Chapter 7

GARCH Option Pricing and Hedging

7.1 Introduction

This chapter builds on the results of chapter 6. European option on stocks with GARCH volatility is priced under the LRNVR. The delta hedge for such options is also derived.

Delta hedging is defined in Hull [23] as a hedging scheme that is designed to make the price of a portfolio of derivatives insensitive to small changes in the price of the underlying.

In the last section some of the properties of the most widely used GARCH process, the GARCH$(1,1)$ process is discussed.

7.2 Option Pricing under the LRNVR

The stock price process under LRNVR was discussed in the previous chapter. The machinery to model stocks with GARCH volatility can also be adapted to price European options.

Theorem 7.2.1 *GARCH option price. The price of a European call option on a non-dividend paying stock, S_t, expiring at T under LRNVR at time t is*

$$C_t^G = e^{-(T-t)r} E^Q \left[\max \left(S_T - K \right)_+ \mid \mathcal{F}_{t-1} \right]$$

where $\max(x)_+$ is the maximum between x and 0. Note that the parameters T and t in this context is in terms of time i.e. fractions of with the days of the year as the denominator, not as the discrete position of a variable or element of a process.

Proof. See theorem 5.4.2, since $e^{-rt}S_t$ is a martingale under the Q measure. ■

Theorem 7.2.2 *Delta hedge under LRNVR. The delta hedge for a stock with a European call option is*

$$\Delta_t^G = e^{-(T-t)r} E^Q \left[\frac{S_T}{S_t} 1_{[S_T \geq K]} \mid \mathcal{F}_{t-1} \right]$$

where $1_{[S_T \geq K]}$ is an indicator function and K the exercise price of the option.

Proof. *From corollary 6.8.2*

$$S_T = S_t \exp \left((T-t) r - \frac{1}{2} \sum_{s=t+1}^{T} \sigma_s + \sum_{s=t+1}^{T} \xi_s \right)$$

define

$$Y_{t,T} \equiv (T-t) r - \frac{1}{2} \sum_{s=t+1}^{T} \sigma_s + \sum_{s=t+1}^{T} \xi_s$$

then

$$E^Q [S_T] = E^Q [S_t \exp(Y_{t,T})]$$

The GARCH option price for a European option proved in theorem 7.2.1 now is

$$C_t^G (S_t) = e^{-(T-t)r} E^Q \left[\max \left(S_t e^{Y_{t,T}} - K \right)_+ \mid \mathcal{F}_{t-1} \right]$$

The delta hedge is the first partial derivative of the option price with respect to the underlying asset price. The strategy is to approximate this derivative with the function C_t^G. For an arbitrary $h > 0$

$$\begin{aligned}
& C_t^G (S_t + h) - C_t^G (S_t) \\
= \; & e^{-(T-t)r} E^Q \Big[\max \left((S_t + h) e^{Y_{t,T}} - K \right)_+ \\
& - \max \left(S_t e^{Y_{t,T}} - K \right)_+ \mid \mathcal{F}_{t-1} \Big] \\
= \; & e^{-(T-t)r} \int_{-\infty}^{\infty} \max \left((S_t + h) e^{y} - K \right)_+ \\
& - \max \left(S_t e^{y} - K \right)_+ dF (y \mid \mathcal{F}_t)
\end{aligned} \tag{7.1}$$

where $F(y \mid \mathcal{F}_t)$ is the cdf of $Y_{t,T}$ under Q. With an indicator function we can express the max function

$$\max \left((S_t + h) e^{Y_{t,T}} - K \right)_+ = \left((S_t + h) e^{Y_{t,T}} - K \right) I_{[(S_t+h) e^{Y_{t,T}} - K > 0]}$$

