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Solution of conservation laws via convergence space completion

by

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Declaration

I, the undersigned, hereby declare that the thesis submitted herewith for the degree Master of Science to the University of Pretoria contains my own, independent work and has not previously been submitted by me or any other person for any degree at this or any other University.

Name: Dennis Ferdinand Agbebaku

Date: August 2011

To the memory of my late aunty Mrs Roseline Segun Okhuiegbe, and my late step-mother Mrs Philomina Agbebaku, both of whom passed on during the course of this work. Memory is the one thing death cannot destroy.

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Summary

In this thesis we consider generalized solutions of scalar conservation laws. In this regard, the Order Completion Method for systems of nonlinear PDEs is modified in a suitable way. In particular, with a given Cauchy problem for scalar conservation law, we associate an injective mapping $T : \mathcal{M} \longrightarrow \mathcal{N}$, where $\mathcal{M}$ and $\mathcal{N}$ are suitable spaces of sufficiently smooth functions, independent of the given conservation law, so that the initial value problem may be expressed as one equation

$$Tu = h \tag{1}$$

for a suitable $h \in \mathcal{N}$.

Uniform convergence structures are introduced on the spaces $\mathcal{M}$ and $\mathcal{N}$ in such a way that the mapping T is a uniformly continuous embedding. Thus there exists a unique, injective uniformly continuous mapping $T^\sharp : \mathcal{M}^\sharp \longrightarrow \mathcal{N}^\sharp$, where $\mathcal{M}^\sharp$ and $\mathcal{N}^\sharp$ denote the completions of $\mathcal{M}$ and $\mathcal{N}$, respectively, that extends T. Thus we arrive at a generalized version of the equation (1), namely,

$$T^\sharp u^\sharp = h \tag{2}$$

where the unknown function u is supposed to belong to $\mathcal{M}$. Any solution of (2), if it exists, is interpreted as a generalized solution of (1). Note that due to the injectivity of $T^\sharp$, the equation (2) has at most one solution. Furthermore, the space $\mathcal{M}$ of generalized functions may be identified in a natural way with a set of Hausdorff continuous interval valued functions. Therefore the solution of (2) has a solution, which agrees with the well known entropy solution.

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Contents

Chapter 1

Introduction

The mathematical models for real-world problems occurring in Physics, Chemistry, Economics, Engineering and Biology are usually expressed in the form of partial differential equations (PDEs) with associated initial and/or boundary values.

We consider only initial value problems, consisting of a PDE

$$T(x,t,D)u(x,t) = h(x,t),\ x \in \Omega,\ t > 0$$

and an initial condition

$$u(x,0) = u_0(x),\ x \in \Omega$$

where $\Omega \subseteq \mathbb{R}^n$ is open and h a suitable function. The fundamental mathematical question, concerns the well-posedness of the problem. Recall [58] that the given initial value problem of a PDE is well-posed if the problem has a solution, if the solution is unique and if the solution depends continuously on the data given in the problem. Each of the three issues involve in the concept of well-posedness is nontrivial in its own right.

It is well known that an initial value problem may fail to have a classical solution on the whole domain of definition of the equation. Indeed, a nonlinear analytic PDE will, according to the well known Cauchy-Kowalevskaia Theorem [71], admit an analytic solution which is defined on a neighborhood of any noncharacteristic hyper surface on which analytic initial data is specified. However, outside this neighborhood the solution may fail to exist. In particular, the solution will typically exhibit singularities outside the mentioned neighborhood of analyticity. Thus, the solutions cannot be guaranteed to exists on the *whole* domain of definition of the given PDE. In fact, even a linear equation without initial conditions may fail to have a solution, as shown by the following example due to Lewy [83].

Example 1.1. Consider the linear operator

$$A(u) = u_x + iu_y - 2i(x+iy)u_t \quad (x,y,t) \in \mathbb{R}^3.$$

There exist C^∞-smooth functions h for which the equation $A(u) = h$ has no solution in $\mathcal{D}'$-distributions in any neighborhood of any point in $\mathbb{R}^3$, see [83] for details. □

In view of the local nature of solutions of an initial value problems, in general, it is clear that there is an interest in solutions to PDEs that may fail to be classical on the whole domain of definition of the respective PDE. Such *generalized or weak solutions* to PDEs are obtained as elements of suitable spaces of generalized functions, that is, objects which retain certain essential features of the usual real or complex valued functions.

Many mathematicians specializing in nonlinear partial differential equations (PDEs) believe that there is no general and type independent theory for the existence and basic regularity properties of generalized solutions of PDEs. In fact, the first chapter of the book [14] by V. I. Arnold starts with the following statement:

> "In contrast to ordinary differential equations, there is no unified theory of partial differential equations. Some equations have their own theories, while others have no theory at all. The reason for this complexity is a more complicated geometry ..."

This seeming inability of mathematical theories to deal with PDEs in a unified way may be attributed to the inherent limitations of the customary, linear topological theories for the solution of PDEs themselves, rather than to any fundamental conceptual obstacles. In this regard we may note that the spaces of generalized functions that are typically used in the study of solutions of linear and nonlinear PDEs cannot deal with sufficiently large classes of singularities. Indeed, due to the celebrated Sobolev Embedding Theorem [112], none of the Sobolev spaces can deal with the most simple singular functions, such as the Heaviside step function

$$H(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ \\ 1 & \text{if} \quad x \geq 0 \end{cases}$$

Furthermore, Colombeau generalized functions [50] as well as distributions, cannot handle an analytic function with an essential singularity at one single point, such as $f(z) = e^{1/z}$. The great Picard Theorem states that such a function will assume every complex number, with possibly one exception, as a value in every neighborhood of the singular point. It will therefore violate the polynomial growth conditions that are imposed on the Colombeau generalized functions near singularities.

However, there are two recent theories that provide general and type independent results regarding the existence and basic regularity properties of large classes of PDEs, namely, the Order Completion Method(OCM) [93] and the generalized method of Steepest Descent in suitably constructed Hilbert spaces, introduced by Neuberger [89, 90, 91, 92].

The Order Completion Method is based on the Dedekind order completion of suitable spaces of (piecewise) smooth functions, and applies to what may be considered all continuous nonlinear PDEs. Furthermore, the solution so obtained satisfies a blanket regularity property. In particular, the solutions may be assimilated with Hausdorff continuous interval valued functions [10]. The recent reformulation and enrichment of the OCM in terms of suitable uniform convergence spaces and their completions has greatly improved the regularity properties as well as the understanding of the structure of solutions, [118, 119, 120, 121].

The underlying ideas upon which the method of Steepest Descent is based does not depend on the particular form of the PDE involved, and is therefore type independent. However, the relevant techniques involve several highly technical aspects which have, as of yet, not been resolved for a class of equations comparable to that to which the OCM applies. However, the numerical computation of solutions, based on this theory, has advanced beyond the proven scope of the underlying analytical techniques, see for instance [92].

In this Thesis we study a class of first order PDEs that may serve as mathematical descriptions of physical *conservation laws*, such as the laws of gas dynamics and the laws of electromagnetism. In particular, we apply the Order Completion Method, as formulated in the context of Convergence spaces as well as uniform convergence spaces completion [119, 120, 121] to the first order nonlinear Cauchy problem of conservation laws. Furthermore, we show how the Convergence Space Completion Method can be applied to solve the initial value problem of the Burgers equation. We construct the entropy solution of the Burgers equation and show how it can be assimilated with the space of $\mathbb{H}$-continuous functions.

In the rest of Chapter 1, some of the concepts and theories that are used in obtaining our results are discussed. The existence and uniqueness of weak solutions of the Cauchy problem of conservation laws is discussed in Section 1.1. Some of the admissibility conditions for singling out a unique solution are considered, namely, the Lax, Oleinik and entropy conditions. Other techniques, namely, the vanishing viscosity method and the compensated compactness technique, for obtaining the entropy solution of a conservation law are also discussed. Some existence and uniqueness results for solutions of conservation laws obtained by Hopf, Lax, Oleinik and Kruzhkov, are also discussed. An introduction to spaces of Hausdorff continuous functions is presented in Section 1.3. Section 1.2 is an introduction to the theory of convergence spaces. Section 1.4 addresses the main ideas underlying the Order Completion Method. Chapter 1 ends with a summary of the main results in this thesis.

1.1 Nonlinear Hyperbolic Conservation Laws

1.1.1 Introduction to Scalar conservation laws

A conservation law states that a particular measurable property of an isolated physical system does not change as the system evolves. In particular, any change in such a conserved quantity can only occur as a result of an "influx" or an "outflow" of this quantity into or out of the system respectively.

The exact mathematical model for a single conservation law in one spatial dimension is given by the first order PDE

$$u_t + (f(u))_x = 0. \tag{1.1}$$

Here u is the conserved quantity while f is the flux. Integrating equation (1.1) over some interval $[a, b]$ leads to

$$\begin{aligned}\frac{d}{dt}\int_a^b u(x,t)dx &= \int_a^b u_t(x,t)dx \\ &= -\int_a^b f(u(x,t))_x dx \\ &= f(u(x,a)) - f(u(x,b)) \\ &= [\text{ in flow at a}] - [\text{ out flow at b}]\end{aligned}$$

In other words, the quantity u is neither created nor destroyed. In particular, the total amount of u contained in the interval $[a, b]$ can only change due to the flow of u across the two endpoints.

In general, if $\mathbf{u} = (u_1, \cdots u_k)$ is a vector of conserved quantities, depending on time t and n independent variables $x_1, \cdots, x_n$, then the flux of $\mathbf{u}$ out of any bounded region $\Omega \subseteq \mathbb{R}^n$ is given by

$$\int_{\partial\Omega} \mathbf{F}(\mathbf{u}) \cdot \mathbf{n} dS.$$

Here $\mathbf{F} : \mathbb{R}^k \longrightarrow \mathbb{M}^{k\times n} = \{A : A \text{ is a matrix of order } k \times n\}$ is the flux, $\mathbf{n}$ denotes the outward unit normal to $\partial\Omega$ and dS the surface element on $\partial\Omega$. Since any change in $\mathbf{u}$ in such a domain Ω over time can only be due to the 'in flow' or 'out flow' of $\mathbf{u}$ into or out of Ω, it follows that

$$\frac{d}{dt}\int_\Omega \mathbf{u} dx = -\int_{\partial\Omega} \mathbf{F}(\mathbf{u}) \cdot \mathbf{n} dS. \tag{1.2}$$

Note that the integral on the right of (1.2) measures the flow out of Ω, hence the minus sign. Assuming that $\mathbf{F}$, $\mathbf{u}$ and $\partial\Omega$ are sufficiently smooth, we may apply the Divergence Theorem to equation (1.2) so that

$$\frac{d}{dt}\int_\Omega \mathbf{u} dx = -\int_\Omega \nabla \cdot \mathbf{F}(\mathbf{u}) dx.$$

Taking the derivatives with respect to t under the integral sign we obtain

$$\int_{\Omega}[\mathbf{u}_t + \nabla \cdot \mathbf{F}(\mathbf{u})]dx = 0.$$

the Mean Value Theorem implies the differential form of conservation laws, which is given by

$$\mathbf{u}_t + \nabla \cdot \mathbf{F}(\mathbf{u}) = 0.$$

The study presented in this thesis is concerned mainly with the Cauchy problem for *strictly hyperbolic* systems in one spatial dimension. That is,

$$\mathbf{u}_t + (\mathbf{F}(\mathbf{u}))_x = 0 \qquad \text{in } \mathbb{R} \times (0, \infty) \tag{1.3}$$

$$\mathbf{u}(x, 0) = \mathbf{u}_0(x) \qquad x \in \mathbb{R}, \tag{1.4}$$

where $u = (u_1, \cdots, u_k)$, $F : \mathbb{R}^k \longrightarrow \mathbb{R}^k$ and $\mathbf{u}_0 = (u_0^1, \cdots, u_0^k)$ is the initial value of $\mathbf{u}$. If $A(\mathbf{u}) = \mathcal{J}_{\mathbf{u}}\mathbf{F}(\mathbf{u})$ is the $k \times k$ Jacobian matrix of the function $\mathbf{F}$ at the point $\mathbf{u}$, then the system (1.3) can be written in the form

$$\mathbf{u}_t + A(\mathbf{u})\mathbf{u}_x = 0. \tag{1.5}$$

Definition 1.2. *We say that a system of conservation laws is* strictly hyperbolic *if the matrix $A(\mathbf{u})$ has k real, distinct eigenvalues, say*

$$\lambda_1(\mathbf{u}) < \cdots < \lambda_k(\mathbf{u}). \tag{1.6}$$

for every $\mathbf{u}$.

1.1.2 Examples of Conservation laws

As mentioned, systems of conservation laws such as (1.3) may serve as mathematical models for certain real-world phenomena. In particular, such equations appear as precise mathematical descriptions of physical conservation laws. In this section we mention a few examples of conservation laws that arise in applications.

Example 1.3 (Traffic Flow)**.** Let $u(x, t)$ denote the density of cars on a highway at point x at time t. For example, u may be the number of cars per kilometer. Assume that u is continuous and that the speed s of cars depends only on their density, that is, $s = s(u)$. We also assume that the speed s of the cars decreases as the density u increases, that is $\frac{ds}{du} < 0$. Given any two points a and b on the highway, the number of cars between a and b varies according to the law

$$\frac{d}{dt}\int_a^b u(x, t)dx = -\int_a^b [s(u)u]_x dx. \tag{1.7}$$

Since (1.7) holds for all $a,\ b \in \mathbb{R}$ this leads to the conservation law

$$u_t + [s(u)u]_x = 0$$

Here the flux is given by $F(u) = s(u)u$. In practice the flux F is often taken to be

$$F(u) = a_1(\ln(\frac{a_2}{u}))u, \qquad 0 < u < a_2,$$

for suitable constants a_1 and a_2.

Example 1.4 (The p-system)**.** The p - system is a simple model for isentropic (constant entropy) gas dynamics. If v is the specific volume and u the velocity of the gas, then the equations are

$$\begin{aligned} v_t - u_x &= 0 \\ u_t + (p(v))_x &= 0 \end{aligned}$$

The flux p is given as

$$p(v) = kv^{-\lambda}, \quad k \geq 0, \lambda \geq 1$$

where k and λ are constants. In applications λ is chosen such that $\lambda \in [1, 3]$ for most gases; in particular $\lambda = \frac{7}{5}$ for air. In the region $v > 0$, the system is strictly hyperbolic. Indeed

$$A = \mathcal{J}\mathbf{F} = \begin{pmatrix} 0 & -1 \\ p'(v) & 0 \end{pmatrix}$$

has real distinct eigenvalues $\lambda = \pm\sqrt{-p'(v)}$. □

Example 1.5 (Gas dynamics)**.** The Euler equations for the dynamics of a compressible, non-viscous gas is given by

$$\begin{aligned} v_t - u_x &= 0 && \text{(conservation of mass)} \\ u_t + p_x &= 0 && \text{(conservation of momentum)} \\ (\nu + \frac{u^2}{2})_t + (pu)_x &= 0 && \text{(conservation of energy).} \end{aligned}$$

Here $v = \rho^{-1}$, where ρ is the density and v is the specific volume. The velocity in the gas is u, while ν is the internal energy and p the pressure. The system is closed by an additional equation $p = p(\nu, v)$ called the equation of state, which depend on the particular gas under consideration. □

Example 1.6 (Electromagnetism)**.** Let E be the electric intensity, D the electric induction, H the magnetic intensity, B the magnetic induction, I the electric current and q the heat flux in an electromagnetic system. The conservation laws of electromagnetism are

$$\begin{aligned} \partial_t B + \nabla \times E &= 0 && \text{(Faraday's law)} \\ \nabla \cdot B &= 0 \\ \partial_t D - \nabla \times H + I &= 0 && \text{(Ampere's law)} \\ \partial_t E + \nabla \cdot (E \times H + q) &= 0 && \text{(conservation of energy).} \end{aligned}$$

□

1.1.3 Solutions to Scalar Conservation Laws

In this section we are concerned with initial value problems for scalar conservation laws in one spatial dimension

$$u_t + (f(u))_x = 0 \text{ in } \mathbb{R} \times (0, \infty) \tag{1.8}$$

$$u(x, 0) = u_0(x) \quad x \in \mathbb{R}. \tag{1.9}$$

Here $u : \mathbb{R} \times [0, \infty) \longrightarrow \mathbb{R}$ is the unknown conserved quantity, $f \in C^\infty(\mathbb{R})$ is the flux and $u_0 : \mathbb{R} \longrightarrow \mathbb{R}$ is the initial value.

When solving the Cauchy problem (1.8) - (1.9), one is typically confronted with the following difficulties: Even in the case of a C^∞ - smooth initial condition u_0, the initial value problem (1.8) - (1.9) may not have a classical solution on the whole domain of definition of the equation (1.8). Indeed, solutions of (1.8) - (1.9) may develop discontinuities after a finite time.

Classical Solutions

A *classical solution* of the Cauchy problem (1.8) - (1.9) is a continuously differentiable function satisfying equations (1.8) - (1.9). One can obtain the classical solution of equation (1.8) - (1.9) by the method of characteristics. To do this, let the flux function f be given, and assume that equation (1.8) is genuinely nonlinear. That is, $f'(u) \neq$ constant for all u, which further implies

$$f''(u) > 0 \quad \text{for all} \quad u. \tag{1.10}$$

If $u \in C^1(\mathbb{R} \times [0, \infty))$ is a solution of the Cauchy problem, then we define the characteristic curves in $\mathbb{R} \times [0, \infty)$ as the level curves of u. That is, for any $y \in \mathbb{R}$ the characteristic curve through the point $(y, 0)$ consists of the set of points where $u(x, t) = u(y, 0) = u_0(y)$. At every point (x, t) on the characteristic curve through $(y, 0)$, (1.8) and (1.9) imply that

$$\nabla u(x, t) \cdot \langle f'(u_0(y)), 1\rangle = 0.$$

Therefore

$$\langle 1, -f'(u_0(y))\rangle$$

is tangent to the curve at every point. Thus the characteristic through $(y, 0)$ is a straight line with equation

$$x(t) = y + tf'(u_0(y)).$$

Since $u(x, t) = u_0(y)$ for every point (x, t) on the curve, we may express the solution of (1.8) - (1.9) *implicitly* as

$$u = u_0(x - tf'(u)).$$

The Implicit Function Theorem may now be used to solve for u. The classical solution of (1.8) - (1.9) found above is unique, but may fail to exist for all $t > 0$ as the following theorem shows

Theorem 1.7. *[109, Proposition 2.1.1] Assume that* $u_0 \in C^1(\mathbb{R})$, *together with its derivative, is bounded on* $\mathbb{R}$. *Set*

$$T^* = \begin{cases} +\infty & \text{if } f'(u_0) \text{ is an increasing function} \\ -(\inf \frac{d}{dx} f'(u_0))^{-1} & \text{otherwise.} \end{cases} \tag{1.11}$$

Then (1.8) - (1.9) *has a unique solution* $u \in C^1(\mathbb{R} \times (0, T^*))$. *For* $T > T^*$, (1.8) - (1.9) *has no classical solutions on* $\mathbb{R} \times [0, T)$.

We now give an example to illustrate the nonexistence of classical solution for some time $t > 0$.

Example 1.8. Consider the initial value problem for Burger's equation

$$u_t + (\frac{u^2}{2})_x = 0 \text{ in } \mathbb{R} \times (0, \infty) \tag{1.12}$$

$$u(x, 0) = u_0(x) \quad x \in \mathbb{R}. \tag{1.13}$$

Using the method of characteristics discussed above we see that for a C^1-smooth function u_0, a classical solution u is given by the implicit equation

$$u(x, t) = u_0(x - tu(x, t)), \quad t \geq 0, \quad x \in \mathbb{R}. \tag{1.14}$$

By the Implicit Function Theorem, we can obtain $u(y, s)$ from (1.14) for y and s in suitable neighborhoods of x and t respectively, whenever

$$1 + tu_0'(x - tu(x, t)) \neq 0. \tag{1.15}$$

If $u_0'(x) \geq 0$ for all $x \in \mathbb{R}$, then condition (1.15) is clearly satisfied for all (x, t), so that the Cauchy problem (1.12) - (1.13) has a unique solution on $\mathbb{R} \times (0, \infty)$. However, if $u_0'(x_0) < 0$ for some $x_0 \in \mathbb{R}$ then for certain values of $t > 0$ the condition (1.15) may fail. Therefore, violation of condition (1.15) implies that the classical solution u fails to exist for the respective values of t and x.

If we take

$$u_0(x) = \begin{cases} 1 & \text{if } x \leq 0 \\ 1 - x & \text{if } 0 \leq x \leq 1 \\ 0 & \text{if } x \geq 1. \end{cases} \tag{1.16}$$

then the unique classical solution of (1.12) - (1.13) is given by

$$u(x, t) = \begin{cases} 1 & \text{if } x < t \\ \frac{1-x}{1-t} & \text{if } t \leq x \leq 1, \\ 0 & \text{if } x \geq 1. \end{cases} \qquad t < 1$$

Clearly, the classical solution of (1.12) - (1.13), with u_0 as in (1.16), breaks down at $t = 1$. It should be noted that the breakdown of the solution $u(x, t)$ at $t = 1$ for initial data u_0 given in (1.16) is not due to the lack of smoothness of u_0, but to the fact that $u_0'(x) = -1 < 0$ for $x \in [0, 1]$. □

In view of the nonexistence, in general, of global classical solutions, one is forced to consider suitable generalized solutions of (1.8) - (1.9).

Weak solutions and non-uniqueness

One well known and much studied generalized formulation of (1.8) - (1.9) is the weak form of the initial value problem. Let us assume temporarily that u is a classical solution of (1.8) - (1.9). The idea is to multiply equation (1.8) with a smooth function ϕ and integrate by parts. More precisely, let ϕ be a test function, that is,

$$\phi : \mathbb{R} \times [0, \infty) \to \mathbb{R} \tag{1.17}$$

has compact support and is C^∞ - smooth. We denote the set of all such test functions by $C_0^\infty(\mathbb{R} \times [0, \infty))$. Multiply equation (1.8) by ϕ and integrate by parts to get

$$\begin{aligned} 0 &= \int_0^\infty \int_{-\infty}^\infty (u_t + (f(u))_x)\phi dx dt \\ &= -\int_0^\infty \int_{-\infty}^\infty u\phi_t dx dt - \int_0^\infty \int_{-\infty}^\infty f(u)\phi_x dx dt - \int_{-\infty}^\infty u\phi|_{t=0} dx. \end{aligned}$$

In view of the initial condition (1.9), we obtain

$$\int_0^\infty \int_{-\infty}^\infty u\phi_t + f(u)\phi_x dx dt + \int_{-\infty}^\infty u_0\phi|_{t=0} dx = 0. \tag{1.18}$$

In contradistinction with equations (1.8) - (1.9), equation (1.18) does not involve any derivative of u, thus equation (1.18) makes sense not only for smooth functions, but also for bounded and measurable functions u and u_0. We thus arrive at the following definition of a weak solution of (1.8) -(1.9).

Definition 1.9. *We say that* $u \in L_\infty(\mathbb{R} \times (0, \infty))$ *is a* weak solution *of* (1.8)-(1.9) *if the equation* (1.18) *holds for each test function* $\phi \in C_0^\infty(\mathbb{R} \times [0, \infty))$.

If $u \in C^1(\mathbb{R} \times (0, \infty))$ is a weak solution of (1.8) - (1.9) then u satisfies (1.8) - (1.9). That is, a C^1-smooth weak solution is a classical solution of equation (1.8) - (1.9). Thus the concept of weak solution of (1.8) - (1.9) is a generalization of the classical notion of solution.

Remark 1.10. Equation (1.8) can also be written in the form:

$$u_t + a(u)u_x = 0, \qquad \text{with } a(u) = f'(u). \tag{1.19}$$

At the level of classical solutions, equations (1.8) and (1.19) are equivalent. That is, $u \in C^1(\mathbb{R} \times [0, \infty))$ is a solution of (1.8) if and only if u is a solution of (1.19). However, if u has a discontinuity, then the left hand side of equation (1.19) may contain a product of a discontinuous function $a(u)$ with the distributional derivative u_x. Such a product is typically not well defined, see for instance [103]. Working with the equation in the form of (1.8) avoids this difficulty when dealing with weak solutions as defined in Definition 1.9.

We give some examples to illustrate the non-uniqueness of solution to the initial value problem (1.8) - (1.9).

Examples 1.11.

(i) Consider the initial value problem of the Burgers equation (1.12) - (1.13). If we take initial data to be (1.16) then it can be shown that the function

$$u_1(x,t) = \begin{cases} 1 & \text{if} \quad x < \frac{1+t}{2} \\ 0 & \text{if} \quad x > \frac{1+t}{2} \end{cases}$$

is a weak solution to the initial value problem (1.8), (1.16).

(ii) Again consider the initial value problem (1.12) - (1.13) with initial data

$$u_0(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ 1 & \text{if} \quad x > 0. \end{cases} \tag{1.20}$$

The function

$$u_1(x,t) = \begin{cases} 0 & \text{if} \quad x < \frac{t}{2} \\ 1 & \text{if} \quad x > \frac{t}{2} \end{cases}$$

is a weak solution to the initial value problem (1.8), (1.20). However, the function

$$u_2(x,t) = \begin{cases} 0 & \text{if} \quad x < 0 \\ \frac{x}{t} & \text{if} \quad 0 < x < t \\ 1 & \text{if} \quad x > t \end{cases}$$

is a solution to the initial value problem (1.8), (1.20), but cannot be classified as a weak solution according to Definition 1.9.

(iii) A more spectacular example of the loss of uniqueness of solution is the following. Consider the Burgers equation (1.12) with the initial data

$$u_0(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x > 0. \end{cases} \tag{1.21}$$

For every $\alpha \in [1, \infty)$ the function

$$u_\alpha(x,t) = \begin{cases} -1 & \text{if } x < \frac{(1-\alpha)}{2}t \\ -\alpha & \text{if } \frac{(1-\alpha)}{2}t < x < 0 \\ +\alpha & \text{if } 0 < x < \frac{(\alpha-1)}{2}t \\ +1 & \text{if } \frac{(\alpha-1)}{2}t < x \end{cases} \tag{1.22}$$

is a solution of (1.12) - (1.13) with u_0 as in (1.21). It can be shown that only the solution for which $\alpha = 1$ satisfies the definition of a weak solution. □

One difficulty that arises in the study of weak solutions of (1.8) - (1.9) is related to the uniqueness of such solutions. In contradistinction with classical solution of (1.8) - (1.9), weak solutions are not unique as shown in the following,

Example 1.12. Consider the initial value problem of the Burgers equation (1.12) - (1.13) with initial condition

$$u_0(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0. \end{cases} \tag{1.23}$$

For every $\alpha \in (0, 1)$, the function u_α defined as

$$u_\alpha(x,t) = \begin{cases} 0 & \text{if } x < \frac{\alpha t}{2}, \\ \alpha & \text{if } \frac{\alpha t}{2} \leq x < \frac{(1+\alpha)t}{2} \\ 1 & \text{if } x \geq \frac{(1+\alpha)t}{2}, \end{cases}$$

is a weak solution of the initial value problem.

The underlying physical laws that are modeled as mathematical conservation laws are deterministic in nature. That is, the future state of a system that evolves according to (1.8) is completely determined by the initial condition (1.9) of the system. From this point of view, the non uniqueness of weak solutions of (1.8) - (1.9), as demonstrated in Example 1.12, is unacceptable. In particular, in the context of physical systems that may be modeled through (1.8) - (1.9), the non uniqueness of weak solutions of the Cauchy problem may be interpreted as follows: The state of the system at time $t > 0$ is not completely determined by the weak formulation of (1.8) - (1.9) alone. Therefore further additional conditions, motivated by physical consideration, must be imposed on the weak solutions of (1.8) - (1.9) in order to obtain the unique solution that describes the evolution of the underlying physical system.

In this regard, let u be a weak solution of (1.8) - (1.9). Assume that u has continuous first order partial derivatives everywhere in the open set $\Omega \subseteq \mathbb{R}\times[0,\infty)$ except on a smooth curve $\mathcal{C}$ in Ω with equation $x = x(t)$.

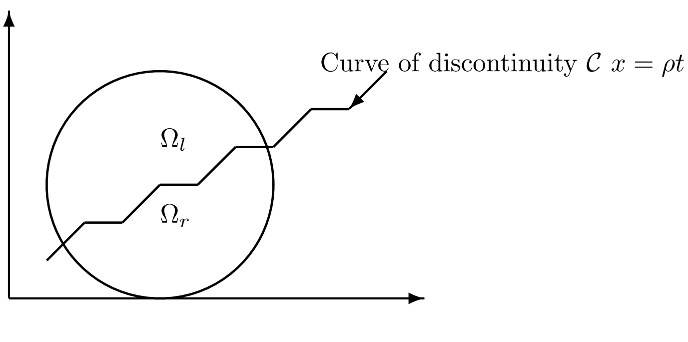


Figure 1.1

That is, limits of u from left and from right of curve $\mathcal{C}$ exist. Hence u has a jump discontinuity across $\mathcal{C}$. Let Ω_l and Ω_r be the parts of Ω on the left and on the right of curve $\mathcal{C}$ respectively, see Figure 1.1.

Furthermore, since u is smooth on either side of the curve $\mathcal{C}$, it is smooth in Ω_l

and Ω_r. Because u is a weak solution of (1.8) - (1.9), we have

$$\int\int_{\Omega} u\phi_t + (f(u))\phi_x dxdt = 0, \tag{1.24}$$

for all $\phi \in C_0^\infty(\Omega)$. Thus, if supp$\phi \subset \Omega_r$, then

$$0 = \int\int_{\Omega} u\phi_t + (f(u))\phi_x dxdt = -\int\int_{\Omega_r} [u_t + (f(u))_x]\phi dxdt. \tag{1.25}$$

which implies

$$u_t + (f(u))_x = 0 \quad \text{in } \Omega_r. \tag{1.26}$$

Similarly,

$$u_t + (f(u))_x = 0 \quad \text{in } \Omega_l. \tag{1.27}$$

From (1.24) we get

$$\begin{aligned} 0 &= \int\int_{\Omega} u\phi_t + f(u)\phi_x dxdt \\ &= \int\int_{\Omega_l} u\phi_t + f(u)\phi_x dxdt + \int\int_{\Omega_r} u\phi_t + f(u)\phi_x dxdt. \end{aligned} \tag{1.28}$$

Now using the fact that u is C^1-smooth in Ω_r and Green's Theorem, we find that

$$\begin{aligned} \int\int_{\Omega_r} (u\phi_t + f(u)\phi_x)dxdt &= \int\int_{\Omega_r} [(u\phi)_t + (f(u)\phi)_x]dxdt \\ &= \int_{\partial\Omega_r} (-u\phi)dx + (f(u)\phi)dt \\ &= \int_{\partial\Omega} (-u\phi)dx + (f(u)\phi)dt + \int_{\mathcal{C}} (-u\phi)dx + (f(u)\phi)dt \end{aligned}$$

Since $\phi = 0$ on $\partial\Omega$, we have

$$\int\int_{\Omega_r} (u\phi_t + f(u)\phi_x)dxdt = \int_{\mathcal{C}} (-u_r\phi)dx + (f(u_r)\phi)dt. \tag{1.29}$$

where u_r the right limit of u on the curve $\mathcal{C}$. Similarly,

$$\int\int_{\Omega_l} (u\phi_t + f(u)\phi_x)dxdt = -\int_{\mathcal{C}} (-u_l\phi)dx + (f(u_l)\phi)dt \tag{1.30}$$

where u_l is the left limit of u on the curve $\mathcal{C}$. Substituting equations (1.29) and (1.30) into equation (1.28) we have

$$\begin{aligned} 0 &= \int\limits_{\mathcal{C}} (-u_l + u_r)\phi dx + (f(u_l) - f(u_r))\phi dt \\ &= \int\limits_{\mathcal{C}} \phi[-(u_l - u_r)dx + (f(u_l) - f(u_r))dt] \end{aligned} \tag{1.31}$$

which further implies

$$[-(u_l - u_r)dx + (f(u_l) - f(u_r))dt] = 0.$$

This implies

$$(u_l - u_r)\frac{dx}{dt} = (f(u_l) - f(u_r))$$

in Ω along the curve $\mathcal{C}$, which may be expressed as

$$(f(u_l) - f(u_r)) = \dot{x}(u_l - u_r). \tag{1.32}$$

We write this as

$$\rho[[u]] = [[f(u)]], \tag{1.33}$$

where $[[u]] = u_l - u_r$ is the jump in u across the curve $\mathcal{C}$, $[[f(u)]] = f(u_l) - f(u_r)$ is the jump in $f(u)$ and $\rho = \frac{dx}{dt}$ is the speed of curve $\mathcal{C}$. Relation (1.33) is known as the *jump condition*. Equation (1.33) is popularly known as Rankine Hugoniot condition.

Remark 1.13. We remark here that if u is a piecewise smooth solution to the initial value problem (1.8) - (1.9), then u satisfies the jump condition if and only if it is a weak solution. In other words, every weak solution to the initial value problem (1.8) - (1.9) satisfies the jump condition. Conversely, every piecewise smooth solution to the initial value problem that satisfies the jump condition is a weak solution to the initial value problem (1.8) - (1.9). This follows from the above derivation of the jump condition. However, if u is a weak solution which is bounded and measurable, then u_l and u_r in condition (1.32) have to be interpreted as

$$\begin{aligned} u_l(x,t) &= \liminf_{y \longrightarrow x,} u(y,t) \\ u_r(x,t) &= \limsup_{y \longrightarrow x,} u(y,t). \end{aligned}$$

Examples 1.14.

(i) Applying the jump condition to the Burgers' equation (1.12) where $f(u) = \frac{1}{2}u^2$, we find that the speed of propagation of the discontinuities is $\frac{dx}{dt} = \rho = \frac{1}{2}(u_l + u_r)$.

(ii) Again, applying the jump condition to the solutions u_α of Example 1.12, we see that by the jump condition, $\rho = \frac{\alpha}{2}$ and $\rho = \frac{(1+\alpha)}{2}$ along the lines of discontinuity $x = \frac{\alpha t}{2}$ and $x = \frac{(1+\alpha)t}{2}$ respectively for each $\alpha \in (0,1)$. Thus, the jump condition alone is not sufficient to determine the unique, physically relevant solution of the Cauchy problem (1.8) - (1.9). □

Admissibility conditions and the Entropy Condition

From Example 1.12, it is clear that the set of weak solutions of a given initial value problem (1.8)-(1.9) may include various solutions which are not physically relevant. In order to single out a *unique* solution that is physically and/or mathematically relevant, suitable additional requirements, which we shall call *admissibility conditions*, are imposed on such solutions, see for instance [52, 79]. These admissibility conditions, such as *entropy conditions*, are typically motivated by some physical considerations. In the literature, various admissibility conditions have been introduced. In this section, we recall some of these conditions.The main results in which these admissibility conditions are employed to single out the unique, physically relevant solution to the Cauchy problem (1.8) -(1.9) are also discussed.

