CHAPTER 5

BOND OPTION PRICING MODELS

Options on long-term bonds are popular derivative instruments used to hedge a fixed income portfolio against the movement of interest rates. An option on a long-term bond gives the holder the right, but not the obligation, to buy or sell the bond at a certain future time at a predetermined strike price or exercise price.

The valuation of options on interest rate instruments, such as bonds, is more complex than options on stocks and commodities, since it involves not only one underlying instrument, but also a subset of instruments which relies on the term structure of interest rates. Several models have been developed over the years to price options on long-term bonds. They can be divided into the following three categories:

- conventional models;
- equilibrium models; and
- no-arbitrage models.

The price of a bond is determined by several factors – its maturity date, coupon rate, ex- or cum-status and yield-to-maturity. The yield-to-maturity is the interest rate or rate-of-return for the bond, commonly referred to as the yield. For short-dated options, it is assumed that the price P of the bond follows a Brownian motion. Conventional models use the stochastic process of the particular underlying bond price to determine a fair value for the price of the option. The behaviour of the remainder of the term structure is not taken into account. These models are widely used in all markets.

An equilibrium model first defines a process for the instantaneous short rate, r. It produces a term structure of interest rates from the value of r at the current time t, and a risk-neutral process for r. Equilibrium models produce a term structure of interest rates as an *output*, using the stochastic process of the short rate r. This does not necessarily fit today's term structure. It can certainly fit the term structure approximately, but in some cases an exact fit is not possible, resulting in significant errors, which are discussed in Section 5.2.2.

A no-arbitrage model, on the other hand, uses the initial term structure as an input and is therefore exactly consistent with today's term structure.

Some of the interest rate models in the above-mentioned three categories are discussed below, and then the analytical solution of the Hull-White model is examined in more detail.

5.1 Conventional models

5.1.1 The Black-Scholes model

The Black-Scholes model is a popular tool to value almost any derivative security. It is easily adjusted to price an option on a bond price. If P is the spot price of a discount bond, or zero-coupon bond, the behaviour of the bond price, P, can be described by the stochastic process

$$dP = \mu P dt + \sigma_P P dW \tag{1}$$

where μ is the expected return, σ_P is the volatility of the bond price and W is a Wiener process.

If X is the exercise price, T the time to expiry of the option and R_T the zero-coupon continuously compounded risk-free interest rate for maturity T, and one uses the Black-Scholes model, then the price c of a European call and the price p of a European put option on a zero-coupon bond (following the process in equation (1)) are given by:

$$c = PN(d_1) - e^{-R_T \cdot T} XN(d_2) \tag{2}$$

and

$$p = e^{-R_T \cdot T} XN(-d_2) - PN(-d_1) \tag{3}$$

where

$$d_1 = \frac{\ell n(P/X) + (R_T + \sigma_P^2/2)T}{\sigma_P\sqrt{T}}$$

$$d_2 = d_1 - \sigma_P\sqrt{T}$$

For a coupon bond where coupons are payable during the life of the option, the coupons can be treated as the dividends on a stock. The spot price of the bond should therefore exclude the present value of the coupons. The volatility parameter, σ_P, should be the volatility of the bond price without the present value of the applicable coupons.

5.1.2. The Black model

The Black version of the Black-Scholes model has proved to be more suitable for the valuation of coupon-bearing bond options, because it uses the forward price. The forward price of the bond already excludes any coupons paid during the life of the option. The Black model is the most popular method for valuing ordinary options on coupon bonds.

The Black model assumes that the price of the underlying instrument is lognormally distributed on the expiry date of the option. If F is the forward price of the underlying bond on the expiry date of the option, the price of a call and put are then given by:

$$c = e^{-R_T \cdot T}\left[F N(d_1) - X N(d_2)\right]$$

$$p = e^{-R_T \cdot T}\left[X N(-d_2) - F N(-d_1)\right] \qquad (4)$$

where

$$d_1 = \frac{\ell n(F/X) + \frac{1}{2}\sigma_F^2 T}{\sigma_F \sqrt{T}}$$

$$d_2 = d_1 - \sigma_F \sqrt{T}$$

For exchange traded options, where an interest-bearing margin is paid and the option is cash-settled only on the expiry date, equation (4) still holds, but with R_T set equal to zero[1].

