An analysis of the term structure of interest rates and bond options in the South African capital market

by

Linda Smit

Presented in partial fulfilment of the degree

Philosophiae Doctor in Applied Mathematics

in the Faculty of Science

University of Pretoria

Pretoria

Supervisor: Prof. FD van Niekerk

May 2000

Acknowledgements

Throughout my studies I received help, advice, guidance and support from many people, and I would like to thank all of them.

Firstly, I would like to thank my mentor and supervisor *Prof. Frik van Niekerk*. In his own unique way, he encouraged and inspired me to complete this thesis. He did not only supervise and guide my work - he formed an integral part of this research. I thank him for all the hours we worked together.

Furthermore, I would like to thank all the *academics* both locally and internationally who gave me guidance and advice, particularly Prof. Pradeep Jadav, Dr Paul Wilmott, Prof. Peter Medvegyev and Prof. Dawie de Jongh.

I would also like to thank the management of *Absa Corporate Bank*, my employer during most of my studies, as well as my colleagues, who supported me in many ways.

Finally, I want to thank my family and friends for all their love and support over the years. To my mother and father who encouraged and inspired me since I began school, and to my sisters, Carin and Sonja, and brother Gustav, who gave me their full support - thank you. To my husband, Wynand, for his patience, and to my son, Marco, who was born in the midst of my studies, all my love.

CONTENTS

List of tables

List of figures

CHAPTER 1

INTRODUCTION

Barings Bank, one of the oldest British banks, went bankrupt in February 1995. A sum twice as large as Barings' capital was lost on the derivatives market by its futures trader, Nick Leeson. A year later, Daiwa Bank lost $1 billion, and a few months later, Sumitomo Corporation lost $1.7 billion in the copper market. In 1997, National Westminster Bank lost £90 million on options trading. In 1998, Long-Term Capital Management had to be bailed out due to its over-exposure in emerging markets. Several other companies worldwide have suffered large losses over the past few years because of their speculative trading in derivatives.

Since most of the losses in these cases were incurred as a result of derivative positions, these collapses caused great concern in financial markets over the world. Derivatives have now been recognized as dangerous instruments. Nevertheless, there has been explosive growth in the derivatives market over the past 25 years. This growth is related to the fact that the trade volume of derivatives is usually much higher than that of the underlying instruments. Notwithstanding various losses by banks and funds which have failed to manage the financial risk adequately, derivatives can be successfully used to hedge or reduce financial risk *or* to create highly leveraged speculative positions in the market. According to the office of the Controller of the Currency (OCC) (Dashtidar, 2000:11), in the third quarter of 1999, the notional value of derivatives held in US commercial banks rose to a record high of $35.7 trillion. One of the most dominant forms of derivatives is interest rate contracts, representing 79% of all activity.

This study concentrates on the South African fixed income market, where the bond market ranks as one of the most liquid emerging bond markets in the world, with a daily turnover in excess of R40 billion (www.bondex.co.za). The South African fixed income market ranks in size in the top 25 fixed income markets in the world and ranks second in terms of market turnover related to market size (www.bondindex.co.za).

1.1 Aim of the study

The enormous impact of derivatives in the financial world necessitates the use of accurate valuation and risk-forecast models. The aim of this study is to focus on the South African fixed income market and evaluate current models and procedures. In order to add value, alternatives are proposed where necessary.

All valuation models depend on certain assumptions and therefore have certain limitations.

However, many participants in the derivatives markets use a 'black box' approach, without realizing the consequences of applying an inappropriate model. Both the awareness of risk and the challenge of making a profit encourage traders to take positions that correspond to their appetites for risk. As a trader, one soon realises the benefit of having more accurate valuation models which enable one to take on more risk with greater confidence. A better understanding of the pricing model, the risk and variables involved gives traders more confidence which, in turn, allows them to take bigger positions.

When one realises how imperfect models are, choosing a viable valuation method becomes difficult. It is for this reason that the study analyses the existing models and procedures used in the South African fixed income market, and, where applicable, tries to find an alternative. Throughout the study there is an attempt to reconcile the knowledge obtained by means of theoretical research with the practical problems experienced in the market.

Although many academics and practitioners have developed methods for valuing and managing interest rate instruments and derivatives, little work has been done with regard to the specific characteristics of the South African fixed income market. The aim of this study is to address certain inefficiencies experienced in the South African fixed income market in the area of term structure analysis and bond option valuation.

1.2 The term structure of interest rates

Since the fixed income market is entirely based on the term structure of interest rates, it remains the most important input in the pricing of any fixed income derivative security. (It also influences other derivatives, as it determines the discount factor for discounting the expected payoff.) Analysing the yield curve is thus a very important aspect of decisionmaking for managers of fixed income portfolios and hedge funds. The yield curve contains information about future market expectations of interest rates. An essential aspect for managing or trading fixed income instruments is to understand the derivation of a zero-coupon yield curve, a swap curve and a forward curve.

In South Africa, where only coupon bonds are traded, the Johannesburg Stock Exchange (JSE) Actuarial Yield Curve has been conventionally accepted as the benchmark yield curve. This curve is available on a daily basis and is seen as the South African yield curve. It is, however, merely a fit through the yield-to-maturities of South African government bonds. It can be seen as an approximation to a par-bond curve, although the bonds are *not* par-bonds. A zero-coupon yield curve gives a homogeneous function of yield against term-to-maturity.

The South African bond market trades mainly in coupon bonds, and little or no data is available for zero-coupon instruments. Thus it is necessary to do bootstrapping. This is, however, a time-consuming process. When one uses this method every day, one becomes convinced that there has to be a quicker and more efficient way to get the same, or even better results. An iterative bootstrap method was therefore developed. It starts with a first guess for the zero-coupon yield curve and then converges to the actual zero-coupon yield curve. Since the publication of this method in *RISK* (Smit & Van Niekerk, 1997), the technique has been used by several practitioners and academics both locally and internationally.

1.3 The bond option market

The South African bond options market is largely driven by the over-the-counter (OTC) market. The options are mainly American options which can be early-exercised. The importance of understanding the risk characteristics of an option and realizing their profit potential serves as an incentive to search for alternative ways of valuing options in order to use all opportunities to optimize profits. Using the Black model (Black, 1976), which was developed for commodity futures, was clearly insufficient.

This study discusses the theory underlying the most popular bond option pricing models, and concentrates on the Hull-White model (Hull & White, 1990). The numerical solution of the Hull-White model applied to South African OTC bond options is discussed in depth. The reasons for choosing the Hull-White model are the following:

- It incorporates mean reversion of interest rates.
- It determines the pull-to-par effect analytically.
- It is exactly consistent with the initial term structure of interest rates.
- It incorporates the early-exercise value of American options.
- It addresses implicitly the risk of a change in the cost-of-carry.

The Hull-White model has had to be adjusted for its application to South African bond options, as these options are traded on the yield-to-maturity of the bond, rather than the price. Because the numerical solution to the Hull-White model uses the current term structure of interest rates as an input, the zero-coupon curve is used as an input. Although the options are American, holders of these options seldom early-exercise them, since it is generally believed that the time-value is lost if one does so. The conditions in which OTC bond options are early-exercised, are therefore discussed in more detail.

The complexity of the Hull-White model encouraged the development of a simplified model for exchange-traded options (see Chapter 7). The new model could also stimulate bond option trade on the South African Futures Exchange (SAFEX), a market which is illiquid at present. An exchange-traded bond option has no short-term risk-free rate component, as the underlying instrument is the bond future, and interest is earned on the margin account. Therefore, instead of using the short-rate or the price of the bond as the stochastic variable, it is possible to assume that the *yield-to-maturity* of the bond follows a Brownian motion. A pricing model for options on the future yield of a bond is in many ways similar to the Black model (see Chapter 5). However, the yield-based model addresses most of the disadvantages of the Black model.

1.4 Structure of the study

The study is structured as follows: Chapter 2 discusses the most important concepts of the valuation of derivative securities. In order to understand the valuation of derivative securities sufficiently, it is necessary also to have a good grasp of the concepts of arbitrage, martingales and partial differential equations. The remainder of the study is divided into two fields of research: first, the term structure of interest rates (Chapters 3 and 4) and, second, bond option valuation models (Chapters 5, 6 and 7).

Chapter 4 aims to develop an improved bootstrapping method in order to obtain a zero-coupon yield curve. A yield curve gives the relation between the yield of a fixed income investment and its term. The zero-coupon yield curve is the basis for pricing all vanilla products (bonds, swaps, forward swaps, etc.) in the fixed income market and serves as an important input in pricing bond options using a no-arbitrage model.

Chapter 5 then discusses several existing bond option valuation models. In Chapters 6 and 7 there is an attempt to improve on existing methods to value South African bond options. In Chapter 8 there is a summary of the findings, followed by conclusions and recommendations.

CHAPTER 2

DERIVATIVE SECURITIES – THE THEORY

The valuation of derivative securities has drawn the attention of mathematicians across the world and has become a field of research for many. In order to obtain the fair value of any derivative security, it is important to understand the concepts of a stochastic process, arbitrage, martingales and partial differential equations. In this chapter, the fundamentals of option pricing theory are briefly set out.

Since 1973, the Black-Scholes model (Black & Scholes, 1973) has been the most popular option pricing model. This model can be adjusted in order to price options on various underlying instruments. The theory of option pricing can be applied to derive the two Black-Scholes option pricing formulas for call and put options, given by equations (19) and (20) (in this chapter).

2.1 Basic theory

2.1.1 *Introduction*

One could consider a probability space $(\Psi, \mathscr{F}, P)$, where Ψ is a sample space; $\mathscr{F}$ is a σ-field on Ψ, and $\mathscr{F}$ consists of a collection of subsets of Ψ, called events; and P denotes a probability measure on $(\Psi, \mathscr{F})$. The measure P is a countable additive set function assigning a non-negative number $P(A)$ to each set $A \in \Psi$.

A *random variable*, called u, is a measurable mapping of Ψ into $\mathbb{R}$. A sequence (u_n) of random variables is called a discrete time stochastic process. Let $\mathscr{F}_n$ be the set of events known at time t_n. A *filtration* of the probability space is an increasing sequence of σ-algebras $\mathscr{F}_1 \subseteq \mathscr{F}_2 \subseteq \ldots \subseteq \mathscr{F}_n \subseteq \ldots \mathscr{F}$.

If each u_n is measurable with regard to the corresponding member of $\mathscr{F}_n$ of the filtration, then the stochastic process is said to be *adapted*. This means that u_n is measurable with regard to $\mathscr{F}_n$, but not necessarily in respect of $\mathscr{F}_{n-1}$. If an event $\mathscr{F}_n$ is not known, then one can find a $\mathscr{F}_{n-1}$ measurable approximation to u_n. This approximation is denoted by $E(u_n | \mathscr{F}_{n-1})$ and is called the conditional expectation of u_n in respect of $\mathscr{F}_{n-1}$.

An adapted sequence (u_n) of random variables is *predictable* if u_n is $\mathscr{F}_{n-1}$ measurable for all $n \geq 1$.

The integrable random variables are the subset of random variables for which the integral with regard to P exists, and

$$\int u\,dP < \infty$$

The integral of u is the unconditional expected value, denoted by $E(u)$. For any event Q in $\mathcal{F}$, and B_Q (the set of points in Ψ for which Q occurs), $B_Q \in \mathcal{F}$. The expected value may be defined as follows:

$$E(Q) = P(B_Q)$$

Assume that the price S_t of an asset is a stochastic variable and follows an Ito process described by the following stochastic differential equation:

$$dS_t = \mu(S,t)\,dt + \sigma(S,t)\,dW \tag{1}$$

where W is a Wiener process with a drift rate of 0 and a variance rate of 1.0. A variance rate of 1.0 means that the variance of the change in W in an interval of length T equals T. The variable S_t has an expected drift rate (average drift per unit time) of μ and a variance rate (variance per unit of time) of σ^2 and satisfies the equations

$$P\left[\int_0^T |\mu|\,dt < \infty\right] = 1$$

$$P\left[\int_0^T \sigma^2\,dt < \infty\right] = 1$$

If N assets are traded in a market and the i-th asset is defined as a risky asset and priced at $S_t^{\,i}$ at time t, the riskless asset can be defined as an investment at the risk-free rate, r, which gives a price of $S_t^{\,0} = e^{rt}$, at time t, where $S_0^{\,0} = 1$. This is the 'zero-th' share. The market price of all assets is given by

$$S_t = (S_t^0, S_t^1, ..., S_t^N)^T$$

A *trading strategy* is a predictable N-dimensional stochastic process,

$$\Theta_t = (q_t^0, q_t^1, ..., q_t^N)^T$$

denoting the holding or position in each asset at time t. The value of a portfolio Π at time t is given by the following equation:

$$\Pi_t = \Theta_t \cdot S_t = q_t^0 S_t^0 + \sum_{n=1}^{N} q_t^n S_t^n \tag{2}$$

For two different periods in time, the strategic position of a portfolio is given by the equation

$$\Theta_t = \begin{cases} \Theta_0, & 0 \le t \le t_1 \\ \Theta_{t_1}, & t_1 < t \le t_2 \end{cases}$$

The change in value of the portfolio at time t_1 is therefore

$$(\Theta_{t_1} - \Theta_0) \cdot S_{t_1}$$

If this product is zero, the portfolio is defined to be self-financing or is called a *self-financing trading strategy*. The strategy is represented by the following equation

$$\Theta_{t_1} \cdot (S_{t_1} - S_0) = \Theta_{t_1} \cdot S_{t_1} - \Theta_0 \cdot S_0$$

or, generally

$$\Pi_{t_{i+1}} - \Pi_{t_i} = \Theta_{t_i} \cdot (S_{t_{i+1}} - S_{t_i})$$

A self-financing trading strategy in continuous time is therefore a strategy where

$$\Pi_t - \Pi_0 = \int_0^t \Theta_t dS_t \tag{3}$$

A strategy Θ is *admissible* if it is self-financing and if $\Pi_t(\Theta) \geq 0$ for any t. An admissible strategy with zero initial value and non-zero final value is called an *arbitrage strategy*. In such a strategy a riskless profit can be made, without initially investing anything .

A *derivative security* is defined as an $\mathscr{F}_T$-measurable random variable $u(T)$. The derivative is *attainable* if there is a self-financing trading strategy Θ_u such that $\Pi_T(\Theta_u) = u(T)$ with a probability of one. This self-financing trading strategy is then called a *replicating strategy*. If all derivative securities in an economy are attainable, the economy is called *complete*. If there are no arbitrage opportunities in an economy, the value of an attainable derivative $u(T)$, is given by the value of the unique replicating strategy.

Any tradable asset which has a strictly positive price (and pays no dividends) for all $t \in [0,T]$ is called a *numeraire*. Generally, the riskless money-market account is the numeraire, although the choice of numeraire is arbitrary. The price of any tradable asset (S^i) can be denominated in terms of a numeraire, for example S^0. The relative price is denoted by $(S^i)' = S^i/S^0$.

2.1.2 *Markov chains*

If S_t is an Ito process satisfying equation (1) and $f(\cdot)$ is any bounded function, and if the information set $\mathscr{F}_t$ contains all information about S_t until time t, then S_t satisfies the Markov property, provided that

$$E[f(S_{t+h}) \mid \mathscr{F}_t] = E[f(S_{t+h}) \mid S_t], \qquad \text{where } h > 0$$

A Markov chain is a stochastic process where the only information useful for predicting future values is the current value. The stochastic process S_t is a Markov chain if it satisfies the Markov property.

2.1.3 *Brownian motion*

A Brownian motion is a real-valued continuous stochastic process, (S_t), $t \in [0,T]$ (also called a Wiener process) with independent increments, such that the increments

$$x = S_{t_2} - S_{t_1}$$

have a normal distribution with mean zero and variance $|t_2 - t_1|$:

$$S_{t_2} - S_{t_1} \sim N(0, t_2 - t_1)$$

with $S_0 = 0$.

A Brownian motion is *standard* if

$$S_0 = 0 \qquad E(S_t) = 0 \qquad E(S_t^2) = t$$

In this case, the density function of a variable x is given by

$$f(x) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t} \right)$$

2.1.4 *Martingales*

Consider a filtration $\{\mathscr{F}_t\}_{t\geq 0}$, where any information is generated by all observed events up to time t. Assume that S is a stochastic process where S is adapted to the filtration $\{\mathscr{F}_t\}_{t\geq 0}$ and for all t we have

$$E(|S(t)|) < \infty$$

A martingale is defined as a zero-drift stochastic process. Therefore, $S(t)$ is a martingale if $E(S(t)|\mathscr{F}_t)$ is defined and for each t and $h > 0$ the following relation holds

$$E(S(t+h)|\mathscr{F}_t) = S(t)$$

If

$$E(S(t+h)|\mathscr{F}_t) \leq S(t)$$

$S(t)$ is called a super-martingale and if

$$E(S(t+h)|\mathscr{F}_t) \geq S(t)$$

$S(t)$ is called a sub-martingale.

Consider the set $\wp$ which contains all probability measures P^* such that

- P^* and P have the same null-sets and are therefore equivalent; and
- the relative price processes $(S^i)'$ are martingales under P^* for all i, therefore

$$E^*[(S^i)'(T)|\mathscr{F}_t] = (S^i)'(t) \qquad \text{for } t \leq T \tag{4}$$

The measures $P^* \in \wp$ are called equivalent martingale measures.

Derivative securities are defined as those securities for which the expectation of the payoff is well-defined. A derivative security is therefore an $\mathscr{F}_T$-measurable random variable, $u(T)$, such that

$$E^{P^*}[|u(T)|] < \infty$$

A continuous trading economy is free of arbitrage opportunities and every derivative security is attainable if $\wp$ contains only one equivalent martingale measure. This was proved by Harrison and Pliska (1981).

For a given numeraire M with a unique equivalent martingale measure P_M, the value of a self-financing trading strategy

$$\Pi_t'(\Theta_u) = \frac{\Pi_t(\Theta_u)}{M(t)}$$

is a P_M-martingale. For a replicating strategy Θ_u that replicates the derivative security $u(T)$, it holds that

$$E^{P_M}\left(\frac{u(T)}{M(T)}\middle|\mathcal{F}_t\right) = E^{P_M}\left(\frac{\Pi_T(\Theta_u)}{M(T)}\middle|\mathcal{F}_t\right)$$

$$= \frac{\Pi_t(\Theta_u)}{M(t)}$$

Therefore,

$$\Pi_t(\Theta_u) = M(t)\, E^{P_M}\left(\frac{u(T)}{M(T)}\middle|\mathcal{F}_t\right) \tag{5}$$

2.2 Principles

2.2.1 Girsanov's theorem

Girsanov's theorem can be used to determine equivalent martingale measures by changing the probability measure and therefore the drift of a Brownian motion.

Theorem: For any stochastic process $\omega(t)$ such that with a probability of 1,

$$\int_0^t \omega(s)^2 ds < \infty$$

one can state that under the measure $dP^* = \rho dP$ the process

$$W^*(t) = W(t) - \int_0^t \omega(s)ds$$

is also a Brownian motion, where the Radon-Nikodym derivative is given by

$$\rho(t) = \exp\left\{\int_0^t \omega(s)du(s) - \frac{1}{2}\int_0^t \omega(s)^2 ds\right\}$$

It therefore follows that

$$dW = dW^* + \omega(t)dt \tag{6}$$

■

2.2.2 *Ito's lemma*

Theorem: X_t is an $\mathbb{R}$-valued Ito process if the following relation holds for all $t \geq 0$,

$$X_t = X_0 + \int_0^t \mu ds + \int_0^t \sigma dW_s \tag{7}$$

where μ and σ are functions of X and t. This stochastic integral is usually interpreted as the stochastic differential equation

$$dX(t) = \mu(t)dt + \sigma(t)dW(t) \tag{8}$$

Then, for a sufficiently differentiable function, $(t,X(t)) \rightarrow f(t,X(t))$ of the process X, for which the partial derivatives are continuous with respect to $(t,X(t))$, the function f has a stochastic differential given by the following equation (Björk, 1999)

$$df(t,X(t)) = \sigma\frac{\partial f}{\partial x}dW(t) + \left(\mu\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2\frac{d^2 f}{dx^2} + \frac{\partial f}{\partial t}\right)dt \tag{9}$$

The Ito-formula can also be written as follows:

$$f(X_t) = f(X_0) + \int_0^t f'(X_u)du + \frac{1}{2}\int_0^t f''(X_u)d\langle X,X\rangle_u \tag{10}$$

where, by definition

$$\langle X,X\rangle_t = \int_0^t \sigma^2 X_u^2 du$$

■

2.2.3 *The Feynman-Kač proposition*

If one assumes that *f* is a solution to the boundary value problem

$$\frac{\partial f}{\partial t}(t,x) + \mu(t,x)\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 f}{\partial x^2} = rf, \qquad f(T,x) = \Phi(X)$$

and one furthermore assumes that the process $\sigma(s,X_s)\frac{\partial F}{\partial x}(s,X_s)$ is in $\mathcal{L}^2$, then f can be represented as

$$f(t,x) = e^{r(T-t)}E_{(t,x)}[\Phi(X_T)] \tag{11}$$

where X satisfies the stochastic differential equation

$$dX_s = \mu(s,X_s)dt + \sigma(s,X_s)dW_s$$

$$X_t = x$$

The process is fully described by Björk (1999). ■

2.2.4 *The Ornstein-Uhlenbeck process*

If X solves the stochastic differential equation

$$dX_t = -\frac{1}{2}X_t dt + dW_t$$

then X is an Ornstein-Uhlenbeck process. Such a process has the normal distribution as its invariant measure. ■

2.3 The Black-Scholes model

2.3.1 *An exact solution for European options*

In the Black-Scholes economy it is assumed that there are two tradable instruments: the riskless money market instrument $M(t)$ (where $M(0) = 1$) and a stock $S(t)$. The value of the money market instrument is strictly positive and can therefore serve as a numeraire. Since the money market instrument is assumed to be riskless if it has a constant risk-free interest rate and no stochastic term, its price is described by

$$dM = rMdt \qquad (12)$$

One can assume that the stock price follows a geometric Brownian motion

$$dS = \mu S dt + \sigma S dW \qquad (13)$$

with constant drift μ and volatility σ.

Since the stock price can be expressed in terms of the numeraire,

$$S' = \frac{S}{M}$$

it follows from Ito's Lemma that:

$$dS' = \left[\frac{M\frac{dS}{dt} - \frac{dM}{dt}S}{M^2}\right]dt + \frac{\mu S}{M}dt + \frac{\sigma S}{M}dW$$

$$= -\frac{rS}{M}dt + \frac{\mu S}{M}dt + \frac{\sigma S}{M}dW$$

$$= (\mu - r)S'dt + \sigma S'dW$$

For $\sigma \neq 0$ Girsanov's theorem can be used to turn the relative stock price into a martingale. Therefore a unique measure P^* is used, where $\partial P^* = \rho dP$ with $\omega(t) = -(\mu - r)/\sigma$, to obtain

$$dW = dW^* - \frac{\mu - r}{\sigma}dt$$

Therefore,

$$dS' = \sigma S' dW^* \tag{14}$$

The stochastic process S' is therefore a martingale, and, consequently, this economy is arbitrage-free and complete for $\sigma \neq 0$. The original price process S follows, under the measure P^*, the process

$$dS = \mu S dt + \sigma S\left(dW^* - \frac{\mu - r}{\sigma}dt\right) \tag{15}$$

or

$$dS = rSdt + \sigma S dW^* \tag{16}$$

Equation (16) shows that under the equivalent martingale measure, the drift μ is replaced by the risk-free rate r. The equivalent form is

$$S(t) = S(0) + \int_0^t rS(u)du + \int_0^t \sigma S(u)dW^* \qquad (17)$$

If

$$f(S_t) = \ell n(S_t)$$

where S_t is an Ito process and a solution of equation (17), and the Ito formula is applied to this equation, the following equation results:

$$\ell n(S_t) = \ell n(S_0) + \int_0^t \frac{dS_u}{S_u} + \frac{1}{2}\int_0^t \left(\frac{-1}{S_u^2}\right)\sigma^2 S_u^2 du$$

Using equation (16), it follows that

$$\begin{aligned} \ell n(S_t) &= \ell n(S_0) + \int_0^t (r - \sigma^2/2)du + \int_0^t \sigma dW^* \\ &= \ell n(S_0) + (r - \frac{1}{2}\sigma^2)t + \sigma W^* \end{aligned}$$

Consequently,

$$S(t) = S(0)\exp\left[(r - \frac{1}{2}\sigma^2)t + \sigma W^*\right] \qquad (18)$$

is a solution of equation (17), and therefore a solution of equation (16). The random variable $W^*(t)$ has a normal distribution with mean 0 and variance t.

If one defines a contingent claim of maturity T by giving its payoff $u \geq 0$, which is $\mathcal{F}_t$ -

measurable, then a European call option on the underlying price of the stock, S, with strike K, at the exercise time T has a payoff of

$$u(T) = \max\{S(T) - K, 0\}$$

In this case, u is a function of the underlying price at time T only. Some options depend on the whole path of the underlying asset, for instance Asian options. From equation (5) it follows that the price of a call option c, at time 0 is given by

$$c = E\left[\max\{S(T) - K, 0\}/M(T)\right]$$

If one uses the explicit solution of $S(T)$ given in equation (18), one gets

$$c = \int_{-\infty}^{\infty} e^{-rT} \max\{S(0)e^{(r - \sigma^2/2)T + \sigma x} - K, 0\} \frac{e^{-\frac{1}{2}\frac{x^2}{T}}}{\sqrt{2\pi T}} dx$$

The payout is non-zero if

$$S(0)\, e^{(r - \sigma^2/2)T + \sigma x} - K > 0$$

$$\therefore \quad \ell n \frac{S(0)}{K} > -(r - \tfrac{1}{2}\sigma^2)T - \sigma x$$

$$\therefore \quad x > -\left[\frac{\ell n\left(\frac{S(0)}{K}\right) + (r - \frac{1}{2}\sigma^2)T}{\sigma}\right]$$

Therefore, it follows that

$$c = \int_{-\frac{\ell n(S(0)/K) + (r-\sigma^2/2)T}{\sigma}}^{\infty} e^{-rT}\{S(0)e^{(r - \sigma^2/2)T+\sigma x} - K\}\frac{e^{-\frac{1}{2}\frac{x^2}{T}}}{\sqrt{2\pi T}} dx$$

$$= S(0) \int_{-\frac{\ell n(S(0)/K) + (r-\sigma^2/2)T}{\sigma}}^{\infty} \frac{1}{\sqrt{2\pi T}} e^{-\frac{1}{2}\sigma^2 T + \sigma x - \frac{1}{2}\frac{x^2}{T}} dx$$

$$- e^{-rT}K \int_{-\frac{\ell n(S(0)/K) + (r-\sigma^2/2)T}{\sigma}}^{\infty} \frac{1}{\sqrt{2\pi T}} e^{-\frac{1}{2}\frac{x^2}{T}} dx$$

$$= I_1 - I_2$$

If one changes variables in I_1 and I_2, this results in

$$I_1 = 1 + S(0) \int_{-\infty}^{\frac{\ell n(S(0)/K) + (r+\sigma^2/2)T}{\sigma\sqrt{T}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\rho^2} d\rho$$

and

$$I_2 = 1 + e^{-rT}K \int_{-\infty}^{\frac{\ell n(S(0)/K) + (r-\sigma^2/2)T}{\sigma\sqrt{T}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\omega^2} d\omega$$

Therefore,

$$c = S(0)\,N(d) - e^{-rT}K\,N(d - \sigma\sqrt{T}) \qquad (19)$$

where

$$d = \frac{\ell n\left(\frac{S(0)}{K}\right) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

which is the well-known Black-Scholes call option pricing formula. The price of a put option is given by

$$p = e^{-rT}K\,N(-d + \sigma\sqrt{T}) - S(0)\,N(-d) \qquad (20)$$

2.3.2 *The Black-Scholes partial differential equation*

In the case of path-dependent options, one cannot use the exact solution, and therefore it is necessary to use a numerical solution of the Black-Scholes partial differential equation. If one assumes that a stock price, S, follows a Wiener process, where the drift and volatility are dependent on the level of the stock price,

$$dS = \mu S dt + \sigma S dW \tag{21}$$

Then the variable S has a lognormal distribution, where ℓn S follows a generalized Wiener process.