Consider that $h > 0$ then

$$(S_t + h) e^{Y_{t,T}} - K > 0$$

can be rewritten as

$$\begin{aligned} e^{Y_{t,T}} &> \frac{K}{(S_t+h)} \\ Y_{t,T} &= \ln e^{Y_{t,T}} > \ln \frac{K}{(S_t+h)} \end{aligned}$$

similarly

$$S_t e^{Y_{t,T}} - K > 0$$

can be rewritten as

$$Y_{t,T} = \ln e^{Y_{t,T}} > \ln \frac{K}{S_t}.$$

Equation 7.1 then becomes

$$\begin{aligned} & e^{-(T-t)r} \int_{\ln \frac{K}{(S_t+h)}}^{\infty} (S_t+h)\, e^{y} - K\, dF\,(y \mid \mathcal{F}_t) \\ & -e^{-(T-t)r} \int_{\ln \frac{K}{S_t}}^{\infty} S_t e^{y} - K\, dF\,(y \mid \mathcal{F}_t) \\ = \; & e^{-(T-t)r} \int_{\ln \frac{K}{(S_t+h)}}^{\ln \frac{K}{S_t}} S_t e^{y} - K\, dF\,(y \mid \mathcal{F}_t) \\ & +e^{-(T-t)r} \int_{\ln \frac{K}{(S_t+h)}}^{\infty} h e^{y}\, dF\,(y \mid \mathcal{F}_t)\,. \end{aligned}$$

Since

$$\lim_{h \to 0} e^{-(T-t)r} \int_{\ln \frac{K}{(S_t+h)}}^{\ln \frac{K}{S_t}} S_t e^{y} - K\, dF\,(y \mid \mathcal{F}_t) = 0$$

the

$$\begin{aligned} & \lim_{h \to 0} \frac{C_t^G\,(S_t+h) - C_t^G\,(S_t)}{h} \\ = \; & e^{-(T-t)r} \int_{\ln \frac{K}{S_t}}^{\infty} e^{y}\, dF\,(y \mid \mathcal{F}_t) \\ = \; & e^{-(T-t)r} E^{Q} \left[e^{y} 1_{[S_T > K]} \right] \end{aligned}$$

This argument could similarly have been proven from the left for $h < 0$. *Thus*

$$\begin{aligned} \frac{dC_t}{dS_t} &= \Delta_t \\ &= e^{-(T-t)r} E^{Q} \left[e^{y} 1_{[S_T > K]} \right]' \end{aligned}$$

This completes the proof. ■

The delta hedge of a stock and a European put option can be derived similarly. The delta hedge is

$$\Delta_t^G = e^{-(T-t)r} E^Q \left[\frac{S_T}{S_t} 1_{[K \geq S_T]} \mid \mathcal{F}_{t-1} \right]$$

7.3 Some Properties of the GARCH$(1,1)$ Process under LRNVR

Theorem 7.3.1 *Under measure Q innovations of the GARCH process is $\chi^2(1)$ distributed with non-centrality parameter λ.*

Proof. *From theorem 6.8.1 we have*

$$\sigma_t^2 = \alpha_0 + \alpha\left(\xi_{t-1} - \lambda\sigma_{t-1}\right)^2 + \beta\sigma_{t-1}^2$$

where

$$\xi_t | \mathcal{F}_{t-1} \sim N\left(0, \sigma_t^2\right)$$

thus

$$\frac{\xi_t}{\sigma_t} | \mathcal{F}_{t-1} \sim N(0,1)$$

The innovations of the GARCH process under LRNVR is

$$\begin{aligned} \sigma_t^2 &= \alpha_0 + \alpha\left(\xi_{t-1} - \lambda\sigma_{t-1}\right)^2 + \beta\sigma_{t-1}^2 \\ &= \alpha_0 + \alpha\sigma_{t-1}^2\left(\frac{\xi_{t-1}}{\sigma_{t-1}} - \lambda\right)^2 + \beta\sigma_{t-1}^2 \end{aligned}$$

then

$$\frac{1}{\alpha}\frac{\sigma_t^2 - \alpha_0}{\sigma_{t-1}^2} - \frac{\beta}{\alpha} = \left(\frac{\xi_{t-1}}{\sigma_{t-1}} - \lambda\right)^2$$

