CHAPTER 6

THE HULL-WHITE MODEL APPLIED TO SOUTH AFRICAN OTC BOND OPTIONS

South Africa has an actively traded bond market where options are traded on the most liquid government bonds. Bond turnover in 1998 was $1.7 trillion equivalent in nominal terms, according to the Bond Exchange of South Africa. A significant over-the-counter (OTC) bond options market has established itself, although option liquidity is concentrated in current government funding bonds. Both options and bonds are traded on yield-to-maturity (referred to as yield). Almost all bond options that are traded are American options.

The effects of the 1998 emerging market crisis, which generated significant losses for banks and hedge funds, again raised several questions concerning the accurate valuation of derivative instruments. The crisis emphasized the imperfections of the Black-Scholes model previously identified by Black (1988).

The nature of American options necessitates the use of numerical models for valuation purposes. The numerical solution of the Hull-White model (1990a) addresses most of the disadvantages of other bond option pricing models and can be successfully applied to South African bond options. Instead of using the bond price as the stochastic variable, the model assumes that the short rate, $r(t)$, follows a mean reverting stochastic process and has a lognormal distribution:

$$dr = (\theta(t) - ar)dt + \sigma_r dW \qquad (1)$$

It is further assumed that the term structure implies a certain expectation of future short rates, and that the expected short rate *process* can therefore be used to determine any bond price. Hull and White (1993) use a trinomial tree to model the short rate process and ensure that the initial term structure is matched before the bond option is valued.

The key characteristics of the numerical solution of the Hull-White model are the following:

- It incorporates mean reversion of interest rates.
- The pull-to-par effect is determined analytically for both the exact solution and the numerical solution.
- The model is consistent with the initial term structure of interest rates.
- It incorporates the early-exercise value of American options.

6.1 Characteristics of South African bond options

The major difference between South African OTC bond options and options in other countries is the fact that South African bonds are traded on the yield-to-maturity and settled on the price[1] while most other bonds are traded on the price. The strike of a South African bond option is, therefore, also given as a yield-to-maturity. Options on many different maturity bonds are traded. Of these, the R153 government bond is the most liquid.

Since the Black model is generally used to price South African bond options, the implied forward *price* is used at the current forward yield. In practice, however, the delta hedge is done in the spot market. The forward price is calculated using the spot yield and the equivalent risk-free rate. The strike price used in the Black pricing formula is the price on the expiry date at the strike rate.

When an option is early-exercised, the actual strike price can be different from the price when the option is exercised at a later stage or on the expiry date of the option. The actual maturity date of the bond exercised stays the same, irrespective of the date when it is exercised. This practice differs from that in countries where the bonds are traded on the price and options are traded on a specific strike price. In such markets, the holder of an American option has the right to buy a fixed term bond, say a 10 year bond, before or on expiry of the option at a certain strike price. When a German bond option, for instance, is early-exercised, the strike price of the original contract stays the same, for example at DM 90.00, while the maturity date of the bond is adjusted in order for the term-to-maturity to stay the same as on the original negotiated contract.

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[1]This is also the case in some European countries, for instance, Finland.

In this study, the pricing of bond options using the Hull and White (1990) analytical solution discussed in the previous chapter is evaluated, as well as the numerical solution using a trinomial tree. The analytical results are then compared to the numerical solution of the Hull and White trinomial tree (1993) for European and American options. The difference between pricing the option with a strike price versus pricing it with a strike yield is evaluated. Finally, the influence of the term structure on the pricing of the options is shown.

6.2 The Hull-White trinomial tree

The analytical solution of the Hull-White model, as discussed in Chapter 5, gives suitable results for European style options. For American style options, a trinomial tree, as constructed by Hull and White, gives more accurate results than the analytical solution, due to the fact that the numerical solution does provide for the early-exercise of options.

The trinomial tree uses discrete time steps to construct a tree of possible values for the short rate, *r*, in the future. (See also the Rendleman-Bartter model discussed in Chapter 5). At time *t*, the price of a bond maturing at time *s* can be determined in terms of the short rate *r* by using the Hull and White analytical formula explained in Chapter 5:

$$P(t,s,r) = A(t,s)\, e^{-B(t,s)r} \tag{2}$$

where

$$B(t,s) = \frac{1 - e^{-a(s-t)}}{a}$$

and

$$\ell n\, A(t,s) = \ell n \frac{P(0,s)}{P(0,t)} - B(t,s)\frac{\partial \ell n\, P(0,t)}{\partial t} - \frac{\sigma^2}{4a^3}(e^{-as} - e^{-at})^2(e^{2at} - 1)$$

Using the short rate given at each node in the tree, the possibility of early exercising can be evaluated for American options.

6.2.1 *The procedure*

The interest rate tree is constructed in two stages. First, one supposes that there is a variable x which is initially zero and follows the process described in Chapter 5, Section 5.3.2:

$$dx = -ax\,dt + \sigma dW \tag{3}$$

which forms a tree which is symmetrical around the $x = 0$ line. The variable

$$x(t+\Delta t) - x(t)$$

is normally distributed and, if terms of a higher order than Δt are ignored, the expected value is

