CHAPTER 3

THE TERM STRUCTURE OF INTEREST RATES

Interest rates play an important role in the economy, whether on the global or national level. It is both a determinant and a result of economic growth. The interest rate term structure is the most important input in pricing almost all fixed income instruments. The term structure, or yield curve, provides a way of measuring the relationship between the rate of interest, or yield and time to maturity. The interest rate associated with an investment gives the return on that investment.

Fixed-income investors have several alternative securities to invest in. In choosing the security to invest in, they consider the following three factors: return, risk and liquidity. The longer the term-to-maturity of the instrument, the larger the price risk. Unless there is a strong expectation that interest rates are going to fall, investors would only invest in a longer-term security if the return is higher. This usually leads to an upward-sloping yield curve.

There are three main theories that are used to explain the shape of the yield curve (Bodie, Kane & Marcus, 1993), namely

- the expectations hypothesis;
- the liquidity preference theory; and
- the market segmentation and preferred habitat theories.

The expectations hypothesis states that the forward rate for a period in the future equals the market consensus expectation of the future interest rate. Therefore, for example, the six month interest rate is determined by the current three month interest rate and the expectation of the three month rate in three months time. The yield curve is therefore determined by expected future changes in interest rates.

The liquidity preference theory argues that there are more short-term investors than long-term investors and therefore short-term investors require a premium to induce them to buy longer-term securities. This implies that the forward rate should exceed the expected spot rate by the liquidity premium. If the liquidity premium is, however, higher than investors feel is fair, they would exploit the abnormal profit opportunities – bringing it back to normal.

The market segmentation theory argues that long- and short-term bonds are traded in segmented markets. Borrowers and lenders tend to operate in different maturity ranges. The interest rate for a particular maturity is therefore determined solely by supply and demand in that area of the yield curve. The preferred habitat theory argues, however, that lenders would leave their preferred maturity area if there is significant inducement offered in another area, thereby eliminating some of the inconsistencies in the yield curve.

3.1 The term structure and forward rates

One could assume that the current time is zero. The T_n-year yield given by the term structure is the interest rate $y(T_n)$ on an investment that is made today, lasting for T_n years, also known as the T_n-year spot interest rate, or zero-coupon yield. The principal and interest are repaid to the investor at the end of T_n years. The forward interest rate $f(T_n,T_m)$ is the rate implied by current spot rates for the period between year T_n and year T_m in the future.

It can be assumed that interest rates are compounded continuously. If investors invest 1 unit today, they will obtain a future value of

$$v = 1 \cdot e^{y(T_n)T_n}$$

in T_n years time.

If investors invest 1 unit today for a period of T_n years at a rate $y(T_n)$, and after T_n years reinvest the money for another (T_m-T_n) years at a forward rate $f(T_n,T_m)$, the future value after T_m years would be

$$v = 1 \cdot e^{y(T_n)T_n} \cdot e^{f(T_n,T_m)\cdot(T_m-T_n)}$$

However, if investors invest the money for a period of T_m years at a rate $y(T_m)$ instead, the future value is

$$v = 1 \cdot e^{y(T_m)T_m}$$

For the no-arbitrage principle to hold, it follows that

$$e^{y(T_m)T_m} = e^{y(T_n)T_n} \cdot e^{f(T_n,T_m)\cdot(T_m - T_n)}$$

Therefore the forward rate for the period $[T_n, T_m]$ is given by

$$f(T_n,T_m) = \frac{y(T_m)T_m - y(T_n)T_n}{T_m - T_n}$$

or

$$f(T_n,T_m) = y(T_m) + T_n\left[\frac{y(T_m) - y(T_n)}{T_m - T_n}\right]$$

If there is a continuous yield curve and limits are taken as T_m approaches T_n, it is clear that $y(T_m)$ approaches $y(T_n)$. The forward rate for a very short period of time, beginning in T_n years, (known as the instantaneous forward rate in T_n years), can be expressed as

$$f_i(T_n) = y(T_n) + T_n \frac{\partial r}{\partial T_n} \qquad (1)$$

where r is called the instantaneous interest rate or short-term spot rate. In the rest of thestudy this rate will be referred to as the short-rate.