where

$$\frac{\xi_{t-1}}{\sigma_{t-1}} | \mathcal{F}_{t-2} \sim N(0,1)$$

which completes the proof. ■

Theorem 7.3.2 *Stationary (unconditional) variance of a GARCH process. If*

$$|\lambda| < \sqrt{\frac{(1-\alpha-\beta)}{\alpha}}$$

under probability measure Q then

1. *The stationary variance of ξ_t,*

$$var(\xi_t) = \frac{\alpha_0}{1 - (1+\lambda^2)\alpha - \beta}$$

2. *ξ_t is leptokurtic*

3. *The*

$$cov^Q\left(\frac{\xi_t}{\sigma_t}, \sigma_{t+1}^2\right) = \frac{-2\lambda\alpha_0\alpha}{1 - (1+\lambda^2)\alpha - \beta}$$

Proof. Proof of part 1.
Under the Q probability measure

$$\begin{aligned}\sigma_t^2 &= \alpha_0 + \alpha(\xi_{t-1} - \lambda\sigma_{t-1})^2 + \beta\sigma_{t-1}^2 \\ &= \alpha_0 + \alpha\sigma_{t-1}^2\left(\frac{\xi_{t-1}}{\sigma_{t-1}} - \lambda\right)^2 + \beta\sigma_{t-1}^2\end{aligned}$$

let

$$z_t \equiv \frac{\xi_t}{\sigma_t} - \lambda$$

then

$$\begin{aligned}\sigma_t^2 &= \alpha_0 + \alpha\sigma_{t-1}^2 z_{t-1}^2 + \beta\sigma_{t-1}^2 \\ &= \alpha_0 + \sigma_{t-1}^2(\alpha z_{t-1}^2 + \beta)\end{aligned}$$

Using this relationship

$$\sigma_{t-1}^2 = \alpha_0 + \sigma_{t-2}^2(\alpha z_{t-2}^2 + \beta)$$

thus

$$\begin{aligned}\sigma_t^2 &= \alpha_0 + (\alpha_0 + \sigma_{t-2}^2(\alpha z_{t-2}^2 + \beta))(\alpha z_{t-1}^2 + \beta) \\ &= \alpha_0 + \alpha_0(\alpha z_{t-1}^2 + \beta) + \sigma_{t-2}^2(\alpha z_{t-2}^2 + \beta)(\alpha z_{t-1}^2 + \beta)\end{aligned}$$

and further

$$\begin{aligned}\sigma_t^2 &= \alpha_0 + \alpha_0(\alpha z_{t-1}^2 + \beta) \\ &\quad + (\alpha_0 + \sigma_{t-3}^2(\alpha z_{t-3}^2 + \beta))(\alpha z_{t-2}^2 + \beta)(\alpha z_{t-1}^2 + \beta) \\ &= \alpha_0 + \alpha_0(\alpha z_{t-1}^2 + \beta) + \alpha_0(\alpha z_{t-2}^2 + \beta)(\alpha z_{t-1}^2 + \beta) \\ &\quad + \sigma_{t-3}^2(\alpha z_{t-3}^2 + \beta)(\alpha z_{t-2}^2 + \beta)(\alpha z_{t-1}^2 + \beta) \\ &= \alpha_0\left[1 + \sum_{k=1}^{2}\prod_{i=1}^{k}(\alpha z_{t-i}^2 + \beta)\right] + \sigma_{t-3}^2\prod_{k=1}^{3}(\alpha z_{t-i}^2 + \beta)\end{aligned}$$

