Chapter 2

Review of Relevant Array Synthesis Techniques

2.1 Introductory Remarks

Several radiating elements can be arranged in space and interconnected to produce a directional radiation pattern. Such a group of antenna elements is referred to as an array antenna, or simply, an array. The radiation pattern of the array is dependent on the radiation pattern of each of the individual antenna elements, the relative excitation of each element and the geometry of the array.

The primary goal of antenna synthesis is to obtain a radiation pattern with specified characteristics. The best way to learn about synthesis is to learn all you can about analysis. The antenna characteristics or performance indices must be unambiguously defined before any synthesis can be attempted. This chapter therefore begins with a discussion of the relevant array analysis techniques, followed by a discussion of various array performance indices. The development of array synthesis techniques relevant to the thesis is then reviewed. Array synthesis started with equi-spaced linear arrays, and this forms the first part of the review. This is followed by an investigation of the continuous line source distributions synthesis methods. Next, the existing synthesis techniques for planar arrays are discussed. The synthesis of any array can be viewed as a numerical optimisation problem, this approach forms the topic of Section 2.8. The chapter concludes with a summary of the synthesis problems which have not been adequately dealt with in the literature and which form the subject of this thesis.

2.2 Array Analysis

Before a synthesis problem can be attempted, means of analysing an array must be available. Such analysis is treated in some detail in references [8, 9, 10, 11, 12, 13] and a complete treatment is not intended here. Instead, only the most relevant material will be considered, and some concepts expressed in a concise form. In the sections to follow, background is given and statements made without proof.

2.2.1 The Radiation Pattern of an Arbitrary Array

An array consists of a collection of discrete radiating elements mounted on some surface. The discrete distribution of excitations is called the aperture distribution of the array. Each element's radiation pattern is influenced by the the structure on which it is mounted. In many cases the surface, and the location of the elements on the surface, are chosen in such a way that the influence of the structure is kept to a minimum, for example the array is assumed to be mounted on an infinite ground plane. This simplifies the analysis of the element patterns considerably. However, in the most general case of a conformal array, the structure's influence is marked, and the complete structure must be included when the element's radiation pattern is calculated. The element pattern of each element can then be evaluated about its phase reference, using a method like geometrical theory of diffraction (GTD) [14] or method of moments (MoM) [15]. The arbitrary array can be modelled as an array of point source elements located in space. Each of these point source elements is located at the phase reference of the corresponding original element, and has the element pattern of the original element. In this way the effect of the structure is accounted for during the determination of the element patterns prior to application of the synthesis procedure, and then simply used as if they were closed form expressions for the element patterns.

Consider the general array geometry shown in Figure 2.1. It is convenient to use a Cartesian coordinate system for the positions of the array elements, while a spherical coordinate system would be most prudent for the radiation pattern expressions. The n-th element is at position (x_n, y_n, z_n). Using the origin of the coordinate system as a reference point, the contribution of the n-th element to the far-field, in the direction (θ, ϕ), is:

$$f(\theta,\phi) = E_n(\theta,\phi)\; e^{\mathrm{jk}(x_n \sin\theta\cos\phi \,+\, y_n \sin\theta\sin\phi \,+\, z_n\cos\theta)} \tag{2.1}$$

where k is the free space wavenumber, and $E_n(\theta, \phi)$ represents the co-polarised far-field radiation pattern (or element pattern) associated with the n-th element. The array co-polarised radiation pattern, $F(\theta, \phi)$, is a superposition of the co-polar contribution from each element in the array to the far-field

$$F(\theta,\phi) = \sum_{n=1}^{N} a_n\; E_n(\theta,\phi)\; e^{\mathrm{jk}(x_n \sin\theta\cos\phi \,+\, y_n \sin\theta\sin\phi \,+\, z_n\cos\theta)} \tag{2.2}$$

The relative complex excitation (amplitude and phase) of the n-th element is denoted by a_n and N is the total number of elements in the array. Similar expressions can be written for the cross-polarised radiation pattern. Since the co-polarised pattern is usually considered, "radiation pattern" in the text refers to the co-polarised radiation pattern.

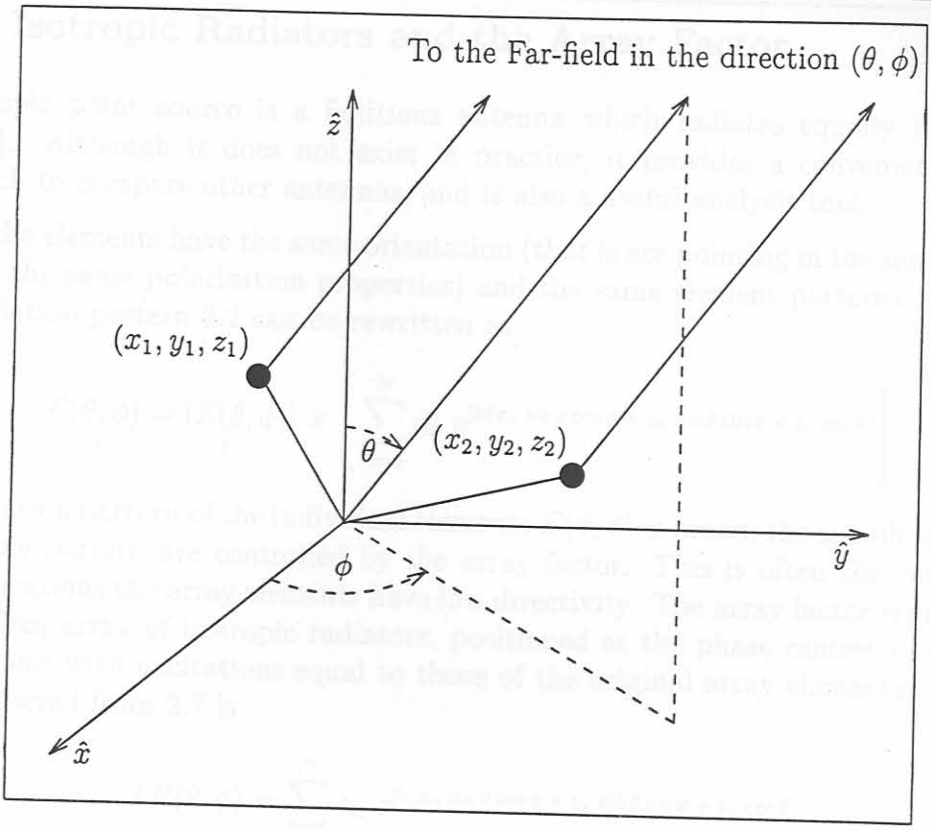

Figure 2.1: Arbitrary array geometry.

If the radiation pattern is evaluated in a number of far-field directions, the radiation pattern can be expressed as a vector

$$\vec{F} = [f_1, f_2, \dots, f_m, \dots, f_M]^T \tag{2.3}$$

where f_m is the value of the radiation pattern in the (θ_m, ϕ_m) direction and M is the total number of field points. $[\cdot]^T$ denotes the transpose. The radiation pattern 2.2 in matrix notation then is

$$\vec{F} = \tilde{B}\,\vec{A} \tag{2.4}$$

where $\vec{A}$ is the excitation column vector

$$\vec{A} = [a_1, a_2, \dots, a_n, \dots, a_N]^T \tag{2.5}$$

and $\tilde{B}$ is the radiation matrix. The mn-th component of the radiation matrix is the contribution of the n-th element to the far-field in the m-th direction,

$$b_{mn} = E_n(\theta_m, \phi_m)\; e^{\mathrm{j}k(x_n \sin\theta_m \cos\phi_m + y_n \sin\theta_m \sin\phi_m + z_n \cos\theta_m)} \tag{2.6}$$

2.2.2 Isotropic Radiators and the Array Factor

An isotropic point source is a fictitious antenna which radiates equally in all directions [16]. Although it does not exist in practice, it provides a convenient reference with which to compare other antennas, and is also a useful analysis tool.

If all the elements have the same orientation (that is are pointing in the same direction and have the same polarisation properties) and the same element patterns $E(\theta,\phi)$, the array radiation pattern 2.2 can be rewritten as

$$F(\theta,\phi) = [E(\theta,\phi)] \times \left[\sum_{n=1}^{N} a_n \, e^{jk(x_n \sin\theta\cos\phi + y_n \sin\theta\sin\phi + z_n\cos\theta)} \right] \tag{2.7}$$

If the radiation pattern of the individual elements $E(\theta,\phi)$ is broad, the significant features of the array pattern are controlled by the array factor. This is often the case since in most applications the array elements have low directivity. The array factor represents the pattern of an array of isotropic radiators, positioned at the phase centres of the actual elements, and with excitations equal to those of the original array elements. The array factor extracted from 2.7 is

$$AF(\theta,\phi) = \sum_{n=1}^{N} a_n \, e^{jk(x_n \sin\theta\cos\phi + y_n \sin\theta\sin\phi + z_n\cos\theta)} \tag{2.8}$$

Expression 2.7 represents the principle of pattern multiplication, which states that the radiation pattern of an array consisting of identical elements is the product of the element pattern of the actual array elements and the array factor.

2.2.3 Linear Array Factor

If all the elements of an array lie along a straight line, they form a linear array. This geometry is not only important in its own right, but is also an essential building block of the majority of planar arrays. The synthesis of such linear arrays is therefore fundamental to all array design. Consider the linear array along the x-axis with uniform inter element spacing d, so that the n-th element is located at position $(nd, 0, 0)$ in a Cartesian system, as shown in Figure 2.2(a).