3.2 The term structure and the short-rate

If one assumes that the current time is denoted by t, and as explained in the previous section, the short-rate, r, at time t is the interest rate for an infinitesimally short period of time Δt, then the value of an interest-rate derivative that provides a payoff of h at time t_n is determined by the expected risk-free rate of return for the period $T = t_n - t$:

$$E\left[e^{-\bar{r}T}h\right] \tag{2}$$

where $\bar{r}$ is the average value of r in the time interval between t and t_n, and E is the expected value in a risk-neutral world. If $P(t\ ,\ t_n)$ is the price at time t of a discount bond that pays a maturity value of 1 unit at time t_n, then equation (2) implies that

$$P(t,t_n) = E\left[e^{-\bar{r}T}\right] \tag{3}$$

If $y(t,T)$ is the continuously compounded spot interest rate at time t for a T- year investment, then

$$P(t,t_n) = e^{-y(t,T)T} \tag{4}$$

or

$$y(t,T) = -\frac{1}{t_n - t}\ell n P(t,t_n) \tag{5}$$

From equation (3) it is clear that

$$y(t,T) = -\frac{1}{t_n - t}\ell n E\left[e^{-\bar{r}(t_n - t)}\right] \tag{6}$$

This equation shows that the term structure of interest rates can be obtained from the initial value of r at time t and the risk-neutral process for r for $t \le t_n$. It is therefore clear that by developing a model of the risk-neutral process for r, the term structure of interest rates could be modelled.

3.3 The relation between the short-rate, bond prices and forward rates

If one assumes that the price $P(t,t_n)$ of a bond is determined by the market's assessment, at time t, of the behaviour of interest rates over the life of the bond, and the yield to maturity for the period $[t,t_n]$ is equivalent to the average forward rate for the period, then it follows that the instantaneous forward rate $f_i(t,t_n)$ is defined by

$$y(t,T) = \frac{1}{t_n - t} \int_t^{t_n} f_i(t,\tau)\, d\tau$$

Therefore,

$$f_i(t,t_n) = \frac{\partial}{\partial t_n}[(t_n - t)\; y(t,T)]$$

If the short-rate is defined as the instantaneous interest rate, then

$$r(t) = y(t,0) = \lim_{T \to 0} y(t,T)$$

If one assumes that the short-rate is a continuous function of time and follows a Markov process, then the spot interest rate r follows the following stochastic differential equation:

$$dr = \mu(t,r)dt + \sigma(t,r)dW \tag{7}$$

where $\mu(t,r)$ and $\sigma(t,r)$ are the instantaneous drift and standard deviation respectively of the process $r(t)$.

If a financial instrument's value $P(t,r)$ is determined directly by the level of the spot interest rate $r(t)$, it follows from Ito's lemma that

$$dP = M(t,r)dt + \Sigma(t,r)dW \tag{8}$$

where

$$M(t,r) = \mu(t,r)\frac{\partial P}{\partial r} + \frac{1}{2}\sigma(t,r)^2\frac{d^2P}{dr^2} + \frac{\partial P}{\partial t}$$

$$\Sigma(t,r) = \sigma(t,r)\frac{\partial P}{\partial r}$$

A locally riskless portfolio Π can be constructed by hedging a derivative P_1 with a Δ-amount of another interest rate derivative P_2:

$$\Pi = P_1(t,r) - \Delta P_2(t,r)$$

where P_1 and P_2 both follow stochastic processes as described above. The portfolio Π is a linear combination of these processes:

$$d\Pi = (M_1(t,r) - \Delta M_2(t,r))dt + (\Sigma_1(t,r) - \Delta\Sigma_2(t,r))dW$$

