Part III

Option pricing under the Local Risk-Neutral Valuation Relationship

Chapter 6

Local Risk-Neutral Valuation

6.1 Introduction

One of the properties of Brownian motion is that equally spaced increments are stationary, that is, it can be assumed that they are independently and identically distributed. The vast majority of empirical studies show that this is generally not the case.

Stochastic volatility in stock prices complicates the pricing of derivative instruments. The assumption of a complete market model and therefore the risk-neutral probability measure derived in chapter 5 no longer holds. This is because we cannot completely hedge away the risk posed by stochastic volatility.

Jin-Chuan Duan (1995) [10] defined a new measure, the *local risk-neutral probability measure*. He showed that an economic agent maximizes its expected utility by using this measure. In this incomplete market, extra assumptions are made about the consumer (its utility function) and the risk premium demanded by the market for taking additional risk. Duan named the properties of the measure, the *local risk-neutral valuation relationship* (LRNVR).

In this chapter the GARCH, EGARCH and GJR-GARCH processes are considered in the GARCH-M framework. The GARCH processes are in discrete time, thus unlike the risk-neutral pricing framework which forms the basis for Black-Scholes framework, the LRNVR is in discrete time.

In section 6.2, the continuous-time option pricing model discussed in chapter 5 is converted into a discrete time model. The goal of this section is to translate and compare some of the well-known continuous time finance concepts into discrete time statistical concepts. For example the continuous time concept of Brownian motion is converted in discrete time to that of expected returns.

The GARCH-in-Mean model for the volatility of a discrete time stock price process used by Duan for option pricing, is introduced in section 6.3.

Utility functions and the risk aversion of economic agents are discussed in section 6.4. A general consumption-investment strategy is maximized in section 6.5.

The LRNVR is defined in section 6.6 after which the local risk-neutral measure is derived in section 6.7. The stock price process under the new measure is discussed in section 6.8.

6.2 The Stock Price Process in Discrete Time

Recall the stock price process of section 5.13 with solution

$$\tilde{S}_t = S_0 \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t\right]$$

In discrete time, with equally spaced observations,

$$\tilde{S}_{t-1} = S_0 \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right) \times (t-1) + \sigma W_{t-1}\right]$$

which gives

$$S_0 = \tilde{S}_{t-1} \exp\left[-\left(\mu - \frac{1}{2}\sigma^2\right) \times (t-1) - \sigma W_{t-1}\right]$$

At time t, the value of

$$\begin{aligned} \tilde{S}_t &= S_0 \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t\right] \\ &= \tilde{S}_{t-1} \exp\left[-\left(\mu - \frac{1}{2}\sigma^2\right)(t-1) - \sigma W_{t-1}\right] \\ &\quad \times \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t\right] \end{aligned}$$

Since

$$W_t - W_{t-1} \sim W_{t-t+1} \sim N(0,1)$$

let

$$\varepsilon_t | \mathcal{F}_{t-1} \overset{P}{\sim} N(0,1)$$

where ε_t is $\mathcal{F}_{t-1}$ measurable.

The one period ahead stock price is defined by

$$\tilde{S}_t = \tilde{S}_{t-1} \exp\left(\mu - \frac{1}{2}\sigma^2 + \sigma\varepsilon_t\right) \tag{6.1}$$

where

$$\varepsilon_t|\mathcal{F}_{t-1} \overset{P}{\sim} N(0,1)$$

If we let

$$\lambda = \frac{\mu - r}{\sigma}$$

where r is the risk-free rate of interest, equation 6.1 would become

$$\tilde{S}_t = \tilde{S}_{t-1} \exp\left(r - \frac{1}{2}\sigma^2 + \lambda\sigma + \sigma\varepsilon_t\right)$$

In the discrete case where the information on time $t-1$ is known, we could just as well have considered a volatility process which is constant between time $t-1$ and t.

6.3 The Stock Price Model under certain GARCH Volatility

Jin-Chuan Duan proposed the following conditional, lognormally distributed stock price process, with stochastic volatility, under the P measure

$$S_t = S_{t-1} \exp\left(r^*\Delta t - \frac{1}{2}\sigma_t^2 + \lambda\sigma_t + \sigma_t\varepsilon_t\right) \tag{6.2}$$

where

$$\varepsilon_t|\mathcal{F}_{t-1} \overset{P}{\sim} N(0,1)$$

is the conditional error process, σ_t^2 is the conditional variance (GARCH process) and λ[1] the unit risk premium. $\mathcal{F}_{t-1}$ is the σ-algebra of information up to time t. The yearly risk-free rate of return is r^*. Henceforth define r over period Δt, the same time period over which the conditional variance is taken. From this point on the period is daily.

[1] A possible interpretation of the unit risk premium follows from section 6.2 which deals with the market price of risk. Define the risk premium as

$$\lambda^* = \frac{\mu - r}{\sigma}$$

where σ is the long term or unconditional standard deviation of the series $\{X_t\}$.We can simplify the term with the risk premium from equation 6.3 to get

$$\lambda^*\sigma_t = (\mu - r)\frac{\sigma_t}{\sigma}.$$

$(\mu - r)$ can be seen as a fixed (positive) premium. $\lambda^*\sigma_t$ increases as the predicted conditional volatility σ_t increases over the long term volatility σ. The economic interpretation is that the market agent demands a higher premium as the expected volatility increases.

