CHAPTER 7

AN ALTERNATIVE PRICING MODEL FOR SOUTH AFRICAN EXCHANGE TRADED BOND OPTIONS[1]

Options on South African government bond future rates are traded through SAFEX. Although there used to be reasonable liquidity in the past, the trading volumes for SAFEX bond options have declined significantly, in comparison with trading in the OTC market.

[1]Results of work done in this chapter were published in *RISK* (Smit & Van Niekerk, 1999)

Trade is usually based on price-volatility which gives a certain option price as calculated using the Black model (1976) where the strike is given as the exercise yield. Options are marked to market daily to establish the margin payment. Hedging SAFEX options with OTC options, where volatility smiles or skews are used, becomes a complex exercise. There was a need to develop a model that prices SAFEX options more accurately and that can address the disadvantages of the Black model, taking into account the simplified nature of SAFEX options.

The standard pricing models for South African bond options are the well-known Black-Scholes model (1973) and Black model (1976). It is generally assumed that the Black model (discussed in Chapter 5) is sufficiently accurate to value options with less than a year to expiry. However, when the Black model is applied to SAFEX options, the pricing of in-the-money and out-the-money options by the Black model, and the valuation of different maturity bonds using price-volatility give rise to concern. These concerns are discussed in the next section.

Since the early-exercise value of SAFEX options is very small[2], marginal benefits accrue from using a no-arbitrage model such as the Hull-White model. It can be argued that the short-term risk-free rate only plays an indirect role in option pricing[3]. Consequently the study propose that the future rate, rather than the short-term risk-free rate is used as the stochastic variable.

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2 Hedging is done in the future, and not in the physical instrument, and therefore does not have any carry-cost.

3 The risk-free rate plays a role only in determining the future rate.

7.1 The Black model – a review

The popularity of the Black model can certainly be ascribed to its simplicity. The model is computationally efficient, requiring only a few basic parameters to calculate a reasonably accurate value for the option. Although there are several disadvantages, it is still, after 25 years, the most popular pricing model in most markets. It is preferred to models that are more accurate, but which are also much more complex and require the estimation of several parameters in order to obtain a more accurate fair value for the option. The calibration of more sophisticated models to the traded market value is a time-consuming process and therefore, many believe that the Black model is sufficiently accurate, especially for short-dated options (see Chapter 6).

In order to compete with the advantages of the Black model, any other model should, therefore, have the same ease of implementation and simplicity of use. Most important, though, it should give a more accurate estimation of the fair value of the option, especially of out-the-money options.

When using Black's model, it is necessary to calculate the forward bond price, the strike price and the price-volatility, using the forward yield and the strike yield. It is then assumed that the forward price, F, follows a geometric Brownian motion:

$$dF = \mu F dt + \sigma_F F dW \tag{1}$$

where μ is the drift and σ_F is the volatility of the forward bond price.

For the purposes of this chapter, a bond is discussed which pays a coupon m times a year at time t_i at a rate of c% and a nominal N at maturity time t_{mn}. The forward bond price, F, in equation (1) is given by the following non-linear function of the forward yield-to-maturity, Y:

$$F(Y) = \frac{c \cdot N}{m} \sum_{i=1}^{mn} e^{-Yt_i} + Ne^{-Yt_{mn}} \tag{2}$$

where all cash flows are discounted to the forward date.

The main disadvantages of the Black model and a motivation for the use of a yield-based model are discussed below.

7.1.1 Distribution of yield and price

The first problem with Black's model is the assumption that the underlying variable is lognormally distributed. A variable has a lognormal distribution if the natural logarithm of the variable is normally distributed. According to this assumption, the bond price can take any value between zero and infinity. In practice, however, the price of a zero-coupon bond is *bounded* and cannot have a value greater than its nominal value. Since the yield is non-negative and unbounded from above, it is therefore more accurate to assume that the *yield* of the bond has a lognormal distribution and follows a geometric Brownian motion.

In order to evaluate the lognormal distribution assumption empirically, an analysis was done on closing prices for the last nine years (data source: INet Bridge). Figure 7.1 gives the distribution of the logarithm of the R150 yield based on closing rates with 10-day intervals. Figure 7.2 shows the distribution of the 10-day *price*-returns of the R150 over the same period.

