The theory of integrated empathies

by

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Dedicated to the memory of my father.

To the Glory of God

The integrated empathy

by Thomas John Brown

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Degree: **PhD**

Abstract

The traditional study of dynamical systems places cause and effect in the same descriptive framework. However, situations occur in science where the observable effects are less complex or different from the events that cause them. For such situations cause and effect should be described in different frameworks. The appropriate causal relationship was discovered and first investigated by N. Sauer. It was termed the empathy relation, given by $S(t+s) = S(t)E(s)$, where $\langle S, E\rangle$ is a double family of evolution operators. This equation is an extension of the familiar semigroup relation. In analogy to the concept of integrated semigroups, a more complex integrated form of the empathy causality is introduced and studied in this work, given by the integrated empathy relation,

$$S(t)E(s) = \int_0^t [S(s+\rho) - S(\rho)]\, d\rho.$$

Presently the focus of our scientific investigation into evolution equations is in the study of a systems of partial differential equations in the form of an implicit Cauchy problem,

$$\frac{d}{dt}[Bu(t)] = Au(t), \quad \lim_{t\to 0+} Bu(t) = y; \quad A, B\colon X \to Y.$$

Here X and Y are Banach spaces, and A and B are unbounded linear operators. The solution is given by $u(t) = S(t)y$, where S satisfies the empathy relation, while the integrated empathy solves an integrated form of this Cauchy problem. Our double family approach does not depend on the closability of B which is very desirable in practice.

The basic tool is the Laplace transform, first used to the define the generator $\langle A, B\rangle$ of the integrated empathy in terms of resolvents, the Laplace transforms of the double family $\langle S, E\rangle$.

The approach is from a dynamic systems viewpoint, and assumptions on the trajectories are added only as needed. They range from assuming initial continuity of S, and then strong continuity of $S(\cdot)y$ and $E(\cdot)y$ to the Lipschitz continuity of $S(\cdot)$ and $E(\cdot)$ in the operator norm. A Hille-Yosida type characterization is obtained for Lipschitz continuous integrated empathies, where the integrated form of Widder's theorem (due to W. Arendt) is employed. It is not possible to use Widder's theorem because it requires the Radon-Nikodým property, which we do not assume.

Implicit Cauchy problems often arise in the mathematical study of problems in physics involving partial differential equations satisfying dynamic boundary conditions. Two worked applications are given, involving the heat equation and the wave equation. They are analysed in a continuous function setting where the Radon-Nikodým property does not hold, (also the domains of A and B are not dense in X). This demonstrates the need to introduce the integrated empathy.

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Chapter 1

Introduction

1.1 Semigroups, integrated semigroups, and the Laplace transform

1.1.1 Linear evolution equations

The law of cause and effect observed in nature can be summarized in the concept that the complete state of a closed system at any given moment determines all subsequent states. This is the case if the laws which govern the dynamics are unchanging with time. Any moment in the evolution of such a natural process can thus be taken to be an initial state from which all subsequent states evolve. This idea can be formulated very simply in mathematical terms in a linear operator setting. Let X be a Banach space and $E = \{E(t) \mid X \to X \colon t > 0\}$ a family of bounded linear operators. The defining property of a *semigroup* E is given by the causal relation $E(t+s) = E(t)E(s)$ where t and s are positive numbers signifying moments in time and $E(t)x_0$ is a curve (trajectory) in the Banach space X giving the state of a system at time t, with initial state x_0.