The disadvantages of the Black model are discussed in Chapter 7 and an alternative model is proposed.

5.2 Equilibrium models

5.2.1 *The Rendleman-Bartter model*

Rendleman and Bartter (1980) developed a model where the short rate, r, is described in a risk-neutral world by an Ito process

$$dr = \mu r dt + \sigma r dW \tag{5}$$

where μ is the drift and σ is the volatility of the short rate. This model assumes that the short rate, r, follows a geometric Brownian motion.

The process for r can be modelled by using a binomial tree, where the parameters are given by

[1]The interest paid on borrowed money is equal to the interest received on the margin account.

$$u = e^{\sigma\sqrt{\Delta t}}$$

$$d = e^{-\sigma\sqrt{\Delta t}}$$

$$p = \frac{a - d}{u - d}$$

where

$$a = e^{\mu\Delta t}$$

The short-term interest rate is chosen to be the rate for the length of the time-interval. Interest rate movements in a risk-neutral world are given by the binomial tree:

$$r_{ij} = r_0 u^j d^{i-j} \tag{6}$$

where r_o is the initial short-term interest rate. An interest rate tree (Rendleman and Bartter, 1979) for the full term of the bond, until it matures, can be constructed using equation (6). The value of the bond P_{ij} at each node is then given by

$$P_{ij} = e^{-r_{ij}\Delta t}\left[p P_{i+1,j+1} + (1 - p)P_{i+1,j} + c\right] \tag{7}$$

where c is the coupon paid at the end of each time-interval. At the maturity date of the bond, the bond price equals the bond's nominal value, which is then the boundary condition for equation (7).

Once the bond price at each node is known, one can continue to determine the option value. In order to calculate the value of an American call option at each node, one starts at the time-step, N, which coincides with the expiry date of the option, and then calculates the intrinsic value of the option:

$$f_{Nj} = \max[P_{Nj} - X, 0]$$

where X is the exercise price of the option. For $i < N$,

$$f_{ij} = \max[P_{ij} - X, e^{-r_{ij}\Delta t}(pf_{i+1,j+1} + (1 - p)f_{i+1,j})]$$

where the first term in the equation tests for the early-exercise value at each node. By rolling back through the tree, the value of the option at the first node is determined, which is the price of the option.

To illustrate the approach, suppose that $\Delta t = 1, \mu = 0.08, \ \sigma = 0.2$. One can suppose the initial value of r is 10% per annum and the aim is to value a 4 year American call option on a 5 year bond that pays a 12% coupon at the end of each year and has a face value of R1000.00.

In order to determine the option price, one first determines the short rate tree, next, the bond price tree and then works backward through the tree to obtain the option price. Figure 5.1 shows the numerical results, giving an option price of R28.28.