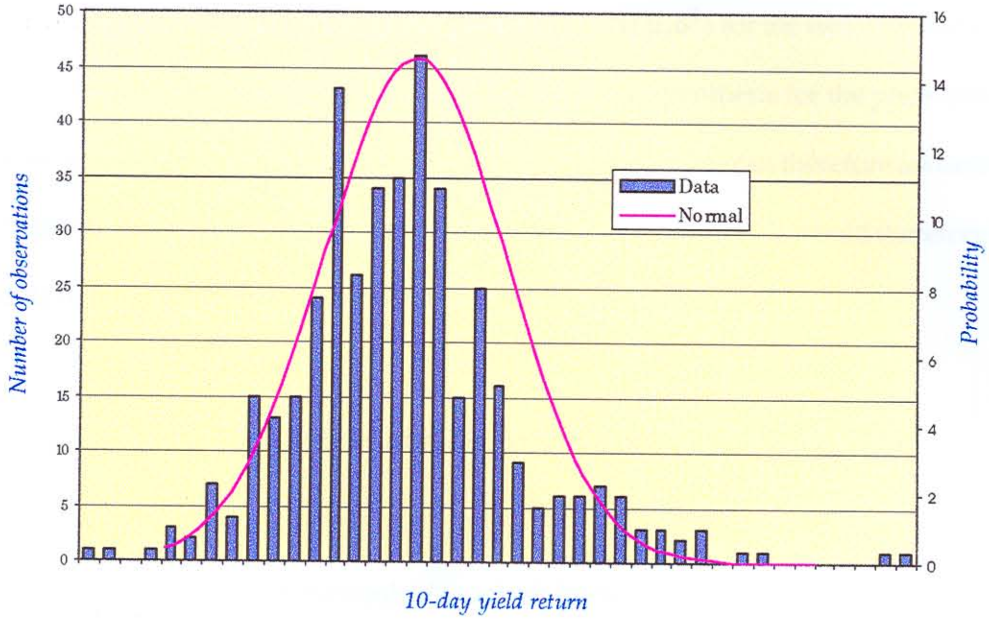

Figure 7.1: Distribution of 10-day yield returns for the R150 bond over a 9-year period

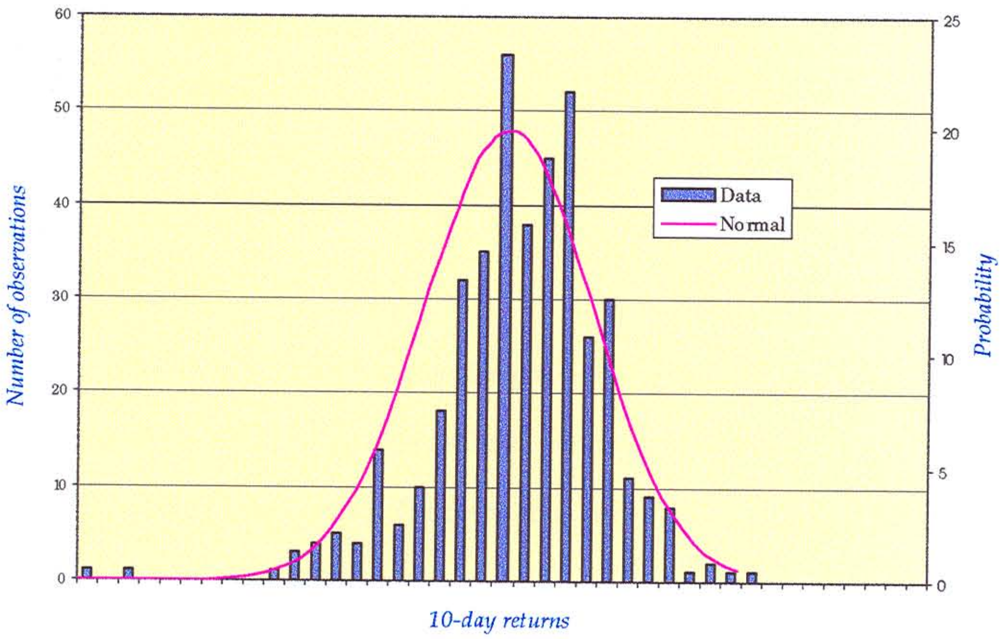

Figure 7.2: Distribution of 10-day price returns of the R150 bond over a 9-year period

A goodness-of-fit test, to test for normality, $H_0 : X \sim N(\mu, \sigma^2)$ for the two distributions, gave the following results: for a significance level of 0.01 the hypothesis for the price-distribution was rejected, and for the yield-distribution it was accepted. One can therefore assume that the yield-distribution is closer to lognormal than the price-distribution. It would therefore be more accurate to use the yield as the stochastic variable in a pricing model.