Then by substituting previous equations for the variance from time $t-1$ to time 0 we obtain

$$\sigma_t^2 = \alpha_0 \left[1 + \sum_{k=1}^{t-1} \prod_{i=1}^{k} \left(\alpha z_{t-i}^2 + \beta\right)\right] + \sigma_0^2 \prod_{k=1}^{t} \left(\alpha z_{t-i}^2 + \beta\right) \tag{7.2}$$

$$\equiv \alpha_0 \sum_{k=0}^{t-1} G_k + \sigma_0^2 G_t \tag{7.3}$$

where

$$G_k \equiv \prod_{i=1}^{k} \left(\alpha z_{t-i}^2 + \beta\right) \tag{7.4}$$

$$= G_{k-1} \left(\alpha z_{t-k}^2 + \beta\right)$$

$$G_0 \equiv 1$$

From theorem 7.3.1 and the discussion on the chi-square distribution in section 2.5.3

$$z_t^2 = \left(\frac{\xi_t}{\sigma_t} - \lambda\right)^2$$

is chi-square distributed with one degree of freedom and non-centrality parameter λ, since

$$z_t \mid \mathcal{F}_{t-1} \overset{Q}{\sim} N\left(0, \sigma_t^2\right)$$

Thus from the tower property of conditional expectation and theorem 2.5.8

$$E^Q\left[E^Q\left[z_t^2 \mid \mathcal{F}_{t-1}\right] \mid \mathcal{F}_0\right] = E^Q\left[z_t^2 \mid \mathcal{F}_0\right]$$

$$= 1 + \lambda^2$$

Now from equation 7.4 for $t > k$

$$G_k = \prod_{i=1}^{k} \left(\alpha z_{t-i}^2 + \beta\right)$$

and the conditional expected value of G_k

$$E^Q\left[G_k \mid \mathcal{F}_0\right] = E^Q\left[\prod_{i=1}^{k} \left(\alpha z_{t-i}^2 + \beta\right) \mid \mathcal{F}_0\right]$$

Since z_r and z_s are independently distributed for all applicable r, s. z_r^2 and z_s^2 are also independent (see theorem 2.4.9). This allows us to write

$$cov\left(\alpha z_{t-i}^2 + \beta, \alpha z_{t-j}^2 + \beta\right) = 0$$

which follows from theorem 2.4.9 such that

$$\begin{aligned} E^Q\left[G_k \mid \mathcal{F}_0\right] &= \prod_{i=1}^{k} E^Q\left[\left(\alpha z_{t-i}^2 + \beta\right) \mid \mathcal{F}_0\right] \\ &= \prod_{i=1}^{k} a\left(1+\lambda^2\right)+\beta \\ &= \left[a\left(1+\lambda^2\right)+\beta\right]^k \end{aligned}$$

Using this result we can write the conditional expectation of equation 7.2

$$\begin{aligned} & E^Q\left[\sigma_t^2 \mid \mathcal{F}_0\right] \\ = \;& E^Q\left[\alpha_0\left[1+\sum_{k=1}^{t-1}\prod_{i=1}^{k}\left(\alpha z_{t-i}^2+\beta\right)\right]+\sigma_0^2\prod_{k=1}^{t}\left(\alpha z_{t-i}^2+\beta\right) \mid \mathcal{F}_0\right] \\ = \;& \alpha_0\sum_{k=0}^{t-1}\left[a\left(1+\lambda^2\right)+\beta\right]^k+\sigma_0^2\left[a\left(1+\lambda^2\right)+\beta\right]^t \end{aligned}$$

Using the condition that

$$|\lambda| < \sqrt{\frac{(1-\alpha-\beta)}{\alpha}}$$

the term

$$a\left(1+\lambda^2\right)+\beta < a\left(1+\frac{(1-\alpha-\beta)}{\alpha}\right)+\beta = 1$$

The stationary variance is the limit of t to infinite of $E\left[\sigma_t^2\right]$. By again using the tower property of conditional expectation

$$E^Q\left[E^Q\left[\sigma_t^2 \mid \mathcal{F}_0\right]\right] = E^Q\left[\sigma_t^2\right]$$

$$\begin{aligned} \lim_{t\to\infty} E^Q\left[\sigma_t^2\right] &= \alpha_0\sum_{k=0}^{\infty}\left[a\left(1+\lambda^2\right)+\beta\right]^k \\ &= \frac{\alpha_0}{a\left(1+\lambda^2\right)+\beta} \end{aligned}$$