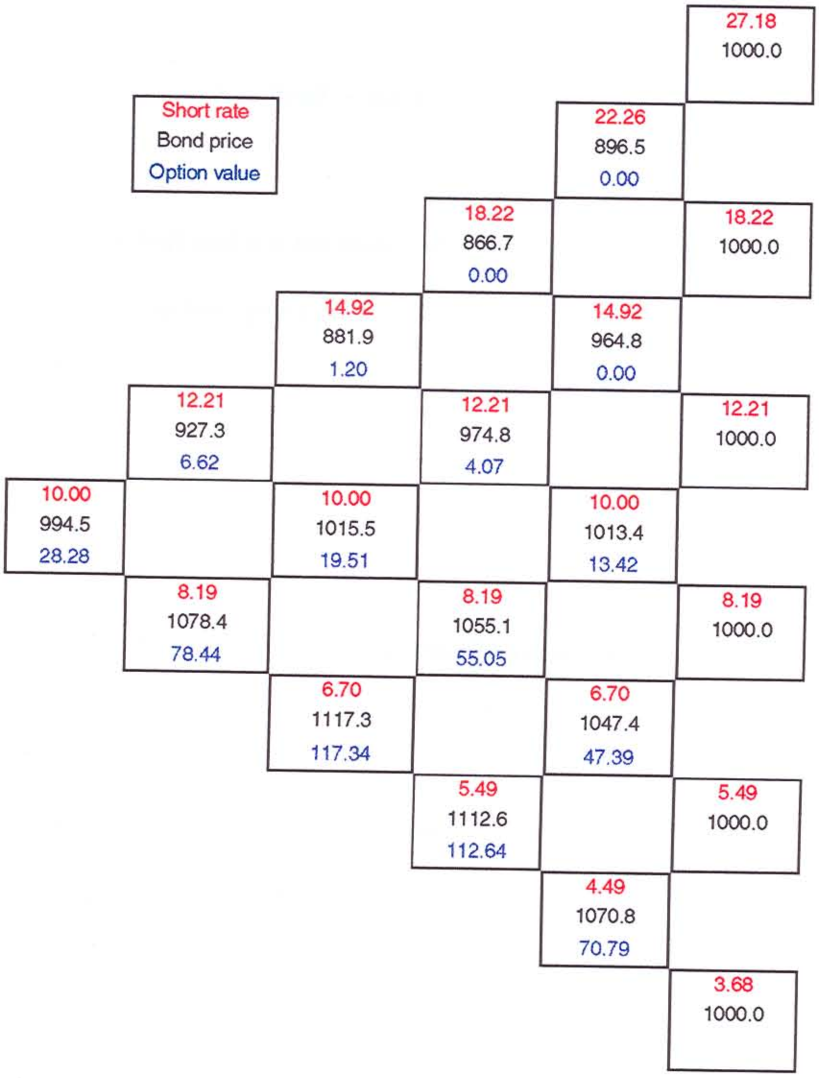

Figure 5.1: Example of a Rendleman-Bartter tree

5.2.2 *The Vasicek model*

Vasicek's model (1977) assumes that the short rate, r, follows a continuous Markov process. The risk-neutral process for r is given by the stochastic differential equation

$$dr = f(r)dt + \sigma dW \tag{8}$$

where $f(r)$ is the instantaneous drift and σ is the standard deviation or volatility of the spot rate process $r(t)$. The parameter $f(r)$ can be expressed in such a form that it includes mean reversion:

$$f(r) = a(b-r)$$

where the short rate, r is pulled to a level b at a rate a.

Vasicek obtained the following analytic formula for the price of a discount bond at time t, paying 1 unit at maturity time t_n:

$$P(t,t_n) = A(t,t_n)e^{-B(t,t_n)r(t)} \tag{9}$$

where, for $a \neq 0$,

$$B(t,t_n) = \frac{1 - e^{-a(t_n - t)}}{a} \tag{10}$$

and

$$A(t,t_n) = \exp\left[\frac{(B(t,t_n) - (t_n - t))(a^2 b - \sigma^2/2)}{a^2} - \frac{\sigma^2 B(t,t_n)^2}{4a}\right] \qquad (11)$$

From the above equations it is possible to obtain the whole term structure as a function of r, once a, b and σ have been chosen. The term structure can be upward-sloping, downward-sloping or humped. The possible shape of the term structure is, however, limited, which causes the assumed term structure to differ significantly from the actual term structure. Figure 5.2 shows an example of a best fit for a term structure, using the Vasicek model. It is clear that the method results in large errors.