Admissibility condition 1 (The Oleinik inequality)

Oleinik [95] introduced the Lipschitz condition, with respect to x, for fixed t for a genuinely nonlinear single conservation law (1.8) given by

$$\frac{u(x+a,t) - u(x,t)}{a} \leq \frac{E}{t}. \qquad a > 0, \qquad t > 0. \tag{1.34}$$

Here $E = \frac{1}{\inf f''}$ is independent of x, t, and a. Using the Lax-Friedreich finite difference scheme, Oleinik showed that if f is convex, which implies that $f'' > 0$, then there exists precisely one weak solution of the Cauchy problem (1.8) - (1.9) satisfying (1.34). Note that the weak solutions of (1.8) - (1.9) that satisfies (1.34) will, for any fixed $t > 0$ have x-difference quotient bounded from above. As t tends to 0, the upper bound for the x-difference quotients may tend to plus infinity.

The Oleinik inequality (1.34) was motivated by the fact that if $u_0' \geq 0$, a classical solution u of (1.8) - (1.9) exists with

$$u_x = \frac{u_0'}{1 + tf''(u_0)u_0'} \qquad .$$

So that

$$u_x \leq \frac{1}{tf''(u_0)} \leq \frac{K}{t}, \quad t > 0 \text{ and } K > 0 \text{ a constant}$$

which is an limiting version of the Oleinik inequality (1.34). It is therefore reasonable for a solution of the Cauchy problem (1.8) - (1.9) to satisfy the inequality (1.34). The basic idea of the finite difference scheme in PDE is to replace derivatives with appropriate finite differences. The main result, concerning solutions satisfying (1.34), which is also found in [111], is given below.

Theorem 1.15. *[111, Theorem 16. 1] Let $u_0 \in L_\infty(\mathbb{R})$, and let $f \in C^2(\mathbb{R})$ with $f''(u) > 0$ on $\{u : |u| \leq \|u_0\|_\infty\}$. Let $M = \|u_0\|_{L_\infty}$, $\mu = \inf\{f''(u) : |u| \leq \|u_0\|_\infty\}$ and $A = \sup\{|f'(u)| : |u| \leq \|u_0\|_\infty\}$. Then there exists exactly one weak solution u of (1.8)-(1.9) satisfying the following:*

(a) There exists a constant $E > 0$, depending only on M, μ and A, such that for every $a > 0$, $t > 0$, and $x \in \mathbb{R}$, the inequality

$$\frac{u(x+a,t) - u(x,t)}{a} < \frac{E}{t}. \tag{1.35}$$

holds.

(b) $|u(x,t)| \leq M$, $\forall\ (x,t) \in \mathbb{R} \times [0,\infty)$.

(c) u is stable and depends continuously on u_0 in the following sense: If $u_0, v_0 \in L_\infty(\mathbb{R}) \cap L_1(\mathbb{R})$ with $\|v_0\|_\infty \leq \|u_0\|_\infty$, and v is the solution of (1.8) with initial data v_0 satisfying (1.35), then for every $x_1, x_2 \in \mathbb{R}$, with $x_1 < x_2$, and every $t > 0$,

$$\int_{x_1}^{x_2} |u(x,t) - v(x,t)|dx \leq \int_{x_1 - At}^{x_2 - At} |u_0(x) - v_0(x)|dx. \tag{1.36}$$

Remark 1.16. (i) An immediate consequence of (1.35) is that for any $t > 0$, the solution $u(\cdot,t)$ is of locally bounded total variation, that is $u \in BV_{loc}$, which means the total variation of u is bounded in every compact subset of $\mathbb{R} \times [0,\infty)$. To see this, let us define a function

$$v(x,t) = u(x,t) - \frac{E}{t}x.$$

Then if $a > 0$ (1.35) implies

$$v(x+a,t) - v(x,t) = u(x+a,t) - u(x,t) - \frac{E}{t}a < 0.$$

That is, v is a decreasing function with respect to x and thus has locally bounded total variation with respect to x. Hence u is of locally bounded total variation since a linear function is also of locally bounded total variation. Thus even though u_0 is only L_∞, the solution $u(\cdot,t)$ is fairly regular. In fact, we can conclude that it has at most a countable number of jump discontinuities, and it is differentiable almost everywhere.

(ii) Theorem 1.15 is limited to single conservation laws in one spatial dimension. An analogue of the Oleinik inequality (1.35) has not been found for systems of conservation laws.

(iii) The Oleinik inequality (1.35) implies that $u_l > u_r$ as we move across a curve of discontinuity. To see this, note that the function $v(x,t) = u(x,t) - \frac{E}{t}x$ is bounded in a domain $(x_1, x_2) \times [0, \infty)$ containing the curve of discontinuity. Then v has left and right limits with respect to x at each point since it is decreasing with respect to x as noted in (i). Consequently $u(x,t)$ has left and right limits at each point. For any point c on the line of discontinuity we have

$$u_r - u_l = \lim_{x \longrightarrow c^+} u(x,t) - \lim_{x \longrightarrow c^+} \frac{E}{t}x - \lim_{x \longrightarrow c^-} u(x,t) + \lim_{x \longrightarrow c^-} \frac{E}{t}x$$

$$= \lim_{x \longrightarrow c^+} v(x,t) - \lim_{x \longrightarrow c^-} v(x,t) < 0,$$

which implies $u_l > u_r$ as we move across a curve of discontinuity.

Admissibility condition 2 (The Lax inequality)

The inequality

$$f'(u_l) > \rho > f'(u_r) \qquad \text{for all } t > 0. \tag{1.37}$$

was introduced by Lax [78]. The inequality (1.37) implies that the characteristics starting on either sides of the curve of discontinuity should intersect each other on the curve, see figure 1.2. At this point of intersection, u has two values which is impossible, so that there is a jump discontinuity at that point. Indeed, if $u_0' < 0$, there are two points $y_1, y_2 \in \mathbb{R}$ such that $y_1 < y_2$ and $u_1 = u_0(y_1) > u_0(y_2) = u_2$. If (1.37) holds then $f'(u_0(y_1)) > f'(u_0(y_2))$ so that the characteristics drawn from points $(y_1, 0)$ and $(y_2, 0)$ intersect at the point when $t = \frac{y_2 - y_1}{f'(u_0(y_1)) - f'(u_0(y_2))}$ with u having values $u(y_1)$ and $u(y_2)$ at that point.

The Lax inequality can be obtained from the Jump condition. To see this, let f be a convex function. Then $f'' > 0$ which implies that f' is increasing. Thus if $u_l > u_r$, then

$$f'(u_l) > f'(u_r).$$

By the Mean Value Theorem there exists $\zeta \in [u_r, u_l]$ such that

$$f'(\zeta) = \frac{f(u_l) - f(u_r)}{u_l - u_r} = \rho.$$

Since f' is increasing we have that

$$f'(u_l) > f'(\zeta) > f'(u_r),$$

which leads to the Lax inequality (1.37). However, not all weak solutions of equations (1.8) - (1.9) satisfying the jump condition (1.33) will also satisfy the

Lax condition (1.37). For example, the lines of discontinuity in the solutions obtained in Example 1.12 that are shown to satisfy the jump condition do not satisfy the Lax inequality (1.37). If all the discontinuities of a weak solution satisfy condition (1.37), then no characteristics drawn backward will intersects the curve of discontinuity, see figure 1.2. A discontinuity satisfying both the jump condition (1.33) and the Lax inequality (1.37) is called a *shock*. A weak solution having only shocks as discontinuity is called shock wave solution. Lax showed that there is exactly one shock wave solution u of the equations (1.8) if the initial condition is such that the left initial state is greater than the right initial state, that is $u_l > u_r$. If we consider the Burger's equation and take the initial condition to be

$$u_0(x) = \begin{cases} 1 & \text{if } x < 0, \\ 0 & \text{if } x > 0, \end{cases}$$

then the shock wave solution is expressed as

$$u(x,t) = \begin{cases} 1 & \text{if } x < \rho t, \\ 0 & \text{if } x > \rho t. \end{cases}$$

Clearly, the jump condition $\rho = \frac{(u_1+u_2)}{2}$ and the Lax condition $f'(u_1) > \rho > f'(u_2)$ are both satisfied if and only if $\rho = \frac{1}{2}$.

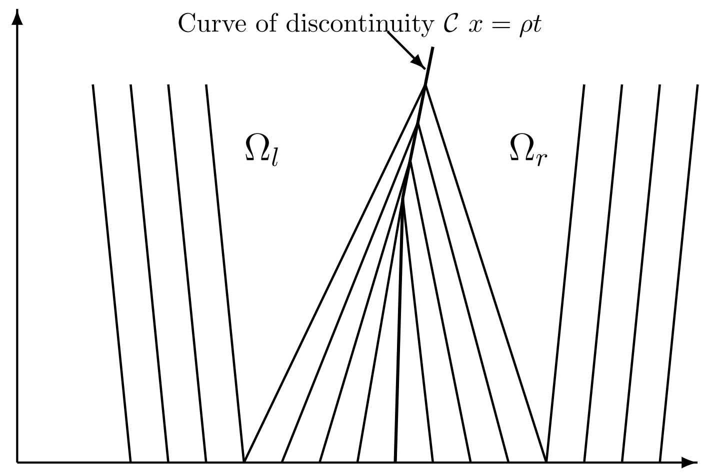


Figure 1.2

A more general example is the Riemann's problem illustrated below.

Example 1.17 (The Riemann's Problem)**.** The Riemann's problem is the Cauchy problem

$$u_t + (f(u))_x = 0 \text{ in } \mathbb{R} \times (0, \infty)$$
$$u(x,0) = u_0(x) = \begin{cases} u_l & \text{if } x < 0 \\ u_r & \text{if } x > 0. \end{cases}$$

Here, $u_l,\ u_r \in \mathbb{R}$ are the left and right initial states of u. Note that $u_l \neq u_r$. If

$u_l > u_r$ the shock wave solution to the Riemann's problem is

$$u(x,t) := \begin{cases} u_l & \text{if } x < \rho t, \\ u_r & \text{if } x > \rho t. \end{cases}$$

□

The main result by Lax is summarized in the following

Theorem 1.18. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a C^2- smooth and convex function. If $u_0 \in L_1(\mathbb{R})$ then there exists a weak solution u of the Cauchy problem* (1.8) *-* (1.9) *which satisfies* (1.37)*. The solution u is defined as*

$$u(x,t) = b\left(\frac{x - y_0}{t}\right) \qquad \text{for each} \quad t > 0 \quad \text{and } a.e.\ x \in \mathbb{R} \tag{1.38}$$

where $y_0 = y_0(x,t)$ is the the value of y that minimizes

$$K(x,y,t) = U_0(y) + tG(\frac{x-y}{t}).$$

Here the function $b(s)$ is defined as $b(s) = (f'(s))^{-1}$ and $G(s)$ is defined as the solution of

$$\frac{dG(s)}{ds} = b(s), \quad G(c) = 0, \text{ with } c = f'(0),$$

and

$$U_0(y) = \int_{-\infty}^{y} u_0(s)ds.$$

The discontinuity of the solution u constructed in Theorem 1.18 are shocks, which means u satisfies the Lax inequality (1.37). In addition, u has the semi group property which means that if $u(x,t_1)$ is taken as a new initial value, the solution $u(x,t_2)$ at $t_2 > t_1$ corresponding to the initial condition $u(x,t_1)$ equals $u(x,t_1+t_2)$.

Remark 1.19. (i) For fixed $t > 0$, the function $y_0(x,t)$ is an increasing function of x, see [79, Lemma 3.3].
(ii)The shock wave solution constructed in Theorem 1.18 satisfies the Oleinik inequality (1.34). Indeed, since b and $y(x,t)$ are increasing functions, then for

$x_2 > x_1$ we have that

$$u(x_1,t) = b\left(\frac{x_1 - y_0(x_1,t)}{t}\right)$$

$$\geq b\left(\frac{x_1 - y_0(x_2,t)}{t}\right)$$

$$\geq b\left(\frac{x_2 - y_0(x_2,t)}{t}\right) - \alpha\frac{x_2 - x_1}{t}$$

$$= u(x_2,t) - \alpha\frac{x_2 - x_1}{t};$$

which implies

$$\frac{u(x_2,t) - u(x_1,t)}{x_2 - x_1} \leq \frac{\alpha}{t},$$

here $\alpha > 0$ is a Lipschitz constant for the function b.

A generalized form of the Lax condition (1.37) was given by Oleinik [96]. For $0 \leq \alpha \leq 1$,

$$f(\alpha u_r + (1-\alpha)u_l) \leq \alpha f(u_r) + (1-\alpha)f(u_l) \qquad \text{if } u_l > u_r, \tag{1.39}$$

$$f(\alpha u_r + (1-\alpha)u_l) \geq \alpha f(u_r) + (1-\alpha)f(u_l) \qquad \text{if } u_l < u_r. \tag{1.40}$$

The inequality (1.39) implies that f is convex. Geometrically this means that the graph of f over the interval $[u_r, u_l]$ lies below the chord of f drawn from the point $(u_l, f(u_l))$ to the point$(u_r, f(u_r))$. On the other hand, the inequality (1.40) implies that f is concave, which means that the graph of f over the interval $[u_l, u_r]$ lies above the chord of f drawn from the point $(u_l, f(u_l))$ to the point$(u_r, f(u_r))$.

We now discuss the relationship between the Lax inequality (1.37) and the generalized Oleinik inequality (1.39). To start with, the convexity of f implies that the inequality (1.39) is equivalent to

$$\frac{f(u^*) - f(u_l)}{u^* - u_l} \geq \frac{f(u_r) - f(u^*)}{u_r - u^*}. \tag{1.41}$$

for every $u^* = \alpha u_r + (1-\alpha)u_l$, with $0 < \alpha < 1$. Combining the inequality (1.41) with the Mean Value Theorem, we have that there exists $\zeta \in [u_r, u_l]$ such that

$$f'(\zeta) = \frac{f(u_l) - f(u_r)}{u_l - u_r} = \rho$$

and

$$\frac{f(u^*) - f(u_l)}{u^* - u_l} \geq f'(\zeta) \geq \frac{f(u_r) - f(u^*)}{u_r - u^*}. \tag{1.42}$$

Now Taking limits as $u^* \longrightarrow u_l$ and $u^* \longrightarrow u_r$ in (1.42), we have

$$f'(u_l) \geq \rho \geq f'(u_r),$$

which is the Lax inequality. Thus, the generalized inequality (1.39) implies the Lax inequality. On the other hand, if the flux function f, is a convex function, which imply $f'' > 0$, then the Lax inequality (1.37) would implies the inequality (1.39).

Essentially, all the conditions considered so far require that the flux function f be convex or concave. Krushkov [74] introduced a more general admissibility condition for a flux function f that is not necessarily convex or concave. One advantage of the Kruzkov condition is that it is also applicable to systems of conservation laws in more than one dimension, whereas the Oleinik condition is limited to scalar conservation laws in one dimension. Although the Lax inequality is applicable to systems of conservation laws, it still requires the convexity of the flux function f, moreover the Lax inequality is limited to systems in one spatial dimension. Kruzkov's admissibility condition is given below.

Admissibility condition 3 (The Entropy condition)

The admissibility condition discussed in this section was first introduced by Kruzhkov [74]. It is formulated in terms of entropy/entropy flux pairs. A pair (Φ, Ψ) of real C^∞ - smooth functions is called an entropy/entropy flux pair for the conservation law (1.8) if

$$\Psi'(z) = \Phi'(z) f'(z), \quad \text{for all } z \in \mathbb{R}. \tag{1.43}$$

The function Ψ is called an entropy flux function for the entropy function Φ. For every convex function Φ we can find a corresponding entropy flux function Ψ given by

$$\Psi(z) = \int_{z_0}^{z} \Phi'(z) f'(z), \quad z \in \mathbb{R}. \tag{1.44}$$

For each entropy/entropy flux pair (Φ, Ψ) one may formulate an admissibility condition, known as *entropy condition* given by

$$\begin{gathered} \int\int_{\Omega} \Phi(u)\phi_t + \Psi(u)\phi_x dxdt \geq 0, \ \Omega \subseteq \mathbb{R} \times [0, \infty) \\ \forall \phi \in C_0^\infty(\mathbb{R} \times (0, \infty)), \ \phi \geq 0, \ \text{supp } \phi \subset \Omega. \end{gathered} \tag{1.45}$$

Definition 1.20 (Entropy solution). *The function* $u \in C([0, \infty), L^1(\mathbb{R})) \cap L_\infty(\mathbb{R} \times (0, \infty))$ *is called an entropy solution of the Cauchy problem* (1.8) - (1.9) *if it satisfies the entropy condition* (1.45) *for each entropy/entropy-flux pair* (Φ, Ψ), *and* $u(., t) \to u_0$ *in* $L_1(\mathbb{R})$ *as* $t \to 0$.

The main result on the existence and uniqueness of entropy solution is given below.

Theorem 1.21. *[109, Theorem 2.3.5] For every function* $u_0 \in L_\infty(\mathbb{R})$, *there exists one and only one entropy solution* $u \in L_\infty(\mathbb{R} \times [0,T)) \bigcap C([0,T); L^1_{loc}(\mathbb{R}))$ *of* (1.8) -(1.9). *The entropy solution* u *satisfies the maximum principle*

$$\|u\|_{L_\infty(\mathbb{R}\times[0,T))} = \|u_0\|_{L_\infty(\mathbb{R})}.$$

Remark 1.22. Theorem 1.21 is valid in several spatial dimension, [75], [109]. By Theorem 1.21, one can construct a semi group operator $S(t)$, associated with the entropy solution $u(x,t)$ with respect to the initial data u_0 and time $t > 0$ written as,

$$S_t u_0(x) = u(x,t).$$

The semi group $S : \mathcal{D} \times [0,\infty) \longrightarrow \mathcal{D}$ with $\mathcal{D} \subset L_1(\mathbb{R})$ a closed domain containing all functions with bounded total variation, has the following properties [30, 102]:

(i) $S_0 u = u, \quad S_{t+s} u = S_t S_s u,$

(ii) S_t is uniformly Lipschitz continuous w.r.t time and initial data: There exists $L,\ L' > 0$ such that

$$\|S_t u_0 - S_s v_0\| \leq L\|u_0 - v_0\| + L'|t-s|.$$

In his proof, Kruzhkov considered a family of entropy-entropy flux pairs $(\Phi_k, \Psi_k)_{k\in\mathbb{N}}$, but Panov [98] has shown that it is not necessary to consider the whole family of entropy/entropy flux pair. A single pair of entropy/entropy flux pair (Φ, Ψ) is sufficient to characterize entropy solutions of (1.8) - (1.9).

We now show that the entropy solution u of the Cauchy problem (1.8) - (1.9) is a weak solution, see [59]. In this regard, assume u is C^1 - smooth in the left subregion Ω_l and right subregion Ω_r of some region $\Omega \subseteq (\mathbb{R} \times [0,\infty))$ divided by a smooth curve $\mathcal{C}$. Let u also satisfy the entropy condition. If we take $\Phi(u) = \pm u$ and $\Psi(u) = F(u)$ in (1.45) we see that

$$u_t + f(u)_x = 0 \qquad \text{in} \ \ \Omega_l, \Omega_r.$$

Integrating (1.45) by parts we get

$$\int\!\!\int_{\Omega_l} \Phi(u)\phi_t + \Psi(u)\phi_x dxdt + \int\!\!\int_{\Omega_r} \Phi(u)\phi_t + \Psi(u)\phi_x dxdt \geq 0$$

from where we deduce

$$\int_{\mathcal{C}} \phi[(\Phi(u_l) - \Phi(u_r))n_2 + (\Psi(u_l) - \Psi(u_r))n_1]dS \geq 0 \tag{1.46}$$

where $\mathbf{n} = (n_1, n_2)$ is the unit normal to $\mathcal{C}$ pointing from Ω_l to Ω_r. Suppose that the curve $\mathcal{C}$ is given by $x = s(t)$ for some smooth function $s : [0,\infty) \longrightarrow \mathbb{R}$. Then $\mathbf{n} = (n_1, n_2) = \frac{(1,-\dot{s})}{\sqrt{1+\dot{s}^2}}$. Consequently (1.46) becomes

$$\dot{s}(\Phi(u_r) - \Phi(u_l)) \geq \Psi(u_r) - \Psi(u_l) \quad \text{along} \ \ \mathcal{C}, \tag{1.47}$$

which leads to the jump condition

$$\dot{s}[[u]] = [[f(u)]]. \tag{1.48}$$

Thus the entropy condition (1.45) satisfies the jump condition and thus a weak solution.

Suppose $u_l > u_r$. Fix u^* such that $u_l > u^* > u_r$ and define the entropy/entropy flux pair as

$$\Phi(z) = \begin{cases} (z - u^*) & \text{if} z - u^* > 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\Psi(z) = \int_{u_l}^{z} \text{sgn}_+(v - u) f'(v) dv.$$

Then

$$\Phi(u_r) - \Phi(u_l) = u^* - u_l$$

and

$$\Psi(u_r) - \Psi(u_l) = f(u^*) - f(u_l).$$

Consequently (1.47) implies

$$\dot{s}(u^* - u_l) \geq f(u^*) - f(u_l)$$

which, when combined with (1.48), gives

$$\dot{s} \leq \frac{f(u^*) - f(u_l)}{u^* - u_l}. \tag{1.49}$$

Similarly, if $u_r > u_l$ and $u_r > u^* > u_l$ then

$$\dot{s} \geq \frac{f(u_r) - f(u^*)}{u_r - u^*}. \tag{1.50}$$

Conditions (1.49) and (1.50) gives condition (1.41). Thus the entropy condition (1.45) implies the Lax condition.

Remark 1.23. (i) Note that any entropy solution of (1.8) - (1.9) is also a weak solution of (1.8). This follows if we set $\Phi(z) = z, z \in \mathbb{R}$, in which case $\Psi = f$.
(ii) If $u \in C^1(\Omega)$ is a classical solution of the initial value problem, then

$$\Phi'(u)(u_t + (f(u))_x) = 0$$

for any convex function Φ. This further implies

$$0 = \Phi'(u)u_t + \Phi'(u)f'(u)u_x = \Phi'(u)u_t + \Psi'(u)u_x,$$

with Ψ any entropy flux associated with Φ. This verifies that a classical solution is also an entropy solution.

The theory of hyperbolic conservation laws has developed in a number of directions. One major approach consists of considering weak solutions in suitable spaces of functions with bounded variation (BV functions). The problem, and a very difficult one, is to prove that various approximating schemes such as the vanishing viscosity methods, the Glimm scheme, wave front tracking etc, converge to the entropy solution, see [3, 16, 26, 27, 64] for details. The BV approach consists of proving convergence of these schemes under assumption on the initial condition u_0 related to its total variation. Typically, one assumes that the total variation satisfies a smallness condition, see [22]. Another approach is to construct weak solutions through weak convergence and compensated compactness arguments, see for instance [87, 114, 130] and Section 1.1.5. In the next section we discuss the vanishing viscosity method for conservation laws.

1.1.4 Solutions of Scalar Conservation Laws via Vanishing Viscosity

The role of the entropy condition in conservation laws is to distinguish between the physically relevant weak solution and other, possibly irrelevant weak solutions. One method for obtaining and analyzing entropy solutions to hyperbolic conservation laws is to modify the given conservation law by adding a small perturbation term to the right-hand side of the equation, for example, εu_{xx}, with $\varepsilon << 1$, to obtain from (1.8) a regularized equation

$$u_t^\varepsilon + (f(u^\varepsilon))_x - \varepsilon u_{xx} = 0 \tag{1.51}$$

The motivation that is often given for the study of the Cauchy problem (1.8) - (1.9) through the regularized problem

$$u_t + (f(u))_x = \varepsilon u_{xx} \text{ in } \mathbb{R} \times (0, \infty), \quad \varepsilon > 0 \tag{1.52}$$

$$u(x, 0) = u_0(x) \quad x \in \mathbb{R}. \tag{1.53}$$

is that a physically and mathematically correct solution of (1.8) - (1.9) should arise as the limit of the solution u^ε of (1.52) - (1.53), as the parameter ε tends to zero. This method is generally known as the vanishing viscosity method [21, 23, 52, 111, 109, 113, 128, 129].

In this regard, we may recall that the model for thermoelastic materials under adiabatic conditions is a first order system of hyperbolic PDEs, while that for thermoviscoelastic, heat-conducting materials is a second order PDE, containing a diffusive term, see [52]. Every material has a degree of viscous response and conducts heat. Classifying a material as an elastic nonconductor of heat simply means that viscosity and heat conductivity are negligible, but not totally absent. The consequence of this is that the theory of adiabatic thermoelasticity may be physically meaningful only as a limiting case of thermoviscoelasticity, with viscosity and heat conductivity tending to zero see [52, 111]. In the same way hyperbolic conservation laws are considered as a limiting case of the parabolic equation (1.52).

We note here that solutions of nonlinear PDEs are in general highly unstable with respect to small perturbations of the equation. Thus in spite of the physical intuition underlying such viscosity methods, the rigorous mathematical analysis of the limiting behavior of solutions of equations like (1.52) - (1.53) as ε tends to 0 is highly non trivial.

It is well known that for any $\varepsilon > 0$, and for bounded and measurable initial data, there exists a unique classical solution u^ε of the parabolic equation (1.52) - (1.53), see [56, 95, 130]. This unique solution u^ε of equation (1.52) - (1.53) is called a viscosity solution of (1.52) - (1.53). The following general theorem guarantees the existence of a sequence of solutions to the parabolic problem (1.52) - (1.53).

Theorem 1.24. *[130, Theorem 1.0.2] (i) For any fixed* $\varepsilon > 0$, *the Cauchy problem* (1.52) - (1.53) *with* $u_0 \in L_\infty$ *has a local classical solution* $u^\varepsilon \in C^\infty(\mathbb{R} \times (0, \tau))$ *for a small time* τ, *which depends only on the* L_∞ *norm of the initial data* u_0.
(ii) If the solution u^ε *has an a priori* L_∞ *bound* $\|u^\varepsilon(\cdot, t)\|_{L_\infty} \leq M(\varepsilon, T)$ *for* $t \in [0, T]$, *then the solution exists on* $\mathbb{R} \times [0, T]$.
(iii) The solution u^ε *satisfies:*

$$\lim_{|x| \longrightarrow \infty} u^\varepsilon(x, t) = 0, \quad if \lim_{|x| \longrightarrow \infty} u_0(x) = 0.$$

Following the standard theory for parabolic equations, the global existence of a solution can easily be obtained by applying the contraction mapping principle to an integral representation of the solution. Whenever there is a local solution with a priori L_∞ bound, the domain of existence of solution can be extended, step by step, to any further time T since the step-time depends only on the L_∞ norm of the initial condition. Details of the proof can be found in [77, 111].

The two fundamental questions concerning the solution u^ε of (1.52) - (1.53) are the following.

(i) In what sense does the sequence of functions u^ε converge to a limit function u as ε tends to 0?

(ii) Given that u^ε converges to some u in a specified way, in what sense can we interpret u as a solution of the Cauchy problem (1.8) - (1.9)? In particular, if u^ε is the unique classical solution of (1.52)-(1.53) and u^ε converges to some function u as ε tends to 0, is u an entropy solution of the Cauchy problem (1.8) - (1.9)?

A partial answer to the above questions is given in the following Theorem, see [52, 58, 75].

Theorem 1.25. *[52, Theorem 6.3.1] Suppose* u^ε *is the solution of* (1.52), (1.53), *and assume that for some sequence* $\{\varepsilon_n\}$ *with* $\varepsilon_n \to 0$ *as* $n \to \infty$, *we have that* u^{ε_n} *is norm bounded in* L_∞ *and* $u_{\varepsilon_n}(x, t) \to u(x, t)$ as $\varepsilon_n \longrightarrow 0$ *in* $\mathbb{R}$ *for almost all*

$(x,t) \in \mathbb{R} \times [0,\infty)$. *In other words,* $u^{\varepsilon_n} \to u$ *boundedly a.e on* $\mathbb{R} \times [0,\infty)$. *Then* u *is an entropy solution of* (1.8)-(1.9) *on* $\mathbb{R} \times [0,\infty)$.

Remark 1.26. (i) Since the weak solutions of (1.8)-(1.9) are in L_∞, and are typically not continuous, it may happen that as the smooth function u^ε approaches u the functions u^ε_x and u^ε_{xx} become unbounded, in a neighborhood of a point of discontinuity of u. Thus establishing the convergence $u^\varepsilon \to u$ is a highly nontrivial issue.
(ii)If u^ε converges to u in the weak sense only; the sequence $F(u^\varepsilon)$ will converge in the weak sense but not to $F(u)$. In this regard, we have the following

Theorem 1.27 ([78], [80])**.** *If the sequence of functions* u_n *converges in the weak sense to a limit* u*, then* $f(u_n)$ *converges in the weak sense to* $f(u)$ *if and only if* $u_n \to u$ *strongly in* L_1.

The theory of scalar conservation laws via vanishing viscosity was initiated by E. Hopf in his 1950 paper [69]. In that paper, Hopf considered the viscous Burgers equation

$$u^\varepsilon_t + \left(\frac{(u^\varepsilon)^2}{2}\right)_x = \varepsilon u^\varepsilon_{xx} \quad \text{in } \mathbb{R} \times (0,\infty) \tag{1.54}$$

$$u^\varepsilon(x,0) = u^0(x) \qquad \mathbb{R} \times \{t=0\}, \tag{1.55}$$

and showed that the solution to the Cauchy problem (1.54) - (1.55) can be expressed via an explicit formula

$$u^\varepsilon(x,t) = \frac{\int_{-\infty}^{\infty} \frac{x-y}{t} e^{-\frac{1}{2\varepsilon}K(x,y,t)}dy}{\int_{-\infty}^{\infty} e^{-\frac{1}{2\varepsilon}K(x,y,t)}dy} \tag{1.56}$$

where

$$K(x,y,t) = \frac{(x-y)^2}{2t} + \int_0^y u_0(s)ds. \tag{1.57}$$

The result by Hopf is stated below.

Theorem 1.28. *[69, Hopf E.] Suppose* $u_0 \in L^1_{loc}(\mathbb{R})$ *is such that*

$$\int_0^x u_0(\xi)d\xi = o(x^2) \quad \text{for} \quad |x| \text{ large.} \tag{1.58}$$

Then there exists a classical solution of equation (1.54)-(1.55) *given by* (1.56). *The solution* u^ε *satisfies the initial condition: For all* $a \in \mathbb{R}$,

$$\int_0^x u^\varepsilon(\xi,t)d\xi \longrightarrow \int_0^a u_0(\xi)d\xi \quad \text{as} \quad x \longrightarrow a, \quad t \longrightarrow 0. \tag{1.59}$$

If, in addition, $u_0(x)$ *is continuous at* $x = a$ *then*

$$u^\varepsilon(x,t) \longrightarrow u_0(a) \quad \text{as} \quad x \longrightarrow a, \quad t \longrightarrow 0. \tag{1.60}$$

A solution of (1.54) - (1.55) *which is* C^2*-smooth in the interval* $0 < t < T$ *and satisfies* (1.59) *for each value of* a *necessarily coincides with* (1.56) *in the interval.*

In his proof Hopf's technique was to first transform the equation (1.54) - (1.55) into the heat equation

$$z_t - \varepsilon z_{xx} = 0, \quad (1.61)$$
$$z(x,0) = z_0(x) = e^{-\frac{1}{2\varepsilon}\int_0^x (u^0(\nu))d\nu} \quad (1.62)$$

using the transformation equation

$$z = e^{-(\frac{1}{2\varepsilon}\int u^\varepsilon dx)}$$

with inverse

$$u^\varepsilon = -2\varepsilon(\log z)_x = -2\varepsilon(\frac{z_x}{z}). \quad (1.63)$$

The solution of (1.61) - (1.62) is then obtained as

$$z(x,t) = \frac{1}{\sqrt{4\pi\varepsilon t}}\int_{-\infty}^{\infty} u^\varepsilon(y,0)e^{-\frac{(x-y)^2}{4\varepsilon t}}. \quad (1.64)$$

Substituting the expression (1.64) for $z(x,t)$ in (1.63) one obtains the formula (1.56) as the unique solution of equations (1.54) - (1.55). The condition (1.58) on the initial value is necessary to guarantee the convergence of the definite integral (1.64), thus also those in the expression (1.56) for the solution of (1.54) - (1.55).

The function $K(x,y,t)$ has the following properties which are used in the sequel

(P1) $K(x,y,t)$ is a continuous function of y with x,t being fixed.

(P2) For any fixed value of x and t, the function $K(x,y,t)$ attains its minimum value at one or several values of y. Furthermore, the set

$$\{y : K(x,y,t) = \min_{z\in\mathbb{R}} K(x,z,t)\} \quad (1.65)$$

is a compact set.

(P3) $K(x, y_{min}(x,t), t) = K(x, y_{max}(x,t), t) = m(x,t)$ is a continuous function of x,t in the half plane $t > 0$, where y_{min} and y_{max} denote the minimum and maximum values of y, respectively, for which $K(x,y,t)$ attains its minimum.

Note that the assumption (1.58) is essential for obtaining property (P2). Indeed, using (1.58) one can see that

$$\lim_{|y|\longrightarrow\infty}\frac{K}{y^2} = \frac{1}{2t} > 0$$

which implies that the set (1.65) is bounded. The fact that it is closed follows from the continuity of K.

It follows from (P2) that the set (1.65) is bounded in $\mathbb{R}$. Hence the functions

$$y_{\min}(x,t) = \min\{y : K(x,y,t) = \min_{z\in\mathbb{R}} K(x,z,t)\}$$

and

$$y_{\max}(x,t) = \max\{y : K(x,y,t) = \min_{z\in\mathbb{R}} K(x,z,t)\}.$$

are well defined. The functions $y_{\min}$ and $y_{\max}$ have the following properties, [69, Lemma 1 and Lemma 3].

(Y1) If $x_1 < x_2$ then $y_{\max}(x_1,t) \le y_{\min}(x_2,t)$.

(Y2) $y_{\min}(x^-,t) = y_{\min}(x,t), \quad y_{\max}(x^+,t) = y_{\max}(x,t)$, where $y_{\min}(x^-,t)$ is the left limit of $y_{\min}$ and $y_{\max}(x^+,t)$ the right limit of $y_{\max}$, for fixed t.

(Y3) $\lim_{x\longrightarrow+\infty} y_{\min}(x,t) = +\infty, \quad \lim_{x\longrightarrow-\infty} y_{\max}(x,t) = -\infty.$

(Y4) As functions of x and t, $y_{\min}$ and $y_{\max}$ are lower semi-continuous and upper semi-continuous respectively. Therefore both functions are continuous at all points (x,t) where $y_{\min}(x,t) = y_{\max}(x,t)$.