The conditional expected rate of return is defined as

$$\begin{aligned}\ln \frac{S_t}{S_{t-1}} &= r - \frac{1}{2}\sigma_t^2 + \lambda\sigma_t + \sigma_t\varepsilon_t \\ &\sim N\left(r - \frac{1}{2}\sigma_t^2 + \lambda\sigma_t, \sigma_t^2\right)\end{aligned} \tag{6.3}$$

This is derived by transforming equation 6.2.

The GARCH option pricing model prices options under conditional heteroscedasticity. This means that conditional variance is allowed to change over time while keeping unconditional variance constant. In this dissertation, options whose variance follows (vanilla) GARCH, GJR-GARCH and EGARCH process will be investigated. The main focus will be on the GARCH(p,q) process and specifically GARCH$(1,1)$ process.

The GARCH(p,q) conditional variance process is

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2 + \sum_{i=1}^{p} \beta_i \sigma_{t-i}^2$$

where $\alpha_0 > 0$ and $\alpha_i, \beta_i \geq 0$ for all applicable numbers i. Notice that σ_t^2 is predictable at time $t-1$.

The GJR-GARCH variance process is

$$\sigma_t^2 = \alpha_0 + \beta\sigma_{t-1}^2 + \alpha\varepsilon_{t-1}^2 + \gamma \max\left(-\varepsilon_t, 0\right)^2$$

where $\gamma > 0$.

The EGARCH variance process is

$$\ln \sigma_t^2 = \alpha_0 + \beta_1 \ln \sigma_{t-1}^2 + \beta_2 \left(|\varepsilon_{t-1}| - \gamma\varepsilon_{t-1}\right)$$

where $\beta_2, \gamma > 0$.

6.4 Consumer Utility Essentials

6.4.1 Utility Functions

The satisfaction (utility) an economic agent gets from consumption can often not be described on a monetary scale. A utility function represents an economic agent's welfare from consumption.

In this dissertation we assume that utility is measurable and possible to represent in a function. This function is called a (cardinal) utility function. Define the utility function by

$$u(x) : \mathbb{R}^+ \to \mathbb{R}^+$$

that is

$$u\,(\text{monetary cost of consumption}) = \text{“welfare” from that consumption}$$

It is generally assumed that a utility function has the following three properties:

1. $u(x)$ is twice differentiable
2. $u(x)$ is an increasing function of x, $u'(x) > 0$
3. $u(x)$ is a concave function of x, $u''(x) < 0$.

Property 2 is due to the fact that an economic agent prefers to consume more than less. Property 3 can be interpreted in terms of additional consumption. The utility that an economic agent gains from additional consumption g, in excess of an existing consumption x,

$$u(x+g) - u(x) < u(g)$$

Each economic agent has its own unique preferences and thus utility function.

6.4.2 Risk Aversion

Absolute Risk Aversion

For a given utility function $u(x)$, in continuous time, we can associate an absolute risk aversion function defined by

$$R(x) = \frac{-u''(x)}{u'(x)} = -\frac{d}{dx}\ln u'(x) \tag{6.4a}$$

Properties 2 and 3 of section 6.4.1 insures that $R(x) > 0$ for all x. The bigger $R(x)$ is, the less risk the economic agent is willing to take for additional consumption. The discrete time version of equation 6.4a

$$\begin{aligned} \widetilde{R}(x) &= -\frac{\ln u'(x_t) - \ln u'(x_{t-1})}{x_t - x_{t-1}} \\ &= -\frac{\ln \frac{u'(x_t)}{u'(x_{t-1})}}{x_t - x_{t-1}} \end{aligned}$$

Relative Risk Aversion

The relative risk aversion for a utility function $u(x)$ is defined by

$$\begin{aligned} r(x) &= x\widetilde{R}(x) = -\frac{u''(x)}{u'(x)}x \\ &= -\frac{\frac{d}{dx}\ln u'(x)}{\frac{d}{dx}\ln x} \end{aligned}$$

The discrete time risk aversion function is

$$\begin{aligned}\widetilde{r}(x) &= -\frac{\ln u'(x_t)-\ln u'(x_{t-1})}{x_t-x_{t-1}} \div \frac{\ln x_t-\ln x_{t-1}}{x_t-x_{t-1}} \\ &= -\frac{\ln u'(x_t)-\ln u'(x_{t-1})}{\ln x_t-\ln x_{t-1}} \\ &= -\frac{\ln \frac{u'(x_t)}{u'(x_{t-1})}}{\ln \frac{x_t}{x_{t-1}}}\end{aligned}$$

6.5 A General Consumption-Investment Strategy

Consider an investor with the following discrete time consumption-investment plan: The investor maximizes its differentiable utility function, $u(x)$: $\mathbb{R}^+ \rightarrow \mathbb{R}^+$, at any point in time $t-1$ by either consuming, $C_{t-1} \in \mathbb{R}^+$ or by investing, $H_{t-1} \in \mathbb{R}$, in a portfolio with random payoff $S_t \in \mathbb{R}^+$ at time t. At time t the investor again has the same choice between consumption and investment from the payoff from time $t-1$. Like most investors, this investor gets more satisfaction out of consuming immediately than waiting for the next period, hence define the impatience factor $\rho \in \mathbb{R}$. At time $t-1$ this plan can be written as

$$\max E^P\left[u(C_{t-1})+\exp(-\rho)\,u(C_t) \mid \mathcal{F}_{t-1}\right] \tag{6.5}$$

subject to

$$v = C_{t-1}+H_{t-1}S_{t-1} \tag{6.6}$$

$$H_{t-1}S_t = C_t+H_tS_t \tag{6.7}$$

v is the payoff of the investment made at time $t-2$. Take note that at time $t-1$ the only choices this investor make is to consume now or invest for one period ahead, thus the expected utility of consumption of periods after time t isn't of concern. Since C_{t-1} is predictable at time $t-1$ problem (6.5) can be rewritten as

$$\max u(C_{t-1})+\exp(-\rho)\,E^P\left[u(C_t) \mid \mathcal{F}_{t-1}\right] \tag{6.8}$$