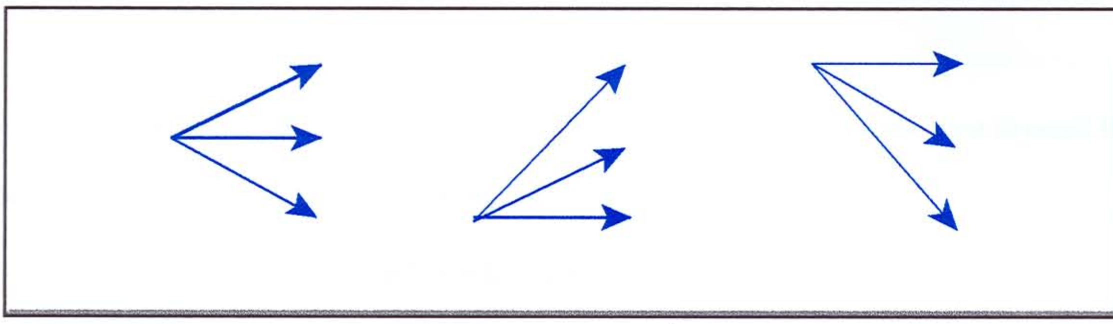
$$-a\,x(t)$$

and the variance is

$$\sigma^2 \Delta t$$

If the length of each time step, Δt, is known, the change of the variable x is normally set equal to

$$\Delta x = \sigma\sqrt{3\Delta t}$$



(*a*) (*b*) (*c*)

Figure 6.1: Possible movements for the short-term rate

Node (i,j) on a trinomial tree, is where $t = i\Delta t$ and $x = j\,\Delta x$. For $j_{min} < j < j_{maz}$, the standard branching process (*a*) in Figure 6.1 is chosen. For a sufficiently large positive j, the branching process in (*c*) is chosen, while (*b*) is chosen when j is sufficiently negative. Hull and White show that $j_{\max}$ should be set equal to the smallest integer greater than

$$0.184 / (a\Delta t)$$

and then

$$j_{min} = -\, j_{\max}$$

If p_u, p_m and p_d are defined as the probabilities of following the highest, middle and lowest branches at each node, the probabilities should match the expected change and variance in x over the next interval Δt. The sum of the probabilities must also equal one. For node (i,j) the three necessary equations for p_u, p_m and p_d are

$$p_d + p_m + p_u = 1$$

$$p_u(k+1)\Delta x + p_m k\Delta x + p_d(k-1)\Delta x = E(\Delta x) = -aj\Delta x\Delta t$$

$$p_u(k+1)^2\Delta x^2 + p_m k^2\Delta x^2 + p_d(k-1)^2\Delta x^2 = E(\Delta x^2) = \sigma^2\Delta t + a^2 j^2\Delta x^2\Delta t^2$$

These equations can be solved in order to obtain p_u, p_d, and p_m for each branching process.

The second stage involves converting the x-tree into a tree for r. It is important to recall the following transformation in Chapter 5:

$$\alpha(t) = r(t) - x(t), \qquad x(0) = 0$$

and

$$d\alpha = [\theta(t) - a\alpha(t)]dt$$

Using an integration factor and $\alpha(0) = r(0)$, it follows that

$$\alpha(t) = e^{-at}\left[r(0) + \int_0^t e^{aq}\,\theta(q)\,dq\right] \tag{4}$$

The $\theta(q)$ function in the above equation can be calculated from the initial term structure:

$$\theta(t) = F_t(0,t) + aF(0,t) + \frac{\sigma^2}{2a}(1 - e^{-2at}) \tag{5}$$

Integrating and substituting this equation into equation (4), this yields the following[2]

$$\alpha(t) = F(0,t) + \frac{\sigma^2}{2a^2}(1 - e^{-at})^2 \tag{6}$$

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[2]Note that $r_0 = F(0,0)$

If one defines α_i as the value of r at time $i\Delta t$ (on the r-tree) minus the corresponding value of x at time $i\Delta t$ on the x-tree, and one defines $Q_{i,j}$ as the present value of a security that has a payoff of 1 unit if node (i,j) is reached, and zero otherwise, then the value of $Q_{i,j}$ would be given as the discounted value of the expectation of reaching node (i,j).

The next step is to use a forward induction procedure which ensures that the initial term structure is matched exactly. The α_i- and $Q_{i,j}$ - values should be calculated in such a way that the initial term structure is matched exactly. This can be done by forward induction:

$$\alpha_m = \frac{\ell n \sum_{j=-n_m}^{n_m} Q_{m,j} \, e^{-j\Delta r \Delta t} - \ell n P_{m+1}}{\Delta t}$$

where P_{m+1} is the price of a discount bond at time $(m + 1)\Delta t$ and n_m is the number of nodes below and above the centre. The value for $Q_{i,j}$, with $i = m + 1$ can then be calculated:

$$Q_{m+1,j} = \sum_k Q_{m,k} \, q(k,j) \exp[-(\alpha_m + k\Delta r)\Delta t]$$

where $q(k,j)$ is the probability of moving from node (m,k) to node $(m+1,j)$ and the summation is taken over all values of k for which this probability is non-zero.

The above Hull-White numerical procedure was programmed in a Fortran computer programme and verified using the data given in Hull (1997) and also the data in Pelsser (1996). Since South African bond options are options to buy or sell bonds at a certain strike *rate* and not price, allowance was made for this trading convention by adjusting the model. Therefore, the study evaluated bond options for both a price strike and a yield strike. The results are discussed in the sections below for both discount bonds and coupon bonds.

6.3 Influence of the strike convention

There are two ways of expressing the strike of a bond option – either as a bond price, X, or as a bond yield, $x\%$. The particular convention used does *not* affect the price of *European* options, but it does affect the price of American options. American options on zero-coupon bonds and coupon-bearing bonds are compared for both conventions.