If f is the value of a derivative security dependent on S, it follows from Ito's Lemma that

$$df = \left(\mu S \frac{\partial f}{\partial S} + \frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{d^2 f}{dS^2} \right) dt + \sigma S \frac{\partial f}{\partial S} dW \tag{22}$$

The discrete versions of equations (21) and (22) for a small interval Δt are

$$\Delta S = \mu S \Delta t + \sigma S \Delta W \tag{23}$$

and

$$\Delta f = \left(\mu S \frac{\partial f}{\partial S} + \frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{d^2 f}{dS^2} \right) \Delta t + \sigma S \frac{\partial f}{\partial S} \Delta W \tag{24}$$

where ΔS and Δf are the changes for a small time interval Δt. If one chooses a portfolio of the stock and the derivative as follows:

- short 1 derivative, and
- long K shares,

then the value of the portfolio, Π, is

$$\Pi = -f + KS \tag{25}$$

and the change in the value of the portfolio in time Δt is

$$\Delta\Pi = -\Delta f + K\Delta S \tag{26}$$

Substituting equations (23) and (24) gives

$$\Delta\Pi = -\left(\mu S\frac{\partial f}{\partial S} + \frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{d^2 f}{dS^2}\right)\Delta t - \sigma S\frac{\partial f}{\partial S}\Delta W + K\mu S\Delta t + K\sigma S\Delta W \tag{27}$$

Choosing $K = \frac{\partial f}{\partial S}$ eliminates the Wiener process and results in

$$\Delta\Pi = \left(-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)\Delta t \tag{28}$$

The portfolio is therefore riskless for the short period of time Δt. In order to agree with the principle of no-arbitrage, it follows that the portfolio should earn the risk-free rate, r, in this period:

$$\Delta\Pi = r\Pi\Delta t$$

Substituting for ΔΠ and Π, gives

$$\left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)\Delta t = r\left(f - \frac{\partial f}{\partial S}S\right)\Delta t$$

or

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf \tag{29}$$

Equation (29) is known as the Black-Scholes partial differential equation (Black & Scholes, 1973). When Black and Scholes published this equation in 1973, they made a major breakthrough in the pricing of any derivative dependent on non-dividend paying stock. Equation (29) can be solved using the Feynman-Kač proposition, to give the exact solution in Section 2.3.1.

2.3.2.1 Black's model

In 1976, Black published a paper describing an adjustment to the Black-Scholes model in order to price options on futures. Options on commodities, say beef, can be difficult to deliver at expiry of the option, therefore it is easier to have an option on the future, and have cash settlement at expiry. Since options on futures tend to be more attractive to investors than options on spot prices, Black's model became widely used in the option market.

If one assumes that the underlying instrument of the option is the future price, F, of the stock on the expiry date of the option, and one assumes that the futures price, F, follows a geometric Brownian motion, then

$$dF = \mu F dt + \sigma F dW \tag{30}$$

Since f is a function of F and t, it follows from Ito's lemma that

$$df = \left(\mu F \frac{\partial f}{\partial F} + \frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 F^2 \frac{d^2 f}{dF^2} \right) dt + \sigma F \frac{\partial f}{\partial F} dW \tag{31}$$

Consider a portfolio consisting of

- short one option, and
- long K futures contracts.

Since it costs nothing to enter into a futures contract, the cash value of the portfolio at $t = 0$ is given by the price of the option contract

$$\Pi = -f \tag{32}$$

The *wealth* of the portfolio can change in time Δt by the amount

$$\Delta\Pi = K\Delta F - \Delta f$$

Using the discrete versions of equations (30) and (31), it follows that by choosing $K = \frac{\partial f}{\partial F}$,

$$\Delta\Pi = \left(-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial F^2}\sigma^2 F^2 \right) \Delta t \tag{33}$$

This change is riskless, therefore to ensure that the arbitrage-free assumption holds, the return should be equal to the risk-free rate of interest

$$\Delta\Pi = r\Pi\Delta t \tag{34}$$

If one substitutes equations (32) and (33), this gives a partial differential equation for the price of an option on a futures price:

$$\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial F^2}\sigma^2 F^2 = rf \tag{35}$$

In the case of exchange traded options where a margin is paid, $\Delta\Pi$ in equation (33) equals 0, and

$$\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial F^2}\sigma^2 F^2 = 0 \tag{36}$$

This equation can be solved analytically for European options and numerically for American options.

2.4 Numerical methods

The exact solution of the Black-Scholes model gives the price of a European option, which can only be exercised on the expiry date of the option. American options can be exercised at any time before or on the expiry date of the option. This implies that whenever the intrinsic value of the option is more than the value of the option, it would be profitable to exercise the option early. The problem with the exact solution of the Black-Scholes model, as set out in section 2.3 above, is that it does not provide for American options with an early-exercise value. Two numerical methods that solve the partial differential equations in Section 2.3 and which support American options are the binomial method and the finite difference method.

2.4.1 *The binomial method*

If S is the price of a non-dividend paying stock, and f is the value of an option on the stock, and the life of the option is divided into intervals of length Δt, then in each time-interval the stock price moves from its initial value of S to either Su or Sd with a probability of p and $1-p$ respectively. This process is shown in Figure 2.1.

In a risk-neutral world, the expected rate of return from an investment should be the risk-free rate, r. Therefore

$$Se^{r\Delta t} = pSu + (1-p)Sd$$

which gives

$$e^{r\Delta t} = pu + (1-p)d \tag{37}$$

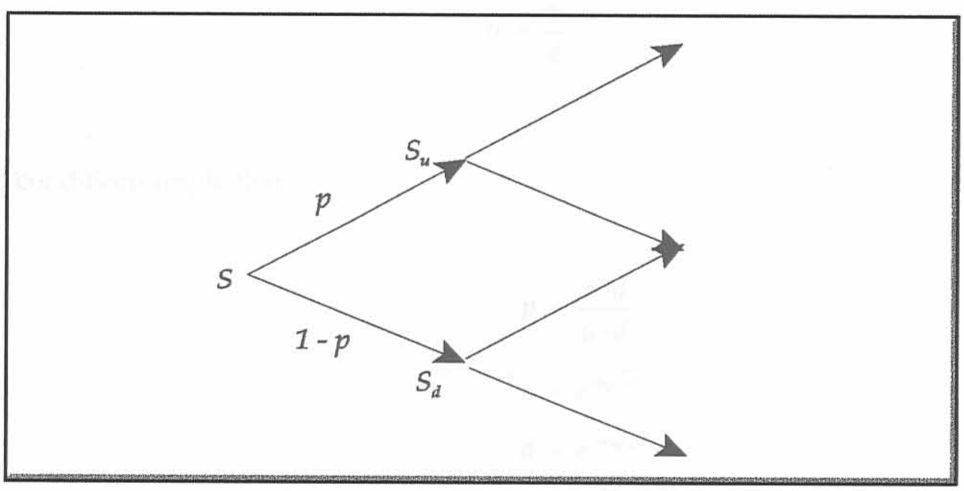


Figure 2.1: The binomial tree for stock price movement

The variance of a parameter S is given by

$$Var(S) = E(S^2) - [E(S)]^2 \tag{38}$$

where

$$Var(S) = S^2 e^{2r\Delta t}(e^{\sigma^2 \Delta t} - 1)$$
$$E(S) = Se^{r\Delta t}$$
$$E(S^2) = pS^2u^2 + (1-p)S^2d^2$$

or

$$e^{2r\Delta t + \sigma^2 \Delta t} = pu^2 + (1 - p)d^2 \tag{39}$$

Equations (37) and (39) give two conditions for u, d, and p. Cox, Ross and Rubinstein (1979) proposed a third condition:

$$u = \frac{1}{d}$$

These conditions imply that:

$$p = \frac{a-d}{u-d}$$
$$u = e^{\sigma\sqrt{\Delta t}}$$
$$d = e^{-\sigma\sqrt{\Delta t}}$$
$$\text{where} \quad a = e^{r\Delta t}$$

A tree of stock prices can be constructed, starting at time zero, and calculating the possible stock prices at time Δt, $2\Delta t$ and so on. In general, at time $i\Delta t$, the i+1 stock prices are:

$$Su^{j}d^{i-j} \qquad j = 0,1,\ldots,i$$

The value of the call option at time T is given by

$$\max(S_T - X, 0)$$

and for a put option by

$$\max(X - S_T, 0)$$

The value of the option is then calculated by working back through the tree. In a risk-neutral world, the value of the option at time $T - \Delta t$ can be calculated by discounting the value at time T at the short term rate r. The same is done for the following time steps. For American options one must check at each node that the early-exercise value is not bigger than the value of the option.

2.4.2 *The finite difference method*

A finite difference method solves a partial differential equation by converting it into a set of difference equations, which are then solved through an iterative process. Consider the differential equation for the value of an option:

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf \tag{40}$$

Since the time t and the stock price S are the two variables in equation (40), N equally spaced time intervals can be chosen between zero and T, the expiry date of the option, and M price intervals can be chosen between zero and S_{max}. This results in a finite difference grid of (M+1) x (N+1) points. The (i,j) point corresponds to time $i\Delta t$ and stock price $j\Delta S$ and $f_{i,j}$ is the value of the option at the (i,j) point.

2.4.2.1 The Implicit finite difference method

The value of $\frac{\partial f}{\partial S}$ at (i,j) is given by an average of the forward and backward differences:

$$\frac{\partial f}{\partial S} = \frac{f_{i,j+1} - f_{i,j-1}}{2\Delta S} \tag{41}$$

The value of $\frac{\partial f}{\partial t}$ at (i,j) is given by the forward difference approximation:

$$\frac{\partial f}{\partial t} = \frac{f_{i+1,j} - f_{i,j}}{\Delta t} \tag{42}$$

The finite difference approximation for $\frac{\partial^2 f}{\partial S^2}$ at the (i,j) point is

$$\frac{\partial^2 f}{\partial S^2} = \left(\frac{f_{i,j+1} - f_{i,j}}{\Delta S} - \frac{f_{i,j-1} - f_{i,j}}{\Delta S} \right) \bigg/ \Delta S$$

or

$$\frac{\partial^2 f}{\partial S^2} = \frac{f_{i,j+1} - 2f_{i,j} + f_{i,j-1}}{\Delta S^2} \tag{43}$$

Substituting equations (41), (42) and (43) into equation (40) gives, after some manipulation

$$a_j f_{i,j-1} + b_j f_{i,j} + c_j f_{i,j+1} = f_{i+1,j} \tag{44}$$

where

$$a_j = \frac{1}{2}rj\Delta t - \frac{1}{2}\sigma^2 j^2 \Delta t$$

$$b_j = 1 + \sigma^2 j^2 \Delta t + r\Delta t$$

$$c_j = -\frac{1}{2}rj\Delta t - \frac{1}{2}\sigma^2 j^2 \Delta t$$

The value of a put option at time T is $\max[X - S_T, 0]$ or $\max[S_T - X, 0]$ for a call option, therefore

$$f_{N,j} = \max[k(X - j\Delta S), 0] \qquad j = 0,1,\ldots,M \tag{45}$$

where $k = 1$ for a put and $k = -1$ for a call option.

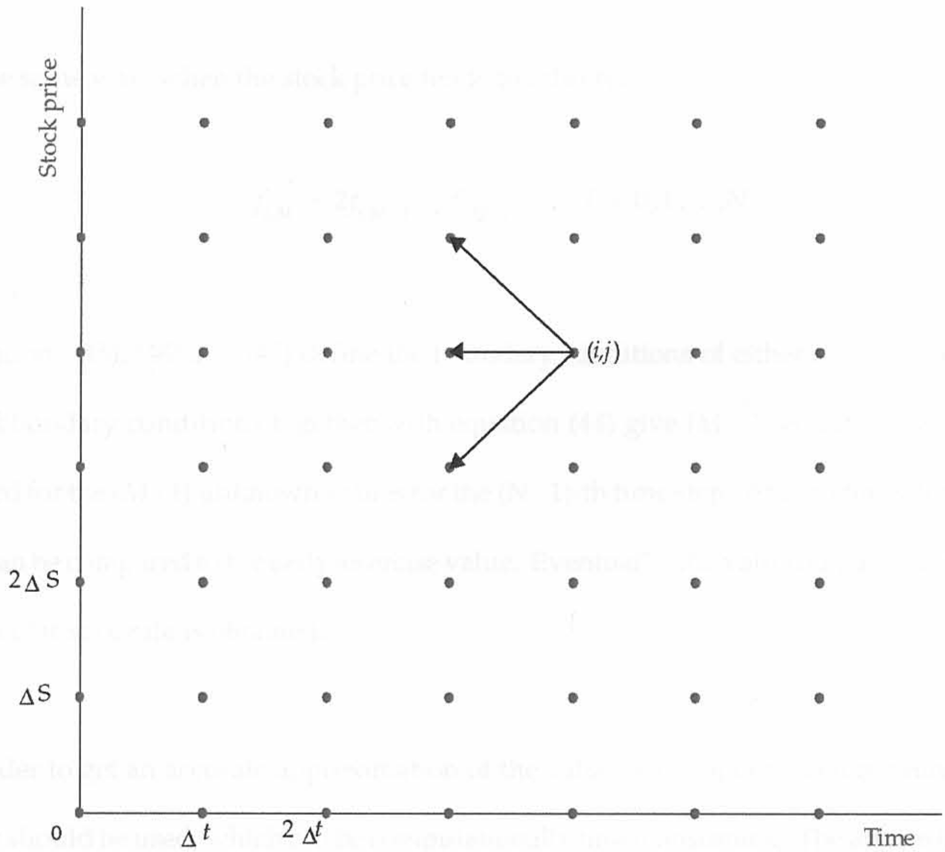


Figure 2.2: Finite difference grid

When the stock price is zero or tends to infinity, the delta of the option tends to zero or 1 respectively. In order to find the value of the option at zero and infinity, one assumes therefore that the second derivative at these points is approximately zero. Hence, when the stock price is zero,

$$\frac{\partial^2 f}{\partial S^2} \approx 0 = \frac{f_{i,2} + f_{i,0} - 2f_{i,1}}{\Delta S^2}$$

or

$$f_{i,0} = 2f_{i,1} - f_{i,2} \qquad i = 0,1,\ldots,N \tag{46}$$

In the same way, when the stock price tends to infinity:

$$f_{i,M} = 2f_{i,M-1} - f_{i,M-2} \qquad i = 0,1,\ldots,N \tag{47}$$

Equations (45), (46) and (47) define the boundary conditions of either a put or a call option. The boundary conditions together with equation (44) give (M - 1) equations which can be solved for the (M - 1) unknown values for the (N - 1)-th time step. At each time step, the value of f can be compared to the early-exercise value. Eventually, the value for f at time $t = 0$ for the particular spot rate is obtained.

In order to get an accurate approximation of the value of the option, a large number of time steps should be used, which can be computationally time-consuming. The approximate value for a very small time step can be obtained by solving the problem for two different time steps, say $\Delta t = 0.1$ and $\Delta t = 0.01$. These values are then linearly extrapolated to give an approximate value for $\Delta t \sim 0$.

The control variate technique can be used when there is an analytic solution to a similar problem, as with a European option. The approximation error is therefore calculated and can be used as a correction term to adjust the numerical value obtained for problems where there is no analytical solution available, for instance, for American options.

An explicit finite difference method can also be used if the implicit scheme is found to be time-consuming. The explicit method is similar to a trinomial tree approach. Unfortunately, often one or more of the two probabilities are negative, which can result in instability and inconsistencies in the solution. For the purposes of this study, an implicit finite difference method which is unconditionally stable is used.

This chapter provides the basic theory for pricing derivative securities. It forms the basis for pricing bond options in Chapters 5, 6 and 7. Before one can value options on interest rates accurately, it is, however, necessary to understand the underlying instrument. Therefore, Chapters 3 and 4 discuss the theory of the term structure of interest rates.

CHAPTER 3

THE TERM STRUCTURE OF INTEREST RATES

Interest rates play an important role in the economy, whether on the global or national level. It is both a determinant and a result of economic growth. The interest rate term structure is the most important input in pricing almost all fixed income instruments. The term structure, or yield curve, provides a way of measuring the relationship between the rate of interest, or yield and time to maturity. The interest rate associated with an investment gives the return on that investment.

Fixed-income investors have several alternative securities to invest in. In choosing the security to invest in, they consider the following three factors: return, risk and liquidity. The longer the term-to-maturity of the instrument, the larger the price risk. Unless there is a strong expectation that interest rates are going to fall, investors would only invest in a longer-term security if the return is higher. This usually leads to an upward-sloping yield curve.

There are three main theories that are used to explain the shape of the yield curve (Bodie, Kane & Marcus, 1993), namely

- the expectations hypothesis;
- the liquidity preference theory; and
- the market segmentation and preferred habitat theories.

The expectations hypothesis states that the forward rate for a period in the future equals the market consensus expectation of the future interest rate. Therefore, for example, the six month interest rate is determined by the current three month interest rate and the expectation of the three month rate in three months time. The yield curve is therefore determined by expected future changes in interest rates.

The liquidity preference theory argues that there are more short-term investors than long-term investors and therefore short-term investors require a premium to induce them to buy longer-term securities. This implies that the forward rate should exceed the expected spot rate by the liquidity premium. If the liquidity premium is, however, higher than investors feel is fair, they would exploit the abnormal profit opportunities – bringing it back to normal.

The market segmentation theory argues that long- and short-term bonds are traded in segmented markets. Borrowers and lenders tend to operate in different maturity ranges. The interest rate for a particular maturity is therefore determined solely by supply and demand in that area of the yield curve. The preferred habitat theory argues, however, that lenders would leave their preferred maturity area if there is significant inducement offered in another area, thereby eliminating some of the inconsistencies in the yield curve.

3.1 The term structure and forward rates

One could assume that the current time is zero. The T_n-year yield given by the term structure is the interest rate $y(T_n)$ on an investment that is made today, lasting for T_n years, also known as the T_n-year spot interest rate, or zero-coupon yield. The principal and interest are repaid to the investor at the end of T_n years. The forward interest rate $f(T_n,T_m)$ is the rate implied by current spot rates for the period between year T_n and year T_m in the future.

It can be assumed that interest rates are compounded continuously. If investors invest 1 unit today, they will obtain a future value of

$$v = 1 \cdot e^{y(T_n)T_n}$$

in T_n years time.

If investors invest 1 unit today for a period of T_n years at a rate $y(T_n)$, and after T_n years reinvest the money for another $(T_m - T_n)$ years at a forward rate $f(T_n,T_m)$, the future value after T_m years would be

$$v = 1 \cdot e^{y(T_n)T_n} \cdot e^{f(T_n,T_m)\cdot(T_m - T_n)}$$

However, if investors invest the money for a period of T_m years at a rate $y(T_m)$ instead, the future value is

$$v = 1 \cdot e^{y(T_m)T_m}$$

For the no-arbitrage principle to hold, it follows that

$$e^{y(T_m)T_m} = e^{y(T_n)T_n} \cdot e^{f(T_n,T_m)\cdot(T_m - T_n)}$$

Therefore the forward rate for the period $[T_n, T_m]$ is given by

$$f(T_n,T_m) = \frac{y(T_m)T_m - y(T_n)T_n}{T_m - T_n}$$

or

$$f(T_n,T_m) = y(T_m) + T_n\left[\frac{y(T_m) - y(T_n)}{T_m - T_n}\right]$$

If there is a continuous yield curve and limits are taken as T_m approaches T_n, it is clear that $y(T_m)$ approaches $y(T_n)$. The forward rate for a very short period of time, beginning in T_n years, (known as the instantaneous forward rate in T_n years), can be expressed as

$$f_i(T_n) = y(T_n) + T_n\frac{\partial r}{\partial T_n} \quad (1)$$

where r is called the instantaneous interest rate or short-term spot rate. In the rest of thestudy this rate will be referred to as the short-rate.

3.2 The term structure and the short-rate

If one assumes that the current time is denoted by t, and as explained in the previous section, the short-rate, r, at time t is the interest rate for an infinitesimally short period of time Δt, then the value of an interest-rate derivative that provides a payoff of h at time t_n is determined by the expected risk-free rate of return for the period $T = t_n - t$:

$$E\left[e^{-\bar{r}T}h\right] \tag{2}$$

where $\bar{r}$ is the average value of r in the time interval between t and t_n, and E is the expected value in a risk-neutral world. If $P(t, t_n)$ is the price at time t of a discount bond that pays a maturity value of 1 unit at time t_n, then equation (2) implies that

$$P(t,t_n) = E\left[e^{-\bar{r}T}\right] \tag{3}$$

If $y(t,T)$ is the continuously compounded spot interest rate at time t for a T- year investment, then

$$P(t,t_n) = e^{-y(t,T)T} \tag{4}$$

or

$$y(t,T) = -\frac{1}{t_n - t}\ell n P(t,t_n) \tag{5}$$

From equation (3) it is clear that

$$y(t,T) = -\frac{1}{t_n - t}\ell n E\left[e^{-\bar{r}(t_n - t)}\right] \tag{6}$$

This equation shows that the term structure of interest rates can be obtained from the initial value of r at time t and the risk-neutral process for r for $t \leq t_n$. It is therefore clear that by developing a model of the risk-neutral process for r, the term structure of interest rates could be modelled.

3.3 The relation between the short-rate, bond prices and forward rates

If one assumes that the price $P(t,t_n)$ of a bond is determined by the market's assessment, at time t, of the behaviour of interest rates over the life of the bond, and the yield to maturity for the period $[t,t_n]$ is equivalent to the average forward rate for the period, then it follows that the instantaneous forward rate $f_i(t,t_n)$ is defined by

$$y(t,T) = \frac{1}{t_n - t} \int_t^{t_n} f_i(t,\tau)\, d\tau$$

Therefore,

$$f_i(t,t_n) = \frac{\partial}{\partial t_n}[(t_n - t)\; y(t,T)]$$

If the short-rate is defined as the instantaneous interest rate, then

$$r(t) = y(t,0) = \lim_{T \to 0} y(t,T)$$

If one assumes that the short-rate is a continuous function of time and follows a Markov process, then the spot interest rate r follows the following stochastic differential equation:

$$dr = \mu(t,r)dt + \sigma(t,r)dW \qquad (7)$$

where $\mu(t,r)$ and $\sigma(t,r)$ are the instantaneous drift and standard deviation respectively of the process $r(t)$.

If a financial instrument's value $P(t,r)$ is determined directly by the level of the spot interest rate $r(t)$, it follows from Ito's lemma that

$$dP = M(t,r)dt + \Sigma(t,r)dW \qquad (8)$$

where

$$M(t,r) = \mu(t,r)\frac{\partial P}{\partial r} + \frac{1}{2}\sigma(t,r)^2\frac{d^2P}{dr^2} + \frac{\partial P}{\partial t}$$

$$\Sigma(t,r) = \sigma(t,r)\frac{\partial P}{\partial r}$$

A locally riskless portfolio Π can be constructed by hedging a derivative P_1 with a Δ-amount of another interest rate derivative P_2:

$$\Pi = P_1(t,r) - \Delta P_2(t,r)$$

where P_1 and P_2 both follow stochastic processes as described above. The portfolio Π is a linear combination of these processes:

$$d\Pi = (M_1(t,r) - \Delta M_2(t,r))dt + (\Sigma_1(t,r) - \Delta\Sigma_2(t,r))dW$$

If one chooses $\Delta=\Sigma_1/\ \Sigma_2$, then the random component in $d\Pi$ is eliminated. Using arbitrage arguments, the portfolio should earn a riskless return in a small period of time, leading to

$$d\Pi = r\Pi dt$$

Using substitution, the following equation is obtained

$$\left(M_1(t,r) - \frac{\Sigma_1(t,r)}{\Sigma_2(t,r)}M_2(t,r)\right)dt = r\left(P_1(t,r) - \frac{\Sigma_1(t,r)}{\Sigma_2(t,r)}P_2(t,r)\right)dt$$

Algebraic manipulation gives

$$\frac{M_1(t,r) - rP_1(t,r)}{\Sigma_1(t,r)} = \frac{M_2(t,r) - rP_2(t,r)}{\Sigma_1(t,r)}$$

which should hold for any pair of derivatives P_1 and P_2. The ratio $(M - rP)/\Sigma$ must therefore be a function of r and t only, which is denoted by $\lambda(r,t)$. For any derivative security P, it follows that

$$\frac{\partial P}{\partial t} + (\mu(t,r) - \lambda(t,r)\,\sigma(t,r))\frac{\partial P}{\partial r} + \frac{1}{2}\sigma(t,r)^2\frac{\partial^2 P}{\partial r^2} - rP = 0 \tag{9}$$

This equation describes the price of a security in a one-factor yield-curve model and is called the term structure equation.

The parameters μ and σ of the short-rate process, and the market price of risk, λ, must be determined from the market. The former two quantities can be obtained from the process $r(t)$, while λ can be determined empirically (Vasicek, 1977) from the equation

$$\left.\frac{\partial P}{\partial t}\right|_{T=0} = \frac{1}{2}(\mu - \sigma\lambda) \tag{10}$$

The spot interest rate, r, can be a function of time and some underlying process u, for example $F(t,u)$. The process u has a normal distribution. This function can take various forms, for instance, linear, quadratic, logarithmic or exponential. The choice of the function determines whether the model has a normally distributed fundamental solution. Pelsser (1996) has proven that the function $F(t,u)$ must be either a linear or a quadratic function in u to give a normally distributed fundamental solution.

If one assumes that instead of the short-rate, the price of the bond follows a Wiener process (as described in Black's model applied to bond options in Chapter 5), then the risk neutral process for the price P of a zero-coupon bond can be described by the following stochastic differential equation:

$$dP(t,t_n) = r(t)\,P(t,t_n)dt + \sigma(t,t_n)\,P(t,t_n)\,dW \tag{11}$$

The expected return, μ, is given in this case by the risk-free rate for that period, since a zero-coupon bond provides no income throughout the life of the bond. The pull-to-par phenomenon states that, at the maturity date of the bond, the bond price must equal its face value. Therefore, instead of constant price volatility, $\sigma(t)$, upon maturity of the bond, the price volatility should equal zero and it can be assumed that:

$$\sigma(t,t_n) \underset{t_n}{\overset{t}{\to}} 0 \tag{12}$$

The forward rate at time t for the period t_n to t_m can be written in the following form:

$$f(t,t_n,t_m) = \frac{\ell n[P(t,t_n)] - \ell n[P(t,t_m)]}{t_m - t_n} \tag{13}$$

Using equation (11) and Ito's lemma, with $g_n = \ell n(P(t,t_n))$, $g_m = \ell n(P(t,t_m))$, it follows that

$$dg_n = \left[r(t) - \frac{\sigma(t,t_n)^2}{2} \right] dt + \sigma(t,t_n)\, dW$$

and

$$dg_m = \left[r(t) - \frac{\sigma(t,t_m)^2}{2} \right] dt + \sigma(t,t_m)\, dW$$

It follows that

$$df(t,t_n,t_m) = \frac{\sigma(t,t_m)^2 - \sigma(t,t_n)^2}{2(t_m - t_n)} dt + \frac{\sigma(t,t_n) - \sigma(t,t_m)}{t_m - t_n} dW$$

It becomes clear that the risk-neutral process for f depends only on the volatility σ. If $t_n = s$ and $t_m = s + \Delta t$ and Δt tends to zero, the forward rate, $f(t,t_n,t_m)$ becomes the instantaneous forward rate $f_i(t,s)$ and

$$df_i(t,s) = \sigma(t,s)\frac{\partial\sigma(t,s)}{\partial s}dt - \frac{\partial\sigma(t,s)}{\partial s}dW$$

The sign of dW can be changed without loss of generality, and therefore the equation can be written as

$$df_i(t,s) = \sigma(t,s)\sigma_s(t,s)\,dt + \sigma_s(t,s)\,dW \tag{14}$$

where σ_s denotes the first derivative. Equation (14) shows that the drift is given by

$$m(t,s) = \sigma(t,s)\,\sigma_s(t,s)$$

and therefore the instantaneous forward rate depends on its standard deviation $v(t,s)$, where

$$v(t,s) = \sigma_s(t,s)$$

If one integrates σ_s between $\tau = t$ and $\tau = s$, the result is

$$\begin{aligned}\int_t^s \sigma_s(t,\tau)\,d\tau &= \sigma(t,s) - \sigma(t,t)\\ &= \sigma(t,s)\end{aligned}$$

Therefore, it follows from equation (14) that the drift-term is given by

$$\begin{aligned}m(t,s) &= \sigma_s(t,s)\,\sigma(t,s)\\ &= v(t,s)\int_t^s v(t,\tau)\,d\tau\end{aligned} \tag{15}$$

Since the short-rate r is given by

$$r(t) = f_i(t,t)$$

and

$$\int_0^t df_i(\tau,t) = f_i(t,t) - f_i(0,t)$$

it follows from equation (14) that

$$r(t) = f_i(0,t) + \int_0^t \sigma(\tau,t)\sigma_t(t,\tau)d\tau + \int_0^t \sigma_t(\tau,t)dW \tag{16}$$

If one differentiates to t, the following process is obtained for r

$$\begin{aligned} dr(t) &= [f_i]_t(0,t)dt + \left\{ \int_0^t [\sigma(\tau,t)\sigma_{tt}(\tau,t) + \sigma_t(\tau,t)^2]d\tau \right\} dt \\ &+ \left\{ \int_0^t \sigma_{tt}(\tau,t)dW \right\} dt + [\sigma_t(\tau,t)|_{\tau=t}]dW \end{aligned} \tag{17}$$

This equation gives the stochastic process for the short-rate where the terms containing dt give the drift in r, and the last term gives the standard deviation of r. The first term is in fact the initial slope of the forward rate curve. The above equation demonstrates the relation between the stochastic process for the bond price and the process for the short-rate. This concept is used in various option pricing models.