Let us recall [4] the definition of lower semi-continuous and upper semi-continuous functions. In this regard, let $\mathbb{R}^* = \mathbb{R}\cup\{\pm\infty\}$ be the set of extended real numbers. A function $u : \Omega \longrightarrow \mathbb{R}^*$ is called *lower semi-continuous* at point $(\alpha,\theta) \in \Omega$ if for every $m < u(\alpha,\theta)$ there exist $\eta > 0$ such that

$$|x-\alpha| < \eta,\ |t-\theta| < \eta, \quad \Longrightarrow \quad m < u(x,t).$$

A function $u : \Omega \longrightarrow \mathbb{R}^*$ is called *upper semi-continuous* at point $(\alpha,\theta) \in \Omega$ if for every $m > u(\alpha,\theta)$ there exist $\eta > 0$ such that

$$|x-\alpha| < \eta,\ |t-\theta| < \eta, \quad \Longrightarrow \quad m > u(x,t).$$

A function $u : \Omega \longrightarrow \mathbb{R}^*$ is called lower(upper) semi-continuous in Ω if it is lower(upper) semi-continuous at every point of Ω.

It is clear, from property (Y1), that y_{min} and y_{max} are monotone functions in x. Since a monotone function has only a denumerable number of discontinuities one can conclude that, for any $t > 0$, $y_{min}(x,t) = y_{max}(x,t)$ for all x except at some denumerable set of values of x where $y_{min} < y_{max}$. That is,

$$\text{for } t > 0 \text{ the set } \{x : y_{min}(x,t) < y_{max}(x,t)\} \text{ is countable.} \tag{1.66}$$

The convergence Theorem for the solution u^ε of (1.54) - (1.55) is stated as follows

Theorem 1.29. *[69, Theorem 3] Let $u^\varepsilon(x,t)$ be the solution of* (1.54) - (1.55) *with u_0 satisfying* (1.58). *Then for all x and $t > 0$,*

$$\frac{x - y_{\max}(x,t)}{t} \le \liminf_{\substack{\alpha\to x\\ \theta\to t\\ \varepsilon\to 0}} u^\varepsilon(\alpha,\theta) \le \limsup_{\substack{\alpha\to x\\ \theta\to t\\ \varepsilon\to 0}} u^\varepsilon(\alpha,\theta) \le \frac{x - y_{\min}(x,t)}{t}.$$

In particular,

$$\lim_{\substack{\alpha \to x \\ \theta \to t \\ \varepsilon \to 0}} u^\varepsilon(\alpha, \theta) = \frac{x - y_{\max}(x,t)}{t} = \frac{x - y_{\min}(x,t)}{t}$$

holds at every point (x, t)*,* $t > 0$*, at which* $y_{\max}(x,t) = y_{\min}(x,t)$*.*

Define the function u as

$$u(x,t) := \lim_{\substack{\alpha \to x \\ \theta \to t \\ \varepsilon \to 0}} u^\varepsilon(\alpha, \theta) \tag{1.67}$$

at every point (x, t), $t > 0$ where this limit exists. By Theorem 1.29, the limit (1.67) will exist at every point where $y_{\min}(x,t) = y_{\max}(x,t)$. At these points, u is well defined and continuous. Furthermore, for each $t > 0$, u has a denumerably many discontinuities, as we noted above. Therefore we can conclude that the set of discontinuities has measure zero. Thus the convergence of u^ε to u as given by (1.67) is almost everywhere.

We define $\overline{u}$ and $\underline{u}$ on $\mathbb{R} \times [0, \infty)$, by

$$\begin{aligned} \underline{u}(x,t) &= \liminf_{\substack{\alpha \to x \\ \theta \to t \\ \varepsilon \to 0}} u^\varepsilon(\alpha, \theta) \\ &= \sup\{\inf\{u^\varepsilon(\alpha,\theta) : |\alpha - x| < \eta,\ |\theta - t| < \eta,\ \varepsilon < \eta\} : \eta > 0\} \end{aligned} \tag{1.68}$$

and

$$\begin{aligned} \overline{u}(x,t) &= \limsup_{\substack{\alpha \to x \\ \theta \to t \\ \varepsilon \to 0}} u^\varepsilon(\alpha, \theta) \\ &= \inf\{\sup\{u^\varepsilon(\alpha,\theta) : |\alpha - x| < \eta,\ |\theta - t| < \eta,\ \varepsilon < \eta\} : \eta > 0\} \end{aligned} \tag{1.69}$$

Note that

$$\underline{u}(x,t) \le \overline{u}(x,t) \quad (x,t) \in \mathbb{R} \times [0, \infty).$$

and

$$\underline{u}(x,t) = \overline{u}(x,t) = u(x,t) \ \text{ whenever } u(x,t) \text{ is defined for } (x,t) \in \mathbb{R} \times [0, \infty).$$

The function $u(x,t)$ defined by (1.67) is a weak solution to the Cauchy problem of the inviscid Burgers equation (1.12) - (1.13). To see this, let us note first that a solution u^ε of (1.54) - (1.55) satisfies the equation

$$\int_0^\infty \int_{-\infty}^\infty \{u^\varepsilon \phi_t + \frac{(u^\varepsilon)^2}{2}\phi_x\} dxdt + \int_{-\infty}^\infty u^0 \phi|_{t=0} dx + \varepsilon \int_0^\infty \int_{-\infty}^\infty u^\varepsilon \phi_{xx} dxdt = 0, \qquad (1.70)$$

for each test function $\phi \in C_0^\infty(\Omega)$. From the convergence Theorem 1.29 it follows that every point (x,t) has a neighborhood in which the solutions u^ε of (1.54) - (1.55) are uniformly bounded as ε tends to 0. As a result of this, one can pass through the limit $\varepsilon \longrightarrow 0$ in (1.70) with ϕ being fixed. Thus we have

$$\int_0^\infty \int_{-\infty}^\infty \{u\phi_t + \frac{u^2}{2}\phi_x\} dxdt + \int_{-\infty}^\infty u^0 \phi|_{t=0} dx = 0,$$

which shows that $u(x,t)$ is a weak solution of the Cauchy problem for the inviscid Burgers equation (1.12).

Furthermore, the limit function $u(x,t)$ defined by (1.67) is an entropy solution of the equation (1.12) - (1.13). To see this, let (Φ, Ψ) be any given pair of entropy/entropy-flux pair for the equation (1.8). Multiply equation (1.54) by $\Phi'(u^\varepsilon)$

$$\begin{aligned} u_t^\varepsilon \Phi'(u^\varepsilon) + u^\varepsilon u_x^\varepsilon \Phi'(u^\varepsilon) &= \varepsilon u_{xx}^\varepsilon \Phi'(u^\varepsilon) \\ &= \varepsilon((\Phi(u^\varepsilon))_{xx} - \Phi''(u^\varepsilon)(u^\varepsilon)^2). \end{aligned}$$

Using (1.43) we get

$$(\Phi(u^\varepsilon))_t + (\Psi(u^\varepsilon))_x = \varepsilon((\Phi(u^\varepsilon))_{xx} - \Phi''(u^\varepsilon)(u^\varepsilon)^2). \qquad (1.71)$$

Now multiply equation (1.71) by $\phi \in C_0^\infty(\mathbb{R} \times (0,\infty))$, $\phi \geq 0$ and integrate over $\mathbb{R} \times [0,\infty)$.

$$\begin{aligned} &\int_{-\infty}^\infty \int_0^\infty (\Phi(u^\varepsilon)\phi_t + \Phi(u^\varepsilon)\phi_x) dxdt \\ &= \varepsilon \int_{-\infty}^\infty \int_0^\infty \Phi(u^\varepsilon)\phi_{xx} dxdt - \varepsilon \int_{-\infty}^\infty \int_0^\infty \Phi''(u^\varepsilon)(u^\varepsilon)^2 \phi dxdt. \\ &\geq \varepsilon \int_{-\infty}^\infty \int_0^\infty \Phi(u^\varepsilon)\phi_{xx} dxdt \qquad (1.72) \end{aligned}$$

since $\Phi''(u^\varepsilon)(u^\varepsilon)^2\phi \geq 0$.

Again from the convergence Theorem 1.29 it follows that every point (x,t) has a neighborhood in which the solutions u^ε of (1.54) - (1.55) are uniformly bounded as ε tends to 0. Moreover, the function $\Phi(u)$ is convex and thus continuous. As such one can pass to the limit as $\varepsilon \longrightarrow 0$ in (1.72) with ϕ being fixed. Thus we have

$$\int_{-\infty}^\infty \int_0^\infty (\Phi(u)\phi_t + \Phi(u)\phi_x) dxdt \geq 0.$$

which shows that $u(x,t)$ is an entropy solution of the Cauchy problem for the inviscid Burgers equation (1.12).

Lax [78] obtained a result similar to that of Hopf by showing that the weak solution

$$u(x,t) = b\left(\frac{x-y_0}{t}\right) \qquad \text{for each} \quad t>0 \quad \text{and } a.e.\ x \in \mathbb{R}$$

obtained in Theorem 1.18 can be written as the limit of the solution u^ε of the Burgers equations(1.12) - (1.13). To see this, consider the equation

$$u_t + (f(u))_x = \frac{1}{2n}u_{xx}, \quad f(u) = \frac{u^2}{2}. \tag{1.73}$$

with the initial condition

$$u_n(x,0) = u_0(x) \tag{1.74}$$

Then similar to Hopf's result, we see that the function

$$u_n = \frac{\int_{-\infty}^{\infty} b(\frac{x-y}{t})e^{-nK(x,y,t)}dy}{\int_{-\infty}^{\infty} e^{-nK(x,y,t)}dy} \tag{1.75}$$

is a solution to the equation (1.73) - (1.74) and we define

$$u(x,t) = \lim_{n\to\infty} u_n.$$

As it was in case of Hopf's result, the convergence of u_n to u as n tends to ∞ is almost everywhere, see [79]. Likewise, define

$$f_n = \frac{\int_{-\infty}^{\infty} f(b(\frac{x-y}{t}))e^{-nK}dy}{\int_{-\infty}^{\infty} e^{-nK}dy}. \tag{1.76}$$

Then

$$f = \lim_{n\to\infty} f_n, \ \text{a.e.}$$

Here,

$$K(x,y,t) = U_0(y) + tG(\frac{x-y}{t}),$$

the function $b(s)$ is defined as $b(s) = (f'(s))^{-1}$, $G(s)$ is defined as the solution of

$$\frac{dG(s)}{ds} = b(s), \quad G(c) = 0, \ \text{with } f'(0) = c,$$

and

$$U_0(y) = \int_0^y u_0(s)ds.$$

If we denote by V_n the function

$$V_n = \log \int_{-\infty}^{\infty} e^{-nK(x,y,t)}dy$$

then

$$u_n = -\frac{1}{n}\frac{\partial}{\partial x}V_n$$

and

$$f_n(x,t) = -\frac{1}{n}\frac{\partial}{\partial t}V_n$$

provided that $f(b(z)) = zb(z) - G(z)$. It then follows that

$$(u_n)_t + (f_n)_x = 0. \tag{1.77}$$

Multiply equation (1.77) by a test function $\phi \in C_0^\infty(\Omega)$ and integrate to get

$$\int\int_\Omega u_n\phi_t + f_n\phi_x = 0,$$

letting $n \to \infty$ we obtain the limit relation

$$\int\int_\Omega u\phi_t + f(u)\phi_x = 0,$$

This shows that u is a weak solution of equations (1.8) - (1.9), see [78, Theorem 2.1].

For an arbitrary function f, there are no explicit formula for the solution to the viscous equation. However, Oleinik [94] proved that for a general convex or concave function, the solutions of the parabolic problem (1.52) - (1.53) tends to a weak solution of (1.8). A simpler proof was given by Ladyzhenskaya in [76].

As mentioned in Section 1.1.3, see in particular Theorem 1.15, Oleinik [95, 96] showed that there exists a unique solution of (1.8)-(1.9) that satisfies the admissibility condition (1.34), provided that the flux function f is convex. This solution is constructed as a limit of solutions u^ε of equation (1.52) -(1.53) obtained through a *finite difference scheme* introduced by Lax in [80], see also [81, 82]. It was subsequently shown that u is in fact the unique solution of (1.8) - (1.9) satisfying (1.34), see [111, Theorem 16.11].

Kruzhkov [74], [75] introduced a new method to apply the vanishing viscosity method to a larger class of equations. For initial data $u_0 \in L_\infty$, he proved existence and uniqueness of the classical solution $u^\varepsilon(x,t)$ of (1.52)-(1.53). Using a family of entropy-entropy flux pairs $(\Phi_k, \Psi_k)_{k\in\mathbb{R}}$ where

$$\Phi_k(u) = |u-k| \text{ and } \Psi_k(u) := sgn(u-k)(f(u) - f(k)),$$

he showed that the solution $u^\varepsilon(x,t)$ of equations (1.52) - (1.53) converges as ε tends to 0 almost everywhere to a weak solution $u(x,t)$ of the Cauchy problem (1.8) -(1.9).

Theorem 1.30. *[75, Kruzhkov]] Let* $u_0 \in L_\infty(\mathbb{R})$. *Then the solution* $u^\varepsilon(x,t)$ *of problem* (1.52) - (1.53) *converges as* $\varepsilon \longrightarrow 0$ *almost everywhere in* $\mathbb{R}\times[0,T)$ *to a function* $u(x,t)$ *which is a weak solution of the problem* (1.8) - (1.9).

In the proof of the above theorem, a priori bounds (independent of ε) were obtained for the solutions $u^\varepsilon(x,t)$ which ensures the compactness of the family of functions $\{u^\varepsilon(x,t) : t > 0\}$ with respect to the L_1- norm. This in turn guarantees the existence of a subsequence u^{ε_n} of u^ε that converges almost everywhere to the weak solution $u(x,t)$. Thus a weak solution of the Cauchy problem (1.8) - (1.9) is constructed as the limit of solution u^ε of the parabolic problem (1.52) - (1.53).

The following theorem shows that the weak solution constructed above is an entropy solution.

Theorem 1.31. *Let $u_0 \in L_\infty(\mathbb{R})$. If $u^\varepsilon(x,t)$ converges to a function $u(x,t)$ almost everywhere as $\varepsilon \longrightarrow 0$ in $\mathbb{R} \times [0,T)$. Then the solution u is the entropy solution of the Cauchy problem* (1.8) - (1.9)*.*

The properties of the solution $u(x,t)$ of problem (1.8) - (1.9) is addressed in the following result, see also [74, 75].

Theorem 1.32. *[109, Proposition 2.3.6] Let $u_0, v_0 \in L_\infty$ and u and v be the entropy solutions of* (1.8) -(1.9) *associated with u_0 and v_0 respectively. Let*

$$M = \sup\{|f'(s)| \ : \ s \in [\inf(u_0(x), v_0(x)), \sup(u_0(x), v_0(x))]\}.$$

Then the following properties are satisfied:

(P1) For all $t > 0$ and every interval $[a,b]$, we have

$$\int_a^b |v(x,t) - u(x,t)|dx \leq \int_{a+Mt}^{b+Mt} |v_0(x) - u_0(x)|dx.$$

(P2) If u_0 and v_0 coincide on $[x_0-\delta, x_0+\delta]$ for some $\delta >, 0$ then u and v coincide on the triangle $\{(x,t) : |x - x_0| + Mt < \delta\}$.

(P3) If $u_0 - v_0 \in L^1(\mathbb{R})$, then $u(t) - v(t) \in L^1(\mathbb{R})$ $\forall$ $t > 0$, where $u(t) := u(\cdot,t)$ and $v(t) := v(\cdot,t)$. Moreover,

$$\|v(t) - u(t)\|_{L^1(\mathbb{R})} \leq \|v_0 - u_0\|_{L^1(\mathbb{R})},$$

and

$$\int_{\mathbb{R}} (v(x,t) - u(x,t))dx = \int_{\mathbb{R}} (v_0(x) - u_0(x))dx.$$

(P4) If $u_0 \in L^1(\mathbb{R})$, then $u(t) \in L^1(\mathbb{R})$, for all $t > 0$, and

$$\|u(t)\|_{L^1(\mathbb{R})} \leq \|u_0\|_{L^1(\mathbb{R})}, \qquad \int_{\mathbb{R}} u(x,t)dx = \int_{\mathbb{R}} u_0(x)dx.$$

(P5) If $u_0(x) \leq v_0(x)$ for almost all $x \in \mathbb{R}$, then $u(x,t) \leq v(x,t)$ for almost all $(x,t) \in \mathbb{R} \times [0,\infty)$.

(P6) If u_0 has bounded total variation, then $u(t)$ has bounded total variation for all $t > 0$ and

$$TV(u(t)) \leq TV(u_0).$$

Remark 1.33. The proof of the above Theorem 1.32 is based on the fact that the semigroup operator S_t of (1.8) is a contraction in $L^1(\mathbb{R}) \cap L_\infty(\mathbb{R})$ with respect to the L^1 -norm. This fact is expressed in property (P3), which implies that if $u_0 \in BV$, then $u \in BV$ for all $t > 0$ as stated in property (P6). Property (P4) is a consequence of property (P3), and it leads to property (P5).

1.1.5 Compensated Compactness Methods for Nonlinear Conservation Laws

As mentioned, the vanishing viscosity method involves the construction of physically meaningful solution of the Cauchy problem (1.8) - (1.9) as the limit of the solutions of the parabolic problem (1.52) - (1.53). The strategy usually adopted in the literature is to obtain a priori bounds on solutions u^ε, of (1.52) - (1.53), that is, to show that

$$\|u^\varepsilon\|_{L_\infty} \leq C, \qquad \text{with } C \text{ a constant independent of } \varepsilon.$$

Such an estimate is then used to show that u^ε converges to some function u, in an appropriate sense, as $\varepsilon \longrightarrow 0$. The final step is usually to show, using a suitable compactness argument, that the limit function u is the entropy solution to the Cauchy problem (1.8) - (1.9). From the forgoing discussion it is clear that an essential part of the vanishing viscosity method is the study of the compactness of the set $\{u^\varepsilon(x,t) : \varepsilon > 0\}$ of solutions of the viscous problem (1.52) - (1.53) with respect to the L_1 topology. That is, the possibility of obtaining the strong convergence of a subsequence u^{ε_n}, where $\varepsilon_n \longrightarrow 0$ as $n \longrightarrow \infty$. These compactness arguments are closely related to the decay of entropy solution for large time, see for instance [41, 42, 43, 44, 45, 46] and [79].

The issue of compactness of the set $\{u^\varepsilon : \varepsilon > 0\}$ has been addressed mainly in the following ways:

(i) The compactness approach is based on a priori BV bounds, which has proven to be more useful in the study of scalar conservation laws or systems of conservation laws, see Section 1.1.4, [21] and [52]. A wide range of numerical examples on the applications of this compactness approach to finite difference approximations exist, see for instance [65], [66], [78], [94]. However, the approach is essentially limited to systems in one spatial dimension, except for the Kruzkov's multidimensional BV-based existence result which rely on the translation invariance of the underlying solution operator.

(ii) Compensated Compactness, developed by Tartar [114], [116] and Murat [87], [88], is based on certain L^2-type, H^{-1}−compact entropy production bounds, which replaces the BV bounds in the BV compactness method, see [47, 49, 130]. So far existence results based on compensated compactness arguments are limited to a system of two conservation laws in two spatial dimension.

In the light of the importance of compactness arguments in the study of nonlinear conservation laws, we discuss briefly some of the main point related to such compactness methods and their applications to nonlinear conservation laws.

BV Compactness

If a collection of functions $\{u^{\varepsilon}(x,t) : \varepsilon > 0\}$ satisfies

$$\|u^{\varepsilon}\|_{L_{\infty}} \leq C_1$$

and

$$TV\{\|u^{\varepsilon}\|\} \leq C_2$$

for all $\varepsilon > 0$, where C_1 and C_2 are constants independent of ε, then by Helly's Compactness Theorem [28, Theorem 2.3] there exists a sequence $\varepsilon_k \longrightarrow 0$ such that the sequence $\{u^{\varepsilon_k}\}$ converges almost everywhere to some $u \in L_{\infty}$.

The use of BV compactness framework in proving existence and uniqueness of solution of (1.8) - (1.9) can be found in [21], [64], [95] and [124], see [51] for other applications.

Compactness in L^1

Suppose that $\{u^{\varepsilon} : \varepsilon > 0\}$ satisfies

(i) $\|u^{\varepsilon}\|_{L^1} \leq C, \qquad C > 0$ a constant independent of ε.

(ii) $\{u^{\varepsilon}(x,t) : \varepsilon > 0\}$ is equicontinuous in $L^1_{loc}(\mathbb{R} \times [0,\infty))$, that is, for any compact subset $\Omega \subseteq \mathbb{R} \times [0,\infty)$,

$$\|u^{\varepsilon}(x+\Delta x, t+\Delta t) - u^{\varepsilon}(x,t)\|_{L^1} \longrightarrow 0 \text{ uniformly on } \Omega$$

as $\Delta x, \Delta t \longrightarrow 0$.

Then there exists a sequence $\varepsilon_k \longrightarrow 0$ such that the sequence

$$u^{\varepsilon_k} \longrightarrow u \qquad \text{in } L^1_{loc}(\mathbb{R} \times [0,\infty).$$

The above compactness in L_1 was applied by Kruzhkov to prove existence and uniqueness of entropy solution to the conservation laws [75].

Remark 1.34. (i) Existence of C^{∞} solution u^{ε} follows from the boundedness of the initial data and Theorem 1.24. The proof of the compactness of the entropy bound is similar to the case where solutions are in L_{∞}, which was briefly shown above.

(ii) We remark here that the concept of generalized solutions of hyperbolic systems of conservation laws is a straight forward generalization of scalar conservation laws discussed above, we therefore omit it and refer the reader [2, 13, 29, 30, 31, 34, 35, 36, 37, 55, 60, 67, 110, 115] for details.

1.2 Convergence Spaces

The Hausdorff-Kuratowski-Bourbaki concept of general topology has proved to be very useful in analysis .One such useful application is the powerful methods of linear functional analysis initiated by Banach [18] within the setting of metric spaces. However, several deficiencies of the Hausdorff-Kuratowski-Bourbaki topology emerged in the middle of the twentieth century. The most serious of these deficiencies is the fact that there is in general no natural topological structure for function spaces.

Recall that if X, Y and Z are sets, then the exponential law

$$Z^{X\times Y} \simeq (Z^X)^Y \tag{1.78}$$

holds. This means there is a canonical one-to-one mapping between the spaces of functions

$$f : X \times Y \longrightarrow Z \tag{1.79}$$

and

$$g : Y \longrightarrow Z^X = \{h : X \longrightarrow Z\} \tag{1.80}$$

That is, with any function (1.79) one can associate the function

$$\tilde{f} : Y \ni y \longmapsto f(\cdot, y) \in Z^X \tag{1.81}$$

defined through

$$\tilde{f}(y) : X \ni x \longmapsto f(x, y) \in Z.$$

Conversely, with the function (1.80) one may associate the function

$$\tilde{g} : X \times Y \longrightarrow Z$$

defined by

$$\tilde{g}(x, y) = g(y)(x) \in Z.$$

For topological spaces X, Y and Z, the exponential law can be written as

$$C(X \times Y, Z) \simeq C(Y, C(X, Z)). \tag{1.82}$$

Consider a continuous function

$$f : X \times Y \longrightarrow Z, \tag{1.83}$$

with the mapping (1.83) we associate the mapping

$$F_f : Y \ni y \longmapsto F_f(y) \in C(X, Z) \tag{1.84}$$

defined by

$$F_f(y) : X \ni x \longmapsto f(x, y) \in Z. \tag{1.85}$$

Conversely, with a continuous function

$$F : Y \longrightarrow C(X, Z) \tag{1.86}$$

associate the mapping

$$f_F : X \times Y \longrightarrow Z \tag{1.87}$$

defined as

$$f_F : X \times Y \ni (x, y) \longmapsto (F(y))(x) \in Z.$$

Let $C(X, Y)$ be equipped with the compact open topology, which has a subbasis

$$\left\{ S(K, U) \,\middle|\, \begin{array}{l} K \subseteq X \text{compact} \\ U \subseteq Y \text{open} \end{array} \right\}$$

where

$$S(K, U) = \{f \in C(X, Y) : f(K) \subseteq U\}.$$

If X is locally compact and Hausdorff, then the mapping (1.87) associated with the mapping (1.86) is continuous whenever the mapping (1.86) is continuous . Hence, whenever X is locally compact and Hausdorff , the mappings (1.83) - (1.87) define a bijection

$$\chi : C(X \times Y, Z) \ni f \longmapsto F_f \in C(Y, C(X, Z)). \tag{1.88}$$

Moreover, if Y and Z are also locally compact then the mapping (1.88) is a homeomorphism. Thus (1.82) holds for locally compact spaces X, Y and Z and the compact open topology on the relevant spaces of continuous functions. However, if the assumptions of local compactness on any of the spaces X, Y or Z are relaxed, then either the mapping (1.87) fails to be continuous, or the mapping (1.88) will no longer be a homeomorphism. Thus, unless all the spaces X, Y and Z are locally compact, there is no topology on $C(X, Y)$ so that the above construction holds, see for instance [25, 86, 122].

Another failure of the Hausdorff-Kuratowski-Bourbaki concept of topology, from the perspective of applications to analysis, concerns the issue of generality. In this regards, we may mention that there are several natural and important notions of convergence that cannot be associated with the Hausdorff-Kuratowski-Bourbaki topology. For instance, the point-wise almost everywhere convergence is not topological. To see this, we recall the following example, see [97, 101, 123].

Example 1.35. Let $M(\mathbb{R})$ denote the set of real Lebesgue measurable functions on $\mathbb{R}$. Consider the sequence (u_n^m), where

$$u_n^m(x) = \begin{cases} 1 & \frac{m-1}{n} \leq x \leq \frac{m}{n} \\ 0 & \text{otherwise.} \end{cases}$$

For any $m, , n \in \mathbb{N}$ and $\varepsilon > 0$ we have

$$A_n^m(\varepsilon) = \{x \in \mathbb{R} : u_n^m(x) \geq \varepsilon\} \subseteq [\frac{m-1}{n}, \frac{m}{n}].$$

Thus $\lim_{m,n \longrightarrow \infty} mes(A_n^m(\varepsilon)) = 0$ So that (u_n^m) converges to 0 in measure. However, (u_n^m)does not converge to 0 almost everywhere. Indeed, for all $a \in \mathbb{R}, a > 0$ and $N \in \mathbb{N}$, there exists $m, n \geq N$ such that $u_n^m(a) = 1$. Now suppose that there is some topology τ_{ae} on $M(\mathbb{R})$ so that the sequence that converges with respect to τ_{ae} are precisely those that converge almost everywhere. Since (u_n^m) does not converge to 0 almost everywhere, it follows that there is a τ_{ae} neighborhood $\mathcal{V}$ of 0 such that

$$\begin{array}{ll} \forall & k \in \mathbb{N} : \\ \exists & m_k, n_k \geq k : \\ & u_{n_k}^{m_k} \notin \mathcal{V}. \end{array}$$

In this way, we obtain a subsequence $(u_{m_k}^{n_k})$ of (u_m^n) so that

$$\begin{array}{ll} \forall & k \in \mathbb{N} \\ & u_{m_k}^{n_k} \notin \mathcal{V}. \end{array} \tag{1.89}$$

Thus no subsequence of $(u_{m_k}^{n_k})$ converges almost everywhere to 0. However, a well known result [68, Theorem 11.26] states that every sequence which converges in measure has a subsequence which converges almost everywhere to the same limit. Therefore $(u_{m_k}^{n_k})$ has a subsequence which converges almost everywhere to 0, which is a contradiction. Thus there is no topology which induces convergence almost everywhere on the set of measurable functions. □

One solution of the above mentioned limitations of Hausdorff-Kuratowski-Bourbaki topology is provided by the theory of convergence spaces, see [20, 25, 123], which is a more general notion of topology. Our main focus in this section is to introduce some of the fundamental concepts related to convergence spaces. A convergence space is a set together with a designated collection of filters. Recall that a filter $\mathcal{F}$ on a set X is a nonempty collection of subsets of X such that

(i) The empty set does not belong to $\mathcal{F}$.

(ii) For all $F \in \mathcal{F}$ and for all $G \subseteq X$, if $G \supseteq F$, then $G \in \mathcal{F}$

(iii) If $F, G \in \mathcal{F}$, then $F \cap G \in \mathcal{F}$.

A subset $\mathcal{B} \subseteq \mathcal{F}$ is a filter basis of $\mathcal{F}$ if each set in $\mathcal{F}$ contains a set in $\mathcal{B}$. The filter $\mathcal{F}$ is said to be generated by $\mathcal{B}$. We then write $\mathcal{F} = [\mathcal{B}]$. If $A \subseteq X$, the filter generated A is written as $[A]$. That is

$$[A] = \{F \subseteq X : F \supseteq A\}.$$

In particular for $x \in X$, $[x]$ is the filter generated by $\{x\}$. The filter $[x]$ is called the principal ultrafilter generated by x. Recall that a filter $\mathcal{G}$ on X is called an ultrafilter if $\mathcal{G} \not\subset \mathcal{F}$ for all filter $\mathcal{F}$ on X. The intersection of two filters $\mathcal{F}$ and $\mathcal{G}$ on X is defined as

$$\mathcal{F} \cap \mathcal{G} = [\{F \cup G : F \in \mathcal{F}, G \in \mathcal{G}\}]$$

If $\mathcal{F}$ is a filter on X, and $\mathcal{G}$ is a filter on Y, then the product of the filters $\mathcal{F}$ and $\mathcal{G}$ is a filter on $X \times Y$ which is defined as

$$\mathcal{F} \times \mathcal{G} = [\{F \times G : F \in \mathcal{F}, G \in \mathcal{G}\}]$$

If filters $\mathcal{F}$ and $\mathcal{G}$ on X are such that $\mathcal{G} \subseteq \mathcal{F}$, then we say that $\mathcal{F}$ is finer than $\mathcal{G}$, or alternatively $\mathcal{G}$ is coarser than $\mathcal{F}$. If $\mathcal{F}$ and $\mathcal{G}$ are filters on X, the filter $\mathcal{F} \vee \mathcal{G}$ may not exist. However, if $A \cap B \neq \emptyset$ for all $A \in \mathcal{F}$ and all $B \in \mathcal{G}$ then

$$\mathcal{F} \vee \mathcal{G} = [\{A \cap B : A \in \mathcal{F}, B \in \mathcal{G}\}]$$

exists. If $f : X \longrightarrow Y$ is a mapping then we define the image filter of $\mathcal{F}$ under f as

$$f(\mathcal{F}) = [\{f(F) : F \in \mathcal{F}\}].$$

If (x_n) is a sequence in X, then we define the Frechét filter associated with (x_n) as

$$\langle (x_n) \rangle = [\{\{x_n : n \geq k\} : k \in \mathbb{N}\}].$$

Recall [20, 123] that a given topological space (X, τ) may be completely described by specifying the convergence associated with the topology τ. In particular, a filter $\mathcal{F}$ on X converges to $x \in X$ if and only if $\mathcal{F} \supseteq \mathcal{V}_\tau(x)$, where $\mathcal{V}_\tau(x)$ denotes the τ-neighborhood filter at $x \in X$. For each $x \in X$ we may denote the set of all filters converging to x with respect to τ by $\lambda_\tau(x)$. That is,

$$\lambda_\tau(x) = \{\mathcal{F} \text{ a filter on } X : \mathcal{F} \supseteq \mathcal{V}_\tau(x)\} \tag{1.90}$$

A sequence (x_n) converges to $x \in X$ with respect to the topology τ if

$$\left\{ \begin{array}{ll} \forall & V \in \mathcal{V}_\tau(x) : \\ \exists & N_V \in \mathbb{N} : \\ & x_n \in V \;\forall\; n \geq N_V \end{array} \right. \tag{1.91}$$

This implies that

$$\langle (x_n) \rangle \supseteq \mathcal{V}_\tau(x).$$

Conversely, if $\langle (x_n) \rangle \supseteq \mathcal{V}_\tau(x)$, then (1.91) must hold. Thus the definition of filter convergence in a topological space is a straight forward generalization of the corresponding notion of convergence for sequences in a topological space.

A convergence structure on a set X is a generalization of the topological convergence (1.90) and is defined as follows

Definition 1.36. *Let X be a nonempty set. A convergence structure on X is the mapping λ from X to the power set of the set of all filters on X that satisfies the following for all $x \in X$:*

(i) $[x] \in \lambda(x)$

(ii) If $\mathcal{F}, \mathcal{G} \in \lambda(x)$, *then* $\mathcal{F} \cap \mathcal{G} \in \lambda(x)$.

(iii) If $\mathcal{F} \in \lambda(x)$, *then* $\mathcal{G} \in \lambda(x)$, *for all filters* $\mathcal{G} \supseteq \mathcal{F}$.

The pair (X, λ) *is called a convergence space. Whenever* $\mathcal{F} \in \lambda(x)$ *we say* $\mathcal{F}$ *converges to* x *and write "*$\mathcal{F} \longrightarrow x$*".*

Remark 1.37. Let λ and μ be two convergence structures on the same set X. Then λ is finer than μ (or μ is coarser than λ) if for every $x \in X$, $\lambda(x) \subseteq \mu(x)$. That is, λ has fewer convergent filters than μ.

As mentioned, convergence spaces are more general than topological spaces. However the concepts of continuity, embedding, homeomorphisms, open set and closure of a set generalize to the more general context of convergence spaces. In this regard, let X and Y be convergence spaces with convergence structures λ_X and λ_Y respectively. A mapping $f : X \longrightarrow Y$ is said to be *continuous* at a point $x \in X$ if

$$f(\mathcal{F}) = [\{f(F) : F \in \mathcal{F}\}] \longrightarrow f(x) \text{ whenever } \mathcal{F} \longrightarrow x \in X.$$

The mapping f is continuous if it is continuous at every point of X. Furthermore, f is called a *homeomorphism* if it is a bijection with both f and f^{-1} are continuous. It is an *embedding* if it is a homeomorphism onto its co-domain.

Clearly the topological convergence (1.90) satisfies the conditions of Definition 1.36. Examples of non-topological convergence structures include the following.

Example 1.38 (Almost every where convergence structure)**.** Let X be the set of real-valued measurable functions on a measure space $(\Omega, \mathcal{A}, \mu)$. Let a convergence structure λ_{ae} be define on X as follows: a filter $\mathcal{F}$ converges to f in (X, λ_{ae}) if $\mathcal{F}$ converges to f almost everywhere in Ω. Then λ_{ae} is a convergence structure. In particular, a sequence (u_n) in X converges almost everywhere to $u \in X$ if and only if $\langle (u_n) \rangle$ converges to u with respect to λ_{ae}. As we have shown in Example 1.35, almost everywhere convergence is not topological, see also [20].