The array factor is rotationally symmetric about the x-axis, thus it is necessary to determine the array factor in only one plane through the x-axis. If we choose the xz-plane, we can use angle θ as positive for positive x values ($\phi = 0$) and negative for negative values of x ($\phi = \pi$) without loss of generality. Although it would be simpler to select the xy-plane and use ϕ as the angular variable, the choice made will become apparent in Chapter 3

The path difference between two consecutive sources to a distant point in space gives rise to an electrical phase difference. It is thus convenient to define an additional variable,

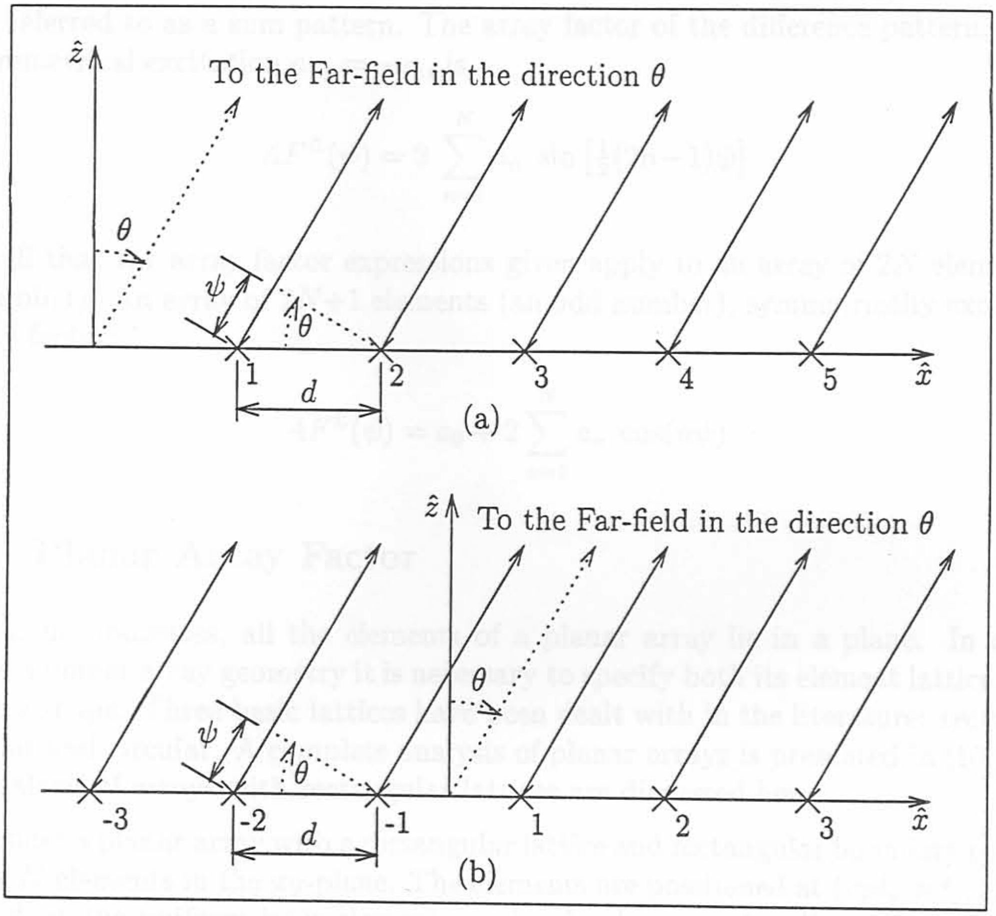

Figure 2.2: Uniformly spaced linear array geometry; (a) general linear array and (b) symetrical even element number linear array.

namely the path length difference,

$$\psi = \mathrm{k}d\sin\theta \tag{2.9}$$

Using this new variable, the linear array factor is expressed as

$$AF(\psi) = \sum_{n=1}^{N} a_n\, e^{jn\psi} \tag{2.10}$$

Consider the $2N$ element uniformly spaced linear array geometry in Figure 2.2(b), with the element numbering scheme and array phase reference as indicated. The distributions for which $|a_{-n}| = |a_n|$ are of particular importance; with symmetrical excitation $a_{-n} = a_n$ the array factor becomes,

$$AF^{\Sigma}(\psi) = 2\sum_{n=1}^{N} a_n \cos\left[\tfrac{1}{2}(2n-1)\psi\right] \tag{2.11}$$

This is referred to as a sum pattern. The array factor of the difference pattern, with an anti-symmetrical excitation $a_{-n}=-a_n$, is

$$AF^{\Delta}(\psi) = 2\sum_{n=1}^{N} a_n \, \sin\left[\tfrac{1}{2}(2n-1)\psi\right] \tag{2.12}$$

Recall that the array factor expressions given apply to an array of $2N$ elements (an even number). An array of $2N+1$ elements (an odd number), symmetrically excited, has an array factor:

$$AF^{\Sigma}(\psi) = a_0 + 2\sum_{n=1}^{N} a_n \, \cos(n\psi) \tag{2.13}$$

2.2.4 Planar Array Factor

As the name indicates, all the elements of a planar array lie in a plane. In order to describe a planar array geometry it is necessary to specify both its element lattice and the boundary shape. Three basic lattices have been dealt with in the literature: rectangular, triangular and circular. A complete analysis of planar arrays is presented in [10, 17, 18]. Only analysis of arrays with rectangular lattices are discussed here.

Consider a planar array with a rectangular lattice and rectangular boundary consisting of M by N elements in the xy-plane. The elements are positioned at $(md_x, nd_y, 0)$, where d_x and d_y is the uniform inter element spacing in the x- and y-directions respectively. Using the substitution

$$\begin{aligned} u &= \mathrm{k}d_x \sin\theta \, \cos\phi \\ v &= \mathrm{k}d_y \sin\theta \, \sin\phi \end{aligned} \tag{2.14}$$

the planar array factor can be written as

$$AF(u,v) = \sum_{m=1}^{M}\sum_{n=1}^{N} a_{mn} \, e^{\mathrm{j}\mathrm{k}(mu+nv)} \tag{2.15}$$

with a_{mn} the normalised complex excitation of the mn-th element.

A planar array of $2M+1$ by $2N+1$ elements in the xy-plane, with a rectangular lattice and rectangular boundary is shown in Figure 2.3. If these elements are excited with quadrantal symmetry, $a_{mn}=a_{-mn}=a_{-m-n}=a_{m-n}$ a sum pattern results. Note that $(-m-n)$ represents a quantity with two subscripts, namely "$-m$" and "$-n$", and not a single subscript quantity with subscript "$-m-n$"; and so on. The planar array factor becomes,

$$AF(u,v) = 4\sum_{m=0}^{M}\sum_{n=0}^{N} \zeta_m\zeta_n a_{mn} \, \cos(mu)\cos(nv) \tag{2.16}$$

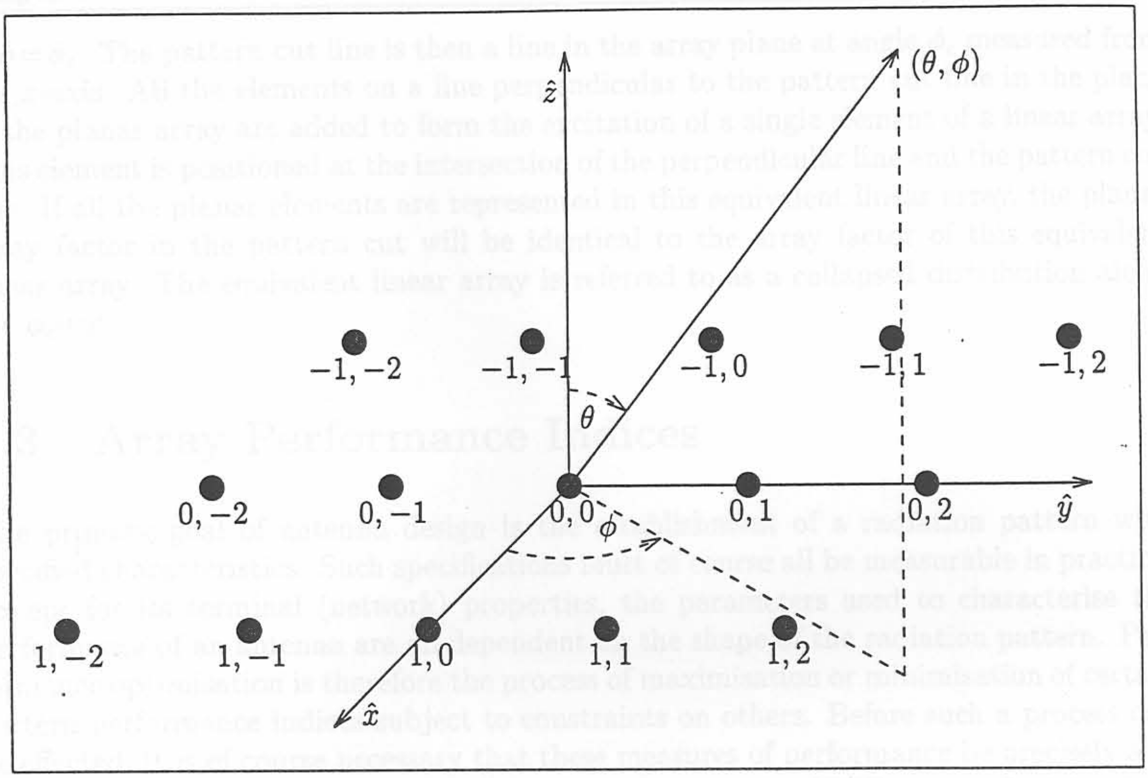

Figure 2.3: Uniformly spaced rectangular planar array geometry.

with

$$\zeta_i = \begin{cases} \frac{1}{2} & \text{for } i=0, \\ 1 & \text{for } i \geq 1. \end{cases} \tag{2.17}$$

The array factor expression of a planar array consisting of $2M$ by $2N$, with a similar numbering scheme and quadrantally symmetric excitation, is

$$AF(u,v) = 4\sum_{m=1}^{M}\sum_{n=1}^{N} a_{mn} \cos\left[\tfrac{1}{2}(2m-1)u\right] \cos\left[\tfrac{1}{2}(2n-1)v\right] \tag{2.18}$$

If the radiation pattern of a planar array is required along one of its principal planes, say $\phi = 90°$, the excitations of each row (all the elements with the same y position coordinate) of the planar array can be represented by a single element with an excitation equal to the sum of all the elements of that row. A linear array along the y-axis is then obtained; we refer to this linear array as the collapsed distribution along the y-axis. Similarly, if the excitations of the elements of each column (elements with the same x position) are added ($\phi = 0°$ principal plane), a collapsed distribution along the x-axis is obtained [19].