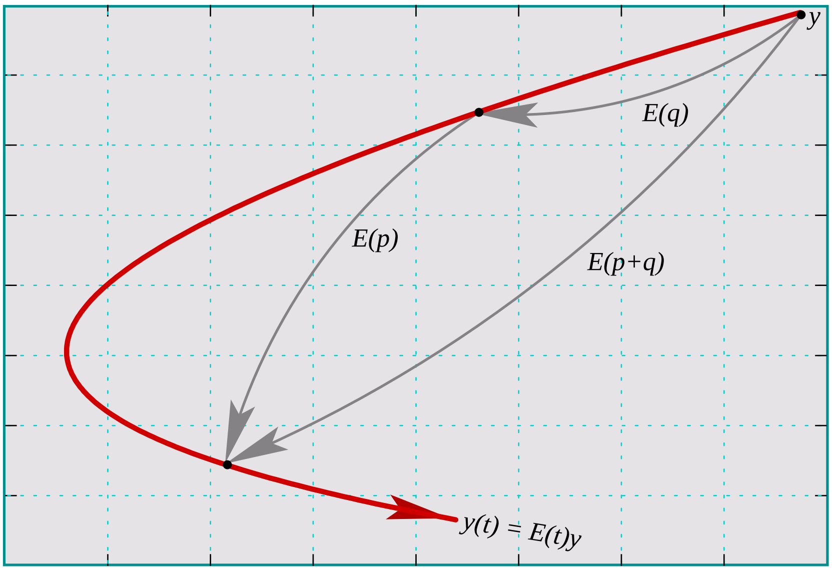

Figure 1.1: *A trajectory of a semigroup,* $E(p+q)y = E(p)E(q)y$

The notion of a semigroup is closely related to the study of evolution equations, notably the classical *Cauchy problem*

$$\frac{d}{dt}u(t) = Au(t); \tag{1.1}$$

$$\lim_{t\to 0+} u(t) = x_0 \tag{1.2}$$

with $x_0 \in X$ and $A\colon \mathcal{D}_A \subset X \to X$ a linear operator.

Remark 1.1 *All derivatives will be in the norm topology.*

The laws that govern the evolution in time of an initial state x_0 are contained in the operator A, its domain $\mathcal{D}_A$, and the choice of the Banach space X.

In fact, if A is closed and the domain of A, $\mathcal{D}_A$ is dense in X with $x_0 \in \mathcal{D}_A$, and A is the so-called infinitesimal generator of E, then $u(t) = E(t)x_0$ is a solution of (1.1–1.2). In this case E has the additional property that $\lim_{t\to 0+} E(t)x$ exists in the norm for each $x \in X$, or E is a C_0-semigroup.

The usual approach to semigroup theory is of an operator theoretic nature and involves the study of difference quotients of E, leading to the Hille-Yosida characterization of the infinitesimal generator ([HP], [Y2]). This is a condition on the resolvent of A, $R(\lambda, A) = (\lambda I - A)^{-1}$, stating that A generates a uniformly bounded C_0-semigroup E with $\|E(t)\| \leq M$ for an $M > 0$ if and only if $\mathcal{D}_A$ is dense in X, $(0, \infty)$ is in the resolvent set of A and

$$\|\lambda^n R^n(\lambda, A)\| \leq M \quad \text{for each } \lambda > 0. \tag{1.3}$$

The present chapter introduces the reader to the methods of studying evolution equations by means of the concepts of semigroups, integrated semigroups and empathies. The discussion is informal.

1.1.2 The role of the Laplace transform

Another way of dealing with semigroups is via the vector-valued Laplace transform of E, defined by $R(\lambda)x = \int_0^\infty e^{-\lambda t} E(t)x\, dt$, with the integral in the sense of Bochner. Now it turns out that the Laplace transform of E is a pseudo-resolvent given by $R(\lambda, A)$, ([HP], [Y2]), see Appendix B, Section 6.2. In the traditional approach the Laplace transform is used only in a very restricted manner, (to show that the operator A is closed), but in [HP] it is used somewhat more extensively.

However the theorem of Widder, with the Laplace transform given by a Lebesgue integral, characterizing the transforms of real-valued functions holds if and only if X has the Radon-Nikodým property. For Banach spaces in general there is an integrated version due to Wolfgang Arendt, (see Appendix B, Section 6.4 and [A]), which characterizes functions F such that $\frac{1}{\lambda}F(\lambda)$ is a transform.