Example 1.39 (Order convergence structure)**.** [12] Let X be an Archimedean vector lattice [24, 84, 117]. A filter $\mathcal{F}$ on X converges to u in X with respect to the order convergence structure λ_0 if and only if

$$\left\{ \begin{array}{ll} \exists & (\alpha_n), (\beta_n) \subset X : \\ & (i)\ \alpha_n \leq \alpha_{n+1} \leq \beta_{n+1} \leq \beta_n,\ n \in \mathbb{N} \\ & (ii)\ \sup\{\alpha_n : n \in \mathbb{N}\} = u = \inf\{\beta_n : n \in \mathbb{N}\} \\ & (iii)\ [\{[\alpha_n, \beta_n] : n \in \mathbb{N}\}] \subseteq \mathcal{F}. \end{array} \right.$$

A sequence (u_n) in X converges to $u \in X$ with respect to λ_0 if and only if (u_n)

order converges to u. That is,

$$\begin{cases} \exists & (\alpha_n), (\beta_n) \subset X : \\ & (i)\ \alpha_n \leq \alpha_{n+1} \leq u_n \leq \beta_{n+1} \leq \beta_n,\ n \in \mathbb{N} \\ & (ii)\ \sup\{\alpha_n : n \in \mathbb{N}\} = u = \inf\{\beta_n : n \in \mathbb{N}\}. \end{cases}$$

The order convergence structure is not topological. To see this, consider the Archimedean vector lattice $C(\mathbb{R})$, and the sequence $(u_n) \subset C(\mathbb{R})$ given by

$$u_n(x) = \begin{cases} 1 - n|x - q_n| & \text{if } |x - q_n| < \frac{1}{n} \\ \\ 0 & \text{if } |x - q_n| \geq \frac{1}{n} \end{cases} \tag{1.92}$$

where $\{q_n \mid n \in \mathbb{N}\} = [0,1] \cap \mathbb{Q}$. The complement of any subset of $\mathbb{Q} \cap [0,1]$ is dense in $[0,1]$. The sequence (u_n) does not order converge to 0. For any $N_0 \in \mathbb{N}$ we have

$$\beta_{N_0}(x) = \sup\{u_n(x) : n \geq N_0\} = 1, x \in [0,1)$$

This means that a sequence $(\beta_n) \subseteq C(\mathbb{R})$ such that $u_n \leq \beta_n$ for all $n \in \mathbb{N}$ cannot decrease to 0. Thus if there is a topology τ on $C^0(\mathbb{R})$ that induces order convergence, then there is some τ-neighborhood V of 0 and a subsequence (u_{n_k}) of (u_n) which is always outside of V. Let (q_{n_k}) denote the sequence of rational numbers associated with the subsequence (u_{n_k}) according to (1.92). Since the sequence (q_{n_k}) is bounded, there exist a subsequence $(q_{n_{k_i}})$ of (q_{n_k}) that converges to some $q \in [0,1]$. Let $(u_{n_{k_i}})$ be the sequence associated with the sequence of rational numbers $(q_{n_{k_i}})$. Then

$$\begin{array}{l} \forall\ \varepsilon > 0 : \\ \exists\ N_\varepsilon \in \mathbb{N} : \\ u_{n_{k_i}}(x) = 0,\ \text{whenever} |x - q_n| > \varepsilon \text{ and } n_{k_i} > N_\varepsilon. \end{array}$$

For each $j \in \mathbb{N}$ set $\varepsilon_j = \frac{1}{j}$ and let the sequence $(\mu_{n_{k_i}}) \subseteq C^0(\mathbb{R})$ be defined as

$$\mu_{n_{k_i}} = \begin{cases} 0 & if |x - q| \geq 2\varepsilon_j \\ 1 & if |x - q| \leq \varepsilon_j \\ \frac{|x-q|}{\varepsilon_j} + 2 & if \varepsilon_j < |x - q| < 2\varepsilon_j \end{cases} \tag{1.93}$$

whenever $N_{\varepsilon_j} < n_{k_i} < N_{\varepsilon_{j+1}}$. The sequence $(\mu_{n_{k_i}})$ decreases to 0, and $0 \leq u_{n_{k_i}} \leq \mu_{n_{k_i}}$, This means that the sequence $(u_{n_{k_i}})$ order converges to 0. Therefore it must eventually be in V, a contradiction. Thus the topology τ cannot exist.

Example 1.40 (Continuous convergence structure)**.** [20] Let X and Y be convergence spaces, $C(X,Y)$ the space of all continuous functions from X to Y and

$$\omega_{X,Y} : C(X,Y) \times X \longrightarrow Y$$

the evaluation mapping. That is, $\omega_{X,Y}(f,x) = f(x)$ for all $f \in C(X,Y)$ and all $x \in X$. A filter $\mathcal{H}$ converges to $f \in C(X,Y)$ with respect to the continuous convergence structure λ_c if and only if

$$\omega_{X,Y}(\mathcal{H} \times \mathcal{F}) \longrightarrow f(x) \ \text{ for all } x \in X \text{ and all } \mathcal{F} \longrightarrow x \in X.$$

The universal property of the continuous convergence structure is states as follows: Let X, Y, Z be convergence spaces. Then the mapping $h : Z \longrightarrow C(X, Y)$ is continuous if and only if the associated mapping

$$\tilde{h} : Z \times X \longrightarrow Y(z, x)$$

defined by $\tilde{h}(z, x) = h(z)(x)$ is continuous.

For more examples and a detailed exposition on convergence spaces see [20, 38, 40, 48, 57, 73].

One method for constructing new convergence spaces from given ones is to make use of initial and final convergence structures. Subspaces, product spaces, projective limits, quotient spaces and inductive limits are examples of initial or final convergence structure.

Let X be a set, $(X_i)_{i\in I}$ a collection of convergence spaces and, for each $i \in I$, $f_i : X \longrightarrow X_i$ a mapping. A filter $\mathcal{F}$ on X converges to x in the initial convergence structure λ_X with respect to the family of mapping $(f_i)_{i\in I}$ if and only if

$$f_i(\mathcal{F}) \longrightarrow f_i(x) \text{ in } X_i \ \text{ for all } i \in I.$$

To see that λ_X is a convergence structure on X, note the following

(i) For each $i \in I$ we have

$$\begin{aligned} f_i([x]) &= [\{f_i(\{x\}) : \{x\} \in [x]\}] \\ &= [\{f_i(x) : \{x\} \in [x]\}] \\ &= [f_i(x)]_{X_i} \longrightarrow f_i(x) \end{aligned}$$

which shows that $[x] \in \lambda_X(x)$.

(ii) Let $\mathcal{F}, \mathcal{G} \in \lambda_X(x)$. Then for each $i \in I$, we have

$$\begin{aligned} f_i(\mathcal{F} \cap \mathcal{G}) &= [\{f_i(F \cup G) : F \in \mathcal{F}, G \in \mathcal{G}\}] \\ &= [\{f_i(F) \cup f_i(G) : F \in \mathcal{F}, G \in \mathcal{G}\}] \\ &= f_i(\mathcal{F}) \cap f_i(\mathcal{G}) \longrightarrow f_i(x) \end{aligned}$$

thus $\mathcal{F} \cap \mathcal{G} \in \lambda_X(x)$.

(iii) Let $\mathcal{F} \in \lambda_X(x)$ and $\mathcal{F} \subseteq \mathcal{G}$. Then

$$\begin{aligned} f_i(\mathcal{G}) &= [\{f_i(G) : G \in \mathcal{G}\}] \\ &\supseteq [\{f_i(F) : F \in \mathcal{F}\}] \\ &= f_i(\mathcal{F}) \longrightarrow f_i(x) \end{aligned}$$

thus for each $i \in I$, $\mathcal{G} \in \lambda_X(x)$.

The initial convergence structure λ_X with respect to the family of mapping $(f_i)_{i\in I}$ is the coarsest convergence structure on X making each of the mapping $f_i : X \longrightarrow X_i$ continuous. That is, for any other convergence structure λ on X such that each f_i is continuous we have

$$\lambda(x) \subseteq \lambda_X(x),\ x \in X.$$

Example 1.41. Let $(X_i)_{i\in I}$ be a family of convergence spaces, and let X be the Cartesian product of the family (X_i). That is

$$X = \prod_{i\in I} X_i.$$

The product convergence structure on X is the initial convergence structure with respect to the projection mapping

$$\pi_i : X \longrightarrow X_i,\ i \in I$$

defined as

$$\pi_i((x_j)_{j\in I}) = x_i \in X_i.$$

A filter $\mathcal{F}$ on X converges to $x = (x_i)$, in X if and only if , for each $i \in I$

$$\pi_i(\mathcal{F}) \longrightarrow \pi_i(x) \in X_i$$

That is

$$\begin{array}{ll} \forall & i \in I \\ \exists & \mathcal{F}_i \in \lambda_{X_i}(x_i) : \\ & \prod_{i\in I} \mathcal{F}_i \subseteq \mathcal{F}. \end{array}$$

Here

$$\prod_{i\in I} \mathcal{F}_i = \left[\left\{ \prod_{i\in I} F_i \,\middle|\, \begin{array}{l} F_i \in \mathcal{F}_i\ i \in I \\ F_i = X_i \text{ for all but finitely many } i \in I \end{array} \right\}\right]$$

denotes the Tychonoff product of the family of filters $(\mathcal{F}_i)_{i\in I}$. □

Example 1.42. Let X be a convergence space and M a subset of X. The subspace convergence structure λ_M on M is the initial convergence structure with respect to the inclusion mapping

$$i_M : M \longrightarrow X$$

given by

$$i_M(x) = x \in X, x \in M.$$

A filter $\mathcal{F}$ on M converges to x in M if and only if

$$[\mathcal{F}]_X = \left[\left\{ G \subseteq X \,\middle|\, \begin{array}{ll} \exists & F \in \mathcal{F} : \\ & F \subseteq G \end{array} \right\}\right]$$

converges to x in X.

Let X be a set, $(X_i)_{i\in I}$ a collection of convergence spaces and, for each $i \in I$, $f_i : X_i \longrightarrow X$ a mapping. A filter $\mathcal{F}$ on X converges to a point x in the final convergence structure with respect to the family of mapping $(f_i)_{i\in I}$ if and only if $\mathcal{F} = [x]$ or

$$\begin{array}{ll} \exists & \text{indices } i_1 \cdots i_k \in I : \\ \exists & \text{point } x_n \in X_{i_n}, n = 1, \cdots, k : \\ \exists & \text{filters } \mathcal{F}_n \in \lambda_{X_{i_n}}(x_n),\ n = 1 \cdots k : \\ & 1) f_i(x_n) = x,\ i = 1 \cdots, k \\ & 2) f_{i_1}(\mathcal{F}_1) \cap \cdots \cap f_{i_k}(\mathcal{F}_k) \subseteq \mathcal{F} \end{array} \tag{1.94}$$

Example 1.43. [20] Let X be a convergence space, Y a set and $q : X \longrightarrow Y$ a surjective mapping. The quotient convergence structure λ_q on Y is the final convergence structure with respect to the mapping q. A filter $\mathcal{F}$ on Y converges to $y \in Y$ if and only if

$$\begin{array}{ll} \exists & \text{points } x_1, \cdots, x_k \in X : \\ \exists & \text{filters } \mathcal{F}_1, \cdots, \mathcal{F}_k \text{ on } X : \\ & \left(\begin{array}{ll} 1) & \mathcal{F}_i \in \lambda_X(x_i), i = 1, \cdots, k : \\ 2) & q(x_i) = y, i = 1, \cdots, k : \\ 3) & q(\mathcal{F}_1) \cap \cdots \cap q(\mathcal{F}_k) \subseteq \mathcal{F}. \end{array} \right. \end{array} \tag{1.95}$$

If X and Y are convergence spaces, and $q : X \longrightarrow Y$ a surjection so that Y carries the quotient convergence structure with respect to q, then q is called a convergence quotient mapping. $\square$

The final convergence structure is the finest convergence structure making all the mapping $(f_i)_{i\in I}$ continuous. That is for any other convergence structure λ in X such that the mapping f_i is continuous we have

$$\lambda_X(x) \subseteq \lambda(x)$$

Let X be a convergence space. For any $x \in X$ a set $V \subseteq X$ is a neighborhood of x if V belongs to every filter that converges to x. That is,

$$V \in \mathcal{V}_{\lambda_X}(x) \Longleftrightarrow \left(\begin{array}{cc} \forall & \mathcal{F} \in \lambda_X(x) \\ & V \in \mathcal{F} \end{array} \right)$$

where $\mathcal{V}_{\lambda_X}(x)$ denotes the neighborhood filter at x. A set $V \subseteq X$ is open if and only if it is a neighborhood of each of its elements.

The concept of adherence in the context of convergence spaces is the generalization of the closure of a subset A of a topological space X. In a topological space X, the closure of a set $A \subseteq X$ consists of A, together with all cluster points of A. That is,

$$cl_\tau(A) = \left\{ x \in X \,\middle|\, \begin{array}{cc} \forall & V \in \mathcal{V}_\tau(x) \\ & V \cap A \neq \emptyset \end{array} \right\}.$$

Therefore for each $x \in cl(A)$, the filter

$$\mathcal{F} = [\{V \cap A : V \in \mathcal{V}_\tau(x)\}]$$

converges to x and $A \in \mathcal{F}$. Conversely, if there is a filter $\mathcal{F} \in \lambda_\tau(x)$ such that $A \in \mathcal{F}$, then it follows from (1.90) that A intersects every neighborhood of x so that $x \in cl(A)$. This means that the closure of a set $A \subseteq X$ is he set of of all points $x \in X$ such that A belongs to some filter $\mathcal{F}$ that converges to x with respect to τ.

In a convergence space the adherence of a set $A \subseteq X$ is the set

$$a_{\lambda_X}(A) = \left\{ x \in X \;\middle|\; \begin{array}{l} \exists \;\; \mathcal{F} \in \lambda_X(x) : \\ \quad A \in \mathcal{F} \end{array} \right\}.$$

That is, $x \in a_{\lambda_X}(A)$ if there is a filter that converges to x and contains A. Where there is no confusion we shall simply denote the adherence of A by $a(A)$. The set $A \subseteq X$ is closed if $a(A) = A$.

Many of the familiar properties of the closure operator of a topological space also hold for the adherence operator in a convergence space. Some of these properties are stated in the following.

Proposition 1.44. *[20] Let X be a convergence space. Then the following hold:*

(i) $a(A) \subseteq a(B)$ if $A \subseteq B$ for all $A, B \subset X$

(ii) $a(\emptyset) = \emptyset$

(iii) $A \subseteq a(A)$ for all $A \subseteq X$

(iv) $a(A \cup B) = a(A) \cup a(B)$ for all $A, B \subseteq X$.

(v) $f(a(A)) \subseteq a(f(A))$ for all $A \subseteq X$, and $f : X \longrightarrow Y$ continuous.

In a non-topological convergence space, the adherence operator is, in general, not idempotent. That is, for some $A \subset X$ $a(A) \neq a(a(A))$. If a convergence space X is such that

$$\begin{array}{l} \forall \;\; x \in X \\ \quad \mathcal{V}_{\lambda_x}(x) \in \lambda_X(x) \end{array}$$

then the convergence space is called pre-topological and the convergence structure λ_X is called a pre-topology. Every topological space is pre-topological but the converse is not true, see for instance [20]. Indeed, one of the characterization of topological convergence spaces is the following

Proposition 1.45. *[20] A convergence space X is topological if and only if X is pre-topological and the adherence operator is idempotent.*

The notions of Hausdorff, T_1 and regular spaces in convergence spaces coincide with the usual ones in the case of a topological space. A convergence space X is called a Hausdorff space if every convergent filter converges to a unique limit. It is called a T_1 space if every finite subset of X is closed , and it is called a regular space if

$$\mathcal{F} \in \lambda_X(x) \Longrightarrow a(\mathcal{F}) = [\{a(F) : F \in \mathcal{F}\}] \in \lambda_X(x)$$

Note that a Hausdorff space is a T_1 space. To see this, let X be a Hausdorff space and let $A \subseteq X$ be a finite set. The set A is a finite union of singleton sets. Therefore it suffices to show that the singleton set $\{y\}$, for $y \in X$, is closed. If $x \in a(\{y\})$ then there exists a filter $\mathcal{F} \longrightarrow x$ and $\{y\} \in \mathcal{F}$. This implies that $\mathcal{F} \subseteq [y]$. Therefore $[y] \longrightarrow x$. Since X is Hausdorff it follows that $x = y$. Thus $a(\{y\}) = \{y\}$.

Conversely, a regular T_1 space is Hausdorff. This is because if X is regular and T_1, and a filter $\mathcal{F}$ converges to x and y then $a(\mathcal{F})$ also converges to x, and to y. Then $x, y \in a(F)$ for all $F \in \mathcal{F}$. So that $a(\mathcal{F}) = [\{a(F) : F \in \mathcal{F}\}] \subseteq [x]$. Hence $[x]$ converges to y. This implies that $y \in a(\{x\})$. But X is T_1, hence $x = y$. Thus X is Hausdorff.

Subspaces, product and projective limits of T_1 spaces, Hausdorff spaces and regular spaces are also T_1, Hausdorff and regular, respectively, as shown in [20, Proposition 1.4.2].

1.2.1 Uniform Convergence Structure

In this section we discuss some of the basic aspects of the theory of uniform convergence spaces which is a generalization of the theory of uniform spaces. Recall [20] that a uniformity on a set X is a filter $\mathcal{U}$ on $X \times X$ such that the following conditions are satisfied.

(i) $\Delta \subseteq \mathcal{U}$ for each $U \in \mathcal{U}$.

(ii) If $U \in \mathcal{U}$, then $U^{-1} \in \mathcal{U}$.

(iii) For each $U \in \mathcal{U}$ there are some $V \in \mathcal{U}$ such that $V \circ V \subseteq U$.

Here $\Delta = \{(x, x) : x \in X\}$ denotes the diagonal in $X \times X$. If U and V are subsets of $X \times X$ then

$$U^{-1} = \{(x, y) \in X \times X : (y, x) \in U\}$$

and the composition of U and V is defined as

$$U \circ V = \left\{ (x, y) \in X \times X \;\middle|\; \begin{array}{ll} \exists & z \in X : \\ & (x, z) \in V \text{ and } (z, y) \in U \end{array} \right\}.$$

A uniformity $\mathcal{U}_X$ on X induces a topology on X in the following way: A set $A \subseteq X$ is open in X if

$$\begin{array}{ll} \forall & x \in A : \\ \exists & U \in \mathcal{U}_X : \\ & U[x] \subseteq A \end{array}$$

where $U[x] = \{y \in X | (x, y) \in U\}$. A filter $\mathcal{F}$ on X is a Cauchy filter if and only if

$$\mathcal{U}_X \subseteq \mathcal{F} \times \mathcal{F}.$$

A uniform space is complete if and only if every Cauchy filter on X converges to some point $x \in X$. We recall [70] the definition of uniform continuity of function defined on a uniform space. Let X and Y be uniform spaces. A mapping $f : X \longrightarrow Y$ is uniformly continuous if and only if

$$\begin{array}{ll} \forall & U \in \mathcal{U}_Y \\ & (f^{-1} \times f^{-1})(U) \in \mathcal{U}_X. \end{array}$$

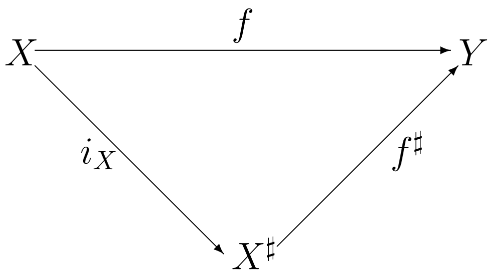
The mapping f is uniform embedding if it is injective and its inverse f^{-1} is uniformly continuous on the subspace $f(X)$ of Y. Furthermore, f is a uniform isomorphism, if it is a uniform embedding which is surjective. The main result due to Weil [126], in connection with completeness of uniform spaces assert that for any Hausdorff uniform space X, one can find a complete Hausdorff uniform space $X^\sharp$ and a uniform embedding

$$i_X : X \longrightarrow X^\sharp$$

such that $i_X(X)$ is dense in $X^\sharp$. Moreover, for any complete, Hausdorff uniform space Y and any uniformly continuous mapping $f : X \longrightarrow Y$ there is a uniformly continuous mapping

$$f^\sharp : X^\sharp \longrightarrow Y$$

such that the diagram



(1.96)

commutes.

Remark 1.46. Note that not every topology τ_X on a set X is induced by a uniformity $\mathcal{U}_X$. In fact, it was shown in [126] that a given topology τ_X on X is induced by a uniformity $\mathcal{U}_X$ if and only if the topology τ_X is completely regular. Hence the class of uniform spaces is rather small in comparison to the class of all topological spaces.

Definition 1.47. *Let X be a set. A family $\mathcal{J}_X$ of filters on $X \times X$ is called a uniform convergence structure if the following holds:*

(i) $[x] \times [x] \in \mathcal{J}_X$ *for every* $x \in X$

(ii) If $\mathcal{U} \in \mathcal{J}_X$ *and* $\mathcal{U} \subseteq \mathcal{V}$, *then* $\mathcal{V} \in \mathcal{J}_X$

(iii) If $\mathcal{U}, \mathcal{V} \in \mathcal{J}_X$*, then* $\mathcal{U} \cap \mathcal{V} \in \mathcal{J}_X$.

(iv) If $\mathcal{U} \in \mathcal{J}_X$*, then* $\mathcal{U}^{-1} \in \mathcal{J}_X$.

(v) If $\mathcal{U}, \mathcal{V} \in \mathcal{J}_X$*, then* $\mathcal{U} \circ \mathcal{V} \in \mathcal{J}_X$ *whenever* $\mathcal{U} \circ \mathcal{V}$ *exists.*

The pair $(X, \mathcal{J}_X)$ *is called a uniform convergence space.*

If $\mathcal{U}$ and $\mathcal{V}$ are filters on $X \times X$ then $\mathcal{U}^{-1}$ is defined as

$$\mathcal{U}^{-1} = [\{U^{-1} : U \in \mathcal{U}\}].$$

If $U \circ V \neq \emptyset$ for all $U \in \mathcal{U}$ and $V \in \mathcal{V}$ then the filter $\mathcal{U} \circ \mathcal{V}$ exists and it is defined as

$$\mathcal{U} \circ \mathcal{V} = [\{U \circ V : U \in \mathcal{U}, V \in \mathcal{V}\}].$$

Uniform convergence spaces generalizes the concept of a uniform space in the sense that every uniformity $\mathcal{U}_X$ on X give rise to a unique uniform convergence structure $\mathcal{J}_{\mathcal{U}_X}$ defined through

$$\mathcal{U} \in \mathcal{J}_{\mathcal{U}_X} \Longrightarrow \mathcal{U}_X \subseteq \mathcal{U}.$$

Every uniform convergence structure $\mathcal{J}_X$ on X induces a convergence structure $\lambda_{\mathcal{J}_X}$ on X defined by

$$\begin{array}{ll} \forall & x \in X \\ \forall & \mathcal{F} \text{ a filter on } X \\ & \mathcal{F} \in \lambda_{\mathcal{J}_X}(x) \Longleftrightarrow \mathcal{F} \times [x] \in \mathcal{J}_X \end{array}$$

The convergence structure $\lambda_{\mathcal{J}_X}$ is called the induced convergence structure. The induced convergence structure need neither be topological nor completely regular, but rather satisfies more general separation properties, see [20]. Every reciprocal convergence structure λ_X is induced by a uniform convergence structure. Recall that a convergence structure is called *reciprocal* if

$$\begin{array}{l} \forall\ x, y \in X \\ \lambda_X(x) = \lambda_X(y) \text{ or } \lambda_X(x) \cap \lambda_X(y) = \emptyset \end{array} \tag{1.97}$$

Note that if a convergence space is Hausdorff then it is reciprocal but the converse is not true. Given a reciprocal convergence structure λ_X on X, the *associated uniform convergence structure* $\mathcal{J}_{\lambda_X}$ on $X \times X$, defined by

$$\mathcal{U} \in \mathcal{J}_{\lambda_X} \Longleftrightarrow \left(\begin{array}{ll} \exists & x_1 \cdots x_k \in X \\ \exists & \mathcal{F}_1 \cdots \mathcal{F}_k \text{ filters on } X : \\ & (1)\quad \mathcal{F}_i \in \lambda_X(x_i) \text{ for } i = 1 \cdots k \\ & (2)\quad (\mathcal{F}_1 \times \mathcal{F}_1) \cap \cdots \cap (\mathcal{F}_k \times \mathcal{F}_k) \subseteq \mathcal{U} \end{array} \right. \tag{1.98}$$

is a uniform convergence structure that induces a convergence structure λ_X. In particular, every Hausdorff convergence structure is induced by the associated uniform convergence structure (1.98). A Hausdorff uniform convergence space is characterized by the following [20, Proposition2.1.10]

Proposition 1.48. *A uniform convergence space* $(X, \mathcal{J}_X)$ *is a Hausdorff uniform convergence space if and only if*

$$\begin{array}{ll} \forall & \mathcal{U} \in \mathcal{J}_X \\ \forall & x, y \in X, x \neq y : \\ \exists & U \in \mathcal{U} : \\ & \quad (x, y) \notin U. \end{array}$$

As mentioned in Section 1.2, new convergence spaces can be constructed from existing ones using the initial and final convergence structure. This is also true of uniform convergence spaces. The initial uniform convergence structure is constructed as follows: Let X be a set and $(X_i, \mathcal{J}_i)_{i \in I}$ a family of convergence spaces. For each $i \in I$ let $f_i : X \longrightarrow X_i$ be a mapping. The initial uniform convergence structure $\mathcal{J}$ on $X \times X$ with respect to the mapping f_i is defined as

$$\mathcal{U} \in \mathcal{J} \Longleftrightarrow \left(\begin{array}{ll} \forall & i \in I \\ & (f_i \times f_i)(\mathcal{U}) \in \mathcal{J}_i. \end{array} \right. \tag{1.99}$$

The initial uniform convergence structure $\mathcal{J}$ induces the initial convergence structure $\lambda_{\mathcal{J}}$, see [20, Proposition 2.2.2]. Subspaces and product uniform convergence spaces are typical example of initial uniform convergence structure.

Let X be a set and $(X_i, \mathcal{J}_i)_{i \in I}$ a family of convergence spaces. For each $i \in I$ let $f_i : X_i \longrightarrow X$ be a mapping. The final uniform convergence structure $\mathcal{J}$ on $X \times X$ with respect to the mapping f_i is defined as

$$\mathcal{U} \in \mathcal{J} \Longleftrightarrow \left(\begin{array}{ll} \exists & \mathcal{U}_1 \cdots \mathcal{U}_n \in \mathcal{J}_0 : \\ \exists & x_1 \cdots x_k \in X : \\ & \mathcal{U}_1 \cap \cdots \cap \mathcal{U}_n \cap ([x_1] \times [x_1]) \cap \cdots \cap ([x_k] \times [x_k]) \subseteq \mathcal{U}. \end{array} \right. \tag{1.100}$$

where $\mathcal{J}_0$ is a family of filters $\mathcal{V}$ on $X \times X$ defined by

$$\mathcal{V} \in \mathcal{J}_0 \Longleftrightarrow \left(\begin{array}{ll} \exists & i_1 \cdots i_n \in I : \\ \exists & \mathcal{V}_k \in \mathcal{J}_{i_k} : \\ & (f_{i_1} \times f_{i_1})(\mathcal{V}_1) \circ \cdots \circ (f_{i_1} \times f_{i_1})(\mathcal{V}_k) \subseteq \mathcal{V}. \end{array} \right.$$

Quotient uniform convergence structure is an example of final uniform convergence structure. We remark here that the final uniform convergence structure does not induce the final convergence structure, refer to [20, 62] for more details.

The concepts of uniform continuity, Cauchy filters, completeness and completion extend to uniform convergence spaces in a natural way. In this regard let X and Y be uniform convergence spaces. A mapping $f : X \longrightarrow Y$ is *uniformly continuous* if

$$\begin{array}{ll} \forall & \mathcal{U} \in \mathcal{J}_X \\ & (f \times f)(\mathcal{U}) \in \mathcal{J}_Y. \end{array}$$

A uniformly continuous mapping f is called a uniformly continuous embedding if it is injective and f^{-1} is uniformly continuous on the subspace $f(X)$ of Y. A uniformly continuous embedding is a uniformly continuous isomorphism if it is also surjective.

A filter $\mathcal{F}$ on X is called a *Cauchy filter* if

$$\mathcal{F} \times \mathcal{F} \in \mathcal{J}_X.$$

Some important properties of Cauchy filters are stated in the following, see [20, Proposition 2.3.2 - 2.2.3].

Proposition 1.49. *Let $(X, \mathcal{J}_X)$ be a uniform convergence space. Then the following hold:*

(i) Each convergent filter is a Cauchy filter.

(ii) If $\mathcal{F}$ is a Cauchy filter and $\mathcal{F} \subseteq \mathcal{G}$ then $\mathcal{G}$ is a Cauchy filter.

(iii) Let $\mathcal{F}$ be a Cauchy filter and let $\mathcal{F} \subseteq \mathcal{G}$. If $\mathcal{G} \longrightarrow x \in X$, then $\mathcal{F} \longrightarrow x$.

(iv) If $\mathcal{F}$ and $\mathcal{G}$ are Cauchy filters and $\mathcal{F} \vee \mathcal{G}$ exists then $\mathcal{F} \cap \mathcal{G}$ is a Cauchy filter.

(v) If $\mathcal{F}$ is a Cauchy filter, $\mathcal{G}$ a filter on X such that $\mathcal{F} \times \mathcal{G} \in \mathcal{J}_X$ then $\mathcal{G}$ is a Cauchy filter.

(vi) If (Y, λ_Y) is a uniform convergence space, $f : X \longrightarrow Y$ is a uniformly continuous mapping, and $\mathcal{F}$ is a Cauchy filter on X then $f(\mathcal{F})$ is a Cauchy filter on Y.

A uniform convergence space X is said to be *complete* if every Cauchy filter converges to a point in X.

Proposition 1.50. *Let $(X, \mathcal{J}_X)$ be a complete uniform convergence space. Then the following hold:*

(i) Each closed subspace of a complete uniform convergence space is complete.

(ii) If $(X, \mathcal{J}_X)$ is Hausdorff, then a subspace of $(X, \mathcal{J}_X)$ is complete if and only if it is closed.

(iii) The product of complete uniform convergence spaces is complete.

Example 1.51. The associated uniform convergence space of a reciprocal convergence space is complete. □

The Weil concept of completion of uniform spaces has been extended to the more general setting of uniform convergence spaces, see [127]. Indeed, if X is a Hausdorff uniform convergence space, then there exists a complete, Hausdorff uniform convergence space $X^\sharp$ and a uniformly continuous embedding

$$i_X : X \longrightarrow X^\sharp$$

such that $i_X(X)$ is dense in $X^\sharp$. Moreover, the completion $X^\sharp$ of X satisfies the *universal property*: If Y is a complete Hausdorff uniform convergence space and

$f : X \longrightarrow Y$ is uniformly continuous, then there exists a uniformly continuous mapping

$$f^\sharp : X^\sharp \longrightarrow Y$$

such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle i_X}\downarrow & \nearrow{\scriptstyle f^\sharp} & \\ X^\sharp & & \end{array} \tag{1.101}$$

commutes. $X^\sharp$ is called the Wyler completion of X. This completion is unique up to uniformly continuous isomorphism. In general, the Wyler completion of uniform convergence spaces does not preserve subspaces. Indeed, the completion of a subspace of a uniform convergence space X will, in general, not be a subspace of the completion $X^\sharp$, [121]. The following Theorems gives the characterization of the completion of a subspace X of a uniform convergence space Y.

Theorem 1.52. *Let X be a subspace of the uniform convergence space Y. Let $i : X \longrightarrow Y$ be the inclusion mapping. Then there exists an injective, uniformly continuous mapping $i^\sharp : X^\sharp \longrightarrow Y^\sharp$, which extends the mapping i. In particular, $i^\sharp(X^\sharp) = a_{Y^\sharp}(i_Y(X))$, where $a_{Y^\sharp}$ denote the adherence operator in $X^\sharp$ and i_Y denote the uniformly continuous embedding associated with the completion $Y^\sharp$. Furthermore, the uniform convergence structure on $Y^\sharp$ is the smallest complete, Hausdorff uniform continuous structure on $a_{Y^\sharp}(X)$, with respect to inclusion so that X is contained in $X^\sharp$ as a dense subspace.*

Theorem 1.53. *Let X and Y be uniformly convergence spaces, and $\varphi : X \longrightarrow Y$ a uniformly continuous embedding. Then there exists an injective uniformly continuous mapping $\varphi^\sharp : X^\sharp \longrightarrow Y^\sharp$, where $X^\sharp$ and $Y^\sharp$ are the completions of X and Y respectively, which extends φ.*

1.2.2 Convergence vector spaces

Let V be a vector space over the scalar field $\mathbb{K}$ of real or complex numbers. A convergence structure λ_V on V is called a vector space convergence structure if the vector space operations

$$+ : (V, \lambda_V) \times (V, \lambda_V) \longrightarrow (V, \lambda_V)$$

and

$$\cdot : \mathbb{K} \times (V, \lambda_V) \longrightarrow (V, \lambda_V)$$

are continuous. In this case V is called a convergence vector space.

Examples 1.54.

1. Every topological vector space is a convergence vector space. Recall [99, 106] that a vector space V over the scalar field $\mathbb{K}$ of real or complex numbers is called a topological vector space if V is endowed with a topology τ_V such that
$$+ : (V, \tau_V) \times (V, \tau_V) \longrightarrow (V, \tau_V)$$
and
$$\cdot : \mathbb{K} \times (V, \tau_V) \longrightarrow (V, \tau_V)$$
are (jointly) continuous.

2. Let X be a convergence space and V a convergence vector space. Then $C_c(X, V)$ is a convergence vector space. In particular $C_c(X) = C_c(X, \mathbb{R})$ is a convergence vector space.

3. Let X and Y be convergence vector spaces. Then $\mathcal{L}_c(X, Y)$, which is the set $\mathcal{L}(X, Y)$ of all continuous linear mapping between X and Y endowed with the subspace convergence structure from $C_c(X, Y)$, is a convergence vector space. In particular, $\mathcal{L}_c(X) = \mathcal{L}_c(X, \mathbb{R})$ is a convergence vector space. The space $\mathcal{L}_c(X)$ is the continuous dual space of the convergence vector space X. It is the canonical dual in the setting of convergence vector spaces.

$\square$

The following Lemma gives some properties of convergence vector spaces which are well-known in the topological case, see [20].

Lemma 1.55. *Let V be a convergence vector space. Then the following statements hold.*

(i) For each $a \in V$ the translation mapping
$$T_a : V \ni x \mapsto a + x \in V$$
is a homeomorphism.

(ii) For all $x \in V$
$$\mathcal{F} \in \lambda_V(x) \Longleftrightarrow \mathcal{F} - x \in \lambda_V(0)$$

(iii) If W is another convergence vector space then a linear mapping $f : V \longrightarrow W$ is continuous if and only if it is continuous at 0.

A standard procedure for constructing a vector space convergence structure or for showing that a given convergence structure is a vector space convergence structure, is given by the following proposition [20].