6.3.1 Options on zero-coupon bonds

6.3.1.1 Price strike

One can consider an option on a zero-coupon bond maturing at time s. If one assumes that the option expires at time T and that the strike, X, is given in terms of the bond price, for a put option, one gets the following payoff at expiry time T:

$$\max[X - P(T,s,r),0] \tag{7}$$

A T-term *American* put option on early exercise at time t (where $t < T$) gives the holder the right to sell a (s-T)-year zero-coupon bond (maturing at time $t+s-T$), for a price X, resulting a profit of

$$X - P(t,t+s-T,r)$$

If f_{ij} denotes the value of the option at time $t_i < T$ at the j-th vertical point on the spot interest rate tree, when the interest rate is r_{ij}, then

$$f_{ij} = \max\left[X - P(t_i,t_i+s-T,r_{ij})\ ,\ e^{-r_{ij}(t_{i+1}-t_i)}\left(\sum_{q=-1}^{1} p_{i,j,q+\delta} f_{i+1,j+q+\delta}\right)\right] \tag{8}$$

where p denotes the appropriate probability and δ = -1, 0 or +1, according to the branching process in the trinomial tree (see Figure 6.1). It is clear that the option should only be early-exercised when the profit is greater than the intrinsic value of the option. The term-to-maturity of the underlying bond being exercised is always $(s-T)$, irrespective of the time t it is exercised. This implies that the early-exercise value of an option with a price-based strike depends only on the short-term interest rate r, and is not directly dependent on the particular time t. Figure 6.2 illustrates the payoff as a function of the short-term rate, which holds for any $t \leq T$.

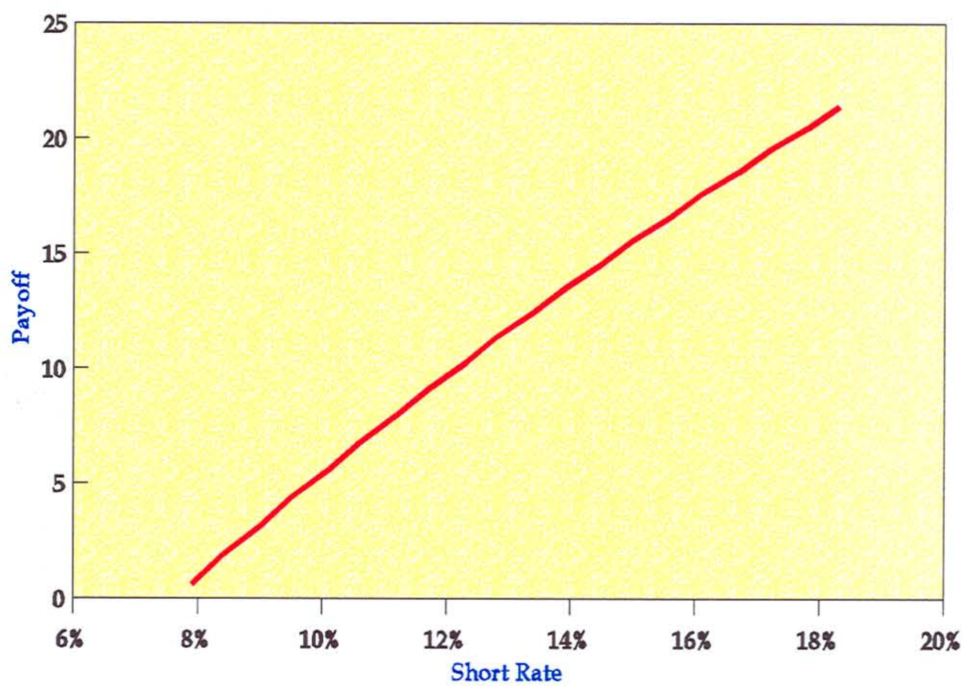

Figure 6.2: **Payoff as a function of the short rate for a 5 year put option on a 9-year bond with a strike price of 0.72 for any $t \leq T$**

In order to compare the prices of European and American bond options, the same example is used as that given by Pelsser (1996:70), but shorter dated options have been added. From Table

6.1 it is evident that the prices of *short dated* European and American put options corresponds to two decimal places. The error becomes significant only for longer-term options.

Table 6.1: Prices for put option on a 9-year zero-coupon bond (price strike) (using 50 time steps, a = 0.10, σ = 0.01 and zero-curve given by $z(t) = 0.08 - 0.05e^{-0.18\,t}$)

Option term	Strike price	European option		American option	Eur vs Amer difference(%)
		Analytical	Numerical		
0.25	0.54	81.01	80.74	80.74	0.00
0.50	0.54	77.77	77.74	77.74	0.00
0.75	0.55	116.56	116.93	116.93	0.00
1.00	0.55	97.91	98.20	98.20	0.00
2.00	0.58	126.51	126.45	126.49	-0.03
3.00	0.63	192.97	192.99	194.00	-0.52
5.00	0.72	135.84	137.36	145.02	-5.58
7.00	0.85	97.34	97.89	114.11	-16.57

6.3.1.2 Yield strike

Next, one can consider a put option where the strike is given in terms of the yield-to-maturity, for example $x\%$. The option gives the owner of a European-style option the right to sell a discount bond (maturing at time s) at expiry of the option at a rate of $x\%$. When American style options are early- exercised at time t, the owner also has the right to sell a discount bond with maturity s priced at a yield-to-maturity of $x\%$. On the expiry date of a European or American option, $t{=}T$, the payoff is exactly the same as for the equivalent price-based option given in equation (7), where x was chosen so that

$$X = e^{-x(s-T)}$$

A European option price is therefore not affected by the yield-strike convention. On the other hand, when early-exercising an American option, there is a difference in the payoff since the yield-strike convention gives an additional advantage by adding a time-dimension to the early-exercise value. The payoff at time t, $t<T$, is given by:

$$e^{-x(s-t)} - P(t,s,r)$$

The payoff is therefore not only affected by the short rate, but it is also affected by the time t when the option is early-exercised, as is illustrated in Figure 6.3.