For $F(\lambda) = \lambda \int_0^\infty e^{-\lambda t} E(t) dt$ to be a pseudo-resolvent, [A], E must have the integrated semigroup property

$$E(t)E(s) = \int_s^{t+s} E(\rho)\, d\rho - \int_0^t E(\rho)\, d\rho, \tag{1.4}$$

(the special case $n = 1$ of the n-times integrated semigroup [A] is taken). But this corresponds to the derivative of E being a semigroup, should it exist. This is to be expected and can be verified easily. We have also found the form

$$E(t)E(s) = \int_0^t [E(\rho + s) - E(\rho)]\, d\rho, \tag{1.5}$$

of (1.4) very handy for calculations. The systematic study of integrated semi-

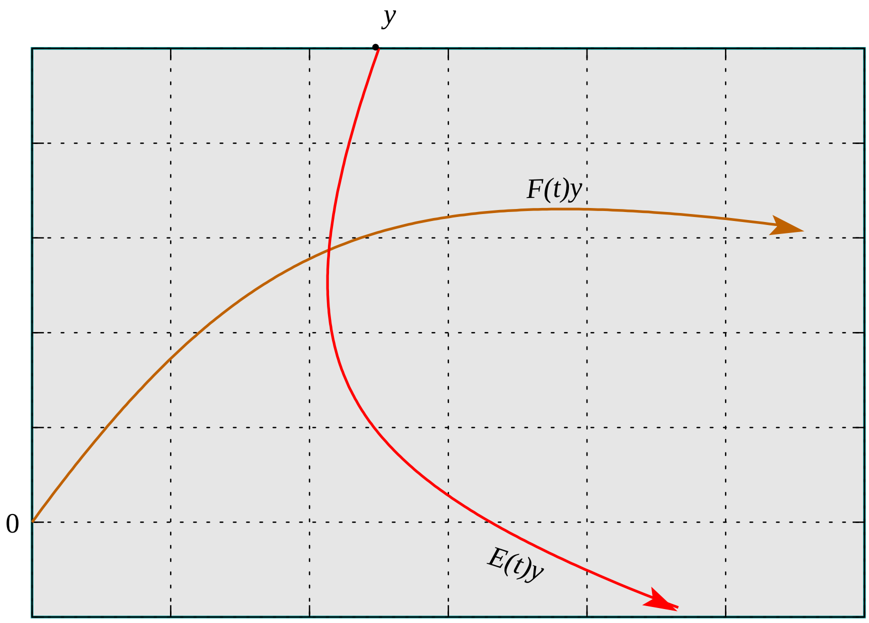

Figure 1.2: *A trajectory of a semigroup, $E(t)y$ and of the associated integrated semigroup, $F(t)y = \int_0^t E(t)y\,dt$*

groups with the aid of the Laplace transform turns out to be very useful in understanding evolution equations, and is a worthwhile pursuit on it's own [A], [ABHN].

The power of this approach is that the hard analysis was really done in the variety of existence and inversion theorems, [ABHN], (some are quoted in Sections 6.1–6.4). Many of the results for integrated empathies follow by applying the known operational properties given in Section 6.5.

1.2 The dynamical systems approach

We are also interested in the study of different types of trajectories under evolution families and call this the dynamical systems approach. Properties that may for instance be investigated and give rise to different families, are strong continuity and differentiability of the trajectories. It has been preferred not to assume that the families are defined for $t = 0$ since this is a subtle but strong condition on the behaviour of E that is often not warranted in physical situations.

In the study of semigroups and integrated semigroups the focus should not only be the solutions of the evolution equations, but also the structure of the families of evolution operators. This aspect is often neglected in mainstream research, but many results follow in the absence of continuity assumptions, or with only a few. For instance, provided $t \mapsto E(t)x$ for each $x \in X$ is measurable, the semigroup property alone guarantees strong continuity on $(0, \infty)$. However, this is not the case for the integrated counterpart. On the other hand the C_0-property has far-reaching consequences for a semigroup. The approach we give is reminiscent of the treatment by Hille and Phillips of semigroups ([HP], Chapter 10, Section 10.6, pp.323–356).

The following system, a hierarchy of assumptions (compare with Section 2.2), is given regarding behaviour at zero for a semigroup E with minimal assumptions being the semigroup property and measurability of the function $t \mapsto E(t)x$ for each $x \in X$.

The strongest is the C_0 class, where $E(t) \to I$ in the strong operator topology as $t \to 0+$. Then there is a classification in terms of the existence of the integrals:

(0) $\int_0^1 \|E(\tau)x\| \, d\tau < \infty$, for all $x \in X$;

(1) $\int_0^1 \|E(\tau)\| \, d\tau < \infty$,

and regarding the type of continuity at zero: The semigroup is

C-summable if $\lim_{t\to 0+} C(t)x = \lim_{t\to 0+} \frac{1}{t}\int_0^t E(\tau)x\, d\tau = x$ and

A-summable if $\lim_{\lambda\to\infty} \lambda R(\lambda)x = x$, for all $x \in X$.