The above comment can be generalised to any planar array geometry as well as any pattern cut. Let us assume the planar array is in the xy-plane and the pattern cut angle is $\phi = \phi_c$. The pattern cut line is then a line in the array plane at angle ϕ_c measured from the x-axis. All the elements on a line perpendicular to the pattern cut line in the plane of the planar array are added to form the excitation of a single element of a linear array. This element is positioned at the intersection of the perpendicular line and the pattern cut line. If all the planar elements are represented in this equivalent linear array, the planar array factor in the pattern cut will be identical to the array factor of this equivalent linear array. The equivalent linear array is referred to as a collapsed distribution along the ϕ_c-cut.

2.3 Array Performance Indices

The primary goal of antenna design is the establishment of a radiation pattern with specified characteristics. Such specifications must of course all be measurable in practice. Except for its terminal (network) properties, the parameters used to characterise the performance of an antenna are all dependent on the shape of the radiation pattern. Performance optimisation is therefore the process of maximisation or minimisation of certain pattern performance indices subject to constraints on others. Before such a process can be effected, it is of course necessary that these measures of performance be precisely and unambiguously defined. This will be done here, using the *IEEE* standards definitions of terms for antennas [20]. Though all of the performance specifications considered are applicable to antennas in general, the terminology here is specifically directed toward arrays.

2.3.1 Nomenclature and Radiation Pattern Characteristics

A radiation pattern is generally divided into two regions, the main lobe region (the maximum radiation is in this region) and the sidelobe region consisting of a number of smaller sidelobes. A number of specifications relating to the radiation pattern (or array factor) are illustrated in Figure 2.4. The radiation pattern properties are dependent on the shape of the pattern, not the absolute levels, thus the pattern is usually normalised to its maximum value and plotted in decibels (dB). This practice will be followed in this thesis. The following characteristics can be defined:

- The sidelobe level (SLL) is the level of the highest sidelobe with respect to the pattern maximum. The sidelobe ratio (SLR) is the reciprocal of the sidelobe level. Therefore the sidelobe level, in decibels, will be a negative number and the sidelobe ratio positive.

- The root-mean-square sidelobe level (RMS-SLL) is the root of the average value of the relative power of the normalised pattern of an antenna taken over a specified angular region, which excludes the main beam.

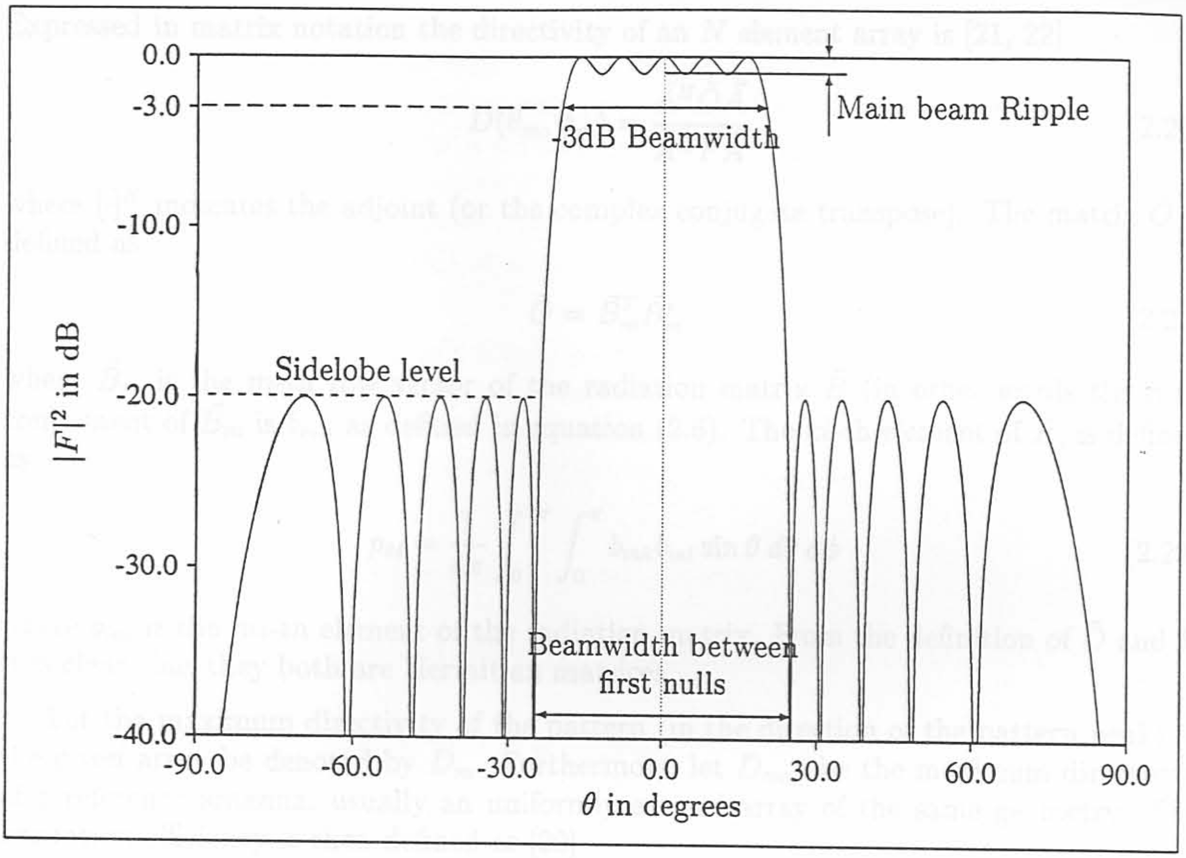

Figure 2.4: Radiation pattern characteristics.

- The beamwidth of the main beam is the angular difference between two selected points on the radiation pattern. Two commonly used beamwidths are the -3dB (or half power) beamwidth and the beamwidth between first nulls.
- In the shaped beam case, the mainbeam ripple is defined as the maximum peak-to-peak deviation of the radiation pattern from the ideal behaviour in the shaped beam region.

2.3.2 Directivity, Gain and Efficiency

The directivity $D(\theta_m, \phi_m)$ in a direction (θ_m, ϕ_m) is defined as the ratio of the radiation intensity from the antenna in that direction, to the average radiation intensity in all directions [20].

$$D(\theta_m, \phi_m) = \frac{4\pi\,|F(\theta_m, \phi_m)|^2}{\int_0^{2\pi}\int_0^{\pi} |F(\theta, \phi)|^2 \sin\theta \, d\theta \, d\phi} \tag{2.19}$$

Expressed in matrix notation the directivity of an N element array is [21, 22]

$$D(\theta_m, \phi_m) = \frac{\vec{A}^H \tilde{O} \vec{A}}{\vec{A}^H \tilde{P} \vec{A}} \tag{2.20}$$

where $[\cdot]^H$ indicates the adjoint (or the complex conjugate transpose). The matrix $\tilde{O}$ is defined as

$$\tilde{O} = \vec{B}_m^T \vec{B}_m^* \tag{2.21}$$

where $\vec{B}_m$ is the m-th row vector of the radiation matrix $\tilde{B}$ (in other words the n-th component of $\vec{B}_m$ is b_{mn} as defined in equation (2.6). The kl-th element of $\tilde{P}$, is defined as

$$p_{kl} = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} b_{mk} b_{ml} \sin\theta \, d\theta \, d\phi \tag{2.22}$$

where b_{mi} is the mi-th element of the radiation matrix. From the definition of $\tilde{O}$ and $\tilde{P}$ it is clear that they both are Hermitian matrices

Let the maximum directivity of the pattern (in the direction of the pattern peak) of the given array be denoted by D_m. Furthermore, let D_{max} be the maximum directivity of a reference antenna, usually an uniformly excited array of the same geometry. The excitation efficiency is then defined as [20]

$$\eta_e = \frac{D_m}{D_{max}} \tag{2.23}$$

The gain, or absolute gain, of an antenna is the ratio of the radiation intensity, in a given direction, to the radiation intensity that would be obtained if the power accepted by the antenna were radiated isotropically. Gain does not include losses due to impedance and polarisation mismatches [20]. Gain does include conductor and dielectric losses. Directivity is a measure of the directional properties of an antenna and depends only on the radiation pattern of the antenna, while gain takes the radiation efficiency into account. The radiation or conversion efficiency of an antenna is therefore

$$\eta_r = \frac{G_m}{D_m} \tag{2.24}$$

If a single value of directivity or gain is given, the maximum value is implied.

2.3.3 Shaped Beam and Contoured Beam Patterns

There can be no general method of specifying shaped beam (or contoured beams) patterns since each is shaped in its own general way. It is perhaps best to name some typical shaped beam patterns which might be desired by systems designers and to suggest the reasons why this might be so.