7.1.2 Yield-price correlation and the volatility skew

The Black model uses bond *price-volatility* which leads to a fundamental problem. Equation (1) implies that the instantaneous variance rate of the forward bond price, F, is equal to $(\sigma_F F)^2$, and is therefore proportional to the bond price. Since the bond price has a *negative* relationship with the yield (see equation (2)), the Black model therefore implies that the variance rate of the bond price or price-volatility is negatively correlated with the yield. Empirical data show, however, that the opposite is true. Correlation analysis of the daily volatility of the South African government R150 bond was done for the last nine years (data Source: INet Bridge). Figure 7.3 shows the results for the last three years, using 40-day price-volatility and daily closing yields.

A *positive* correlation was found between the price-volatility and the yield of the bond. The absolute value of the yield change over a 40-day period also showed a positive correlation, as well as yield-volatility against yield. The correlation coefficient, ρ, between the price-volatility and yield data was equal to 0.7605.

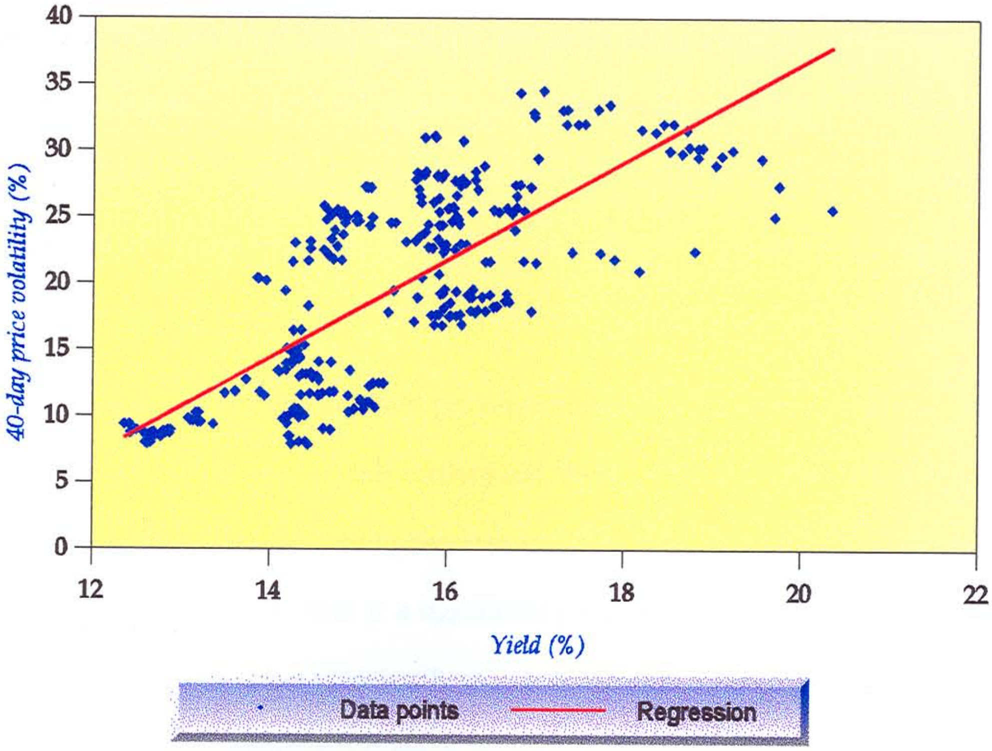

Figure 7.3: Correlation between price-volatility and yield-to-maturity

In order to estimate whether the correlation coefficient is significantly different from zero, a *t*-test was done, which resulted in a *t*-value of 20.05 for the above data. For a sample out of a population, a value of $t = 2.576$ ($\alpha = 0.005$) would occur only once in 100 random samples when drawn from a universe with a value of ρ equal to zero. The probability of a value of t equal to 20.05 is extremely small if the value of ρ is equal to zero. The conclusion is therefore that the correlation coefficient (ρ) is positive in the universe from which the sample was taken.

Based on the empirical evidence, one can conclude that the Black model fails to give a true reflection of the variance of the underlying instrument.

