Part IV

Implementation and Numerical Results

Chapter 8

Implementation of GARCH Option Pricing

8.1 Introduction

In this chapter, methods to implement the GARCH option pricing model is discussed. Two separate numerical procedures are required in the implementation of this model, the first is the calibration of the parameters to the stock or option data and the second is the forecast of the option price.

8.2 Calibrating the GARCH Process to Empirical Data

8.2.1 Historical Data

In the GARCH option pricing procedure proposed by Jin-Chuan Duan, the GARCH process is "fitted" to the process of the underlying stock or index. This means that the parameters of the GARCH-M process under the P measure is fitted to the returns series of the underlying by maximizing its loglikelihood function.

For the (vanilla) GARCH$(1, 1) - M$ process under the P measure

$$S_t = S_{t-1} \exp\left(r\Delta t - \frac{1}{2}\sigma_t^2 + \lambda\sigma_t + \varepsilon_t\right)$$

where ε_t is the returns at time t, the rest of the parameters and variables are as in section 4.6. The vanilla GARCH$(1, 1)$ process is[1]

$$\hat{\sigma}_t^2 = \hat{\alpha}_0 + \hat{\alpha}\varepsilon_t^2 + \hat{\beta}\hat{\sigma}_{t-1}^2.$$

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[1] Estimates are written with hats.

From section 4.4.3, equation 4.9 the likelihood function for the variance up to time t is

$$
\begin{aligned}
f\left(0, \hat{\sigma}_t^2\right) &= \sum_{i=1}^{t} -\frac{1}{2} \ln \hat{\sigma}_i^2 - \frac{1}{2} \frac{\varepsilon_i^2}{\hat{\sigma}_i^2} \\
&= \sum_{i=1}^{t} l_i\left(\alpha_0, \alpha, \beta, \sigma_0^2\right)
\end{aligned}
$$

The optimization problem for the variance is as follows

$$
\max_{\hat{\alpha}_0, \hat{\alpha}, \hat{\beta}, \hat{\sigma}_0^2} f\left(0, \hat{\sigma}_t^2\right)
$$

where the likelihood function $f\left(0, \hat{\sigma}_t^2\right)$ is maximized in terms of the parameters $\hat{\alpha}_0, \hat{\alpha}$ and $\hat{\beta}$. Since the value of $\hat{\sigma}_0^2$ isn't known, it forms part of the optimization problem.

The value of parameter $\hat{\lambda}$ is then estimated by minimizing the sum of squares between the actual and estimated stock prices up to time t

$$
\begin{aligned}
&\min_{\hat{\lambda}} \sum_{i=1}^{t} \left(S_i - \hat{S}_i\right)^2 \\
&= \min_{\hat{\lambda}} \sum_{i=1}^{t} \left(S_i - \hat{S}_{i-1} \exp\left(r\Delta t - \frac{1}{2}\hat{\sigma}_i^2 + \hat{\lambda}\hat{\sigma}_i + \varepsilon_i\right)\right)^2
\end{aligned}
$$

$\hat{S}_i$ is an estimate of the of the stock price at time i and S_i is the actual stock price.

Both of the optimization problems are due to overdetermined systems. This means that there are more equations than variables. Tim Bollerslev (1986) suggests the use of the Berndt, Hall, Hall and Hausman algorithm for the estimation of the variance optimization problem. A similar algorithm can also be used for the mean optimization problem.

Many new statistical computer packages have built-in GARCH algorithms. Often, the problem with these built-in algorithms are that they are designed to solve only certain types of GARCH parameter estimation problems.

The GARCH toolbox available with Matlab R 12 is only limited to solving vanilla GARCH problems. Fortunately the optimization toolbox of Matlab is excellent. The procedure `fmincon` can be used to fit a tailor-made GARCH and means process.

8.2.2 Implied Volatility

As mentioned in section 1.3 of the introduction, the levels of implied volatility of warrants are substantially higher than that of the historical volatility

of the underlying. This means that there is little use in pricing an option with a model based on the history of the underlying.

In this dissertation we investigate the calibration of a GARCH process to the implied volatility of warrant..