Proof of part 2.
We need to prove that $E^Q\left[\xi_t^4\right] > 3\left(E^Q\left[\xi_t^2\right]\right)^2$ since

$$\xi_t \mid \mathcal{F}_0 \overset{Q}{\sim} N\left(0,\sigma_t^2\right)$$

In theorem 2.5.9 it was proved that

$$E^Q\left[z_t^4 \mid \mathcal{F}_0\right] = 3+6\lambda^2+\lambda^4$$

thus for $t > k$

$$\begin{aligned} & E^Q\left[G_k^2 \mid \mathcal{F}_0\right] \\ = \; & E^Q\left[\left(\prod_{i=1}^{k} \alpha z_{t-i}^2 + \beta\right)^2 \mid \mathcal{F}_0\right] \\ = \; & E^Q\left[\prod_{i=1}^{k} \left(\alpha z_{t-i}^2 + \beta\right)^2 \mid \mathcal{F}_0\right] \end{aligned}$$

Since

$$z_t^2 = \left(\frac{\xi_t}{\sigma_t} - \lambda\right)^2$$

is noncentral chi-square distributed with 1 degree of freedom and non-centrality parameter λ. *Again as in part 1, it follows from theorem 2.4.9 that*

$$cov\left(\left(\alpha z_{t-i}^2 + \beta\right)^2, \left(\alpha z_{t-j}^2 + \beta\right)^2\right) = 0$$

for all $i, j \in \{0, 1, ..., k\}$ *and* $i \neq j$. *Then*

$$\begin{aligned} & E^Q\left[\prod_{i=1}^{k} \left(\alpha z_{t-i}^2 + \beta\right)^2 \mid \mathcal{F}_0\right] \\ = \; & \prod_{i=1}^{k} E^Q\left[\left(\alpha z_{t-i}^2 + \beta\right)^2 \mid \mathcal{F}_0\right] \\ = \; & \prod_{i=1}^{k} E^Q\left[\alpha^2 z_{t-i}^4 + 2\alpha\beta z_{t-i}^2 + \beta^2 \mid \mathcal{F}_0\right] \\ = \; & \prod_{i=1}^{k} \left[\alpha^2\left(3 + 6\lambda^2 + \lambda^4\right) + 2\alpha\beta\left(1 + \lambda^2\right) + \beta^2\right] \\ = \; & \left[\alpha^2\left(3 + 6\lambda^2 + \lambda^4\right) + 2\alpha\beta\left(1 + \lambda^2\right) + \beta^2\right]^k \end{aligned}$$

For notational purposes define

$$\begin{aligned} u & \equiv \alpha^2\left(3 + 6\lambda^2 + \lambda^4\right) + 2\alpha\beta\left(1 + \lambda^2\right) + \beta^2 \\ v & = \alpha\left(1 + \lambda^2\right) + \beta \end{aligned}$$

then

$$u \equiv v^2 + 2\alpha^2\left(1 + 2\lambda^2\right)\alpha^2 \tag{7.5}$$

and

$$u > v$$

since all terms of equation 7.5 are positive.

For $k > j$

$$\begin{aligned}
& E^Q\left[G_k G_j \mid \mathcal{F}_0\right] \\
= \; & E^Q\left[\prod_{i=1}^{k}\left(\alpha z_{t-i}^2+\beta\right)\prod_{i=1}^{j}\left(\alpha z_{t-i}^2+\beta\right) \mid \mathcal{F}_0\right] \\
= \; & E^Q\left[\prod_{i=1}^{j}\left(\alpha z_{t-i}^2+\beta\right)^2\prod_{i=j+1}^{k}\left(\alpha z_{t-i}^2+\beta\right) \mid \mathcal{F}_0\right]
\end{aligned}$$