The aim here is to maximize utility in terms of consumption and investment. From equations (6.6) and (6.7) consumption in subject (6.8) can be rewritten in terms of investment as

$$\max\ u(v-H_{t-1}S_{t-1})+\exp(-\rho)\,E^P\left[u(H_{t-1}S_t-H_tS_t) \mid \mathcal{F}_{t-1}\right] \tag{6.9}$$

If we then maximize problem (6.9) in terms of H_{t-1} we get

$$\begin{aligned} 0 &= \frac{\partial}{\partial H_{t-1}} u\left(v - H_{t-1}S_{t-1}\right) \\ &\quad + \frac{\partial}{\partial H_{t-1}} \exp\left(-\rho\right) E^P\left[u\left(H_{t-1}S_t - H_t S_t\right) \mid \mathcal{F}_{t-1}\right] \\ &= -S_{t-1}u'\left(v - H_{t-1}S_{t-1}\right) \\ &\quad + \exp\left(-\rho\right) E^P\left[S_t u'\left(H_{t-1}S_t - H_t S_t\right) \mid \mathcal{F}_{t-1}\right] \end{aligned}$$

which by further simplification and equations (6.6) and (6.7) yield

$$S_{t-1} = E^P\left[\exp\left(-\rho\right) \frac{u'\left(C_t\right)}{u'\left(C_{t-1}\right)} S_t \mid \mathcal{F}_{t-1}\right] \tag{6.10}$$

Thus the price of the portfolio at time $t-1$ is written in terms of the expected value of the economic agent's utility, its impatience and the expected future portfolio value.

6.6 The Local Risk-Neutral Valuation Relationship

The conventional risk-neutral valuation doesn't accommodate heteroscedasticity of stock returns. The Local Risk-neutral Valuation Relationship (LRNVR) is a way to generalize risk-neutral valuation to accommodate heteroscedasticity.

Definition 6.6.1 $(\Omega, \mathcal{F}, P)$. *A probability measure Q is said to be a local risk-neutral probability measure if*

1. *Q is equivalent to measure P*
2. $E^Q\left[\ln\left(\frac{S_t}{S_{t-1}}\right) |\mathcal{F}_{t-1}\right] = r$ *for all* $t \in \mathbb{R}^+$
3. $Var^Q\left[\ln\left(\frac{S_t}{S_{t-1}}\right) |\mathcal{F}_{t-1}\right] = Var^P\left[\ln\left(\frac{S_t}{S_{t-1}}\right) |\mathcal{F}_{t-1}\right] \qquad P$ *a.s.*

Remark 6.6.2 *Condition 1 is the same as in 5.2.7. Condition 2 is also similar but only defined over one period. The expected return doesn't locally depend on preferences. The one period conditional variance of the returns are invariant almost surely under the equivalent measures.*

The rest of this chapter focuses on the possible characteristics of an economic agent and the distribution of returns for which the LRNVR will hold.

6.7 The Local Risk-Neutral Probability Measure

Theorem 6.7.1 *Let a process Y_t be such that $Y_t \mid \mathcal{F}_{t-1} \overset{P}{\sim}$ is normally distributed with constant mean and variance under the P-measure. Define Q as*

$$dQ = \exp\left((r-\rho)T + \sum_{s=1}^{T} Y_s\right) dP$$

then Q is a measure and is equivalent to P.

Proof. Measure. *Q is a measure by Corollary 4.9 of Bartle [3], since*

$$\exp\left((r-\rho)T + \sum_{s=1}^{T} Y_s\right)$$

is a nonnegative $\mathcal{F}_{t-1}$ measurable function from Ω to $\mathbb{R}$.

Equivalence. *Let $A \in \mathcal{F}$ be a set such that*

$$P(A) = 0$$

Then

$$\begin{aligned} P(A) &= 0 \\ &\Leftrightarrow \int_A dP = \int_\Omega I_A dP = 0 \end{aligned}$$

where I_A is a characteristic function for A, I_A is a measurable and nonnegative function.