If one chooses $\Delta = \Sigma_1 / \Sigma_2$, then the random component in $d\Pi$ is eliminated. Using arbitrage arguments, the portfolio should earn a riskless return in a small period of time, leading to

$$d\Pi = r\Pi dt$$

Using substitution, the following equation is obtained

$$\left(M_1(t,r) - \frac{\Sigma_1(t,r)}{\Sigma_2(t,r)}M_2(t,r) \right) dt = r\left(P_1(t,r) - \frac{\Sigma_1(t,r)}{\Sigma_2(t,r)}P_2(t,r) \right) dt$$

Algebraic manipulation gives

$$\frac{M_1(t,r) - rP_1(t,r)}{\Sigma_1(t,r)} = \frac{M_2(t,r) - rP_2(t,r)}{\Sigma_1(t,r)}$$

which should hold for any pair of derivatives P_1 and P_2. The ratio $(M - rP)/\Sigma$ must therefore be a function of r and t only, which is denoted by $\lambda(r,t)$. For any derivative security P, it follows that

$$\frac{\partial P}{\partial t} + (\mu(t,r) - \lambda(t,r)\,\sigma(t,r))\frac{\partial P}{\partial r} + \frac{1}{2}\sigma(t,r)^2\frac{\partial^2 P}{\partial r^2} - rP = 0 \tag{9}$$

This equation describes the price of a security in a one-factor yield-curve model and is called the term structure equation.

The parameters μ and σ of the short-rate process, and the market price of risk, λ, must be determined from the market. The former two quantities can be obtained from the process $r(t)$, while λ can be determined empirically (Vasicek, 1977) from the equation

$$\left.\frac{\partial P}{\partial t}\right|_{T=0} = \frac{1}{2}(\mu - \sigma\lambda) \tag{10}$$

The spot interest rate, r, can be a function of time and some underlying process u, for example $F(t,u)$. The process u has a normal distribution. This function can take various forms, for instance, linear, quadratic, logarithmic or exponential. The choice of the function determines whether the model has a normally distributed fundamental solution. Pelsser (1996) has proven that the function $F(t,u)$ must be either a linear or a quadratic function in u to give a normally distributed fundamental solution.

If one assumes that instead of the short-rate, the price of the bond follows a Wiener process (as described in Black's model applied to bond options in Chapter 5), then the risk neutral process for the price P of a zero-coupon bond can be described by the following stochastic differential equation:

$$dP(t,t_n) = r(t)\,P(t,t_n)dt + \sigma(t,t_n)\,P(t,t_n)\,dW \tag{11}$$

The expected return, μ, is given in this case by the risk-free rate for that period, since a zero-coupon bond provides no income throughout the life of the bond. The pull-to-par phenomenon states that, at the maturity date of the bond, the bond price must equal its face value. Therefore, instead of constant price volatility, $\sigma(t)$, upon maturity of the bond, the price volatility should equal zero and it can be assumed that:

$$\sigma(t,t_n) \underset{t_n}{\overset{t}{\rightarrow}} 0 \tag{12}$$

The forward rate at time t for the period t_n to t_m can be written in the following form:

$$f(t,t_n,t_m) = \frac{\ell n[P(t,t_n)] - \ell n[P(t,t_m)]}{t_m - t_n} \tag{13}$$

Using equation (11) and Ito's lemma, with $g_n = \ell n(P(t,t_n))$, $g_m = \ell n(P(t,t_m))$, it follows that

$$dg_n = \left[r(t) - \frac{\sigma(t,t_n)^2}{2}\right]dt + \sigma(t,t_n)dW$$

and

$$dg_m = \left[r(t) - \frac{\sigma(t,t_m)^2}{2}\right]dt + \sigma(t,t_m)dW$$

It follows that

$$df(t,t_n,t_m) = \frac{\sigma(t,t_m)^2 - \sigma(t,t_n)^2}{2(t_m - t_n)}dt + \frac{\sigma(t,t_n) - \sigma(t,t_m)}{t_m - t_n}dW$$