3.4 The term structure – coupon vs zero-coupon

A discount bond is an instrument that provides a single cashflow at a time s in the future. The price of the discount bond is determined by the s-term yield in the market at the time of purchase. Coupon-bearing bonds pay a stream of certain payments at times $\{t_i\}$, called coupons, as well as a notional payment at the end of the term of the bond. A coupon bond can be seen as a combination of discount bonds. The relation between the yield-to-maturity and the term-to-maturity of discount bonds describes the term structure of interest rates, which are used in the pricing of any fixed income instrument. The term structure implies the market consensus of forward rates and forward curves, often used for hedging purposes.

In term structure analysis it is essential that each observation used as a data point produces a yield with an unambiguous relationship with the term of the security. This is the case only with pure discount securities such as zero-coupon bonds. A coupon bond, on the other hand, can be seen as a composite of pure discount instruments – one for each of the bond's remaining cashflows – while an interest rate swap can be seen as a par yield bond. Opportunities to restore equilibrium between the markets for coupon bonds, zero-coupon bonds and swaps exist through arbitrage. The zero-coupon yield curve serves as the instrument to discount the cashflows of any interest rate security, in order to obtain the fair value of a security when selecting fixed income securities for an investment portfolio.

Consider, for example, a market in which zero-coupon bonds as well as coupon-bearing bonds are traded. Depending on whether the coupons are worth more (or less) than the actual bond, participants in the market will either strip the coupons (separate the coupons from the nominal amount of a bond), or reconstitute the bonds (by re-bundling zero-coupon bonds). The value of coupons and bonds should be determined from a single curve to ensure that no arbitrage opportunities occur.

In South Africa the JSE Actuarial Yield curve is seen as the benchmark curve. This curve is a fit through the yield-to-maturities of all government bonds. It therefore approximates a par-bond curve. No official zero-coupon yield curve is available in South Africa.

3.5 Constructing the initial term structure: the standard bootstrap method

In liquid fixed income markets, zero-coupon bonds and money market rates are typically used to construct a zero-coupon yield curve. In markets where a limited number of zero-coupon bonds are traded, usually, a sufficient number of coupon-bearing bonds are traded to apply standard bootstrap procedures. In the South African fixed income market, however, only a limited number of liquid instruments are available to construct a zero-coupon yield curve.

In the South African fixed income market, bonds are traded on a yield-to-maturity basis (Faure *et al.*, 1991). The yield-to-maturity of a bond can be defined as the internal rate of return of the investment. When a particular bond is priced using its yield-to-maturity, it is assumed that all cashflows are discounted at the same yield.

If P_k denotes the price of a coupon bond (bond k), and if continuously compounded interest rates are used, the price for a South African bond is calculated by discounting all cashflows at the quoted yield-to-maturity:

$$P_k = \sum_{i=1}^{n-1} \gamma_k e^{-t_i^{(k)}\eta_k} + (1 + \gamma_k) e^{-t_n^{(k)}\eta_k}. \qquad (18)$$

where, for bond P_k

- η_k is the continuously compounded yield to maturity;
- γ_k is the periodic coupon paid;
- $t_i^{(k)}$ is the time to a coupon dat;,
- n is the number of coupons to be paid to maturity; and
- $t_n^{(k)}$ is the term-to-maturity, and there is a repayment of 1 unit at this time.

The term-to-maturity, $t_n^{(k)}$, and yield-to-maturity rates, η_k, give an array which serves as the input for term structure analysis. The ambiguity in the relationship between the yield-to-maturity and the term-to-maturity may be rectified by determining the underlying zero-coupon yields by sequentially stripping off coupons (Hull, 1997).

3.5.1 *Example of bootstrapping*

A practical example illustrates the process of bootstrapping. One can assume that the interest rates for 3, 6 and 12 month periods are known, but after that one only has the yields for coupon bonds maturing in 1.5 years, 2.0 years and 2.75 years, where coupons are paid every six months, as shown in Table 3.1.

Table 3.1: Data for bootstrap method

Term-to-maturity (years)	Annual coupon (%)	Continuously compounded yield (%)
0.25	0	10.13
0.5	0	10.68
1.0	0	11.43
1.5	10	11.74
2.0	12	11.84
2.75	13	11.76

In order to obtain the term structure for the period from 3 months to 2.75 years, it is necessary to do bootstrapping. The price of the bond can be split up into the price of the 6-monthly coupons, and then the price for the nominal plus the coupon at maturity. Since the interest rates for the first two coupon periods are known, the 1.5 year zero-coupon rate, $z(1.5)$, can be determined from the price of the 1.5 year bond:

$$P_{1.5} = ce^{-z(0.5)\times 0.5} + ce^{-z(1.0)\times 1.0} + (N+c)e^{-z(1.5)\times 1.5}$$

where c is the coupon-payment. Since $z(1.5)$ is the only unknown, it can easily be calculated as 11.8%. A similar calculation results in the 2-year rate, $z(2)$, from the 2-year bond, as 11.90%.

Although $z(2.75)$ is still unknown, one can use linear interpolation to find the $z(2.25)$ in terms of $z(2.75)$ and $z(2)$:

$$z(2.25) \equiv \frac{2}{3}z(2) + \frac{1}{3}z(2.75)$$

Using this equation in the pricing formula then gives

$$P_{2.75} = ce^{-z(0.25)\times 0.25} + ce^{-z(0.75)\times 0.75} + ce^{-z(1.25)\times 1.25}$$

$$+ ce^{-z(1.75)\times 1.75} + ce^{-\left(\frac{2}{3}z(2) + \frac{1}{3}z(2.75)\right)\times 2.25} + (N+c)e^{-z(2.75)\times 2.75}$$

The zero-coupon rates, $z(0.25)$, $z(0.75)$, $z(1.25)$, $z(1.75)$ and $z(2)$, are known, or can be interpolated from the rates already known. Numerical procedures such as the Newton-Raphson method, can then be used to establish the 2.75 year rate, or $z(2.75)$, for this bond, which is 11.90%. Continuing this process results in the term structure of interest rates. The process of calculating the spot interest rates by stripping off coupons is called bootstrapping. The curve calculated in this example is shown in Figure 3.1 below.

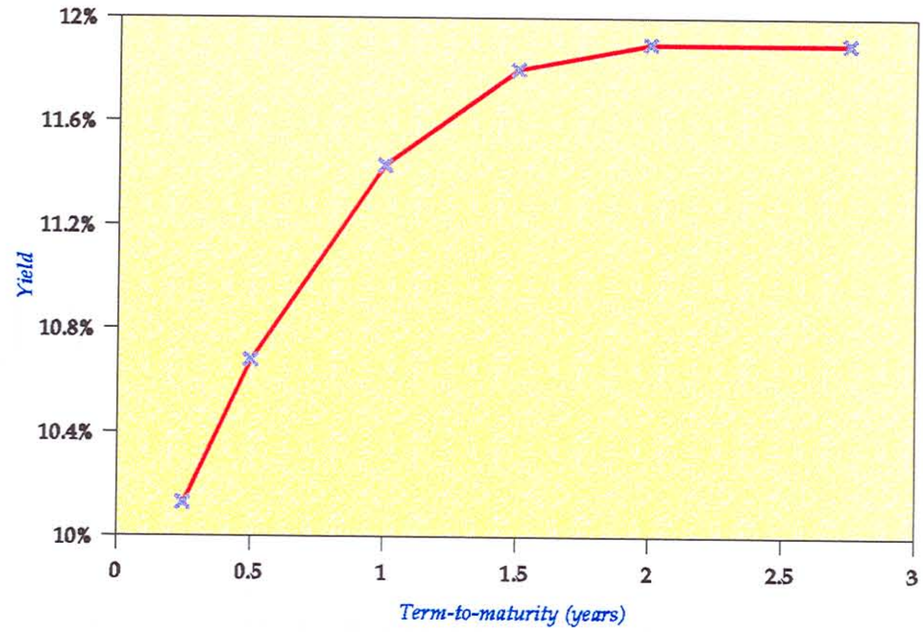


Figure 3.1: Zero-coupon yield curve

According to Vasicek and Fong (1982), the objective of the empirical estimation of the term structure is to fit a zero-coupon curve (or spot rate curve) that both fits the data sufficiently well and is a sufficiently smooth function.

The latter requirement is particularly important, as it will determine the smoothness of the forward curve, derived from the spot rate curve. Because financial markets are dynamic and volatile, the term structure changes periodically to comply with changing perspectives. The objective is therefore to find a method of estimating a zero-coupon curve that both fulfils the above requirements and can be easily adjusted to accommodate a volatile market. A method that complies to these requirements is discussed in the next chapter.

CHAPTER 4

THE ITERATIVE BOOTSTRAP METHOD[1]

The determination of a smooth zero-coupon yield curve in a market where only coupon bonds are traded can be a difficult and time-consuming process. When only a few data points are available, it is especially difficult to obtain a smooth forward curve.

The standard bootstrap technique was evaluated empirically, using South African yield curve data, which motivated the formulation of a more efficient technique. In this chapter, the formulation of the iterative bootstrap method is discussed, and the convergence of the iterative sequence is proved. Empirical results illustrate the use of the method.

[1]The results of the research discussed in this chapter were published in *RISK* (Smit & Van Niekerk, 1997).

4.1 Disadvantages using the standard bootstrap method

The problem with the standard bootstrap procedure is that it is assumed that sufficient data are available to determine the present value of all coupons. As such data are not available in the South African market, interpolation techniques (such as the Newton Raphson technique) must be used to find intermediate data points (for bootstrap purposes) before fitting the final curve. There can be any number of intermediate data points, even twenty or more, depending on the number of coupons between two data points. If the data points do not form a smooth curve, it is possible that the curve from which coupons are discounted will differ from the final fitted curve, causing a discrepancy. Another disadvantage is that the interpolation of data points in the standard bootstrap technique is time-consuming.

Once the zero-coupon rates have been determined, the question arises as to which approximation technique to use. Polynomial approximation and spline fitting are the most commonly used techniques, but they are not always suitable for the South African yield curve, due to structural inefficiencies in the fixed income market and the resultant dispersion of data points.

A solution to these problems was developed in this study. This solution involves constructing a zero-coupon curve using an iterative bootstrap method (IBS-method), where the entire curve is *simultaneously* bootstrapped, starting with a first guess. Each iteration results in a sequence of implied zero-coupon rates which are then fitted using least squares approximation, and used again in the next iteration. This approach is described in the sections below.

4.2 Iterative bootstrapping – introduction

A standard bootstrap procedure follows a process where the coupons of each individual bond in the data set is bootstrapped to obtain a fixed zero-coupon rate for a specific term. This rate is again used in the bootstrap process for the next bond. The method therefore progresses along the time-axis to find the discrete zero-coupon rates, which are then approximated by a curve. Interpolation methods are used to discount coupons at intermediate maturity dates.

In order to overcome problems with the standard bootstrap method, a method is suggested that follows an iteration process. The entire data set is bootstrapped simultaneously, using implied zero-coupon rates obtained in the previous iteration, by starting with a first guess for the zero-coupon yield curve. For each iteration, this again results in a set of implied zero-coupon rates (one data point for each coupon bond). A least squares approximation technique is used to obtain a smooth curve which is employed to discount cashflows for the next iteration. These iterations converge and ultimately yield a unique zero-coupon curve for the particular approximation technique[2]. The iterative bootstrap method is a dynamic method compared to the more static standard bootstrap method.

The advantage of bootstrapping the bonds simultaneously in the iteration process is that, for each iteration, different cashflows are discounted from the same smooth curve to find the implied zero-coupon rates for the next iteration. The final fitted zero-coupon curve is therefore obtained by bootstrapping from the same curve. Therefore, there is no discrepancy between the curve that has been used for bootstrapping, and the final fitted zero-coupon yield curve. The replacement of the interpolation of data points with a method where a fitted curve

[2]Different approximation techniques result in slight differences in the resultant term structure.

determines the points speeds up the whole process. The use of numerical methods, such as the Newton Raphson method, also becomes unnecessary.

4.3 The iterative bootstrap method

The following assumptions are made:

- It is possible for the yield curve $z(t)$ to have any shape (positive, negative).
- All interest rates are positive.
- A bond pays a nominal value of 1 unit at the end of its term.
- The term-to-maturity, t, is given in years.
- Continuously compounded interest rates are used.
- Market participants take advantage of arbitrage opportunities as they occur.

If all fixed income securities meet the no-arbitrage principle, the price P_k of an arbitrary coupon-bearing bond, k, should equal the sum of the n cashflows, discounted at the particular zero-coupon rate, $z(t)$:

$$P_k = \sum_{i=1}^{n-1} \gamma_k e^{-t_i^{(k)} z(t_i^{(k)})} + (1 + \gamma_k) e^{-t_n^{(k)} z(t_n^{(k)})} \tag{1}$$

where γ_k is the coupon payment. The price of the bond can also be determined using the market yield-to-maturity, η_k, for the bond k (as traded in the market), therefore

$$P_k = \sum_{i=1}^{n-1} \gamma_k e^{-t_i^{(k)} \eta_k} + (1 + \gamma_k) e^{-t_n^{(k)} \eta_k} \tag{2}$$

Since the final zero-coupon yield curve, $z(t)$ is not yet known and the IBS-method does not use the interpolation of data points, a curve from which to bootstrap is needed. If it is assumed that $y_j(t)$ is the j-th approximate fit for the zero-coupon yield curve (for the j-th iteration), starting at $y_1(t)$ as a first guess, equation (1) can be reformulated as follows:

$$P_k = \sum_{i=1}^{n-1} \gamma_k e^{-t_i^{(k)} y_j(t_i^{(k)})} + (1 + \gamma_k) e^{-t_n^{(k)} y_j^*(t_n^{(k)})} \qquad (3)$$

where P_k is known from equation (2) (the market price) and $y_j^*(t_n^{(k)})$ is the implied zero-coupon yield for the term $t_n^{(k)}$ years and the only unknown parameter. Equation (3) holds at any time throughout the iteration process. If the curve $y_j(t)$ is different from the correct zero-coupon curve, the point $y_j^*(t_n^{(k)})$ deviates from this curve, in order to give the correct price, P_k.

Coupons are bootstrapped simultaneously using $y_j(t)$ for each iteration j and all bonds k, $k=1,...,m$, where m is the number of bonds. From equation (3), it is possible to solve $y_j^*(t_n^{(k)})$ for all bonds in the data set, to get an implied array of zero-coupon yields for each term $t_n^{(k)}$ in each iteration:

$$y_j^*(t_n^{(k)}) = -\frac{1}{t_n^{(k)}} \ell n \left[\frac{P_k - \sum_{i=1}^{n-1} \gamma_k e^{-t_i^{(k)} y_j(t_i^{(k)})}}{1 + \gamma_k} \right] \qquad (4)$$

where it is assumed that $t_n^{(1)} \neq t_n^{(2)}$.

To serve as input for the next iteration, the zero-coupon rates $y_j^*(t_n^{(k)})$ are approximated by a fit $y_{j+1}(t_n^{(k)})$. By repeating the process for $j = 1,2,...$ a sequence of implied zero-coupon rates is found for each term, $t_n^{(k)}$, in the data set. The theorem discussed below states that these implied zero-coupon curves converge to the zero-coupon curve implied by the coupon bond market.

4.4 Theorem

Given an arbitrary guess for the function, $y_1(t)$, the approximate linear interpolated fit $y_j(t)$ will converge to the zero-coupon yield curve, $z(t)$, on condition that

$$0 \leq \gamma < (e^{\tau z(\tau)} - 1)^{-1}$$

for any bond maturing at time $\tau < t_m$, where t_m is the maximum range of the term structure, and $y_j(t)$ interpolates the implied zero-coupon rates $y_{j-1}^{*}(t_i)$, $j > 1$, $0 < t_i < t_m$.

4.4.1 *Proof*

If P is the price of a bond maturing at time t_3, paying coupons, γ, at time t_1, t_2 and t_3, and at time t_1 the zero rate $z(t_1)$ is known, it is possible to prove that the theorem holds for this bond. If the theorem holds for time t_3, it is possible to demonstrate that it will hold for any time $t_i \leq t_m$, where t_m is the maximum term of the term structure.

Using the first guess $y_1(t)$ and assuming that $y_1(t_2) > z(t_2)$, it follows from equations (1) and (3) that

$$z(t_3) > y_1^{*}(t_3) \tag{5}$$

If one assumes that $y_1(t_2) < z(t_2)$, it implies that

$$z(t_3) < y_1^{*}(t_3) \tag{6}$$

According to equations (1) and (3)

$$\gamma e^{-t_2 y_j(t_2)} + (\gamma + 1)e^{-t_3 y_j^{*}(t_3)} = \gamma e^{-t_2 z(t_2)} + (\gamma + 1)e^{-t_3 z(t_3)} \tag{7}$$

Therefore, for $j = 1$,

$$\frac{e^{-t_2 z(t_2)} - e^{-t_2 y_1(t_2)}}{e^{-t_3 y_1^*(t_3)} - e^{-t_3 z(t_3)}} = \frac{1 + \gamma}{\gamma}$$

or

$$\frac{1 - e^{-t_2[z(t_2) - y_1(t_2)]}}{1 - e^{-t_3[y_1^*(t_3) - z(t_3)]}} = \frac{1 + \gamma}{\gamma}\left[e^{-t_3 z(t_3) + t_2 y_1(t_2)}\right] \tag{8}$$

Next, if one assumes that convergence does not occur, and

$$z(t_2) - y_1(t_2) \le y_1^*(t_3) - z(t_3) \tag{9}$$

then, for $t_2 < t_3$,

$$[z(t_2) - y_1(t_2)]\, t_2 < [y_1^*(t_3) - z(t_3)]\, t_3$$

Since e^{-x} is a decreasing function, it follows that

$$e^{-t_3[y_1^*(t_3) - z(t_3)]} < e^{-t_2[z(t_2) - y_1(t_2)]}$$

Therefore,

$$\frac{1 - e^{-t_2[z(t_2) - y_1(t_2)]}}{1 - e^{-t_3[y_1^*(t_3) - z(t_3)]}} < 1 \tag{10}$$

If equation (10) holds, it follows from equation (8) that

$$\frac{1 + \gamma}{\gamma}\, e^{-t_3 z(t_3) + t_2 y_1(t_2)} < 1$$

Therefore,

$$-t_3 z(t_3) + t_2 y_1(t_2) < \ln\left(\frac{\gamma}{1 + \gamma}\right)$$

$$\Rightarrow \quad y_1(t_2) < \frac{1}{t_2}\left[t_3 z(t_3) + \ln\left(\frac{\gamma}{1 + \gamma}\right)\right]$$

Since $y_1(t) > 0, \ \forall\, t > 0$, it follows that

$$z(t_3) > \frac{1}{t_3}\ln\left(\frac{1+\gamma}{\gamma}\right)$$

$$\therefore \gamma > (e^{t_3 z(t_3)} - 1)^{-1}$$

However, this violates the assumption that

$$\gamma < (e^{t z(t)} - 1)^{-1}$$

This implies that equation (9) does not hold, therefore

$$y_1^*(t_3) - z(t_3) < z(t_2) - y_1(t_2) \tag{11}$$

On the other hand, since $y_1(t_2) < z(t_2)$, and $y_1^*(t_3) > z(t_3)$, it is possible to say

$$-(z(t_2) - y_1(t_2)) < y_1^*(t_3) - z(t_3) \tag{12}$$

From equations (11) and (12), it follows that

$$|y_1^*(t_3) - z(t_3)| < z(t_2) - y_1(t_2) \tag{13}$$

Since the function $y_{j+1}(t)$ interpolates $y_j^*(t_i)$, $\forall\, j$, one can substitute $y_1^*(t_i)$ with $y_2(t_i)$, i = 1,2,3.... The iteration process therefore results in the following:

$$\begin{aligned} |z(t_2) - y_1(t_2)| &> |y_1^*(t_3) - z(t_3)| \\ &= |y_2(t_3) - z(t_3)| \end{aligned} \tag{14}$$

Although many sophisticated interpolation techniques can be used to interpolate the implied zero-coupon yields $y_j^*(t)$, it is possible to assume for the purposes of the proof that one interpolates linearly between the points $(t_1, z(t_1))$ and $(t_3, y_1(t_3))$. Then

$$y_1(t_2) = \frac{t_2 - t_1}{t_3 - t_1}\left(y_1(t_3) - z(t_1)\right) + z(t_1) \tag{15}$$

and

$$z(t_2) = \frac{t_2 - t_1}{t_3 - t_1}\left(z(t_3) - z(t_1)\right) + z(t_1) \tag{16}$$

Therefore,

$$|z(t_2) - y_1(t_2)| = \frac{(t_2 - t_1)}{(t_3 - t_1)}|z(t_3) - y_1(t_3)|$$

Since $t_2 < t_3$, it follows from equation (14) that

$$|y_2(t_3) - z(t_3)| < |z(t_3) - y_1(t_3)| \tag{17}$$

Finally, it is important to show that

$$|y_3(t_3) - z(t_3)| < |y_2(t_3) - z(t_3)|$$

Since $y_1^*(t_2) = y_2(t_2) > z(t_2)$, it follows as in equation (5) that $y_2^*(t_3) < z(t_3)$. If one supposes that

$$y_2(t_2) - z(t_2) \leq z(t_3) - y_2^*(t_3)$$

then, as in equation (10), it follows that

$$\frac{1 - e^{-t_2[y_2(t_2) - z(t_2)]}}{1 - e^{-t_3[z(t_3) - y_2^*(t_3)]}} < 1 \tag{18}$$

Using equation (7) one can write:

$$\frac{1 - e^{-t_2[y_2(t_2) - z(t_2)]}}{1 - e^{-t_3[z(t_3) - y_2^*(t_3)]}} = \frac{1 + \gamma}{\gamma}\left[e^{-t_3 y_2^*(t_3) + t_2 z(t_2)}\right]$$

Therefore,

$$\frac{1+\gamma}{\gamma}\left[e^{-t_3 y_2^*(t_3) + t_2 z(t_2)}\right] < 1$$

$$\Rightarrow \quad y_2^*(t_3) > \frac{1}{t_3}\left[t_2 z(t_2) - \ln\left(\frac{\gamma}{\gamma+1}\right)\right]$$

Since $z(t_3) > y_2^*(t_3)$, it follows that

$$z(t_3) > \frac{1}{t_3}\left[t_2 z(t_2) - \ln\left(\frac{\gamma}{\gamma+1}\right)\right]$$

$$\therefore \quad \frac{\gamma+1}{\gamma} < \frac{e^{t_3 z(t_3)}}{e^{t_2 z(t_2)}} \le e^{t_3 z(t_3)}$$

Therefore,

$$\gamma > (e^{t_3 z(t_3)} - 1)^{-1} \tag{19}$$

Equation (19) again violates the assumption. Hence,

$$z(t_3) - y_2^*(t_3) < y_2(t_2) - z(t_2)$$

Since $y_1(t_2) > z(t_2)$, $y_2^*(t_3) < z(t_3)$ and $y_2^*(t_3) = y_3(t_3)$, it follows that

$$|y_3(t_3) - z(t_3)| < |y_2(t_2) - z(t_2)| \tag{20}$$

Linear interpolation between $(t_1, z(t_1))$ and $(t_3, y_2(t_3))$ gives equations (15) and (16) with y_2 instead of y_1. Therefore,

$$|y_3(t_3) - z(t_3)| < |y_2(t_3) - z(t_3)| \tag{21}$$

In general:

$$|y_j(\tau) - z(\tau)| < |z(\tau) - y_{j-1}(\tau)| \quad \forall\, \tau < t_n,\ j \ge 1$$

which proves that the sequence $\{|\, y_j(\tau) - z(\tau)|\}$ converges to zero, which proves that

$$\lim_{j \to \infty} y_j(\tau) = z(\tau)$$

□

4.5 An illustration of the method

The following example illustrates the use of the iterative process to determine a zero-coupon yield curve. One could suppose that the interest rates for three risk-free securities in the money market are known (maturing in 1, 6 and 12 months):

$$y_1(0.08) = 13.95\%; \; y_1(0.5) = 14.48\%; \; y_1(1.0) = 14.88\%$$

Since the money market instruments are zero-coupon rates, it follows that

$$y_j(t) = z(t) \qquad \forall\, j, \quad t \leq 1 \tag{22}$$

If four different coupon-bearing bonds are traded in the market, maturing in 3, 5, 8 and 10 years respectively, and the bonds, with a nominal value of 1 unit, pay semi-annual coupons of γ_k units and are priced at present at P_k:, then

$$P_1 = 0.9751097, \qquad \gamma_1 = 0.075$$

$$P_2 = 0.9845960, \qquad \gamma_2 = 0.08$$

$$P_3 = 0.8766290, \qquad \gamma_3 = 0.07$$

$$P_4 = 0.8080316, \qquad \gamma_4 = 0.065$$

To start the iteration process, a continuous extrapolation is guessed for $y_1(t)$, $1 < t \leq 10$, where $y_1(3) = 15.3\%$ and $y_1(5) = 15.6\%$ and $y_1(8) = 15.9\%$ and $y_1(10) = 16.1\%$.

Each of the four bonds in the example implies a zero-coupon yield $y_j^*(t_n)$, where $t_n = 3, 5, 8$ and 10 respectively for each iteration *j*. For example, the first bond (maturing in three years), gives, for $j = 1$,

$$y_1^*(3) = -\frac{1}{3}\,\ell n\left[\frac{0.9751097 - 0.075\left(e^{-0.5z(0.5)} + e^{-z(1)} + e^{-1.5y_1(1.5)} + e^{-2y_1(2)} + e^{-2.5y_1(2.5)}\right)}{1.075}\right]$$

In the same way $y_1^*(5)$, $y_1^*(8)$ and $y_1^*(10)$ can be found. Using these results as well as the data points $y_1(t_i)$, t_i = 0.08, 0.5, 1.0, a second approximate fit, $y_2(t)$, is found. Repeating this process, results in a sequence $y_j(t)$ as shown graphically in Figure 4.1 (overleaf). Table 4.1 shows the numerical results for each iteration. The results show clearly that the sequence $y_j^*(t)$ converges.

Table 4.1: Implied zero-coupon yields for five iterations starting with a first guess y_1

Term, t_i	y_1 (%)	y_1^* (%)	y_2^* (%)	y_3^* (%)	y_4^* (%)	y_5^* (%)
0.08	13.95	13.95	13.95	13.95	13.95	13.95
0.5	14.48	14.48	14.48	14.48	14.48	14.48
1	14.88	14.88	14.88	14.88	14.88	14.88
3	15.30	15.55	15.53	15.53	15.53	15.53
5	15.60	16.07	15.97	15.98	15.98	15.98
8	15.90	16.79	16.42	16.51	16.50	16.50
10	16.10	17.30	16.62	16.85	16.79	16.80

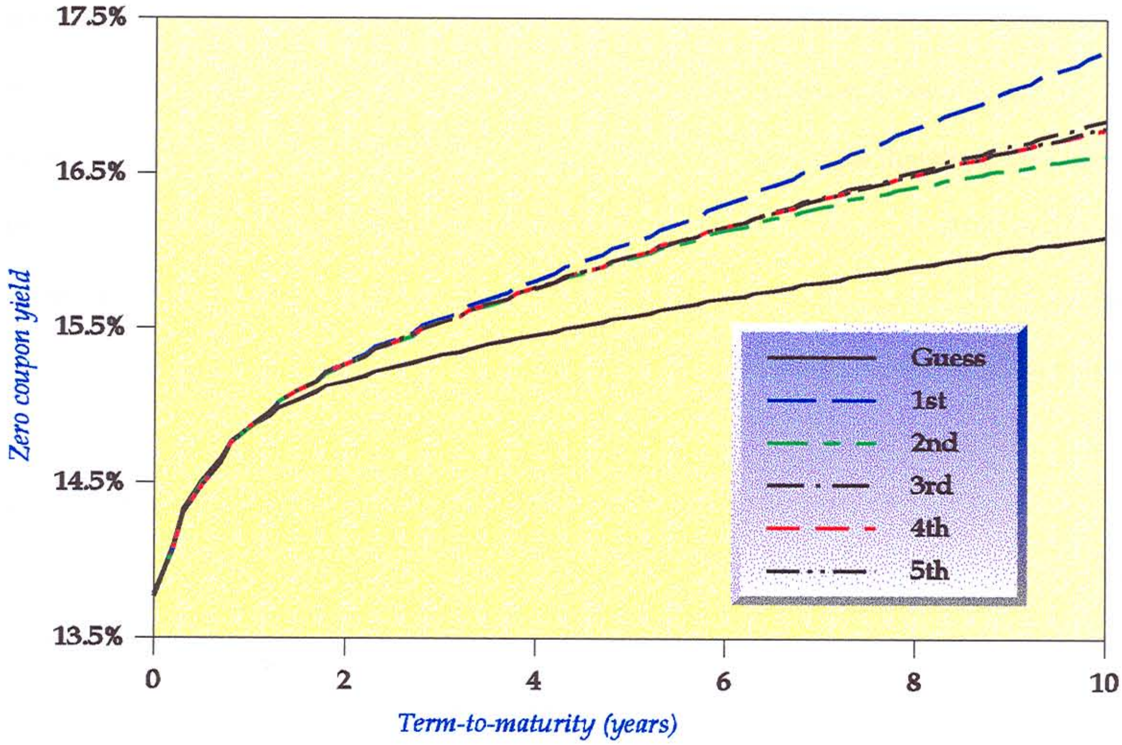


Figure 4.1: Results of the iterative process

4.6 Empirical results

The IBS-method developed in the previous sections was used to derive a zero-coupon yield curve for the South African fixed income market empirically. Daily closing rates over a three-year period were used to evaluate the method. Money market instruments were used to obtain data points between $t = 0$ and $t = 1$, while actively traded bonds were used to obtain information for the remainder of the term structure.