The most common shaped beam pattern is probably the cosecant-squared elevation beam. The antenna pattern has a $\csc^2\theta$ beam shape in the vertical principal plane and a pencil beam in the horizontal principal plane. This beam shape is used in airport beacons and ground mapping radars. At a certain ground distance away from the beacon, the actual distance to the aircraft varies with the altitude of the aircraft. This beam shape radiates more electromagnetic power at a higher elevation angle to compensate for the difference in actual distance to the aircraft, resulting in a round trip signal that is θ independent.

Flat-top patterns with various footprint contour shapes, for example a bean shaped footprint for hemispheric coverage or a footprint in the shape of man-made boundaries (say the USA), are widely used in satellite applications where a certain area must be illuminated by an electromagnetic field of uniform strength.

2.3.4 Dynamic Range and Smoothness

Dynamic range is defined as the ratio of the maximum excitation magnitude to minimum excitation magnitude,

$$R = \frac{|a_{max}|}{|a_{min}|} \tag{2.25}$$

An excitation with a large dynamic range will be difficult the realize due to difficulties in building power dividers with large power division ratios in the feed network.

Excitations in close proximity with large difference in magnitude and/or phase are very difficult to implement due to the mutual coupling between the array elements. No explicit definition of the smoothness of an array excitation has been found in the literature. A measure of the smoothness of an array may by defined as the maximum element-to-element ratio variation

$$S_m = \left[\frac{|a_n|}{|a_{n\pm1}|}\right]_{max}. \tag{2.26}$$

To alleviate mutual coupling effects, the smoothness must be as close to unity as possible. This definition is used in most publications [23], but some publications [24] define the smoothness as the maximum Euclidean distance between any two adjacent array element excitations

$$S_m = |a_n - a_{n\pm1}|_{max}. \tag{2.27}$$

With this definition the smoothness must be as small as possible to minimize mutual coupling effects.

2.3.5 Sensitivity

In an engineering problem an important question is that concerning the sensitivity of a particular synthesised array excitation. Since the practical realization of the aperture distribution is never exact, it is important to ascertain how such imperfections will affect the array factor. Tolerance sensitivity is a measure of the effect imperfections in the aperture distribution will have on the radiation pattern array factor [25:p.744]. A small tolerance sensitivity is always desirable. The tolerance sensitivity S is defined as the ratio of variance of the radiation pattern peak produced by the aperture errors,

$$S = \frac{\vec{A}^T \vec{A}^*}{\vec{A}^T \tilde{B} \vec{A}^*} \tag{2.28}$$

where $[\cdot]^*$ denotes the complex conjugate, and with the excitation vector $\vec{A}$ and the radiation matrix $\tilde{B}$ as defined in Section 2.2.1 .

2.3.6 Visible Space and Grating Lobes

Examination of equation (2.10) reveals that the linear array factor is periodic in the variable ψ. In the linear array case visible space is the part of the infinite periodic function that corresponds to a variation in the directional angle θ from $-90°$ to $90°$. Invisible space is all the values of ψ outside the visible space. As the inter element spacing increases, more of the periodic array factor will be mapped into visible space. If the spacing is large enough the radiation from the elements will add constructively at more than one θ angle. This additional main lobe is called a grating lobe. All array geometries have this phenomenon.

In most cases the grating lobes are unwanted. Close scrutiny of equation (2.19) shows that grating lobes adversely affect the directivity. The directional patterns of the array elements may suppress grating lobes. Larger inter element spacing allows for larger elements which will be more directive and will suppress the grating lobes to a greater extent. Grating lobes can be suppressed in conformal arrays by adjusting the element excitation of the element (if any) pointing in the direction of the grating lobe.

Visible space for planar arrays in the xy-plane is $0° \leq \theta \leq 90°$ and $-180° \leq \phi \leq 180°$. For conformal arrays visible space is $0° \leq \theta \leq 180°$ and $-180° \leq \phi \leq 180°$.

2.4 Classification of Array Synthesis Techniques

The antenna array synthesis problem can be succinctly stated as that of finding the excitations which will produce a radiation pattern with certain performance indices maximised subject to specified constraints on the pattern and possibly even the excitations themselves. Such constraints cannot be completely arbitrary and must be consistent with the basic physical properties of the array.

Array synthesis is, from a mathematical point of view, a problem of optimisation theory, and many engineers apply this approach. It can on the other hand in certain cases also be approached from the point of view of approximation theory. Early work was based almost entirely on the latter type of consideration, and research in this area continues. Optimisation and approximation theory are disciplines which are of course inextricably linked.

Work on the synthesis of antenna arrays may be divided in a number of ways. For instance, one could classify synthesis techniques according to the geometries being considered. Alternatively, synthesis methods could be separated into those which deal with continuous sources (from which array excitations are obtained by some form of sampling) or discrete elements directly (for which the array element excitations can be obtained in a direct manner without the need for any further approximation, sampling or perturbation procedures). Synthesis techniques could also be grouped in terms of whether they are applicable to single lobe high directivity or shaped beam patterns. We could also classify methods according to whether they are analytical in character or rely on some numerical optimisation procedure.

Lastly, whether the radiated field pattern or the radiated power pattern are of interest in the synthesis problem can also be used as a basis for classification. Field synthesis (where the phase of the radiated fields are fixed) is less complicated as there exist a unique answer to the problem. In practical arrays the phase of the far-field is usually set by the way the arrays is implemented, for example any centre-fed symmetrical array will always have a linear far-field phase. With a correctly chosen optimisation technique convergence is guaranteed.

In power pattern synthesis only the power, or square of the radiated fields, are of interest. The phase distribution of the array pattern is important and may be arbitrary. Thus the power pattern synthesis problem, in general terms, is an ill-conditioned inverse problem with no unique solution. Although this gives a large set of possible solutions, allowing for a solution that makes engineering sense (more effective array or one easier to realise), it makes the synthesis problem much more difficult. Engineering judgement is also necessary to set constraints that will give the optimum result, not only from a synthesis viewpoint, but also from the point of view of the physical realisation of the synthesised distribution.

Most of the work on array synthesis methods described in the literature deals with the linear array problem. These methods are often incorporated in some way to synthesise planar arrays. However, this does not always provide the best results, and more recently attempts have been made to devise approaches which tackle the planar array problem more effectively. Their use often assumes a knowledge of the standard linear array methods. Relatively little has been published on conformal synthesis.

2.5 Direct Synthesis of Equi-spaced Linear Arrays

Consider a linear array of $N+1$ elements, positioned along the x-axis (starting at $x=0$). By the introduction of the substitution $w = e^{j\psi}$ the linear array factor (2.10) can be

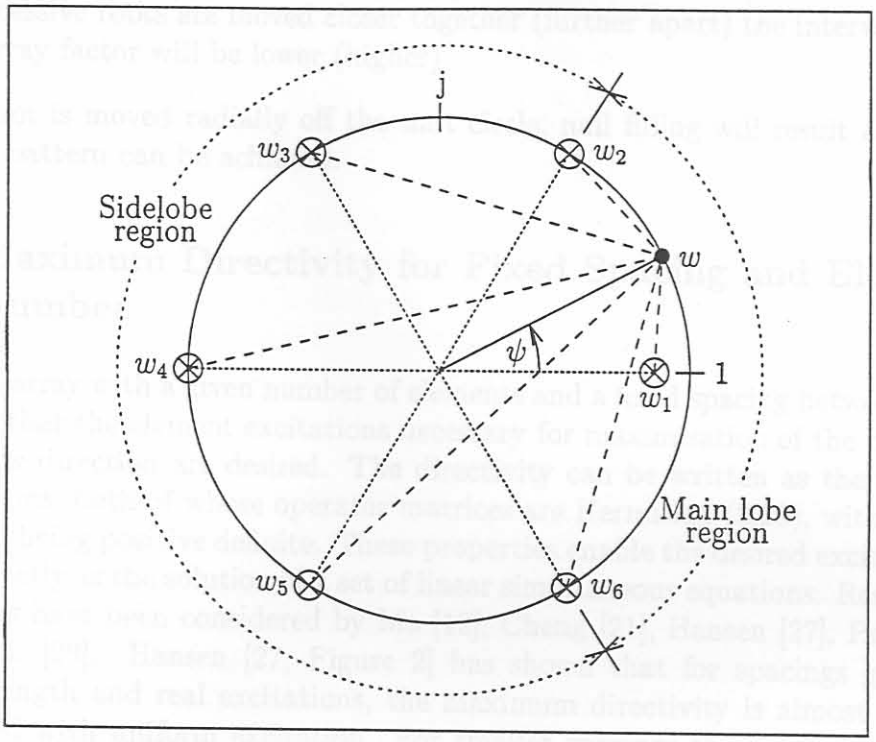

Figure 2.5: Schelkunoff unit circle representation of an eight element linear array

written in the form of a polynomial of degree N, with N distinct roots w_n as

$$AF(w) = \sum_{n=0}^{N} a_n \, w^n = \prod_{n=1}^{N} (w - w_n) \tag{2.29}$$

Schelkunoff [26] introduced the useful concept of constructing a unit circle in the complex w plane. Since ψ is real, the absolute value of $w = e^{j\psi}$ is always unity. Plotted in the complex plane, w (but not w_n) is always on the circumference of the unit circle as depicted in Figure 2.5. For a half wavelength inter element space $d = \frac{\lambda}{2}$, the range of ψ is 2π radians, and so w describes a complete circle as θ varies from 0° to 180° . If the spacing is greater than half wavelength the range of ψ is greater than 2π radians, and the range of w will overlap itself. For $d = \lambda$ spacing w will circumvent the unit circle twice and grating lobes appear (the second pass through the main lobe region is a grating lobe.