Using the *price* as the underlying variable in the Black model influences the valuation of out-the-money bond options. Participants in the market currently compensate for this mispricing by using a *volatility skew*. There is, however, usually under- or over-compensation, due to the uncertainty of where in- and out-the-money options should be trading.

7.1.3 Price-volatility

The Black model assumes a *constant* price-volatility for the life of the option. The price-volatility, however, depends mainly on the yield, the time to maturity and the convexity of the bond price curve. The pull-to-par-phenomenon of the bond price can have a big influence, especially when the option term is a significant proportion of the term of the bond. For an option on a future, the effect is insignificant, since the term-to-maturity from the future date stays constant. When hedging is done in the spot-market, however, this becomes a problem.

Since price-volatility is used in the Black model, different volatilities are used to price options on different bonds. One must therefore ensure that the volatility relations are always consistent, for example with a parallel shift in the yield curve. Using a yield-based model would solve this problem, since the same yield-volatility could be used for parallel shifts in the yield curve.

7.1.4 Yield versus short-rate and price

The work done by Longstaff (1990) on caps, floors and T-bills, suggests that the yield can be seen as the underlying variable that determines the price of a bond. Vasicek (1977) was the first to develop a term structure model which assumes that the price of a discount bond is determined by the assessment of the short-term rate process over the term of the bond. The bond price obtained in this way can then be converted to obtain the yield-to-maturity by goal-seeking equation (2). This yield-to-maturity rate therefore implies a certain expectation of future short- term rates.

Although term structure models use the short-rate, one can argue that it is better to model the stochastic process of the instrument in which hedging is being done, in order to stay delta neutral (Wilmott, 1998:441). One can therefore make the assumption that the yield-to-maturity of a bond contains all available information about the market expectation of the short-rate, and can therefore be seen as the underlying variable determining the price of a bond.

The particular problem to solve here is the valuation of an exchange-traded American option on the future yield of a long-term bond. A model is suggested where the value of the option is derived from the stochastic process followed by the future yield, using a constant yield-volatility. The model is solved numerically in order to provide for early-exercise.

7.2 The proposed yield-based model

A bond can be traded on its yield-to-maturity, instead of price, with a T-year option on the future T-year yield of the bond. If one assumes that the T-term future yield, Y, of the bond follows a stochastic process, then

$$dY = \mu Y dt + \sigma Y dW \qquad (3)$$

where μ is the drift and σ is the volatility of the yield.

If F is the T-term future price of the bond, and since F depends only on the future yield, Y, on that date, it follows from Ito's lemma (Björk, 1999) that

$$\begin{aligned} dF &= \left(F_Y\mu Y + \tfrac{1}{2}\sigma^2 Y^2 F_{YY}\right)dt + \sigma Y F_Y dW \\ &= \alpha_F F dt + \sigma_F F dW \end{aligned} \tag{4}$$

where

$$\begin{aligned} \alpha_F &= \frac{\mu Y F_Y + \tfrac{1}{2}\sigma^2 Y^2 F_{YY}}{F} \\ \sigma_F &= \frac{\sigma Y F_Y}{F} \end{aligned} \tag{5}$$

and where F_Y denotes the first derivative and F_{YY} denotes the second derivative to Y. If one defines V as the value of a contingent claim dependent on the level of the future yield of the bond, since V is a function of Y and t, it also follows from Ito's lemma that

$$\begin{aligned} dV &= \left(V_Y\mu Y + V_t + \tfrac{1}{2}\sigma^2 Y^2 V_{YY}\right)dt + \sigma Y V_Y dW \\ &= \alpha_V V dt + \sigma_V V dW \end{aligned} \tag{6}$$

where

$$\begin{aligned} \alpha_V &= \frac{\mu Y V_Y + V_t + \tfrac{1}{2}\sigma^2 Y^2 V_{YY}}{V} \\ \sigma_V &= \frac{\sigma Y V_Y}{V} \end{aligned} \tag{7}$$

One can set up a portfolio, Σ, consisting of two assets:

- the contingent claim, V; and
- the underlying bond future, with a price F at the future yield Y.