Proposition 1.56. *Let V be a vector space over $\mathbb{K}$ and let $\mathcal{V}(0)$ be the zero neighborhood filter on $\mathbb{K}$. Let $\mathcal{S}$ be a family of filters on V satisfying the the following conditions:*

(i) If $\mathcal{F} \in \mathcal{S}$ and $\mathcal{G} \in \mathcal{S}$ then $\mathcal{F} \cap \mathcal{G} \in \mathcal{S}$.

(ii) If $\mathcal{F} \in \mathcal{S}$ then $\mathcal{G} \in \mathcal{S}$ for all filters $\mathcal{G} \supseteq \mathcal{F}$.

(iii) If $\mathcal{F} \in \mathcal{S}$ and $\mathcal{G} \in \mathcal{S}$ then $\mathcal{F} + \mathcal{G} \in \mathcal{S}$.

(iv) If $\mathcal{F} \in \mathcal{S}$ then $\mathcal{V}(0)\mathcal{F} \in \mathcal{S}$.

(v) If $\mathcal{F} \in \mathcal{S}$ then $\alpha\mathcal{F} \in \mathcal{S}$ for all $\alpha \in \mathbb{K}$

(vi) For all $x \in V$, $\mathcal{V}(0)x \in \mathcal{S}$.

Then the mapping λ_V from V to the power set of all the set of filters on V defined by

$$\mathcal{F} \in \lambda_V(x) \Longleftrightarrow \mathcal{F} - x \in \mathcal{S}.$$

is a vector space convergence structure on V.

As mentioned, a convergence space X is topological if and only if it is pre-topological and the adherence operator is idempotent. However, for a convergence vector space to be topological it is sufficient for V to be pre-topological. That is, a convergence vector space is topological if and only if it is pre-topological. Also, a convergence vector space is Hausdorff if and only if the set $\{0\}$ is closed, see [20].

A convergence vector space is equipped with a natural uniform convergence structure, called the *induced uniform convergence structure*, which is denoted as $\mathcal{J}_V$. In this regard, let V be a convergence vector space, and let $\mathcal{U}$ be a filter on $V \times V$. Then

$$\mathcal{U} \in \mathcal{J}_V \Longleftrightarrow \left(\begin{array}{l} \exists \ \ \mathcal{F} \text{ a filter on } V : \\ \quad (1)\mathcal{F} \longrightarrow 0 \\ \quad (2)\Delta(\mathcal{F}) \subseteq \mathcal{U}. \end{array} \right. \tag{1.102}$$

Here $\Delta(\mathcal{F}) = [\{\Delta(F) : F \in \mathcal{F}\}]$ and for any set $F \subseteq V$

$$\Delta(F) = \{(x, y) \in V \times V : x - y \in F\}. \tag{1.103}$$

Lemma 1.57. *Let V be a convergence vector space. Then for all $A, B \subseteq V$ and for all filters $\mathcal{F}, \mathcal{G}$ on V we have*

(i) $\Delta(A \cap B) = \Delta(A) \cap \Delta(B)$.

(ii) $\Delta(A \cup B) = \Delta(A) \cup \Delta(B)$.

(iii) $\Delta(\mathcal{F} \cap \mathcal{G}) = \Delta(\mathcal{F}) \cap \Delta(\mathcal{G})$.

(iv) If $\mathcal{U}$ is a filter on $V \times V$, then the filter $[\{A \subseteq V : \Delta(A) \in \mathcal{U}\}]$ is an ultrafilter if $\mathcal{U}$ is an ultrafilter.

(v) For any $x \in V$,

$$\mathcal{F} \times [x] \supseteq \Delta(\mathcal{G}) \Longrightarrow \mathcal{F} \supseteq \mathcal{G} + x$$

(vi) $\Delta(\mathcal{F}+\mathcal{G}) \subseteq \Delta(\mathcal{F}) \circ \Delta(\mathcal{G})$. *Here*

$$\begin{aligned}\mathcal{F}+\mathcal{G} &= [\{F+G : F\in\mathcal{F}, G\in\mathcal{G}\}] \\ &= [\{\{x+y : x\in F, y\in G\} : F\in\mathcal{F}, G\in\mathcal{G}\}]\end{aligned}$$

(vii) $a(\Delta(\mathcal{F})) \supseteq \Delta(a(\mathcal{F}))$.

The convergence structure induced by the uniform convergence structure $\mathcal{J}_V$ agrees with the vector space convergence structure λ_V, that is, $\lambda_{\mathcal{J}_V} = \lambda_V$. If V and W are convergence vector space and a linear mapping $f : V \longrightarrow W$ is continuous then f is uniformly continuous, see for instance [20, Proposition 2.5.3]. Note that the induced uniform convergence structure of a reciprocal convergence vector space is not in general the associated uniform convergence structure and hence need not be complete.

In a convergence vector space Cauchy filters are characterized as follows: A filter $\mathcal{F}$ on V is a *Cauchy filter* if and only if $\mathcal{F}-\mathcal{F}$ converges to 0. The Wyler completion of unform convergence spaces does not preserve algebraic structures. If V is a convergence vector space carrying its induced uniform convergence structure, then its completion $V^\sharp$ is naturally a convergence vector space. However the uniform structure does not induce a vector convergence structure and so one has to consider its "convergence vector space modification" which has all the desired properties of a convergence vector space completion, see [20, 61], see also [63, 100]. If V is a Hausdorff convergence vector space, it is not possible in general to embed it into a complete convergence vector space, since V may contain unbounded Cauchy filters. However it is possible to modify its completion $V^\sharp$ so that it contains V as a dense subspace, satisfies the universal extension property for linear mappings, and every bounded Cauchy filter converges. See for instance [62]. However if every Cauchy filter $\mathcal{F}$ in V is bounded, that is, there is some $F \in \mathcal{F}$ so that $\mathcal{V}(0)F \longrightarrow 0$, then there is a complete, Hausdorff convergence vector space $V^\sharp$ and a linear embedding $i_V : V \longrightarrow V^\sharp$. such that i_V is dense in $V^\sharp$. Furthermore, for every complete Hausdorff convergence vector space W and every continuous linear mapping $f : V \longrightarrow W$ there exists a continuous linear mapping $f^\sharp : V^\sharp \longrightarrow W$ so that the diagram

$$\begin{array}{ccc} V & \xrightarrow{\quad f \quad} & W \\ {\scriptstyle i_V}\big\downarrow & \nearrow{\scriptstyle f^\sharp} & \\ V^\sharp & & \end{array} \tag{1.104}$$

commutes.

Below are some important examples of complete convergence vector spaces [20].

Examples 1.58.

(i) If X is any convergence space, then $C_c(X)$ is a complete convergence vector space.

(ii) If V is a convergence vector space, then $\mathcal{L}_c(V)$ is a complete convergence vector space.

1.3 Hausdorff Continuous Functions

In this section we discuss *Hausdorff continuous* (H-continuous) extended real interval valued functions defined on a metric space X, see [5, 7, 8, 9, 108, 123]. Interval valued functions are traditionally associated with validated computing, where they naturally appear as error bounds for numerical and theoretical computations, see for instance [1, 72]. Sendov [107], see also [6], introduced the concept of H-continuous functions in connection with Hausdorff approximations of real functions of real a variable.

We now recall the basic notations and concepts involve in H-continuous functions. In this regard, let $\mathbb{IR}$ denote the set of all closed real intervals $[\underline{a}, \overline{a}] = \{x \in \mathbb{R} : \underline{a} \leq x \leq \overline{a}\}$. That is,

$$\mathbb{IR} = \{a = [\underline{a}, \overline{a}] : \underline{a}, \overline{a} \in \mathbb{R}\},$$

and let $\mathbb{IR}^*$ denote the set of all extended, closed real intervals. That is,

$$\mathbb{IR}^* = \{a = [\underline{a}, \overline{a}] : \underline{a}, \overline{a} \in \mathbb{R}^*\},$$

where $\mathbb{R}^* = \mathbb{R} \cup \{\pm\infty\}$ denote the extended real line with the usual ordering. Clearly, $\mathbb{IR} \subset \mathbb{IR}^*$. Given an interval $a = [\underline{a}, \overline{a}] \in \mathbb{IR}^*$, the number $w(a) = \overline{a} - \underline{a}$ is called the width of a, and $|a| = \max\{|\underline{a}|, |\overline{a}|\}$ is called the modulus of a. An interval a is a *proper* interval if $w(a) > 0$ and a *point* interval, if $w(a) = 0$. If we identify $a \in \mathbb{R}^*$ with the point interval $[a, a] \in \mathbb{IR}^*$, then $\mathbb{R}^* \subset \mathbb{IR}^*$. On $\mathbb{IR}^*$ we define the partial order through

$$a \leq b \Longleftrightarrow \underline{a} \leq \underline{b} \text{ and } \overline{a} \leq \overline{b}.$$

Let X be a metric space. Denote by $\mathbb{A}(X)$ the set of all interval valued function defined on X. That is,

$$\mathbb{A}(X) = \{u : X \longrightarrow \mathbb{IR}^*\}.$$

Since $\mathbb{R}^* \subset \mathbb{IR}^*$ we have that

$$\mathcal{A}(X) \subset \mathbb{A}(X),$$

where $\mathcal{A}(X) = \{u : X \longrightarrow \mathbb{R}^*\}$.

On $\mathbb{A}(X)$ we define the pointwise partial order through

$$u \leq v \Longleftrightarrow \left(\begin{array}{ll} \forall & x \in X \\ & u(x) \leq v(x) \end{array} \right. \tag{1.105}$$

Note that if $u \in \mathbb{A}(X)$ then for all $x \in X$ the value of u at x is the interval $[\underline{u}(x), \overline{u}(x)]$. Hence the function u can be written as $u = [\underline{u}, \overline{u}]$ where $\underline{u}, \overline{u} \in \mathcal{A}(X)$ and $\underline{u} \leq \overline{u}$. The concept of a H-continuous function is formulated in terms of extended Baire operators. The extended Baire operators are defined as follows: Let $D \subseteq X$ be dense. For $u \in \mathbb{A}(X)$ and $\eta > 0$ we denote by $I(\eta, D, u)$ the function

$$I(\eta, D, u)(x) = \inf\{\underline{u}(y) | y \in B_\eta(x) \cap D\},\ x \in X,$$

and $S(\eta, D, u)$ the function

$$S(\eta, D, u)(x) = \sup\{\overline{u}(y) | y \in B_\eta(x) \cap D\},\ x \in X.$$

The function $I(D, \cdot) : \mathbb{A}(X) \longrightarrow \mathcal{A}(X)$ is defined by

$$I(D, u)(x) = \sup_{\eta > 0} I(\eta, D, u)(x), \quad x \in X \tag{1.106}$$

and the function $S(D, \cdot) : \mathbb{A}(X) \longrightarrow \mathcal{A}(X)$ is defined by

$$S(D, u)(x) = \inf_{\eta > 0} S(\eta, D, u)(x), \quad x \in X. \tag{1.107}$$

In fact, since

$$I(\eta, D, u)(x) < I(\delta, D, u)(x),\ x \in X$$

and

$$S(\eta, D, u)(x) > S(\delta, D, u)(x),\ x \in X$$

whenever $\eta < \delta$, it follows that

$$I(D, u)(x) = \lim_{\eta \to 0} I(\eta, D, u)(x)\ x \in X$$

and

$$S(D, u)(x) = \lim_{\eta \to 0} S(\eta, D, u)(x)\ x \in X$$

The operators $I(D, \cdot)$ and $S(D, \cdot)$ are called Lower and Upper extended Baire operators respectively.

The operator $F(D, \cdot) : \mathbb{A}(X) \longrightarrow \mathbb{A}(X)$ defined by

$$F(D, u)(x) = [I(D, u)(x), S(D, u)(x)], \quad x \in X \tag{1.108}$$

is called the Graph Completion Operator. In the case when $D = X$ the set D is omitted from the argument and we write

$$I(u)(x) = I(X, u)(x),\ S(u)(x) = S(X, u)(x),\ F(u)(x) = F(X, u)(x)$$

The operators (1.106), (1.107) and (1.108) satisfy the following properties.

(C_1) $I(u) \leq u \leq S(u),\ u \in \mathbb{A}(X)$

(C_2) I, S, F and their compositions are idempotent. That is, for all $u \in \mathbb{A}(X)$,

(i) $I(I(u)) = I(u)$
(ii) $S(S(u)) = S(u)$
(iii) $F(F(u)) = F(u)$
(iv) $(I \circ S)((I \circ S)(u)) = (I \circ S)(u)$

(C_3) I, S, F and their compositions are monotone. That is, for all $u, v \in \mathbb{A}(X)$,

$$u \leq v \Longrightarrow \left(\begin{array}{ll} (i) & I(u) \leq I(v) \\ (ii) & S(u) \leq S(v) \\ (iii) & F(u) \leq F(v) \\ (iv) & (I \circ S)(u) \leq (I \circ S)(v) \end{array} \right.$$

The operator F is monotone with respect to inclusion, that is

$$u(x) \subseteq v(x),\ x \in X \quad \Longrightarrow F(u)(x) \subseteq\ F(v)(x),\ x \in X.$$

Furthermore, it is easy to see that, for $u \in \mathbb{A}(X)$, the functions $I(u)$ and $S(u)$ are lower and upper semi-continuous functions, respectively, on X.

We now define the set of Hausdorff continuous function.

Definition 1.59. *A function $u \in \mathbb{A}(X)$ is called H-continuous if for every function $v \in \mathbb{A}(X)$ which satisfies the inclusion $v(x) \subseteq u(x)$, $x \in X$, we have $F(v)(x) = u(x)$, $x \in X$.*

Denote by $\mathbb{H}(X) \subseteq \mathbb{A}(X)$ the set of all H-continuous functions on X. Clearly all continuous real valued functions are H-continuous, that is $C(X) \subseteq \mathbb{H}(X)$. Indeed, if u is continuous then u is both upper and lower semi-continuous and hence

$$F(u) = [I(u), S(u)] = [u, u] = u$$

Furthermore, let $v \in \mathbb{A}(X)$ be such that $v(x) \subseteq u(x)$. Then $v(x) = u(x), x \in X$ and hence $F(v)(x) = F(u)(x) = u(x), x \in X$ which shows that u is H-continuous. The set $\mathbb{H}(X)$ inherits the partial order (1.105). Equipped with this partial order, the set $\mathbb{H}(X)$ is a complete lattice. That is,

$$\begin{array}{l} \forall \quad A \subseteq \mathbb{H}(X) \\ \exists \quad u_0, v_0 \in \mathbb{H}(X) : \\ \qquad (i) \quad u_0 = \sup A \\ \qquad (ii) \quad v_0 = \inf A \end{array} \tag{1.109}$$

The supremum and infimum in (1.109) may be describe as follows: If

$$\phi : X \ni x \mapsto \sup\{\overline{u}(x) : u \in A\} \in \mathbb{R}^*$$

and

$$\psi : X \ni x \mapsto \inf\{\underline{u}(x) : u \in A\} \in \mathbb{R}^*$$

then

$$u_0 = F(I(S(\phi))), \quad v_0 = F(S(I(\psi)))$$

Below are some examples of H-continuous functions which are not continuous.

Examples 1.60.

(i) Let $X = \mathbb{R}$. The function

$$u(x) = \begin{cases} 1 & \text{if } \ x > 0 \\ [-1, 1] & \text{if } \ x = 0 \\ -1 & \text{if } \ x < 0 \end{cases}$$

is H-continuous.

(ii) Let $X = \{(x,t) : t \geq 0\} \subseteq \mathbb{R}^2$. For $(x,t) \in X$ the function

$$u(x,t) = \begin{cases} 1 & \text{if } \ t \in [0,1), \ \ x < t-1 \\ \frac{x}{t-1} & \text{if } \ t \in [0,1), \ \ x \in [t-1, 0] \\ 0 & \text{if } \ t \in [0,1) \ \ x > 0 \\ 1 & \text{if } \ t \geq 1, \ \ x < \frac{t-1}{2} \\ [0,1] & \text{if } \ t \geq 1, \ \ x = \frac{t-1}{2} \\ 0 & \text{if } \ t \geq 1, \ \ x > \frac{t-1}{2} \end{cases}$$

is H-continuous. This function arises as a shockwave solution of the nonlinear conservation law.

The lower and upper Baire operators can be written in terms of $\underline{u}$ and $\overline{u}$. Indeed, it is clear that

$$I(u) = I(\underline{u}) \text{ and } S(u) = S(\overline{u})$$

Hence

$$F(u) = [I(\underline{u}), S(\overline{u})]$$

Therefore we have that,

$$F(u) = u \Longleftrightarrow \left(\begin{array}{c} \underline{u} = I(\underline{u}), \\ \overline{u} = S(\overline{u}) \end{array} \right) \Longleftrightarrow \begin{cases} \underline{u} \text{ is lower semi} - \text{continuous}, \\ \overline{u} \text{ is upper semi} - \text{continuous} \end{cases}$$

H-continuous functions are characterized as follows:

Theorem 1.61. *[4] Let $u = [\underline{u}, \overline{u}] \in \mathbb{A}(X)$. The following conditions are equivalent:*

(a) The function u is H-continuous.

(b) $F(\underline{u}) = F(\overline{u}) = u$

(c) $S(\underline{u}) = \overline{u}, \;\; I(\overline{u}) = \underline{u}$.

H-continuous functions may be constructed as follows:

Theorem 1.62. *Let* $u \in \mathbb{A}(X)$*. The functions*

$$F(S(I(u)))$$

and

$$F(I(S(u)))$$

are H-continuous.

The set $\mathbb{H}(X)$ of H-continuous functions contains the following three important subsets. The set

$$\mathbb{H}_{ft}(X) = \left\{ u \in \mathbb{H}(X) \;\middle|\; \begin{array}{ll} \forall & x \in X : \\ & u(x) \in \mathbb{IR} \end{array} \right\}, \tag{1.110}$$

of all finite H-continuous functions, the set

$$\mathbb{H}_{nf}(X) = \left\{ u \in \mathbb{H}(X) \;\middle|\; \begin{array}{ll} \exists & \Gamma \subset X \text{ closed nowhere dense} : \\ & x \in X \backslash \Gamma \Longrightarrow u(x) \in \mathbb{IR} \end{array} \right\}, \tag{1.111}$$

of nearly finite H-continuous functions, and the set

$$\mathbb{H}_{b}(X) = \left\{ u \in \mathbb{H}_{ft}(X) \;\middle|\; \begin{array}{ll} \exists & [\underline{a}, \overline{a}] \in \mathbb{IR} : \\ & u(x) \subseteq [\underline{a}, \overline{a}], x \in X \end{array} \right\}, \tag{1.112}$$

of bounded H-continuous functions. Since the functions in $C(X)$ assume values which are finite real numbers, we have the following inclusions:

$$C(X) \subseteq \mathbb{H}_{ft}(X) \subseteq \mathbb{H}(X)$$

and

$$C_b(X) \subseteq \mathbb{H}_b(X) \subseteq \mathbb{H}_{ft}(X) \subseteq \mathbb{H}(X).$$

Here $C_b(X)$ denotes the space of all bounded continuous functions. It has been shown, see [4], that the set $\mathbb{H}_{ft}(X)$ is Dedekind order complete and thus contains the Dedekind order completion of $C(X)$ if X is an arbitrary topological space. If X is a metric space then the space $\mathbb{H}_{ft}(X)$ is the Dedekind order completion of $C(X)$.

1.4 The Order Completion Method

In this section we discuss the Order Completion Method (OCM) for nonlinear PDEs. The OCM is a type independent theory for the existence and basic regularity of solutions to nonlinear PDEs, based on the order completion of partially ordered sets of functions. This theory yields the existence and uniqueness of generalized solutions to arbitrary continuous nonlinear PDEs.

Let us consider a nonlinear PDE of order at most m of the form

$$T(x,D)u(x) = h(x), x \in \Omega. \tag{1.113}$$

Here $\Omega \subseteq \mathbb{R}^n$ is open, and $h \in C^0(\Omega)$. The nonlinear operator $T(x,D)$ is defined in terms of a jointly continuous function

$$F : \Omega \times \mathbb{R}^m \longrightarrow \mathbb{R}$$

by setting

$$T(x,D)u(x) = F(x, u(x), \cdots, D^\alpha u(x), \cdots), \ |\alpha| \leq m, \ x \in \Omega, \tag{1.114}$$

for any $u \in C^m(\Omega)$. We assume that the PDE (1.113) satisfies

$$h(x) \in \text{int}\{F(x,\zeta) | \zeta \in \mathbb{R}^m\}, \ x \in \Omega. \tag{1.115}$$

Under this condition, the following fundamental approximation result holds [120].

Theorem 1.63. *Suppose that* (1.115) *holds. Then for all* $\varepsilon > 0$ *there exists* $\Gamma_\varepsilon \subset \Omega$ *closed and nowhere dense and* $u^\varepsilon \in C^m(\Omega \backslash \Gamma_\varepsilon)$ *such that*

$$h(x) - \varepsilon < T(x,D)u^\varepsilon(x) \leq h(x), \qquad x \in \Omega \backslash \Gamma_\varepsilon.$$

The OCM consists of using the Theorem 1.63, interpreted in appropriate function spaces, to construct solutions of the PDE (1.113). We summarize this construction below, see [10, 11, 93, 103, 104] for a detailed exposition. In this regard, consider the space $C^m_{nd}(\Omega)$ defined as follows: For any integer $0 \leq m < \infty$, set

$$C^m_{nd}(\Omega) = \left\{ u \in \mathcal{A}(\Omega) \;\middle|\; \begin{array}{l} \exists \;\; \Gamma \subset \Omega \text{ closed, nowhere dense :} \\ \quad 1) \;\; u : \Omega \backslash \Gamma \longrightarrow \mathbb{R} \\ \quad 2) \;\; u \in C^m(\Omega \backslash \Gamma) \end{array} \right\} \tag{1.116}$$

Clearly, $C^m(\Omega) \subseteq C^m_{nd}(\Omega), 0 \leq m \leq \infty$. Since the mapping F that defines $T(x,D)$ through (1.114) is continuous, it follows that if $u \in C^m(\Omega \backslash \Gamma)$ with $\Gamma \subset \Omega$ closed nowhere dense, then $T(x,D)u \in C^0(\Omega \backslash \Gamma)$. That is, with the operator $T(x,D)$ we can associate a mapping

$$T(x,D) : C^m_{nd}(\Omega) \longrightarrow C^0_{nd}(\Omega). \tag{1.117}$$

On $C^0_{nd}(\Omega)$ we define an equivalence relation as follows: For any $u, v \in C^0_{nd}(\Omega)$, we have

$$u \sim v \Longleftrightarrow \left\{ \begin{array}{l} \exists \;\; \Gamma \subset \Omega \text{ closed nowhere dense } : \\ \quad 1) \;\; u, v \in C^0(\Omega \backslash \Gamma) \\ \quad 2) \;\; u(x) = v(x), \; x \in \Omega \backslash \Gamma \end{array} \right. \tag{1.118}$$

the quotient space $C^0_{nd}(\Omega)/\sim$ is denoted by $\mathcal{M}^0(\Omega)$. We also introduce an equivalence relation on $C^m_{nd}(\Omega)$ in the following way: For any $u, v \in C^m_{nd}(\Omega)$,

$$u \sim_T v \Longleftrightarrow Tu \sim Tv. \tag{1.119}$$

The space $\mathcal{M}_T^m(\Omega)$ is defined as the quotient space $C_{nd}^m(\Omega)/\sim_T$. The mapping (1.117) induces an injective mapping

$$\widehat{T} : \mathcal{M}_T^m(\Omega) \longrightarrow \mathcal{M}^0(\Omega) \tag{1.120}$$

in a canonical way, so that the diagram

$$\begin{array}{ccc} C_{nd}^m(\Omega) & \xrightarrow{\quad T \quad} & C_{nd}^0(\Omega) \\ \Big\downarrow q_1 & & \Big\downarrow q_2 \\ \mathcal{M}_T^m(\Omega) & \xrightarrow{\quad \widehat{T} \quad} & \mathcal{M}^0(\Omega) \end{array} \tag{1.121}$$

commutes, with q_1 and q_2 canonical quotient mappings associated with the equivalence relations (1.118) and (1.119) respectively. The mapping $\widehat{T}$ is defined as follows: If $U \in \mathcal{M}_T^m(\Omega)$ is the $\sim_T$- equivalence class generated by $u \in C_{nd}^m(\Omega)$, then $\widehat{T}(U)$ is the $\sim_T$- equivalence class generated by Tu.

On the space $\mathcal{M}^0(\Omega)$, define a partial order as follows: For any $H, G \in \mathcal{M}^0(\Omega)$,

$$H \leq G \Longleftrightarrow \left\{ \begin{array}{ll} \exists \;\; h \in H,\; g \in G,\; \Gamma \subset \Omega \text{ closed nowhere dense } : \\ \qquad (1) \;\; h, g \in C^0(\Omega\backslash\Gamma) \\ \qquad (2) \;\; h \leq g \text{ on } \Omega\backslash\Gamma \end{array} \right. \tag{1.122}$$

On the space $\mathcal{M}_T^m(\Omega)$ define a partial order $\leq_T$ through the mapping $\widehat{T}$ as follows: For any $U, V \in \mathcal{M}_T^m(\Omega)$

$$U \leq_T V \Longleftrightarrow \widehat{T}U \leq \widehat{T}V \text{ in } \mathcal{M}^0(\Omega). \tag{1.123}$$

With respect to the partial orders (1.122) and (1.123) on $\mathcal{M}^0(\Omega)$ and $\mathcal{M}_T^m(\Omega)$, respectively, the mapping $\widehat{T}$ is an *order isomorphic embedding* [93]. That is, $\widehat{T}$ is injective and

$$\begin{array}{l} \forall \;\; U, V \in \mathcal{M}_T^m(\Omega) : \\ \quad U \leq_T V \Longleftrightarrow \widehat{T}U \leq \widehat{T} \end{array}$$

According to the McNeille completion Theorem [85], see also [93, Appendix], there exists a unique Dedekind complete partially ordered sets $(\mathcal{M}^0(\Omega)^\sharp, \leq)$ and $(\mathcal{M}_T^m(\Omega)^\sharp, \leq_T)$, and order isomorphic embeddings

$$i_{\mathcal{M}_T^m(\Omega)} : \mathcal{M}_T^m(\Omega) \longrightarrow \mathcal{M}_T^m(\Omega)^\sharp$$

and

$$i_{\mathcal{M}^0(\Omega)} : \mathcal{M}^0(\Omega) \longrightarrow \mathcal{M}^0(\Omega)^\sharp$$

so that the following *universal property* is satisfied: For every order isomorphic embedding

$$S : \mathcal{M}_T^m(\Omega) \longrightarrow \mathcal{M}^0(\Omega)$$

there exists a unique order isomorphic embedding $S^\sharp : \mathcal{M}_T^m(\Omega) \longrightarrow \mathcal{M}^0(\Omega)$ so that the diagram

$$\begin{array}{ccc} \mathcal{M}_T^m(\Omega) & \xrightarrow{\quad S \quad} & \mathcal{M}^0(\Omega) \\ \Big\downarrow i_{\mathcal{M}_T^m(\Omega)} & & \Big\downarrow i_{\mathcal{M}^0(\Omega)} \\ \mathcal{M}_T^m(\Omega)^\sharp & \xrightarrow{\quad S^\sharp \quad} & \mathcal{M}^0(\Omega)^\sharp \end{array} \tag{1.124}$$

commutes. In particular, there exists a unique order isomorphic embedding

$$\widehat{T}^\sharp : \mathcal{M}_T^m(\Omega)^\sharp \longrightarrow \mathcal{M}^0(\Omega)^\sharp,$$

which is an extension of the mapping $\widehat{T}$ so that the diagram

$$\begin{array}{ccc} \mathcal{M}_T^m(\Omega) & \xrightarrow{\quad T \quad} & \mathcal{M}^0(\Omega) \\ \Big\downarrow i_{\mathcal{M}_T^m(\Omega)} & & \Big\downarrow i_{\mathcal{M}^0(\Omega)} \\ \mathcal{M}_T^m(\Omega)^\sharp & \xrightarrow{\quad \widehat{T}^\sharp \quad} & \mathcal{M}^0(\Omega)^\sharp \end{array} \tag{1.125}$$

commutes. In this way we arrive at an extension of the nonlinear PDE (1.113). In this regard, any solution $U^\sharp \in \mathcal{M}_T^m(\Omega)^\sharp$ of the equation

$$\widehat{T}^\sharp U^\sharp = f$$

is considered a generalized solution of (1.113).

The main existence and uniqueness result for solutions of the PDE (1.113) is stated below.

Theorem 1.64. *[93] If the PDE* (1.113) *satisfies the condition* (1.115) *then there exists a unique solution $U^\sharp \in \mathcal{M}_T^m(\Omega)^\sharp$ such that*

$$\widehat{T}^\sharp U^\sharp = f$$

As shown in [4], this generalized solution to the PDE (1.113) may be assimilated with usual Hausdorff continuous functions in $\mathbb{H}_{nf}(\Omega)$. Indeed, the Dedekind order completion $\mathcal{M}^0(\Omega)^\sharp$ of $\mathcal{M}^0(\Omega)$ is order isomorphic with the space $\mathbb{H}_{nf}$ of nearly finite H-continuous functions on Ω. Thus, since

$$\widehat{T}^\sharp : \mathcal{M}_T^m(\Omega) \longrightarrow \mathcal{M}^0(\Omega)^\sharp$$

is an order isomorphic embedding, one may obtain an order isomorphic embedding

$$\widehat{T}_0^\sharp : \mathcal{M}_T^m(\Omega) \longrightarrow \mathbb{H}_{nf}(\Omega)$$

so that $\mathcal{M}_T^m(\Omega)^\sharp$ is order isomorphic with a subspace of $\mathbb{H}_{nf}(\Omega)$.

1.4.1 Main Ideas of Convergence space Completion

One major deficiency of the OCM, as formulated in Section 1.4, is that the spaces of generalized functions containing solutions of a PDE (1.113) may to a large extent depend on the particular nonlinear operator $T(x, D)$. Furthermore, there is no concept of generalized partial derivative for generalized functions.

Recently, [119], [120], [121] these issues were resolved by introducing suitable uniform convergence spaces. Here we recall briefly the main ideas underlying this new approach.

To illustrate the convergence space completion method we introduce normal lower and upper semi-continuous functions which are defined through,

$$u \in \mathcal{A}(\Omega) \text{ is normal lower semi} - \text{continuous at } x_0 \in \Omega \;\Leftrightarrow\; I(S(u(x_0))) = u(x_0),$$

$$u \in \mathcal{A}(\Omega) \text{ is normal upper semi} - \text{continuous at } x_0 \in \Omega \;\Leftrightarrow\; S(I(u(x_0))) = u(x_0).$$

A function is normal lower or normal upper semi-continuous on Ω if it is normal lower or normal upper semi-continuous at every point $x_0 \in \Omega$, [4], [54]. Every continuous function is both normal upper semi continuous and normal lower semi continuous.

Definition 1.65. *A normal lower semi-continuous function is called* nearly finite *whenever the set* $\{x \in \Omega : u(x) \in \mathbb{R}\}$ is open and dense in Ω.

We denote by, $\mathcal{NL}(\Omega)$, the set of all nearly finite normal lower semi-continuous functions on Ω. That is,

$$\mathcal{NL}(\Omega) = \left\{ u \in \mathcal{A}(\Omega) \;\middle|\; \begin{array}{l} (1)(I \circ S)u(x) = u(x) \\ (2)\ \{x \in \Omega : u(x) \in \mathbb{R}\} \text{ is open and dense in } \Omega \end{array} \right\}$$

Note that every continuous, real valued function is nearly finite normal lower semi-continuous. Thus we have that

$$C(\Omega) \subseteq \mathcal{NL}(\Omega).$$

Now consider the space

$$\mathcal{ML}^m(\Omega) = \{u \in \mathcal{NL}(\Omega) : u \in C^m_{nd}(\Omega)\}. \tag{1.126}$$

The space $\mathcal{ML}^m(\Omega)$ is a sublattice of $\mathcal{NL}(\Omega)$. In particular, the space

$$\mathcal{ML}^0(\Omega) = \{u \in \mathcal{NL}(\Omega) : u \in C^0_{nd}(\Omega)\},$$

is σ-order dense in $\mathcal{NL}(\Omega)$. This means for each $u \in \mathcal{NL}(\Omega)$

$$\begin{array}{ll} \exists & (\lambda_n), (\mu_n) \subset \mathcal{ML}^0(\Omega) : \\ & (i) \quad \lambda_n \leq \lambda_{n+1} \leq \mu_{n+1} \leq \mu_n \quad n\mathbb{N}, \\ & (ii) \quad \sup\{\lambda_n : n \in \mathbb{N}\} = u = \inf\{\mu_n : n \in \mathbb{N}\}. \end{array} \tag{1.127}$$

On the space $\mathcal{ML}^0(\Omega)$ we define a uniform convergence structure as follows:

Definition 1.66. Let Λ consists of all nonempty order intervals in $\mathcal{ML}^0(\Omega)$. Let $\mathcal{J}_0$ denote the family of filters on $\mathcal{ML}^0(\Omega) \times \mathcal{ML}^0(\Omega)$ defined as follows:

$$\mathcal{U} \in \mathcal{J}_0 \Longleftrightarrow \left\{ \begin{array}{ll} \exists & k \in \mathbb{N} : \\ \forall & j = 1, \cdots, k : \\ \exists & \Lambda_j = \{I_n^j\} \subseteq \Lambda : \\ \exists & u_j \in \mathcal{NL}(\Omega) : \\ & \begin{array}{ll} (i) & I_{n+1}^j \subseteq I_n^j, \quad n \in \mathbb{N} \\ (ii) & \liminf\limits_{n \longrightarrow \infty} \{I_n^j\} = u_j = \limsup\limits_{n \longrightarrow \infty} \{I_n^j\} \\ (iii) & ([\Lambda_1] \times [\Lambda_1]) \cap \cdots \cap ([\Lambda_k] \times [\Lambda_k]) \subseteq \mathcal{U}. \end{array} \end{array} \right. \tag{1.128}$$

The uniform convergence structure $\mathcal{J}_0$ is uniformly Hausdorff, first countable and induces the convergence structure $\lambda_{\mathcal{J}_0}$ on $\mathcal{ML}^0(\Omega)$ given by

$$\mathcal{F} \in \lambda_{\mathcal{J}_0} \Longleftrightarrow \left\{ \begin{array}{ll} \exists & (\lambda_n), (\mu_n) \subset \mathcal{ML}^0(\Omega) : \\ & \begin{array}{ll} (i) & \lambda_n \leq \lambda_{n+1} \leq \mu_{n+1} \leq \mu_n \quad n\mathbb{N}, \\ (ii) & \sup\{\lambda_n : n \in \mathbb{N}\} = u = \inf\{\mu_n : n \in \mathbb{N}\} \\ (iii) & [\{[\lambda_n, \mu_n] : n \in \mathbb{N}\}] \subset \mathcal{F} \end{array} \end{array} \right. \tag{1.129}$$

We now consider the PDE (1.113). With the operator $T(x, D)$ one may associate a mapping

$$T : \mathcal{ML}^m(\Omega) \longrightarrow \mathcal{ML}^0(\Omega) \tag{1.130}$$

defined by

$$T(u)(x) = (I \circ S)(F(x, u, \cdots, \mathcal{D}^\alpha u, \cdots))(x), \quad x \in \Omega \tag{1.131}$$

where

$$\mathcal{D}^\alpha u = (I \circ S)(D^\alpha u).$$

On the space $\mathcal{ML}^m(\Omega)$ consider the equivalence relation $\sim_T$ induced by T through

$$\begin{array}{ll} \forall & u, v \in \mathcal{ML}^m(\Omega) : \\ & u \sim_T v \Longleftrightarrow Tu = Tv \end{array} \tag{1.132}$$

Denote by $\mathcal{ML}_T^m(\Omega)$ the quotient space $\mathcal{ML}_T^m(\Omega) \backslash \sim_T$. With the mapping (1.130) one may associate in a canonical way an injective mapping

$$\widehat{T} : \mathcal{ML}_T^m(\Omega) \longrightarrow \mathcal{ML}^0(\Omega) \tag{1.133}$$

such that the diagram

$$\begin{array}{ccc} \mathcal{ML}^m(\Omega) & \xrightarrow{\quad T \quad} & \mathcal{ML}^0(\Omega) \\ {\scriptstyle q_T} \Big\downarrow & & \Big\downarrow {\scriptstyle id} \\ \mathcal{ML}_T^m(\Omega) & \xrightarrow{\quad \widehat{T} \quad} & \mathcal{ML}^0(\Omega) \end{array} \tag{1.134}$$

commutes. Here, q_T denotes the canonical quotient map associated with the equivalence relation (1.132) and id is the identity map on $\mathcal{ML}^0(\Omega)$.