$$\int_\Omega I_A dP = 0 \Leftrightarrow I_A = 0 \quad P-a.s.$$

This means that

$$I(\omega) = 0$$

for all $\omega \in A \equiv \Omega \backslash M$ where $P(M) = 0$. This holds if and only if

$$I(\omega) f(\omega) = 0$$

for all $\omega \in A \equiv \Omega \backslash M$ where $P(M) = 0$. $f(\omega)$ is a measurable positive continuous function from Ω to $\mathbb{R}^+$. The product of real measurable functions $f_A I_A$ is also measurable. This is the same as

$$fI_A = 0 \quad P-a.s.$$

It is also equivalent to [3]

$$\int_\Omega f_A I_A dP = \int_A f dP$$

It is clear that

$$\tilde{f}(\omega) = \exp\left((r-\rho)T + \sum_{s=1}^{T} Y_s(\omega)\right)$$

is a nonnegative, measurable continuous function from Ω *to* $\mathbb{R}^+$ *(* $Y_s : \Omega \to \mathbb{R}$ *for every* s*).*

$$\tilde{f}(\omega) = 0$$

if and only if $Y_s(\omega) = -\infty$ *for any* s*. Fortunately* $P\{Y_s = -\infty\} = 0$ *since* Y_s *is normal.*

Thus

$$\begin{aligned} P(A) &= 0 \\ &\Leftrightarrow \int_\Omega I_A dP = 0 \\ &\Leftrightarrow \int_\Omega \exp\left((r-\rho)T + \sum_{s=1}^{T} Y_s(\omega)\right) I_A dP = 0 \\ &\Leftrightarrow Q(A) = 0 \end{aligned}$$

Thus the measure Q *is equivalent to measure* P*.* ■

The measure Q isn't, in general, a probability measure. In the next theorem conditions under which Q is a probability measure will be defined and a desirable property of Q will be derived.

Theorem 6.7.2 *If*

$$S_{t-1} = E^P\left[S_t \exp(-\rho + Y_t) \mid \mathcal{F}_{t-1}\right] \tag{6.11}$$

then

1. Q *is a probability measure*
2. *If* W_t *is* $\mathcal{F}_t$ *measurable then*

$$E^Q\left[W_t \mid \mathcal{F}_{t-1}\right] = E^P\left[W_t \exp((r-\rho) + Y_t) \mid \mathcal{F}_{t-1}\right]$$

Proof. *From the definition of* Q

$$dQ = \exp\left((r-\rho)T + \sum_{s=1}^{T} Y_s\right) dP$$

In integral notation

$$
\begin{aligned}
\int_\Omega dQ &= \int_\Omega \exp\left((r-\rho)T + \sum_{s=1}^{T} Y_s\right) dP \\
&= E^P\left[\exp\left((r-\rho)T + \sum_{s=1}^{T} Y_s\right)\right] \\
&= E^P\left[\exp\left((r-\rho)T + \sum_{s=1}^{T} Y_s\right) \mid \mathcal{F}_0\right] \\
&= E^P\left[\exp\left((r-\rho)(T-1) + \sum_{s=1}^{T-1} Y_s\right) \exp(r-\rho+Y_T) \mid \mathcal{F}_0\right] \\
&= E^P\left[e^{\left((r-\rho)(T-1)+\sum_{s=1}^{T-1} Y_s\right)} e^r E^P\left[e^{\rho+Y_T} \mid \mathcal{F}_{T-1}\right] \mid \mathcal{F}_0\right]
\end{aligned}
$$

This last step is due to the tower property of conditional expectation. Assumption 6.11 states that

$$E^P\left[\exp(-\rho+Y_t) S_t \mid \mathcal{F}_{t-1}\right] = S_{t-1}$$

thus

$$E^P\left[\exp(-\rho+Y_T) \mid \mathcal{F}_{T-1}\right] = \exp(-r) \tag{6.12}$$

for a risk-free asset. The result is that

$$
\begin{aligned}
\int dQ &= E^P\left[\exp\left((r-\rho)(T-1) + \sum_{s=1}^{T-1} Y_s\right) \exp(r)\exp(-r) \mid \mathcal{F}_0\right] \\
&= E^P\left[\exp\left((r-\rho)(T-1) + \sum_{s=1}^{T-1} Y_s\right) \mid \mathcal{F}_0\right]
\end{aligned}
$$

The tower property can again be invoked and an argument similar to 6.12 can be derived.

$$
\begin{aligned}
\int_\Omega dQ &= E^P\left[e^{\left((r-\rho)(T-2)+\sum_{s=1}^{T-2} Y_s\right)} e^r E^P\left[e^{(\rho+Y_{T-1})} \mid \mathcal{F}_{T-2}\right] \mid \mathcal{F}_0\right] \\
&= E^P\left[\exp\left((r-\rho)(T-2) + \sum_{s=1}^{T-2} Y_s\right) \mid \mathcal{F}_0\right]
\end{aligned}
$$

This can be repeated until we have, at filtration $\mathcal{F}_0$,

$$\int_\Omega dQ = E^P\left[\exp((r-\rho)+Y_1) \mid \mathcal{F}_0\right] = 1$$

Thus

$$Q(\Omega) = 1$$

with this property, the measure Q is a probability measure. We also have that

$$Q(\Omega) = E^P\left[\exp\left((r-\rho)+Y_1\right)\right] = 1$$

it is clear that

$$\exp\left((r-\rho)T + \sum_{s=1}^{T} Y_s\right) \geq 0$$

and we proved in theorem 6.7.1 that Q is equivalent to P. The Radon-Nikodym theorem can be invoked thus

$$\exp\left((r-\rho)T + \sum_{s=1}^{T} Y_s\right)$$

is $P-a.s.$ unique and for any $\mathcal{F}_t$ measurable set W_t,

$$E^Q\left[W_t \mid \mathcal{F}_{t-1}\right] = E^P\left[W_t \exp\left((r-\rho)+Y_t\right) \mid \mathcal{F}_{t-1}\right]$$