To compensate for market data that do not form a smooth curve, a least squares approximation technique was used in order to obtain a reasonably accurate fit of the data points which then served as input in order to interpolate for the next iteration. Appendix A sets out a discussion of the least squares approximation technique. It is important to realise that the success of the iterative method depends on a reasonably accurate interpolation of data points for bootstrapping purposes in each iteration. It is possible, for instance, to obtain an implied negative interest rate if the curve fitting technique oscillates or diverges from the data points.[3]

The empirical results of the study show that the technique yields a smooth spot rate curve and that the curve approximates the data points sufficiently well. The iterative method was compared to the standard bootstrapping technique, using a least squares fit. Market data from 1996 were used, which resulted in similar results for both methods, as is shown in Figure 4.2. The iterative method, however, provides a more accurate result in the region where data points do not form a smooth curve, due to interpolation discrepancies when the standard bootstrap technique is used. The difference between the two curves in Figure 4.2 increases when the data points are less smooth. The iterative method also proved to be computationally more efficient.

Figure 4.3 shows the results of a par-bond curve in November 1999, as derived from the zero-coupon curve. The forward curves implied by the zero-coupon curve in the above examples are sufficiently smooth. The implied forward swap curves for a 10-year and a 5-year swap are shown in Figure 4.4. When the standard bootstrap method is used, these curves are usually irregular with sudden changes in the slope of the curve.

---

[3]This is why the theorem assumes a linear interpolation.

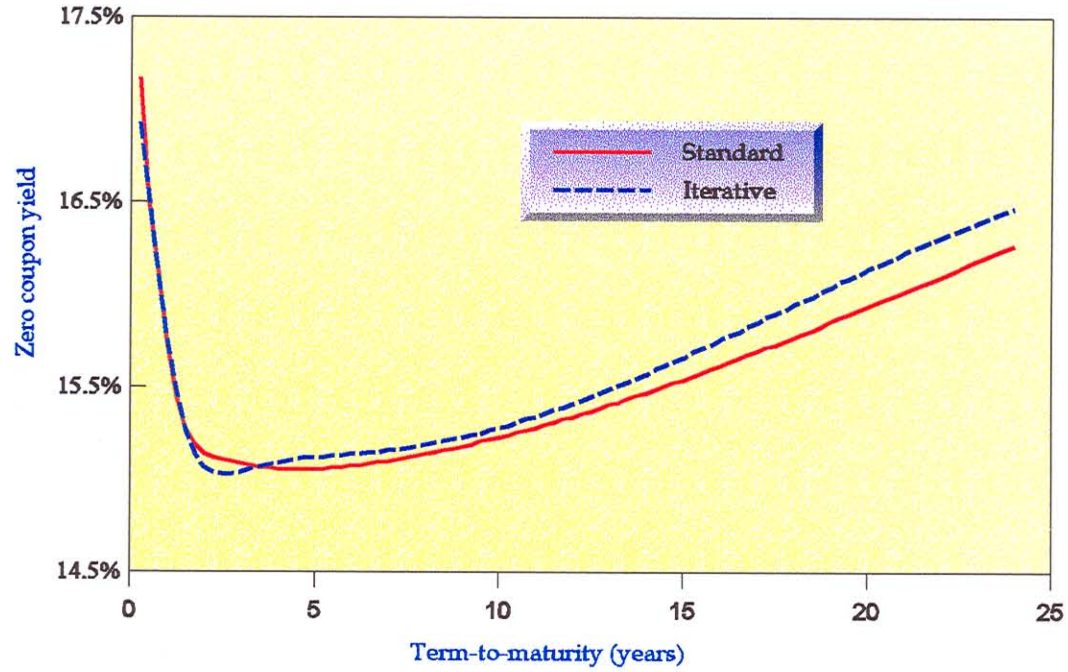


Figure 4.2: Comparison between standard bootstrap method and iterative bootstrap method

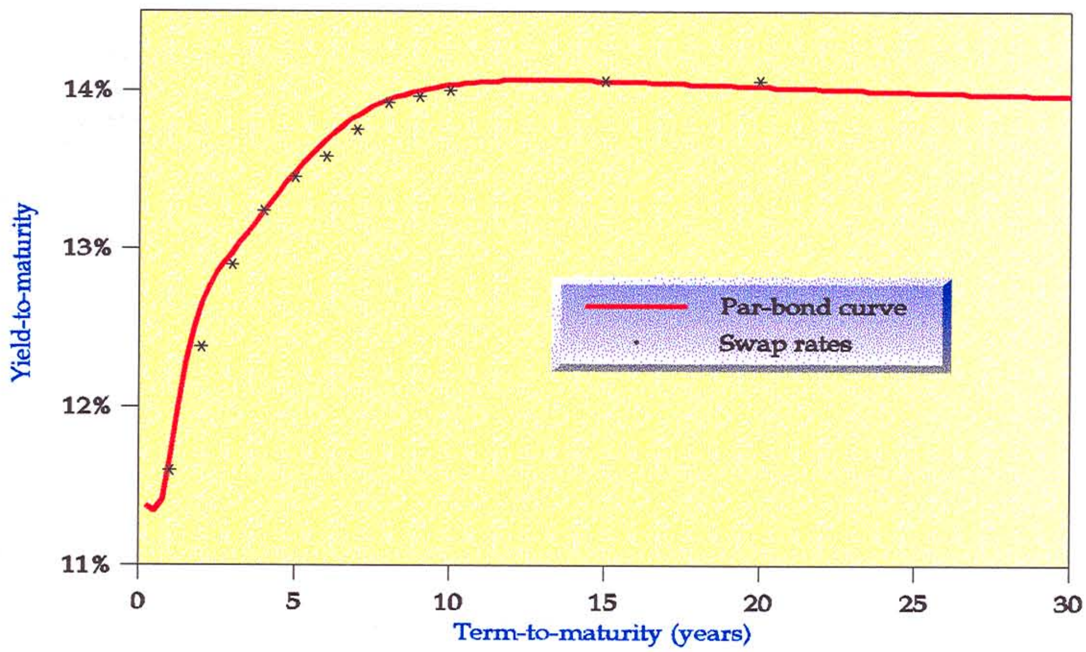


Figure 4.3: Par-bond yield curve in November 1999

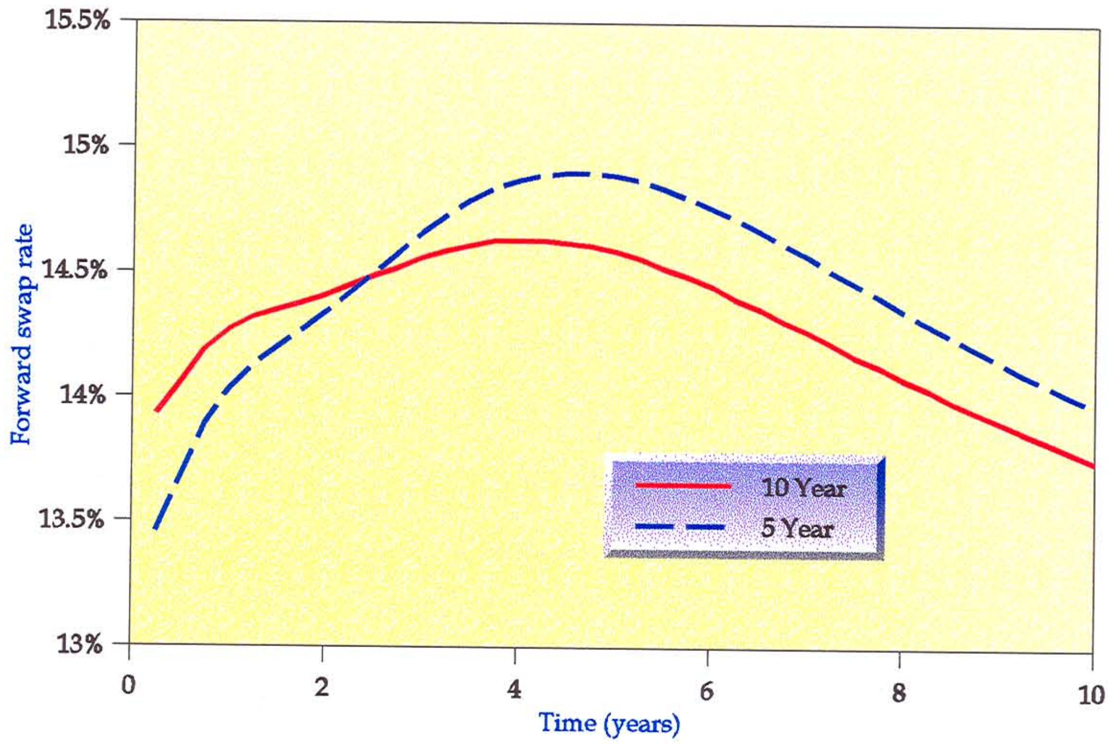


Figure 4.4: Implied forward swap curves for the yield curve in Figure 4.3

The empirical results from the South African market show that the method performs sufficiently well and yields better results than alternative techniques. Some of the advantages of the method are that it produces a smooth term structure, a smooth forward curve and that it is flexible and computationally efficient. It can therefore be applied in volatile and illiquid emerging fixed income markets to identify mispricings and arbitrage opportunities.

In order to evaluate the accuracy of the approximation of the zero-coupon yield curve, the zero-coupon yield curve obtained is used to calculate the implied yield-to-maturity, based on this curve. The sum of the squares of the errors in these rates was in the order of 1.4×10^{-4}. The individual deviations from the actual market rates varied between zero basis points for the more liquid bonds and 15 basis points for less liquid bonds containing a liquidity premium.

4.7 Credit premium

For the valuation of most derivatives, it is usually assumed that there is no risk of counter-party default. The no-default assumption does not, however, apply for bonds, and therefore the risk of a default on the coupons and/or nominal must be accounted for. This is done by adding basis points to the yield of the bond, in order to compensate for the credit risk. Bonds that are less tradable, on the other hand, also trade at a liquidity spread to the more liquid bonds.

The South African government bonds have the highest credit rating in the country. The government bonds can therefore be used to give a homogeneous zero-coupon yield curve with the same credit rating. All other bonds are priced from this curve to determine their yield (plus the credit and/or liquidity premium). Non-government organisations, for example Transnet, Eskom and Telkom, have a fairly big credit spread to the government curve, although some have government guarantees (Brown, 1999).

One advantage of the zero-coupon yield curve is that any bond can be priced from the zero-coupon yield curve, as determined from government bonds. An implied yield-to-maturity can therefore be found for any other bond. The difference between this *implied* yield and the market yield equals the credit spread added to compensate for the credit risk. (It is assumed that bonds with similar liquidity are compared in this example, in order to be able to ignore the liquidity premium.)

The evolution of the credit spread was investigated for some non-government bonds over time. Table 4.2 shows the credit spread over a 3-year period for three Transnet bonds, maturing in 2002, 2008 and 2010 respectively. The liquidity of these bonds is comparable to the smaller RSA government issues, and one can therefore ignore the liquidity spread. It is evident that the credit spread increased over the last three years, and must be taken into account when pricing these bonds. The 1998 emerging market crisis emphasised the importance of appropriate credit spreads for non-government bonds.

Table 4.2: Credit spreads for Transnet bonds

Date	Credit spread above government yield curve (basis points)		
	T001	T004	T011
November 1996	11	10	4
October 1997	12	22	15
February 1998	10	20	20
November 1998	32	37	41
May 1999	32	32	43
November 1999	43	29	37
May 2000	21	23	36

4.8 Concluding Remarks

The standard bootstrap method displays some inefficiencies when it is applied to a yield curve where there are only coupon bonds and irregular data points. An iterative method was therefore developed, which starts with a first guess, and then converges to the actual zero-coupon yield curve. This method is more efficient than the standard method.

The IBS-method developed in this chapter can be used to price all vanilla fixed income instruments. It can also serve as input in pricing many derivative securities, for instance options on fixed income vanilla products.

The next few chapters concentrate on the valuation of options on fixed income products, where the zero-coupon yield curve is an important input for some models.

CHAPTER 5

BOND OPTION PRICING MODELS

Options on long-term bonds are popular derivative instruments used to hedge a fixed income portfolio against the movement of interest rates. An option on a long-term bond gives the holder the right, but not the obligation, to buy or sell the bond at a certain future time at a predetermined strike price or exercise price.

The valuation of options on interest rate instruments, such as bonds, is more complex than options on stocks and commodities, since it involves not only one underlying instrument, but also a subset of instruments which relies on the term structure of interest rates. Several models have been developed over the years to price options on long-term bonds. They can be divided into the following three categories:

- conventional models;
- equilibrium models; and
- no-arbitrage models.

The price of a bond is determined by several factors – its maturity date, coupon rate, ex- or cum-status and yield-to-maturity. The yield-to-maturity is the interest rate or rate-of-return for the bond, commonly referred to as the yield. For short-dated options, it is assumed that the price P of the bond follows a Brownian motion. Conventional models use the stochastic process of the particular underlying bond price to determine a fair value for the price of the option. The behaviour of the remainder of the term structure is not taken into account. These models are widely used in all markets.

An equilibrium model first defines a process for the instantaneous short rate, r. It produces a term structure of interest rates from the value of r at the current time t, and a risk-neutral process for r. Equilibrium models produce a term structure of interest rates as an *output*, using the stochastic process of the short rate r. This does not necessarily fit today's term structure. It can certainly fit the term structure approximately, but in some cases an exact fit is not possible, resulting in significant errors, which are discussed in Section 5.2.2.

A no-arbitrage model, on the other hand, uses the initial term structure as an input and is therefore exactly consistent with today's term structure.

Some of the interest rate models in the above-mentioned three categories are discussed below, and then the analytical solution of the Hull-White model is examined in more detail.

5.1 Conventional models

5.1.1 The Black-Scholes model

The Black-Scholes model is a popular tool to value almost any derivative security. It is easily adjusted to price an option on a bond price. If P is the spot price of a discount bond, or zero-coupon bond, the behaviour of the bond price, P, can be described by the stochastic process

$$dP = \mu P dt + \sigma_P P dW \tag{1}$$

where μ is the expected return, σ_P is the volatility of the bond price and W is a Wiener process.

If X is the exercise price, T the time to expiry of the option and R_T the zero-coupon continuously compounded risk-free interest rate for maturity T, and one uses the Black-Scholes model, then the price c of a European call and the price p of a European put option on a zero-coupon bond (following the process in equation (1)) are given by:

$$c = PN(d_1) - e^{-R_T \cdot T} XN(d_2) \tag{2}$$

and

$$p = e^{-R_T \cdot T} XN(-d_2) - PN(-d_1) \tag{3}$$

where

$$d_1 = \frac{\ell n(P/X) + (R_T + \sigma_P^2/2)T}{\sigma_P\sqrt{T}}$$

$$d_2 = d_1 - \sigma_P\sqrt{T}$$

For a coupon bond where coupons are payable during the life of the option, the coupons can be treated as the dividends on a stock. The spot price of the bond should therefore exclude the present value of the coupons. The volatility parameter, σ_P, should be the volatility of the bond price without the present value of the applicable coupons.

5.1.2. The Black model

The Black version of the Black-Scholes model has proved to be more suitable for the valuation of coupon-bearing bond options, because it uses the forward price. The forward price of the bond already excludes any coupons paid during the life of the option. The Black model is the most popular method for valuing ordinary options on coupon bonds.

The Black model assumes that the price of the underlying instrument is lognormally distributed on the expiry date of the option. If F is the forward price of the underlying bond on the expiry date of the option, the price of a call and put are then given by:

$$c = e^{-R_T \cdot T}\left[F N(d_1) - X N(d_2)\right]$$
$$p = e^{-R_T \cdot T}\left[X N(-d_2) - F N(-d_1)\right] \qquad (4)$$

where

$$d_1 = \frac{\ell n(F/X) + \frac{1}{2}\sigma_F^2 T}{\sigma_F\sqrt{T}}$$

$$d_2 = d_1 - \sigma_F\sqrt{T}$$

For exchange traded options, where an interest-bearing margin is paid and the option is cash-settled only on the expiry date, equation (4) still holds, but with R_T set equal to zero[1].

The disadvantages of the Black model are discussed in Chapter 7 and an alternative model is proposed.

5.2 Equilibrium models

5.2.1 The Rendleman-Bartter model

Rendleman and Bartter (1980) developed a model where the short rate, r, is described in a risk-neutral world by an Ito process

$$dr = \mu r dt + \sigma r dW \qquad (5)$$

where μ is the drift and σ is the volatility of the short rate. This model assumes that the short rate, r, follows a geometric Brownian motion.

The process for r can be modelled by using a binomial tree, where the parameters are given by

[1]The interest paid on borrowed money is equal to the interest received on the margin account.

$$u = e^{\sigma\sqrt{\Delta t}}$$

$$d = e^{-\sigma\sqrt{\Delta t}}$$

$$p = \frac{a - d}{u - d}$$

where

$$a = e^{\mu\Delta t}$$

The short-term interest rate is chosen to be the rate for the length of the time-interval. Interest rate movements in a risk-neutral world are given by the binomial tree:

$$r_{ij} = r_0 u^j d^{i-j} \tag{6}$$

where r_o is the initial short-term interest rate. An interest rate tree (Rendleman and Bartter, 1979) for the full term of the bond, until it matures, can be constructed using equation (6). The value of the bond P_{ij} at each node is then given by

$$P_{ij} = e^{-r_{ij}\Delta t}\left[p P_{i+1,j+1} + (1 - p)P_{i+1,j} + c\right] \tag{7}$$

where c is the coupon paid at the end of each time-interval. At the maturity date of the bond, the bond price equals the bond's nominal value, which is then the boundary condition for equation (7).

Once the bond price at each node is known, one can continue to determine the option value. In order to calculate the value of an American call option at each node, one starts at the time-step, N, which coincides with the expiry date of the option, and then calculates the intrinsic value of the option:

$$f_{Nj} = \max[P_{Nj} - X, 0]$$

where X is the exercise price of the option. For $i < N$,

$$f_{ij} = \max[P_{ij} - X, e^{-r_{ij}\Delta t}(pf_{i+1,j+1} + (1 - p)f_{i+1,j})]$$

where the first term in the equation tests for the early-exercise value at each node. By rolling back through the tree, the value of the option at the first node is determined, which is the price of the option.

To illustrate the approach, suppose that $\Delta t = 1, \mu = 0.08, \ \sigma = 0.2$. One can suppose the initial value of r is 10% per annum and the aim is to value a 4 year American call option on a 5 year bond that pays a 12% coupon at the end of each year and has a face value of R1000.00.

In order to determine the option price, one first determines the short rate tree, next, the bond price tree and then works backward through the tree to obtain the option price. Figure 5.1 shows the numerical results, giving an option price of R28.28.

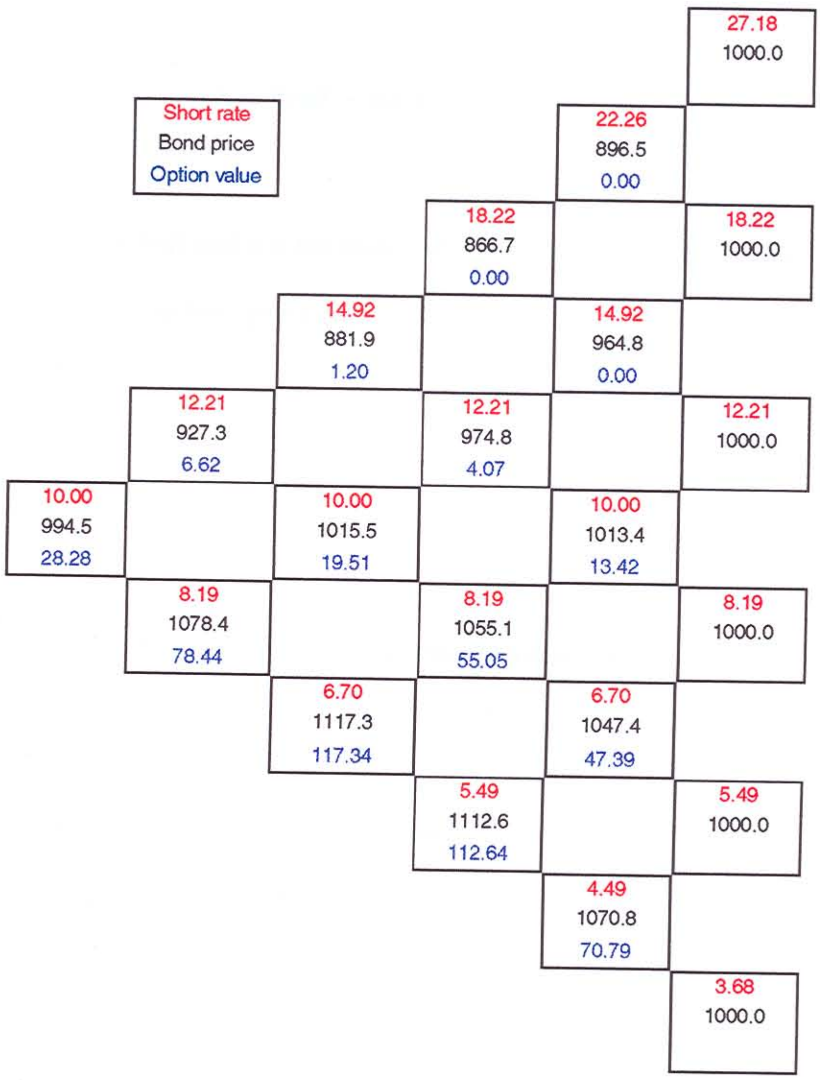


Figure 5.1: Example of a Rendleman-Bartter tree

5.2.2 *The Vasicek model*

Vasicek's model (1977) assumes that the short rate, r, follows a continuous Markov process. The risk-neutral process for r is given by the stochastic differential equation

$$dr = f(r)dt + \sigma dW \tag{8}$$

where $f(r)$ is the instantaneous drift and σ is the standard deviation or volatility of the spot rate process $r(t)$. The parameter $f(r)$ can be expressed in such a form that it includes mean reversion:

$$f(r) = a(b-r)$$

where the short rate, r is pulled to a level b at a rate a.

Vasicek obtained the following analytic formula for the price of a discount bond at time t, paying 1 unit at maturity time t_n:

$$P(t,t_n) = A(t,t_n)e^{-B(t,t_n)r(t)} \tag{9}$$

where, for $a \neq 0$,

$$B(t,t_n) = \frac{1 - e^{-a(t_n - t)}}{a} \tag{10}$$

and

$$A(t,t_n) = \exp\left[\frac{(B(t,t_n) - (t_n - t))(a^2b - \sigma^2/2)}{a^2} - \frac{\sigma^2 B(t,t_n)^2}{4a}\right] \qquad (11)$$

From the above equations it is possible to obtain the whole term structure as a function of r, once a, b and σ have been chosen. The term structure can be upward-sloping, downward-sloping or humped. The possible shape of the term structure is, however, limited, which causes the assumed term structure to differ significantly from the actual term structure. Figure 5.2 shows an example of a best fit for a term structure, using the Vasicek model. It is clear that the method results in large errors.

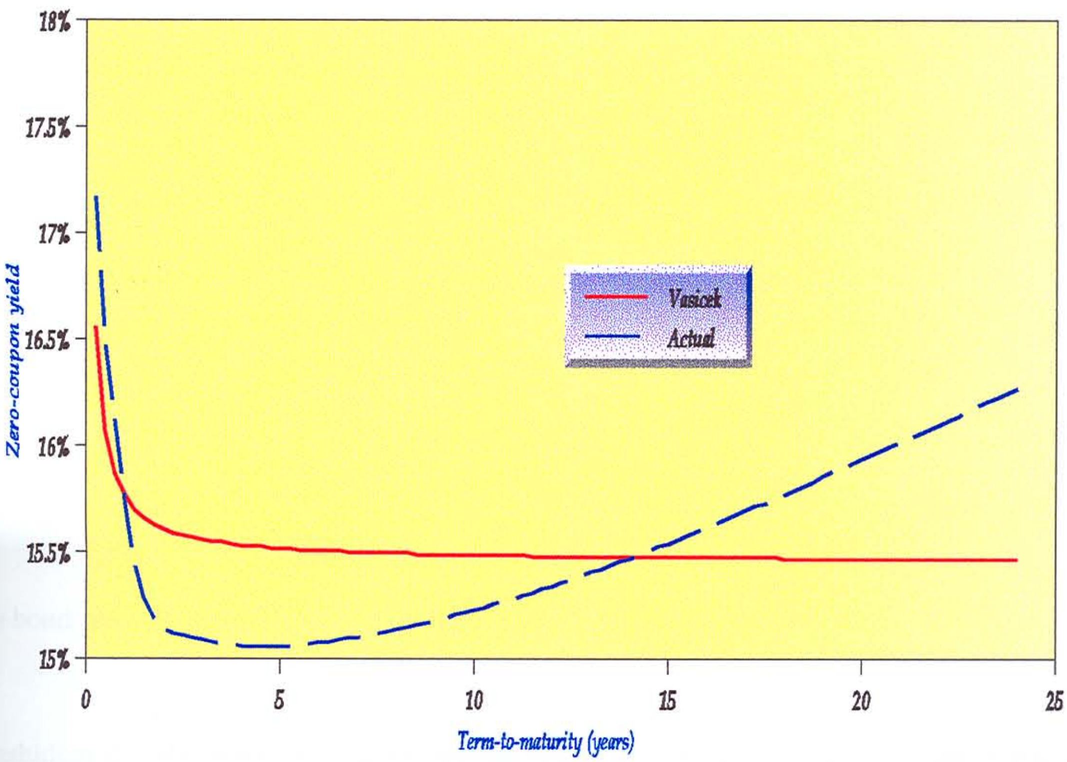


Figure 5.2: **Difference between the best fit using a Vasicek term structure and a 1997 South African yield curve**

5.2.3 *Jamshidian's model*

Jamshidian (1989) has demonstrated how to determine the value of an option on a discount bond using the Vasicek model. The value of a European call option at time t, expiring at time T, on a discount bond with a 1 unit principal maturing at time t_n, is given by

$$c = P(t,t_n)N(h) - XP(t,T)N(h - \sigma_p) \tag{12}$$

where

$$h = \frac{1}{\sigma_p}\ell n\frac{P(t,t_n)}{P(t,T)X} + \frac{\sigma_P}{2}$$

$$\sigma_p = v(t,T)B(T,t_n)$$

$$v(t,T)^2 = \frac{\sigma^2(1 - e^{-2a(T-t)})}{2a}$$

and σ_P is the price volatility and X is the strike price. The price of a European put option on the bond is

$$p = XP(t,T)N(-h + \sigma_p) - P(t,t_n)N(-h) \tag{13}$$

The bond price, P, in the above equations, is given by Vasicek's model.