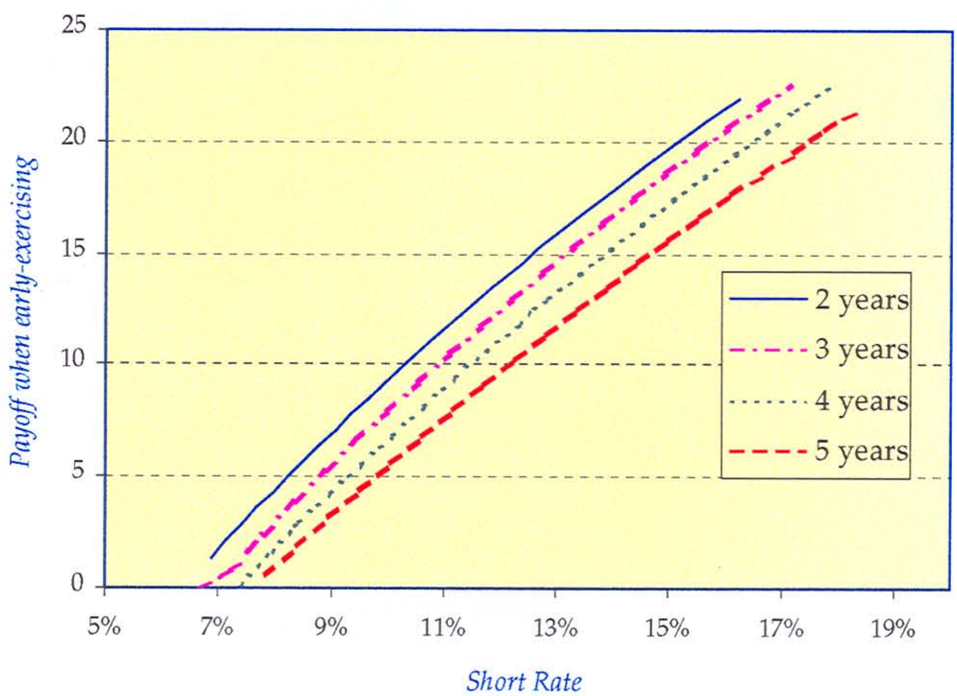

Figure 6.3: **Payoff as a function of the short rate for a 5-year put option on a 9-year bond with a strike yield of 8.21% for various t≤T**

For the same short rate, the payoff for early-exercising the option is greater for smaller t. This should influence the pricing of an American option significantly, as is shown in Table 6.2. The yield-strike was chosen to give, at the expiry of the option, approximately the same price strike as in Table 6.1.

Table 6.2: Prices for put option on a 9-year zero-coupon bond (yield strike) (using 50 time steps, $a = 0.10$, $\sigma = 0.01$ and zero-curve given by $z(t) = 0.08 - 0.05e^{-0.18\,t}$)

Option term	Strike yield (%)	European option Analytical	European option Numerical	American option	Eur vs Amer difference(%)
0.25	7.042	81.05	80.78	80.78	0.00
0.50	7.249	77.82	77.79	77.79	0.00
0.75	7.247	116.44	116.81	116.81	0.00
1.00	7.473	97.90	98.20	98.20	0.00
2.00	7.782	126.47	126.42	126.89	-0.37
3.00	7.701	192.89	192.91	199.48	-3.41
5.00	8.213	135.80	137.32	169.80	-23.66
7.00	8.126	97.33	97.88	204.73	-109.16

6.3.2 Options on coupon-bearing bonds

6.3.2.1 Price strike

When pricing an option on a coupon-bearing bond, a slightly different approach than given in Section 6.3.1 is followed. One can consider a T-term option on an s_n-term coupon-bearing bond, based on a strike *price*. A coupon bond can be seen as a portfolio of discount bonds, one discount bond for every coupon maturing at time s_i, plus one discount bond for the principal maturing at time s_n. An European option on a coupon bond can therefore be treated as a portfolio of options on the individual discount bonds in the portfolio as described by Jamshidian (1989). Only cashflows that are due after the expiry of the option are considered in the pricing of the option. In the example, a total of n cashflows is assumed.

Since the strike price is also equivalent to the sum of discounted cashflows, the original strike price is used to solve for the 'strike' short-term interest rate, r^*, where the Hull-White analytical formulas are used to price the individual discount bonds. One then uses r^* to obtain the individual strike prices, X_i for the underlying discount bonds maturing at time s_i with a 1 unit nominal. The individual options on the different cashflows are therefore separated in order to price them independently according to the method used for discount (zero-coupon) bonds. The individual option prices are then added to obtain a single price for the option on the coupon-bearing bond. It follows that the payoff for a put-option at the expiry of the option when the spot short-term interest rate equals r is

$$\sum_{i=1}^{n} c_i \max[X_i - P(T,s_i,r),\ 0] \tag{9}$$

where the i-th cash flow is given by c_i. Therefore, the price of an option on a coupon-bearing bond equals the sum of n options on the underlying discount bonds. The price of a put option is then given by

$$p = \sum_{i=1}^{n} p_i \tag{10}$$

where p_i denotes the price of the individual options. When American options are priced, the individual options on discount bonds are priced using the trinomial tree approach and evaluating for the desirability of early exercising, using equation (8). Equation (10) therefore holds for European and American options, where the strike is given in terms of the price.

The results in Table 6.3, for example, give the prices of a put option on an 8% coupon bond, where coupons are paid semi-annually and the strike is given in terms of a bond price. When one compares the difference between American and European option prices for a coupon bond in Table 6.3 to the results for a zero-coupon bond in Table 6.1, one sees that there is an almost insignificant change in the differences.