■

Theorem 6.7.3 *If*

$$S_{t-1} = E^P\left[\exp\left(-\rho+Y_t\right) S_t \mid \mathcal{F}_{t-1}\right]$$

then

1. $\ln\left(\frac{S_t}{S_{t-1}}\right) \mid \mathcal{F}_{t-1} \overset{Q}{\sim} normal$

2. $E^Q\left[\frac{S_t}{S_{t-1}} \mid \mathcal{F}_{t-1}\right] = e^r$ *for all* $t \in \mathbb{R}^+$

3. $Var^Q\left[\ln\left(\frac{S_t}{S_{t-1}}\right) \mid \mathcal{F}_{t-1}\right] = Var^P\left[\ln\left(\frac{S_t}{S_{t-1}}\right) \mid \mathcal{F}_{t-1}\right] \qquad P\ a.s.$

Proof. Lemma 2.
From theorem 6.7.2 we have

$$\begin{aligned} & E^Q\left[\frac{S_t}{S_{t-1}} \mid \mathcal{F}_{t-1}\right] \\ = \ & E^P\left[\frac{S_t}{S_{t-1}} \exp\left((r-\rho)+Y_t\right) \mid \mathcal{F}_{t-1}\right] \\ = \ & \frac{e^r}{S_{t-1}} E^P\left[S_t \exp\left(-\rho+Y_t\right) \mid \mathcal{F}_{t-1}\right] \\ = \ & e^r \end{aligned}$$

Proof of lemmas 1 and 3.

In theorem 6.7.2 we proved that

$$E^Q[W_t \mid \mathcal{F}_{t-1}] = E^P[W_t \exp((r-\rho)+Y_t) \mid \mathcal{F}_{t-1}]$$

for all $\mathcal{F}_t$ measurable sets W_t. If W_t is $\mathcal{F}_t-$measurable, so is W_t^c for all $c \in \mathbb{R}$. From theorem 6.7.2 we have that

$$E^Q[S_t^c \mid \mathcal{F}_{t-1}] = E^P\left[S_t^c e^{((r-\rho)+Y_t)} \mid \mathcal{F}_{t-1}\right]$$

then

$$\begin{aligned} E^Q\left[\frac{S_t^c}{S_{t-1}^c} \mid \mathcal{F}_{t-1}\right] &= E^P\left[\frac{S_t^c}{S_{t-1}^c} e^{((r-\rho)+Y_t)} \mid \mathcal{F}_{t-1}\right] \\ E^Q\left[e^{c\ln\frac{S_t}{S_{t-1}}} \mid \mathcal{F}_{t-1}\right] &= E^P\left[e^{c\ln\frac{S_t}{S_{t-1}}} e^{((r-\rho)+Y_t)} \mid \mathcal{F}_{t-1}\right] \\ E^Q\left[e^{cX_t} \mid \mathcal{F}_{t-1}\right] &= E^P\left[e^{cX_t} e^{((r-\rho)+Y_t)} \mid \mathcal{F}_{t-1}\right] \end{aligned}$$

if we define

$$X_t = \ln\frac{S_t}{S_{t-1}}.$$

Throughout this chapter there's been assumed that $X_t \mid \mathcal{F}_{t-1}$ is normally distributed under P, say

$$X_t \mid \mathcal{F}_{t-1} \overset{P}{\sim} N\left(\mu_t, v_t^2\right)$$

In theorem 6.7.1 we assumed that Y_t is also conditionally normal. Y_t can thus be written in terms of X_t, a constant α and another random variable with zero mean U_t, which is independent of X_t. Then

$$Y_t = \alpha + \beta X_t + U_t$$

with $\beta \in \mathbb{R}$. Thus

$$\begin{aligned} & E^Q\left[e^{cX_t} \mid \mathcal{F}_{t-1}\right] \\ = {} & E^P\left[e^{cX_t} e^{((r-\rho)+Y_t)} \mid \mathcal{F}_{t-1}\right] \\ = {} & E^P\left[e^{cX_t+\beta X_t+\alpha+U_t+(r-\rho)} \mid \mathcal{F}_{t-1}\right] \\ = {} & e^{\alpha+r-\rho} E^P\left[e^{(c+\beta)X_t+U_t} \mid \mathcal{F}_{t-1}\right] \end{aligned} \tag{6.13}$$

The joint variance of $(c+\beta)X_t$ and U_t under P is

$$var\left((c+\beta)X_t + U_t\right) = (c+\beta)^2 v_t^2 + E^P\left[U_t^2\right]$$

since U_t is of zero mean. By the moment generating function

$$\begin{aligned} & E^P\left[e^{(c+\beta)X_t+U_t} \mid \mathcal{F}_{t-1}\right] \\ = \; & e^{\mu_t(c+\beta)+\frac{1}{2}\left((c+\beta)^2 v_t^2+E^P\left[U_t^2\right]\right)} \\ = \; & e^{\mu_t(c+\beta)+\frac{1}{2}\left(c^2+2c\beta+\beta^2\right)v_t^2+\frac{1}{2}E^P\left[U_t^2\right]} \\ = \; & e^{\frac{1}{2}\beta^2 v_t^2+\mu_t\beta+\frac{1}{2}E^P\left[U_t^2\right]+\frac{1}{2}c^2 v_t^2+c\left(\mu_t+\beta v_t^2\right)} \end{aligned}$$