The approach is as follows:

1. In section 5.4.5, the discussion of the Black-Scholes formula and implied volatility, it was mentioned that implied volatility is annualized. Implied volatility at time t, say σ_t must thus by multiplied by the square root of the relevant time fraction, for instance if the available returns series is daily then the new adjusted series must be $\sigma_t\sqrt{1/252}$.

2. The GARCH process is a variance process, not a standard deviations process, thus the square of the new series in point 1 must be taken, which gives $\sigma_t^2/252 \equiv I_t$.

3. Implied volatility is used to price options, thus it is already under the Q measure. This means that the unit risk premium λ is already "absorbed" into the GARCH process.

4. The parameter estimation for the Asymmetric GARCH(p,q) process is as follows,

$$\min_{\hat{\alpha}_0,\hat{\alpha},\hat{\beta}} \sum_{i=1}^{t} \left(I_t - \sigma_t^2\right)^2$$

$$= \min_{\hat{\alpha}_0,\hat{\alpha},\hat{\beta},\lambda} \sum_{i=1}^{t}$$

$$\left(I_t - \alpha_0 + \sum_{j=1}^{q}\alpha_j\left(\varepsilon_{i-j} - \lambda\sigma_{i-j}\right)^2 + \sum_{j=1}^{p}\beta_j\sigma_{i-j}^2\right)^2 \qquad (8.1)$$

Unlike the parameter estimation in section 8.2.1 above, the value of σ_0^2 here isn't part of the minimization problem. That is because if we let σ_0^2 equal I_0, the value at time $i = 0$ in equation 8.1 is zero.

5. Optimization here is again done with the `fmincon` procedure of Matlab.

8.3 Monte Carlo Simulations

Monte Carlo simulations is a method to solve stochastic integrals numerically. This is done by simulating N sample paths of a stochastic processes, say f by the generation of random variables from the underlying probability

distribution. All the versions of f are then added together and divided by the amount of simulations. By the law of large numbers we can write

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(X_n) = \int_{R} f(x) q(x) dx \tag{8.2}$$

where (X_n) are independently drawn from the distribution with pdf q.

In this dissertation q is the pdf of the normal distribution discussed in section 2.5.

8.3.1 European Option with Constant Volatility

The pricing theorem for a European option in the Black-Scholes sense, theorem 5.4.2, yields

$$
\begin{aligned}
V_t &= e^{-r(T-t)} E^Q \left[V_T \mid \mathcal{F}_t \right] \\
&= e^{-r(T-t)} E^Q \left[f_T \mid \mathcal{F}_t \right] \\
&= e^{-r(T-t)} \int_{R} f_T(x) q(x) dx
\end{aligned}
$$

where q is a pdf. For a put option, $f_T = (X - S_T)_+$, where X and S_T are the strike price and the stock price at time T respectively. S_T and thus f_T is a function of Brownian motion. By equation 5.12,

$$S_T = S_0 \exp \left(\left(\mu - \frac{1}{2} \sigma^2 \right) T + \sigma W_t \right)$$

In discrete time, this can be estimated by

$$\widehat{S_T} = \widehat{S_0} \exp \left(\left(\mu - \frac{1}{2} \sigma^2 \right) T + \sigma \xi_t \sqrt{\Delta t} \right)$$

where

$$\xi_t | \mathcal{F}_{t-1} \overset{Q}{\sim} N(0,1)$$

8.3.2 European Options with GARCH Volatility

The aim is again to estimate the value of f_T at time T. This time it must be remembered the the GARCH-M process used in this dissertation is defined in discrete time, we are thus not solving an integral. The stock price process, under the LRNVR with GARCH conditional volatility, as defined in theorem 6.8.1 is

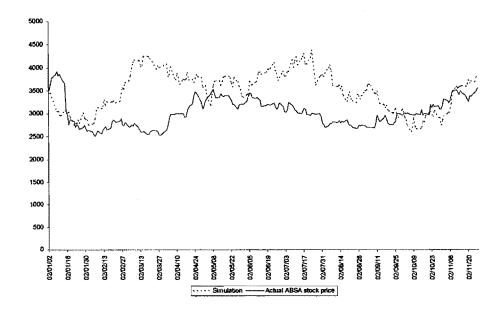$$S_T = S_t \exp \left((T-t) r - \sum_{i=t}^{T} \left(\frac{1}{2} \sigma_i^2 + \xi_i \sigma_i \right) \right)$$