The relative portfolio can be denoted by (u_F , u_V). The bond future and the derivative are exchange-traded and interest is paid on the margin account (which is seen as a security for the contracts entered into). It initially costs nothing to enter into an exchange-traded option or future contract, therefore the initial investment is zero, while the portfolio value is given by

$$\Sigma \neq 0 \tag{8}$$

An immediate change in the value of the underlying instrument (the future yield) would result in a change in the value of the derivative and the future price of the bond. Therefore:

$$\begin{aligned} d\Sigma &= \Sigma\left[u_F(\alpha_F dt + \sigma_F dW) + u_V(\alpha_V dt + \sigma_V dW)\right] \\ &= \Sigma\left[(u_F\alpha_F + u_V\alpha_V)dt + (u_F\sigma_F + u_V\sigma_V)dW\right] \end{aligned}$$

For the relative portfolio,

$$u_F + u_V = 1 \tag{9}$$

For the dW-term to vanish, the following condition can be introduced:

$$u_F\sigma_F + u_V\sigma_V = 0 \tag{10}$$

Therefore,

$$d\Sigma = \Sigma\left[u_F\alpha_F + u_V\alpha_V\right]dt \tag{11}$$

which is a linear riskless portfolio. Since there is no initial investment, the principle of no-arbitrage states that

$$d\Sigma = 0 \tag{12}$$

Therefore,

$$u_F\alpha_F + u_V\alpha_V = 0 \tag{13}$$

From equations (9) and (10) it is clear that

$$u_V = \frac{\sigma_F}{\sigma_F - \sigma_V}$$
$$u_F = \frac{-\sigma_V}{\sigma_F - \sigma_V}$$

Using equations (5) and (7), one can then write the following:

$$u_V = \frac{F_Y V}{F_Y V - FV_Y}$$
$$u_F = \frac{-FV_Y}{F_Y V - FV_Y}$$

Substituting in equation (13), one gets

$$-\frac{FV_Y}{F_Y V - FV_Y}\left[\frac{\mu Y F_Y + \frac{1}{2}\sigma^2 Y^2 F_{YY}}{F}\right]$$

$$+\frac{F_Y V}{F_Y V - FV_Y}\left[\frac{\mu Y V_Y + V_t + \frac{1}{2}\sigma^2 Y^2 V_{YY}}{V}\right] = 0$$

Therefore,

$$V_t + \frac{1}{2}\sigma^2 Y^2\left[V_{YY} - \frac{F_{YY}}{F_Y}V_Y\right] = 0 \tag{14}$$

This gives a partial differential equation for the value of a derivative security dependent on the future yield of a bond. In order to provide for the early-exercise value for American options, one can therefore solve the above differential equation with an implicit finite difference method. In order to evaluate the results, the results are compared with that of the Black model.

7.3 Numerical solution

In order to solve equation (14), an implicit finite difference method was used, with

$$\frac{\partial V}{\partial t} = \frac{V_{i+1,j} - V_{i,j}}{\Delta t}$$

$$\frac{\partial V}{\partial Y} = \frac{V_{i,j+1} - V_{i,j-1}}{2\Delta Y}$$

$$\frac{\partial F}{\partial Y} = \frac{F_{i,j+1} - F_{i,j-1}}{2\Delta Y}$$

$$\frac{\partial^2 V}{\partial Y^2} = \frac{V_{i,j+1} + V_{i,j-1} - 2V_{i,j}}{\Delta Y^2}$$

$$\frac{\partial^2 F}{\partial Y^2} = \frac{F_{i,j+1} + F_{i,j-1} - 2F_{i,j}}{\Delta Y^2}$$

Substituting these equations into equation (14), results in the implicit scheme

$$a_j V_{i,j-1} + b_j V_{i,j} + c_j V_{i,j+1} = V_{i+1,j} \tag{15}$$

where

$$a_j = -\frac{1}{2}\sigma^2 j^2 \Delta t \left[\left(\frac{V_{i,j+1} + V_{i,j-1} - 2V_{i,j}}{V_{i,j+1} - V_{i,j-1}} \right) + 1 \right]$$

$$b_j = 1 + \sigma^2 j^2 \Delta t$$

$$c_j = \frac{1}{2}\sigma^2 j^2 \Delta t \left[\left(\frac{V_{i,j+1} + V_{i,j-1} - 2V_{i,j}}{V_{i,j+1} - V_{i,j-1}} \right) + 1 \right]$$