With the self-explanatory notation the different combinations of these assumptions then give rise to the class inclusions

$$C_0 \subset (1, C_1) \subset (0, C_1) \subset (0, A) \subset A,$$

$$C_0 \subset (1, C_1) \subset (1, A) \subset (0, A) \subset A.$$

(The subscript "1" in C_1 indicates that the semigroup is strongly measurable.) There is a similarity between the C-condition for a semigroup and the initial continuity assumption given in Section 2.2.

1.3 Dynamic boundary conditions and implicit evolution equations

The physical evolution in a closed system is usually described by a partial differential equation involving the value of a variable and its derivatives on the domain together with given values on the boundary. In many practical situations the boundary values are in flux, and change in some way "in empathy" with values of the variable in the domain (the term was first used in [SS2], and then [S1] in the present context). A treatment of this kind of modelling as it relates to physical science can be found in the doctoral thesis of Wessel Rossouw, [R]. The resulting mathematical model is in the form of an implicit Cauchy problem,

$$\frac{d}{dt}\big[Bu(t)\big] = Au(t); \tag{1.6}$$

$$\lim_{t\to 0+} Bu(t) = y_0, \tag{1.7}$$

with $A, B\colon X \to Y$ linear operators and X, Y Banach spaces.

Now the study of solutions to an implicit Cauchy problem can be tackled directly, with existing methods, as by Favini and others, [AF], [FY], [N]. Those results presently all rely on A and B being closed operators. This is however often not the case in practice especially not for B. See [SvdM] for examples.

Usually though, it is found that $\langle A, B\rangle\colon X \supset \mathcal{D} \to Y \times Y$ is closed, which is equivalent to $A - \lambda B$ being closed for any $\lambda \in R$ greater than a fixed a, [vdM]. This in turn only requires verification for two distinct values of λ. Our approach is ideal for this situation.

1.4 Empathy and the Radon-Nikodým property

Niko Sauer discovered the analogy of the semigroup causality for double families which is applicable to evolution problems of the type (1.6–1.7), [S2]. In this theory X and Y are Banach spaces and the double family $\langle S, E\rangle$ is a pair of families of bounded linear operators with

$$E = \{E(t) \mid Y \to Y\colon t > 0\}, \ \ S = \{S(t) \mid Y \to X\colon t > 0\}$$

that satisfies the *empathy* relation

$$S(t + s) = S(t)E(s). \tag{1.8}$$

The way we think of this, is that "cause" lies in the space X and "effect" in Y, E is suggestive of "evolution" or "exponential" while S could be thought of as the "solution" family. The family E is thought of as describing the evolution of the state of the boundary.

In this context the generator turns out to be the *pair of operators* $\langle A, B\rangle$ that occur in (1.6–1.7).

The idea in this formulation is that $u(t) = S(t)y_0$ would satisfy (1.6–1.7). The relation is studied as it stands in [S2] and [S3], the biggest shortcoming being