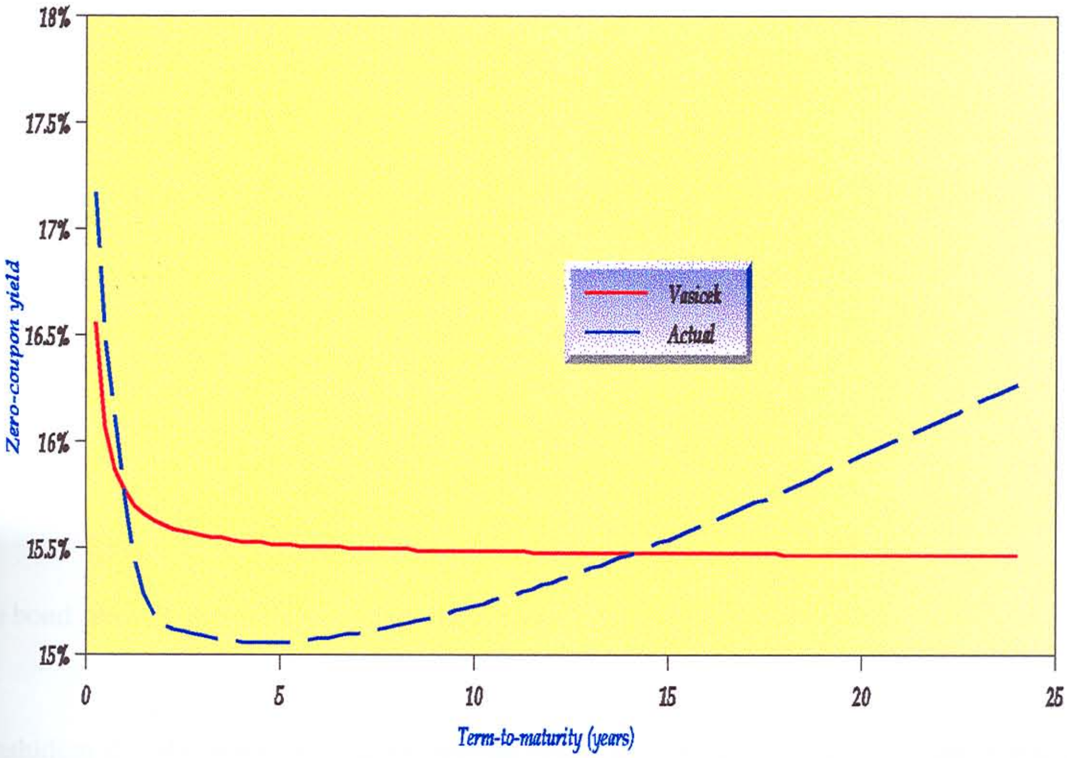

Figure 5.2: **Difference between the best fit using a Vasicek term structure and a 1997 South African yield curve**

5.2.3 *Jamshidian's model*

Jamshidian (1989) has demonstrated how to determine the value of an option on a discount bond using the Vasicek model. The value of a European call option at time t, expiring at time T, on a discount bond with a 1 unit principal maturing at time t_n, is given by

$$c = P(t,t_n)N(h) - XP(t,T)N(h - \sigma_p) \qquad (12)$$

where

$$h = \frac{1}{\sigma_p}\ell n\frac{P(t,t_n)}{P(t,T)X} + \frac{\sigma_P}{2}$$

$$\sigma_p = v(t,T)B(T,t_n)$$

$$v(t,T)^2 = \frac{\sigma^2(1 - e^{-2a(T-t)})}{2a}$$

and σ_P is the price volatility and X is the strike price. The price of a European put option on the bond is

$$p = XP(t,T)N(-h + \sigma_p) - P(t,t_n)N(-h) \qquad (13)$$

The bond price, P, in the above equations, is given by Vasicek's model.

Jamshidian also showed that a coupon-bearing bond can be regarded as a composite of discount bonds, one for each cash flow. An option on a coupon-bearing bond can then be seen as a combination of options on discount bonds, one for each remaining cash flow of the bond after the option expires. If r^* is the particular short rate that causes the coupon-bearing bond price to equal the strike price (which is found by using an iterative procedure, such as the Newton Raphson method), and X_i is the resultant strike price for each individual option, using r^*, and if $P(T,t_i)$ is the price at time T of a zero-coupon bond maturing at time t_i, then the payoff of a call option is given by

$$\sum_{i=1}^{n} c_i \max[0, P(T,t_i) - X_i] \tag{14}$$

which is the sum of n options on the underlying discount bonds.

The Cox, Ingersoll and Ross model (1985) is similar to Vasicek's model. The Cox, Ingersoll and Ross model provides for non-negative interest rates, by adding a $\sqrt{r}$-factor to the second term in equation (8).