Since the value of the option at the expiry date is just the payoff of the option, the problem can be solved backwards. The value of the option at the expiry date is determined by the payoff given by a difference in price of $max[F(X) - F(Y_T), 0]$ for a call option, and $max[F(Y_T) - F(X), 0]$

for a put option, where Y_T is the yield at time T and X the strike rate. To obtain the value of the option at the boundaries of the finite difference grid, where Y reaches its minimum and maximum, it can be assumed that the gamma of the option at these points should be zero (see also Chapter 2, Section 2.4.2.1). Since the implicit scheme results in a tri-diagonal system, the procedure of *LU* decomposition (Wilmott, Dewynne & Howison, 1993) was used to solve the system. For the longer-dated options, 1000 time steps were used, with fewer for the shorter-dated options.

7.4 Empirical results

A comparison between the Black model and the yield-based model was done for the government R150 bond, maturing in February 2005 with a coupon of 12%, as well as the government R153 bond, maturing in August 2010 with a coupon of 13%. In-the-money, as well as out-the-money options were compared for different expiry dates. The same yield-volatility was used for both bonds.

7.4.1 Price differences

Figures 7.4 and 7.5 show the results of the R150 and R153 bonds respectively, using a yield-volatility of 20%. The data for these examples are set out in Tables 7.1 and 7.2. The results show clearly that, compared to the yield-based model, out-the-money call options are overvalued by the Black model, while out-the-money put options are undervalued.