By theorem 2.4.9

$$cov\left[\left(\alpha z_{t-i}^2+\beta\right)^2,\left(\alpha z_{t-i}^2+\beta\right)\right]=0$$

for all $i,j \in \{0,1,...,k\}$, $k>j$. Thus

$$\begin{aligned}
& E^Q\left[\prod_{i=1}^{j}\left(\alpha z_{t-i}^2+\beta\right)^2\prod_{i=j+1}^{k}\left(\alpha z_{t-i}^2+\beta\right) \mid \mathcal{F}_0\right] \\
= \; & \prod_{i=1}^{j}E^Q\left[\left(\alpha z_{t-i}^2+\beta\right)^2 \mid \mathcal{F}_0\right]\prod_{i=j+1}^{k}E^Q\left[\left(\alpha z_{t-i}^2+\beta\right) \mid \mathcal{F}_0\right] \\
= \; & u^j v^{k-j}
\end{aligned}$$

Then the conditional expected value of σ_t^4, the square of the GARCH process under LRNVR at time t follows from equation 7.3

$$\begin{aligned}
& E^Q\left[\sigma_t^4 \mid \mathcal{F}_0\right] \\
= \; & E^Q\left[\left(\alpha_0\sum_{k=0}^{t-1}G_k+\sigma_0^2G_t\right)^2 \mid \mathcal{F}_0\right] \\
= \; & E^Q\left[\left(\alpha_0\sum_{k=0}^{t-1}G_k\right)^2+2\sigma_0^2G_t\left(\alpha_0\sum_{k=0}^{t-1}G_k\right)+\left(\sigma_0^2G_t\right)^2 \mid \mathcal{F}_0\right] \\
= \; & \alpha_0^2E^Q\left[\left(\sum_{k=0}^{t-1}G_k\right)^2 \mid \mathcal{F}_0\right]+2\alpha_0\sigma_0^2\sum_{k=0}^{t-1}E^Q\left[G_tG_k \mid \mathcal{F}_0\right] \\
& +\sigma_0^4E^Q\left[G_t^2 \mid \mathcal{F}\right] \\
= \; & \sigma_0^4u^t+2\alpha_0\sigma_0^2\sum_{k=0}^{t-1}u^kv^{t-k}+\alpha_0^2\left[\sum_{k=0}^{t-1}u^k+2\sum_{k=0}^{t-1}\sum_{j=0}^{k-1}u^jv^{k-j}\right] \qquad (7.6)
\end{aligned}$$

where the third term is a common mathematical expansion. The properties of geometric series (see Haggarty [21]) are used to simplify equation 7.6:

1. *Geometric series*

$$\sum_{k=0}^{t-1} u^k = \frac{1-u^t}{1-u}$$

2. *Geometric series where v^t is independent of the summation*

$$\begin{aligned}\sum_{k=0}^{t-1} u^k v^{t-k} &= v^t \sum_{k=0}^{t-1} \left(\frac{u}{v}\right)^k \\ &= v^t \frac{\left(1-\left(\frac{u}{v}\right)^t\right)}{1-\frac{u}{v}} \\ &= v\frac{u^t - v^t}{u-v}\end{aligned}$$

3. *Geometric series using point 2, where $\frac{v}{u-v}$ is independent of the summation*

$$\begin{aligned}\sum_{k=0}^{t-1}\sum_{j=0}^{k-1} u^j v^{k-j} &= \sum_{k=0}^{t-1} v\frac{u^k - v^k}{v-u} \\ &= \frac{v}{u-v}\sum_{k=0}^{t-1} u^k - \sum_{k=0}^{t-1} v^k \\ &= \frac{v}{u-v}\left(\frac{1-u^t}{1-u} - \frac{1-v^t}{1-v}\right)\end{aligned}$$

Equation 7.6 is simplified such that

$$\begin{aligned}& E^Q\left[\sigma_t^4 \mid \mathcal{F}_0\right] \\ = \; & \sigma_0^4 u^t + 2\alpha_0\sigma_0^2 v\frac{u^t - v^t}{u-v} \\ & +\alpha_0^2\left[\frac{1-u^t}{1-u} + 2\frac{v}{u-v}\left(\frac{1-u^t}{1-u} - \frac{1-v^t}{1-v}\right)\right]\end{aligned}$$