5.3 No-arbitrage models

5.3.1 The Ho-Lee model

Ho and Lee (1986) proposed the first no-arbitrage Markov model by extending Vasicek's model. They showed how an interest rate model can be designed so that it is automatically consistent with the initial term structure. The short rate r is described by the stochastic differential equation

$$dr = \theta(t)dt + \sigma dW \tag{15}$$

where σ is the constant instantaneous standard deviation of the short rate, and the drift $\theta(t)$ defines the average direction in which r moves and ensures that the model fits the initial term structure:

$$\theta(t) = F_t(0,t) + \sigma^2 t \tag{16}$$

where $F(0,t)$ is the forward rate at time t and F_t denotes the first derivative. The advantage of the Ho and Lee model is that the model is a Markov analytically tractable model. It does not, however, make provision for the mean reversion of interest rates. This, together with the assumption that interest rates are normally distributed, leads to a relatively high probability that interest rates will become negative.

The Ho-Lee model's analytic expression for the price of a discount bond at time t in terms of the short rate is

$$P(t,t_n) = A(t,t_n)e^{-r(t)(t_n-t)} \tag{17}$$

where

$$\ell n A(t,t_n) = \ell n \frac{P(0,t_n)}{P(0,t)} - (t_n - t)\frac{\partial \ell n P(0,t)}{\partial t} - \frac{1}{2}\sigma^2 r(t_n - t)^2$$

The Ho-Lee analytical value at time zero for a European call option expiring at time T on a discount bond maturing at time t_n, with a face value of 1 unit, is given by

$$c = P(0,t_n)N(h) - XP(0,T)N(h-\sigma_p) \tag{18}$$

where

$$h = \frac{1}{\sigma_p}\ell n\frac{P(0,t_n)}{P(0,T)X} + \frac{\sigma_p}{2}$$

$$\sigma_p = \sigma(t_n - T)\sqrt{T}$$

While the Ho-Lee model is a Markov model, Heath, Jarrow and Morton (1992) developed a model where the short rate, r, is non-Markov. In order to determine the stochastic process for r over a short period of time, dt, one needs to know what the value of r was at the beginning of the period, as well as the path it followed to reach this value, which makes the Heath, Jarrow and Morton model a non-Markov model. The model specifies the volatilities of all instantaneous forward rates at all future times, which is called a volatility structure. This method leads to a non-recombining tree which is computationally extremely time-consuming since there are 2^n nodes after n time steps. The Hull-White model, by contrast, has a recombining tree that speeds up computer time.

5.3.2 The Hull-White model

The mean reversion of interest rates is a phenomenon that is not captured by the Ho-Lee model. There are compelling arguments in favour of mean reversion. When interest rates are high, investments decline and the economy slows down. The opposite occurs when interest rates are low. The Ho and Lee model was extended by Hull and White (1990), who added mean-reversion to the short-term interest rate, r, in the stochastic process:

$$dr = (\theta(t) - ar)dt + \sigma dW \qquad (19)$$

where a and σ, are constants and $\theta(t)$ is a function of time chosen in such a way that the model is consistent with the initial term structure. The coefficient of dt is approximately equal to the slope of the forward rate curve at time zero. When the short-rate moves away from this curve, it reverts back to the curve at a rate a. The mean reversion component reduces the probability of negative interest rates, compared to the Ho-Lee model.

The Hull-White model is exactly consistent with the latest term structure of interest rates, and is therefore known as a no-arbitrage model. The spot rate in the Hull-White model is a linear function of the underlying process. The value of an interest rate derivative, f (which depends on the process in (19)) is given by the partial differential equation:

$$\frac{\partial f}{\partial t} + \left(\theta(t) - ar\right)\frac{\partial f}{\partial r} + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial r^2} - rf = 0 \qquad (20)$$

In order to solve the above partial differential equation, one first has to simplify the stochastic process. If the following transformation is considered in order to obtain the short rate change in a stochastic world for a flat term structure,

$$x = r - \alpha(t) \qquad (21)$$

where $\alpha(0)$ is chosen so that $x(0) = 0$ and x follows a process symmetrical around $x = 0$:

$$dx = -ax\,dt + \sigma dW \qquad (22)$$

then, from equations (19) and (22), one can say that

$$d\alpha = [\theta(t) - a\alpha(t)]\,dt$$

If one solves this differential equation with an integration factor, one gets

$$\alpha(t) = e^{-at}\left[r(0) + \int_0^t e^{aq}\,\theta(q)\,dq\right] \tag{23}$$

The price of an interest rate derivative security in terms of the new variable x can be written as $g(t,x)$.