Each root w_n is described by its angular position as well as its radial distance off the unit circle. The array factor is the product of the distances between each of the roots and the angular position of w. The array factor is thus completely controlled by the placement of the roots w_n. The synthesis of any desired pattern can be viewed as a problem of finding the proper positions of the roots; in fact many of the successful synthesis methods rely on some zero placement scheme. Three obvious features can be noted, viz:

- if the roots are placed on the unit circle, a pattern with deep nulls will result

- if successive roots are moved closer together (further apart) the intervening lobe of the array factor will be lower (higher)
- if a root is moved radially off the unit circle, null filling will result and a shaped beam pattern can be achieved.

2.5.1 Maximum Directivity for Fixed Spacing and Element Number

Consider an array with a given number of elements and a fixed spacing between elements, and assume that the element excitations necessary for maximisation of the directivity in the broadside direction are desired. The directivity can be written as the ratio of two quadratic forms, both of whose operator matrices are Hermitian (2.20), with that in the denominator being positive definite. These properties enable the desired excitations to be obtained directly as the solution of a set of linear simultaneous equations. Results of such computations have been considered by Ma [12], Cheng [21], Hansen [27], Pritchard [28] and Lo et al. [29]. Hansen [27, Figure 2] has shown that for spacings greater than a half wavelength and real excitations, the maximum directivity is almost identical to that obtained with uniform excitation. For smaller spacings the maximum directivity obtainable is greater than that of a uniform array; this phenomenon is called super directivity. However, the excitations required for super directivity have large oscillatory variations in amplitude and phase from one element to another [25:p.761], and are always associated with an enormously large tolerance sensitivity. For this reason fabrication difficulties are usually prohibitive and super directivity is avoided in most instances. Experimentally it is difficult to produce an array with a directivity much in excess of that produced by a uniformly excited array. In general the sidelobes of a uniformly excited array will be too high for practical use; hence the synthesis techniques to be described immediately below.

2.5.2 Dolph-Chebyshev Synthesis

Rather than maximise the directivity of the array, consider the problem of minimisation of beamwidth; the two approaches are not necessarily equivalent. beamwidth minimisation subject to a constraint on the sidelobe ratio is the classical array synthesis problem solved by C. L. Dolph in his monumental 1946 paper [30]. The underlying argument behind Dolph's approach has been put concisely by Hansen [27]:

> "A symmetrically tapered (amplitude) distribution over the array or aperture is associated with a pattern having lower sidelobes than those of the uniform (amplitude) array. Lowering the sidelobes broadens the beamwidth and lowers the excitation efficiency. ... Some improvement in both beamwidth and efficiency is obtained by raising the farther out sidelobes. Intuitively one might expect equal level sidelobes to be optimum for a given sidelobe level."

In order to synthesise such a pattern for broadside arrays with inter element spacing greater than or equal to a half wavelength, Dolph made use of the Chebyshev polynomials. Expressions for computing the required excitations can be derived from these polynomials. Such formulas have been derived by Barbiere [31], Stegen [32, 33], van der Maas [34] and Bresler [35].

Dolph [30] was able to prove that the array so synthesised is optimum in the sense that for the specified sidelobe ratio and element number, the beamwidth (between first nulls) is the narrowest possible. Alternatively, for a specified first null beamwidth, the sidelobe level is the lowest obtainable from the given array geometry. This means that it is impossible to find another set of excitation coefficients yielding better performance, in both beamwidth and sidelobe ratio, for the given element number and uniform spacing d. It represents a closed form solution to the optimisation problem of first null beamwidth minimisation subject to sidelobe constraints.

The Dolph-Chebyshev theory is indispensable and serves as a firm foundation for sum pattern synthesis. It provides a means of understanding array principles and indicates upper bounds on the performance that can be achieved under certain circumstances. However, it does have a number of drawbacks with regard to its use as a practical distribution. There is a tendency of equal sidelobe level distributions such as the Dolph-Chebyshev to have large excitation peaks at the array ends (a non-monotonic distribution) for certain element number/sidelobe ratio combinations. For a given number of elements there will be a certain sidelobe ratio for which the distribution of excitations is "just" monotonic. If the number of elements is increased but this same sidelobe ratio is desired, the required distribution will be non-monotonic. Increasing the sidelobe ratio (lower sidelobes) will once more allow a monotonic distribution.

The peaks in the distribution at the array ends (called edge brightening) are not only disadvantagious in that they are difficult to implement and make an array more susceptible to edge effects, but they are also indicative of an increase in the tolerance sensitivity [25].

Optimum beamwidth arrays do not necessarily provide optimum directivity, especially if the array is large [36:p.91]. To see this, one can consider a Dolph-Chebyshev array with a fixed sidelobe ratio. Let the array size increase (increase the number of elements while keeping the spacing fixed), at each stage keeping the sidelobe ratio constant and normalising the radiation pattern. This is permissible because the directivity to be found at each stage is only dependent on the angular distribution of the radiation and not on any absolute levels. It is then observed that the denominator of the directivity expression is dominated by the power in the sidelobes after a certain array size is reached, and remains roughly constant thereafter. This is called directivity compression [36:p.91]. Thus it is found that the Dolph-Chebyshev distribution has a directivity limit [33] because of its constant sidelobe level property. For a given array size and maximum sidelobe level it may not be optimum from a directivity point of view. To remove this limitation a taper must be incorporated into the far out sidelobes.

2.5.3 Villeneuve Distributions

As recently as 1986 Hansen [25:p.698] could correctly state that there were "no discrete distributions that yield a highly efficient tapered sidelobe pattern" directly and that in designing most arrays a continuous distribution had to be sampled in some manner. For the narrow beam, low sidelobe pattern, this is no longer the case as a result of an ingenious approach devised by Villeneuve [37]. The method utilises the important principle of synthesising aperture distributions by correct positioning of the space factor zeros. The Villeneuve distribution is the discrete equivalent of the highly desirable Taylor $\bar{n}$ distribution of continuous line sources (to be discussed in Section 2.6.2). The array element excitations can be obtained in a direct manner with the Villeneuve approach.

The application of the Villeneuve procedure consists of the following: Select an array of a specified number of elements with the maximum sidelobe level specified. The first step consists of determining the space factor zeros for a Dolph-Chebyshev distribution with the same sidelobe level. Next determine the zeroes of a uniformly excited array of the same number of elements (the array factor of which will have a tapered sidelobe envelope). Then alter the zeroes of the Chebyshev array so that all except the first $\bar{n}-1$ zeroes now coincide with those of the uniform array. In addition, multiply each of the first $\bar{n}-1$ Chebyshev zeros by a dilation factor σ [25:p.722]. Use these final zeroes of the perturbed Chebyshev array to determine the final element excitations. Villeneuve [37] has devised efficient ways of doing this. These excitations are those of a discrete "Taylor-like" distribution (the latter is discussed in Section 2.6.2), with the close-in sidelobes close to the design maximum specified, and the further out ones decreasing at the rate $\frac{1}{u}$ (where $u = \frac{Nd}{\lambda}\sin\theta$) in amplitude as their position becomes more remote from the main beam. As with the continuous Taylor distribution, $\bar{n}$ is a design variable. A comparison of the excitation efficiencies of the Villeneuve (discrete) and Taylor (continuous) distributions has been published by Hansen [38]. The above technique is now generally referred to as the Villeneuve $\bar{n}$ distribution [38].

Finally, a generalised Villeneuve $\bar{n}$ distribution has been developed by McNamara [39] which can be used to directly synthesise discrete array distributions for high efficiency sum patterns of arbitrary sidelobe level and envelope taper. The Dolph-Chebyshev distribution serves as a "parent" space factor. The correct perturbation of the zeros of this "parent" space factor serves to incorporate the sidelobe behaviour desired, while at the same time keeping the excitation efficiency and beamwidths as close to their optimum values as is possible under the required sidelobe ratio and envelope taper conditions. The level of the first sidelobe is set by the parent Dolph-Chebyshev distribution, the envelope taper rate controlled by a parameter ν, and the point at which the required taper proper begins determined by the transition index $\bar{n}$. The excitations are obtained from the perturbed space factor zeros by solving a set of simultaneous linear equations. The synthesis procedure is rapid and consequently design trade off studies are feasible. The proper choice of the values of $\bar{n}$ and ν for a particular application will depend on the relative importance of the peak sidelobe level compared to that of the farther out sidelobes, the root-mean-square sidelobe ratio desired [40],, and their effect on the excitation efficiency.

2.5.4 Difference Pattern Synthesis

A difference pattern has a null at bore sight with two main lobes of equal height, but 180° phase difference, on each side of the bore sight null, as well as some sidelobes. A difference pattern is obtained by an anti-symmetrical (or out of phase) excitation. For a difference pattern to be optimum in the Dolph-Chebyshev sense it must have the narrowest possible difference lobes and the largest bore sight slope for a specified sidelobe level. Price and Hyneman [41] listed the properties of the polynomial which will provide the optimum difference excitations. McNamara [42] identified the Zolotarev polynomials as the appropriate set of polynomials. The Zolotarev polynomial distribution [42, 43, 44] or its modification [45, 46] can be used for the synthesis of linear arrays with optimum difference pattern performance. The Zolotarev distribution is the difference pattern counterpart for the Dolph-Chebyshev sum pattern, while the modified Zolotarev distribution is the difference analogy of the generalised Villeneuve $\bar{n}$ distribution.

The maximisation of directivity and the maximisation of normalised bore sight slope for difference arrays (without sidelobe constraints) do not result in the same excitation. Difference patterns with maximum directivity and maximum normalised bore sight slope have been discussed in references [47] and [48] respectively. Simultaneous sum and difference pattern synthesis employing quadratic programming is described in reference [49].

2.5.5 Synthesis of Linear Arrays with Shaped Beams

The array factor can be divided into two regions, the shaped beam region and the sidelobe region. The shaping function represents the ideal behaviour of the array factor in the shaped beam region.