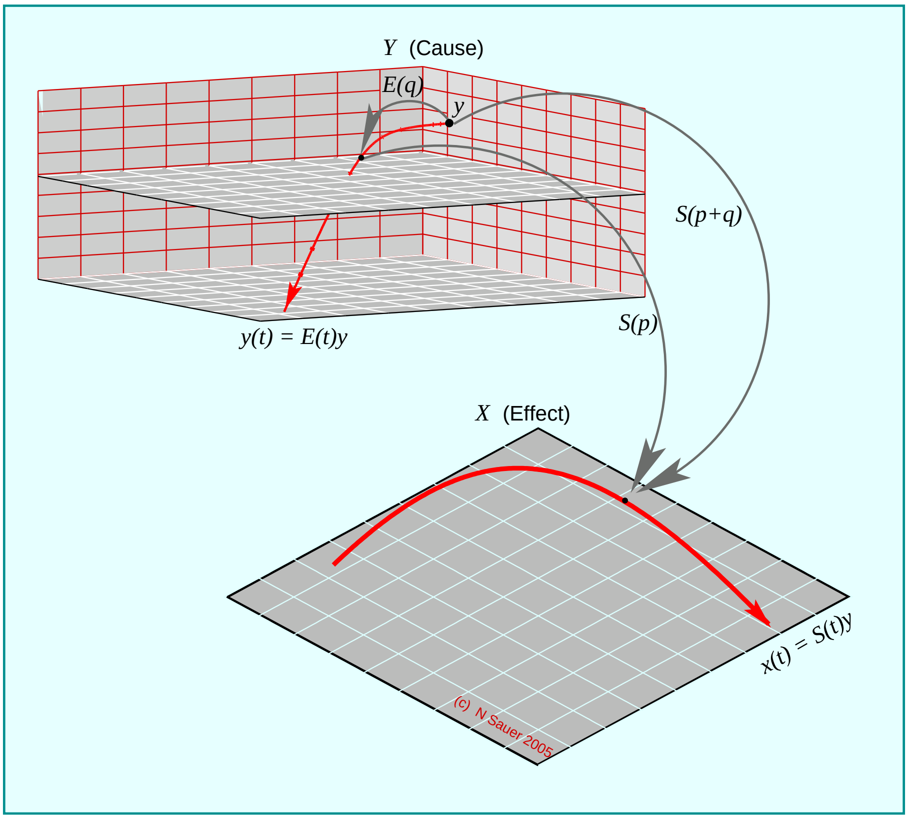

Figure 1.3: *An empathy,* $S(p+q)y = S(p)E(q)y$

that Y must have the Radon-Nikodým property which is very restrictive on the result characterizing the generator, (Theorem 8.2, [S2]).

These results have been used with success in applications of the type in the previous section [GvD].

In order to find a more general result an approach similar to that of Arendt, [A], was followed, naturally giving rise to the concept of an *integrated empathy.* The causality (1.8) is replaced by

$$S(t)E(s) = \int_s^{t+s} S(\rho)d\rho - \int_0^t S(\rho)\, d\rho. \tag{1.9}$$

This method allows us to harness the powerful existence theorem of Arendt

(Theorem 1.1, [A]). See Appendix B regarding the exact formulation of Arendt's theorem.

This approach allows one to study the evolution operators in a much more general setting, albeit at the cost of the implicit Cauchy problem having a weaker *integrated* form (2.29).

1.5 Conventions and notation

If $F = \{F(t) \mid Y \to X \colon t \in (-\infty, +\infty)\}$ where X and Y are Banach spaces is a family of bounded linear operators we shall adopt the following convention: The operator $\int_a^b F(\tau)\,d\tau$ is defined pointwise by $\big(\int_a^b F(\tau)\,d\tau\big)y := \int_a^b F(\tau)y\,d\tau$ for $y \in Y$, provided the integral converges in the norm topology of X as a Bochner integral (Lebesgue integral, [DS], Chapter 3, pp.95–232). The linearity of the integral ensures that linearity is preserved. The integral operator thus defined is not necessarily bounded.

We will freely make use of the fact that if F and G are such families, equalities (whenever the expressions are defined) such as $G(t)\int_s^r F(\tau)\,d\tau = \int_s^r G(t)F(\tau)\,d\tau$ hold (the result of $G(t)$ being a bounded operator).

We observe that from the definition of the integral of a family of operators we also have $\int_0^t F(\tau)\,d\tau\,E(t) = \int_s^r F(\tau)E(t)\,d\tau$, (which means $\int_0^t F(\tau)\,d\tau\,E(t)y = \int_s^r F(\tau)E(t)y\,d\tau$ for each $y \in Y$). Similar use is made of these facts when Laplace transforms are taken.

The *Laplace transform* of F is defined pointwise (if it exists) as the integral,

$$\mathcal{L}_\lambda\{F(t)\}y \colon= \int_0^\infty e^{-\lambda t}F(t)y\,dt \tag{1.10}$$

The notation $\mathcal{L}_{\lambda,t}\{F(t,s)\}y \colon= \int_0^\infty e^{-\lambda t}F(t,s)y\,dt$ is used when there are more variables, and we say the transform is taken with respect to t at λ.

This requires that the function $t \mapsto e^{-\lambda t}F(t)y \in L^1\big((0,\infty), X\big)$. It follows immediately that the restriction to a bounded interval of the function $t \mapsto e^{-\lambda t} F(t)y$ is integrable and an application of the dominated convergence theorem gives $t \mapsto F(t)y$ is integrable on bounded intervals.