Therefore,

$$f(t,r) \equiv g(t,x) = g(t,r - \alpha(t))$$

Then,

$$f_t = g_t + g_x\left(\frac{dx}{dt}\right)$$

$$= g_t - (-a\alpha(t) + \theta(t))\,g_x$$

where the subscripts denote the relevant derivatives. Similarly one gets

$$f_r = g_x$$

$$f_{rr} = g_{xx}$$

Substituting into equation (20) one gets

$$g_t - axg_x + \frac{1}{2}\sigma^2 g_{xx} - (x + \alpha(t))g = 0 \tag{24}$$

This partial differential corresponds to an economy where, under the equivalent martingale measure Q^*, the spot interest rate is generated by

$$dx = -axdt + \sigma dW \tag{25}$$

and

$$x(t) = r(t) - \alpha(t)$$

The stochastic process for x in equation (25) is therefore independent of the function $\alpha(t)$.

The process for x is assumed to follow an Ornstein-Uhlenbeck process. Therefore, given a value $x(t)$ at any point t, the probability distribution for $x(T)$ for $T > t$, is a normal distribution with mean

$$e^{-a(T-t)}x(t)$$

and variance

$$\frac{\sigma^2}{2a}\left(1 - e^{-2a(T-t)}\right)$$

Using the Feynman-Kač formula (see Section 2.2.3) and the T-forward- risk-adjusted measure Q_T, the solution for equation (24) can be expressed as

$$g(t,x) = P(t,T,x)\, E^{Q_T}(h(T,x(T))\,|\,\mathcal{F}_T) \tag{26}$$

where $h(T,x(T))$ is the boundary condition at time T, and $P(t,T,x)$ is the price of a discount bond with maturity T at time t.

In order to determine the price $P(t,T,x)$ and the distribution of x under Q_T, the Fourier transform $\tilde{g}$ of the fundamental solution g^{δ} is used. Pelsser (1996) has shown that $\tilde{g}$ must take the form

$$\tilde{g}(t,x;T,\psi) = \exp\{A(t;T,\psi) + B(t;T,\psi)x\} \tag{27}$$

where the boundary condition is given by

$$\tilde{g}(T,x;T,\psi) = e^{i\psi x}$$

Then, equation (24) becomes

$$x(B_t - aB - 1) + A_t + \frac{1}{2}\sigma^2 B^2 - \alpha(t) = 0$$

which is solved if A and B satisfy the system

$$B_t - aB - 1 = 0$$

$$A_t + \frac{1}{2}\sigma^2 B^2 - \alpha(t) = 0$$

subject to $A(T;T,\psi) = 0$ and $B(T;T,\psi) = i\psi$.

Using an integration factor, one obtains

$$B(t;T,\psi) = i\psi e^{-a(T-t)} - \frac{1 - e^{-a(T-t)}}{a}$$

and by integration the result is

$$A(t;T,\psi) = \frac{\sigma^2}{2a^3}\left(a(T - t) - 2(1 - e^{-a(T-t)} + \frac{1}{2}(1 - e^{-2a(T-t)})\right)$$

$$- i\psi\frac{\sigma^2}{2a^2}(1 - e^{-a(T-t)})^2 - \frac{1}{2}\psi^2\frac{\sigma^2}{2a}(1 - \exp^{-2a(T-t)})$$

$$- \int_t^T \alpha(s)ds$$

Substituting A and B into equation (27) yields

$$\tilde{g}(t,x;T,\psi) = \exp\{A(t,T) - B(t,T)x + i\psi M(t,T,x) - \frac{1}{2}\psi^2\Sigma(t,T)\} \tag{28}$$