The array factor summation (2.10) is very similar to a Fourier series of the same order as the number of elements in the array. The Fourier series synthesis method [9:p.531] equate the Fourier series coefficients, calculated from the desired pattern, to the element excitations. Due to the limited extent (harmonics) of the Fourier series, the resulting array factor exhibits very large ripple in the shaped region with high sidelobes in the sidelobe region.

An alternate particularly convenient way to synthesise a shaped radiation pattern is to sample the ideal pattern at various points. The Woodward-Lawson [50, 9:p.526] technique is the most popular of these methods. The technique relies on the fact that any realizable pattern, of an N-element array, can be analysed as the weighted summation of a set of N orthogonal beams, each having a basic $\frac{\sin(Nx)}{\sin(x)}$ shape. The maximum of each of the $\frac{\sin(Nx)}{\sin(x)}$-shaped beams coincides with a zero in all the other beams. Woodward's synthesis technique samples the ideal pattern at increments of $\frac{2\pi}{N}$. The synthesised array is thus the superposition of array factors from N uniform amplitude, linear progressive phase excitations with the phase increment of each excitation such that the array factor peak is placed at the sampling point. Although the array factor is equal to the ideal pattern at the sample points, it exhibits large ripples in the shaped region and high uncontrollable sidelobes in the sidelobe region.

An objection to the Woodward method is that it only uses data form N points, whereas there are $2N-2$ degrees of freedom ($N-1$ relative amplitudes and phases) available. Woodward's method makes inefficient use of the zeroes on the Schelkunoff unit circle, placing them in radial pairs. Milne published a method [51] that effectively takes $2N-1$ sample points, spaced at intervals of $\frac{2\pi}{(2N-1)}$, to determine a realizable array factor. The method essentially is a convolution of a "scanning function" (usually a Dolph-Chebychev pattern) and the shaping function.

By a trial-and-error adjustment of the angular positions of the roots of the factorised array factor, on the Schelkunoff unit circle, Elliott [52] obtained any sidelobe topography. Later, Elliott and Stern [53] introduced the concept of null filling to produce shaped beam patterns. This is achieved by moving the roots in the shaped beam region radially from the Schelkunoff unit circle. Orchard et al. [54] presented an iterative method by which the angular and radial positions of all the roots are simultaneously adjusted so that the amplitude of each ripple in the shaped region and the height of each sidelobe in the non-shaped region are individually controlled.

The method was extended by Kim and Elliott [55] to produce pure real distributions for symmetric patterns, such as flat-top patterns. The pure real distributions are obtained by placing the roots in the shaped beam region in conjugate pairs. By pairing the roots in the shaped beam region, both roots at the same angular position but one root inside and the other outside the unit circle, a centre-fed symmetric distribution can be obtained [56]. Ares et al. [57] extended the technique to allow the synthesis of patterns with prescribed nulls.

2.6 Analytical Synthesis of Continuous Line Source Distributions

Although arrays of discrete radiating elements are being dealt with in this work, no review of synthesis techniques would be complete without reference to similar work on the synthesis of continuous line source distributions. Line source synthesis is important in the array context for several reasons. Firstly, some general principles are equally applicable to arrays. Secondly, continuous distributions can be sampled for use with discrete arrays. Furthermore, the direct discrete array synthesis methods have generally developed "under the guidance" of the theory on continuous distributions.

Bouwkamp and de Bruyn [58] showed that with a continuous line source of fixed length it is possible (in theory) to achieve any desired directivity. Although this implies that there is no limit to the directivity, any directivity increase above that obtained from the aperture when it is uniformly excited is accompanied by a sharp increase in the net reactive power required at the source to produce it [18:p.3], and thus a large tolerance sensitivity. Practical considerations therefore makes it unacceptable, as in the case of the unconstrained maximisation of the directivity of the discrete array discussed earlier (in Section 2.5.1). To be realizable physically, some constraint has to be placed on the proportion of reactive to radiative power, or equivalently on the quality factor [36:pp.3-4].

It is customary, when dealing with continuous line-source distributions, to use the variable $u = \frac{L \sin\theta}{\lambda}$, where L is the length of the source and λ is free-space wavelength. This practice will be followed in the discussions to follow.

2.6.1 Maximisation of Directivity Subject to a Sidelobe Level Constraint

The next question regarding continuous distributions is that of determining a distribution which provides the narrowest beamwidth for a given sidelobe level, and vice versa. This was answered by Taylor [59], who used the Dolph-Chebyshev theory as starting point. Using an asymptotic relationship for the Chebyshev polynomials given by van der Maas [34], Taylor derived the continuous equivalent of the Dolph-Chebyshev distribution. This distribution produces a pattern where all the sidelobes are of an equal level, and is optimum in the sense that it provides the narrowest beamwidth for a given sidelobe ratio of any non-super directive distribution. Taylor called this the "ideal" line source distribution. "Ideal" because of the fact that it is not realizable, having a singularity at each end.

2.6.2 Maximisation of Directivity Subject to a Tapered Sidelobe Level Constraint

A solution to the problem of singularities at the ends of the distribution was also devised by Taylor [59]. He recognised that the synthesis problem is one of correctly positioning the zeros of the space factor. Taylor observed that close-in zeros should maintain their spacings to keep the close-in sidelobes suitably low and the beamwidth narrow, but that at the same time the further out sidelobes should decay as $\frac{1}{u}$ [36:p.55, 25:p.720]. Such sidelobe decay is found in the space factor of a uniform line-source distribution. Taylor stretched the u scale by a dilation factor σ slightly greater than unity (so that the close-in zero locations are not shifted much) and chosen such that at some point a shifted zero is made to coincide with an integer $\bar{n}$. From this transition point, the zeros of the ideal line-source are replaced by those of the uniform line-source. In this pattern the first few sidelobes (the number is determined by the position of the transition point selected) are roughly equal, with a $\frac{1}{u}$-sidelobe decay thereafter. The corresponding aperture distribution is then found as a Fourier series obtained from the above mentioned information on the zeros [8:p.58]. The final result is a distribution (referred to as the Taylor $\bar{n}$ distribution) which, for a given sidelobe ratio, gives both a narrower beamwidth and higher directivity than any comparable continuous line-source distribution. Information relating the sidelobe ratio, dilation factor and $\bar{n}$ values have been given by Hansen [8:p.57]. Also given are expressions for the aperture distribution itself [8:p.58]. Too large a value for $\bar{n}$ (exactly how large depends on the specified sidelobe ratio) implies that the ideal line-source distribution is "being approached too closely." The aperture distribution then becomes non-monotonic with peaks at the aperture ends (although the singularities of the ideal source do not occur), and an accompanying increase in excitation tolerances. The Taylor $\bar{n}$ distribution was generalised by Rhodes [18:pp.129-137] to a distribution dependent on the transition variable $\bar{n}$, and an additional variable, which controls the taper rate of the sidelobe envelope for a selected transition zero. The "ideal" line-source and Taylor $\bar{n}$ distribution approaches just described are special cases of this generalised distribution.

A third continuous distribution due to Taylor is his one-parameter line-source distribution [8:p.58]. The Taylor $\bar{n}$ distribution essentially selects a design between the "ideal" and one-parameter cases. However, for the same first sidelobe ratio, the Taylor $\bar{n}$ distribution has a higher excitation efficiency (and hence directivity), and is therefore used more often. The reason for this is that the $\bar{n}$ distribution tends to flatten out at the ends of the line source while the one-parameter case does not. The Taylor one-parameter distribution was generalised by Bickmore and Spellmire, whose work has been reported in and [25:pp.731-733], into a two-parameter continuous line-source distribution. One of the parameters selects the starting sidelobe ratio, while the other selects the rate of decay of the sidelobes. These two parameters are completely independent. As expected, the Taylor one-parameter distribution can be shown to be a special case of the Bickmore-Spellmire distribution.

2.6.3 General Pattern Synthesis

The need for sidelobe suppression may vary as a function of direction. Elliott proposed an iterative perturbation method that gives control over each sidelobe [10:pp.162-172, 60]. The Taylor distribution is used as a starting point. If the perturbations are small the expression for the new pattern can be truncated. This truncation results in a set of deterministic linear equations. The perturbed root positions are then computed by matrix inversion. If the required pattern differs too much from the starting Taylor pattern an interim desired pattern can be postulated to ensure convergence.

Ares [61] adapted the general method of Orchard et al. [54] for discrete linear arrays to synthesise continuous line sources. As with linear arrays synthesis the roots can be moved radially from the unit circle to produce filled-in nulls in the shaped region. The method allows the synthesis of shaped beams with any arbitrary sidelobe topography.

2.6.4 Continuous Difference Distributions

An analogous synthesis problem exists for difference patterns: find the line source distribution that will produce an antisymmetric difference pattern with two main beams surrounded by sidelobes of a specified maximum height. The problem was addressed by Bayliss [62]; an ideal space factor was not found from which to determine the null positions, but Bayliss undertook a numerical parametric study to find formulas for the root placement.

The perturbation used to obtain an arbitrary sidelobe level topography from a Taylor pattern can also be used to synthesise a difference pattern with arbitrary controlled sidelobe levels [63, 10:pp.185-187].

2.6.5 Discrete Versus Continuous Line Source Synthesis

It is clear from the previous section that the theory of continuous aperture line source distributions for sum patterns is extensive and well developed. If these are to be used with arrays of discrete elements, some form of discretization process must be performed. The earliest discretization methods simply sampled the continuous distributions at the element locations. Unless the arrays are extremely large (that is, larger than most practical arrays) a badly degraded pattern may be obtained [64, 65].

An alternative technique was proposed by Winter [66]. The initial array element excitations are determined by sampling the continuous distribution and then iteratively adjusting these using Newton-Raphson minimisation of an error expression comprised of the sum of the squares of the differences between calculated (discrete) and specified (continuous) levels for a selected number of sidelobes. He reported successful discretization of arrays with very low sidelobe levels [67].