Now, to derive the value of the unconditional kurtosis of ξ_t, we take the limit of $E^Q\left[\sigma_t^4 \mid \mathcal{F}_0\right]$. If we remember that $u > v$ and assume that $u \geq 1$ then

$$E^Q\left[\sigma_t^4\right] = \lim_{t\to\infty} E^Q\left[\sigma_t^4 \mid \mathcal{F}_0\right] = \infty \tag{7.7}$$

and if $u < 1$

$$\lim_{t\to\infty} E^Q\left[\sigma_t^4 \mid \mathcal{F}_0\right] = \alpha_0^2\frac{(1-v)}{(1-u)(1-v)}$$

since

$$\begin{aligned}
& \lim_{t\to\infty} \alpha_0^2 \left[\frac{1-u^t}{1-u} + 2\frac{v}{u-v}\left(\frac{1-u^t}{1-u} - \frac{1-v^t}{1-v}\right)\right] \\
= & \lim_{t\to\infty} 2\alpha_0^2 \left[\frac{v}{u-v}\left(\frac{1-u^t}{1-u} - \frac{1-v^t}{1-v}\right)\right] + \frac{\alpha_0^2}{1-u} \\
= & 2\alpha_0^2 \left[\frac{v}{u-v}\left(\frac{1}{1-u} - \frac{1}{1-v}\right)\right] + \frac{\alpha_0^2}{1-u} \\
= & \alpha_0^2 \left[\frac{2v}{u-v}\frac{(1-v)-(1-u)}{(1-u)(1-v)} + \frac{1}{1-u}\right] \\
= & \alpha_0^2 \left[\frac{2v}{u-v}\frac{u-v}{(1-u)(1-v)} + \frac{1}{1-u}\right] \\
= & \alpha_0^2 \left[\frac{2v+1-v}{(1-u)(1-v)}\right] = \frac{\alpha_0^2(1+v)}{(1-u)(1-v)} \\
= & E^Q\left[\sigma_t^4\right]
\end{aligned} \tag{7.8}$$

Since σ_t^2 *is* $\mathcal{F}_{t-1}$ *measurable under* Q *and*

$$\frac{\xi_t}{\sigma_t} \mid \mathcal{F}_0 \overset{Q}{\sim} N(0,1)$$

the

$$\begin{aligned}
E^Q\left[\xi_t^4\right] &= E^Q\left[E^Q\left[\xi_t^4 \mid \mathcal{F}_{t-1}\right]\right] \\
&= E^Q\left[\sigma_t^4 E^Q\left[\left(\frac{\xi_t}{\sigma_t}\right)^4 \mid \mathcal{F}_{t-1}\right]\right] \\
&= 3E^Q\left[\sigma_t^4\right],
\end{aligned}$$

where

$$E^Q\left[\left(\frac{\xi_t}{\sigma_t}\right)^4 \mid \mathcal{F}_{t-1}\right]$$

is the kurtosis under Q.

Finally, from equation 7.7 it is clear that ξ_t *is leptokurtic if* $u \geq 1$. *If* $u < 1$ *then*

$$\begin{aligned}
E^Q\left[\xi_t^4\right] &= 3\frac{\alpha_0^2(1+v)}{(1-u)(1-v)} \\
&= 3\frac{1-v^2}{1-u}\left(E^Q\left[\xi_t^2\right]\right)^2
\end{aligned}$$

from the definition of v. *Since* $u > v > 0$

$$E^Q\left[\xi_t^4\right] > 3\left(E^Q\left[\xi_t^2\right]\right)^2$$

Proof of part 3.