Jamshidian also showed that a coupon-bearing bond can be regarded as a composite of discount bonds, one for each cash flow. An option on a coupon-bearing bond can then be seen as a combination of options on discount bonds, one for each remaining cash flow of the bond after the option expires. If r^* is the particular short rate that causes the coupon-bearing bond price to equal the strike price (which is found by using an iterative procedure, such as the Newton Raphson method), and X_i is the resultant strike price for each individual option, using r^*, and if $P(T,t_i)$ is the price at time T of a zero-coupon bond maturing at time t_i, then the payoff of a call option is given by

$$\sum_{i=1}^{n} c_i \max[0, P(T,t_i) - X_i] \tag{14}$$

which is the sum of n options on the underlying discount bonds.

The Cox, Ingersoll and Ross model (1985) is similar to Vasicek's model. The Cox, Ingersoll and Ross model provides for non-negative interest rates, by adding a $\sqrt{r}$-factor to the second term in equation (8).

5.3 No-arbitrage models

5.3.1 The Ho-Lee model

Ho and Lee (1986) proposed the first no-arbitrage Markov model by extending Vasicek's model. They showed how an interest rate model can be designed so that it is automatically consistent with the initial term structure. The short rate r is described by the stochastic differential equation

$$dr = \theta(t)dt + \sigma dW \tag{15}$$

where σ is the constant instantaneous standard deviation of the short rate, and the drift $\theta(t)$ defines the average direction in which r moves and ensures that the model fits the initial term structure:

$$\theta(t) = F_t(0,t) + \sigma^2 t \tag{16}$$

where $F(0,t)$ is the forward rate at time t and F_t denotes the first derivative. The advantage of the Ho and Lee model is that the model is a Markov analytically tractable model. It does not, however, make provision for the mean reversion of interest rates. This, together with the assumption that interest rates are normally distributed, leads to a relatively high probability that interest rates will become negative.

The Ho-Lee model's analytic expression for the price of a discount bond at time t in terms of the short rate is

$$P(t,t_n) = A(t,t_n)e^{-r(t)(t_n-t)} \tag{17}$$

where

$$\ell n A(t,t_n) = \ell n \frac{P(0,t_n)}{P(0,t)} - (t_n - t)\frac{\partial \ell n P(0,t)}{\partial t} - \frac{1}{2}\sigma^2 r(t_n - t)^2$$

The Ho-Lee analytical value at time zero for a European call option expiring at time T on a discount bond maturing at time t_n, with a face value of 1 unit, is given by

$$c = P(0,t_n)N(h) - XP(0,T)N(h-\sigma_p) \tag{18}$$

where

$$h = \frac{1}{\sigma_p} \ell n \frac{P(0,t_n)}{P(0,T)X} + \frac{\sigma_p}{2}$$

$$\sigma_p = \sigma(t_n - T)\sqrt{T}$$

While the Ho-Lee model is a Markov model, Heath, Jarrow and Morton (1992) developed a model where the short rate, r, is non-Markov. In order to determine the stochastic process for r over a short period of time, dt, one needs to know what the value of r was at the beginning of the period, as well as the path it followed to reach this value, which makes the Heath, Jarrow and Morton model a non-Markov model. The model specifies the volatilities of all instantaneous forward rates at all future times, which is called a volatility structure. This method leads to a non-recombining tree which is computationally extremely time-consuming since there are 2^n nodes after n time steps. The Hull-White model, by contrast, has a recombining tree that speeds up computer time.

5.3.2 *The Hull-White model*

The mean reversion of interest rates is a phenomenon that is not captured by the Ho-Lee model. There are compelling arguments in favour of mean reversion. When interest rates are high, investments decline and the economy slows down. The opposite occurs when interest rates are low. The Ho and Lee model was extended by Hull and White (1990), who added mean-reversion to the short-term interest rate, r, in the stochastic process:

$$dr = (\theta(t) - ar)dt + \sigma dW \tag{19}$$

where a and σ, are constants and $\theta(t)$ is a function of time chosen in such a way that the model is consistent with the initial term structure. The coefficient of dt is approximately equal to the slope of the forward rate curve at time zero. When the short-rate moves away from this curve, it reverts back to the curve at a rate a. The mean reversion component reduces the probability of negative interest rates, compared to the Ho-Lee model.

The Hull-White model is exactly consistent with the latest term structure of interest rates, and is therefore known as a no-arbitrage model. The spot rate in the Hull-White model is a linear function of the underlying process. The value of an interest rate derivative, f (which depends on the process in (19)) is given by the partial differential equation:

$$\frac{\partial f}{\partial t} + \left(\theta(t) - ar\right)\frac{\partial f}{\partial r} + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial r^2} - rf = 0 \tag{20}$$

In order to solve the above partial differential equation, one first has to simplify the stochastic process. If the following transformation is considered in order to obtain the short rate change in a stochastic world for a flat term structure,

$$x = r - \alpha(t) \tag{21}$$

where $\alpha(0)$ is chosen so that $x(0) = 0$ and x follows a process symmetrical around $x = 0$:

$$dx = -axdt + \sigma dW \tag{22}$$

then, from equations (19) and (22), one can say that

$$d\alpha = [\theta(t) - a\alpha(t)]dt$$

If one solves this differential equation with an integration factor, one gets

$$\alpha(t) = e^{-at}\left[r(0) + \int_0^t e^{aq}\,\theta(q)\,dq\right] \tag{23}$$

The price of an interest rate derivative security in terms of the new variable x can be written as $g(t,x)$.

Therefore,

$$f(t,r) \equiv g(t,x) = g(t,r - \alpha(t))$$

Then,

$$f_t = g_t + g_x\left(\frac{dx}{dt}\right)$$

$$= g_t - (-a\alpha(t) + \theta(t))\,g_x$$

where the subscripts denote the relevant derivatives. Similarly one gets

$$f_r = g_x$$

$$f_{rr} = g_{xx}$$

Substituting into equation (20) one gets

$$g_t - axg_x + \frac{1}{2}\sigma^2 g_{xx} - (x + \alpha(t))g = 0 \tag{24}$$

This partial differential corresponds to an economy where, under the equivalent martingale measure Q^*, the spot interest rate is generated by

$$dx = -ax\,dt + \sigma dW \tag{25}$$

and

$$x(t) = r(t) - \alpha(t)$$

The stochastic process for x in equation (25) is therefore independent of the function $\alpha(t)$.

The process for x is assumed to follow an Ornstein-Uhlenbeck process. Therefore, given a value $x(t)$ at any point t, the probability distribution for $x(T)$ for $T > t$, is a normal distribution with mean

$$e^{-a(T-t)}x(t)$$

and variance

$$\frac{\sigma^2}{2a}\left(1 - e^{-2a(T-t)}\right)$$

Using the Feynman-Kač formula (see Section 2.2.3) and the T-forward- risk-adjusted measure Q_T, the solution for equation (24) can be expressed as

$$g(t,x) = P(t,T,x)\, E^{Q_T}(h(T,x(T))\,|\,\mathscr{F}_T) \tag{26}$$

where $h(T,x(T))$ is the boundary condition at time T, and $P(t,T,x)$ is the price of a discount bond with maturity T at time t.

In order to determine the price $P(t,T,x)$ and the distribution of x under Q_T, the Fourier transform $\tilde{g}$ of the fundamental solution g^{δ} is used. Pelsser (1996) has shown that $\tilde{g}$ must take the form

$$\tilde{g}(t,x;T,\psi) = \exp\{A(t;T,\psi) + B(t;T,\psi)x\} \tag{27}$$

where the boundary condition is given by

$$\tilde{g}(T,x;T,\psi) = e^{i\psi x}$$

Then, equation (24) becomes

$$x(B_t - aB - 1) + A_t + \frac{1}{2}\sigma^2 B^2 - \alpha(t) = 0$$

which is solved if A and B satisfy the system

$$B_t - aB - 1 = 0$$

$$A_t + \frac{1}{2}\sigma^2 B^2 - \alpha(t) = 0$$

subject to $A(T;T,\psi) = 0$ and $B(T;T,\psi) = i\psi$.

Using an integration factor, one obtains

$$B(t;T,\psi) = i\psi e^{-a(T-t)} - \frac{1 - e^{-a(T-t)}}{a}$$

and by integration the result is

$$A(t;T,\psi) = \frac{\sigma^2}{2a^3}\left(a(T - t) - 2(1 - e^{-a(T-t)} + \frac{1}{2}(1 \; - e^{-2a(T-t)})\right)$$

$$- i\psi\frac{\sigma^2}{2a^2}(1 - e^{-a(T-t)})^2 - \frac{1}{2}\psi^2\frac{\sigma^2}{2a}(1 - \exp^{-2a(T-t)})$$

$$- \int_t^T \alpha(s)ds$$

Substituting A and B into equation (27) yields

$$\tilde{g}(t,x;T,\psi) = \exp\{A(t,T) - B(t,T)x + i\psi M(t,T,x) - \frac{1}{2}\psi^2\Sigma(t,T)\} \tag{28}$$

where

$$A(t;T) = \frac{\sigma^2}{2a^3}\Big(a(T - t) - 2(1 - e^{-a(T-t)}$$

$$+ \tfrac{1}{2}(1 - e^{-2a(T-t)}\Big) - \int_t^T \alpha(s)ds$$

$$B(t,T) = \frac{1 - e^{-a(T-t)}}{a} \qquad (29)$$

$$M(t,T,x) = xe^{-a(T-t)} - \frac{\sigma^2}{2a^2}(1 - e^{-a(T-t)})^2$$

$$\Sigma(t,T) = \frac{\sigma^2}{2a}(1 - e^{-2a(T-t)})$$

Equation (28) can also be written as the product of the discount bond price and the characteristic function of the probability density function under the T-forward-risk-adjusted measure:

$$\tilde{g}(t,x;T,\psi) = P(t,T,x)\,\{\exp(i\psi M(t,T,x) - \tfrac{1}{2}\psi^2\Sigma(t,T))\} \qquad (30)$$

The probability density function has a mean $M(t,T,x)$ and a variance $\Sigma(t,T)$.

Using the above results, one can determine the value of a European call option on a discount bond. If $c(t,T,s,X,x)$ is the value of a call option at time t, that gives the owner the right to buy a discount bond with maturity s at time T, $t<T<s$, for a price X, then the payoff, h, of the option is given by

$$h(T,x(T)) = \max\{P(T,s,x(T)) - X,0\}$$

The expected payoff of the option can be expressed under the T-forward-risk-adjusted measure Q_T as follows:

$$s = E^{Q_T}(\max\{P(T,s,x(T)) - X,0\}|\mathcal{F}_t)$$

If the value of $x(T)$ equals z, then the expectation is given by

$$\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi\Sigma(t,T)}}\max\{e^{A(T,s)-B(T,s)z}-X,0\}\exp\left\{-\frac{1}{2}\frac{(z-M(t,T,x))^2}{\Sigma(t,T)}\right\}dz \tag{31}$$

The payoff is non-zero if

$$z<\frac{A(T,s)-\ell n X}{B(T,s)}$$

Therefore,

$$\begin{aligned} s &= \int_{-\infty}^{\frac{A-\ell nX}{B}}\frac{e^{A(T,s)-B(T,s)z}-X}{\sqrt{2\pi\Sigma(t,T)}}\exp\left\{-\frac{1}{2}\frac{(z-M(t,T,x))^2}{\Sigma(t,T)}\right\}dz \\ &= \int_{-\infty}^{\frac{A-\ell nX}{B}}\frac{1}{\sqrt{2\pi\Sigma(t,T)}}\exp\left\{-\frac{1}{2}\frac{(z-M(t,T,x))^2}{\Sigma(t,T)}+A(T,s)-B(T,s)z\right\}dz \\ &\quad -\int_{-\infty}^{\frac{A-\ell nX}{B}}\frac{X}{\sqrt{2\pi\Sigma(t,T)}}\cdot\exp\left\{-\frac{1}{2}\frac{(z-M(t,T,x))^2}{\Sigma(t,T)}\right\}dz \end{aligned} \tag{32}$$

If the first integral is called I_1 and the second integral is called I_2, then by completing the square, it follows that

$$I_2 = X\int_{-\infty}^{\frac{A(t,s)-\ell nX}{B(T,s)}}\exp\left[-\tfrac{1}{2}\{(z-M)/\sqrt{\Sigma}\}^2\right]dz$$

It can easily be shown that

$$A(T,s) - B(T,s)M = [A(t,s) - A(t,T)] - [B(t,s) - B(t,T)]y - \tfrac{1}{2}B(T,s)^2\Sigma$$

Therefore, substituting and expressing the equation in terms of cumulative normal distribution functions one gets

$$\begin{aligned} I_2 &= X \int_{-\infty}^{h_2} e^{\frac{1}{2}\rho^2}\, d\rho \\ &= XN(h_2) \end{aligned}$$

where

$$h_2 = \frac{\ell n(P(t,s,x)/P(t,T,x)X) - B(T,s)^2\Sigma(t,T)}{B(T,s)\sqrt{\Sigma(t,T)}}$$

Similarly, for I_1 the result is

$$I_1 = \int_{-\infty}^{\frac{A-\ell nX}{B}} \frac{1}{\sqrt{2\pi\Sigma(t,T)}} \exp\left\{-\frac{1}{2}\left\{\frac{z - (M - B(T,s)\Sigma)}{\sqrt{\Sigma(t,T)}}\right\}^2 + A(T,s) - B(T,s)M + \frac{1}{2}B(T,s)^2\Sigma\right\} dz$$

$$= \int_{-\infty}^{\frac{A-\ell nX}{B}} \frac{1}{\sqrt{2\pi\Sigma(t,T)}} \exp\left\{-\frac{1}{2}\left\{\frac{z - (M - B(T,s)\Sigma)}{\sqrt{\Sigma(t,T)}}\right\}^2\right\} \exp\{A(t,s) - A(t,T) - [B(t,s) - B(t,T)]x\}\, dz$$

$$= \frac{\exp[A(t,s) - B(t,s)x]}{\exp[A(t,T) - B(t,T)x]} \int_{-\infty}^{\frac{A-\ell nX}{B}} \frac{1}{\sqrt{2\pi\Sigma(t,T)}} \exp\left\{-\frac{1}{2}\left\{\frac{z - (M - B(T,s)\Sigma)}{\sqrt{\Sigma(t,T)}}\right\}^2\right\} dz$$

which gives

$$I_1 = \frac{P(t,s,x)}{P(t,T,x)} \int_{-\infty}^{h_1} e^{\frac{1}{2}\omega^2} d\omega$$
$$= \frac{P(t,s,x)}{P(t,T,x)} N(h_1)$$

where

$$h_1 = \frac{\ell n(P(t,s,x)/P(t,T,x)X) + B(T,s)^2\Sigma(t,T)}{B(T,s)\sqrt{\Sigma(t,T)}}$$

Therefore,

$$s = \frac{P(t,s,x)}{P(t,T,x)} N(h_1) - X N(h_2) \tag{33}$$

Discounting the above expected value to the current date, t, gives the price of a call option:

$$c = P(t,T,x)\left(\frac{P(t,s,x)}{P(t,T,x)} N(h_1) - X N(h_2)\right)$$
$$= P(t,s,x)\, N(h_1) - P(t,T,x) X N(h_2) \tag{34}$$

where

$$h_1 = \frac{\ell n(P(t,s,x)/P(t,T,x)X)}{\sigma^*} + \frac{\sigma^*}{2}$$

$$h_2 = h_1 - \sigma^* \tag{35}$$

$$\sigma^* = \frac{\sigma_r}{a}\left(1 - e^{-a(s-T)}\right)\sqrt{\frac{1-e^{-2a(T-t)}}{2a}}$$

The price of a put option is given by

$$p = P(t,T,x)XN(-h_2) - P(t,s,x)N(-h_1) \tag{36}$$

The equation for σ^* takes into account the pull-to-par effect of a bond. The analytical solution to the Hull-White model overcomes the volatility problem of the Black model, discussed in Section 3.3, since it takes the pull-to-par effect into account. However, the solution does not hold for American options. To address this problem, a numerical solution is necessary, as discussed in the next chapter.

5.3.3 Other Markov models

The tree procedure used by Hull and White can also be used to construct other one-factor Markov models. For example, a tree can be constructed in $\ell n\, r$ rather than r, as described by Black, Derman and Toy (1990) or Black and Karasinski (1991) where

$$d\,\ell n r = [\theta(t) - a\,\ell n r]dt + \sigma dz$$

The procedure suggested by Black, Derman and Toy (1990) matches the volatilities of all rates at time zero. The trinomial tree procedure is explained by Hull and White (1993). Black and Karasinski (1991) suggested a binomial tree procedure involving time steps of varying lengths.

The next chapter discusses the numerical solution of the Hull-White model applied to South African bond options.

CHAPTER 6

THE HULL-WHITE MODEL APPLIED TO SOUTH AFRICAN OTC BOND OPTIONS

South Africa has an actively traded bond market where options are traded on the most liquid government bonds. Bond turnover in 1998 was $1.7 trillion equivalent in nominal terms, according to the Bond Exchange of South Africa. A significant over-the-counter (OTC) bond options market has established itself, although option liquidity is concentrated in current government funding bonds. Both options and bonds are traded on yield-to-maturity (referred to as yield). Almost all bond options that are traded are American options.

The effects of the 1998 emerging market crisis, which generated significant losses for banks and hedge funds, again raised several questions concerning the accurate valuation of derivative instruments. The crisis emphasized the imperfections of the Black-Scholes model previously identified by Black (1988).

The nature of American options necessitates the use of numerical models for valuation purposes. The numerical solution of the Hull-White model (1990a) addresses most of the disadvantages of other bond option pricing models and can be successfully applied to South African bond options. Instead of using the bond price as the stochastic variable, the model assumes that the short rate, $r(t)$, follows a mean reverting stochastic process and has a lognormal distribution:

$$dr = (\theta(t) - ar)dt + \sigma_r dW \tag{1}$$

It is further assumed that the term structure implies a certain expectation of future short rates, and that the expected short rate *process* can therefore be used to determine any bond price. Hull and White (1993) use a trinomial tree to model the short rate process and ensure that the initial term structure is matched before the bond option is valued.

The key characteristics of the numerical solution of the Hull-White model are the following:

- It incorporates mean reversion of interest rates.
- The pull-to-par effect is determined analytically for both the exact solution and the numerical solution.
- The model is consistent with the initial term structure of interest rates.
- It incorporates the early-exercise value of American options.

6.1 Characteristics of South African bond options

The major difference between South African OTC bond options and options in other countries is the fact that South African bonds are traded on the yield-to-maturity and settled on the price[1] while most other bonds are traded on the price. The strike of a South African bond option is, therefore, also given as a yield-to-maturity. Options on many different maturity bonds are traded. Of these, the R153 government bond is the most liquid.

Since the Black model is generally used to price South African bond options, the implied forward *price* is used at the current forward yield. In practice, however, the delta hedge is done in the spot market. The forward price is calculated using the spot yield and the equivalent risk-free rate. The strike price used in the Black pricing formula is the price on the expiry date at the strike rate.

When an option is early-exercised, the actual strike price can be different from the price when the option is exercised at a later stage or on the expiry date of the option. The actual maturity date of the bond exercised stays the same, irrespective of the date when it is exercised. This practice differs from that in countries where the bonds are traded on the price and options are traded on a specific strike price. In such markets, the holder of an American option has the right to buy a fixed term bond, say a 10 year bond, before or on expiry of the option at a certain strike price. When a German bond option, for instance, is early-exercised, the strike price of the original contract stays the same, for example at DM 90.00, while the maturity date of the bond is adjusted in order for the term-to-maturity to stay the same as on the original negotiated contract.

---

[1]This is also the case in some European countries, for instance, Finland.

In this study, the pricing of bond options using the Hull and White (1990) analytical solution discussed in the previous chapter is evaluated, as well as the numerical solution using a trinomial tree. The analytical results are then compared to the numerical solution of the Hull and White trinomial tree (1993) for European and American options. The difference between pricing the option with a strike price versus pricing it with a strike yield is evaluated. Finally, the influence of the term structure on the pricing of the options is shown.

6.2 The Hull-White trinomial tree

The analytical solution of the Hull-White model, as discussed in Chapter 5, gives suitable results for European style options. For American style options, a trinomial tree, as constructed by Hull and White, gives more accurate results than the analytical solution, due to the fact that the numerical solution does provide for the early-exercise of options.

The trinomial tree uses discrete time steps to construct a tree of possible values for the short rate, *r*, in the future. (See also the Rendleman-Bartter model discussed in Chapter 5). At time *t*, the price of a bond maturing at time *s* can be determined in terms of the short rate *r* by using the Hull and White analytical formula explained in Chapter 5:

$$P(t,s,r) = A(t,s)\, e^{-B(t,s)r} \tag{2}$$

where

$$B(t,s) = \frac{1 - e^{-a(s-t)}}{a}$$

and

$$\ell n\, A(t,s) = \ell n \frac{P(0,s)}{P(0,t)} - B(t,s)\frac{\partial \ell n\, P(0,t)}{\partial t} - \frac{\sigma^2}{4a^3}(e^{-as} - e^{-at})^2(e^{2at} - 1)$$

Using the short rate given at each node in the tree, the possibility of early exercising can be evaluated for American options.

6.2.1 *The procedure*

The interest rate tree is constructed in two stages. First, one supposes that there is a variable x which is initially zero and follows the process described in Chapter 5, Section 5.3.2:

$$dx = -ax\,dt + \sigma dW \qquad (3)$$

which forms a tree which is symmetrical around the $x = 0$ line. The variable

$$x(t+\Delta t) - x(t)$$

is normally distributed and, if terms of a higher order than Δt are ignored, the expected value is

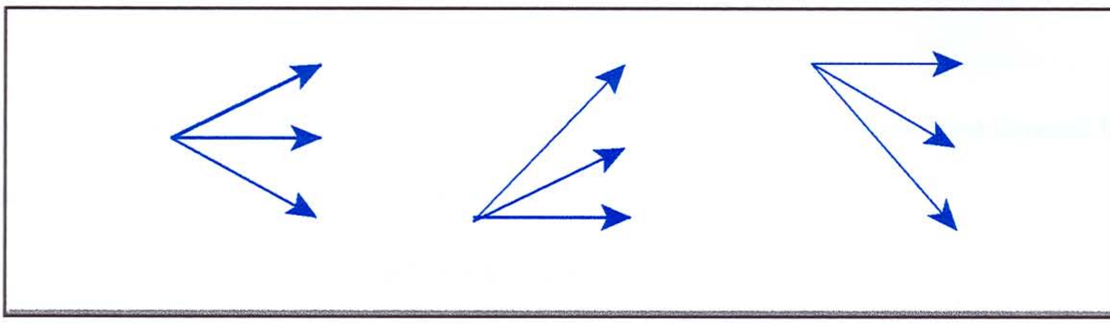
$$-a\,x(t)$$

and the variance is

$$\sigma^2 \Delta t$$

If the length of each time step, Δt, is known, the change of the variable x is normally set equal to

$$\Delta x = \sigma\sqrt{3\Delta t}$$



(*a*) (*b*) (*c*)

Figure 6.1: Possible movements for the short-term rate

Node (i,j) on a trinomial tree, is where $t = i\Delta t$ and $x = j\ \Delta x$. For $j_{min} < j < j_{maz}$, the standard branching process (*a*) in Figure 6.1 is chosen. For a sufficiently large positive j, the branching process in (*c*) is chosen, while (*b*) is chosen when j is sufficiently negative. Hull and White show that $j_{\max}$ should be set equal to the smallest integer greater than

$$0.184 / (a\Delta t)$$

and then

$$j_{min} = -\,j_{\max}$$

If p_u, p_m and p_d are defined as the probabilities of following the highest, middle and lowest branches at each node, the probabilities should match the expected change and variance in x over the next interval Δt. The sum of the probabilities must also equal one. For node (i,j) the three necessary equations for p_u, p_m and p_d are

$$p_d + p_m + p_u = 1$$

$$p_u(k+1)\Delta x + p_m k\Delta x + p_d(k-1)\Delta x = E(\Delta x) = -aj\Delta x\Delta t$$

$$p_u(k+1)^2\Delta x^2 + p_m k^2\Delta x^2 + p_d(k-1)^2\Delta x^2 = E(\Delta x^2) = \sigma^2\Delta t + a^2 j^2\Delta x^2\Delta t^2$$

These equations can be solved in order to obtain p_u, p_d, and p_m for each branching process.

The second stage involves converting the x-tree into a tree for r. It is important to recall the following transformation in Chapter 5:

$$\alpha(t) = r(t) - x(t), \qquad x(0) = 0$$

and

$$d\alpha = [\theta(t) - a\alpha(t)]dt$$

Using an integration factor and $\alpha(0) = r(0)$, it follows that

$$\alpha(t) = e^{-at}\left[r(0) + \int_0^t e^{aq}\,\theta(q)\,dq\right] \tag{4}$$

The $\theta(q)$ function in the above equation can be calculated from the initial term structure:

$$\theta(t) = F_t(0,t) + aF(0,t) + \frac{\sigma^2}{2a}(1 - e^{-2at}) \tag{5}$$

Integrating and substituting this equation into equation (4), this yields the following[2]

$$\alpha(t) = F(0,t) + \frac{\sigma^2}{2a^2}(1 - e^{-at})^2 \tag{6}$$

---

[2]Note that $r_0 = F(0,0)$

If one defines α_i as the value of r at time $i\Delta t$ (on the r-tree) minus the corresponding value of x at time $i\Delta t$ on the x-tree, and one defines $Q_{i,j}$ as the present value of a security that has a payoff of 1 unit if node (i,j) is reached, and zero otherwise, then the value of $Q_{i,j}$ would be given as the discounted value of the expectation of reaching node (i,j).

The next step is to use a forward induction procedure which ensures that the initial term structure is matched exactly. The α_i- and $Q_{i,j}$ - values should be calculated in such a way that the initial term structure is matched exactly. This can be done by forward induction:

$$\alpha_m = \frac{\ln \sum_{j=-n_m}^{n_m} Q_{m,j} e^{-j\Delta r \Delta t} - \ln P_{m+1}}{\Delta t}$$

where P_{m+1} is the price of a discount bond at time $(m + 1)\Delta t$ and n_m is the number of nodes below and above the centre. The value for $Q_{i,j}$, with $i = m + 1$ can then be calculated:

$$Q_{m+1,j} = \sum_k Q_{m,k} q(k,j) \exp[-(\alpha_m + k\Delta r)\Delta t]$$

where $q(k,j)$ is the probability of moving from node (m,k) to node $(m+1,j)$ and the summation is taken over all values of k for which this probability is non-zero.

The above Hull-White numerical procedure was programmed in a Fortran computer programme and verified using the data given in Hull (1997) and also the data in Pelsser (1996). Since South African bond options are options to buy or sell bonds at a certain strike *rate* and not price, allowance was made for this trading convention by adjusting the model. Therefore, the study evaluated bond options for both a price strike and a yield strike. The results are discussed in the sections below for both discount bonds and coupon bonds.

6.3 Influence of the strike convention

There are two ways of expressing the strike of a bond option – either as a bond price, X, or as a bond yield, $x\%$. The particular convention used does *not* affect the price of *European* options, but it does affect the price of American options. American options on zero-coupon bonds and coupon-bearing bonds are compared for both conventions.

6.3.1 Options on zero-coupon bonds

6.3.1.1 Price strike

One can consider an option on a zero-coupon bond maturing at time s. If one assumes that the option expires at time T and that the strike, X, is given in terms of the bond price, for a put option, one gets the following payoff at expiry time T:

$$\max[X - P(T,s,r), 0] \tag{7}$$

A T-term *American* put option on early exercise at time t (where $t < T$) gives the holder the right to sell a (s-T)-year zero-coupon bond (maturing at time $t + s - T$), for a price X, resulting a profit of

$$X - P(t, t+s-T, r)$$

If f_{ij} denotes the value of the option at time $t_i < T$ at the j-th vertical point on the spot interest rate tree, when the interest rate is r_{ij}, then

$$f_{ij} = \max\left[X - P(t_i, t_i+s-T, r_{ij}),\ e^{-r_{ij}(t_{i+1}-t_i)}\left(\sum_{q=-1}^{1} p_{i,j,q+\delta} f_{i+1,j+q+\delta}\right)\right] \tag{8}$$

where p denotes the appropriate probability and δ = -1, 0 or +1, according to the branching process in the trinomial tree (see Figure 6.1). It is clear that the option should only be early-exercised when the profit is greater than the intrinsic value of the option. The term-to-maturity of the underlying bond being exercised is always $(s-T)$, irrespective of the time t it is exercised. This implies that the early-exercise value of an option with a price-based strike depends only on the short-term interest rate r, and is not directly dependent on the particular time t. Figure 6.2 illustrates the payoff as a function of the short-term rate, which holds for any $t \leq T$.