Elliott [10:pp.172-180, 64] devised a more sophisticated yet direct alternative method. This method matches zeros. Instead of sampling the continuous aperture distribution, one requires that the pattern zeros of the continuous case also occur in the starting pattern of the discrete case. If the resulting pattern does not meet the design goal, a linear perturbation of the zero positions is iteratively applied to the discrete distribution in order to bring the final pattern within specification.

2.7 Techniques for Planar Array Synthesis

There is considerable interest in the problem of designing satellite-borne antennas that will beam electromagnetic waves of uniform strength onto a specified portion of the earth's surface while causing negligible radiation to reach the remainder of the earth's surface. In order to best utilise the limited spacecraft transmitter power and reduce possible interference problems with other coverage areas, the ideal antenna would have uniform gain over the coverage region and no radiation elsewhere. Such shaped beams are said to have a *contoured footprint* [19]. The fact that a real antenna is of finite size, along with several other practical limitations, means that there will be some sidelobe radiation outside the defined irregularly shaped coverage region.

The most commonly utilised method of generating contoured beams has been that of an array of feed horns illuminating an offset paraboloidal reflector [68, 69]. Each feed generates a component beam, and these are properly weighted through a beam-forming network to obtain the desired contoured beam. Alternatively, a shaped offset reflector fed by a single feed [70] is also capable of producing fixed contoured beam coverage.

A disadvantage of the component beam approach is the fairly high ripple within the footprint if a rather large number of beams are not used [19], and the relatively poor control of the sidelobe level outside the coverage area. Also, observing that the size, mass and complexity (especially of the feed arrays) of satellite antennas have grown significantly with each new system, Bornemann, Balling and English [71] have argued in favour of eliminating the reflector and using a direct radiating array. Such an approach would seem particularly advantageous for reconfigurable antennas [72], although this is now also possible if reflector antennas are used [70].

The previous sections provide a relatively complete summary of direct methods that can be used to synthesise linear arrays providing high directivity, low sidelobe patterns. Regarding their relation to planar array synthesis, Elliott [10:p.196] notes that: "Under certain circumstances, much of what was developed there (for linear arrays) can be carried over to apply to planar arrays. However, practical considerations will at times require the use of design techniques that are peculiar to planar arrays.". Two-dimensional polynomial roots are not just simple points as in the one-dimensional case, but they can also be contours. One root contour may even cross another. Due to the lack of a general two-dimensional factorisation theorem one dimensional (linear array) root-placement techniques can not in general be extended to the two-dimensional (planar array) case. Synthesis methods for direct radiating planar arrays with fully contoured beams [73, 74, 75, 76] use some form of non-linear numerical optimisation procedure. Albeit successful in some cases, such techniques do not always have guaranteed convergence [77], and are furthermore time consuming and expensive. The method described in [78, 79, 80] represents initial attempts at overcoming these difficulties by utilising root placement based techniques, known to work well in linear array synthesis, for obtaining continuous distributions with elliptical contours. These distributions are subsequently iteratively sampled to obtain the excitations of the actual discrete array. Applying root placement directly to the discrete planar array does not produce satisfactory patterns, but the excitations thus obtained serve well as an initial guess for numerical optimisation techniques [77].

In order to describe a planar array geometry it is necessary to specify both its element lattice and the boundary shape, as mentioned earlier. Again, only those synthesis techniques applicable to the rectangular lattice case are of interest here. In the remainder of this section we will briefly list those methods for planar arrays that can be classed as "analytical", in that no form of iteration, perturbation or numerical optimisation is needed (although a considerable amount of computation may still be required) once the set of specifications for the final pattern has been set. Numerical synthesis methods form the subject of Section 2.8.

2.7.1 Separable Distributions

For rectangular arrays, if the boundary shape is square or rectangular, and if the aperture distribution is separable, the pattern is the product of the patterns of two spatially orthogonal linear arrays (collapsed distributions), and all the methods mentioned for linear arrays in earlier sections can be extended readily. However, separable distributions suffer from some directivity limitations [10:p.211] which are highly undesirable in practice. The separable distribution over achieves in most of the sidelobe region (i. e. gives sidelobe levels far lower than necessary). This reduction is bought at the price of beam broadening, with its concomitant lowering in directivity. These limitations can only be overcome by the use of ϕ-symmetric patterns.

2.7.2 Direct Synthesis of Discrete Arrays

The Dolph-Chebyshev technique has not been extended to general planar arrays. However, Tseng and Cheng [81] have shown how it can be done for a planar array with a rectangular lattice and a rectangular boundary shape, and with an equal number of elements in each principal direction (the $\hat{x}$- and $\hat{y}$-directions). They used the Baklanov transformation (written in terms of the u and v as defined in this chapter) $w = w_0 \cos(\frac{1}{2}u)\cos(\frac{1}{2}v)$ [82] to convert the Chebyshev polynomial, a function of one variable w, to a polynomial of two variables u and v. The resulting Tseng-Cheng distribution is non-separable and gives a Dolph-Chebyshev pattern in every ϕ-cut.

Goto [83] first expressed a symmetrical linear array factor in polynomial form, with the polynomial variable $w = \cos(\frac{1}{2}\psi)$. This allows the synthesis of planar arrays with patterns other than Dolph-Chebyshev patterns. He then used a transformation similar to the Baklanov transformation to synthesise planar arrays with hexagonal lattice; $w = \cos(u)\cos(v)$. Goto extended the technique to planar array with not only a hexagonal lattice but also a hexagonal boundary [84]. To achieve a hexagonal planar array the transformation is $w = \alpha + \beta\cos(\psi) = \frac{1}{2}\cos(u)[\cos(u) + \cos(v)]$. This transformation results in planar arrays with near rotationally symmetric patterns.

The restriction of equal numbers of elements along each array axis of the Tseng-Cheng distribution can be lifted, as shown by Kim and Elliott [85], by using a generalisation of the Baklanov transformation, $w = \cos^p(\frac{1}{2}u)\cos^q(\frac{1}{2}v)$. The ratio of "rows" versus "columns" of the array is fixed by the selection of p and q. They also showed that null-filling is feasible, achieving elliptical footprint contours. A transformation similar to that of Goto [84] was proposed by Kim [86] to synthesise hexagonal arrays to produce almost rotationally symmetric patterns.

Richie and Kritikos [77] applied a root placement technique, known to work well with linear arrays, directly to the discrete planar array. Roots in a two dimensional pattern can be points, lines or more complex contours making root placement difficult. The resulting pattern does not meet all the design criteria, such as sidelobe level and ripple specifications, but the excitation may used as a starting values for numerical optimisation techniques.

2.7.3 The Convolution Synthesis Method

Realizing the array factor of an array with uniform spacing has the form of a discrete Fourier transform, Laxpati [87] proposed the use of discrete convolution of small arrays to obtain the excitation of a planar array. The planar array factor is the product of the array factors of a number of small arrays. Each of these small array factors has its own prescribed null locus; as a result of the multiplication the planar array factor contains all these null loci [88, 89]. The convolution method can be used to synthesise an array with arbitrary prescribed pattern nulls and almost any boundary shape [90]. Since no closed form relation exists between the null loci and the lobe peaks the placement of loci for the synthesis of more general patterns must be done interactively. This would be very time consuming even for relatively small arrays.

Laxpati [90] synthesised a 36-element diamond by the convolution of 5 four-element rhombiod arrays. The convolution of the diamond shaped array with a six-element linear array (along the x-axis) was then used to obtain a five-ring hexagonal array. Using a larger linear array it is possible to obtain an array with a hexagonal lattice and a stretched hexagonal boundary. The use of differently shaped arrays as building blocks result in differently shaped root loci. For instance a four-element rhombiod array factor has near circular shaped inner contours while all the root loci of a linear array factor are lines perpendicular to its axis. In places in the pattern where root loci are closely spaced the sidelobe will be very low and vice versa. Since the root positions are controled only on one principal axis high sidelobe do appear in the examples in his paper.

The method was extended to synthesis symmetrical square and hexagonal arrays where the sidelobe typography, rather than the null loci, is described [91]. This extension requires the small arrays to have the same geometry. The convolution method is computationally simple and can be used to synthesise very large arrays.

2.7.4 The Transformation Based Synthesis Technique

The transformation technique utilises a transformation, first used to design digital filters [92], that divides the planar array synthesis problem into two decoupled sub-problems [93, 94]. In the antenna array context, one sub-problem consists of a linear array synthesis, for which powerful methods of determining appropriate element excitations exist. The other involves the determination of certain coefficients of the transform in order to achieve the required footprint contours. The number of coefficients which need to be used depends on the complexity of the desired contour, but is very small in comparison to the number of planar array elements. The size required for this prototype linear array depends on the number of transformation coefficients used and the planar array size. Recursive formulas then determine the final planar array excitations from the information forthcoming from the above two sub-problem solutions. Thus the method is computationally efficient, making trade-off studies feasible even for large arrays. Simple formulas for the calculation of the transformation coefficients for circular and elliptical contours are given in [95, 94]. However, only patterns with quadrantal or centro symmetry can be synthesised using the transformation method up to the stage to which it was developed in [95, 94]. Extensions to the transformation to allow arbitrary contours, as well as non-rectangular boundaries and lattices, form the subject of Chapter 3 of this thesis and a detailed mathematical treatment of the transformation based synthesis method is given there.

2.7.5 Direct Synthesis of Continuous Planar Source Distributions

Taylor [96] extended his continuous line-source analysis to the case of a planar aperture with circular boundary shape with a ϕ-symmetric aperture distribution. A one-parameter circular distribution has been described by Hansen [97, 98] using the procedure adopted by Taylor for the one-parameter line source distribution mentioned earlier in Section 2.6.2. Although similar to the Taylor distribution, the Hansen one-parameter distribution is "slightly less efficient, but the required distribution is smoother and more robust" [25:p.855].