where

$$A(t;T) = \frac{\sigma^2}{2a^3}\Big(a(T-t) - 2(1 - e^{-a(T-t)}) + \tfrac{1}{2}(1 - e^{-2a(T-t)})\Big) - \int_t^T \alpha(s)ds$$

$$B(t,T) = \frac{1 - e^{-a(T-t)}}{a} \tag{29}$$

$$M(t,T,x) = xe^{-a(T-t)} - \frac{\sigma^2}{2a^2}(1 - e^{-a(T-t)})^2$$

$$\Sigma(t,T) = \frac{\sigma^2}{2a}(1 - e^{-2a(T-t)})$$

Equation (28) can also be written as the product of the discount bond price and the characteristic function of the probability density function under the T-forward-risk-adjusted measure:

$$\tilde{g}(t,x;T,\psi) = P(t,T,x)\{\exp(i\psi M(t,T,x) - \tfrac{1}{2}\psi^2\Sigma(t,T))\} \tag{30}$$

The probability density function has a mean $M(t,T,x)$ and a variance $\Sigma(t,T)$.

Using the above results, one can determine the value of a European call option on a discount bond. If $c(t,T,s,X,x)$ is the value of a call option at time t, that gives the owner the right to buy a discount bond with maturity s at time T, $t<T<s$, for a price X, then the payoff, h, of the option is given by

$$h(T,x(T)) = \max\{P(T,s,x(T)) - X,0\}$$

The expected payoff of the option can be expressed under the T-forward-risk-adjusted measure Q_T as follows:

$$s = E^{Q_T}(\max\{P(T,s,x(T)) - X,0\}|\mathcal{F}_t)$$

If the value of $x(T)$ equals z, then the expectation is given by

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\Sigma(t,T)}} \max\{e^{A(T,s)-B(T,s)z} - X, 0\} \exp\left\{-\frac{1}{2}\frac{(z-M(t,T,x))^2}{\Sigma(t,T)}\right\} dz \tag{31}$$

The payoff is non-zero if

$$z < \frac{A(T,s) - \ell n X}{B(T,s)}$$

Therefore,

$$\begin{aligned} s &= \int_{-\infty}^{\frac{A-\ell n X}{B}} \frac{e^{A(T,s)-B(T,s)z} - X}{\sqrt{2\pi\Sigma(t,T)}} \exp\left\{-\frac{1}{2}\frac{(z-M(t,T,x))^2}{\Sigma(t,T)}\right\} dz \\ &= \int_{-\infty}^{\frac{A-\ell n X}{B}} \frac{1}{\sqrt{2\pi\Sigma(t,T)}} \exp\left\{-\frac{1}{2}\frac{(z-M(t,T,x))^2}{\Sigma(t,T)} + A(T,s)-B(T,s)z\right\} dz \\ &\quad - \int_{-\infty}^{\frac{A-\ell n X}{B}} \frac{X}{\sqrt{2\pi\Sigma(t,T)}} \cdot \exp\left\{-\frac{1}{2}\frac{(z-M(t,T,x))^2}{\Sigma(t,T)}\right\} dz \end{aligned} \tag{32}$$

If the first integral is called I_1 and the second integral is called I_2, then by completing the square, it follows that

$$I_2 = X \int_{-\infty}^{\frac{A(t,s)-\ell n X}{B(T,s)}} \exp\left[-\tfrac{1}{2}\{(z-M)/\sqrt{\Sigma}\}^2\right] dz$$

It can easily be shown that

$$A(T,s) - B(T,s)M = [A(t,s) - A(t,T)] - [B(t,s) - B(t,T)]y - \tfrac{1}{2}B(T,s)^2\Sigma$$

Therefore, substituting and expressing the equation in terms of cumulative normal distribution functions one gets

$$\begin{aligned} I_2 &= X \int_{-\infty}^{h_2} e^{\frac{1}{2}\rho^2}\, d\rho \\ &= XN(h_2) \end{aligned}$$

where

$$h_2 = \frac{\ell n(P(t,s,x)/P(t,T,x)X) - B(T,s)^2\Sigma(t,T)}{B(T,s)\sqrt{\Sigma(t,T)}}$$

