Chapter 1

Introduction

Three categories of financial models prevail in the market[1]. They are the following:

1. **Structural models.** Simplifying assumptions about the underlying market processes and market equilibrium are made to infer equilibrium prices and thus the relationships between underlying instruments and their contingent claims (i.e. options). The Black-Scholes[2] formula is the most famous structural model. The Black-Scholes formula is the result of a method called *risk-neutral* (or arbitrage) pricing. A result of the risk-neutral pricing is that we can infer a unique, correct price of a contingent claim given its underlying stock price. Any other option price would lead to an arbitrage opportunity.

2. **Statistical models.** These models rely on empirical data and their co-dependencies. Fewer assumptions, if any, are made concerning the structure of the market. Examples of statistical models in financial mathematics are the capital asset pricing model and time series processes. Financial time series are used to describe data, to obtain insight into their dynamic patterns and to forecast out-of-sample returns. The *Generalized Autoregressive Conditional Heteroscedastic* (GARCH) process is a famous time series used to model the conditional variance of a process.

3. **Combination of structural and statistical models.** This category of models combines the above categories of models. The GARCH option pricing model under the *local risk-neutral valuation relationship* (LRNVR), discussed in this dissertation, is the combination of GARCH literature and risk-neutral valuation.

[1]See 'Risk Management' by Crouhy, Galai and Mark [9].

[2]The Black-Scholes model was developed by Black and Scholes (1973) and Merton (1973).

Under risk-neutral pricing, the price of a contingent claim is independent of the risk preference and utility functions of buyers and sellers, hence there exists a unique and correct option price. The cost of this model is the simplifying assumptions. Some of the crude assumptions made in the Black-Scholes model are:

1. Stock prices are *lognormally* distributed, thus the continuously compounded stock returns are normally distributed.
2. The mean and volatility under this distribution are constant.
3. The risk-free interest rate is constant or a known function of time.
4. Delta hedging is done continuously (short selling is allowed and securities are perfectly devisable).
5. No transaction costs on the underlying.
6. No arbitrage opportunities.

Empirical evidence shows that none of these assumptions are valid. In this dissertation the assumption of constant volatility is abandoned, for (conditional) *stochastic volatility.*

Volatility has many definitions. It is generally seen as the standard deviation of a random process (i.e. the stock returns process). In the Black-Scholes framework, implied volatility can be inferred from the market price of the option and the underlying. Conditional volatility can be seen as a measure of risk. This is because levels of trade tend to increase in uncertainty in the stock, sector or market in general and hence the standard deviation or price fluctuations increase.[3].

In this dissertation, volatility is seen as the standard deviation of a stochastic process. Implied volatility comes into play in later chapters where the GARCH option pricing model is applied to JSE Exchange traded warrants.

1.1 The Problem of Stochastic Volatility

The Black-Scholes model is a complete market model. A market model is *complete* if and only if all contingent claims are *replicable.* Equivalently, under no arbitrage conditions, a market model is complete if and only if there exists a unique risk-free probability measure.

If stochastic volatility is introduced into a market model, it is no longer complete[4]. This is because there are too much variability in the stock price

[3]For a thorough discussion on market volatility, see Poon & Granger [29].

[4]See Fouque et al [17].

which cannot be hedged away completely, since there are no instruments in the market which is perfectly correlated with the individual stock's volatility. Equivalently there doesn't exist a unique risk-neutral probability measure.

A consequence of stochastic volatility is that the price of the contingent claim depends on the risk preference and utility of investors. This complicates computation of the price of the contingent claim.

1.2 A Proposed Solution

The aim of this dissertation is to discuss a solution too the problem of option pricing in incomplete markets, due to stochastic volatility. The LRNVR was introduced by Jin-Chuan Duan [10] in 1995. Duan proved that the measure

$$dQ = e^{-(r-\rho)T}\frac{U'(C_t)}{U'(C_{t-1})}dP$$

satisfies the LRNVR. In this measure, r is the risk-free interest rate, ρ is an impatience factor and U' is the first derivative of the utility function of consumption C_t at time t. The measure Q is called the local risk-neutral measure.

The volatility process in this dissertation is the GARCH process introduced by Engle (1982) and Bollerslev (1986) [6]. The GARCH process is a discrete time process of the changing variance of the returns of an underlying instrument. This process captures phenomena of returns series coined "stylized facts". These phenomena are heavy-tails[5] of distributions, volatility clustering[6] and mean reversion[7]. GARCH processes have been extended to capture another stylized fact called the leverage effect[8]. Such GARCH processes are called asymmetric GARCH processes.

The GARCH parameters are derived from actual market prices. The stock price, at expiry of a European option, is forecasted with the GARCH process under the local risk-neutral measure. This forecast is done with Monte Carlo simulations.

In this dissertation the GARCH option pricing method is applied to South African put warrants.

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[5] Excess kurtosis above that of the normal distribution.