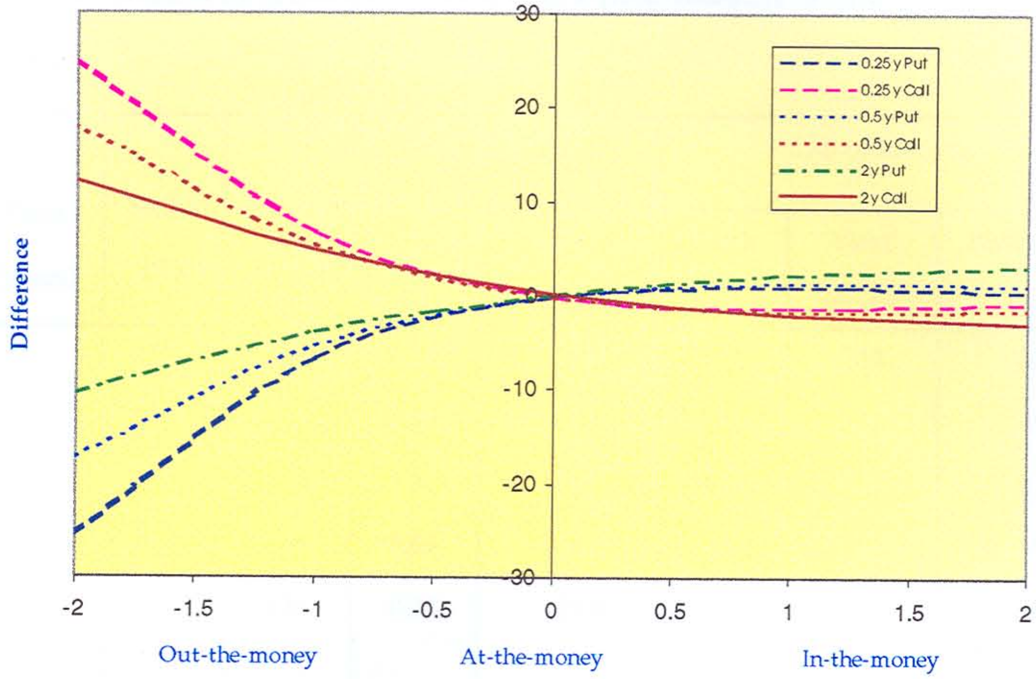

Figure 7.4: Pricing differences for R150 bond with a yield-volatility of 20%

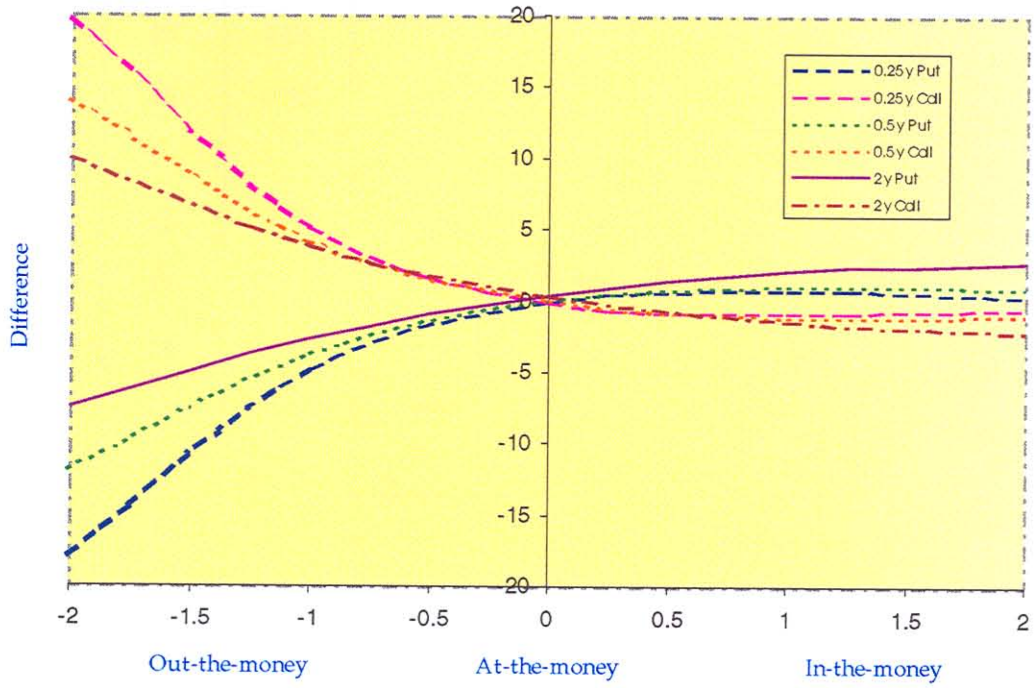

Figure 7.5: Pricing differences for R153 bond with a yield-volatility of 20%

Table 7.1: Results for R150 option prices with a yield-volatility of 20%

Term (years)	Strike (%)	Put option price			Call option price		
		Black model	Yield model	Difference (%)	Black model	Yield model	Difference (%)
0.25	12	7.19	7.14	0.6	0.16	0.12	24.9
	13	4.09	4.05	1.1	0.66	0.62	6.8
	14	1.87	1.87	0.0	1.87	1.87	0.0
	15	0.65	0.69	-6.6	3.92	3.96	-1.1
	16	0.16	0.20	-25.6	6.56	6.60	-0.6
0.5	12	7.31	7.22	1.3	0.53	0.43	18.0
	13	4.59	4.52	1.5	1.27	1.20	5.6
	14	2.56	2.56	0.0	2.56	2.56	0.0
	15	1.24	1.31	-5.2	4.41	4.48	-1.5
	16	0.52	0.61	-17.2	6.72	6.81	-1.3
2	12	7.04	6.80	3.4	1.95	1.71	12.2
	13	5.31	5.18	2.5	2.81	2.68	4.8
	14	3.89	3.87	0.3	3.89	3.87	0.3
	15	2.75	2.85	-3.8	5.17	5.27	-2.0
	16	1.88	2.07	-10.4	6.64	6.84	-2.9

Table 7.2: Results for R153 option prices with a yield-volatility of 20%

Term (years)	Strike (%)	Put option orice			Call option price		
		Black model	Yield model	Difference (%)	Black model	Yield model	Difference (%)
0.25	12	11.54	11.49	0.4	0.24	0.20	19.8
	13	6.44	6.39	0.8	1.03	0.97	5.2
	14	2.90	2.90	-0.0	2.90	2.90	-0.0
	15	1.00	1.05	-4.9	6.00	6.05	-0.8
	16	0.26	0.30	-17.9	9.86	9.91	-0.5
0.5	12	12.01	11.89	1.0	0.83	0.72	14.1
	13	7.39	7.31	1.2	2.03	1.94	4.2
	14	4.06	4.06	0.0	4.06	4.06	0.0
	15	1.97	2.04	-3.7	6.92	6.99	-1.1
	16	0.83	0.93	-11.9	10.36	10.46	-1.0
2	12	14.12	13.74	2.7	3.83	3.45	9.9
	13	10.47	10.25	2.0	5.51	5.29	3.9
	14	7.55	7.52	0.3	7.55	7.52	0.3
	15	5.29	5.43	-2.7	9.91	10.05	-1.4
	16	3.60	3.87	-7.5	12.52	12.79	-2.1

The opposite is true for in-the-money options, as expected. The relative difference is bigger for the short-term R150-bond than for the longer-term R153 bond. The early-exercise value of American options was found to be very small, as can be expected from short-term options on the future yield. Since the yield-model has been calibrated to the Black model, the small difference for at-the-money options is the early-exercise value.