It becomes clear that the risk-neutral process for f depends only on the volatility σ. If $t_n = s$ and $t_m = s + \Delta t$ and Δt tends to zero, the forward rate, $f(t,t_n,t_m)$ becomes the instantaneous forward rate $f_i(t,s)$ and

$$df_i(t,s) = \sigma(t,s)\frac{\partial\sigma(t,s)}{\partial s}dt - \frac{\partial\sigma(t,s)}{\partial s}dW$$

The sign of dW can be changed without loss of generality, and therefore the equation can be written as

$$df_i(t,s) = \sigma(t,s)\sigma_s(t,s)\,dt + \sigma_s(t,s)\,dW \tag{14}$$

where σ_s denotes the first derivative. Equation (14) shows that the drift is given by

$$m(t,s) = \sigma(t,s)\,\sigma_s(t,s)$$

and therefore the instantaneous forward rate depends on its standard deviation $v(t,s)$, where

$$v(t,s) = \sigma_s(t,s)$$

If one integrates σ_s between $\tau = t$ and $\tau = s$, the result is

$$\begin{aligned}\int_t^s \sigma_s(t,\tau)\ d\tau &= \sigma(t,s) - \sigma(t,t)\\ &= \sigma(t,s)\end{aligned}$$

Therefore, it follows from equation (14) that the drift-term is given by

$$\begin{aligned}m(t,s) &= \sigma_s(t,s)\,\sigma(t,s)\\ &= v(t,s)\int_t^s v(t,\tau)\,d\tau\end{aligned} \tag{15}$$

Since the short-rate r is given by

$$r(t) = f_i(t,t)$$

and

$$\int_0^t df_i(\tau,t) = f_i(t,t) - f_i(0,t)$$

it follows from equation (14) that

$$r(t) = f_i(0,t) + \int_0^t \sigma(\tau,t)\sigma_t(t,\tau)d\tau + \int_0^t \sigma_t(\tau,t)dW \qquad (16)$$

If one differentiates to t, the following process is obtained for r

$$dr(t) = [f_i]_t(0,t)dt + \left\{ \int_0^t [\sigma(\tau,t)\sigma_{tt}(\tau,t) + \sigma_t(\tau,t)^2]d\tau \right\} dt$$
$$+ \left\{ \int_0^t \sigma_{tt}(\tau,t)dW \right\} dt + [\sigma_t(\tau,t)|_{\tau=t}]dW \qquad (17)$$

This equation gives the stochastic process for the short-rate where the terms containing dt give the drift in r, and the last term gives the standard deviation of r. The first term is in fact the initial slope of the forward rate curve. The above equation demonstrates the relation between the stochastic process for the bond price and the process for the short-rate. This concept is used in various option pricing models.

3.4 The term structure – coupon vs zero-coupon

A discount bond is an instrument that provides a single cashflow at a time s in the future. The price of the discount bond is determined by the s-term yield in the market at the time of purchase. Coupon-bearing bonds pay a stream of certain payments at times $\{t_i\}$, called coupons, as well as a notional payment at the end of the term of the bond. A coupon bond can be seen as a combination of discount bonds. The relation between the yield-to-maturity and the term-to-maturity of discount bonds describes the term structure of interest rates, which are used in the pricing of any fixed income instrument. The term structure implies the market consensus of forward rates and forward curves, often used for hedging purposes.

In term structure analysis it is essential that each observation used as a data point produces a yield with an unambiguous relationship with the term of the security. This is the case only with pure discount securities such as zero-coupon bonds. A coupon bond, on the other hand, can be seen as a composite of pure discount instruments – one for each of the bond's remaining cashflows – while an interest rate swap can be seen as a par yield bond. Opportunities to restore equilibrium between the markets for coupon bonds, zero-coupon bonds and swaps exist through arbitrage. The zero-coupon yield curve serves as the instrument to discount the cashflows of any interest rate security, in order to obtain the fair value of a security when selecting fixed income securities for an investment portfolio.