Then equation 6.13 becomes

$$\begin{aligned} E^Q\left[e^{cX_t} \mid \mathcal{F}_{t-1}\right] &= e^{\left[(r-\rho)+\frac{1}{2}E^P\left[U_t^2 \mid \mathcal{F}_{t-1}\right]+\mu_t\beta+\beta^2 v_t^2\right]} \times \\ & \quad e^{\left[\frac{1}{2}c^2 v_t^2+c\left(\mu_t+\beta v_t^2\right)\right]} \end{aligned}$$

This equation holds for all $c \in \mathbb{R}$. If we let $c = 0$ then

$$\begin{aligned} 1 &= E^Q\left[1 \mid \mathcal{F}_{t-1}\right] \\ &= e^{(r-\rho)+\frac{1}{2}E^P\left[U_t^2 \mid \mathcal{F}_{t-1}\right]+\mu_t\beta+\beta^2 v_t^2} \end{aligned}$$

so we are left with

$$E^Q\left[e^{cX_t} \mid \mathcal{F}_{t-1}\right] = e^{\frac{1}{2}c^2 v_t^2+c\left(\mu_t+\beta v_t^2\right)}$$

If we let $c = 1$, then by the form of the answer of a moment generating function,

$$X_t = \ln\left(\frac{S_t}{S_{t-1}}\right) \mid \mathcal{F}_{t-1} \overset{Q}{\sim} N\left(\mu_t + \beta v_t^2, v_t^2\right)$$

Which proves 1. The conditional variance under P of X_t is also σ^2 thus lemma 3 is also proved. ∎

Theorem 6.7.4 *An economic agent who's an expected utility maximizer and whose utility function is separable and additive is a LRNVR investor under the following conditions:*

1. *The utility function is of constant relative risk aversion and the changes in the logarithm of the aggregate consumption are conditionally normally distributed with constant mean and variance under the P measure*

2. *The utility function is of constant absolute risk aversion and the changes in the logarithm of the aggregate consumption are conditionally normally distributed with constant mean and variance under the P measure*

3. *The utility function is linear.*

The local risk-neutral measure is

$$dQ = e^{-(r-\rho)T}\frac{U'(C_t)}{U'(C_{t-1})}dP$$

The implied interest rate is assumed constant.

Proof.

1. *From the discussion on the utility function and risk aversion it is possible to define conditions 1 to 3:*

   (a) ***Condition 1:*** *A utility function of constant relative risk aversion is defined by*

$$\begin{aligned}\lambda_1 &= -\frac{d\ln U'(C)}{dC} \div \frac{d\ln C}{dC} \\ &= -\frac{\ln U'(C_t) - \ln U'(C_{t-1})}{\ln C_t - \ln C_{t-1}}\end{aligned}$$

$$\begin{aligned}\ln U'(C_t) - \ln U'(C_{t-1}) &= (-\lambda_1)(\ln C_t - \ln C_{t-1}) \\ \ln\left(\frac{U'(C_t)}{U'(C_{t-1})}\right) &= (-\lambda_1)\ln\left(\frac{C_t}{C_{t-1}}\right) \qquad (6.14)\end{aligned}$$

   Since we assume that $\ln(C_t/C_{t-1})$ is normally distributed with constant mean and variance under P, $\ln(U'(C_t)/U'(C_{t-1}))$ is also normal with constant mean and variance.

   (b) ***Condition 2:*** *A utility function of constant absolute risk aversion is defined by*

$$\begin{aligned}\lambda_2 &= -\frac{d\ln U'(C)}{dC} \\ &= -\frac{\ln U'(C_t) - \ln U'(C_{t-1})}{C_t - C_{t-1}}\end{aligned}$$

   thus

$$\begin{aligned}\ln U'(C_t) - \ln U'(C_{t-1}) &= (-\lambda_2)(C_t - C_{t-1}) \\ \ln\left(\frac{U'(C_t)}{U'(C_{t-1})}\right) &= (-\lambda_2)(C_t - C_{t-1})\end{aligned}$$

   By the assumption that $C_t - C_{t-1}$ is normally distributed with constant mean and variance under P, $\ln(U'(C_t)/U'(C_{t-1}))$ is also normal with constant mean and variance.

   (c) ***Condition 3:*** *A linear utility function is defined by*

$$U(C_t) = aC_t + c$$

   thus

$$U'(C_t) = a$$

and

$$\frac{U'(C_t)}{U'(C_{t-1})} = 1$$

The ratio of marginal utilities

$$\ln\left(\frac{U'(C_t)}{U'(C_{t-1})}\right) = 0 \overset{P}{\sim} N(0,0)$$

From all three conditions it is clear that $\ln\left(\frac{U'(C_t)}{U'(C_{t-1})}\right)$ *is normal with constant mean and variance.*