Similarly, for I_1 the result is

$$\begin{aligned} I_1 &= \int_{-\infty}^{\frac{A-\ell nX}{B}} \frac{1}{\sqrt{2\pi\Sigma(t,T)}} \exp\left\{-\frac{1}{2}\left\{\frac{z - (M - B(T,s)\Sigma)}{\sqrt{\Sigma(t,T)}}\right\}^2 + A(T,s) - B(T,s)M + \frac{1}{2}B(T,s)^2\Sigma\right\} dz \\ &= \int_{-\infty}^{\frac{A-\ell nX}{B}} \frac{1}{\sqrt{2\pi\Sigma(t,T)}} \exp\left\{-\frac{1}{2}\left\{\frac{z - (M - B(T,s)\Sigma)}{\sqrt{\Sigma(t,T)}}\right\}^2\right\} \exp\{A(t,s) - A(t,T) - [B(t,s) - B(t,T)]x\}\, dz \\ &= \frac{\exp[A(t,s) - B(t,s)x]}{\exp[A(t,T) - B(t,T)x]} \int_{-\infty}^{\frac{A-\ell nX}{B}} \frac{1}{\sqrt{2\pi\Sigma(t,T)}} \exp\left\{-\frac{1}{2}\left\{\frac{z - (M - B(T,s)\Sigma)}{\sqrt{\Sigma(t,T)}}\right\}^2\right\} dz \end{aligned}$$

which gives

$$I_1 = \frac{P(t,s,x)}{P(t,T,x)} \int_{-\infty}^{h_1} e^{\frac{1}{2}\omega^2} d\omega$$

$$= \frac{P(t,s,x)}{P(t,T,x)} N(h_1)$$

where

$$h_1 = \frac{\ell n(P(t,s,x)/P(t,T,x)X) + B(T,s)^2\Sigma(t,T)}{B(T,s)\sqrt{\Sigma(t,T)}}$$

Therefore,

$$s = \frac{P(t,s,x)}{P(t,T,x)} N(h_1) - X N(h_2) \tag{33}$$

Discounting the above expected value to the current date, t, gives the price of a call option:

$$c = P(t,T,x)\left(\frac{P(t,s,x)}{P(t,T,x)} N(h_1) - X N(h_2)\right)$$

$$= P(t,s,x)\, N(h_1) - P(t,T,x) X N(h_2) \tag{34}$$

where

$$h_1 = \frac{\ell n(P(t,s,x)/P(t,T,x)X)}{\sigma^*} + \frac{\sigma^*}{2}$$

$$h_2 = h_1 - \sigma^* \tag{35}$$

$$\sigma^* = \frac{\sigma_r}{a}\left(1 - e^{-a(s-T)}\right)\sqrt{\frac{1-e^{-2a(T-t)}}{2a}}$$

The price of a put option is given by

$$p = P(t,T,x)XN(-h_2) - P(t,s,x)N(-h_1) \qquad (36)$$

The equation for σ^* takes into account the pull-to-par effect of a bond. The analytical solution to the Hull-White model overcomes the volatility problem of the Black model, discussed in Section 3.3, since it takes the pull-to-par effect into account. However, the solution does not hold for American options. To address this problem, a numerical solution is necessary, as discussed in the next chapter.

5.3.3 Other Markov models

The tree procedure used by Hull and White can also be used to construct other one-factor Markov models. For example, a tree can be constructed in $\ell n\, r$ rather than r, as described by Black, Derman and Toy (1990) or Black and Karasinski (1991) where

$$d\,\ell n r = [\theta(t) - a\,\ell n r]dt + \sigma dz$$

The procedure suggested by Black, Derman and Toy (1990) matches the volatilities of all rates at time zero. The trinomial tree procedure is explained by Hull and White (1993). Black and Karasinski (1991) suggested a binomial tree procedure involving time steps of varying lengths.

The next chapter discusses the numerical solution of the Hull-White model applied to South African bond options.