From theorem 6.8.1 equation 6.16 we have

$$\sigma_t^2 = \alpha_0 + \alpha\left(\xi_{t-1} - \lambda\sigma_{t-1}\right)^2 + \beta\sigma_{t-1}^2$$

and

$$\xi_t|\mathcal{F}_{t-1} \sim N\left(0, \sigma_t^2\right)$$

Thus

$$\frac{\xi_t}{\sigma_t}\sigma_{t+1}^2 = \frac{\xi_t}{\sigma_t}\alpha_0 + \alpha\frac{\xi_t}{\sigma_t}\left(\xi_t - \lambda\sigma_t\right)^2 + \beta\frac{\xi_t}{\sigma_t}\sigma_t^2$$

and

$$\begin{aligned} E^Q\left[\frac{\xi_t}{\sigma_t}\sigma_{t+1}^2|\mathcal{F}_{t-1}\right] &= E^Q\left[\frac{\xi_t}{\sigma_t}\alpha_0 + \alpha\frac{\xi_t}{\sigma_t}\left(\xi_t - \lambda\sigma_t\right)^2 + \beta\frac{\xi_t}{\sigma_t}\sigma_t^2|\mathcal{F}_{t-1}\right] \\ &= E^Q\left[\alpha\frac{\xi_t}{\sigma_t}\left(\xi_t - \lambda\sigma_t\right)^2|\mathcal{F}_{t-1}\right] \end{aligned}$$

since

$$E^Q\left[\xi_t|\mathcal{F}_{t-1}\right] = 0$$

Then

$$\begin{aligned} & E^Q\left[\alpha\frac{\xi_t}{\sigma_t}\left(\xi_t - \lambda\sigma_t\right)^2|\mathcal{F}_{t-1}\right] \\ = \ & \alpha E^Q\left[\frac{\xi_t}{\sigma_t}\left(\xi_t^2 - 2\lambda\xi_t\sigma_t + (\lambda\sigma_t)^2\right)|\mathcal{F}_{t-1}\right] \\ = \ & \alpha E^Q\left[\frac{\xi_t^3}{\sigma_t} - 2\lambda\xi_t^2 + \lambda^2\xi_t\sigma_t|\mathcal{F}_{t-1}\right] \\ = \ & \frac{\alpha}{\sigma_t}E^Q\left[\xi_t^3|\mathcal{F}_{t-1}\right] - 2\alpha\lambda E^Q\left[\xi_t^2|\mathcal{F}_{t-1}\right] + \alpha\lambda^2\sigma_t E^Q\left[\xi_t|\mathcal{F}_{t-1}\right] \end{aligned}$$

since σ_t is $\mathcal{F}_{t-1}$ measurable, the

$$\begin{aligned} & E^Q\left[\alpha\frac{\xi_t}{\sigma_t}\left(\xi_t - \lambda\sigma_t\right)^2|\mathcal{F}_{t-1}\right] \\ = \ & -2\alpha\lambda E^Q\left[\xi_t^2|\mathcal{F}_{t-1}\right] \\ = \ & -2\alpha\lambda\sigma_t^2. \end{aligned}$$

Finally,

$$\begin{aligned}
& cov^Q\left[\frac{\xi_t}{\sigma_t}, \sigma_{t+1}^2\right] \\
= \; & E^Q\left[\frac{\xi_t}{\sigma_t}\sigma_{t+1}^2\right] - E^Q\left[\frac{\xi_t}{\sigma_t}\right] E^Q\left[\sigma_{t+1}^2\right] \\
= \; & E^Q\left[\frac{\xi_t}{\sigma_t}\sigma_{t+1}^2\right] \\
= \; & E^Q\left[E^Q\left[\frac{\xi_t}{\sigma_t}\sigma_{t+1}^2 | \mathcal{F}_{t-1}\right]\right]
\end{aligned}$$

the tower property of conditional expectation. The

$$\begin{aligned}
& cov^Q\left[\frac{\xi_t}{\sigma_t}, \sigma_{t+1}^2\right] \\
= \; & E^Q\left[E^Q\left[\alpha\frac{\xi_t}{\sigma_t}\left(\xi_t - \lambda\sigma_t\right)^2 | \mathcal{F}_{t-1}\right]\right] \\
= \; & E^Q\left[-2\alpha\lambda\sigma_t^2\right] \\
= \; & -2\alpha\lambda E^Q\left[\sigma_t^2\right] \\
= \; & \frac{-2\lambda\alpha_0\alpha}{1-\left(1+\lambda^2\right)\alpha - \beta}
\end{aligned}$$

by the proof of 2. ∎