Table 6.3: Prices for put option on a 9-year, 8% coupon bond (price strike) (using 50 time steps, a = 0.10, σ = 0.01 and zero-curve given by $z(t) = 0.08 - 0.05e^{-0.18t}$)

Option term	Strike price	European option		American option	Eur vs Amer difference (%)
		Analytical	Numerical		
0.25	1.084	81.52	81.78	81.78	0.00
0.50	1.045	78.05	78.38	78.38	0.00
0.75	1.060	116.34	116.84	116.84	0.00
1.00	1.023	97.33	97.93	97.93	0.00
2.00	0.997	125.16	124.76	124.79	-0.02
3.00	0.997	190.50	191.27	192.16	-0.47
5.00	0.980	134.50	134.93	142.59	-5.68
7.00	0.992	97.58	98.83	115.36	-16.73

6.3.2.2 Yield strike

The strike convention has no influence on European options and equation (10) still holds for coupon-bearing bonds using the yield strike. However, the pricing of an American option should be treated in a different way. Since early-exercising an option on the yield of a bond involves not only those coupons that are due after the expiry date of the option, but also those due between the early-exercise date and the expiry date, it follows that equation (10) does not hold for the price of an American option. The value of the option at time $t_i<T$ in the trinomial tree is therefore

$$f_{ij} = \max\left[\sum_{k=1}^{m} c_k\left(e^{-x(t_i-s_k)} - P(t_i,s_k,r_{ij})\right),\ e^{-r_{ij}(t_{i+1}-t_i)}\left(\sum_{q=-1}^{1} p_{i,j,q+\delta} f_{i+1,j+q+\delta}\right)\right] \tag{11}$$

where m is the number of cashflows from time t_i to the maturity of the bond. The option price is therefore determined by a different process as followed by Jamshidian (1989), since the number of coupons at different time-steps may differ. This adds to the bigger difference already obtained for American yield-strike options on zero-coupon bonds shown in Table 6.2.

When one compares the results in Table 6.3 with similar options, but uses the yield-strike convention, one obtains the results shown in Table 6.4 (overleaf).

Table 6.4: Prices for put option on a 9-year, 8% coupon bond (yield strike) (using 50 time steps, a = 0.10, σ = 0.01 and zero-curve given by $z(t) = 0.08 - 0.05e^{-0.18\,t}$)

Option term	Strike yield (%)	European option		American option	Eur vs Amer difference (%)
		Analytical	Numerical		
0.25	6.89	80.46	80.74	80.74	0.00
0.50	7.15	78.36	78.36	78.68	0.00
0.75	7.20	116.48	116.98	116.98	0.00
1.00	7.47	97.25	97.85	97.86	-0.01
2.00	7.89	126.89	126.61	127.27	-0.52
3.00	7.90	191.72	192.58	200.85	-4.29
5.00	8.42	134.71	135.11	174.26	-28.98
7.00	8.27	97.57	98.82	226.86	-129.57

6.3.2.3 Conclusion

The above tables indicate that there is an insignificant difference between short-dated (less than 1 year) European and American put options. The results show clearly that *short-dated* European and American put options in this example can be priced accurately using the *analytical* Hull and White model. The *Black model* can therefore also be used accurately, by adjusting the volatilities for different option terms. The early-exercise value of American options becomes significant only for longer-dated options (more than 1 year). The pricing difference (or early-exercise value) becomes even more significant for options based on the yield-strike convention, but is still fairly priced for short-dated options. The shape of the term structure, could, however, influence these results.

6.4 Influence of the shape of the term structure

The term structure in the previous examples shows a sharply increasing shape. These results can be compared to an example where a sharply *decreasing* term structure was used. Table 6.5 shows a significant increase in the early-exercise value of the put option (compared to Table 6.4). An obvious reason for this sudden increase in the price of an American option lies in the shape of the term structure.

Table 6.5: Prices for put option on a 9-year, 8% coupon bond (yield strike) (using 50 time steps, $a = 0.1$, $\sigma = 0.01$ and zero-curve given by $z(t) = 0.03+0.05e^{-0.18t}$)

Option Term	Strike Yield (%)	European option		American option	Eur vs Amer difference (%)
		Analytical	Numerical		
0.25	4.0	171.93	172.21	221.15	-28
0.50	4.0	152.82	152.32	227.71	-50
0.75	4.0	138.57	138.88	232.83	-68
1.00	4.0	126.50	127.01	236.34	-86
2.00	3.8	130.89	132.33	386.12	-192
3.00	3.5	149.99	150.15	640.20	-326
5.00	3.2	123.12	124.37	894.88	-620
7.00	2.5	107.64	108.98	1540.02	-1313

For an increasing term structure (where interest rates are expected to rise) early-exercising a put option is not optimal, since the profit is greatest when the interest rate is even higher. The relatively inexpensive rate at which bonds can be carried (financed) makes it profitable to carry the bonds until the expiry date. A short-dated in-the-money put option will therefore not be early-exercised.

A market involving a decreasing term structure (where interest rates are expected to decline) gives the opposite effect. The high cost involved in carrying bonds causes lower future rates. A put option with a certain strike would be less expensive in this market than in a market with an increasing term structure. However, an unexpected increase in interest rates would make the option more likely to be early-exercised than previously. The reason for this is that the holder of a covered (hedged) put option has a long position in bonds and higher rates would therefore increase the cost-of-carry. The expectation of decreasing rates still holds, which makes the profitability of early-exercising an in-the-money put option greater. It is thus evident that, although the European option is cheaper in this market than in the increasing market, the added value for an American option is greater.

It is obvious that the opposite holds for a call option, giving more early-exercise value for a yield curve with a positive slope than with a negatively sloping yield curve. Using the same example as that used in Table 6.4, but for a call option, one obtains the results set out in Table 6.6.

6.4.1 Conclusion

The results show that there is a small pricing difference between short-dated American and European *put* options in a market with an increasing term structure (and *call* options in a market with a decreasing term structure). However, for the opposite scenario, the pricing difference becomes significant, as was shown above.