Our definition is slightly different from the literature ([HP], [ABHN]) that uses the *improper* Bochner integral.We have preferred the Bochner integral on $(0, \infty)$ because it enables one to use Fubini's theorem to prove the more general form of the convolution theorem that is given in Section 6.5.

For the convolution of two functions the notation

$$(f * g)(t) := \int_0^t f(s)g(t-s)\,ds$$

is used and f_a will be the shift of f by a units, i.e. $f_a(t) := f(t+a)$ with a a real constant.

1.6 Outlay of the thesis

Chapter 2 is a systematic treatment of the integrated empathy, written in such a way that a given continuity assumption is made only when really needed. This chapter forms the core of the work. The main result is the characterization of Lipschitz continuous empathies. A result of Sauer, (Theorem 8.2, [S2]), now follows readily, using a method of proof analogous to (Theorem 6.2, [A]). This relates to the differentiability of the integrated family, given the Radon Nikodým property for Y. It is important to note that in these results denseness of the domain of the generator is not assumed.

It is shown in Chapter 3 that the theory may easily be extended to cover the "more-than-once" integrated case. This is not done from the start in order to emphasise the dynamical systems concepts which may be obscured by technical detail.

The only important difference is that the characterization inequalities are now not quite so simple, and the implicit Cauchy problem has more restricted initial conditions.

A discussion of attempted scientific applications is given in Chapter 4, where the examples of [S2] are reworked in a space of continuous functions setting. We have also looked at a few of the many future areas of research suggested by the work.

Finally, in an attempt to make the work self-contained, results on the Laplace transform which are used in the main text are collected in Appendix B, to be referred to as needed. Of special note is the vector-valued versions of inversion and existence theorems: the Post-Widder and Widder theorems; and a theorem on the convolution of a family of operators with a vector-valued function.

It should be noted that the basis of the work is a joint paper of Niko Sauer and Thomas Brown, [BS], and references to it will mostly be omitted. This paper contains all the main results of Chapter 2 and the basic ideas in Chapter 3 are outlined there. The worked examples of Chapter 4 are also given. However, much more complete discussions are to be found in the present work.

Chapter 2

The integrated empathy

The results in [BS] were mostly obtained in combining the ideas of [S2] with those of [A]. A major difficulty in this process was in obtaining the correct vantage points, i.e. definitions and assumptions. A substantial part of the theory does not depend on strong continuity, which is assumed from the outset in [A] and [ABHN].

2.1 Definitions and basic identities

Let X and Y be Banach spaces over the real or complex field. We consider two families of bounded linear operators $E = \{E(t) \mid Y \to Y\colon t > 0\}$ and $S = \{S(t) \mid Y \to X\colon t > 0\}$. The operators $E(t)$ and $S(t)$ are not defined for $t = 0$, corresponding to the fact that in general evolution equations are defined for $t > 0$ and initial conditions are in the form of limits as $t \to 0+$. We assume throughout that the Laplace transforms as defined in (1.10), of S and E exist on Y for every $\lambda > 0$, and use the notation

$$p(\lambda)\colon Y \to X, \quad p(\lambda) := \mathcal{L}_\lambda\{S\}; \tag{2.1}$$

$$r(\lambda)\colon Y \to Y, \quad r(\lambda) := \mathcal{L}_\lambda\{E\}. \tag{2.2}$$

The functions $t \mapsto S(t)y$ and $t \mapsto E(t)y$ are integrable on bounded intervals, see Section 1.5. It will prove very useful to use the notation

$$P(\lambda) := \lambda p(\lambda), \quad R(\lambda) := \lambda r(\lambda).$$

In Section 2.8 it is described how a more general theory, with p and r defined only for $\lambda > a > 0$ can be obtained from this.

Remark 2.1 *In other texts assumptions are usually made to ensure that the Lapace transforms above exist (e.g. exponential boundedness) [A] . Instead, we assume that the transforms exist [S2].*

An *integrated empathy* is defined as a double family, $\langle S, E\rangle$ as above, that satisfies the causality condition

$$S(t)E(s) = \int_0^t [S(\rho + s) - S(\rho)]\, d\rho \tag{2.3}$$

or, equivalently

$$S(t)E(s) = \int_0^{t+s} S(\rho)\, d\rho - \int_0^t S(\rho)\, d\rho - \int_0^s S(\rho)\, d\rho, \tag{2.4}$$

which implies the identity $S(t)E(s) = S(s)E(t)$.