Figure 8.1: A sample path of a Monte Carlo simulation compared with the actual ABSA stock price process.

where

$$\xi_i | \mathcal{F}_{i-1} \overset{Q}{\sim} N(0,1)$$

and

$$\sigma_i^2 = \alpha_0 + \sum_{j=1}^{q} \alpha_j \left(\xi_{i-j} - \lambda \sigma_{i-j} \right)^2 + \sum_{i=1}^{p} \beta_j \sigma_{i-j}^2$$

8.3.3 Notes

1. To simulate possible sample paths of the stock price, a random number ξ_t is generated for each interval $t \in N \cap [1, T]$. The intervals are equally spaced, say of size Δt. If we use an annual risk-free interest rate and daily time intervals, Δt would be $1/252$, since we usually assume 252 trading days in a year.

2. A large number of future paths are simulated. The number depends on the accuracy required. This can vary between a 1000 and 50000 or even more simulations.

8.3.4 Generating Other Distributions from the Uniform Distribution

Many computer packages can only generate uniform random variables between 0 and 1. Most other packages, like Matlab and Excel, generate only random variables from certain famous distributions. The following famous technique is a way to generate random variables from uniform random variables:

To generate random variables from this cdf, use the following famous result: Say we are able to generate a uniform random variable, U, between 0 and 1. Define the inverse of $\widehat{F_Z(z)}$ as

$$\widehat{F_Z(y)}^{-1} = \inf\left\{ z \mid \widehat{F_Z(z)} \leq y \right\}$$

where $0 \leq y \leq 1$, thus

$$\widehat{F_Z(U)}^{-1} = Z.$$

It then simply follows that

$$
\begin{aligned}
\widehat{F_Z(z)} &= P(Z \leq z) \\
&= P\left(\widehat{F_Z(U)}^{-1} \leq z \right) \\
&= P\left(U \leq \widehat{F_Z(z)} \right)
\end{aligned}
$$

Thus by generating a value u from U, calculate $\widehat{F_Z(u)}^{-1}$ which is set equal to z. This yields a $\widehat{F_Z(z)}$ distributed random variable.

8.4 Variance Reduction Techniques

Monte Carlo simulations are computationally expensive. It is practical to employ variance reduction procedures to decrease the number of simulations needed. Hull [23] gives a broad summary of variance reduction procedures.

The variance reduction procedures used in this dissertation are the *antithetic variable* and *moment matching* techniques. The control variate technique[2] can possibly also be used. To be certain of the soundness of the use of the control variate technique for the particular simulations done in this dissertation, further investigation is needed. This is beyond the scope of the dissertation.

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[2]For the simulations in this dissertation, for the option price resultant from a forecasted GARCH-M process, we can do the Controle Variate Technique as follows:

Two simulations, the standard Black-Scholes option pricing integral and the Duan GARCH integral are done in parallel using the same random variables.

The Black-Scholes simulation is then subtracted from the GARCH one and the equivalent analytical Black-Scholes value is added.

8.4.1 The Antithetic Variable Technique

With this technique two values for the derivative is calculated. The first value, f^1 is calculated the normal way with random sample vector $[\xi_t]_{T \times 1}$ taken from the applicable distribution. For the second value f^2, $-1 \times [\xi_t]_{T \times 1}$ is used. The final answer is the average of the two values

$$\frac{f^1 + f^2}{2}$$

The advantage of this technique is that a value above the true value can be "canceled out" by one below and vice versa.

8.4.2 Moment Matching

In this dissertation the standard normal distribution is used. In the moment matching technique all of the sampled random variables for each sample path, say the vector $[\xi_t]_{T \times 1}$ is standardized. This is done by subtracting the mean of the sample m from each element of the sample and then dividing that by the standard deviation s of the sample,

$$y_t = \frac{\xi_t - m}{s}$$

yielding a standard normal random variable.