impossible that the orthogonality constraint can be maintained over all offsets in the asynchronous channel.

3.6 PERFORMANCE OF THE COHERENT SINGLE USER MATCHED FILTER DETECTOR IN FREQUENCY FLAT FADING

Let us now evaluate the single user matched filter where the received signals are subject to frequency flat Rayleigh fading. We assume coherent detection, i.e. that the fading amplitude and phase of the user of interest is perfectly known at the receiver. Let us adopt the synchronous version of the complex valued model in (2.31):

$$y(t) = \sum_{k=1}^{K} \sum_{i=1}^{M} \tilde{A}_k b_k[i] s_k(t - iT) + \sigma n(t), \tag{3.104}$$

where $\tilde{A}_k$ is the complex valued amplitude of user k due to a phase θ_k. The real and imaginary part of $\tilde{A}_k$ are independent and Gaussian with zero mean and standard deviation equal to A_k. First, let us examine the single user case.

3.6.1 THE SINGLE USER CASE IN THE PRESENCE OF FADING

Here we consider the one-shot model without loss of optimality, since we consider the fading coefficients to be perfectly known. In the single-user case we have

$$y(t) = \tilde{A} b s(t) + \sigma n(t), \quad t \in [0, T]. \tag{3.105}$$

The optimum decision rule selects the value of $b = \in \{\pm 1\}$ that minimizes the mean-square distance

$$\int_0^T \left| y(t) - \tilde{A} b s(t) \right|^2 dt = \int_0^T |y(t)|^2 \, dt + \int_0^T \left| \tilde{A} b s(t) \right| - 2\Re \left\{ \int_0^T y^*(t) \tilde{A} b s(t) dt \right\}, \tag{3.106}$$

that is, where the optimum decision rule is given by

$$\hat{b} = \mathrm{sgn} \left(\Re \left\{ \tilde{A} \int_0^T y^*(t) s(t) dt \right\} \right). \tag{3.107}$$

The inner product

$$y = \langle y^*, s \rangle = \int_0^T y^*(t) s(t) dt \tag{3.108}$$

is a sufficient statistic. The decision rule in (3.107) is equal to b only if the angle between the complex values $\tilde{A}$ and y is acute. That is to say that their absolute phase difference is less than $\pi/2$. Let us find the error probability of the decision rule by conditioning on the transmitted bits and the received fading coefficients:

$$\begin{aligned} P[\hat{b}=1|b=-1,\tilde{A}] &= P\left[-\left|\tilde{A}\right|^2 + \sigma\Re\left\{\tilde{A}\int_0^T n^*(t)s(t)dt\right\} > 0\,\middle|\,\tilde{A}\right] \\ &= P\left[-\left|\tilde{A}\right|^2 + \sigma\Re\left\{\tilde{A}\right\}N_\Re + \sigma\Im\left\{\tilde{A}\right\}N_\Im > 0\,\middle|\,\tilde{A}\right] \\ &= Q\left(\frac{\left|\tilde{A}\right|}{\sigma}\right), \end{aligned} \tag{3.109}$$

where

$$N_\Re \stackrel{\text{def}}{=} \Re\left\{\int_0^T n^*(t)s(t)dt\right\}, \tag{3.110}$$

$$N_\Im \stackrel{\text{def}}{=} \Im\left\{\int_0^T n^*(t)s(t)dt\right\}. \tag{3.111}$$

The symbols (3.110) and (3.111) denote Gaussian random variables with zero mean and unit variance. As in previous cases, the probability of error if a one is sent is identical. We assume that the received complex amplitude $\tilde{A}$ has a independent Rayleigh distributed real and imaginary parts. From (2.49) we have

$$f_R(r) = \begin{cases} r\exp\left(-\frac{r^2}{2}\right), & 0 \le r \le \infty, \\ 0, & r < 0, \end{cases} \tag{3.112}$$

where R is the Rayleigh distributed random variable. We may write the received amplitude as the product of the Rayleigh distributed random variable and a deterministic part where $\tilde{A} = AR$. To find the BEP of the single user, we have to average over all values of the Rayleigh faded received amplitude. Subsequently, the BEP is given by