On the space $\mathcal{ML}^m_T(\Omega)$ we consider the initial uniform convergence structure $\mathcal{J}_T$ with respect to the mapping $\widehat{T}$: For any filter $\mathcal{U} \in \mathcal{ML}^m_T(\Omega) \times \mathcal{ML}^m_T(\Omega)$

$$\mathcal{U} \in \mathcal{J}_T \Longleftrightarrow (\widehat{T} \times \widehat{T})(\mathcal{U}) \in \mathcal{J}_0 \tag{1.135}$$

Since the mapping $\widehat{T}$ is injective, it follows that the space $\mathcal{ML}^m_T(\Omega)$ is uniformly isomorphic to the subspace $\widehat{T}(\mathcal{ML}^m_T(\Omega))$ of $\mathcal{ML}^0(\Omega)$, see [120]. Thus the mapping $\widehat{T}$ is a uniformly continuous embedding. The Wyler completion of the space $(\mathcal{ML}^0(\Omega), \mathcal{J}_0)$ is the space $\mathcal{NL}(\Omega)$ equipped with the uniform convergence structure $\mathcal{J}_0^\sharp$ defined as follows, see [120].

Definition 1.67. *Let Λ consists of all nonempty order intervals in $\mathcal{ML}^0(\Omega)$. Let $\mathcal{J}_0^\sharp$ denote the family of filters on $\mathcal{NL}(\Omega) \times \mathcal{NL}(\Omega)$ defined as follows*

$$\mathcal{U} \in \mathcal{J}_0^\sharp \Longleftrightarrow \left\{ \begin{array}{ll} \exists & k \in \mathbb{N} : \\ \forall & i = 1, \cdots, k \\ \exists & \Lambda_i = \{I_n^i : n \in \mathbb{N}\} \subseteq \Lambda : \\ \exists & u_i \in \mathcal{NL}(\Omega) : \\ & \quad (i) \quad I_{n+1}^i \subseteq I_n^i \quad n \in \mathbb{N} \\ & \quad (ii) \quad \liminf\limits_{n\to\infty} \{I_n^i\} = u_i = \limsup\limits_{n\to\infty} \{I_n^i\} \\ & \quad (iii) \quad \bigcap\limits_{i=k}^{k} (([\Lambda_i] \times [\Lambda_i]) \cap ([u_i] \times [u_i])) \subseteq \mathcal{U}. \end{array} \right. \tag{1.136}$$

The completion of the space $\mathcal{ML}^m_T(\Omega)$ is denoted by $\mathcal{NL}_T(\Omega)$, and is realized as a subspace of $\mathcal{NL}(\Omega)$. In particular, the mapping $\widehat{T}$ extends uniquely to an injective uniformly continuous mapping

$$\widehat{T}^\sharp : \mathcal{NL}_T(\Omega) \longrightarrow \mathcal{NL}(\Omega).$$

This is summarized in the following commutative diagram.

$$\begin{array}{ccc} \mathcal{ML}^m_T(\Omega) & \xrightarrow{\quad \widehat{T} \quad} & \mathcal{ML}^0(\Omega) \\ \phi \downarrow & & \downarrow \psi \\ \mathcal{NL}_T(\Omega) & \xrightarrow{\quad \widehat{T}^\sharp \quad} & \mathcal{NL}(\Omega) \end{array} \tag{1.137}$$

Here ϕ and ψ are the canonical uniformly continuous embeddings associated with the completions $\mathcal{NL}_T(\Omega)$ and $\mathcal{NL}(\Omega)$, respectively. A first existence and uniqueness result for the generalized solutions of the PDE (1.113) is given below.

Theorem 1.68. *For every* $f \in C^0(\Omega)$ *satisfying* (1.115), *there exists a unique* $U^\sharp \in \mathcal{NL}_T(\Omega)$ *such that*

$$\widehat{T}^\sharp U^\sharp = f.$$

Theorem 1.68 is essentially a reformulation of Theorem 1.64 in the context of uniform convergence spaces.Thus the mentioned deficiencies of the OCM also applies to Theorem 1.68. However, by introducing a parallel construction of spaces of generalized functions, which is independent of the particular nonlinear operator T we may resolve these difficulties. In this regard, we introduce on $\mathcal{ML}^m(\Omega)$ the initial uniform convergence structure $\mathcal{J}_m$ with respect to the partial derivatives

$$D^\alpha : \mathcal{ML}^m(\Omega) \longrightarrow \mathcal{ML}^0(\Omega). \tag{1.138}$$

That is,

$$\mathcal{U} \in \mathcal{J}_m \Longleftrightarrow \left(\begin{array}{ll} \forall & \alpha \in \mathbb{N}^n, \ |\alpha| \leq m : \\ & (D^\alpha \times D^\alpha)(\mathcal{U}) \in \mathcal{J}_0 \end{array} \right.$$

thus each of the mappings (1.138) is uniformly continuous so that the mapping

$$\mathbf{D} : \mathcal{ML}^m(\Omega) \longrightarrow \mathcal{ML}^0(\Omega).$$

is a uniformly continuous embedding, therefore [121] D extends uniquely to an injective, uniformly continuous mapping

$$\mathbf{D}^\sharp : \mathcal{NL}^m(\Omega) \longrightarrow \mathcal{NL}(\Omega)^\mu. \tag{1.139}$$

where $\mathcal{NL}^m(\Omega)$ denotes the completion of $\mathcal{ML}^m(\Omega)$. This gives a first and basic regularity result: The generalized functions in $\mathcal{NL}^n(\Omega)$ may be represented, through their generalized partial derivatives, as normal lower semi-continuous functions. Indeed, the mapping (1.139) may be represented as

$$\mathbf{D}^\sharp(u) = ((D^\alpha)^\sharp)_{|\alpha| \leq m}$$

where $(D^\alpha)^\sharp$ denotes the extension of D^α to $\mathcal{NL}^m(\Omega)$.

Theorem 1.69. *The mapping*

$$T : \mathcal{ML}^m(\Omega) \longrightarrow \mathcal{ML}^0(\Omega)$$

defined in (1.130) - (1.131) *is uniformly continuous.*

In view of Theorem 1.69 the mapping T extends to a unique uniformly continuous mapping

$$T^\sharp : \mathcal{NL}^m(\Omega) \longrightarrow \mathcal{NL}(\Omega)$$

so that the diagram

$$\begin{array}{ccc} \mathcal{ML}^m(\Omega) & \xrightarrow{\quad T \quad} & \mathcal{ML}^0(\Omega) \\ \varphi \downarrow & & \downarrow \psi \\ \mathcal{NL}^m(\Omega) & \xrightarrow{\quad T^\sharp \quad} & \mathcal{NL}(\Omega) \end{array} \tag{1.140}$$

commutes. Here φ and ψ are the uniformly continuous embeddings associated with the completion $\mathcal{NL}^m(\Omega)$ and $\mathcal{NL}(\Omega)$, respectively. The main existence result for the solutions of (1.113) in $\mathcal{NL}(\Omega)$ is the following

Theorem 1.70. *If for each $x \in \Omega$ there is some $\zeta \in \mathbb{R}^m$ and neighbourhoods V and W of x and ζ so that*

$$F(x, \zeta) = f(x)$$

and

$$F : V \times W :\longrightarrow \mathbb{R}$$

is open, then there exists $u^\sharp \in \mathcal{NL}^m(\Omega)$ such that

$$T^\sharp u^\sharp = f$$

The relationship Between Theorem 1.68 and Theorem 1.70 is summarized as follows: If

$$F(x, \cdot) : \mathbb{R}^m \longrightarrow \mathbb{R}$$

is open and surjective for each $x \in \Omega$, then the PDE (1.113) admits generalized solutions $U^\sharp \in \mathcal{NL}^m_T(\Omega)$ and $u^\sharp \in \mathcal{NL}^m(\Omega)$ for every $f \in C^0(\Omega)$ and

$$U^\sharp = \{u^\sharp \in \mathcal{NL}^m(\Omega) | T^\sharp u^\sharp = f\}.$$

Thus the generalized solution in $\mathcal{NL}^m_T(\Omega)$ may be viewed as the set of all solutions in $\mathcal{NL}^m(\Omega)$.

1.5 Summary of the Main Results

In chapter two of this thesis we present the main results obtained. The Order Completion Method, in particular the formulation of this Theory in terms of uniform convergence spaces presented in Section 1.4.1 is modified for single conservation laws in one spatial dimension. The following points are addressed.

- Suitable convergence vector spaces are introduced for the formulation of question of existence and uniqueness of generalized solution of the mentioned conservation law. The completion of this space is described in terms of the set of finite H-continuous functions.

- The issue of existence of generalized solution of conservation laws is formulated in an operator theoretic context. It is shown that each conservation law, with a given initial condition, admits at most one generalized solution.
- Existence of a generalized solution for the Burgers equation is demonstrated. It is also shown that this solution is the entropy solution of the Burgers equation.

Chapter 2

Hausdorff Continuous Solution of Scalar Conservation laws

2.1 Introduction

In this chapter we study the solutions of the initial value problem

$$u_t + (f(u))_x = 0, \quad \text{in } \mathbb{R} \times (0, \infty) \tag{2.1}$$

$$u(x, 0) = u_0(x), \quad x \in \mathbb{R} \tag{2.2}$$

in the context of Order Completion Method, and in particular the formulation and extension of the theory introduced in [118], [120] and [122], see also Sections 1.4 and 1.4.1 in the introduction. In particular, the general theory developed in [120] is adapted so as to deliver the entropy solution of (2.1) - (2.2). In this regard, we introduce suitable convergence vector spaces $\mathcal{M}$ and $\mathcal{N}$. With the initial value problem (2.1)-(2.2) a mapping

$$T : \mathcal{M} \longrightarrow \mathcal{N} \tag{2.3}$$

is associated so that (2.1)-(2.2) may be written as one single equation

$$Tu = h$$

for a suitable $h \in \mathcal{N}$.

The vector space convergence structure on $\mathcal{M}$ and $\mathcal{N}$ are constructed in such a way that the mapping (2.3) is uniformly continuous. In this way we obtain a canonical uniformly continuous extension

$$T^\sharp : \mathcal{M}^\sharp \longrightarrow \mathcal{N}^\sharp$$

of (2.3) to the completions $\mathcal{M}^\sharp$ and $\mathcal{N}^\sharp$ of $\mathcal{M}$ and $\mathcal{N}$, respectively. Any solution $u^\sharp \in \mathcal{M}^\sharp$ of the equation

$$T^\sharp u^\sharp = h \tag{2.4}$$

is interpreted as a generalized solution of the initial value problem (2.1) -(2.2).

The main result presented in this chapter concerns existence, uniqueness and regularity of the solutions of (2.4). In this regard, we prove the following:

(A) Equation (2.4) has at most one solution $u^\sharp \in \mathcal{M}^\sharp$.

(B) $\mathcal{M}^\sharp$ may be identified with a set of H-continuous functions, thus the solution of (2.4) is H-continuous, if it exists.

(C) There exist a solution $u^\sharp \in \mathcal{M}^\sharp$ for the initial value problem (2.1)-(2.2) with

$$f(u) = \frac{(u)^2}{2}.$$

This solution can be identified with the entropy solution of the Burgers equation.

The main novelty of the approach developed here is that the theory of entropy solution of scalar conservation laws is developed in an operator - theoretic setting. In this regard, we may recall, see for instance [105], that weak solutions methods for the solutions of linear and nonlinear PDEs involve an ad hoc extension of a partial differential operator associated with a given PDE. Given topological vector spaces X and Y of sufficiently smooth functions, and a partial differential operator

$$T : X \longrightarrow Y, \tag{2.5}$$

a Cauchy sequence (u_n) in X is constructed so that the sequence (Tu_n) converges to some $h \in Y$. The sequence (u_n) being a Cauchy sequence in X, converges to some $u^\sharp$ in the completion $X^\sharp$ of X. Now, based on the convergence

$$u_n \longrightarrow u^\sharp,\ Tu_n \longrightarrow h$$

of one single sequence (u_n), $u^\sharp$ is declared to be a generalized solution of the PDE

$$Tu = h.$$

This amounts to an ad hoc extension of the mapping (2.5) to a mapping

$$T^\sharp : X \cup \{u^\sharp\} \longrightarrow Y.$$

In the case of a linear PDE this approach turns out to be well founded, due to the automatic continuity of certain linear mappings on topological vector spaces. However, in the case of non-linear PDEs such methods can, and often do, lead to non-linear stability paradoxes, see for instance [105, Chapter1, Section 8]. The result obtained in this chapter places the theory of entropy solutions of conservation laws on a firm operator - theoretic footing.

The rest of this Chapter is organized as follows. In Section 2.2 we discuss the convergence vector spaces used in our result. The approximation result needed for the existence of a solution is discussed in Section 2.3. Existence and uniqueness results for the Burgers equation are presented in Section 2.4.

2.2 Convergence Vector Spaces for Conservation Laws

As mentioned, the novelty in the approach in this section to the conservation law (2.1) - (2.2) is based mostly on the different way of constructing the operator equation $Tu = h$ associated with the conservation law. It is essential for this development that the classical solution of the problem (2.1) - (2.2) is unique whenever it exists. We present below a precise formulation of the uniqueness result. In this regard, we now define the following convergence vector spaces. Let

$$\mathcal{M} = \{u \in C^1(\mathbb{R} \times (0,\infty)) \cap C^0(\mathbb{R} \times [0,\infty)) : u(\cdot,0) \in \mathcal{U}_0\} \tag{2.6}$$

and

$$\mathcal{N} = C^0(\mathbb{R} \times (0,\infty)) \times \mathcal{U}_0 \tag{2.7}$$

where $\mathcal{U}_0$ is a set of initial conditions. In the literature $\mathcal{U}_0$ is defined in different ways. Here we take

$$\mathcal{U}_0 = \{h \in C^0(\mathbb{R}) : \lim_{x\longrightarrow\infty} h(x), \quad \lim_{x\longrightarrow-\infty} h(x) \text{ exist}\} \tag{2.8}$$

The following result is an extended formulation of Theorem 6.2 in [28].

Theorem 2.1. *Let f be Liptschitz on compacta. For $u, v \in \mathcal{M}$ set*

$$\phi = u_t + (f(u))_x \tag{2.9}$$
$$\psi = v_t + (f(v))_x. \tag{2.10}$$

Then there exists L such that

$$\int_a^b |v(x,t) - u(x,t)|\, dx \le \int_{a-Lt}^{b+Lt} |u(x,0) - v(x,0)|\, dx + \int_0^t \int_{a-Lt}^{b+Lt} |\phi - \psi|\, dxdt. \tag{2.11}$$

Proof. Since $\lim_{x\longrightarrow\infty} u(x,0)$, $\lim_{x\longrightarrow-\infty} u(x,0)$, $\lim_{x\longrightarrow\infty} v(x,0)$ and $\lim_{x\longrightarrow-\infty} v(x,0)$ exist, then u, v are both bounded. Let $u(x,t), v(x,t) \in [-d,d]$, $x \in \mathbb{R}$. Using the fact that f is Liptschitz on compacta there exists L such that

$$|f(w) - f(w')| \le L\,|w - w'| \quad \text{for } w,\ w' \in [-d,d]. \tag{2.12}$$

From (2.9)-(2.10) we have that

$$\psi - \phi = (v-u)_t + (f(v) - f(u))_x. \tag{2.13}$$

Multiply equation (2.13) by the function

$$\operatorname{sgn}(v-u) = \begin{cases} 1 & if\ v-u > 0 \\ -1 & if\ v-u < 0 \\ 0 & if\ u = v \end{cases}$$

to get

$$(\psi - \phi)\text{sgn}(v-u) = |v-u|_t + ((f(v) - f(u))_x)\text{sgn}(v-u),$$

which can be written as

$$|v-u|_t + ((f(v) - f(u))\text{sgn}(v-u))_x = (\psi - \phi)\text{sgn}(v-u). \tag{2.14}$$

Now integrate (2.14) over the trapezium

$$D = \left\{ (x,t) \in \mathbb{R} \times [0,\infty) \;\middle|\; \begin{array}{l} 0 \leq t \leq \tau; \\ a - L(\tau - t) \leq x \leq b + L(\tau - t) \end{array} \right\}, \tag{2.15}$$

for arbitrary fixed $\tau > 0$. Then

$$\iint_D (|v-u|_t)\, dxdt + \iint_D (((f(v) - f(u))\text{sgn}(v-u))_x)\, dxdt$$
$$= \iint_D (\psi - \phi)\text{sgn}(v-u) dxdt. \tag{2.16}$$

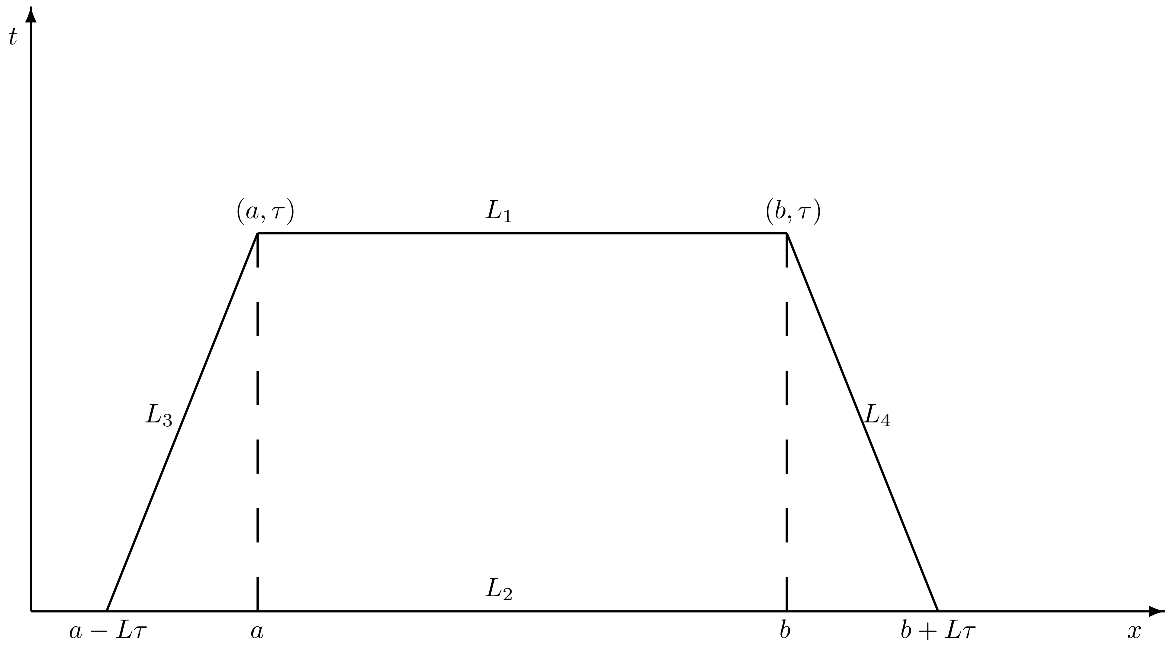


Figure 2.1

Apply Green's Theorem to the left hand side of equation (2.16). Note that, see Figure 2.1, on

$$\begin{aligned} &L_1 : t = \tau, \quad dt = 0 \\ &L_2 : t = 0, \quad dt = 0 \\ &L_3 : x = a + L(t - \tau), \quad dx = Ldt \\ &L_4 : x = b - L(t - \tau), \quad dx = -Ldt. \end{aligned}$$

Therefore

$$\iint\limits_D (|v-u|_t)\,dxdt + \iint\limits_D (((f(v)-f(u))\mathrm{sgn}(v-u))_x)\,dxdt$$
$$= \iint\limits_D (|v-u|_t - (-((f(v)-f(u))\mathrm{sgn}(v-u))_x))\,dxdt$$
$$= \oint\limits_{L_1+L_2+L_3+L_4} |v-u|\,dx - ((f(v)-f(u))\mathrm{sgn}(v-u))dt$$
$$= \int_a^b |v(x,\tau)-u(x,\tau)|\,dx - \int_{a-L\tau}^{b+L\tau} |v_0(x)-u_0(x)|\,dx$$
$$+ \int_0^\tau (L\,|v(C_3(t),t)-u(C_3(t),t)|)dt$$
$$- \int_0^\tau (((f(v((C_3(t),t))) - f(u((C_3(t),t))))\mathrm{sgn}(v-u))dt$$
$$+ \int_0^\tau (L\,|v(C_4(t),t)-u(C_4(t),t)|)dt$$
$$- \int_0^\tau (((f(v(C_4(t),t)) - f(u(C_4(t),t)))\mathrm{sgn}(v-u))dt$$

where

$$C_3(t) = a + L(t-\tau), \qquad C_4(t) = b - L(t-\tau).$$

That is,

$$\iint\limits_D (|v-u|_t)\,dxdt + \iint\limits_D (((f(v)-f(u))\mathrm{sgn}(v-u))_x)\,dxdt$$
$$= \int_a^b |v(x,\tau)-u(x,\tau)|\,dx - \int_{a-L\tau}^{b+L\tau} |v_0(x)-u_0(x)|\,dx$$
$$+ \int_0^\tau (L\,|v(C_3(t),t)-u(C_3(t),t)|)dt$$
$$- \int_0^\tau (((f(v((C_3(t),t))) - f(u((C_3(t),t))))\mathrm{sgn}(v-u))dt$$
$$+ \int_0^\tau (L\,|v(C_4(t),t)-u(C_4(t),t)|)dt$$
$$- \int_0^\tau (((f(v(C_4(t),t)) - f(u(C_4(t),t)))\mathrm{sgn}(v-u))dt$$

Using the inequality (2.12) we see that

$$L\,|v(C_3(t),t)-u(C_3(t),t)|$$
$$-((f(v((C_3(t),t))) - f(u((C_3(t),t))))\mathrm{sgn}(v-u)) \geq 0$$

and

$$L\ |v(C_4(t),t) - u(C_4(t),t)| \\ -\left((f(v(C_4(t),t)) - f(u(C_4(t),t)))\mathrm{sgn}(v-u)\right) \geq 0$$

Therefore

$$\iint\limits_D \left(|v-u|_t\right) dxdt + \iint\limits_D \left(((f(v)-f(u))\mathrm{sgn}(v-u))_x\right) dxdt \\ \geq \int_a^b |v(x,\tau) - u(x,\tau)|\, dx - \int_{a-L\tau}^{b+L\tau} \left|v^0(x) - u^0(x)\right| dx,$$

which further implies that

$$\iint\limits_D (\psi - \phi)\mathrm{sgn}(v-u) dxdt \\ = \iint\limits_D \left(|v-u|_t + ((f(v)-f(u))\mathrm{sgn}(v-u))_x\right) dxdt \\ \geq \int_a^b |v(x,\tau)-u(x,\tau)|\, dx - \int_{a-L\tau}^{b+L\tau} \left|v^0(x) - u^0(x)\right| dx.$$

Thus we obtain the inequality

$$\int_a^b |v(x,\tau) - u(x,\tau)|\, dx \\ \leq \int_{a-L\tau}^{b+L\tau} \left|v^0(x) - u^0(x)\right| dx + \iint_D (\psi-\phi)\mathrm{sgn}(v-u) dxdt \\ \leq \int_{a-L\tau}^{b+L\tau} \left|v^0(x) - u^0(x)\right| dx + \iint_D |\psi - \phi|\, dxdt$$

as required. □

Consider the operator

$$T : \mathcal{M} \longrightarrow \mathcal{N} \tag{2.17}$$

defined by

$$Tu = \begin{pmatrix} u_t + (f(u))_x \\ u(\cdot, 0) \end{pmatrix} = \begin{pmatrix} T_1 u \\ T_2 u \end{pmatrix} \tag{2.18}$$

The mentioned uniqueness of a classical solution of (2.1) -(2.2) is extended in the following way.

Lemma 2.2. *The operator T is injective*

Proof. The injectivity of the operator T follows from Theorem 2.1. Indeed, let

$$Tu = Tv$$

for some $u, v \in \mathcal{M}$. Then for any $t > 0$, $a, b \in \mathbb{R}$, $a < b$ we have

$$\begin{aligned}\int_a^b |v(x,t) - u(x,t)|\, dx &\leq \int_{a-Lt}^{b+Lt} |T_2u - T_2v|\, dx \\ &+ \int_0^t \int_{a-Lt}^{b+Lt} |T_1u - T_1v|\, dxdt \\ &= 0.\end{aligned}$$

By the continuity of u and v this implies $u = v$.

□

Convergence Structures on $\mathcal{M}$ and $\mathcal{N}$

On the considered spaces $\mathcal{M}$ and $\mathcal{N}$ we define the respective convergence structures as follows: On $\mathcal{M}$ we consider the following convergence structure which we denote as λ_1. Given a filter $\mathcal{F}$ on $\mathcal{M}$, we have

$$\mathcal{F} \in \lambda_1(u) \Longleftrightarrow \begin{cases} \exists \;\; (\alpha_n),\;\; (\beta_n) \subseteq C^0(\mathbb{R} \times [0,\infty)) : \\ \qquad (i) \quad \alpha_n \leq \alpha_{n+1} \leq u \leq \beta_{n+1} \leq \beta_n, n \in \mathbb{N} \\ \qquad (ii) \quad \displaystyle\int_a^b (\beta_n(x,t) - \alpha_n(x,t))dx \longrightarrow 0 \\ \qquad\qquad \text{for } t \geq 0,\;\; a, b \in \mathbb{R},\;\; a \leq b \\ \qquad (iii) \quad [\{[\alpha_n, \beta_n] : n \in \mathbb{N}\}] \subseteq \mathcal{F}. \end{cases} \tag{2.19}$$

Here the interval $[\alpha_n, \beta_n]$ is considered in $\mathcal{M}$ with respect to the usual point-wise order, that is, $[\alpha_n, \beta_n] = \{v \in \mathcal{M} : \alpha_n(x,t) \leq v(x,t) \leq \beta_n(x,t),\; x \in \mathbb{R},\; t \in [0,\infty)\}$

Proposition 2.3. *The convergence structure λ_1 is a Hausdorff vector space convergence structure.*

Proof. We first show that λ_1 is a convergence structure on $\mathcal{M}$ by showing that λ_1 satisfies the definition of a convergence structure given in (1.36).

(i) Consider $u \in \mathcal{M}$. In (2.19) set $\alpha_n = \beta_n = u$ for all $n \in \mathbb{N}$. We see that conditions (2.19)(i) and (ii) are satisfied, and $[\{[\alpha_n, \beta_n] : n \in \mathbb{N}\}] = [u]$. Therefore $[u] \in \lambda_1$.

(ii) Let $\mathcal{F}, \mathcal{G} \in \lambda_1(u)$ be filters on $\mathcal{M}$. Then there exist sequences $(\alpha_n^{(1)}), (\beta_n^{(1)})$ and $(\alpha_n^{(2)}), (\beta_n^{(2)})$ on $C^0(\mathbb{R}, [0,\infty))$, converging to the same limit, which can be associated with filters $\mathcal{F}$ and $\mathcal{G}$ respectively according to (2.19). Denote

$\alpha_n = \inf\{\alpha_n^{(1)}, \alpha_n^{(2)}\}$, $\beta_n = \sup\{\beta_n^{(1)}, \beta_n^{(2)}\}, n \in \mathbb{N}$. Clearly, $\alpha_n \leq \alpha_{n+1} \leq u \leq \beta_{n+1} \leq \beta_n$. Moreover, we have

$$\forall \;\; a, b \in \mathbb{R}, a \leq b$$
$$\int_a^b \beta_n(x,t) - \alpha_n(x,t)dx \longrightarrow 0.$$

Furthermore, we have that

$$[\alpha_n^1, \beta_n^1] \subseteq [\alpha_n, \beta_n]$$

and

$$[\alpha_n^2, \beta_n^2] \subseteq [\alpha_n, \beta_n].$$

Therefore,

$$[\alpha_n^1, \beta_n^1] \cup [\alpha_n^2, \beta_n^2] \subseteq [\alpha_n, \beta_n],$$

which implies

$$[\{[\alpha_n, \beta_n] : n \in \mathbb{N}\}] \subseteq \mathcal{F} \cap \mathcal{G}.$$

(iii) Let $\mathcal{F} \in \lambda_1(u)$. Let $\mathcal{G}$ be a filter finer than $\mathcal{F}$. Then there exist sequences $(\alpha_n), (\beta_n)$ on $C^0(\mathbb{R}, [0, \infty))$ satisfying (2.19)(i), (ii) and $[\{[\alpha_n, \beta_n] : n \in \mathbb{N}\}] \subseteq \mathcal{F} \subseteq \mathcal{G}$. Hence $\mathcal{G} \in \lambda_1(u)$.

It follows from (i) - (iii) above that λ_1 is a convergence structure.

Next, we show that addition and scalar multiplication are continuous. In this regard, let $\mathcal{F} \longrightarrow u$ and $\mathcal{G} \longrightarrow v$ with respect to λ_1. Then there exist sequences

$$(\alpha_n^1), (\beta_n^1) \;\; \subset C^0(\mathbb{R} \times [0, \infty))$$

that can be associated with the filter $\mathcal{F}$ according to (2.19) and sequences

$$(\alpha_n^2), (\beta_n^2) \;\; \subset C^0(\mathbb{R} \times [0, \infty))$$

that can be associated with the filter $\mathcal{G}$ according to (2.19). Therefore, we have

(a) from (2.19)(i) we have

$$\alpha_n^1 + \alpha_n^2 \leq \alpha_{n+1}^1 + \alpha_{n+1}^2 \leq u + v \leq \beta_{n+1}^1 + \beta_{n+1}^2 \leq \beta_n^1 + \beta_n^2.$$

(b) From (2.19)(ii) we have

$$\int_a^b (\beta_n^1(x,t) + \beta_n^2(x,t) - \alpha_n^1(x,t) - \alpha_n^2(x,t))dx$$
$$= \int_a^b (\beta_n^1(x,t) - \alpha_n^1(x,t)) + \int_a^b (\beta_n^2(x,t) - \alpha_n^2(x,t))dx \longrightarrow 0.$$

(c) From (2.19)(iii) we have that

$$\begin{aligned} &\forall \;\; n \in \mathbb{N} \\ &\exists \;\; F \in \mathcal{F} \\ &\exists \;\; G \in \mathcal{G} : \\ &\quad\; F \subseteq [\alpha_n^1, \beta_n^1] \text{ and } G \subseteq [\alpha_n^2, \beta_n^2] \end{aligned}$$

Hence,

$$
\begin{aligned}
F + G &\subseteq [\alpha_n^1, \beta_n^1] + [\alpha_n^2, \beta_n^2] \\
&\subseteq [\alpha_n^1 + \alpha_n^2, \beta_n^1 + \beta_n^2 - \alpha_n^1(x,t)]
\end{aligned}
$$

so that

$$[\{[\alpha_n^1 + \alpha_n^2, \beta_n^1 + \beta_n^2 - \alpha_n^1(x,t)] : n \in \mathbb{N}\}] \subseteq \mathcal{F} + \mathcal{G}.$$

It thus follows from (a) - (c) above that $\mathcal{F} + \mathcal{G} \in \lambda_1(u+v)$, which shows that addition is continuous.

To show that scalar multiplication is continuous, let $\mathcal{F} \in \lambda_1(u)$ and let $(\alpha_n), (\beta_n)$ on $C^0(\mathbb{R}, [0,\infty))$ be sequences associated with $\mathcal{F}$ according to (2.19). Then for any constant $c \in \mathbb{R},\ c \geq 0$ we have

(a) $c\alpha_n \leq c\alpha_{n+1} \leq cu \leq c\beta_{n+1} \leq c\beta_n$

(b) $\forall\ t \geq 0,\ a, b \in \mathbb{R},\ a \leq b$

$$\int_a^b (c\beta_n(x,t) - c\alpha_n(x,t))dx = \int_a^b c(\beta_n(x,t) - \alpha_n(x,t))dx \longrightarrow 0$$

(c) $\forall\ n \in \mathbb{N}\ \exists\ F \in \mathcal{F}$ such that $cF \subseteq [c\alpha_n, c\beta_n]$. Which implies $[\{[c\alpha_n, c\beta_n] : n \in \mathbb{N}\}] \subseteq c\mathcal{F}$.

The case $c < 0$ is treated in a similar way. Thus, $c\mathcal{F} \in \lambda_1(cu)$ which implies that scalar multiplication is continuous.