Table 6.6 Prices for call option on a 9-year, 8% coupon bond (yield strike) (using 50 time steps, a = 0.10, σ = 0.01 and zero-curve given by $z(t) = 0.08 - 0.05e^{-0.18\,t}$)

Option term	Strike yield (%)	European option		American option	Eur vs Amer difference (%)
		Analytical	Numerical		
0.25	6.89	100.54	100.61	146.09	-45
0.50	7.15	191.09	191.42	305.92	-60
0.75	7.20	183.18	183.78	338.41	-84
1.00	7.47	253.07	253.72	520.93	-105
2.00	7.89	273.26	273.09	790.81	-190
3.00	7.90	192.66	193.69	787.72	-307
5.00	8.42	167.22	167.81	1099.08	-555
7.00	8.27	69.68	71.01	985.32	-1288

6.5 Calibration of the volatility parameters

The main problem in using more sophisticated models such as the Hull-White model is to estimate the volatility parameters, σ_r and a. In order to derive the full benefit of the more accurate Hull-White model, one must calibrate the Hull-White model to liquid options traded in the market.

Since the market uses the Black model, the Hull-White model should therefore be calibrated to the Black model using short-dated options. In the case of options where the pricing difference between European and American options is small (either put or call options, depending on the market), the prices and implied volatilities of at-the-money options given by the Black model can be used to solve the volatility-parameters (σ_r and a) for the analytical Hull-White model – which is similar to the Black model. These parameters can then be used to price longer-dated options by using the Hull and White trinomial tree approach, thereby reducing the pricing error.

6.5.1 *Estimation of parameters for zero-coupon bonds*

The results in the previous sections show that the pricing error is small when short-dated American *put* options are priced using a European model in a market with an *increasing* (positive sloping) yield curve (and *call* options in a market with a *decreasing* or negative sloping yield curve). Therefore, since short-dated American bond options are priced in the market using the Black model, these prices and implied volatilities can be used to imply the volatility-parameters, σ_r and a in the analytical Hull-White model. Once the parameters have been obtained, they can be used in the numerical solution to give the prices for American options.

If one compares equations (4) and (34) in Chapter 5[3] it indicates that, when valuing the same option with these two models, the option prices can only be equal if

$$\sigma^* = \sigma_F \sqrt{T}$$

Therefore,

$$\sigma_F(s) = \frac{\sigma_r}{a}\left[1 - e^{-a(s-T)}\right]\sqrt{\frac{1 - e^{-2aT}}{2aT}}$$

where $\sigma_F(s)$ is derived from the market price of an option on a discount bond maturing at time

[3]The Black model and Hull-White model respectively.

s. One can solve the parameters σ_r and a by obtaining a best fit for the function $\sigma_F(s)$. This gives a volatility curve that serves as an input for the Hull-White model.

Once the calibrated Hull-White volatility parameters are known, they can be used to obtain a more accurate estimate of the price of American call options in a market with a positive-sloping yield curve, or American put options in a market with a negative-sloping yield curve, by using the Hull-White numerical method. Longer-dated options can also be priced this way, although an estimate of future volatility is required to do so.

6.5.2 Estimation of parameters for coupon-bearing bonds

The volatility curve obtained in Section 6.5.1 indicates the volatility against the maturity date of a *zero-coupon* bond. Since a coupon bond has several cashflows, it cannot give the same result. If one has only option market data for coupon bonds, and no zero-coupon bond data, one must approximate the volatility curve for zero-coupon bonds.

An option price is influenced by the market consensus of the bond price volatility for the particular option term. If one assumes that a bond's price volatility (as used in the Black model) is a combination of the implied volatilities of the individual cashflows, one can express the volatility as a function of the average time of cashflows, or the *duration* of the bond. Using this approximation, the volatilities as a function of duration give an implied volatility curve for a certain option term T. This then implies that the volatility parameters, σ_r and a, result in a volatility curve against *duration*. Figure 6.4 shows an example of a volatility curve for an option term of 6 months using data for November 1999. Fitting the curve through the data points produces the following values: $a = 0.055$ and $\sigma_r = 0.0295$. The graph also shows the implied curve for a 1-year option.

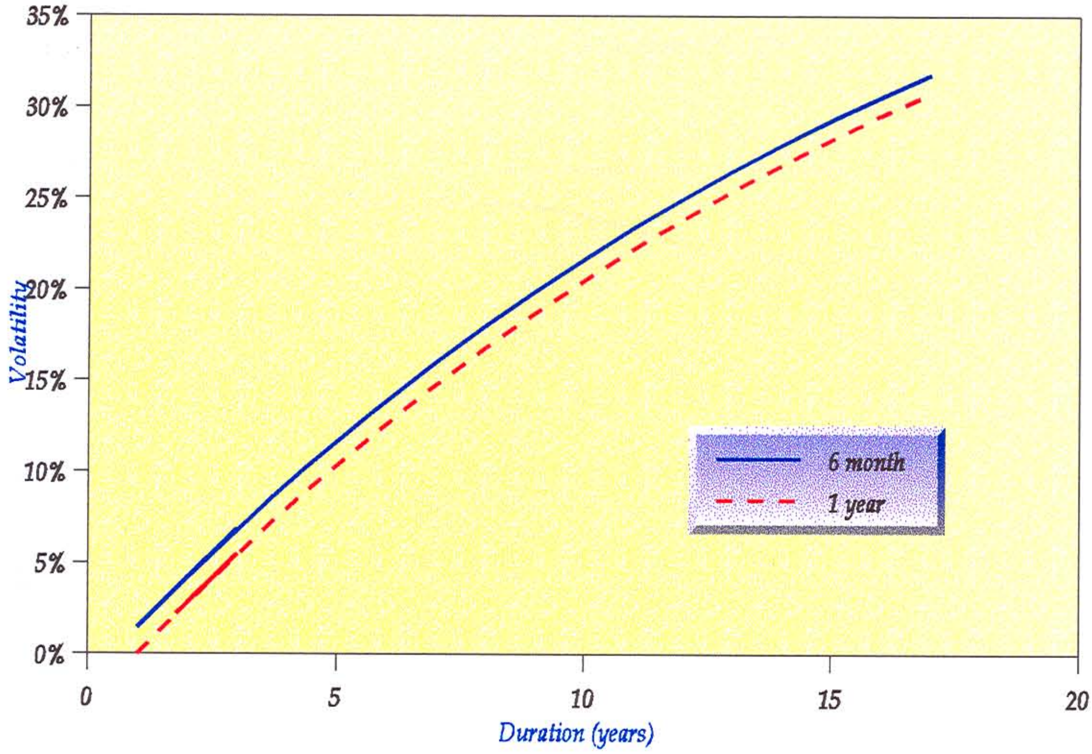

Figure 6.4: A fitted volatility curve for an option term of 0.5 years, and an implied curve for 1 year, using the same parameters