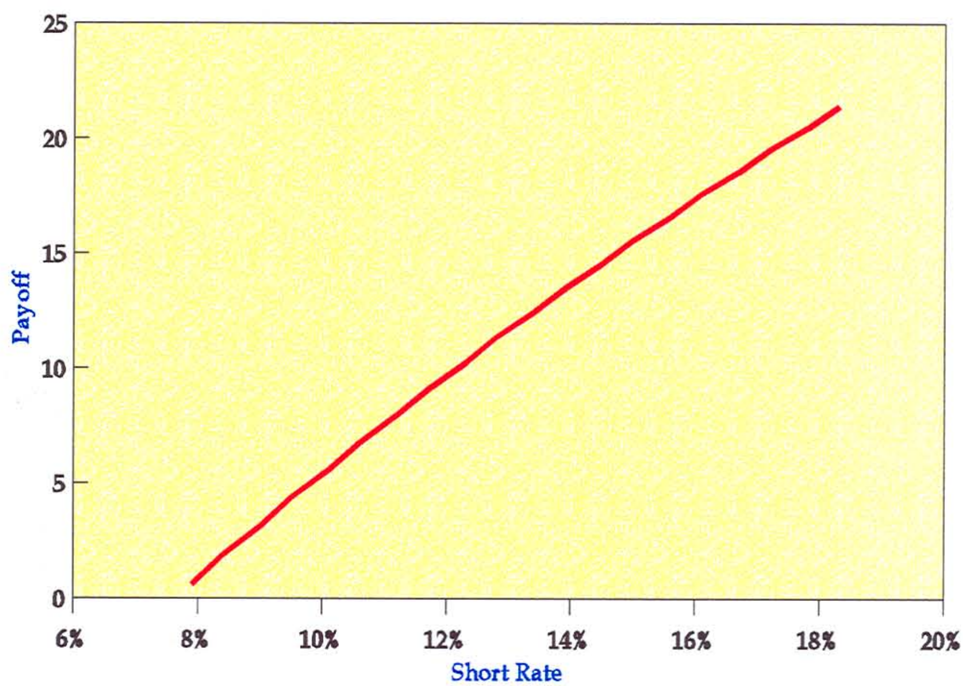


Figure 6.2: **Payoff as a function of the short rate for a 5 year put option on a 9-year bond with a strike price of 0.72 for any t≤T**

In order to compare the prices of European and American bond options, the same example is used as that given by Pelsser (1996:70), but shorter dated options have been added. From Table

6.1 it is evident that the prices of *short dated* European and American put options corresponds to two decimal places. The error becomes significant only for longer-term options.

Table 6.1: Prices for put option on a 9-year zero-coupon bond (price strike) (using 50 time steps, $a = 0.10$, $\sigma = 0.01$ and zero-curve given by $z(t) = 0.08 - 0.05e^{-0.18\,t}$)

Option term	Strike price	European option		American option	Eur vs Amer difference(%)
		Analytical	Numerical		
0.25	0.54	81.01	80.74	80.74	0.00
0.50	0.54	77.77	77.74	77.74	0.00
0.75	0.55	116.56	116.93	116.93	0.00
1.00	0.55	97.91	98.20	98.20	0.00
2.00	0.58	126.51	126.45	126.49	-0.03
3.00	0.63	192.97	192.99	194.00	-0.52
5.00	0.72	135.84	137.36	145.02	-5.58
7.00	0.85	97.34	97.89	114.11	-16.57

6.3.1.2 Yield strike

Next, one can consider a put option where the strike is given in terms of the yield-to-maturity, for example $x\%$. The option gives the owner of a European-style option the right to sell a discount bond (maturing at time s) at expiry of the option at a rate of $x\%$. When American style options are early- exercised at time t, the owner also has the right to sell a discount bond with maturity s priced at a yield-to-maturity of $x\%$. On the expiry date of a European or American option, $t=T$, the payoff is exactly the same as for the equivalent price-based option given in equation (7), where x was chosen so that

$$X = e^{-x(s-T)}$$

A European option price is therefore not affected by the yield-strike convention. On the other hand, when early-exercising an American option, there is a difference in the payoff since the yield-strike convention gives an additional advantage by adding a time-dimension to the early-exercise value. The payoff at time t, $t<T$, is given by:

$$e^{-x(s-t)} - P(t,s,r)$$

The payoff is therefore not only affected by the short rate, but it is also affected by the time t when the option is early-exercised, as is illustrated in Figure 6.3.

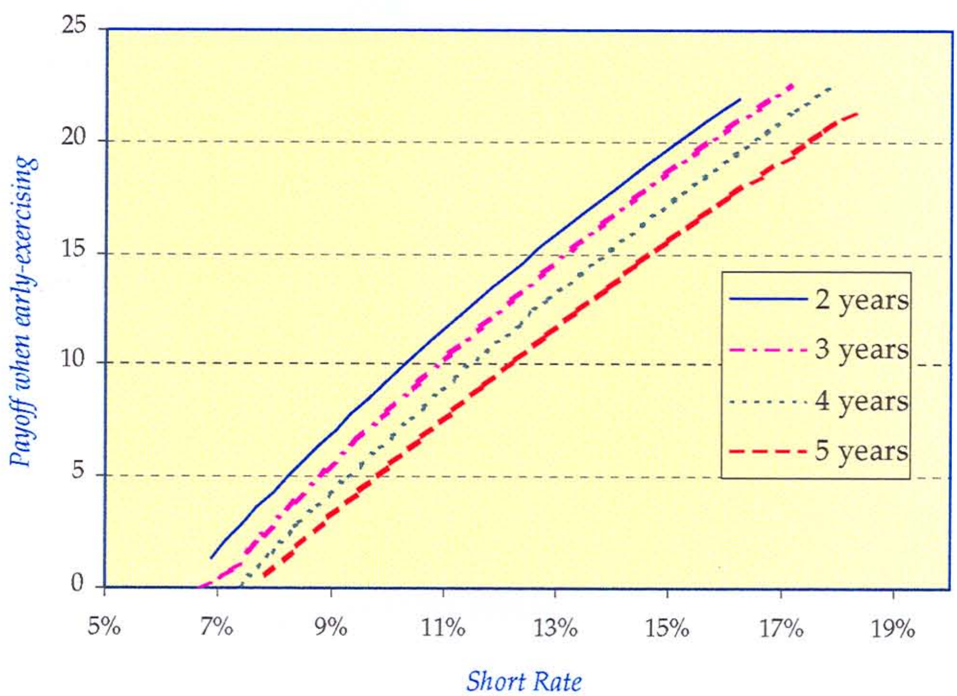


Figure 6.3: **Payoff as a function of the short rate for a 5-year put option on a 9-year bond with a strike yield of 8.21% for various $t \leq T$**

For the same short rate, the payoff for early-exercising the option is greater for smaller t. This should influence the pricing of an American option significantly, as is shown in Table 6.2. The yield-strike was chosen to give, at the expiry of the option, approximately the same price strike as in Table 6.1.

Table 6.2: Prices for put option on a 9-year zero-coupon bond (yield strike) (using 50 time steps, a = 0.10, σ = 0.01 and zero-curve given by z(t) = 0.08 - 0.05e$^{-0.18t}$)

Option term	Strike yield (%)	European option		American option	Eur vs Amer difference(%)
		Analytical	Numerical		
0.25	7.042	81.05	80.78	80.78	0.00
0.50	7.249	77.82	77.79	77.79	0.00
0.75	7.247	116.44	116.81	116.81	0.00
1.00	7.473	97.90	98.20	98.20	0.00
2.00	7.782	126.47	126.42	126.89	-0.37
3.00	7.701	192.89	192.91	199.48	-3.41
5.00	8.213	135.80	137.32	169.80	-23.66
7.00	8.126	97.33	97.88	204.73	-109.16

6.3.2 *Options on coupon-bearing bonds*

6.3.2.1 Price strike

When pricing an option on a coupon-bearing bond, a slightly different approach than given in Section 6.3.1 is followed. One can consider a T-term option on an s_n-term coupon-bearing bond, based on a strike *price*. A coupon bond can be seen as a portfolio of discount bonds, one discount bond for every coupon maturing at time s_i, plus one discount bond for the principal maturing at time s_n. An European option on a coupon bond can therefore be treated as a portfolio of options on the individual discount bonds in the portfolio as described by Jamshidian (1989). Only cashflows that are due after the expiry of the option are considered in the pricing of the option. In the example, a total of n cashflows is assumed.

Since the strike price is also equivalent to the sum of discounted cashflows, the original strike price is used to solve for the 'strike' short-term interest rate, r^*, where the Hull-White analytical formulas are used to price the individual discount bonds. One then uses r^* to obtain the individual strike prices, X_i for the underlying discount bonds maturing at time s_i with a 1 unit nominal. The individual options on the different cashflows are therefore separated in order to price them independently according to the method used for discount (zero-coupon) bonds. The individual option prices are then added to obtain a single price for the option on the coupon-bearing bond. It follows that the payoff for a put-option at the expiry of the option when the spot short-term interest rate equals r is

$$\sum_{i=1}^{n} c_i \max[X_i - P(T,s_i,r),\ 0] \tag{9}$$

where the i-th cash flow is given by c_i. Therefore, the price of an option on a coupon-bearing bond equals the sum of n options on the underlying discount bonds. The price of a put option is then given by

$$p = \sum_{i=1}^{n} p_i \tag{10}$$

where p_i denotes the price of the individual options. When American options are priced, the individual options on discount bonds are priced using the trinomial tree approach and evaluating for the desirability of early exercising, using equation (8). Equation (10) therefore holds for European and American options, where the strike is given in terms of the price.

The results in Table 6.3, for example, give the prices of a put option on an 8% coupon bond, where coupons are paid semi-annually and the strike is given in terms of a bond price. When one compares the difference between American and European option prices for a coupon bond in Table 6.3 to the results for a zero-coupon bond in Table 6.1, one sees that there is an almost insignificant change in the differences.

Table 6.3: Prices for put option on a 9-year, 8% coupon bond (price strike) (using 50 time steps, $a = 0.10$, $\sigma = 0.01$ and zero-curve given by $z(t) = 0.08 - 0.05e^{-0.18\,t}$)

Option term	Strike price	European option		American option	Eur vs Amer difference (%)
		Analytical	Numerical		
0.25	1.084	81.52	81.78	81.78	0.00
0.50	1.045	78.05	78.38	78.38	0.00
0.75	1.060	116.34	116.84	116.84	0.00
1.00	1.023	97.33	97.93	97.93	0.00
2.00	0.997	125.16	124.76	124.79	-0.02
3.00	0.997	190.50	191.27	192.16	-0.47
5.00	0.980	134.50	134.93	142.59	-5.68
7.00	0.992	97.58	98.83	115.36	-16.73

6.3.2.2 Yield strike

The strike convention has no influence on European options and equation (10) still holds for coupon-bearing bonds using the yield strike. However, the pricing of an American option should be treated in a different way. Since early-exercising an option on the yield of a bond involves not only those coupons that are due after the expiry date of the option, but also those due between the early-exercise date and the expiry date, it follows that equation (10) does not hold for the price of an American option. The value of the option at time $t_i<T$ in the trinomial tree is therefore

$$f_{ij} = \max\left[\sum_{k=1}^{m} c_k\left(e^{-x(t_i-s_k)} - P(t_i,s_k,r_{ij})\right),\ e^{-r_{ij}(t_{i+1}-t_i)}\left(\sum_{q=-1}^{1} p_{i,j,q+\delta} f_{i+1,j+q+\delta}\right)\right] \tag{11}$$

where m is the number of cashflows from time t_i to the maturity of the bond. The option price is therefore determined by a different process as followed by Jamshidian (1989), since the number of coupons at different time-steps may differ. This adds to the bigger difference already obtained for American yield-strike options on zero-coupon bonds shown in Table 6.2.

When one compares the results in Table 6.3 with similar options, but uses the yield-strike convention, one obtains the results shown in Table 6.4 (overleaf).

Table 6.4: Prices for put option on a 9-year, 8% coupon bond (yield strike) (using 50 time steps, a = 0.10, σ = 0.01 and zero-curve given by $z(t) = 0.08 - 0.05e^{-0.18\,t}$)

Option term	Strike yield (%)	European option		American option	Eur vs Amer difference (%)
		Analytical	Numerical		
0.25	6.89	80.46	80.74	80.74	0.00
0.50	7.15	78.36	78.36	78.68	0.00
0.75	7.20	116.48	116.98	116.98	0.00
1.00	7.47	97.25	97.85	97.86	-0.01
2.00	7.89	126.89	126.61	127.27	-0.52
3.00	7.90	191.72	192.58	200.85	-4.29
5.00	8.42	134.71	135.11	174.26	-28.98
7.00	8.27	97.57	98.82	226.86	-129.57

6.3.2.3 Conclusion

The above tables indicate that there is an insignificant difference between short-dated (less than 1 year) European and American put options. The results show clearly that *short-dated* European and American put options in this example can be priced accurately using the *analytical* Hull and White model. The *Black model* can therefore also be used accurately, by adjusting the volatilities for different option terms. The early-exercise value of American options becomes significant only for longer-dated options (more than 1 year). The pricing difference (or early-exercise value) becomes even more significant for options based on the yield-strike convention, but is still fairly priced for short-dated options. The shape of the term structure, could, however, influence these results.

6.4 Influence of the shape of the term structure

The term structure in the previous examples shows a sharply increasing shape. These results can be compared to an example where a sharply *decreasing* term structure was used. Table 6.5 shows a significant increase in the early-exercise value of the put option (compared to Table 6.4). An obvious reason for this sudden increase in the price of an American option lies in the shape of the term structure.

Table 6.5: Prices for put option on a 9-year, 8% coupon bond (yield strike) (using 50 time steps, $a = 0.1$, $\sigma = 0.01$ and zero-curve given by $z(t) = 0.03+0.05e^{-0.18t}$)

Option Term	Strike Yield (%)	European option		American option	Eur vs Amer difference (%)
		Analytical	Numerical		
0.25	4.0	171.93	172.21	221.15	-28
0.50	4.0	152.82	152.32	227.71	-50
0.75	4.0	138.57	138.88	232.83	-68
1.00	4.0	126.50	127.01	236.34	-86
2.00	3.8	130.89	132.33	386.12	-192
3.00	3.5	149.99	150.15	640.20	-326
5.00	3.2	123.12	124.37	894.88	-620
7.00	2.5	107.64	108.98	1540.02	-1313

For an increasing term structure (where interest rates are expected to rise) early-exercising a put option is not optimal, since the profit is greatest when the interest rate is even higher. The relatively inexpensive rate at which bonds can be carried (financed) makes it profitable to carry the bonds until the expiry date. A short-dated in-the-money put option will therefore not be early-exercised.

A market involving a decreasing term structure (where interest rates are expected to decline) gives the opposite effect. The high cost involved in carrying bonds causes lower future rates. A put option with a certain strike would be less expensive in this market than in a market with an increasing term structure. However, an unexpected increase in interest rates would make the option more likely to be early-exercised than previously. The reason for this is that the holder of a covered (hedged) put option has a long position in bonds and higher rates would therefore increase the cost-of-carry. The expectation of decreasing rates still holds, which makes the profitability of early-exercising an in-the-money put option greater. It is thus evident that, although the European option is cheaper in this market than in the increasing market, the added value for an American option is greater.

It is obvious that the opposite holds for a call option, giving more early-exercise value for a yield curve with a positive slope than with a negatively sloping yield curve. Using the same example as that used in Table 6.4, but for a call option, one obtains the results set out in Table 6.6.

6.4.1 Conclusion

The results show that there is a small pricing difference between short-dated American and European *put* options in a market with an increasing term structure (and *call* options in a market with a decreasing term structure). However, for the opposite scenario, the pricing difference becomes significant, as was shown above.

Table 6.6 **Prices for call option on a 9-year, 8% coupon bond (yield strike) (using 50 time steps, a = 0.10, σ = 0.01 and zero-curve given by z(t) = 0.08 - 0.05e$^{-0.18\,t}$)**

Option term	Strike yield (%)	European option		American option	Eur vs Amer difference (%)
		Analytical	Numerical		
0.25	6.89	100.54	100.61	146.09	-45
0.50	7.15	191.09	191.42	305.92	-60
0.75	7.20	183.18	183.78	338.41	-84
1.00	7.47	253.07	253.72	520.93	-105
2.00	7.89	273.26	273.09	790.81	-190
3.00	7.90	192.66	193.69	787.72	-307
5.00	8.42	167.22	167.81	1099.08	-555
7.00	8.27	69.68	71.01	985.32	-1288

6.5 Calibration of the volatility parameters

The main problem in using more sophisticated models such as the Hull-White model is to estimate the volatility parameters, σ_r and a. In order to derive the full benefit of the more accurate Hull-White model, one must calibrate the Hull-White model to liquid options traded in the market.

Since the market uses the Black model, the Hull-White model should therefore be calibrated to the Black model using short-dated options. In the case of options where the pricing difference between European and American options is small (either put or call options, depending on the market), the prices and implied volatilities of at-the-money options given by the Black model can be used to solve the volatility-parameters (σ_r and a) for the analytical Hull-White model – which is similar to the Black model. These parameters can then be used to price longer-dated options by using the Hull and White trinomial tree approach, thereby reducing the pricing error.

6.5.1 *Estimation of parameters for zero-coupon bonds*

The results in the previous sections show that the pricing error is small when short-dated American *put* options are priced using a European model in a market with an *increasing* (positive sloping) yield curve (and *call* options in a market with a *decreasing* or negative sloping yield curve). Therefore, since short-dated American bond options are priced in the market using the Black model, these prices and implied volatilities can be used to imply the volatility-parameters, σ_r and a in the analytical Hull-White model. Once the parameters have been obtained, they can be used in the numerical solution to give the prices for American options.

If one compares equations (4) and (34) in Chapter 5[3] it indicates that, when valuing the same option with these two models, the option prices can only be equal if

$$\sigma^* = \sigma_F \sqrt{T}$$

Therefore,

$$\sigma_F(s) = \frac{\sigma_r}{a}\left[1 - e^{-a(s-T)}\right]\sqrt{\frac{1 - e^{-2aT}}{2aT}}$$

where $\sigma_F(s)$ is derived from the market price of an option on a discount bond maturing at time

---

[3]The Black model and Hull-White model respectively.

s. One can solve the parameters σ_r and a by obtaining a best fit for the function $\sigma_F(s)$. This gives a volatility curve that serves as an input for the Hull-White model.

Once the calibrated Hull-White volatility parameters are known, they can be used to obtain a more accurate estimate of the price of American call options in a market with a positive-sloping yield curve, or American put options in a market with a negative-sloping yield curve, by using the Hull-White numerical method. Longer-dated options can also be priced this way, although an estimate of future volatility is required to do so.

6.5.2 *Estimation of parameters for coupon-bearing bonds*

The volatility curve obtained in Section 6.5.1 indicates the volatility against the maturity date of a *zero-coupon* bond. Since a coupon bond has several cashflows, it cannot give the same result. If one has only option market data for coupon bonds, and no zero-coupon bond data, one must approximate the volatility curve for zero-coupon bonds.

An option price is influenced by the market consensus of the bond price volatility for the particular option term. If one assumes that a bond's price volatility (as used in the Black model) is a combination of the implied volatilities of the individual cashflows, one can express the volatility as a function of the average time of cashflows, or the *duration* of the bond. Using this approximation, the volatilities as a function of duration give an implied volatility curve for a certain option term T. This then implies that the volatility parameters, σ_r and a, result in a volatility curve against *duration*. Figure 6.4 shows an example of a volatility curve for an option term of 6 months using data for November 1999. Fitting the curve through the data points produces the following values: $a = 0.055$ and $\sigma_r = 0.0295$. The graph also shows the implied curve for a 1-year option.

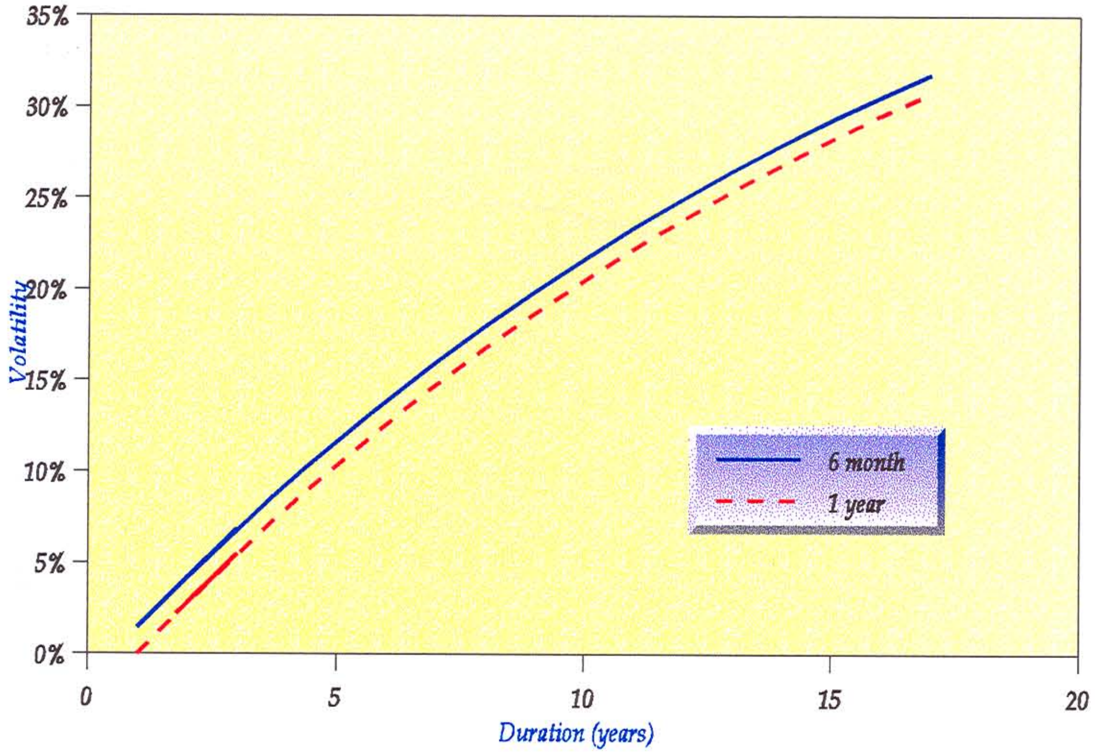


Figure 6.4: **A fitted volatility curve for an option term of 0.5 years, and an implied curve for 1 year, using the same parameters**

The above method gives a reasonable approximation of the volatility curve, but is not perfect, since the volatility of one bond is influenced by the volatility of another bond used in the construction of the curve. If there are any abnormalities in the implied volatilities of different bonds, the fitted curve smoothes out these discrepancies, giving an approximated value.

6.6 Empirical study for South African OTC bond options

An empirical study was done for South African OTC bond options. It was assumed that the most liquid at-the-money government R150 (12% coupon, maturing 28 February 2005) and R153 (13% coupon, maturing 31 August 2010) options traded in the South African market are the benchmark options. The Hull-White model was then calibrated to these options. The yield curve in November 1999 was used, as is shown in Figure 6.5.

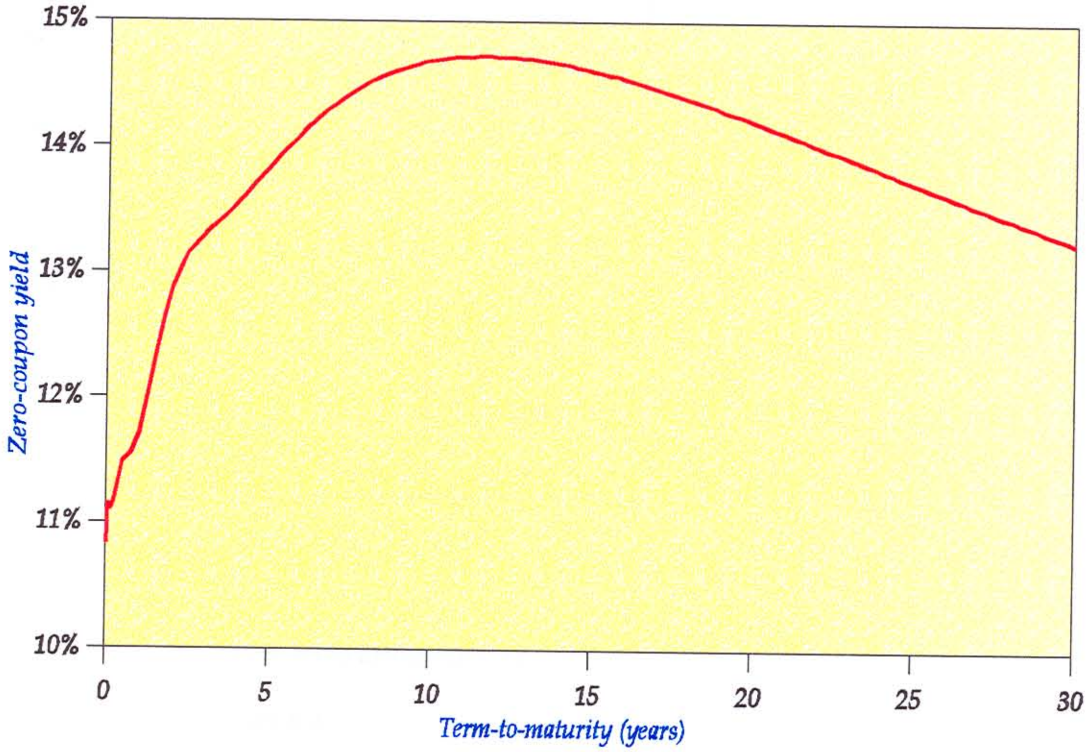


Figure 6.5: South African zero-coupon yield curve in November 1999

The modified Hull-White numerical method was hence used to calculate option prices for the R150 and R153, shown in Tables 6.7 and 6.8 respectively. The difference between the European and American prices in Tables 6.7 and 6.8 is an indication of the early-exercise value, which is larger for call options, due to the shape of the yield curve. Since the yield curve is relatively flat compared to the previous examples, both call and put option prices contain early-exercise value.

Table 6.7: Prices for R150 at-the-money-spot call and put options using the yield curve in Figure 6.4

Option term (years)	European call price	American call price	European/ American difference (%)	European put price	American put price	European/ American difference (%)
0.5	0.805	0.861	-7.0	4.752	4.816	-1.3
1.0	1.152	1.384	-20.1	5.641	5.885	-4.3
2.0	1.557	2.012	-29.3	4.887	6.692	-36.9
3.0	1.362	2.330	-71.0	3.704	6.993	-88.8
4.0	0.819	2.447	-198.9	2.336	7.125	-205.0
5.0	0.208	2.479	-1092.2	0.607	7.169	-1080.7

Table 6.8: Prices for R153 at-the-money-spot call and put options using the yield curve in Figure 6.4

Option term (years)	European call price	American call price	European/ American difference (%)	European put price	American put price	European/ American difference(%)
0.5	2.650	2.845	-7.3	3.216	3.225	-0.3
1.0	3.125	3.648	-16.7	4.841	4.909	-1.4
2.0	3.757	4.613	-22.8	5.618	6.507	-15.8
3.0	3.687	5.167	-40.1	5.719	7.386	-29.2
4.0	3.283	5.469	-66.6	5.513	7.996	-45.0
5.0	2.856	5.643	-97.6	4.893	8.394	-71.5

The longer options on the R150 bond show the effect when the term of the option becomes comparable to the term-to-maturity of the bond. Figure 6.6 shows the effect of the strike yield for a particular expiry date for 6 month options, as well as 1 year options, using the R153 bond as the underlying instrument.