Elementary properties of the integral imply that formulae (2.3) and (2.4) are indeed the same as (1.9).

The following examples are apparent and both are suggestive of the terminology:

Example 2.1 *Let $\langle S', E'\rangle$ be an empathy, (1.8). If $S(t) := \int_0^t S'(\rho)\, d\rho$ and $E(t) := \int_0^t E'(\rho)\, d\rho$ then $\langle S, E\rangle$ is an integrated empathy.*

An elementary calculation shows that the statement of Example 2.1 is true:
We note that $S'(t+s) = S'(t)E'(s)$ means that

$$\int_0^s \int_0^t S'(\sigma)E'(\rho)\, d\sigma\, d\rho \quad = \quad S(t)E(s)$$

$$= \int_0^t \int_0^s S'(\sigma+\rho)\,d\sigma\,d\rho \;=\; \int_0^t [S(\rho+s)-S(\rho)]\,d\rho.$$

The next example is also easy to verify.

Example 2.2 *Suppose that E is an integrated semigroup on Y and $C\colon Y \to X$ is a bounded linear operator. If $S(t)\colon = CE(t)$, then $\langle S, E\rangle$ is an integrated empathy.*

We give an identity for compositions that follows using only (2.3).

Theorem 2.1 *Let $\langle S, E\rangle$ be an integrated empathy, then, with $\xi = \rho_1 + \cdots + \rho_n$,*

$$S(t)E(s_1)\ldots E(s_n) = \int_0^{s_1} \cdots \int_0^{s_n} [S(t+\xi)) - S(\xi)]\,d\rho_n \ldots d\rho_1. \tag{2.5}$$

Proof. The proof of the theorem is by induction, and the basis step, ($n = 1$), is simply the definition.

For the induction step we need an identity, (2.7) below.

First observe that

$$\begin{aligned} &\int_0^t S(\rho+s+a) - S(\rho+a)\,d\rho \\ &= \int_0^{t+s} S(\rho+a)\,d\rho - \int_0^t S(\rho+a)\,d\rho - \int_0^s S(\rho+a)\,d\rho \\ &= \int_0^s S(\rho+t+a) - S(\rho+a)\,d\rho \end{aligned} \tag{2.6}$$

for any positive number a. Then

$$\begin{aligned} &\int_0^s S(t+a+\rho) - S(a+\rho)\,d\rho\,E(r) \\ &= \int_0^s S(t+a+\rho)E(r) - S(a+\rho)E(r)\,d\rho \\ &= \int_0^s \left[\int_0^{t+a+\rho} S(r+\sigma) - S(\sigma)\,d\sigma - \int_0^{a+\rho} S(r+\sigma) - S(\sigma)\,d\sigma\right] d\rho \end{aligned}$$

$$
\begin{aligned}
&= \int_0^s \left[\int_{a+\rho}^{t+a+\rho} S(r+\sigma) - S(\sigma)\, d\sigma \right] d\rho \\
&= \int_0^s \left[\int_0^t S(r+a+\rho+\sigma) - S(a+\rho+\sigma)\, d\sigma \right] d\rho \\
&= \int_0^s \left[\int_0^r S(t+a+\rho+\sigma) - S(a+\rho+\sigma)\, d\sigma \right] d\rho \qquad (2.7)
\end{aligned}
$$

where (2.6) was used in the last step.