The above method gives a reasonable approximation of the volatility curve, but is not perfect, since the volatility of one bond is influenced by the volatility of another bond used in the construction of the curve. If there are any abnormalities in the implied volatilities of different bonds, the fitted curve smoothes out these discrepancies, giving an approximated value.

6.6 Empirical study for South African OTC bond options

An empirical study was done for South African OTC bond options. It was assumed that the most liquid at-the-money government R150 (12% coupon, maturing 28 February 2005) and R153 (13% coupon, maturing 31 August 2010) options traded in the South African market are the benchmark options. The Hull-White model was then calibrated to these options. The yield curve in November 1999 was used, as is shown in Figure 6.5.

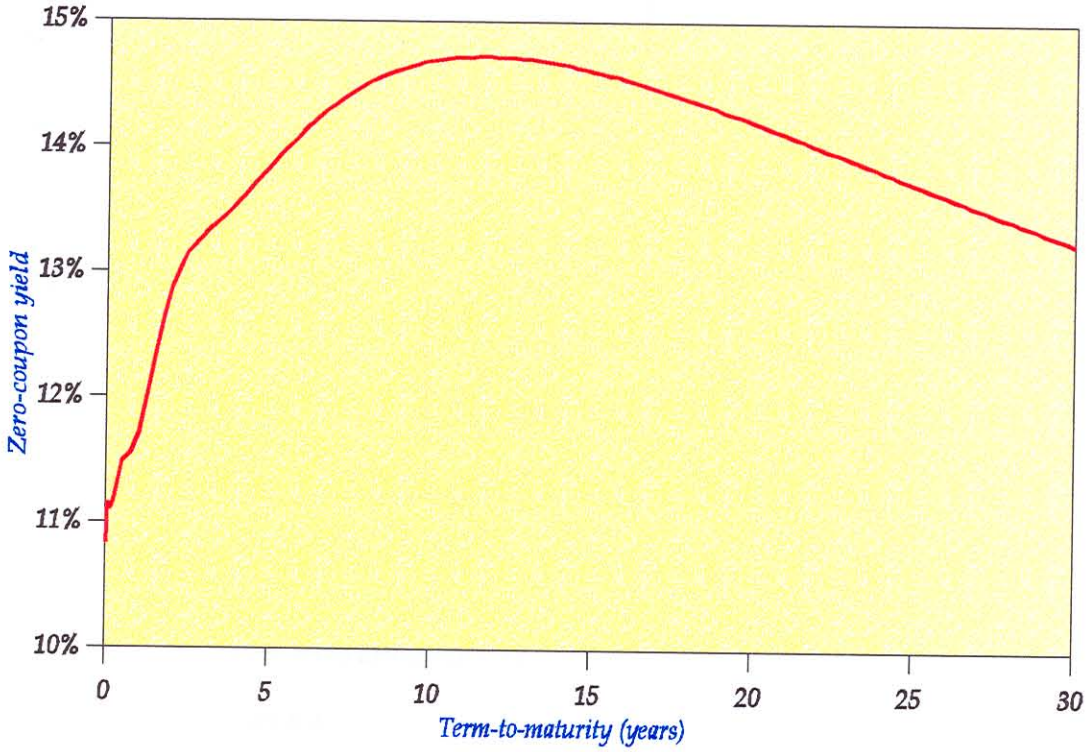

Figure 6.5: South African zero-coupon yield curve in November 1999

The modified Hull-White numerical method was hence used to calculate option prices for the R150 and R153, shown in Tables 6.7 and 6.8 respectively. The difference between the European and American prices in Tables 6.7 and 6.8 is an indication of the early-exercise value, which is larger for call options, due to the shape of the yield curve. Since the yield curve is relatively flat compared to the previous examples, both call and put option prices contain early-exercise value.

Table 6.7: Prices for R150 at-the-money-spot call and put options using the yield curve in Figure 6.4

Option term (years)	European call price	American call price	European/ American difference (%)	European put price	American put price	European/ American difference (%)
0.5	0.805	0.861	-7.0	4.752	4.816	-1.3
1.0	1.152	1.384	-20.1	5.641	5.885	-4.3
2.0	1.557	2.012	-29.3	4.887	6.692	-36.9
3.0	1.362	2.330	-71.0	3.704	6.993	-88.8
4.0	0.819	2.447	-198.9	2.336	7.125	-205.0
5.0	0.208	2.479	-1092.2	0.607	7.169	-1080.7

Table 6.8: Prices for R153 at-the-money-spot call and put options using the yield curve in Figure 6.4

Option term (years)	European call price	American call price	European/ American difference (%)	European put price	American put price	European/ American difference(%)
0.5	2.650	2.845	-7.3	3.216	3.225	-0.3
1.0	3.125	3.648	-16.7	4.841	4.909	-1.4
2.0	3.757	4.613	-22.8	5.618	6.507	-15.8
3.0	3.687	5.167	-40.1	5.719	7.386	-29.2
4.0	3.283	5.469	-66.6	5.513	7.996	-45.0
5.0	2.856	5.643	-97.6	4.893	8.394	-71.5

The longer options on the R150 bond show the effect when the term of the option becomes comparable to the term-to-maturity of the bond. Figure 6.6 shows the effect of the strike yield for a particular expiry date for 6 month options, as well as 1 year options, using the R153 bond as the underlying instrument.