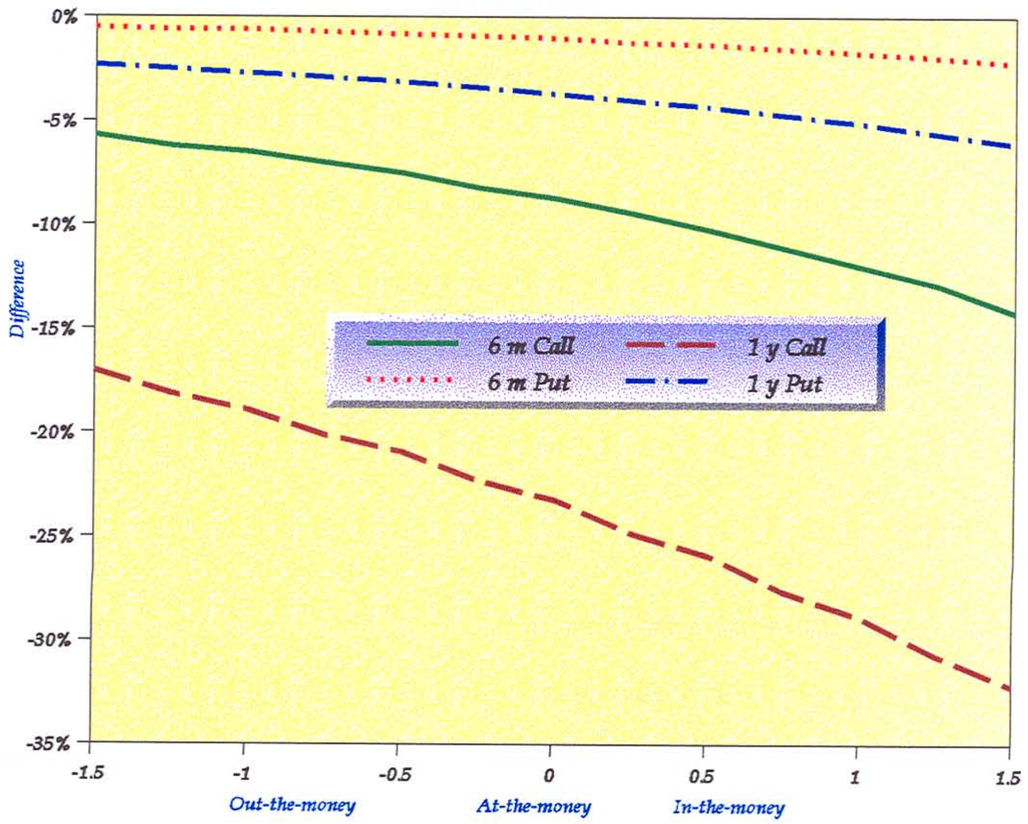


Figure 6.6: **Price difference between European and American options on the R153 bond for different yield strikes**

6.7 Concluding Remarks

Since the market convention is to use the Black model to price bond options, the Hull-White analytical solution (which is similar to the Black model) was compared to the Hull-White numerical solution, with specific reference to South African options. Results were compared using different strike conventions and term structures. Both the strike convention, and the term structure influence the price of American options.

The pricing difference between options on the bond price and options on the yield-to-maturity of a bond becomes significant for longer-dated American options. These pricing differences are also influenced by the term structure of interest rates. Pricing these options with the Black model can therefore lead to significant errors. The results show, however, that the pricing difference is usually small when a European model is used to price the following:

- short-dated American put options in a market with an increasing term structure; and
- short-dated American call options in a market with a decreasing term structure.

For the opposite situation, however, the error becomes significant, even for short-dated American options.

Since short-dated American bond options in the market are usually priced using the Black model, the market prices and implied volatilities for at-the-money options (put or call options, depending on the yield curve) can be used to solve the volatility parameters σ_r and a in the analytical Hull-White model. The parameters can then be used to price longer-dated American options by using the Hull and White trinomial tree approach and therefore reducing the pricing error. Therefore, the use of the Hull-White model is strongly recommended for longer-dated options, as well as in-the-money options, where there is a bigger early-exercise value.

It can be concluded that the primary advantage of the Hull-White model is to value longer-dated American options when using the yield-strike convention. The main advantage is certainly the estimation of the early-exercise value for American options which becomes significant for OTC call options when there is a positive-sloping yield curve, and for OTC put options when there is a negative-sloping yield curve. The early-exercise value is mainly a result of a move in the risk-free rate that affects the carry-cost of the hedge. Since OTC options are hedged in the physical instrument (the bond), the carry-cost can have a big influence.

The main disadvantage of the Hull-White model is, however, the need to calibrate the parameters and the estimation of the yield curve before one is able to price an option. By contrast, the Black model is easy to use and one needs only the volatility and the equivalent risk-free rate to price an option. This is one of the reasons why the Black model remains a popular pricing tool, even though it has several disadvantages.

In addition to OTC bond options, there are also South African Futures Exchange (SAFEX) bond options which are traded on the future yield of a bond. Since the largest volume options are traded on the near contract, the Black model is usually sufficient to estimate a reasonably accurate value, especially for at-the-money options. The biggest concern is out-the-money and in-the-money options where there is uncertainty about the accuracy of the Black model. Therefore, a pricing model that is as easy to use as the Black model, *and* can solve the problem of valuing in-the-money and out-the-money SAFEX bond options is discussed in the next chapter.

CHAPTER 7

AN ALTERNATIVE PRICING MODEL FOR SOUTH AFRICAN EXCHANGE TRADED BOND OPTIONS[1]

Options on South African government bond future rates are traded through SAFEX. Although there used to be reasonable liquidity in the past, the trading volumes for SAFEX bond options have declined significantly, in comparison with trading in the OTC market.

[1]Results of work done in this chapter were published in *RISK* (Smit & Van Niekerk, 1999)

Trade is usually based on price-volatility which gives a certain option price as calculated using the Black model (1976) where the strike is given as the exercise yield. Options are marked to market daily to establish the margin payment. Hedging SAFEX options with OTC options, where volatility smiles or skews are used, becomes a complex exercise. There was a need to develop a model that prices SAFEX options more accurately and that can address the disadvantages of the Black model, taking into account the simplified nature of SAFEX options.

The standard pricing models for South African bond options are the well-known Black-Scholes model (1973) and Black model (1976). It is generally assumed that the Black model (discussed in Chapter 5) is sufficiently accurate to value options with less than a year to expiry. However, when the Black model is applied to SAFEX options, the pricing of in-the-money and out-the-money options by the Black model, and the valuation of different maturity bonds using price-volatility give rise to concern. These concerns are discussed in the next section.

Since the early-exercise value of SAFEX options is very small[2], marginal benefits accrue from using a no-arbitrage model such as the Hull-White model. It can be argued that the short-term risk-free rate only plays an indirect role in option pricing[3]. Consequently the study propose that the future rate, rather than the short-term risk-free rate is used as the stochastic variable.

2 Hedging is done in the future, and not in the physical instrument, and therefore does not have any carry-cost.

3 The risk-free rate plays a role only in determining the future rate.

7.1 The Black model – a review

The popularity of the Black model can certainly be ascribed to its simplicity. The model is computationally efficient, requiring only a few basic parameters to calculate a reasonably accurate value for the option. Although there are several disadvantages, it is still, after 25 years, the most popular pricing model in most markets. It is preferred to models that are more accurate, but which are also much more complex and require the estimation of several parameters in order to obtain a more accurate fair value for the option. The calibration of more sophisticated models to the traded market value is a time-consuming process and therefore, many believe that the Black model is sufficiently accurate, especially for short-dated options (see Chapter 6).

In order to compete with the advantages of the Black model, any other model should, therefore, have the same ease of implementation and simplicity of use. Most important, though, it should give a more accurate estimation of the fair value of the option, especially of out-the-money options.

When using Black's model, it is necessary to calculate the forward bond price, the strike price and the price-volatility, using the forward yield and the strike yield. It is then assumed that the forward price, F, follows a geometric Brownian motion:

$$dF = \mu F dt + \sigma_F F dW \tag{1}$$

where μ is the drift and σ_F is the volatility of the forward bond price.

For the purposes of this chapter, a bond is discussed which pays a coupon m times a year at time t_i at a rate of c% and a nominal N at maturity time t_{mn}. The forward bond price, F, in equation (1) is given by the following non-linear function of the forward yield-to-maturity, Y:

$$F(Y) = \frac{c \cdot N}{m} \sum_{i=1}^{mn} e^{-Yt_i} + Ne^{-Yt_{mn}} \tag{2}$$

where all cash flows are discounted to the forward date.

The main disadvantages of the Black model and a motivation for the use of a yield-based model are discussed below.

7.1.1 Distribution of yield and price

The first problem with Black's model is the assumption that the underlying variable is lognormally distributed. A variable has a lognormal distribution if the natural logarithm of the variable is normally distributed. According to this assumption, the bond price can take any value between zero and infinity. In practice, however, the price of a zero-coupon bond is *bounded* and cannot have a value greater than its nominal value. Since the yield is non-negative and unbounded from above, it is therefore more accurate to assume that the *yield* of the bond has a lognormal distribution and follows a geometric Brownian motion.

In order to evaluate the lognormal distribution assumption empirically, an analysis was done on closing prices for the last nine years (data source: INet Bridge). Figure 7.1 gives the distribution of the logarithm of the R150 yield based on closing rates with 10-day intervals. Figure 7.2 shows the distribution of the 10-day *price*-returns of the R150 over the same period.

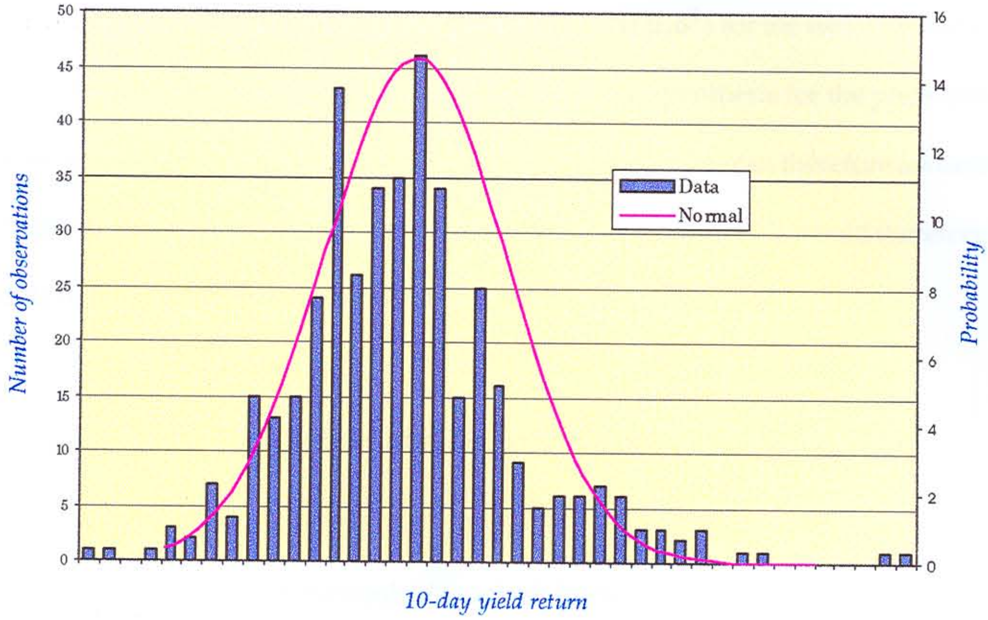


Figure 7.1: Distribution of 10-day yield returns for the R150 bond over a 9-year period

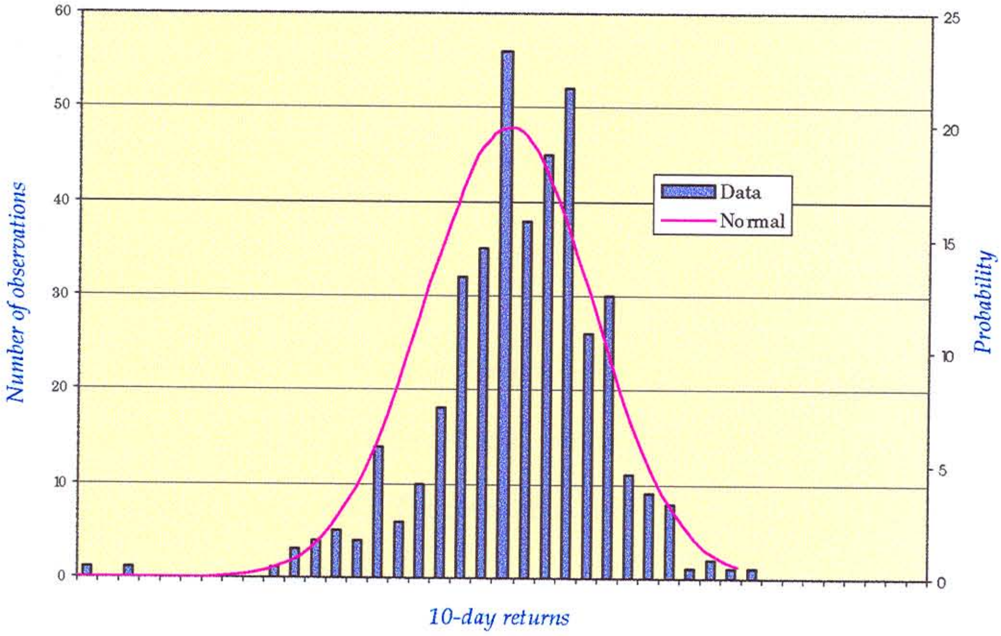


Figure 7.2: Distribution of 10-day price returns of the R150 bond over a 9-year period

A goodness-of-fit test, to test for normality, $H_0 : X \sim N(\mu, \sigma^2)$ for the two distributions, gave the following results: for a significance level of 0.01 the hypothesis for the price-distribution was rejected, and for the yield-distribution it was accepted. One can therefore assume that the yield-distribution is closer to lognormal than the price-distribution. It would therefore be more accurate to use the yield as the stochastic variable in a pricing model.

7.1.2 Yield-price correlation and the volatility skew

The Black model uses bond *price-volatility* which leads to a fundamental problem. Equation (1) implies that the instantaneous variance rate of the forward bond price, F, is equal to $(\sigma_F F)^2$, and is therefore proportional to the bond price. Since the bond price has a *negative* relationship with the yield (see equation (2)), the Black model therefore implies that the variance rate of the bond price or price-volatility is negatively correlated with the yield. Empirical data show, however, that the opposite is true. Correlation analysis of the daily volatility of the South African government R150 bond was done for the last nine years (data Source: INet Bridge). Figure 7.3 shows the results for the last three years, using 40-day price-volatility and daily closing yields.

A *positive* correlation was found between the price-volatility and the yield of the bond. The absolute value of the yield change over a 40-day period also showed a positive correlation, as well as yield-volatility against yield. The correlation coefficient, ρ, between the price-volatility and yield data was equal to 0.7605.

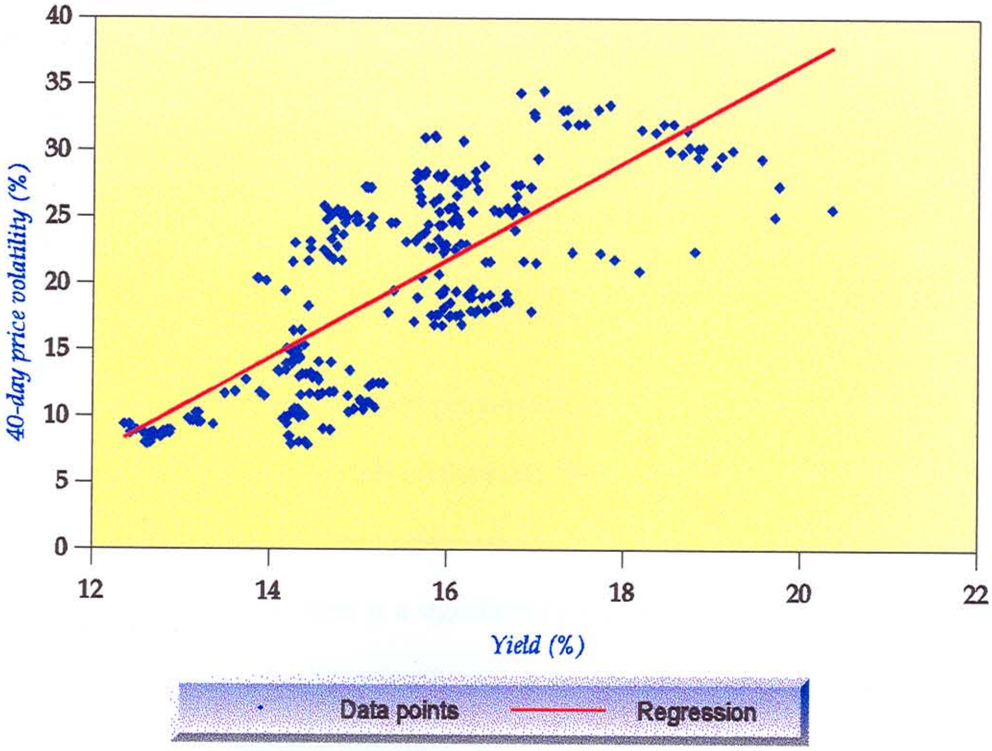


Figure 7.3: Correlation between price-volatility and yield-to-maturity

In order to estimate whether the correlation coefficient is significantly different from zero, a t-test was done, which resulted in a t-value of 20.05 for the above data. For a sample out of a population, a value of $t = 2.576$ ($\alpha = 0.005$) would occur only once in 100 random samples when drawn from a universe with a value of ρ equal to zero. The probability of a value of t equal to 20.05 is extremely small if the value of ρ is equal to zero. The conclusion is therefore that the correlation coefficient (ρ) is positive in the universe from which the sample was taken.

Based on the empirical evidence, one can conclude that the Black model fails to give a true reflection of the variance of the underlying instrument.

Using the *price* as the underlying variable in the Black model influences the valuation of out-the-money bond options. Participants in the market currently compensate for this mispricing by using a *volatility skew*. There is, however, usually under- or over-compensation, due to the uncertainty of where in- and out-the-money options should be trading.

7.1.3 Price-volatility

The Black model assumes a *constant* price-volatility for the life of the option. The price-volatility, however, depends mainly on the yield, the time to maturity and the convexity of the bond price curve. The pull-to-par-phenomenon of the bond price can have a big influence, especially when the option term is a significant proportion of the term of the bond. For an option on a future, the effect is insignificant, since the term-to-maturity from the future date stays constant. When hedging is done in the spot-market, however, this becomes a problem.

Since price-volatility is used in the Black model, different volatilities are used to price options on different bonds. One must therefore ensure that the volatility relations are always consistent, for example with a parallel shift in the yield curve. Using a yield-based model would solve this problem, since the same yield-volatility could be used for parallel shifts in the yield curve.

7.1.4 Yield versus short-rate and price

The work done by Longstaff (1990) on caps, floors and T-bills, suggests that the yield can be seen as the underlying variable that determines the price of a bond. Vasicek (1977) was the first to develop a term structure model which assumes that the price of a discount bond is determined by the assessment of the short-term rate process over the term of the bond. The bond price obtained in this way can then be converted to obtain the yield-to-maturity by goal-seeking equation (2). This yield-to-maturity rate therefore implies a certain expectation of future short- term rates.

Although term structure models use the short-rate, one can argue that it is better to model the stochastic process of the instrument in which hedging is being done, in order to stay delta neutral (Wilmott, 1998:441). One can therefore make the assumption that the yield-to-maturity of a bond contains all available information about the market expectation of the short-rate, and can therefore be seen as the underlying variable determining the price of a bond.

The particular problem to solve here is the valuation of an exchange-traded American option on the future yield of a long-term bond. A model is suggested where the value of the option is derived from the stochastic process followed by the future yield, using a constant yield-volatility. The model is solved numerically in order to provide for early-exercise.

7.2 The proposed yield-based model

A bond can be traded on its yield-to-maturity, instead of price, with a T-year option on the future T-year yield of the bond. If one assumes that the T-term future yield, Y, of the bond follows a stochastic process, then

$$dY = \mu Y dt + \sigma Y dW \qquad (3)$$

where μ is the drift and σ is the volatility of the yield.

If F is the T-term future price of the bond, and since F depends only on the future yield, Y, on that date, it follows from Ito's lemma (Björk, 1999) that

$$\begin{aligned} dF &= \left(F_Y \mu Y + \tfrac{1}{2}\sigma^2 Y^2 F_{YY}\right)dt + \sigma Y F_Y dW \\ &= \alpha_F F dt + \sigma_F F dW \end{aligned} \tag{4}$$

where

$$\begin{aligned} \alpha_F &= \frac{\mu Y F_Y + \tfrac{1}{2}\sigma^2 Y^2 F_{YY}}{F} \\ \sigma_F &= \frac{\sigma Y F_Y}{F} \end{aligned} \tag{5}$$

and where F_Y denotes the first derivative and F_{YY} denotes the second derivative to Y. If one defines V as the value of a contingent claim dependent on the level of the future yield of the bond, since V is a function of Y and t, it also follows from Ito's lemma that

$$\begin{aligned} dV &= \left(V_Y \mu Y + V_t + \tfrac{1}{2}\sigma^2 Y^2 V_{YY}\right)dt + \sigma Y V_Y dW \\ &= \alpha_V V dt + \sigma_V V dW \end{aligned} \tag{6}$$

where

$$\begin{aligned} \alpha_V &= \frac{\mu Y V_Y + V_t + \tfrac{1}{2}\sigma^2 Y^2 V_{YY}}{V} \\ \sigma_V &= \frac{\sigma Y V_Y}{V} \end{aligned} \tag{7}$$

One can set up a portfolio, Σ, consisting of two assets:

- the contingent claim, V; and
- the underlying bond future, with a price F at the future yield Y.

The relative portfolio can be denoted by (u_F , u_V). The bond future and the derivative are exchange-traded and interest is paid on the margin account (which is seen as a security for the contracts entered into). It initially costs nothing to enter into an exchange-traded option or future contract, therefore the initial investment is zero, while the portfolio value is given by

$$\Sigma \neq 0 \tag{8}$$

An immediate change in the value of the underlying instrument (the future yield) would result in a change in the value of the derivative and the future price of the bond. Therefore:

$$\begin{aligned} d\Sigma &= \Sigma\left[u_F(\alpha_F dt + \sigma_F dW) + u_V(\alpha_V dt + \sigma_V dW)\right] \\ &= \Sigma\left[(u_F\alpha_F + u_V\alpha_V)dt + (u_F\sigma_F + u_V\sigma_V)dW\right] \end{aligned}$$

For the relative portfolio,

$$u_F + u_V = 1 \tag{9}$$

For the dW-term to vanish, the following condition can be introduced:

$$u_F\sigma_F + u_V\sigma_V = 0 \tag{10}$$

Therefore,

$$d\Sigma = \Sigma\left[u_F\alpha_F + u_V\alpha_V\right]dt \tag{11}$$

which is a linear riskless portfolio. Since there is no initial investment, the principle of no-arbitrage states that

$$d\Sigma = 0 \tag{12}$$

Therefore,

$$u_F\alpha_F + u_V\alpha_V = 0 \tag{13}$$

From equations (9) and (10) it is clear that

$$u_V = \frac{\sigma_F}{\sigma_F - \sigma_V}$$
$$u_F = \frac{-\sigma_V}{\sigma_F - \sigma_V}$$

Using equations (5) and (7), one can then write the following:

$$u_V = \frac{F_Y V}{F_Y V - F V_Y}$$
$$u_F = \frac{-F V_Y}{F_Y V - F V_Y}$$

Substituting in equation (13), one gets

$$- \frac{F V_Y}{F_Y V - F V_Y}\left[\frac{\mu Y F_Y + \frac{1}{2}\sigma^2 Y^2 F_{YY}}{F}\right]$$

$$+ \frac{F_Y V}{F_Y V - F V_Y}\left[\frac{\mu Y V_Y + V_t + \frac{1}{2}\sigma^2 Y^2 V_{YY}}{V}\right] = 0$$

Therefore,

$$V_t + \frac{1}{2}\sigma^2 Y^2\left[V_{YY} - \frac{F_{YY}}{F_Y}V_Y\right] = 0 \tag{14}$$

This gives a partial differential equation for the value of a derivative security dependent on the future yield of a bond. In order to provide for the early-exercise value for American options, one can therefore solve the above differential equation with an implicit finite difference method. In order to evaluate the results, the results are compared with that of the Black model.

7.3 Numerical solution

In order to solve equation (14), an implicit finite difference method was used, with

$$\frac{\partial V}{\partial t} = \frac{V_{i+1,j} - V_{i,j}}{\Delta t}$$

$$\frac{\partial V}{\partial Y} = \frac{V_{i,j+1} - V_{i,j-1}}{2\Delta Y}$$

$$\frac{\partial F}{\partial Y} = \frac{F_{i,j+1} - F_{i,j-1}}{2\Delta Y}$$

$$\frac{\partial^2 V}{\partial Y^2} = \frac{V_{i,j+1} + V_{i,j-1} - 2V_{i,j}}{\Delta Y^2}$$

$$\frac{\partial^2 F}{\partial Y^2} = \frac{F_{i,j+1} + F_{i,j-1} - 2F_{i,j}}{\Delta Y^2}$$

Substituting these equations into equation (14), results in the implicit scheme

$$a_j V_{i,j-1} + b_j V_{i,j} + c_j V_{i,j+1} = V_{i+1,j} \qquad (15)$$

where

$$a_j = -\frac{1}{2}\sigma^2 j^2 \Delta t \left[\left(\frac{V_{i,j+1} + V_{i,j-1} - 2V_{i,j}}{V_{i,j+1} - V_{i,j-1}}\right) + 1\right]$$

$$b_j = 1 + \sigma^2 j^2 \Delta t$$

$$c_j = \frac{1}{2}\sigma^2 j^2 \Delta t \left[\left(\frac{V_{i,j+1} + V_{i,j-1} - 2V_{i,j}}{V_{i,j+1} - V_{i,j-1}}\right) + 1\right]$$

Since the value of the option at the expiry date is just the payoff of the option, the problem can be solved backwards. The value of the option at the expiry date is determined by the payoff given by a difference in price of $max[F(X) - F(Y_T), 0]$ for a call option, and $max[F(Y_T) - F(X), 0]$

for a put option, where Y_T is the yield at time T and X the strike rate. To obtain the value of the option at the boundaries of the finite difference grid, where Y reaches its minimum and maximum, it can be assumed that the gamma of the option at these points should be zero (see also Chapter 2, Section 2.4.2.1). Since the implicit scheme results in a tri-diagonal system, the procedure of *LU* decomposition (Wilmott, Dewynne & Howison, 1993) was used to solve the system. For the longer-dated options, 1000 time steps were used, with fewer for the shorter-dated options.

7.4 Empirical results

A comparison between the Black model and the yield-based model was done for the government R150 bond, maturing in February 2005 with a coupon of 12%, as well as the government R153 bond, maturing in August 2010 with a coupon of 13%. In-the-money, as well as out-the-money options were compared for different expiry dates. The same yield-volatility was used for both bonds.

7.4.1 Price differences

Figures 7.4 and 7.5 show the results of the R150 and R153 bonds respectively, using a yield-volatility of 20%. The data for these examples are set out in Tables 7.1 and 7.2. The results show clearly that, compared to the yield-based model, out-the-money call options are overvalued by the Black model, while out-the-money put options are undervalued.