Consider, for example, a market in which zero-coupon bonds as well as coupon-bearing bonds are traded. Depending on whether the coupons are worth more (or less) than the actual bond, participants in the market will either strip the coupons (separate the coupons from the nominal amount of a bond), or reconstitute the bonds (by re-bundling zero-coupon bonds). The value of coupons and bonds should be determined from a single curve to ensure that no arbitrage opportunities occur.

In South Africa the JSE Actuarial Yield curve is seen as the benchmark curve. This curve is a fit through the yield-to-maturities of all government bonds. It therefore approximates a par-bond curve. No official zero-coupon yield curve is available in South Africa.

3.5 Constructing the initial term structure: the standard bootstrap method

In liquid fixed income markets, zero-coupon bonds and money market rates are typically used to construct a zero-coupon yield curve. In markets where a limited number of zero-coupon bonds are traded, usually, a sufficient number of coupon-bearing bonds are traded to apply standard bootstrap procedures. In the South African fixed income market, however, only a limited number of liquid instruments are available to construct a zero-coupon yield curve.

In the South African fixed income market, bonds are traded on a yield-to-maturity basis (Faure *et al.*, 1991). The yield-to-maturity of a bond can be defined as the internal rate of return of the investment. When a particular bond is priced using its yield-to-maturity, it is assumed that all cashflows are discounted at the same yield.

If P_k denotes the price of a coupon bond (bond k), and if continuously compounded interest rates are used, the price for a South African bond is calculated by discounting all cashflows at the quoted yield-to-maturity:

$$P_k = \sum_{i=1}^{n-1} \gamma_k e^{-t_i^{(k)}\eta_k} + (1 + \gamma_k) e^{-t_n^{(k)}\eta_k}. \tag{18}$$

where, for bond P_k

- η_k is the continuously compounded yield to maturity;
- γ_k is the periodic coupon paid;
- $t_i^{(k)}$ is the time to a coupon dat;,
- n is the number of coupons to be paid to maturity; and
- $t_n^{(k)}$ is the term-to-maturity, and there is a repayment of 1 unit at this time.

The term-to-maturity, $t_n^{(k)}$, and yield-to-maturity rates, η_k, give an array which serves as the input for term structure analysis. The ambiguity in the relationship between the yield-to-maturity and the term-to-maturity may be rectified by determining the underlying zero-coupon yields by sequentially stripping off coupons (Hull, 1997).

3.5.1 *Example of bootstrapping*

A practical example illustrates the process of bootstrapping. One can assume that the interest rates for 3, 6 and 12 month periods are known, but after that one only has the yields for coupon bonds maturing in 1.5 years, 2.0 years and 2.75 years, where coupons are paid every six months, as shown in Table 3.1.

Table 3.1: Data for bootstrap method

Term-to-maturity (years)	Annual coupon (%)	Continuously compounded yield (%)
0.25	0	10.13
0.5	0	10.68
1.0	0	11.43
1.5	10	11.74
2.0	12	11.84
2.75	13	11.76

In order to obtain the term structure for the period from 3 months to 2.75 years, it is necessary to do bootstrapping. The price of the bond can be split up into the price of the 6-monthly coupons, and then the price for the nominal plus the coupon at maturity. Since the interest rates for the first two coupon periods are known, the 1.5 year zero-coupon rate, $z(1.5)$, can be determined from the price of the 1.5 year bond:

$$P_{1.5} = ce^{-z(0.5)\times 0.5} + ce^{-z(1.0)\times 1.0} + (N+c)e^{-z(1.5)\times 1.5}$$

where c is the coupon-payment. Since $z(1.5)$ is the only unknown, it can easily be calculated as 11.8%. A similar calculation results in the 2-year rate, $z(2)$, from the 2-year bond, as 11.90%.