Various extensions to Taylor's method have been proposed [99, 100, 101]. The most significant contribution was made by Elliott and Stern [79], who succeeded in combining Taylor's method with the linear array synthesis method proposed by Orchard, Elliott and Stern [54]. This development permits the synthesis of rotationally symmetric shaped beam patterns with individually controllable ring sidelobes for continuous planar apertures with circular boundaries. Recently Ares, Monero and Elliot [80] extended this combined Taylor method by linearly stretching the distribution to obtain a arbitrarily shaped contoured footprint. A circular flat-top pattern is first synthesised, then the aperture in every ϕ-cut is stretched inversely proportional to the flat-top beamwidth in that ϕ-cut to obtain the desired footprint pattern.

2.7.6 The Sampling of Continuous Planar Distributions

If the continuous distributions described in Section 2.7.5 are to be used with arrays of discrete elements, some form of discretization must be performed. As indicated for the linear array situation in Section 2.6.5, the naive direct sampling of the continuous distribution may lead to noticeable pattern degradation unless the array is very large. Unfortunately, the rather convenient improved sampling ideas for linear arrays, discussed in Section 2.6.5, do not carry over to planar arrays, reasons for which are given by Elliott [10:pp.243-249]. Less direct approaches (albeit using the continuous distributions as starting points) are required to iteratively alter the sampled distribution in order to improve the array pattern performance, and these are better classified as numerical synthesis procedures to be discussed as part of Section 2.8 which follows immediately.

2.8 Numerical Synthesis of Arrays

Numerical techniques which neither take advantage of the peculiar properties of the array factor nor information available from the analytical solution to related synthesis problems, or both, are usually slow to converge and inefficient. Therefore, although it is always possible to formulate the array synthesis problem very precisely as a nonlinear optimisation problem with nonlinear constraints, such general optimisation approaches prove to be undesirable in practice. More "customised" methods which exploit known characteristics and insight into the array synthesis problem are preferred. Some of the more general methods can be used for conformal array synthesis.

In order to summarise the numerical array synthesis methods it will be convenient to divide these into two groups:

- those that can be considered perturbation methods which begin with results obtained from one of the synthesis procedures discussed in earlier sections
- those that make use of minimisation/maximisation (optimisation algorithms) of some array performance index subject to a set of constraints.

2.8.1 Synthesis of Planar Arrays with Contoured Beam Patterns

All the planar synthesis techniques mentioned in Section 2.7 can be used to synthesise the direct radiating array. Bornemann [71] uses a method similar to Woodward's, applied to a planar array, to achieve various complex beam shapes and contours. Guy [102] proposed an interactive iterative synthesis method which uses far-field pattern only. He has reported successful designs of shaped beam and contoured beam antennas for planar as well as conformal arrays. A sampled continuous distribution is used by Elliott [78] and Ares [80] as starting point for numerical optimisation, as discussed in Section 2.8.3

2.8.2 Use of Collapsed Distributions

The concept of a collapsed distribution was mentioned in Section 2.2.4. A planar array synthesis method which uses linear array synthesis on the two orthogonal collapsed distributions of a given planar array, and then spreads out the collapsed distributions using the conjugate gradient technique or the iterative method of Section 2.8.3, has been described by Elliott [19]. The method is applicable to both sum and difference patterns and can yield non-separable final distributions. It has the advantage of beginning with the use of established linear array techniques. In spite of the claims that such a method is a sort of panacea to all planar array synthesis problems, this is not always the case, as evidenced by attempts at further improvements by Autrey [65] and Elliott and Stern [79].

2.8.3 Iterative Perturbation Methods

In Section 2.7.5 the direct synthesis of continuous planar source distributions was mentioned. It was indicated there that the relatively straightforward altered sampling approaches applicable to linear arrays do not carry over to the planar array case. The minimum source density is required for discretization is determined by the behaviour of the radiation pattern outside the visible region [103].

To improve the pattern characteristics of the sampled distribution an iterative sampling scheme was proposed by Stutzman and Coffey [104]. Elliott [10:pp.243-249] has described an iterative procedure which tries to improve on the conventionally sampled continuous planar distribution, the final set of non-separable excitations being distinctly different from the starting ones. Application of this iterative technique to the sampling of continuous planar distributions is also described [10:pp.256-261, 78].

Fletcher-Powell minimisation was successfully used by Ares [80] to minimise a cost function comprising the sum of the squares of the differences between the actual pattern and the required pattern [80], for contoured beam arrays.

2.8.4 Array Synthesis as an Intersection of Sets

The use of the method of projections for the synthesis of antenna arrays has grown out of the development of techniques in the field of image processing, in particular image restoration. Details of the method are described by Levi and Stark [105]. Prasad [106] applied an early version of the method (called the method of alternating orthogonal projections) to the array synthesis problem. Limitations on the type of constraints that could be used led to improvements resulting in the method of successive projections. This has been applied to arrays by Poulton [107, 108]. The generalised projection technique has been used as the basis for a synthesis method [109, 110, 111] which allows constraints not only on the sidelobe levels but also on the excitations (eg. dynamic range, smoothness of the distribution), something that is most advantageous when having to deal with mutual coupling effects. The method searches for the intersection of two sets by iteratively projecting back and forth between the sets. The technique uses the fast Fourier transform and the inverse fast Fourier transform as a projector pair. The method has recently been extended to circular arc conformal array synthesis by using singular value decomposition as a back projector [112]. Even more recently Bucci et al. [113] achieved success in the synthesis of general conformal arrays by using a self-scaled version of the Broyden-Fletcher-Golfarb-Shanno method [114] one of the projectors used in the search for the intersection of the sets.

Array synthesis (with the emphasis on conformal arrays) as the intersection of sets is further investigated in Chapter 5 of this thesis. A complete overview and the progression of these methods are given there.

2.8.5 Constrainted Minimisation/Maximisation (Optimisation)

Constrainted optimisation techniques differ in :

- the way in which the array problem is formulated as an optimisation problem (e.g. the choice of performance indices to be minimised or maximised)
- the type of constraints which are applied (e.g. limited Q, maximum sidelobe levels allowed)
- the particular optimisation algorithm used (e.g. linear programming, quadratic programming, use of ratios of Hermitian quadratic forms, general nonlinear function maximisation/minimisation).

Hansen summarised general methods of this nature [25:pp.743-748]; as indicated earlier the use of such general methods are not always practical. Detailed overviews of optimisation methods have also been given by Cheng [21], Lo et al. [29] and Sanzgiri and Butler [115]. Wilson [116] applied a linear approximation procedure to derive the arrays excitations of rectangular planar arrays with sum or difference patterns. Quadratic programming was proposed by Einarsson [117] for those cases where the directivity must be maximised subject to a set of sidelobe constraints; the technique maximises the directivity by minimising an associated quadratic form. By writing the sidelobe constraints as a linear set, Einarsson [117] was able to formulate the optimisation problem as a quadratic programming one. By the optimisation, using the method of steepest descent, of a piecewise differentiable objective function DeFord and Gandhi [118] synthesised linear and planar arrays.

The method of simulated annealing, used for computing the properties of systems of interacting molecules, was applied to the synthesis of antenna arrays by Farhat and Bai [119]. Simulated annealing is used to optimise the energy (or cost function) of a system by slowly lowering the "temperature" (control variable) of the system until the system "freezes"; in a manner similar to the annealing of a structurally pure crystal. Due to a probabilistic selection rule the process can get out of a local minimum and proceed to the global minimum. However, the choices for the initial temperature and the probabilistic factor are crucial for speed and success of convergence. Recently good results ware obtained using simulated annealing for the synthesis of circular arc and cylindrical arrays [120, 121].

2.9 Array Synthesis Techniques Developed in this Thesis

The conventional array synthesis methods have been reviewed in the previous sections. Three inadequacies have been identified: firstly, the transformation synthesis technique can be used to synthesise only centro symmetric contours, secondly no difference pattern synthesis method exists for planar arrays and thirdly no effective conformal array synthesis method exists. Due to the non linear nature of conformal array synthesis an effective conformal array synthesis method must have a rapid rate of convergence and a measure of confidence that the result will be close to the optimal solution.

The transformation based synthesis method can be used for the synthesis of rectangular planar arrays with quadrantal or centro symmetrical footprint patterns. In Chapter 3 of this thesis the transformation technique will be extended to allow for the synthesis of:

- arbitrary contoured footprint patterns;
- planar arrays with non rectangular boundaries and
- planar arrays with triangular lattices.

Application of the newly developed and extended transformation based synthesis technique is examined by the use of a number of specific examples.

The principal contribution made in Chapter 4 is the presentation of a well ordered, step by step method for the synthesis of planar arrays with difference patterns. The method uses as one of the steps the extended transformation method developed in Chapter 3, and also utilises the convolution synthesis method. The result is a near-optimum difference pattern for planar arrays.

Due to the nature of conformal arrays no simplification can be made to ease the synthesis problem. Currently no effective synthesis technique exists to produce discrete conformal arrays. Chapter 5 considers the development of a such a conformal array synthesis technique, and explores ways to increase the rate of convergence. Any numerical optimisation is prone to fall into local minima. To avoid local minima a starting point close to the global minimum is needed. A novel selection of the starting point is proposed and investigated. This synthesis method is very flexible, allowing the synthesis of both co- and cross polarisation radiation patterns of arrays with an arbitrary geometry; with constraints on the radiation pattern as well as the excitations.

Finally, Chapter 6 completes the thesis by drawing a number of general conclusions.