$$\begin{aligned} P_e^F(\sigma) &= E\left[Q\left(\frac{\left|\tilde{A}\right|}{\sigma}\right)\right] \\ &= \int_0^\infty r\exp\left(-\frac{r^2}{2}\right)Q\left(\frac{Ar}{\sigma}\right)dr \\ &= \frac{1}{2}\left(1 - \frac{1}{\sqrt{1+\sigma^2/A^2}}\right), \end{aligned} \tag{3.113}$$

where (3.113) follows from (A.7). The BEP exhibits an interesting property when compared to the case of a deterministic amplitude. In the deterministic case, the decay in BEP is exponential. In the Rayleigh faded case, however, the BEP has a much slower hyperbolic decay. This highlights the detrimental effect a Rayleigh fading channel has on a digital communication system, and in our case, a single user CDMA system.

The exact BEP of user k in the case of Rayleigh fading is given in [31] as

$$
\begin{aligned}
P_e^F(\sigma, k) &= \frac{1}{2^{K-1}} \sum_{(b_1,\ldots,b_k)=(\{-1,1\},\ldots,\{-1,1\})} E\left[Q\left(\frac{\left|\tilde{A}_k\right|}{\sigma} + \sum_{j\neq k} b_j \frac{\Re\left\{\tilde{A}_j^* \tilde{A}_k\right\}}{\sigma\left|\tilde{A}_k\right|}\rho_{jk}\right)\right] \\
&= E\left[Q\left(\frac{\left|\tilde{A}_k\right|}{\sigma} + \sum_{j\neq k} \frac{\Re\left\{\tilde{A}_j\right\}}{\sigma}\rho_{jk}\right)\right] \qquad (3.114) \\
&= E\left[Q\left(\frac{\left|\tilde{A}_k\right|}{\sqrt{\sigma^2 + \sum\limits_{j\neq k} A_j^2 \rho_{jk}}}\right)\right] \qquad (3.115) \\
&= \frac{1}{2}\left(1 - \frac{A_k}{\sqrt{\sigma^2 + \sum\limits_{j} A_j^2 \rho_{jk}}}\right), \qquad (3.116)
\end{aligned}
$$

where the phase term $\tilde{A}_k / \left|\tilde{A}_k\right|$ and the binary coefficients b_k have been dropped in (3.114) since they do not affect the distribution of the random variable inside the Q-function. Similar to the single user case, (3.115) follows from (A.7) because $\Re\{\tilde{A}_j\}$ are independent Gaussian random variables. We can obtain (3.116) by solving the averaging integral that led to the single user result. The asymptotic multiuser efficiency in the case of Rayleigh fading is given by [31],

$$
\eta_k^F = \lim_{\sigma\to 0} \frac{\sigma^2}{4A_k^2 P_e^F(\sigma, k)}. \qquad (3.117)
$$

3.7 SUMMARY

The chapter begins by declaring the multiuser detection problem as a hypothesis testing problem. The concept of sufficient statistic is visited, and it is shown that the single user matched filter receiver contains sufficient statistic to make an optimal decision. The optimal (matched filter) single user receiver is analyzed and discussed. The CDMA matched filter detector for multiple users is presented, and is analyzed for the two user case. Performance measures such as BEP and power tradeoff regions are introduced, with the two user channel in mind. The phenomenon of the near-far effect is discussed as a basic limitation of the matched filter CDMA receiver. A useful visualization of the two user matched filter detector is presented in terms of a signal space representation.

The K user matched filter detection case is also analyzed in this chapter. The exact and Gaussian approximated BEP equations are derived and presented as a performance measure for the K user case. The infinite user limit for BEP in a CDMA channel is also visited. Asymptotic multiuser efficiency and related measures such as near-far resistance and signal to interference ratios are also presented and discussed. The chapter is concluded with analysis of the matched filter detector in single and multiuser channels with frequency flat fading.