We now show that λ_1 is Hausdorff. To do this we need to show that the set $\{0\}$ is closed. Let $u \in a(\{0\})$. Then

$$
\begin{aligned}
\exists\ \ &\mathcal{F} \longrightarrow u : \\
&\{0\} \in \mathcal{F}
\end{aligned}
$$

Since $\mathcal{F} \longrightarrow u$, it follows that there exist sequences $(\alpha_n), (\beta_n) \subseteq C^0(\mathbb{R} \times [0,\infty))$ satisfying

$$
\begin{aligned}
&(i) && \alpha_n \leq \alpha_{n+1} \leq u \leq \beta_{n+1} \leq \beta_n, n \in \mathbb{N} \\
&(ii) && \int_a^b (\beta_n(x,t) - \alpha_n(x,t))dx \longrightarrow 0 \\
& && \qquad\qquad \forall\ t \geq 0,\ a, b \in \mathbb{R},\ a \leq b \\
&(iii) && [\{[\alpha_n, \beta_n] : n \in \mathbb{N}\}] \subseteq \mathcal{F}.
\end{aligned}
\tag{2.20}
$$

But $\mathcal{F} \subseteq [0]$, which implies, from (2.20)(iii), that

$$[\{[\alpha_n, \beta_n] : n \in \mathbb{N}\}] \subseteq \mathcal{F} \subseteq [0].$$

Which means,

$$
\begin{aligned}
\forall\ \ &n \in \mathbb{N} \\
\exists\ \ &A \ni 0 : \\
&A \subseteq [\alpha_n, \beta_n].
\end{aligned}
$$

This implies $\alpha_n \leq 0 \leq \beta_n$. Taking limit as $n \longrightarrow \infty$ we have $u = 0$. Hence $a(\{0\}) = \{0\}$, which implies that $\{0\}$ is closed. This completes the proof. □

Let us recall that given a filter $\mathcal{F}$ on $\mathcal{M}$ and $u \in \mathcal{M}$, $\mathcal{F}$ converges to u with respect to the subspace convergence structure induced on $\mathcal{M}$ by the order convergence structure, see Example 1.42, whenever

$$\begin{array}{ll} \exists \text{ sequences } (\alpha_n), (\beta_n) \subset C^0(\mathbb{R} \times [0, \infty)) : & \\ \quad (i) \quad \alpha_n \leq \alpha_{n+1} \leq \beta_{n+1} \leq \beta_n & \\ \quad (ii) \quad \sup\{\alpha_n : n \in \mathbb{N}\} = u = \inf\{\beta_n : n \in \mathbb{N}\} & \\ \quad (iii) \quad [\{[\alpha_n, \beta_n] : n \in \mathbb{N}\}] \subseteq \mathcal{F}. & \end{array} \tag{2.21}$$

where the infimum and the supremum are both taken in $C^0(\mathbb{R} \times [0, \infty))$. Denote this induced convergence structure on $\mathcal{M}$ by λ_s. The convergence structure λ_1 on $\mathcal{M}$ is closely related to λ_s. In this regard, we have the following.

Lemma 2.4. *Let $\mathcal{F}$ converge to u with respect to λ_1. Then $\mathcal{F}$ converge to u with respect to the convergence structure λ_s.*

Proof. Let (α_n) and (β_n) be sequences associated with $\mathcal{F}$ according to (2.19). Conditions (2.21)(i)and (2.21)(iii) follows from (2.19)(i) and (2.19)(iii), respectively. It follows from (2.19)(i) that $u \in \mathcal{M} \subset C^0(\mathbb{R} \times [0, \infty))$ is an upper bound of $\{\alpha_n : n \in \mathbb{N}\}$ and a lower bound of $\{\beta_n : n \in \mathbb{N}\}$ in $C^0(\mathbb{R} \times [0, \infty))$. We need to show that it is the least upper bound for $\{\alpha_n : n \in \mathbb{N}\}$ and the greatest lower bound for $\{\beta_n : n \in \mathbb{N}\}$. Let v be an upper bound for $\{\alpha_n : n \in \mathbb{N}\}$ and w be a lower bound of $\{\beta_n : n \in \mathbb{N}\}$ in $C^0(\mathbb{R} \times [0, \infty))$ such that $v \leq u \leq w$. Then for any $a, b \in \mathbb{R},\ a \leq b$, and $t \geq 0$ we have

$$\int_a^b (u(x,t) - v(x,t))dx \leq \int_a^b (\beta_n(x,t) - \alpha_n(x,t))dx \longrightarrow 0 \text{ as } n \longrightarrow 0.$$

Therefore

$$\int_a^b (u(x,t) - v(x,t))dx = 0.$$

Using the continuity of $u - v$ and the fact that $u(x,t) - v(x,t) \geq 0$ for all $x \in [a, b]$ and $t \geq 0$ we obtain $u = v$. Similarly,

$$\int_a^b (w(x,t) - u(x,t))dx = 0$$

which implies $w = v$. Using the fact that $\sup\{\alpha_n : n \in \mathbb{N}\} \leq u$, $\inf\{\beta_n : n \in \mathbb{N}\} \geq u$ and v, w are arbitrary we obtain $\sup\{\alpha_n : n \in \mathbb{N}\} = u = \inf\{\beta_n : n \in \mathbb{N}\}$. This completes the proof. □

Note that the convergence structure λ_1 is finer than the convergence structure λ_s. Indeed, it follows from Lemma 2.4 that $\lambda_1(u) \subset \lambda_s(u)$.

The convergence vector space $\mathcal{M}$ is equipped with the induced uniform convergence structure $\mathcal{J}_{\mathcal{M}}$ defined as follows, see (1.102): Let $\mathcal{U}$ be a filter on $\mathcal{M} \times \mathcal{M}$.

Then

$$\mathcal{U} \in \mathcal{J}_{\mathcal{M}} \Longleftrightarrow \left(\begin{array}{l} \exists \;\; \mathcal{F} \text{ a filter on } \mathcal{M} : \\ \quad (1)\mathcal{F} \in \lambda_1(0) \\ \quad (2)\Delta(\mathcal{F}) \subseteq \mathcal{U} \end{array} \right. \tag{2.22}$$

Lemma 2.5. *The operator $T : \mathcal{M} \longrightarrow \mathcal{N}$ is uniformly continuous with respect to the vector space convergence structures defined on $\mathcal{N}$ as follows:*

$$\mathcal{F} \longrightarrow (u,h) \Longleftrightarrow \left(\begin{array}{l} \pi_1(\mathcal{F}) \longrightarrow u \text{ weakly in } L^1 \\ \pi_2(\mathcal{F}) \longrightarrow h \text{ in } L^1_{loc}. \end{array} \right.$$

Here π_1 is the projection on $C^0(\mathbb{R} \times (0,\infty))$ and π_2 is the projection on $\mathcal{U}_0$.

Proof. We need to show that

$$\begin{array}{l} \forall \;\; \mathcal{F} \longrightarrow 0 \text{ in } \mathcal{M} \\ \exists \;\; \mathcal{G} \longrightarrow 0 \text{ in } \mathcal{N} : \\ \quad (T \times T)(\Delta(\mathcal{F})) \supseteq \Delta(\mathcal{G}). \end{array} \tag{2.23}$$

Let $\mathcal{F} \longrightarrow 0$ in $\mathcal{M}$ and let (α_n) and (β_n) be sequences associated with $\mathcal{F}$ according to (2.19). It is sufficient to prove that $(T \times T)(\Delta([\{[\alpha_n, \beta_n] : n \in \mathbb{N}\}] \supseteq \Delta(\mathcal{G})$ for some filter $\mathcal{G}$ in $\mathcal{N}$ such that $\mathcal{G} \longrightarrow 0$ in $\mathcal{N}$. Let $\phi \in C_0^\infty(\mathbb{R} \times [0,\infty))$ be a test function. Let $d = \max\{ -\min_{(x,t)\in \text{ supp } \phi} \alpha_1(x,t), \max_{(x,t)\in \text{ supp } \phi} \beta_1(x,t)\}$. Then using (2.19)(i) we have that $\alpha_n(x,t) \in [-d,d]$, $\beta_n(x,t) \in [-d,d]$, for $(x,t) \in$ supp ϕ. Let $-d \leq u,v \leq d$. Then there exists L_ϕ such that

$$|f(u(x,t)) - f(v(x,t))| \leq L_\phi \, |u(x,t) - v(x,t)| \text{ for } (x,t) \in \text{ supp } \phi \tag{2.24}$$

where L_ϕ is the lipschitz constant of f on the compact interval $[-d,d]$.

For any $n \in \mathbb{N}$ define

$$G_n = \left\{ g \in \mathcal{N} \;\middle|\; \begin{array}{l} \left| \int\limits_{-\infty}^{\infty} \int\limits_{0}^{\infty} \pi_1(g)(x,t)\phi dxdt \right| \\ \qquad \leq \iint\limits_{\Omega} (|\phi_t| + L_\phi \, |\phi_x|) \, (\beta_n - \alpha_n) \, dxdt, \\ \qquad\qquad \text{for all } \phi \in C_0^\infty(\mathbb{R} \times [0,\infty)) \; \phi \\ \left| \int\limits_a^b \pi_2(g)(x)dx \right| \leq \int\limits_a^b (\beta_n(x,0) - \alpha_n(x,0))dx \\ \qquad\qquad \text{for any } a,b \in \mathbb{R}, a \leq b \end{array} \right\}. \tag{2.25}$$

Let

$$\mathcal{G} = [\{G_n : n \in \mathbb{N}\}]. \tag{2.26}$$

From (2.25) we see that the filter $\mathcal{G} \longrightarrow 0$ in $\mathcal{N}$. It remains to show that the inclusion in (2.23) holds. Equivalently, we need to show that

$$\begin{array}{ll} \forall & G \in \mathcal{G} \\ \exists & F \in \mathcal{F}: \\ & (T \times T)(\Delta(F)) \subseteq \Delta(G). \end{array}$$

For $n \in \mathbb{N}$ let $(u, v) \in \Delta([\alpha_n, \beta_n])$, that is $u - v \in [\alpha_n, \beta_n]$. Then for every test function ϕ and real intervals $[a, b]$ we have

$$\begin{aligned} \left| \iint\limits_{\Omega} (T_1 v - T_1 u) \phi dx dt \right| &= \left| \iint\limits_{\text{supp } \phi} ((v-u)_t + (f(v) - f(u))_x) \phi dx dt \right| \\ &= \left| \iint\limits_{\text{supp } \phi} (v-u)\phi_t + (f(v) - f(u))\phi_x dx dt \right| \\ &\leq \iint\limits_{\text{supp } \phi} (|v-u|\,|\phi_t| + |f(v) - f(u)|\,|\phi_x|) dx dt. \\ &\leq \iint\limits_{\text{supp } \phi} (|\phi_t| + L\,|\phi_x|)\,|v-u|\, dx dt. \\ &\leq \iint\limits_{\text{supp } \phi} (|\phi_t| + L\,|\phi_x|)(\beta_n - \alpha_n) dx dt \\ &= \iint\limits_{\Omega} (|\phi_t| + L\,|\phi_x|)(\beta_n - \alpha_n) dx dt. \end{aligned}$$

where $\Omega = \mathbb{R} \times [0, \infty)$, and

$$\begin{aligned} \left| \int_a^b (T_2 v - T_2 u) dx \right| &= \left| \int_a^b (v(x,0) - u(x,0)) dx \right| \\ &\leq \int_a^b (|v(x,0) - u(x,0)|) dx \\ &\leq \int_a^b (|v(x,0) - u(x,0)|) dx \\ &\leq \int_a^b (\beta_n(x,0) - \alpha_n(x,0)) dx. \end{aligned}$$

Therefore,

$$(T \times T)(u, v) = (Tu, Tv) \in \{(p, q) : p - q \in G_n\} = \Delta(G_n),$$

which implies that $(T \times T)\Delta([\alpha_n, \beta_n]) \subseteq \Delta(G_n)$. Hence

$$(T \times T)(\Delta([\{[\alpha_n, \beta_n] : n \in \mathbb{N}\}])) \supseteq \Delta([\{g_n : n \in \mathbb{N}\}]) = \Delta(\mathcal{G}).$$

This completes the proof. □

Corollary 2.6. *Let $\mathcal{F}$ be a Cauchy filter on $\mathcal{M}$. Then*

$$\begin{pmatrix} \pi_1(T(\mathcal{F})) \text{ is weakly } L^1 \text{ Cauchy} \\ \pi_2(T(\mathcal{F})) \text{is Cauchy in } L^1_{loc}. \end{pmatrix}$$

On the space $\mathcal{N}$ we consider the final uniform convergence structure $\mathcal{J}_{\mathcal{N},T}$ which is defined as follows:

$$\mathcal{U} \in \mathcal{J}_{\mathcal{N},T} \Longleftrightarrow \left(\begin{array}{ll} \exists & \mathcal{V} \in \mathcal{J}_{\mathcal{M}} : \\ \exists & \phi_1 \cdots \phi_k \in \mathcal{N} : \\ & (T \times T)(\mathcal{V}) \cap ([\phi_1] \times [\phi_1]) \cap \cdots \cap ([\phi_k] \times [\phi_k]) \subseteq \mathcal{U}. \end{array} \right. \tag{2.27}$$

Proposition 2.7. *The uniform convergence structure $\mathcal{J}_{\mathcal{N},T}$ is Hausdorff.*

Proof. We need to show that

$$\begin{array}{ll} \forall & \phi, \psi \in \mathcal{N},\ \phi \neq \psi : \\ \forall & \mathcal{U} \in \mathcal{J}_{\mathcal{N},T} : \\ \exists & U \in \mathcal{U} : \\ & (\phi, \psi) \notin U. \end{array}$$

Let $\phi, \psi \in \mathcal{N}$ be such that $\phi \neq \psi$. Set

$$\mathcal{U} = (T \times T)(\mathcal{V}) \cap ([\phi_1] \times [\phi_1] \cap \cdots \cap [\phi_k] \times [\phi_k])$$

with basis

$$U = (T \times T)(V) \cup ((\phi_1, \phi_1), \cdots, (\phi_k, \phi_k)),\ V \in \mathcal{V}.$$

Suppose $(\phi, \psi) \in U$, then $(\phi, \psi) \notin ((\phi_1, \phi_1), \cdots, (\phi_k, \phi_k))$ which implies $(\phi, \psi) \in (T \times T)(V)$, $V \in \mathcal{V}$. Since T is injective it follows that $(T^{-1}\phi, T^{-1}\psi) \in V$, $V \in \mathcal{V}$. Thus $T^{-1}\phi = T^{-1}\psi$ since $\mathcal{J}_{\mathcal{M}}$ is Hausdorff so that $\phi = \psi$, which is a contradiction. Hence $(\phi, \psi) \notin U$, for some $U \in \mathcal{U}$. This completes the proof. □

Note that the final uniform convergence structure $\mathcal{J}_{\mathcal{N},T}$ is the finest uniform convergence structure on $\mathcal{N}$ making T uniformly continuous, see [20]. Thus we have the following

Corollary 2.8. *If $\mathcal{F}$ is a Cauchy filter on $\mathcal{N}$ with respect to $\mathcal{J}_{\mathcal{N},T}$, then $\pi_1(\mathcal{F})$ is weakly Cauchy in L^1 and $\pi_2(\mathcal{F})$ is Cauchy in L^1_{loc}.*

Proof. The result follows from Corollary 2.6. □

We now apply the completion process. In this regard, the Wyler completion of $\mathcal{M}$ is constructed in the following way. Denote by $C[\mathcal{M}]$ the set of all Cauchy filters on $\mathcal{M}$, and define an equivalence relation on $C[\mathcal{M}]$ through

$$\mathcal{F} \sim_C \mathcal{G} \Leftrightarrow \mathcal{F} \cap \mathcal{G} \in C[\mathcal{M}]. \tag{2.28}$$

Let us denote by $\mathcal{M}^\sharp$ the quotient space $C[\mathcal{M}]/\sim_C$. For $\mathcal{F} \in C[\mathcal{M}]$, denote the equivalence class generated by $\mathcal{F}$ with respect to (2.28) by $[\mathcal{F}]$. One may identify

$\mathcal{M}$ with a subset of $\mathcal{M}^\sharp$ by associating each $u \in \mathcal{M}$ with $\lambda_1(u) \subset C[\mathcal{M}]$. From the definition of a convergence structure given in (1.36) it is clear that $\lambda_1(u)$ is indeed a $\sim_C$-equivalence class. Furthermore, since λ_1 is Hausdorff, the mapping

$$i_{\mathcal{M}} : \mathcal{M} \ni u \mapsto \lambda_1(u) \in \mathcal{M}^\sharp$$

is injective. Thus we may consider the convergence space $\mathcal{M}$ as a subset of $\mathcal{M}^\sharp$

The Wyler completion of $\mathcal{M}$ is the set $\mathcal{M}^\sharp$, equipped with the following vector space convergence structure, see for instance [100]:

$$\mathcal{G} \in \lambda_1^\sharp([\mathcal{F}]) \Leftrightarrow \left(\begin{array}{ll} \exists & \mathcal{F}_1 \cdots \mathcal{F}_n \in [\mathcal{F}] : \\ & i_{\mathcal{M}}(\mathcal{F}_1) \bigcap \cdots \bigcap i_{\mathcal{M}}(\mathcal{F}_n) \bigcap [\mathcal{F}_1] \bigcap \cdots \bigcap [\mathcal{F}_n] \subseteq \mathcal{G} \end{array} \right. \tag{2.29}$$

Similarly, let $C[\mathcal{N}]$ be the set of all Cauchy filters on $\mathcal{N}$, and define an equivalence relation on $C[\mathcal{N}]$ through

$$\mathcal{F} \sim_C \mathcal{G} \Leftrightarrow \mathcal{F} \cap \mathcal{G} \in C[\mathcal{N}]. \tag{2.30}$$

The Wyler completion of $(\mathcal{N}, \mathcal{J}_{\mathcal{N},T})$ is the set $\mathcal{N}^\sharp$ equipped with the uniform convergence structure $\mathcal{J}^\sharp_{\mathcal{N},T}$ which is defined as follows

$$\mathcal{U} \in \mathcal{J}^\sharp_{\mathcal{N},T} \Longleftrightarrow \left(\begin{array}{ll} \exists & \mathcal{V} \in \mathcal{J}_{\mathcal{N},T} : \\ \exists & \text{Cauchy filters } \mathcal{F}_1 \cdots \mathcal{F}_k \in C[\mathcal{N}] \backslash \lambda_{\mathcal{J}_{\mathcal{N},T}} : \\ & (i_{\mathcal{N}} \times i_{\mathcal{N}})(\mathcal{V}) \bigcap [(i_{\mathcal{N}}(\mathcal{F}_1) \times [\mathcal{F}_1]) \cap ([\mathcal{F}_1] \times i_{\mathcal{N}}(\mathcal{F}_1)) \cap \cdots \\ & \cap (i_{\mathcal{N}}(\mathcal{F}_k) \times [\mathcal{F}_k]) \cap (\mathcal{F}_k \times i_{\mathcal{N}}(\mathcal{F}_k))] \subseteq \mathcal{U}, \end{array} \right. \tag{2.31}$$

where $\lambda_{\mathcal{J}_{\mathcal{N},T}}$ denotes the convergence structure induced by the final convergence structure $\mathcal{J}_{\mathcal{N},T}$. In general, $\lambda_{\mathcal{J}_{\mathcal{N},T}}$ is not a final convergence structure, see [20] and Section 1.2.1.

Since $T : \mathcal{M} \longrightarrow \mathcal{N}$ is uniformly continuous there exists a unique uniformly continuous mapping $T^\sharp : \mathcal{M}^\sharp \longrightarrow \mathcal{N}^\sharp$ such that the diagram

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\quad T \quad} & \mathcal{N} \\ {\scriptstyle i_{\mathcal{M}}} \downarrow & & \downarrow {\scriptstyle i_{\mathcal{N}}} \\ \mathcal{M}^\sharp & \xrightarrow{\quad T^\sharp \quad} & \mathcal{N}^\sharp \end{array} \tag{2.32}$$

commutes, where $i_{\mathcal{M}}$ and $i_{\mathcal{N}}$ are the uniformly continuous embeddings associated with the completion $\mathcal{M}^\sharp$ and $\mathcal{N}^\sharp$, respectively. Furthermore, since T is injective, it follows by the definition of $\mathcal{J}_{\mathcal{N},T}$ that T is a uniformly continuous embedding. That is, T^{-1} is uniformly continuous on $T(\mathcal{M}) \subset \mathcal{N}$. Therefore the mapping $T^\sharp$ is injective as well.

We now give a concrete description of the completion $\mathcal{M}^\sharp$ of $\mathcal{M}$ as a subset of the space of finite Hausdorff continuous function $\mathbb{H}$. In this regard, the following characterization of Cauchy filters is essential.

Proposition 2.9. *A filter $\mathcal{F}$ on $\mathcal{M}$ is a Cauchy filter with respect to the vector space convergence structure λ_1 if and only if*

$$\left\{\begin{array}{lll} \exists & (\alpha_n),(\beta_n)\subseteq C^0(\mathbb{R}\times[0,\infty)): & \\ & (i) & \alpha_n\leq\alpha_{n+1}\leq\beta_{n+1}\leq\beta_n \\ & (ii) & \int_a^b(\beta_n(x,t)-\alpha_n(x,t))dx\longrightarrow 0 \\ & & \qquad\qquad \forall\ t>0,\ a,b\in\mathbb{R},\ a\leq b \\ & (iii) & [\{[\alpha_n,\beta_n]:n\in\mathbb{N}\}]\subseteq\mathcal{F}. \end{array}\right. \tag{2.33}$$

Proof. Let (2.33) hold. Then set $\alpha_n^{(1)}=\alpha_n-\beta_n$ and $\beta_n^{(1)}=\beta_n-\alpha_n$ on $C^0(\mathbb{R}\times[0,\infty))$. Therefore the sequences $\alpha_n^{(1)}$ and $\beta_n^{(1)}$ satisfy the following.

(i) $\alpha_n^{(1)}\leq\alpha_{n+1}^{(1)}\leq 0\leq\beta_{n+1}^{(1)}\leq\beta_n^{(1)}$. This follows from (2.33)(i).

(ii) $\displaystyle\int_a^b(\beta_n^{(1)}(x,t)-\alpha_n^{(1)}(x,t))dx\longrightarrow 0$. This is because

$$\begin{aligned}\int_a^b(\beta_n^{(1)}(x,t)-\alpha_n^{(1)}(x,t))dx &= \int_a^b(\beta_n-\alpha_n-\alpha_n+\beta_n)dx \\ &= 2\int_a^b(\beta_n-\alpha_n)dx\longrightarrow 0.\end{aligned}$$

(iii) $[\{[\alpha_n^{(1)},\beta_n^{(1)}]:n\in\mathbb{N}\}]\subseteq\mathcal{F}-\mathcal{F}$. To see this, observe that from (2.33)(iii) we have

$$\begin{array}{ll}\forall & n\in\mathbb{N} \\ \exists & F\in\mathcal{F}: \\ & F\subseteq[\alpha_n,\beta_n].\end{array}$$

It follows that

$$F-F\subseteq[\alpha_n,\beta_n]-[\alpha_n,\beta_n]\subseteq[\alpha_n-\beta_n,\beta_n-\alpha_n]=[\alpha_n^{(1)},\beta_n^{(1)}].$$

Thus,

$$[\{[\alpha_n^{(1)},\beta_n^{(1)}]:n\in\mathbb{N}\}]\subseteq\mathcal{F}-\mathcal{F},$$

which implies $\mathcal{F}-\mathcal{F}$ converges to zero. Hence $\mathcal{F}$ is Cauchy with respect to λ_1.

Conversely, let $\mathcal{F}$ on $\mathcal{M}$ be a Cauchy filter with respect to λ_1. Then $\mathcal{F}-\mathcal{F}\in\lambda_1(0)$. Let $\alpha_n,\beta_n\subseteq C^0(\mathbb{R}\times[0,\infty))$ be sequences associated with $\mathcal{F}-\mathcal{F}$ according to (2.19). It follows from (2.19)(iii) that

$$\begin{array}{ll}\forall & n\in\mathbb{N} \\ \exists & F\in\mathcal{F}: \\ & F-F\subseteq[\alpha_n,\beta_n].\end{array}$$

Choose any $v\in F$. Then $F\subseteq F-F+v$. Since the ultrafilter $[v]\in\lambda_1(v)$, it follows that there exists sequences $\alpha_n^1,\beta_n^1\subseteq C^0(\mathbb{R}\times[0,\infty))$ such that $[\{[\alpha_n^1,\beta_n^1]:$

$n \in \mathbb{N}\}] \subseteq [v]$. Therefore,

$$F \subseteq F - F + v \subseteq [\alpha_n, \beta_n] + [\alpha_n^1, \beta_n^1]$$
$$\subseteq [\alpha_n + \alpha_n^1, \beta_n + \beta_n^1],$$

which implies

$$[\{[\alpha_n + \alpha_n^1, \beta_n + \beta_n^1] : n \in \mathbb{N}\}] \subseteq \mathcal{F}.$$

Denote $\tilde{\alpha}_n = \alpha_n + \alpha_n^1$ and $\tilde{\beta}_n = \beta_n + \beta_n^1$. clearly, $\tilde{\alpha}_n \leq \tilde{\alpha}_{n+1} \leq \tilde{\beta}_{n+1} \leq \tilde{\beta}_n$. Moreover, for all $a, b \in \mathbb{R}, a \leq b$ we have

$$\int_a^b (\tilde{\beta}_n - \tilde{\alpha}_n)dx = \int_a^b (\beta_n + \beta_n^1 - \alpha_n - \alpha_n^1)dx$$
$$= \int_a^b (\beta_n - \alpha_n) + \int_a^b (\beta_n^1 - \alpha_n^1)dx \longrightarrow 0.$$

Hence there exists sequences $(\tilde{\alpha}_n), (\tilde{\beta}_n) \subseteq C^0(\mathbb{R} \times [0, \infty))$ satisfying (2.19). This completes the proof. □

Consider some $p \in \mathcal{M}^\sharp$. Then there exists a Cauchy filter $\mathcal{G}$ on $\mathcal{M}$ such that $\mathcal{G} \longrightarrow p$ in $\mathcal{M}^\sharp$. Then $\mathcal{G}$ is Cauchy with respect to λ_s as well. Therefore there exists $u \in \mathbb{H}$ such that $\mathcal{G} \longrightarrow u$ in $\mathbb{H}$ with respect to the order convergence structure on $\mathbb{H}$. We define the mapping $\eta : \mathcal{M}^\sharp \longrightarrow \mathbb{H}$ via

$$\eta(p) = u \tag{2.34}$$

Theorem 2.10. *The map η is well defined, that is, if $\mathcal{G}, \mathcal{V}$ are Cauchy filters in $\mathcal{M}$ and $\mathcal{G}, \mathcal{V} \longrightarrow p$ in $\mathcal{M}^\sharp$. Then $\mathcal{G}$ and $\mathcal{V}$ converge to the same limit u in $\mathbb{H}$.*

Proof. Let $\mathcal{G}, \mathcal{V} \longrightarrow p$ in $\mathcal{M}^\sharp$. Then $\mathcal{G} \cap \mathcal{V} \longrightarrow p$ in $\mathcal{M}^\sharp$. But $\mathcal{G} \cap \mathcal{V}$ is a Cauchy filter with respect to λ_1. Therefore it converges in $\mathbb{H}$. Let $\mathcal{G} \cap \mathcal{V} \longrightarrow w$ in $\mathbb{H}$, then $\mathcal{G} \supseteq \mathcal{G} \cap \mathcal{V}$ implies that $\mathcal{G} \longrightarrow w$. Similarly, $\mathcal{V} \longrightarrow w$ in $\mathbb{H}$. The proof is complete. □

Theorem 2.11. *The map η is injective.*

Proof. Let $\eta(p) = \eta(q) = u$ for some $p, q \in \mathcal{M}^\sharp$. There exist Cauchy filters $\mathcal{G}_1, \mathcal{G}_2$ on $\mathcal{M}$ such that $\mathcal{G}_1 \longrightarrow p, \mathcal{G}_2 \longrightarrow q$ in $\mathcal{M}^\sharp$ and $\mathcal{G}_1, \mathcal{G}_2 \longrightarrow u \in \mathbb{H}$. Let $(\alpha_n^{(i)})$, $(\beta_n^{(i)})$ be the sequences associated with $\mathcal{G}_i$, $i = 1, 2$ in terms of (2.33). Let $\alpha_n = \inf\{\alpha_n^{(1)}, \alpha_n^{(2)}\}$ in $C^0(\mathbb{R} \times [0, \infty))$, that is, α_n is the point-wise minimum of $\alpha_n^{(1)}$ and $\alpha_n^{(2)}$. Similarly, $\beta_n = \sup\{\beta_n^{(1)}, \beta_n^{(2)}\}$. Clearly, $\alpha_n, \beta_n \in C^0(\mathbb{R} \times [0, \infty))$ and the sequences $(\alpha_n), (\beta_n)$ are monotone increasing and decreasing respectively. It is also easy to see that $[\{[\alpha_n, \beta_n] : n \in \mathbb{N}\}] \subseteq \mathcal{G}_1 \cap \mathcal{G}_2$. In order to associate the sequences (α_n) and (β_n) with $\mathcal{G}_1 \cap \mathcal{G}_2$ in terms of (2.33) we need to show that the property (2.33)(ii) is satisfied. From the definition of the order convergence structure we have $\alpha_n^{(i)}(x,t) \leq \overline{u}(x,t) \leq \beta_n^{(i)}(x,t) \quad i = 1,2$. Using the fact that $\max\{x,y\} \leq x+y$ for $x \geq 0, y \geq 0$ we obtain

$$\begin{aligned}\beta_n(x,t) - \alpha_n(x,t) \\ &= \max\{\beta_n^{(1)}(x,t), \beta_n^{(2)}(x,t)\} - \overline{u}(x,t) \\ &\quad + \overline{u}(x,t) - \min\{\alpha_n^{(1)}(x,t), \alpha_n^{(2)}(x,t)\} \\ &= \max\{\beta_n^{(1)}(x,t) - \overline{u}(x,t), \beta_n^{(2)}(x,t) - \overline{u}(x,t)\} \\ &\quad + \max\{\overline{u}(x,t) - \alpha_n^{(1)}(x,t), \overline{u}(x,t) - \alpha_n^{(2)}(x,t)\} \\ &\leq \beta_n^{(1)}(x,t) - \overline{u}(x,t) + \beta_n^{(2)}(x,t) - \overline{u}(x,t) \\ &\quad + \overline{u}(x,t) - \alpha_n^{(1)}(x,t) + \overline{u}(x,t) - \alpha_n^{(2)}(x,t) \\ &= \beta_n^{(1)}(x,t) - \alpha_n^{(1)}(x,t) + \beta_n^{(2)}(x,t) - \alpha_n^{(2)}(x,t).\end{aligned}$$

Then we have

$$\begin{aligned}\int_a^b (\beta_n(x,t) - \alpha_n(x,t))dx \\ &\leq \int_a^b (\beta_n^{(1)}(x,t) - \alpha_n^{(1)}(x,t))dx \\ &\quad + \int_a^b (\beta_n^{(2)}(x,t) - \alpha_n^{(2)}(x,t))dx \longrightarrow 0.\end{aligned}$$

Therefore $\mathcal{G}_1 \cap \mathcal{G}_2$ is a Cauchy filter with respect to λ_1 in $\mathcal{M}$. This means that $\mathcal{G}_1 \cap \mathcal{G}_2$ converges in $\mathcal{M}^\sharp$. Let $\mathcal{G}_1 \cap \mathcal{G}_2 \longrightarrow w$ in $\mathcal{M}^\sharp$. Then $\mathcal{G}_1$ and $\mathcal{G}_2$ being finer that $\mathcal{G}_1 \cap \mathcal{G}_2$ also converges to w. Hence $p = w = q$. $\square$

2.3 Approximation results

In this section we consider the Cauchy problem for the viscous Burgers equation of the form

$$v_t^{\delta,\varepsilon} + \frac{1}{2}\left(v^{\delta,\varepsilon}\right)_x^2 = \varepsilon v_{xx}^{\delta,\varepsilon} \qquad \text{in } \mathbb{R} \times (0,\infty) \tag{2.35}$$

$$v^\delta(x,0) = u^0(x) - 2\delta, \quad \delta > 0 \quad \text{in } \mathbb{R} \times \{t=0\} \tag{2.36}$$

which is the Cauchy problem of the viscous Burgers equation with a vertical shift by 2δ in the initial condition. Using the auxiliary problem (2.35) - (2.36) and techniques for *problems of monotonic type, [125],* we show how the entropy solution of the inviscid Burgers equation [39, 53, 69]

$$u_t + \frac{1}{2}(u)_x^2 = 0 \quad \text{in } \mathbb{R} \times (0,\infty) \tag{2.37}$$

with the initial condition

$$u(x,0) = u^0(x) \quad \text{in } \mathbb{R} \times \{t=0\}. \tag{2.38}$$

is approximated from below.

Applying Hopf's technique [69], see also Section 1.1.4, to equations (2.35) - (2.36), we have a solution similar to (1.56) where $K(x, y, t)$ is replaced with $K^\delta(x, y, t)$, defined below in (2.40). Theorem 1.28 may be stated as follows

Theorem 2.12. *Suppose $u_0 \in L^1_{loc}(\mathbb{R})$ is such that (1.58) holds. Then there exists a unique classical solution of equation (2.35)-(2.36) given by*

$$v^{\delta,\varepsilon}(x,t) = \frac{\int_{-\infty}^{\infty} \frac{x-y}{t} e^{-\frac{1}{2\varepsilon}K^\delta(x,y,t)} dy}{\int_{-\infty}^{\infty} e^{-\frac{1}{2\varepsilon}K^\delta(x,y,t)} dy} \tag{2.39}$$

where

$$K^\delta(x,y,t) = \frac{(x-y)^2}{2t} + \int_0^y u_0(s)ds - 2\delta y. \tag{2.40}$$

with the following properties:

(i) For all $a \in \mathbb{R}$,

$$\int_0^x v^{\delta,\varepsilon}(\xi,t)d\xi \longrightarrow \int_0^a u_0(\xi)d\xi - 2\delta a \quad \text{as} \quad x \longrightarrow a, \quad t \longrightarrow 0, \tag{2.41}$$

(ii) If $u_0(x)$ is continuous at $x = a$ then

$$v^{\delta,\varepsilon}(x,t) \longrightarrow u_0(a) - 2\delta \quad \text{as} \quad x \longrightarrow a, \quad t \longrightarrow 0. \tag{2.42}$$

Furthermore, a solution of (2.35) - (2.36) which is C^2-smooth in an interval $0 < t < \mathcal{T}$ and satisfies (2.41) for each value of $a \in \mathbb{R}$ necessarily coincides with (2.39) in this interval.