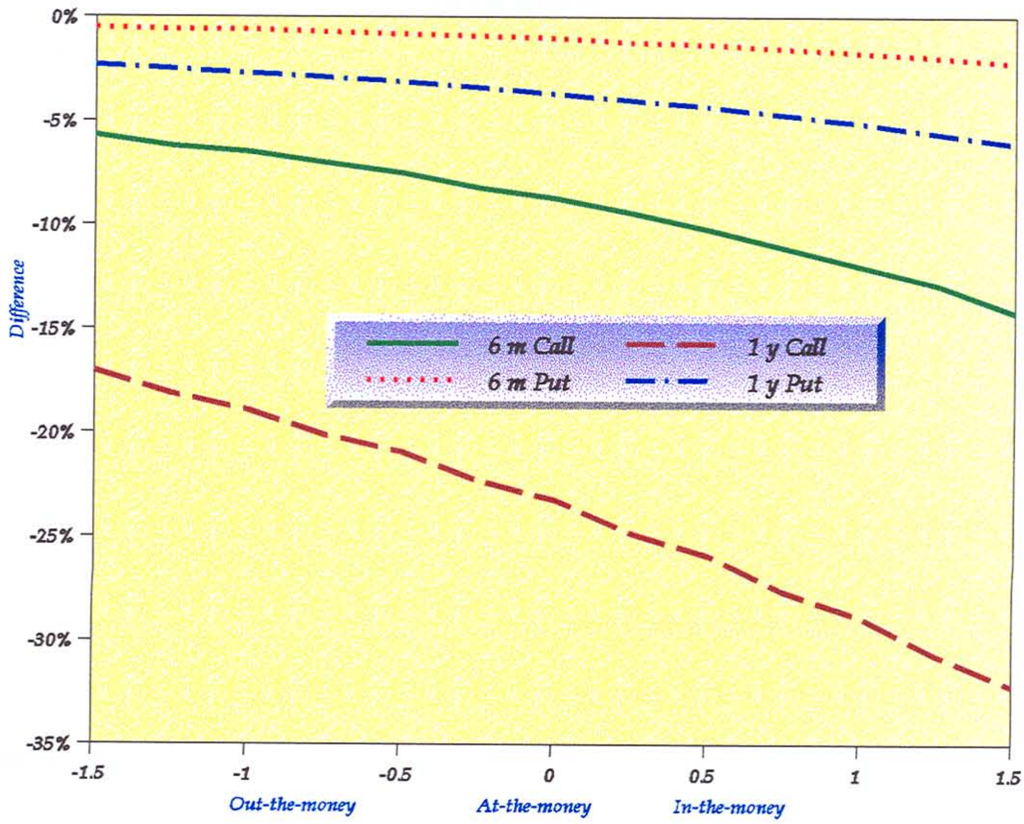

Figure 6.6: **Price difference between European and American options on the R153 bond for different yield strikes**

6.7 Concluding Remarks

Since the market convention is to use the Black model to price bond options, the Hull-White analytical solution (which is similar to the Black model) was compared to the Hull-White numerical solution, with specific reference to South African options. Results were compared using different strike conventions and term structures. Both the strike convention, and the term structure influence the price of American options.

The pricing difference between options on the bond price and options on the yield-to-maturity of a bond becomes significant for longer-dated American options. These pricing differences are also influenced by the term structure of interest rates. Pricing these options with the Black model can therefore lead to significant errors. The results show, however, that the pricing difference is usually small when a European model is used to price the following:

- short-dated American put options in a market with an increasing term structure; and
- short-dated American call options in a market with a decreasing term structure.

For the opposite situation, however, the error becomes significant, even for short-dated American options.

Since short-dated American bond options in the market are usually priced using the Black model, the market prices and implied volatilities for at-the-money options (put or call options, depending on the yield curve) can be used to solve the volatility parameters σ_r and a in the analytical Hull-White model. The parameters can then be used to price longer-dated American options by using the Hull and White trinomial tree approach and therefore reducing the pricing error. Therefore, the use of the Hull-White model is strongly recommended for longer-dated options, as well as in-the-money options, where there is a bigger early-exercise value.

It can be concluded that the primary advantage of the Hull-White model is to value longer-dated American options when using the yield-strike convention. The main advantage is certainly the estimation of the early-exercise value for American options which becomes significant for OTC call options when there is a positive-sloping yield curve, and for OTC put options when there is a negative-sloping yield curve. The early-exercise value is mainly a result of a move in the risk-free rate that affects the carry-cost of the hedge. Since OTC options are hedged in the physical instrument (the bond), the carry-cost can have a big influence.

The main disadvantage of the Hull-White model is, however, the need to calibrate the parameters and the estimation of the yield curve before one is able to price an option. By contrast, the Black model is easy to use and one needs only the volatility and the equivalent risk-free rate to price an option. This is one of the reasons why the Black model remains a popular pricing tool, even though it has several disadvantages.

In addition to OTC bond options, there are also South African Futures Exchange (SAFEX) bond options which are traded on the future yield of a bond. Since the largest volume options are traded on the near contract, the Black model is usually sufficient to estimate a reasonably accurate value, especially for at-the-money options. The biggest concern is out-the-money and in-the-money options where there is uncertainty about the accuracy of the Black model. Therefore, a pricing model that is as easy to use as the Black model, *and* can solve the problem of valuing in-the-money and out-the-money SAFEX bond options is discussed in the next chapter.