Bonds with other maturity dates and coupons were also evaluated and similar results were obtained. These include the R162 maturing on 15 January 2002 (12.5% coupon), the R184 maturing on 21 December 2006 (12.5% coupon), the R157 maturing on 15 September 2015 (13.5% coupon) and the R186 maturing on 21 December 2026 (10.5% coupon). One big advantage of the yield-based model is that a constant yield-volatility can be used for different maturity bonds, while the price-volatility for the Black model must be manually adjusted to compensate for bonds with different maturity dates and coupons.

7.4.2 Delta differences

Since the delta of an option plays as important a role as the option itself (being the hedge), the yield-based model's delta was also compared with the delta given by the Black model. The results showed a relatively big difference between the two models, as displayed in Table 7.3. The delta given by the Black model is larger than the delta given by the yield-based model for call options, and smaller for put options.

The results for the delta indicate that a position will be under-hedged for put options, which is problematic when rates spike up. Call-options, on the other hand, will be over-hedged. The relative difference is larger for short-dated call options and longer-dated put options.

Table 7.3: Delta difference for R153 options with a yield-volatility of 10%

Term (years)	Strike (%)	Put option delta			Call option delta		
		Black model	Yield model	Difference (%)	Black model	Yield model	Difference (%)
0.25	12	-0.92	-0.94	-2.2	0.08	0.06	25.0
	13	-0.76	-0.78	-2.6	0.24	0.22	8.3
	14	-0.49	-0.51	-4.1	0.51	0.49	3.9
	15	-0.23	-0.25	-8.7	0.77	0.75	2.6
	16	-0.08	-0.09	-12.5	0.92	0.91	1.1
0.5	12	-0.84	-0.87	-3.6	0.16	0.13	18.8
	13	-0.68	-0.71	-4.4	0.32	0.29	9.4
	14	-0.49	-0.51	-4.1	0.51	0.49	3.9
	15	-0.3	-0.32	-6.7	0.7	0.68	2.9
	16	-0.15	-0.18	-20.0	0.85	0.82	3.5
2	12	-0.68	-0.73	-7.4	0.32	0.27	15.6
	13	-0.58	-0.63	-8.6	0.42	0.37	11.9
	14	-0.48	-0.53	-10.4	0.52	0.47	9.6
	15	-0.38	-0.43	-13.2	0.62	0.57	8.1
	16	-0.29	-0.34	-17.2	0.71	0.66	7.0

7.5 Summary

The valuation of options on the forward yield of a bond using a model based on the stochastic behaviour of the yield, rather than the price, has several advantages.

Firstly, when the model proposed here is compared with that of Black, the numerical results show that:

- the two models price at-the-money options similarly;
- the Black model overvalues out-the-money call options and undervalues out-the-money put options; and
- a small price difference occurs for in-the-money options.

Secondly, the yield-based model addresses most of the disadvantages of the Black model:

- The yield-based model uses the yield as the underlying instrument, which is closer to a lognormal distribution than the price of the bond.
- The same yield-volatility can be used for any maturity bond and the option price is automatically adjusted for the duration-difference. With the Black-model an independent price-volatility has to be estimated first, and recalculated every time the yield or volatility changes.
- The yield-based model provides for the pull-to-par effect of bonds when pricing long-term options. The Black model does not provide for this and the decline in volatility for longer options has to be adjusted by adjusting the price-volatility.
- The yield-based model can also value options on swaps (swaptions) in a similar way, making arbitrage between bond options and swaptions easier. (See Appendix B for the usual pricing convention of swaptions.) The yield-based model is consistent with the pricing of swaptions using the Black model.

The main disadvantage of this model is that it assumes that hedging will be done using the future yield. The model thus neglects movement in the short-rate for over-the-counter spot hedges. When hedging is done in the spot market, the early-exercise value increases for certain options as described in Chapter 6, since it is dependent on a certain carry or repo rate. The yield-based model is therefore well-suited for options on the SAFEX future. For OTC options, it is important to compensate for the early-exercise value.