The function $K^\delta(x, y, t)$ satisfies properties (P1) - (P3) given in Section 1.1.4. Using this fact we now introduce the functions

$$y^\delta_{min} = \min\{y : K^\delta(x,y,t) = \min_{z\in\mathbb{R}} K^\delta(x,z,t)\}$$

and

$$y^\delta_{max} = \max\{y : K^\delta(x,y,t) = \min_{z\in\mathbb{R}} K^\delta(x,z,t)\}$$

Observe that

$$K^\delta(x,y,t) = K(x,y,t) - 2\delta y \quad \text{for} \quad x\ ,t \quad \text{fixed}, \tag{2.43}$$

Lemma 2.13. *For each $\delta > 0$ we have*

$$y^\delta_{min}(x,t) = y_{min}(x + 2\delta t, t)$$

$$y^\delta_{max}(x,t) = y_{max}(x + 2\delta t, t).$$

Proof. From (1.57) we get

$$\begin{aligned}
K(x+2\delta t, y, t) - 2\delta x - 2\delta^2 t \\
&= \frac{(x+2\delta t-y)^2}{2t} + \int_0^y u_0(s)ds - 2\delta x - 2\delta^2 t \\
&= \frac{(x-y)^2 + 4(x-y)\delta t + 4\delta^2 t^2}{2t} + \int_0^y u_0(s)ds - 2\delta x - 2\delta^2 t \\
&= \frac{(x-y)^2}{2t} + 2\delta x - 2\delta y + 2\delta^2 t + \int_0^y u_0(s)ds - 2\delta x - 2\delta^2 t \\
&= \frac{(x-y)^2}{2t} + \int_0^y u_0(s)ds - 2\delta y \\
&= K^\delta(x,y,t)
\end{aligned}$$

This implies that for fixed x and t, the functions $K^\delta(x,y,t)$ and $K(x,y,t)$ differ by a constant. Therefore they have the same set of minimizers, which implies the statement of the Lemma. □

The functions y^δ_{min} and y^δ_{max} have the properties (Y1) - (Y4) given in Section 1.1.4, see also [69]. Hence, the functions y^δ_{min} and y^δ_{max} are monotone functions in x. Moreover, for every $t \geq 0$, $y^\delta_{min}(x,t) = y^\delta_{max}(x,t)$ for all $x \in \mathbb{R}$ with the possible exception of a denumerable set of values of x where $y^\delta_{min}(x,t) < y^\delta_{max}(x,t)$. It follows from property (Y1) that

$$\frac{x - y^\delta_{min}(x,t)}{t} \leq \frac{x - y_{max}(x+\delta t)}{t} \tag{2.44}$$

From Theorem 1.29 we have that for all x and $t > 0$,

$$\frac{x - y^\delta_{\max}(x,t)}{t} \leq \liminf_{\substack{\alpha \to x \\ \theta \to t \\ \varepsilon \to 0}} v^{\delta,\varepsilon}(\alpha,\theta) \leq \limsup_{\substack{\alpha \to x \\ \theta \to t \\ \varepsilon \to 0}} v^{\delta,\varepsilon}(\alpha,\theta) \leq \frac{x - y^\delta_{\min}(x,t)}{t} \tag{2.45}$$

and, in particular, that

$$v^\delta(x,t) = \lim_{\substack{\alpha \to x \\ \theta \to t \\ \varepsilon \to 0}} v^{\delta,\varepsilon}(\alpha,\theta) = \frac{x - y^\delta_{\max}(x,t)}{t} = \frac{x - y^\delta_{\min}(x,t)}{t} \tag{2.46}$$

holds at every point (x,t) where $y^\delta_{\max}(x,t) = y^\delta_{\min}(x,t)$.

The following Lemma shows that the functions $\underline{u}$ and $\overline{u}$ defined in (1.68) and (1.69) are lower and upper semi-continuous respectively.

Lemma 2.14. *The functions* $\underline{u}$ *and* $\overline{u}$ *defined in* (1.68) *and* (1.69) *are lower semi-continuous and upper semi-continuous respectively.*

Proof. Let $\underline{u} > m$ for some $m \in \mathbb{R}$ and let μ be such that $\underline{u} > m + \mu$. Since

$$\underline{u}(x,t) = \sup\{\inf\{u^\varepsilon(\alpha,\theta) : |\alpha - x| < \eta,\ |\theta - t| < \eta,\ \varepsilon < \eta\} : \eta > 0\},$$

it follows that

$$\begin{aligned} \exists\ \ &\eta > 0 : \\ &\inf\{u^\varepsilon(\alpha,\theta) : |\alpha - x| < \eta,\ |\theta - t| < \eta,\ \varepsilon < \eta\} > m + \mu. \end{aligned}$$

Therefore,

$$u^\varepsilon(\alpha,\theta) > m + \mu \text{ if } |\alpha - x| < \eta,\ |\theta - t| < \eta,\ \varepsilon < \eta.$$

Let

$$\tilde{x} \in (x - \eta, x + \eta) \text{ and } \tilde{t} \in (t - \eta, t + \eta).$$

Then

$$\underline{u}(\tilde{x},\tilde{t}) = \liminf_{\substack{\alpha \to \tilde{x} \\ \theta \to \tilde{t} \\ \varepsilon \to 0}} u^\varepsilon(\alpha,\theta) \geq m + \mu > m.$$

Since the last inequality holds for all $\tilde{x} \in (x - \eta, x + \eta)$ and $\tilde{t} \in (t - \eta, t + \eta)$ it shows that $\underline{u}$ is lower semi- continuous. The proof of upper semi-continuity of $\overline{u}$ is done in a similar way. □

It follows from Lemma 2.14 that the functions

$$\underline{v}^\delta(x,t) = \liminf_{\substack{\alpha \to x \\ \theta \to t \\ \varepsilon \to 0}} v^{\delta,\varepsilon}(\alpha,\theta) \tag{2.47}$$

and

$$\overline{v}^\delta(x,t) = \limsup_{\substack{\alpha \to x \\ \theta \to t \\ \varepsilon \to 0}} v^{\delta,\varepsilon}(\alpha,\theta) \tag{2.48}$$

are lower semi-continuous and upper semi-continuous respectively.

Lemma 2.15. *The functions $\overline{v}^\delta$ defined by* (2.48) *and $\underline{u}$ defined by* (1.69) *satisfy the following inequality*

$$\overline{v}^\delta(x,t) \leq \underline{u}(x + \delta t) - \delta \quad x \in \mathbb{R},\ t \geq 0.$$

Proof. From the inequality (2.45), it follows that

$$\begin{aligned}\overline{v}^{\delta}(x,t) &\le \frac{x - y^{\delta}_{\min}(x,t)}{t} = \frac{x - y_{\min}(x+2\delta t, t)}{t} \\ &\le \frac{x - y_{\max}(x+\delta t, t)}{t} = \frac{x + \delta t - y_{\max}(x+\delta t, t)}{t} - \delta \\ &\le \liminf_{\substack{z \to x \\ \tau \to t \\ \varepsilon \to 0}} u^{\varepsilon}(z + \delta\tau, \tau) - \delta \\ &= \underline{u}(x+\delta t, t) - \delta\end{aligned}$$

as required. □

Consider the viscous problem

$$w_t^{\delta,\rho,\varepsilon} + \frac{1}{2}\left(w^{\delta,\rho,\varepsilon}\right)_x^2 = \varepsilon w_{xx}^{\delta,\rho,\varepsilon} \qquad \mathbb{R} \times (0,\infty) \tag{2.49}$$

$$w^{\delta,\rho,\varepsilon}(x,0) = I(\rho, u^0)(x) - 2\delta \qquad \mathbb{R} \times \{t = 0\} \tag{2.50}$$

Lemma 2.16. *Let* $w^{\delta,\rho,\varepsilon}$ *denote the solution of the Cauchy problem* (2.49) - (2.50). *Then*

$$\overline{w}^{\delta,\rho}(x,t) \le \underline{u}(x,t) - \delta, \quad x \in \mathbb{R}, \ t \in \left[0, \frac{\rho}{\delta}\right]. \tag{2.51}$$

where $\overline{w}^{\delta,\rho}(x,t) = \limsup_{\substack{\alpha \to x \\ \theta \to t \\ \varepsilon \to 0}} w^{\delta,\rho,\varepsilon}(\alpha,\theta)$ *as it is in* (1.69).

Proof. Consider the viscous problem (2.35) - (2.36) with solution $v^{\delta,\varepsilon}$ and the Cauchy problem

$$z_t^{\delta,\sigma,\varepsilon} + \frac{1}{2}(z^{\delta,\sigma,\varepsilon})_x^2 = \varepsilon z_{xx}^{\delta,\sigma,\varepsilon} \tag{2.52}$$

$$z^{\delta,\sigma,\varepsilon}(x,0) = u_0(x+\sigma) - 2\delta = v^{\delta,\varepsilon}(x+\sigma, 0) \tag{2.53}$$

From (2.50) we have that

$$\begin{aligned}w^{\delta,\rho,\varepsilon}(x,0) &= I(\rho, u^{\varepsilon})(x,0) - 2\delta \\ &\le u^{\varepsilon}(x+\sigma, 0) - 2\delta = u_0(x+\sigma) - 2\delta \quad \forall\ |\sigma| \le \rho \\ &= z^{\delta,\sigma,\varepsilon}(x,0) = v^{\delta,\varepsilon}(x+\sigma,0).\end{aligned}$$

It follows from [125, Chapter IV 25II] that

$$w^{\delta,\rho,\varepsilon}(x,t) \le v^{\delta,\varepsilon}(x+\sigma,t) \quad \forall\ t > 0, \ x \in \mathbb{R}, \quad |\sigma| \le \rho.$$

Therefore,

$$\overline{w}^{\delta,\rho}(x,t) = \limsup_{\substack{\varepsilon\to 0\\ \alpha\to x\\ \theta\to t}} w^{\delta,\rho,\varepsilon}(\alpha,\theta) \le \limsup_{\substack{\varepsilon\to 0\\ \alpha\to x\\ \theta\to t}} v^{\delta,\varepsilon}(\alpha+\sigma,\theta) = \overline{v}^{\delta}(x+\sigma,t),$$

that is, By Lemma 2.15

$$\begin{aligned}\overline{w}^{\delta,\rho}(x,t) &\le \overline{v}^{\delta}(x+\sigma,t), \qquad \forall\ |\sigma|<\rho\\ &\le \underline{u}(x+\delta t+\sigma,t)-\delta.\end{aligned}$$

Now, for fixed x and t take $\sigma = -\delta t$. Then

$$\overline{w}^{\delta}(x,t) \le \underline{u}(x,t)-\delta$$

as required. □

2.3.1 Requirements for u_0

Lemma 2.17. *Assume that*

$$\lim_{x\longrightarrow\infty} u_0(x) \quad \text{and} \quad \lim_{x\longrightarrow -\infty} u_0(x) \tag{2.54}$$

exist. Then condition (1.58) *is satisfied.*

Proof. Let $\lim_{x\longrightarrow\infty} u_0(x) = \beta$ then

$$\exists\ M:\ |u_0(x)-\beta|<1 \ \text{ for } \ x>M$$

$$\begin{aligned}\lim_{x\longrightarrow\infty}\frac{|\int_0^x u_0(s)ds|}{x^2} &\le \lim_{x\longrightarrow\infty}\frac{|\int_0^M u_0(s)ds| + |\int_M^x u_0(s)ds|}{x^2}\\ &\le \lim_{x\longrightarrow\infty}\frac{|\int_0^M u_0(s)ds|}{x^2} + \lim_{x\longrightarrow\infty}\frac{(|\beta|+1)(x-M)}{x^2}\\ &= 0\end{aligned}$$

□

The following Lemmas are consequences of condition (2.54) on u_0.

Lemma 2.18. *Suppose conditions* (2.54) *holds. Then for every* $\varepsilon > 0$

$$\lim_{\substack{x\to+\infty\\ t\to\tilde{t}}} u^{\varepsilon}(x,t) = \lim_{x\to+\infty} u_0(x)$$

and

$$\lim_{\substack{x\to-\infty\\ t\to\tilde{t}}} u^{\varepsilon}(x,t) = \lim_{x\to-\infty} u_0(x)$$

where u^ε is the solution of the viscous Burger's equation (1.54) (1.55).

Proof. Let $N > 0$. For any $\varepsilon < 1$ and $x > N$ we have

$$\left|\int_{-\infty}^{N} u_0(y)e^{-\frac{1}{2\varepsilon}K(x,y,t)}dy\right| = \left|\int_{-\infty}^{N} u_0(y)e^{-\frac{1}{2\varepsilon}\left(\frac{(x-y)^2}{4t}+\int_0^y u_0(s)ds\right)}e^{-\frac{1}{2\varepsilon}\frac{(x-y)^2}{4t}}dy\right|$$
$$\leq \max_{y\in(-\infty,N]} e^{-\frac{1}{2\varepsilon}\frac{(x-y)^2}{4t}}\int_{-\infty}^{N}|u_0(y)|e^{-\frac{1}{2\varepsilon}\left(\frac{(x-y)^2}{4t}+\int_0^y u_0(s)ds\right)}dy$$
$$\leq e^{-\frac{1}{2\varepsilon}\frac{(x-N)^2}{4T}}\int_{-\infty}^{N}|u_0(y)|e^{-\frac{1}{2}\left(\frac{(N-y)^2}{4t}+\int_0^y u_0(s)ds\right)}dy \qquad (2.55)$$

Taking the limit as $x \longrightarrow +\infty$ we have that the expression on the right of (2.55) converges to zero, so that

$$\lim_{x\longrightarrow+\infty}\left|\int_{-\infty}^{N} u_0(y)e^{-\frac{1}{2\varepsilon}K(x,y,t)}dy\right| = 0$$

which implies

$$\lim_{x\longrightarrow+\infty}\int_{-\infty}^{N} u_0(y)e^{-\frac{1}{2\varepsilon}K(x,y,t)}dy = 0 \qquad (2.56)$$

Similarly,

$$\lim_{x\longrightarrow+\infty}\left|\int_{-\infty}^{N} e^{-\frac{1}{2\varepsilon}K(x,y,t)}dy\right| = 0$$

which implies

$$\lim_{x\longrightarrow+\infty}\int_{-\infty}^{N} e^{-\frac{1}{2\varepsilon}K(x,y,t)}dy = 0 \qquad (2.57)$$

Now consider the solution u^ε of the viscous Burgers equation (1.54) - (1.55) which is given as

$$u^\varepsilon(x,t) = \frac{\int_{-\infty}^{\infty} u_0(y)e^{-\frac{1}{2\varepsilon}K(x,y,t)}dy}{\int_{-\infty}^{\infty} e^{-\frac{1}{2\varepsilon}K(x,y,t)}dy}$$
$$= \frac{\int_{-\infty}^{N} u_0(y)e^{-\frac{1}{2\varepsilon}K(x,y,t)}dy + \int_{N}^{\infty} u_0(x)e^{-\frac{1}{2\varepsilon}K(x,y,t)}dy}{\int_{-\infty}^{N} e^{-\frac{1}{2\varepsilon}K(x,y,t)}dy + \int_{\infty}^{N} e^{-\frac{1}{2\varepsilon}K(x,y,t)}dy}$$

Let $\sigma > 0$ and N be such that

$$\beta - \sigma < u_0(x) < \beta + \sigma \quad \text{whenever } |x| > N,$$

where $\beta = \lim_{x\to+\infty} u_0(x)$. Then

$$u^\varepsilon(x,t) \leq \frac{\int_{-\infty}^{N} u_0(y)e^{-\frac{1}{2\varepsilon}K(x,y,t)}dy + (\beta+\sigma)\int_{N}^{\infty} e^{-\frac{1}{2\varepsilon}K(x,y,t)}dy}{\int_{-\infty}^{N} e^{-\frac{1}{2\varepsilon}K(x,y,t)}dy + \int_{\infty}^{N} e^{-\frac{1}{2\varepsilon}K(x,y,t)}dy}$$

Using (2.56) and (2.57) we have that

$$\limsup_{\substack{x\to+\infty\\ t\to\tilde{t}}} u^{\varepsilon}(x,t) \leq \frac{\limsup_{\substack{x\to+\infty\\ t\to\tilde{t}}}\left[\int_{-\infty}^{N} u_0(y)e^{-\frac{1}{2\varepsilon}K(x,y,t)}dy + (\beta+\sigma)\int_{N}^{\infty} e^{-\frac{1}{2\varepsilon}K(x,y,t)}dy\right]}{\limsup_{\substack{x\to+\infty\\ t\to\tilde{t}}}\left[\int_{-\infty}^{N} e^{-\frac{1}{2\varepsilon}K(x,y,t)}dy + \int_{\infty}^{N} e^{-\frac{1}{2\varepsilon}K(x,y,t)}dy\right]}.$$

Therefore

$$\limsup_{\substack{x\to+\infty\\ t\to\tilde{t}}} u^{\varepsilon}(x,t) = \frac{\limsup_{\substack{x\to+\infty\\ t\to\tilde{t}}}\left[(\beta+\sigma)\int_{N}^{\infty} e^{-\frac{1}{2\varepsilon}K(x,y,t)}dy\right]}{\limsup_{\substack{x\to+\infty\\ t\to\tilde{t}}}\left[\int_{\infty}^{N} e^{-\frac{1}{2\varepsilon}K(x,y,t)}dy\right]} = \beta+\sigma.$$

Similarly,

$$\limsup_{\substack{x\to+\infty\\ t\to\tilde{t}}} u^{\varepsilon}(x,t) \geq \frac{\limsup_{\substack{x\to+\infty\\ t\to\tilde{t}}}\left[(\beta-\sigma)\int_{N}^{\infty} e^{-\frac{1}{2\varepsilon}K(x,y,t)}dy\right]}{\limsup_{\substack{x\to+\infty\\ t\to\tilde{t}}}\left[\int_{\infty}^{N} e^{-\frac{1}{2\varepsilon}K(x,y,t)}dy\right]} = \beta-\sigma.$$

Since σ is arbitrary, we have

$$\limsup_{\substack{x\to+\infty\\ t\to\tilde{t}}} u^{\varepsilon}(x,t) = \liminf_{\substack{x\to+\infty\\ t\to\tilde{t}}} u^{\varepsilon}(x,t) = \beta = \lim_{x\to+\infty} u_0(x)$$

which implies

$$\lim_{\substack{x\to+\infty\\ t\to\tilde{t}}} u^{\varepsilon}(x,t) = \lim_{x\to+\infty} u_0(x)$$

as required. The proof of the second part of the Lemma is similar. □

As an easy consequence of Lemma 2.18, we obtain

Corollary 2.19. *For any $\tilde{t} \geq 0$ we have*

$$\lim_{\substack{x\to+\infty\\ t\to\tilde{t}}} u(x,t) = \limsup_{\substack{x\to+\infty\\ t\to\tilde{t}}} \underline{u}(x,t) = \liminf_{\substack{x\to+\infty\\ t\to\tilde{t}}} \overline{u}(x,t) = \beta = \lim_{x\to+\infty} u_0(x)$$

and

$$\lim_{\substack{x\to-\infty\\ t\to\tilde{t}}} u(x,t) = \limsup_{\substack{x\to-\infty\\ t\to\tilde{t}}} \underline{u}(x,t) = \liminf_{\substack{x\to-\infty\\ t\to\tilde{t}}} \overline{u}(x,t) = \beta = \lim_{x\to-\infty} u_0(x)$$

where $\underline{u}$ *and* $\overline{u}$ *are defined by* (1.68) *and* (1.68) *respectively.*

2.4 Existence and uniqueness results

In this section we prove existence result for solution of the equation

$$T^\sharp u^\sharp = 0$$

in the case of the Burgers equation. More precisely, for every $u_o \in \mathcal{U}_0$ we construct a Cauchy sequence (w_k) in $\mathcal{M}$ such that $Tw_k \longrightarrow \begin{pmatrix} 0 \\ u_0 \end{pmatrix}$ in $\mathcal{N}$. The approximation results in the previous session are utilized for this purpose. Let $\delta_k = \frac{1}{4^k}$ and $\rho_k = \frac{1}{2^k},\ k\in\mathbb{N}$. For every $k\in\mathbb{N}$ consider the problem (2.49) - (2.50) with $\delta=\delta_k$ and $\rho=\rho_k$. Using Lemma 2.16 we obtain the following inequality

$$\overline{w}^{\delta_k,\rho_k}(x,t) \le \underline{w}^{\delta_{k+1},\rho_{k+1}}(x,t) - (\delta_k-\delta_{k+1}), \qquad x\in\mathbb{R},\ t\in\left[0,\frac{2^{k+1}}{3}\right]. \tag{2.58}$$

Indeed, we have

$$\begin{aligned}\overline{w}^{\delta_k,\rho_k}(x,0) &= I(\rho_k,u_0)(x) - 2\delta_k \\ &= I(\rho_k-\rho_{k+1}, w^{\delta_{k+1},\rho_{k+1}}(\cdot,0))(x) - 2(\delta_k-\delta_{k+1}).\end{aligned}$$

Hence the inequality (2.58) follows from Lemma 2.16 with $\underline{u} = \underline{w}^{\delta_{k+1},\rho_{k+1}}$, $\delta = \delta_k-\delta_{k+1}$ and $\rho=\rho_k-\rho_{k+1}$. The upper bound for the time interval is obtained as follows

$$\frac{\rho}{\delta} = \frac{\rho_k-\rho_{k+1}}{\delta_k-\delta_{k+1}} = \frac{\frac{1}{2^k}-\frac{1}{2^{k+1}}}{\frac{1}{4^k}-\frac{1}{4^{k+1}}} = \frac{\frac{1}{2^{k+1}}(2-1)}{\frac{1}{4^{k+1}}(4-1)} = \frac{2^{k+1}}{3}.$$

The construction of the Cauchy sequence is based on the following

Lemma 2.20. *For every* k *there exists* ε_k *such that* $w^{\delta_{2k},\rho_{2k},\varepsilon_k}$ *satisfies*

$$\overline{w}^{\delta_{2k-1},\rho_{2k-1}}(x,t) \le w^{\delta_{2k},\rho_{2k},\varepsilon_k}(x,t) \le \underline{w}^{\delta_{2k+1},\rho_{2k+1}}(x,t), \qquad \text{for } x\in\mathbb{R},\ \text{and } t\in[0,\frac{4^k}{3}]. \tag{2.59}$$

Proof. Assume the opposite, that is, there exists $k>0$ such that for every $\varepsilon>0$ there exists $(x_\varepsilon,t_\varepsilon)$ with $t_\varepsilon\in[0,\frac{4^k}{3}]$ such that one of the inequalities in (2.59) is violated. Since t_ε is in a compact interval, there exists a sequence (ε_n) such that

(t_{ε_n}) converges. Let $t_{\varepsilon_n} \longrightarrow \tilde{t} \in [0, \frac{4^k}{3}]$. At least one of the inequalities in (2.59) is violated for a subsequence of (ε_n). To avoid too many notations we denote this subsequence by (ε_n). Assume the second inequality is violated. The other case is dealt with in a similar way. Now let us consider the sequence (x_{ε_n}).

Case 1. The sequence (x_{ε_n}) has an accumulation point $\tilde{x} \in \mathbb{R}$. Then there is a subsequence converging to $\tilde{x}$. Without loss of generality we may assume that $x_{\varepsilon_n} \longrightarrow \tilde{x}$. Then

$$\begin{aligned}\overline{w}^{\delta_{2k},\rho_{2k}}(\tilde{x},\tilde{t}) &\geq \limsup_{n\longrightarrow\infty} w^{\delta_{2k},\rho_{2k},\varepsilon_k}(x_{\varepsilon_n}, t_{\varepsilon_n})\\ &\geq \underline{w}^{\delta_{2k+1},\rho_{2k+1}}(\tilde{x},\tilde{t})\end{aligned}$$

which contradicts (2.58).

Case 2. The sequence (x_{ε_n}) is unbounded. Then it has a subsequence diverging to $+\infty$ or $-\infty$. Let us denote this subsequence again by (x_{ε_n}) and let it converge to $+\infty$ (the case of $-\infty$ is treated similarly). Then using Corollary 2.19 we have

$$\begin{aligned}\lim_{n\longrightarrow\infty} w^{\delta_{2k},\rho_{2k},\varepsilon_n}(x_{\varepsilon_n}, t_{\varepsilon_n}) &\geq \lim_{n\longrightarrow\infty} \overline{w}^{\delta_{2k+1},\rho_{2k+1}}(x_{\varepsilon_n}, t_{\varepsilon_n})\\ &= \lim_{n\longrightarrow\infty} I(\rho_{2k+1}, u_0)(x) - 2\delta_{2k+1}\\ &= \lim_{n\longrightarrow\infty} u_0(x) - 2\delta_{2k+1}. \end{aligned} \tag{2.60}$$

On the other hand, by Lemma 2.18

$$\begin{aligned}\lim_{n\longrightarrow\infty} w^{\delta_{2k},\rho_{2k},\varepsilon_n}(x_{\varepsilon_n}, t_{\varepsilon_n}) &= \lim_{n\longrightarrow\infty} I(\rho_{2k}, u_0)(x) - 2\delta_{2k}\\ &= \lim_{n\longrightarrow\infty} u_0(x) - 2\delta_{2k}. \end{aligned} \tag{2.61}$$

The relations (2.60) and (2.61) lead to

$$\lim_{n\longrightarrow\infty} u_0(x) - 2\delta_{2k} \geq \lim_{n\longrightarrow\infty} u_0(x) - 2\delta_{2k+1}$$

which is impossible since $\delta_{2k} > \delta_{2k+1}$. The contradictions obtained in Case 1 and Case 2 proves the statement of the lemma. □

Now we construct an increasing sequence (α_n) in $\mathcal{M}$ as follows. Set

$$\alpha_k(x,t) = w^{\delta_{2k},\rho_{2k},\varepsilon_k(x,t)}(x,t)$$
$$\text{for } x \in \mathbb{R},\ t \in [0, 4^{k-1}].$$

Then α_k is extended for $t \in [4^{k-1}, \infty)$ in such a way that $\alpha_k \in C^1(\mathbb{R} \times [0,\infty))$,

$$\alpha_{k-1}(x,t) \leq \alpha_k(x,t) < \inf_{p>k} w^{\delta_{2p},\rho_{2p},\varepsilon_p}(x,t) \tag{2.62}$$

Note that for every (x,t) the sequence $(w^{\delta_{2p},\rho_{2p},\varepsilon_p}(x,t))$ is eventually monotone increasing so that the infimum in the inequality (2.62) is finite. The inequality $\alpha_{k-1}(x,t) \leq \alpha_k(x,t)$ is obtained from (2.59) for $x \in \mathbb{R}$ and $t \in [0, 4^{k-1}]$ and from (2.62) for $x \in \mathbb{R}$ and $t \in [4^{k-1}, \infty)$. It is also easy to see from Lemma 2.16 that $\alpha_k(x,t) \leq \underline{u}(x,t),\ x \in \mathbb{R},\ t \geq 0$.

Lemma 2.21. *At any point (x,t) we have*

$$\alpha_k(x,t) \geq \overline{u}(x - 3\delta_{2k-1}t - \rho_{2k-1}, t) - 3\delta_{2k-1} - \frac{2\rho_{2k-1}}{t}$$

for sufficiently large k.

Proof. Let the point $(x,t), t > 0$ be fixed. Let k be so large that $4^{k-1} > t$. Then

$$\begin{aligned} \alpha_k(x,t) &= w^{\delta_{2k},\rho_{2k},\varepsilon_k}(x,t) \\ &\geq \overline{w}^{\delta_{2k-1},\rho_{2k-1}}(x,t) \\ &\geq \underline{w}^{\delta_{2k-1},\rho_{2k-1}}(x,t) \\ &\geq \frac{x - y_{\max}^{2k-1}(x,t)}{t} \end{aligned} \tag{2.63}$$

where

$$y_{\max}^{2k-1}(x,t) = \max\{y : K^{2k-1}(x,y,t) = \min_{z\in\mathbb{R}} K^{2k-1}(x,z,t)\}$$

and

$$K^{2k-1}(x,y,t) = \frac{(x-y)^2}{2t} + \int_0^y I(\rho_{2k-1},u_0)(s)ds - 2\delta_{2k-1}y.$$

Then $y_{\max}^{2k-1}(x,t)$ is a solution to $\frac{\partial K^{2k-1}}{\partial y} = 0$. That is,

$$\frac{y_{\max}^{2k-1}(x,t) - x}{t} + I(\rho_{2k-1},u_0)(y_{\max}^{2k-1}(x,t)) - 2\delta_{2k-1} = 0.$$

Then there exists $\gamma = \gamma(x,t),\ |\gamma| \leq \rho_{2k-1}$ such that

$$\frac{y_{\max}^{2k-1}(x,t) + \gamma(x,t) - (x + 2\delta_{2k-1}t + \gamma(x,t))}{t} + u_0(y_{\max}^{2k-1}(x,t) + \gamma(x,t)) = 0$$

or equivalently

$$\frac{\partial K}{\partial y}(x + 2\delta_{2k-1}t + \gamma(x,t), y_{\max}^{2k-1}(x,t) + \gamma(x,t), t) = 0$$

Therefore,

$$y_{\max}(x + 2\delta_{2k-1}t + \gamma(x,t), t) \geq y_{\max}^{2k-1}(x,t) + \gamma(x,t)$$

Furthermore, using the monotonicity of $y_{\max}$ we have

$$\begin{aligned} y_{\max}^{2k-1}(x,t) &\leq y_{\max}(x + 2\delta_{2k-1}t + \gamma(x,t), t) - \gamma(x,t) \\ &\leq y_{\max}(x + 2\delta_{2k-1}t + \rho_{2k-1}, t) + \rho_{2k-1}. \end{aligned}$$

From (2.63) we obtain

$$\begin{aligned} \alpha_k(x,t) &\geq \frac{x - y_{\max}(x + 2\delta_{2k-1}t + \rho_{2k-1}, t)}{t} - \frac{\rho_{2k-1}}{t} \\ &> \frac{x - y_{\min}(x + 3\delta_{2k-1}t + \rho_{2k-1}, t)}{t} - \frac{\rho_{2k-1}}{t} \\ &= \frac{x + 3\delta_{2k-1}t - y_{\min}(x + 3\delta_{2k-1}t + \rho_{2k-1}, t)}{t} - \frac{2\rho_{2k-1}}{t} - 3\delta_{2k-1} \\ &\geq \overline{u}(x + 3\delta_{2k-1}t + \rho_{2k-1}, t) - \frac{2\rho_{2k-1}}{t} - 3\delta_{2k-1}. \end{aligned}$$

This completes the proof. □

In a similar way one constructs a decreasing sequence (β_k) such that at any point (x,t) we have

$$\overline{u}(x,t) \leq \beta_k(x,t) \leq \underline{u}(x - 3\delta_{2k-1}t - \rho_{2k-1}, t) + 3\delta_{2k-1} + \frac{2\rho_{2k-1}}{t}.$$

Clearly, $\alpha_k \leq w^{\delta_{2k},\rho_{2k},\varepsilon_k} \leq \beta_k$. In order to prove that $(w^{\delta_{2k},\rho_{2k},\varepsilon_k})$ is a Cauchy sequence in $\mathcal{M}$, it remains to show that (2.33)(ii) holds. Let $t > 0$ and $a, b \in \mathbb{R}$, such that $a \leq b$. For all sufficiently large k

$$\begin{aligned}
&\int_a^b (\beta_k(x,t) - \alpha_k(x,t))dx \\
&\leq \int_a^b [\underline{u}(x - 3\delta_{2k-1}t - \rho_{2k-1}, t) - \overline{u}(x + 3\delta_{2k-1}t + \rho_{2k-1}, t) + 6\delta_{2k-1} + \frac{4\rho_{2k-1}}{t}]dx \\
&= \int_{a+3\delta_{2k-1}t+\rho_{2k-1}}^{b+3\delta_{2k-1}t+\rho_{2k-1}} \underline{u}(x,t)dx - \int_{a-3\delta_{2k-1}t-\rho_{2k-1}}^{b-3\delta_{2k-1}t-\rho_{2k-1}} \overline{u}(x,t)dx + (6\delta_{2k-1}t + \rho_{2k-1})(b-a) \\
&= \int_{b-3\delta_{2k-1}t-\rho_{2k-1}}^{b+3\delta_{2k-1}t+\rho_{2k-1}} \underline{u}(x,t)dx + \int_{a-3\delta_{2k-1}t-\rho_{2k-1}}^{a+3\delta_{2k-1}t+\rho_{2k-1}} \overline{u}(x,t)dx + (6\delta_{2k-1}t + \rho_{2k-1})(b-a).
\end{aligned}$$

The last expression tends to 0 as $k \longrightarrow \infty$. Hence

$$\int_a^b (\beta_k(x,t) - \alpha_k(x,t))dx \longrightarrow 0 \text{ as } k \longrightarrow \infty.$$

Thus the Frechét filter $\langle(w^{\delta_{2k},\rho_{2}k,\varepsilon_k})\rangle$ associated with the sequence $(w^{\delta_{2k},\rho_{2}k,\varepsilon_k})$ defines an element p of $\mathcal{M}^\sharp$.

Moreover, in the topology of $\mathcal{N}$ we have

$$T_1 w^{\delta_{2k},\rho_{2}k,\varepsilon_k} \longrightarrow 0$$

and

$$T_2 w^{\delta_{2k},\rho_{2}k,\varepsilon_k} \longrightarrow u_0.$$

Therefore

$$T^\sharp p = \begin{pmatrix} 0 \\ u_0 \end{pmatrix}.$$

In this way we have proved the following

Theorem 2.22. *For any $u_0 \in \mathcal{U}_0$ there exists a unique $p \in \mathcal{M}^\sharp$ such that*

$$T^\sharp p = \begin{pmatrix} 0 \\ u_0 \end{pmatrix}.$$

This means that the initial value problem for the Burgers equation has a unique solution in $\mathcal{M}^\sharp$.

It is easy to see that $(w^{\delta_{2k}\rho_{2k}\varepsilon_k})$ order converges to the unique $\mathbb{H}$-continuous function $u = [\underline{u}, \overline{u}]$. Hence we have $\eta(p) = u$. Therefore the Burger's equation has an $\mathbb{H}$-continuous solution which corresponds to the well-known entropy solution. In particular, $\underline{u} = \overline{u}$ almost everywhere, and any real valued function v such that $v(x,t) \in u(x,t)$ for all $(x,t) \in \mathbb{R} \times [0,\infty)$ satisfies the entropy condition for the Burger's equation.

Chapter 3

Concluding Remarks

3.1 Main results

We considered the Cauchy problem of nonlinear conservation law with smooth flux function and continuous initial condition in the context of Convergence Space Completion Method. In particular, the Convergence Space Completion Method was applied to the nonlinear operator equation derived from the Cauchy problem for nonlinear scalar conservation law. In this regard, suitable uniform convergence spaces were introduced. The completions of these uniform convergence spaces were obtained through the Wyler completion process. In addition, a uniformly continuous and injective mapping was obtained as an extension of the nonlinear operator derived from the Cauchy problem.

It was shown that the extended operator equation has at most one generalized solution which can be identified with the entropy solution in the case of the Burgers equation. Thus we obtained an existence and uniqueness result for the operator equation of the Burgers equation. The uniqueness of solution follows from the injectivity of the extended operator. It was further shown that the space of generalized solutions can be identified with the space of Hausdorff continuous functions, thus the unique solution of the Burgers equation so obtained is identified with a Hausdorff continuous function. This provides a further regularity property for the generalized solution of the Burgers equation.

3.2 Topics for further research

In this work we have applied the Order Completion Method, which is a general and type independent theory for existence and regularity of generalized solutions for large class of systems of nonlinear PDEs, to obtained the entropy solution of Burgers equation. The application of the Order Completion Method to the case of a more general flux function is very important and should be considered.

Systems of conservation laws appear very often in real world problems, thus an application of the Order Completion Method to specific systems of conservation laws is an important issue that should be looked into.

Apart from conservation laws, there other types of specific linear and nonlinear PDEs, which the Order Completion Method could be applied to. This is another interesting area for further research.

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