[6] Volatility levels tend to cluster together at the same levels for a certain duration, after which it clusters together at another level.

[7] Volatility levels tens to revert back to a certain long-term level after a shock. The reversion to this level is not neccesarily immediate.

[8] The market tends to react more drastcally to bad news than good news.

1.3 Description of South African Derivative Instruments and Experiment

There are two markets where financial derivatives are traded in South Africa. The one market is the warrants market of the JSE Securities Exchange (JSE) and the other is the South African Futures Exchange (SAFEX). The SAFEX exchange was bought by the JSE on the 1^{st} of July 2001.

Equity options on SAFEX are traded on a limited number of stocks and on some index futures. The SAFEX market tends to be illiquid. In illiquid markets the spread between bid and offer prices tends to be wider than that of a more liquid markets.

On the JSE, warrants[9] are traded. A warrant is an option issued, like a stock, by financial institutions on equities, certain interest rate instruments and some indices. This means that a market player must own a warrant to sell it, thus no short selling is allowed. The warrants market is more liquid than the SAFEX options market, but because no short selling is allowed, there are no way to gain from overpriced warrants. In this market, only market equilibrium (supply and demand) controls price levels. The result is that the implied volatility levels of warrants tend to be higher than the volatility of stock prices. See figure 1.1.

In this dissertation the GARCH option pricing method is applied to equity European put warrants on the JSE. Approximately 30% of traded warrants are European put warrants. The warrants market was selected because it's more liquid than the SAFEX option market. In more liquid markets, option prices reacts more rapidly to changes in the price of the underlying, thus the testing of the GARCH option pricing method is easier to do.

In Duan's 1995 paper the GARCH process is calibrated to the returns series of the underlying equity or index with the maximum likelihood method. Since the implied volatility of warrants are higher than the historical standard deviation of the underlying equity, the GARCH process in this dissertation is fitted to the implied volatility of the warrant.

1.4 Outline of the Dissertation

In the following chapter, essential background to probability theory is discussed. This discussion includes some measure theoretical background, stochastic mathematics and discussions on the normal distribution.

In chapter 3, basic concepts of time series are introduced. Autoregressive Moving Averages time series are the main topic of discussion. Univariate volatility processes literature is reviewed and investigated in section 4 which

[9]Warrants on the JSE must not be confused for an option issued by a company on its own stock which is available in some countries.

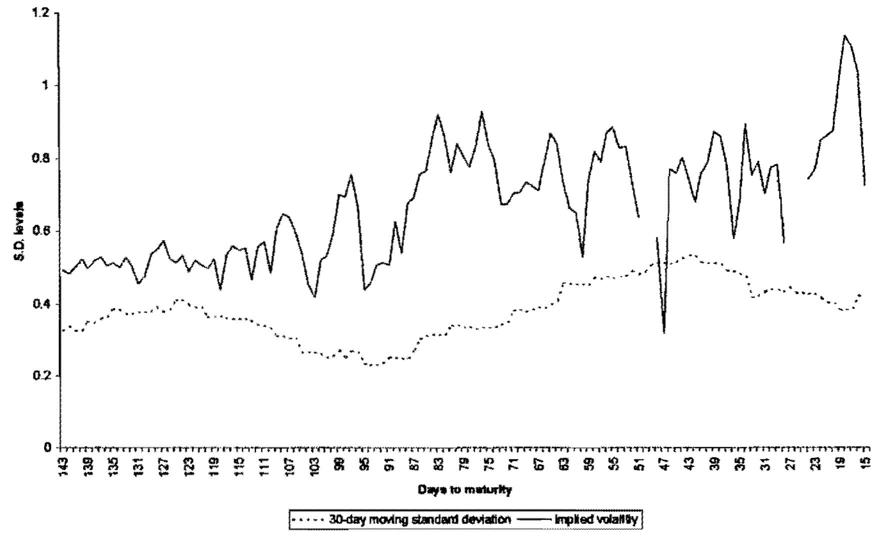

Figure 1.1: The moving 30-day standard deviation against the implied volatility of the warrant: 3SAPIB on Sappi. The breaks in the implied volatility graph is due to market illiquidity. The intrinsic value of the replicating portfolio is more than the value of the option.

builds on the ARMA discussion. The most important univariate volatility process is the (vanilla) GARCH process. Other important GARCH processes are also investigated.

Risk-neutral valuation is the basis of modern option pricing. Risk-neutral valuation and continuous time finance is discussed in chapter 5. This discussion leads to the pricing of options in incomplete markets and the LRNVR investigated in chapter 6.

Chapter 7 is about the application of the LRNVR to option pricing. Delta hedging under LRNVR is also investigated.

Monte Carlo simulations and optimization forms part of chapter 8 where the implementation of GARCH option pricing is discussed.

Results are given in chapter 9 and the conclusion follows in chapter 10. Related literature is discussed in section 11.