2. *In section* 6.5 *we saw that under the* $P-$*measure*

$$\begin{aligned} S_{t-1} &= E^P\left[e^{-\rho}\frac{u'(C_t)}{u'(C_{t-1})}S_t \mid \mathcal{F}_{t-1}\right] \\ &= E^P\left[e^{-\rho+\ln\left(\frac{u'(C_t)}{u'(C_{t-1})}\right)}S_t \mid \mathcal{F}_{t-1}\right] \\ &\equiv E^P\left[e^{-\rho+Y_t}S_t \mid \mathcal{F}_{t-1}\right] \end{aligned} \tag{6.15}$$

where $Y_t = \ln\left(\frac{u'(C_t)}{u'(C_{t-1})}\right)$. Y_t, *as mentioned, is normally distributed under conditions 1 to 3. If we define* Q *as*

$$dQ = e^{(r-\rho)T+\sum_{s=1}^{T} Y_s} dP$$

then from theorem 6.7.1, Q *is a measure which is equivalent to* P. *From theorem 6.7.2 we see that* Q *is a probability measure and*

$$E^Q\left[W_t \mid \mathcal{F}_{t-1}\right] = E^P\left[W_t e^{(r-\rho)T+Y_y} \mid \mathcal{F}_{t-1}\right]$$

for any W_t *which is* $\mathcal{F}_t$ *measurable. Another result from equation 6.15 stated in theorem 6.7.3 is that*

(a) $ln\left(\frac{S_t}{S_{t-1}}\right) \mid \mathcal{F}_{t-1} \overset{Q}{\sim}$ *normal*

(b) $E^Q\left[\frac{S_t}{S_{t-1}} \mid \mathcal{F}_{t-1}\right] = e^r$ *for all* $t \in \mathbb{R}^+$

(c) $Var^Q\left[\ln\left(\frac{S_t}{S_{t-1}}\right) | \mathcal{F}_{t-1}\right] = Var^P\left[\ln\left(\frac{S_t}{S_{t-1}}\right) | \mathcal{F}_{t-1}\right]$ $\quad P$ *a.s.*

3. *Thus for an economic agent who's an expected utility maximizer, whose utility function is separable, additive and fulfills one of the three stated conditions, the Local risk-neutral Valuation Relationship also holds.*

■

6.8 The Stock Price Process under LRNVR

In this section the stock price process under the LRNVR is derived.

Theorem 6.8.1 *Under the $Q-$measure, implied by the LRNVR,*

$$\ln \frac{S_t}{S_{t-1}} = r - \frac{1}{2}\sigma_t^2 + \xi_t \sigma_t$$

where

$$\xi_t | \mathcal{F}_{t-1} \overset{Q}{\sim} N(0,1)$$

and

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{q} \alpha_i \left(\xi_{t-i} - \lambda\sigma_{t-i}\right)^2 + \sum_{i=1}^{p} \beta_i \sigma_{t-i}^2 \tag{6.16}$$

Note that the parameters T and t in this context are in terms of time i.e. fractions with the days of the year as the denominator, not as the discrete index. That is, for 63 days in a 252 day year $t = 0.25$.

Proof. *As proved in theorem 6.7.3, $\ln \frac{S_t}{S_{t-1}} | \mathcal{F}_{t-1}$ is normally distributed under measure Q. It can thus be written in terms of a deterministic and random variable*

$$\ln \frac{S_t}{S_{t-1}} = v_t + \xi_t \tag{6.17}$$

under Q.The random variable is obviously normal with mean zero and variance the same of that of $\ln \frac{S_t}{S_{t-1}}$ under Q. It will be proved that

1. $v_t = r - \frac{1}{2}\sigma_t^2$
2. $\sigma_t^2 = \alpha_0 + \sum_{i=1}^{q} \alpha_i \left(\xi_{t-i} - \lambda\sigma_{t-i}\right)^2 + \sum_{i=1}^{p} \beta_i \sigma_{t-i}^2$

Proof of 1:
From equation 6.17

$$\begin{aligned}
\frac{S_t}{S_{t-1}} &= e^{v_t + \sigma_t \xi_t} \\
E^Q\left[\frac{S_t}{S_{t-1}} | \mathcal{F}_{t-1}\right] &= E^Q\left[e^{v_t + \sigma_t \xi_t} | \mathcal{F}_{t-1}\right] \\
&= e^{v_t} E^Q\left[e^{\sigma_t \xi_t} | \mathcal{F}_{t-1}\right]
\end{aligned}$$

then by the moment generating function for an normally distributed random variable we have

$$E^Q\left[\frac{S_t}{S_{t-1}} | \mathcal{F}_{t-1}\right] = e^{v_t + \frac{1}{2} Var^Q\left[\ln \frac{S_t}{S_{t-1}} | \mathcal{F}_{t-1}\right]} E^Q\left[1 | \mathcal{F}_{t-1}\right]$$

Since

$$\begin{aligned} Var^Q\left[\ln\left(\frac{S_t}{S_{t-1}}\right)|\mathcal{F}_{t-1}\right] &= Var^P\left[\ln\left(\frac{S_t}{S_{t-1}}\right)|\mathcal{F}_{t-1}\right] \\ &= \sigma_t^2 \ P-a.s. \end{aligned}$$

from theorem 6.7.3 we can write

$$E^Q\left[\frac{S_t}{S_{t-1}}|\mathcal{F}_{t-1}\right] = e^{\upsilon_t+\frac{1}{2}\sigma_t^2}$$

It was also proved in theorem 6.7.3 that

$$E^Q\left[\frac{S_t}{S_{t-1}} \mid \mathcal{F}_{t-1}\right] = e^r$$

thus

$$\upsilon_t + \frac{1}{2}\sigma_t^2 = r$$

$$\upsilon_t = r - \frac{1}{2}\sigma_t^2$$

Proof of 2.