Although $z(2.75)$ is still unknown, one can use linear interpolation to find the $z(2.25)$ in terms of $z(2.75)$ and $z(2)$:

$$z(2.25) \equiv \frac{2}{3}z(2) + \frac{1}{3}z(2.75)$$

Using this equation in the pricing formula then gives

$$P_{2.75} = ce^{-z(0.25)\times 0.25} + ce^{-z(0.75)\times 0.75} + ce^{-z(1.25)\times 1.25}$$

$$+ ce^{-z(1.75)\times 1.75} + ce^{-\left(\frac{2}{3}z(2) + \frac{1}{3}z(2.75)\right)\times 2.25} + (N+c)e^{-z(2.75)\times 2.75}$$

The zero-coupon rates, $z(0.25)$, $z(0.75)$, $z(1.25)$, $z(1.75)$ and $z(2)$, are known, or can be interpolated from the rates already known. Numerical procedures such as the Newton-Raphson method, can then be used to establish the 2.75 year rate, or $z(2.75)$, for this bond, which is 11.90%. Continuing this process results in the term structure of interest rates. The process of calculating the spot interest rates by stripping off coupons is called bootstrapping. The curve calculated in this example is shown in Figure 3.1 below.

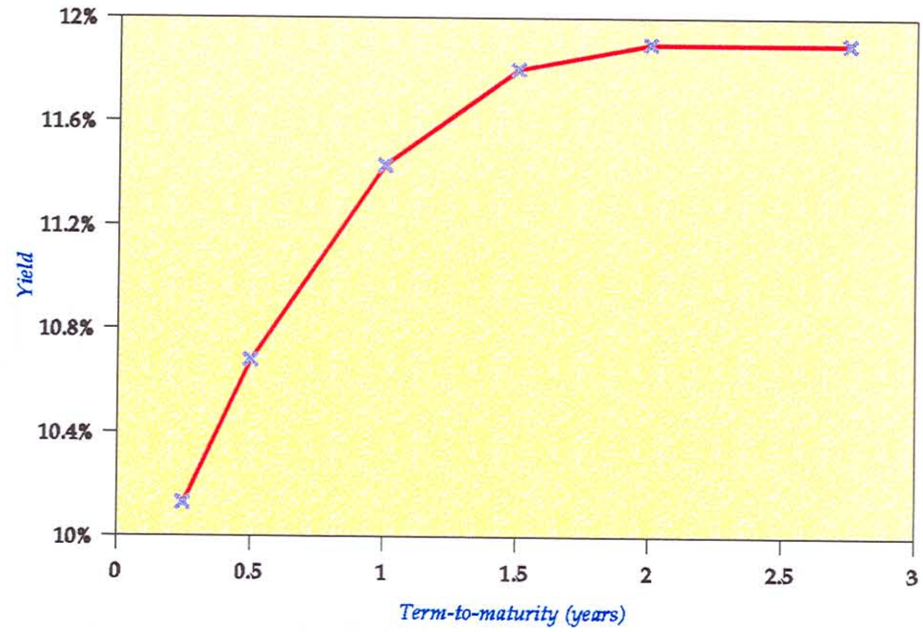

Figure 3.1: Zero-coupon yield curve

According to Vasicek and Fong (1982), the objective of the empirical estimation of the term structure is to fit a zero-coupon curve (or spot rate curve) that both fits the data sufficiently well and is a sufficiently smooth function.

The latter requirement is particularly important, as it will determine the smoothness of the forward curve, derived from the spot rate curve. Because financial markets are dynamic and volatile, the term structure changes periodically to comply with changing perspectives. The objective is therefore to find a method of estimating a zero-coupon curve that both fulfils the above requirements and can be easily adjusted to accommodate a volatile market. A method that complies to these requirements is discussed in the next chapter.