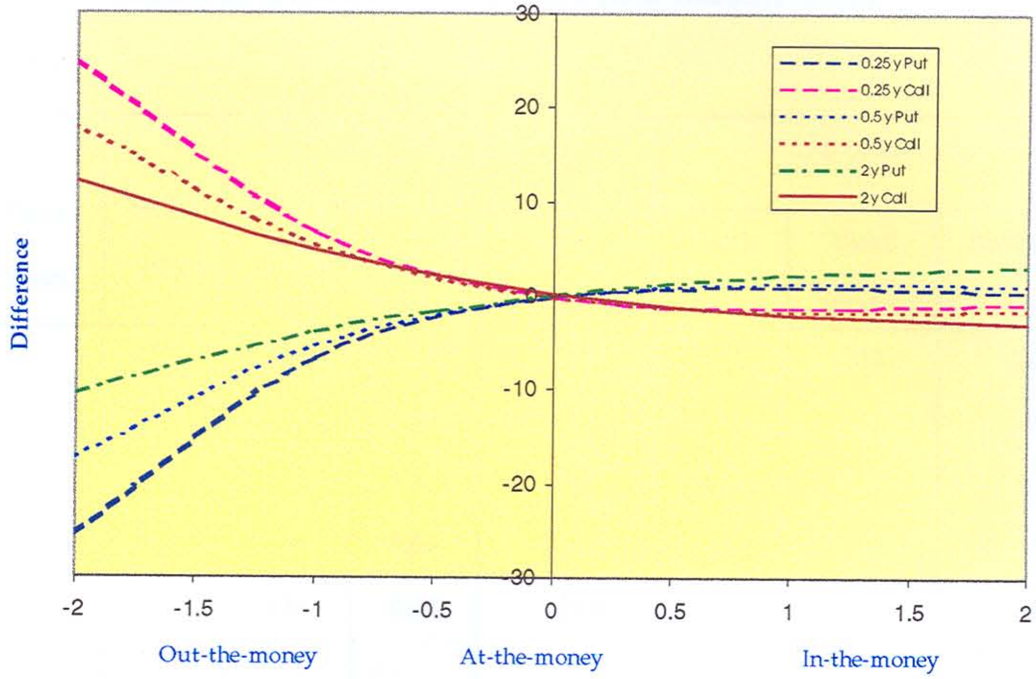


Figure 7.4: Pricing differences for R150 bond with a yield-volatility of 20%

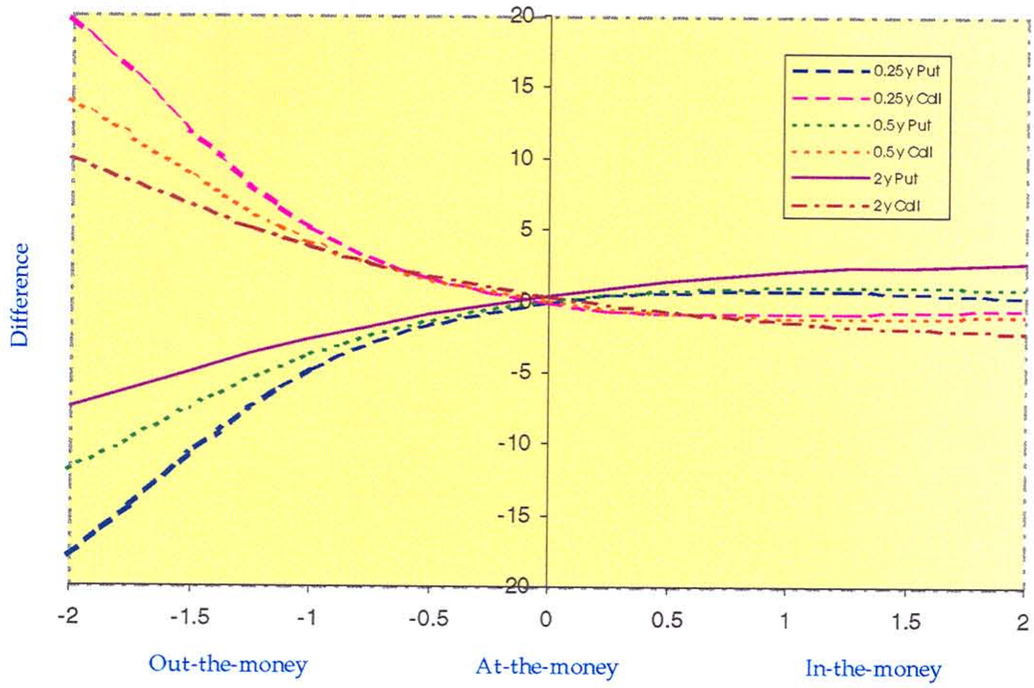


Figure 7.5: Pricing differences for R153 bond with a yield-volatility of 20%

Table 7.1: Results for R150 option prices with a yield-volatility of 20%

Term (years)	Strike (%)	Put option price			Call option price		
		Black model	Yield model	Difference (%)	Black model	Yield model	Difference (%)
0.25	12	7.19	7.14	0.6	0.16	0.12	24.9
	13	4.09	4.05	1.1	0.66	0.62	6.8
	14	1.87	1.87	0.0	1.87	1.87	0.0
	15	0.65	0.69	-6.6	3.92	3.96	-1.1
	16	0.16	0.20	-25.6	6.56	6.60	-0.6
0.5	12	7.31	7.22	1.3	0.53	0.43	18.0
	13	4.59	4.52	1.5	1.27	1.20	5.6
	14	2.56	2.56	0.0	2.56	2.56	0.0
	15	1.24	1.31	-5.2	4.41	4.48	-1.5
	16	0.52	0.61	-17.2	6.72	6.81	-1.3
2	12	7.04	6.80	3.4	1.95	1.71	12.2
	13	5.31	5.18	2.5	2.81	2.68	4.8
	14	3.89	3.87	0.3	3.89	3.87	0.3
	15	2.75	2.85	-3.8	5.17	5.27	-2.0
	16	1.88	2.07	-10.4	6.64	6.84	-2.9

Table 7.2: Results for R153 option prices with a yield-volatility of 20%

Term (years)	Strike (%)	Put option orice			Call option price		
		Black model	Yield model	Difference (%)	Black model	Yield model	Difference (%)
0.25	12	11.54	11.49	0.4	0.24	0.20	19.8
	13	6.44	6.39	0.8	1.03	0.97	5.2
	14	2.90	2.90	-0.0	2.90	2.90	-0.0
	15	1.00	1.05	-4.9	6.00	6.05	-0.8
	16	0.26	0.30	-17.9	9.86	9.91	-0.5
0.5	12	12.01	11.89	1.0	0.83	0.72	14.1
	13	7.39	7.31	1.2	2.03	1.94	4.2
	14	4.06	4.06	0.0	4.06	4.06	0.0
	15	1.97	2.04	-3.7	6.92	6.99	-1.1
	16	0.83	0.93	-11.9	10.36	10.46	-1.0
2	12	14.12	13.74	2.7	3.83	3.45	9.9
	13	10.47	10.25	2.0	5.51	5.29	3.9
	14	7.55	7.52	0.3	7.55	7.52	0.3
	15	5.29	5.43	-2.7	9.91	10.05	-1.4
	16	3.60	3.87	-7.5	12.52	12.79	-2.1

The opposite is true for in-the-money options, as expected. The relative difference is bigger for the short-term R150-bond than for the longer-term R153 bond. The early-exercise value of American options was found to be very small, as can be expected from short-term options on the future yield. Since the yield-model has been calibrated to the Black model, the small difference for at-the-money options is the early-exercise value.

Bonds with other maturity dates and coupons were also evaluated and similar results were obtained. These include the R162 maturing on 15 January 2002 (12.5% coupon), the R184 maturing on 21 December 2006 (12.5% coupon), the R157 maturing on 15 September 2015 (13.5% coupon) and the R186 maturing on 21 December 2026 (10.5% coupon). One big advantage of the yield-based model is that a constant yield-volatility can be used for different maturity bonds, while the price-volatility for the Black model must be manually adjusted to compensate for bonds with different maturity dates and coupons.

7.4.2 Delta differences

Since the delta of an option plays as important a role as the option itself (being the hedge), the yield-based model's delta was also compared with the delta given by the Black model. The results showed a relatively big difference between the two models, as displayed in Table 7.3. The delta given by the Black model is larger than the delta given by the yield-based model for call options, and smaller for put options.

The results for the delta indicate that a position will be under-hedged for put options, which is problematic when rates spike up. Call-options, on the other hand, will be over-hedged. The relative difference is larger for short-dated call options and longer-dated put options.

Table 7.3: Delta difference for R153 options with a yield-volatility of 10%

Term (years)	Strike (%)	Put option delta			Call option delta		
		Black model	Yield model	Difference (%)	Black model	Yield model	Difference (%)
	12	-0.92	-0.94	-2.2	0.08	0.06	25.0
	13	-0.76	-0.78	-2.6	0.24	0.22	8.3
0.25	14	-0.49	-0.51	-4.1	0.51	0.49	3.9
	15	-0.23	-0.25	-8.7	0.77	0.75	2.6
	16	-0.08	-0.09	-12.5	0.92	0.91	1.1
	12	-0.84	-0.87	-3.6	0.16	0.13	18.8
	13	-0.68	-0.71	-4.4	0.32	0.29	9.4
0.5	14	-0.49	-0.51	-4.1	0.51	0.49	3.9
	15	-0.3	-0.32	-6.7	0.7	0.68	2.9
	16	-0.15	-0.18	-20.0	0.85	0.82	3.5
	12	-0.68	-0.73	-7.4	0.32	0.27	15.6
	13	-0.58	-0.63	-8.6	0.42	0.37	11.9
2	14	-0.48	-0.53	-10.4	0.52	0.47	9.6
	15	-0.38	-0.43	-13.2	0.62	0.57	8.1
	16	-0.29	-0.34	-17.2	0.71	0.66	7.0

7.5 Summary

The valuation of options on the forward yield of a bond using a model based on the stochastic behaviour of the yield, rather than the price, has several advantages.

Firstly, when the model proposed here is compared with that of Black, the numerical results show that:

- the two models price at-the-money options similarly;
- the Black model overvalues out-the-money call options and undervalues out-the-money put options; and
- a small price difference occurs for in-the-money options.

Secondly, the yield-based model addresses most of the disadvantages of the Black model:

- The yield-based model uses the yield as the underlying instrument, which is closer to a lognormal distribution than the price of the bond.
- The same yield-volatility can be used for any maturity bond and the option price is automatically adjusted for the duration-difference. With the Black-model an independent price-volatility has to be estimated first, and recalculated every time the yield or volatility changes.
- The yield-based model provides for the pull-to-par effect of bonds when pricing long-term options. The Black model does not provide for this and the decline in volatility for longer options has to be adjusted by adjusting the price-volatility.
- The yield-based model can also value options on swaps (swaptions) in a similar way, making arbitrage between bond options and swaptions easier. (See Appendix B for the usual pricing convention of swaptions.) The yield-based model is consistent with the pricing of swaptions using the Black model.

The main disadvantage of this model is that it assumes that hedging will be done using the future yield. The model thus neglects movement in the short-rate for over-the-counter spot hedges. When hedging is done in the spot market, the early-exercise value increases for certain options as described in Chapter 6, since it is dependent on a certain carry or repo rate. The yield-based model is therefore well-suited for options on the SAFEX future. For OTC options, it is important to compensate for the early-exercise value.

CHAPTER 8

CONCLUSIONS

Although a large body of research already exists in the area of derivative securities, the characteristics of the South African fixed income market pose particular challenges for researchers in this market. This study has set out to add value to a specific area where little work has been done up till now, building on the theoretical work of other practitioners and scientists.

8.1 Summary

The aim of this study was to add value to the South African fixed income market by analysing the existing procedures and models that are being used and, where necessary, to make a contribution by recommending alternatives.

Chapter 2 introduced the basic theory of pricing derivative securities. If a unique probability measure could be found so that the relative prices in an economy become martingales, then a continuous economy would be complete and free of arbitrage opportunities. The no-arbitrage assumption plays an important role in many pricing models. The concepts of no-arbitrage, martingales and partial differential equations serve as the basis for the valuation of derivative securities.

Chapters 3 and 4 were concerned with the term structure of interest rates. The short-term risk-free rate of interest (or spot rate) is the cornerstone of the fixed income market. The short-term risk-free rate and the market consensus on the future change in this rate form the term structure of interest rates. The term structure of interest rates determines the price of fixed deposits, bonds, swaps and other derivative securities in the fixed income market.

The zero-coupon yield curve can serve as the basis for estimating all other fixed income instruments. The estimation of the zero-coupon yield curve is, therefore, fundamental in order to price all other derivatives accurately, including bond options. The standard bootstrap method is cumbersome and the procedure of estimating the zero-coupon rates causes discrepancies. For these reasons, Chapter 4 introduced an iterative bootstrap method. This method starts with a first guess for the zero curve and then uses an iterative procedure which converges to the actual zero-coupon curve. Convergence to the actual yield curve following this method is proved. This method generates a zero-coupon curve in a much smoother and manageable way, without having to use other time-consuming numerical methods such as the Newton Raphson technique.

Chapter 5 discussed relevant bond option pricing models and focussed on the most appropriate model, the Hull-White model. The Hull-White model is based on the stochastic behaviour of the short-term rate and prices European options using the exact solution for the partial differential equation obtained. Hull and White further introduced a trinomial tree numerical approach to obtain a fair value for an American option, as discussed in Chapter 6. In order to use the Hull-White model for South African OTC bond options, the model was adjusted to make provision for a yield-strike convention, rather than a price-strike. The influence of the different strike conventions was shown for both coupon and zero-coupon bonds. The influence of the shape of the term structure became clear when the results of both a sharply increasing and decreasing term structure were compared. The successful use of the Hull-White model in practice depends largely on the estimation of the volatility parameters σ_r and a. The calibration of these parameters was discussed for zero-coupon and coupon bonds.

Since the convention in South Africa is to use a European model for the pricing of American options, the difference between European and American options was determined according to the Hull and White model, in order to establish an error-factor. Empirical results for South African options show a significant difference between the European and American prices.

SAFEX-traded bond options are options on the future yield of a bond, with an initial margin and a margin account on which interest is earned. Since the short-term risk-free rate does not influence these options directly, a model based on the stochastic behaviour of the future yield-to-maturity is more suitable. Chapter 7 discussed an option pricing model for SAFEX-traded bond options. The simplicity of this model makes it comparable to the Black model, which prices bond options using the stochastic behaviour of the *price* of the bond. A specific benefit of the yield-based model is that it addresses the disadvantages of the Black model.

8.2 Conclusions

This study adds value to several areas in the fixed income market, with specific reference to the South African market. The impact of the results is significant. The areas of contribution can be divided into three categories:

- construction of a zero-coupon yield curve using a new method, called the iterative bootstrap method;
- pricing and calibrating longer dated American OTC bond options with the yield-strike convention, using a modified Hull-White model; and
- pricing SAFEX options on the future yield of a bond, using a new methodology.

The development of an iterative bootstrap technique benefits the estimation of a zero-coupon yield curve, first for trading purposes and, secondly, as input to obtain bond option prices using the Hull-White numerical solution.

The Hull-White numerical solution was modified in order to price options on the yield-to-maturity of a bond. The influence of the strike-convention, as well as the shape of the yield curve, is shown. The early-exercise value for American options becomes significant in some cases. The results illustrate the impact of using a European model to price American over-the-counter options. A convenient way of calibrating the Hull-White model to market data is suggested.

The characteristics of exchange-traded bond options made it feasible to develop a simplified model for SAFEX options. Empirical evidence of the correlation between price-volatility and the yield of the bond indicates a fundamental problem in applying the Black model to South African futures options. Since the yield-based method addresses all the major disadvantages of the Black model, it can be used with much more confidence to price future options, especially out-the-money options. More efficient hedging is also possible when one uses the delta of the yield-based model.

8.3 Recommendations

The fixed income market remains an area that requires further research and refinement, especially in South Africa. Historically, the South African yield curve has been one of the most interesting yield curves in emerging markets, because it has many different, and, sometimes unusual shapes. Generally this causes a problem, since it complicates the fit of data points. The success of the iterative bootstrap method is largely due to the approximation technique used to fit the data points. Although success has been achieved in this study in using different combinations of linear functions, these functions require further adjustments, particularly when there is a large change in the shape of the yield curve. The development of a fitting procedure that adjusts automatically to the shape of the yield curve would add more value to the iterative bootstrap technique.

Vanilla European and American options on coupon bonds, which simultaneously address options on swaps, have been considered in this study. Other options on fixed rate instruments, as well as exotic options are, however, areas of research which have drawn little attention so far. Future research in these areas is recommended.

Volatility remains the most important input parameter in any option pricing model. The longer the option term, the bigger the influence of the expected volatility on the option price. Accurate estimation of the volatility parameters used in the Hull-White model numerical method becomes more crucial when there is no benchmark. Intensive research in this area will benefit practical users of the Hull-White model for South African options and will make the model more accessible to practitioners at large.

The distribution of the yield-to-maturity of bonds, as shown in this study, remains an interesting problem. Empirical research in this area to find an improved fit to the distribution of data, and a model to approximate the valuation of an option that satisfies this distribution can add more value.

Bibliography

Björk, T. 1999. *Arbitrage Theory in Continuous Time.* New York: Oxford University Press.

Black, F. 1976. The Pricing of Commodity Contracts. *Journal of Financial Economics,* 3:167-179.

Black, F. 1988. How to Use the Holes in Black-Scholes. *RISK,* March.

Black, F. & Scholes, M. 1973. The Pricing of Options and Corporate Liabilities. *Journal of Political Economy,* 81:637-59.

Black, F., Derman, E. & Toy, W. 1990. A One-Factor Model of Interest Rates and Its Application to Treasury Bond Options. *Financial Analysts Journal,* January-February: 33-39.

Black, F. & Karansinski, P. 1991. Bond and Option Pricing When Short Rates Are Lognormal. *Financial Analysts Journal,* July-August: 52-59.

Bodie, Z., Kane, A. & Marcus, A.J. 1993. *Investments.* Second Edition. Boston: Richard D. Irwin.

Brown, T. 1999. Virgin Territory. *RISK,* June Special Report: 16.

Burden, L. & Faires, D.J. 1989. *Numerical Analysis.* Boston: PWS KENT.

Cox J.C., Ingersoll, J.E. & Ross, S.A. 1985. A Theory of the Term Structure of Interest Rates. *Econometrica,* 53:75-100.

Cox, J., Ross, S.A. & Rubinstein, M. 1979. Option Pricing: A Simplified Approach. *Journal of Financial Economics,* 7:229-264.

Dastidar, R. 2000. Derivatives at record volumes, says OCC. *RISK,* January: 11

Faure, A.P., Falkena, H., Kok, W. & Raine G. 1991. *The Interest-bearing Securities Market.* Johannesburg: Southern.

Harrison, J.M. & Pliska, S. 1981. Martingales and Stochastic Integrals in the Theory of Continuous Trading. *Stochastic Processes and their Applications,* 11:215-260.

Heath, D., Jarrow, R. & Morton, A. 1992. Bond Pricing and the Term Structure of the Interest Rates: A New Methodology. *Econometrica,* 60(1):77-105.

Ho, T.S.Y. & Lee, S.B. 1986. Term Structure Movements and Pricing Interest Rate Contingent Claims. *Journal of Finance,* 41:1011-1029.

Hull, J. C. 1997. *Options, Futures, and other Derivative Securities.* Third Edition. New Jersey: Prentice Hall.

Hull, J. & White, A. 1990. Pricing Interest Rate Derivative Securities. *The Review of Financial Studies,* 3 (4):573-592.

Hull, J. & White, A. 1993. One-Factor Interest Rate Models and the Valuation of Interest Rate Derivative Securities. *Journal of Financial and Quantitative Analysis,* 28(2), June: 235-254.

Hull, J. & White, A. 1994. Numerical Procedures for Implementing Term Structure Models I: Single-Factor Models. *The Journal of Derivatives,* 2(1): 7-16.

Jamshidian, F. 1989. An Exact Bond Option Pricing Formula. *The Journal of Finance,* 44, March: 205-209.

Jamshidian, F. 1996. Sorting out Swaptions. *RISK,* March: 59-60.

Longstaff, F.A. 1990. The valuation of options on yields. *Journal of Financial Economics,* 26: 97-121.

Pelsser, A.A.J. 1996. *Efficient Methods for Valuing and Managing Interest Rate and Other Derivative Securities.* Ph.D. Thesis. Erasmus University, Rotterdam.

Rendleman, R. & Bartter, B. 1979. Two State Option Pricing. *Journal of Finance,* 34: 1092-1110.

Rendleman, R. & Bartter, B. 1980. The Pricing of Options on Debt Securities. *Journal of Financial and Quantitative Analysis,* 15, March:11-24.

Smit, L. & Van Niekerk, F.D. 1997. Curve Building. *RISK,* April Special Report: xiv-xvi.

Smit, L. & Van Niekerk, F.D. 1999. Pricing Risk for South African bond options. *RISK,* June Special Report: 8-10.

Vasicek, O.A. 1977. An Equilibrium Characterization of the Term Structure. *Journal of Financial Economics,* 5: 177-188.

Vasicek, O.A. and Fong, G. 1982. Term Structure Modeling Using Exponential Splines. *Journal of Finance,* 37: 341-348.

Wilmott, P., Dewynne, J. & Howison, S. 1993. *Option Pricing: Mathematical Models and Computation.* Oxford: Oxford Financial Press.

Wilmott, P. 1998. *Derivatives – The Theory and Practice of Financial Engineering.* England: John Wiley and Sons.

Internet: www.bondex.co.za. The South African Bond Exchange. South Africa.

Internet: www.bondindex.co.za. South African Liquid Investment Grade Bond Index. Standard Bank & Alexander Forbes. South Africa.

Inet-Bridge. South Africa. Internet: www.inetbridge.co.za.

APPENDICES

APPENDIX A

The least squares approximation technique

Imprecisions in the financial market may result in prices and/or rates that do not always form a smooth curve. When determining an approximate fit to the term structure of interest rates, it is evident that all data points are not in perfect relation to each other. The data set can be approximated in several ways. However, the approximated curve should:

- be a smooth and continuous function of time; and
- have a smooth and continuous first derivative.

An approximation method that is commonly used to find the maximum likelihood estimate of the model parameters, is least squares approximation (Burden & Faires, 1989:425-451). The set of data points is fitted to a model which is a linear combination of specified functions of the term, t. The general form for this model is

$$y(t) = \sum_{k=1}^{M} a_k \, \xi_k(t)$$

where $\xi_k(t)$ = arbitrary fixed functions of t.

a_k = M adjustable coefficients, $M <$ number of data points

The coefficients a are determined by minimizing the function:

$$\chi^2 = \sum_{i=1}^{N} \left[\frac{y_i - \sum_{k=1}^{M} a_k \, \xi_k(t_i)}{\sigma_i} \right]^2$$

where y_i = discrete data points, each with a term to maturity of t_i years,

σ_i = standard deviation of data point i.

Since the standard deviation serves as a weighting factor, it can be replaced by a weighting factor in order to give a bigger weighting to more tradable bonds. Different functions for ξ can be used in order to accommodate the particular shape of the curve being fitted. The least squares approximation technique works sufficiently well for many curve shapes, especially when more than one function is used.

APPENDIX B

The pricing of swap options

The price of a coupon bond is a non-linear function of the yield-to-maturity, given by

$$P_k = \sum_{i=1}^{n-1} \gamma_k e^{-t_i^{(k)}\eta_k} + (1 + \gamma_k) e^{-t_n^{(k)}\eta_k}.$$

In order to price a European option on a bond yield, it is usually treated as an option on the bond price, using price-volatility in the Black model. The price of swaps, on the other hand, is a *linear* function of the fixed rate. Options on swaps (swaptions) are also valued by using the Black model, but using the yield-volatility.

European swaptions are an example of options that can be priced by an exact solution, since the price is a linear function of the fixed rate (Jamshidian, 1996). The value of an n-year swap paying a fixed rate of R%, making m payments per year, is given by

$$S(R) = \frac{R \cdot N}{m}\sum_{i=1}^{mn} e^{-r_i t_i} + N e^{-r_{mn} t_{mn}} - N$$

which is a linear function of the swap rate R. The rates $\{r_i\}$ are the zero-coupon rates for each payment period. At expiry, the payoff, h, of a swaption is therefore a linear function of the difference in two interest payments:

$$\begin{aligned} h &= \max\{S(R) - S(R_X), 0\} \\ &= \sum_{i=1}^{mn} \frac{N}{m} e^{-r_i t_i} \max\{(R - R_X), 0\} \end{aligned}$$

where R_X is the strike rate. The coefficient of $\max\{(R - R_X),0\}$ is therefore the value of the payoff per 1% gain in the swap rate. The price of an option to receive a fixed rate can therefore be calculated by using the Black exact solution and calculating the expected payoff in percentage terms, and multiplying with the payoff per 1%:

$$\sum_{i=1}^{mn} \frac{N}{m} e^{-r_i t_i} \left(R_F N(d_1) - R_X N(d_2)\right)$$

where R_F is the forward swap rate and

$$d_1 = \frac{\ell n\left(R_F / R_X\right) + \sigma^2 T/2}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

This is the same as using the swap price and price volatility to calculate the option price.

In order to compare the above with a bond, it is necessary to explain the fundamental difference between a bond and a swap. For a *swap*, the change in price for a 1 point change in the swap rate is constant. Due to the convexity of the *bond* price, the change in price for a 1 point move in the yield, changes, depending on the particular base yield. The difference in value per point for an out-the-money strike can be very different from the at-the-money value per point. It is clear that the described method to price a swap option cannot be directly applied to an option on a coupon bond.

Opsomming

'n Analise van die
termynstruktuur van rentekoerse
en opsies op effekte in die
Suid-Afrikaanse kapitaalmark

deur

Linda Smit

Studieleier : Prof. FD van Niekerk
Departement : Wiskunde en Toegepaste Wiskunde
Graad : Ph.D.

Afgeleide instrumente vorm 'n integrerende deel van handel in die finansiële wêreld en maak die gebruik van akkurate waarderingsmodelle en risiko-modelle noodsaaklik. Hierdie studie fokus op die Suid-Afrikaanse vasterentedraende mark, en evalueer bestaande modelle en prosedures. Alle waarderingsmodelle maak staat op sekere aannames en het gevolglik beperkings. Tekortkominge in die Suid-Afrikaanse vasterentedraende mark word derhalwe aangespreek, eerstens die termynstruktuur van rentekoerse en tweedens, die waardering van opsies op effekte.

Aangesien die vasterentedraende mark ten volle afhanklik is van die termynstruktuur van rentekoerse, is dit die belangrikste faktor in die prysberekening van enige vasterentedraende

afgeleide instrument. Die Suid-Afrikaanse effektemark verhandel hoofsaaklik in koepondraende effekte en bykans geen inligting is beskikbaar vir nulkoeponeffekte (wat die termynstruktuur bepaal) nie. 'n Verbeterde weergawe van die optrek-metode (bootstrap-method) vir die bepaling van 'n nulkoepon opbrengskurwe word derhalwe voorgestel. Die nulkoepon opbrengskurwe vorm die fondament vir die prysberekening van enige vanieljeproduk in die vasterentedraende mark en dien as inset vir die prysberekening van opsies op effekte wanneer 'n geen-arbitrage model gebruik word.

Die studie poog vervolgens om te verbeter op bestaande metodes om die waarde van Suid-Afrikaanse opsies op effekte te bepaal. 'n Studie na die eienskappe van die Hull-White model (1990) het gedien as motivering om die model toe te pas op Suid-Afrikaanse opsies, wat Amerikaans van aard is. Die Hull-White model moes egter aangepas word alvorens dit toegepas kon word op Suid-Afrikaanse opsies, omdat laasgenoemde in plaas van die prys van die effek, die opbrengskoers as trefprys gebruik. Aangesien die numeriese oplossing van die model die huidige termynstruktuur van rentekoerse as inset gebruik, is die nulkoepon opbrengskurwe weereens hier aangewend. Optimale omstandighede waaronder opsies op effekte vroeg uitgeoefen word, is bespreek.

Die gekompliseerde aard van die Hull-White model het die ontwikkeling van 'n vereenvoudigde model vir beurs-verhandelde opsies op die South African Futures Exchange (SAFEX), geregverdig. 'n Beurs-verhandelde opsie word nie beïnvloed deur die korttermyn risikovrye rentekoers-veranderlike nie, aangesien die onderliggende instrument die effek se termynkontrak-koers is. Daar kan dus aanvaar word dat, in plaas van die korttermynrentekoers of die prys, die effek se opbrengskoers vir die ooreenstemmende termynkontrak gebruik kan word as die stogastiese veranderlike. 'n Model wat hierdie proses as uitgangspunt gebruik, is soortgelyk aan die Black-model (1976), maar spreek egter meeste van laasgenoemde se nadele aan.

Summary

An analysis of the
term structure of interest rates
and bond options in the
South African capital market

by

Linda Smit

Supervisor : Prof. FD van Niekerk
Department : Mathematics and Applied Mathematics
Degree : Ph.D.

The enormous impact of derivatives in the financial world necessitates the use of accurate valuation and risk-forecast models. This study focuses on the South African fixed income market and evaluates current models and procedures. All valuation models depend on certain assumptions and therefore have limitations. Certain inefficiencies experienced in the South African fixed income market are addressed, firstly, term structure analysis, and, secondly, bond option valuation.

Since the fixed income market is entirely based on the term structure of interest rates, it remains the most important input in the pricing of any fixed income derivative security. The South African bond market trades mainly in coupon bonds, and little or no data is available

for zero-coupon instruments. (The term structure of interest rates is determined by the zero coupon rates.) An improved bootstrap method for the derivation of a zero-coupon yield curve is proposed. The zero-coupon yield curve is the basis for pricing all vanilla products in the fixed income market and serves as an important input in pricing bond options using a no-arbitrage model.

The study hence attempts to improve on existing methods to value South African bond options. An analysis of the characteristics of the Hull-White model (1990) served as motivation to apply the model to South African over-the-counter bond options, which are American options. The Hull-White model has had to be adjusted for its application to South African bond options, as these options are traded on the yield-to-maturity of the bond, rather than the price. Since the numerical solution to the Hull-White model uses the current term structure of interest rates as an input, the zero-coupon curve is used. Optimum conditions for the early exercise of over the counter bond options are discussed.

The complexity of the Hull-White model encouraged the development of a simplified model for exchange-traded options on the South African Futures Exchange (SAFEX). An exchange-traded bond option has no short-term risk-free rate component, as the underlying instrument is the bond future and the only payment is being made to a margin account where interest is earned. Therefore, instead of using the risk-free rate as the stochastic variable, it is possible to assume that the yield-to-maturity of the bond, and not the price, follows a Brownian motion. A pricing model for options on the future yield of a bond is in many ways similar to the Black model (1976). However, the yield-based model addresses most of the disadvantages of the Black model.