Recall the original stock price process with GARCH volatility under the P measure, equation 6.2,

$$\ln\frac{S_t}{S_{t-1}} = r + \lambda\sigma_t - \frac{1}{2}\sigma_t^2 + \varepsilon_t$$

and the process implied by proof 1 above under measure $-Q$

$$\ln\frac{S_t}{S_{t-1}} = r - \frac{1}{2}\sigma_t^2 + \xi_t$$

Again using the result

$$Var^Q\left[\ln\left(\frac{S_t}{S_{t-1}}\right)|\mathcal{F}_{t-1}\right] = Var^P\left[\ln\left(\frac{S_t}{S_{t-1}}\right)|\mathcal{F}_{t-1}\right] \ P-a.s$$

from theorem 6.7.3 we can write

$$r + \lambda\sigma_t - \frac{1}{2}\sigma_t^2 + \varepsilon_t = r - \frac{1}{2}\sigma_t^2 + \xi_t$$

thus

$$\varepsilon_t = \xi_t - \lambda\sigma_t$$

Substituting this result into

$$\ln\frac{S_t}{S_{t-1}} = r + \lambda\sigma_t - \frac{1}{2}\sigma_t^2 + \varepsilon_t$$

yields

$$\ln \frac{S_t}{S_{t-1}} = r - \frac{1}{2}\sigma_t^2 + \xi_t$$

and into the GARCH process, yields

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{q} \alpha_i \left(\xi_{t-i} - \lambda\sigma_{t-i}\right)^2 + \sum_{i=1}^{p} \beta_i \sigma_{t-i}^2 \tag{6.18}$$

under the Q measure. ■

The equivalent GJR-GARCH process under the Q measure is

$$\sigma_t^2 = \alpha_0 + \beta\sigma_{t-1}^2 + \alpha\left(\xi_{t-1} - \lambda\sigma_{t-1}\right)^2 + \gamma \max\left(-\xi_{t-1} + \lambda\sigma_{t-1}, 0\right)^2$$

where $\gamma > 0$.

The EGARCH variance process under the Q measure is

$$\ln \sigma_t^2 = \alpha_0 + \beta_1 \ln \sigma_{t-1}^2 + \beta_2 \left(\left|\xi_{t-1} - \lambda\sigma_{t-1}\right| - \gamma\left(\xi_{t-1} - \lambda\sigma_{t-1}\right)\right)$$

where $\beta_2, \gamma > 0$.

Theorem 6.8.1 can easily be proved for the above two GARCH processes by substituting equation 6.18 with the respective process and replacing the P variable ε_t with the Q variable $\xi_{t-1} - \lambda\sigma_{t-1}$.

Corollary 6.8.2 *Theorem 6.8.1 implies that under the Q measure*

$$S_T = S_t \exp\left((T-t) \times r - \frac{1}{2}\sum_{s=t+1}^{T} \sigma_s + \sum_{s=t+1}^{T} \xi_s\right) \tag{6.19}$$

Proof. *From theorem 6.8.1 we have that*

$$\ln \frac{S_t}{S_{t-1}} = r - \frac{1}{2}\sigma_t^2 + \xi_t$$

for every $t \in \mathbb{R}$ under Q. Thus

$$\begin{aligned} \ln \frac{S_T}{S_t} &= \sum_{s=t+1}^{T} \ln \frac{S_s}{S_{s-1}} \\ &= \sum_{s=t+1}^{T} r - \frac{1}{2}\sigma_s^2 + \xi_s \\ &= r(T-t) - \frac{1}{2}\sum_{s=t+1}^{T} \sigma_s^2 + \sum_{s=t+1}^{T} \xi_s \end{aligned} \tag{6.20}$$

which means that

$$S_T = S_t \exp\left((T-t)\,r - \frac{1}{2}\sum_{s=t+1}^{T}\sigma_s + \sum_{s=t+1}^{T}\xi_s\right)$$

by taking exponents on both sides of equation 6.20. ■

Corollary 6.8.3 *The discounted process $e^{-rt}S_t$ is a martingale under the Q measure.*

Proof. *Corollary 6.8.2 is equivalent to*

$$S_t = S_{t-1}\exp\left(r - \frac{1}{2}\sigma_t^2 + \xi_t\right)$$

thus the conditional expected value of $e^{-rt}S_t$ is

$$\begin{aligned}
& E^Q\left[\exp(-rt)\,S_t \mid \mathcal{F}_{t-1}\right] \\
= \; & E^Q\left[S_{t-1}\exp(-rt)\exp\left(r - \frac{1}{2}\sigma_t^2 + \xi_t\right) \mid \mathcal{F}_{t-1}\right] \\
= \; & S_{t-1}\exp(-r(t-1))\,E^Q\left[\exp\left(-\frac{1}{2}\sigma_t^2 + \xi_t\right) \mid \mathcal{F}_{t-1}\right] \\
= \; & S_{t-1}\exp(-r(t-1))
\end{aligned}$$

because $\xi_t|\mathcal{F}_{t-1} \overset{Q}{\sim} N\left(0, \sigma_t^2\right)$ and by the moment generating function

$$E^Q\left[\exp(\xi_t) \mid \mathcal{F}_{t-1}\right] = \exp\frac{1}{2}\sigma_t^2$$

which completes the proof. ■