We assume the statement (2.5) is true for $n = k$, and use (2.7) to prove it for $n = k+1$ as follows

$$
\begin{aligned}
&[S(t)E(s_1)\ldots E(s_k)]E(s_{k+1}) \\
&\quad = \int_0^{s_1} \ldots \int_0^{s_k} [S(t+\xi)) - S(\xi)]\, d\rho_k \ldots d\rho_1\, E(s_{k+1}) \\
&\quad = \int_0^{s_1} \ldots \int_0^{s_{k-1}} \left[\int_0^{s_k} [S(t+\xi)) - S(\xi)]\, d\rho_k\, E(s_{k+1}) \right] d\rho_{k-1} \ldots d\rho_1 \\
&\quad = \int_0^{s_1} \ldots \left[\int_0^{s_k} \int_0^{s_{k+1}} [S(t+\xi+\rho_{k+1})) - S(\xi+\rho_{k+1})]\, d\rho_{k+1} d\rho_k \right] \ldots d\rho_1 .
\end{aligned}
$$

□

2.2 The hierarchy of assumptions

The assumptions we have made so far, i.e. the existence of the Laplace transforms of the operator-valued functions S and E, and the integrated empathy relation, will be referred to as the *minimal* assumptions. These conditions are embodied in (2.1), (2.2) and (2.3). They will be used throughout. Note that Theorem 2.1 requires only these minimal assumptions.

In the systematic study of the double families of evolution operators additional conditions must be satisfied in order to obtain progressively stronger results. They will be added only as necessary in the sequel. These are:

INVERTIBILITY. There is at least one $\xi > 0$ for which the inverse of the linear operator $p(\xi)$ is defined on the range of $p(\xi)$, i.e. $p(\xi)$ is injective.

INITIAL CONTINUITY. For every $y \in Y$, $\lim_{t\to 0+} S(t)y = 0$.

STRONG CONTINUITY. For every $y \in Y$ the mappings $t \mapsto S(t)y$ and $t \mapsto E(t)y$ are norm continuous on $(0, \infty)$ in X and Y respectively.

LIPSCHITZ CONTINUITY. There exist positive constants M and N such that for all $t, h > 0$:

$$\begin{aligned} \|S(t+h) - S(t)\| &\leq Mh; \\ \|E(t+h) - E(t)\| &\leq Nh. \end{aligned}$$

Remark 2.2 *The Lipschitz continuity condition implies both strong continuity and initial continuity, (Section 2.8). In other cases it appears to be that the conditions are independent, but the natural development of the theory requires that they be added one by one.*

2.3 Resolvent relations and invertibility

As for semigroups (and integrated semigroups) the technique of Laplace transforms allowed one to characterize the empathy relation in terms of pseudo-resolvents [S2]. We will do this for an integrated empathy in the same way.

The following relations are obtained only from the definitions and are of importance.

Theorem 2.2 *Under the minimal assumptions the following identities hold for all positive r, s, t and λ*

$$S(t)E(s) = S(s)E(t); \tag{2.8}$$

$$S(t)E(r)E(s) \text{ is invariant under permutations of } r,\ s \text{ and } t; \tag{2.9}$$

$$S(t)R(\lambda) = P(\lambda)E(t). \tag{2.10}$$

Proof. The first identity, (2.8) is a consequence of (2.4). To prove (2.9) we note that Theorem 2.1 gives

$$S(t)E(r)E(s) = \int_0^r \left[\int_0^s S(\sigma+\rho+t) - S(\sigma+\rho)\, d\sigma \right] d\rho.$$

Fubini's theorem shows that r and s may be interchanged. The claim now follows from the first identity, and (2.10) is obtained from (2.8) by taking Laplace transforms at λ with respect to s.

In the demonstration of the following theorem we freely make use of the convolution theorem and operational properties as set out in Section 6.5. The proof turns out to be an easy calculation.

Theorem 2.3 *If $\langle S, E\rangle$ is an integrated empathy, then the pseudo-resolvent equation*

$$P(\lambda) - P(\mu) = (\mu-\lambda)P(\lambda)R(\mu) = (\mu-\lambda)P(\mu)R(\lambda) \tag{2.11}$$

holds under the minimal assumptions. As a partial converse, if the strong continuity assumption is added for $\langle S, E\rangle$ then (2.11) implies that $\langle S, E\rangle$ satisfies the integrated empathy relation (2.3).

Proof. If we take Laplace transforms on both sides of (2.3) at λ with respect to t we obtain

$$S(s)r(\lambda) = \mathcal{L}_\lambda\big\{(1 * S_s)(t) - (1 * S)(t)\big\},$$

(for the notation see Section 1.5). The convolution theorem gives

$$\begin{aligned} S(s)r(\lambda) &= \frac{1}{\lambda}\mathcal{L}_\lambda\{S_s\} - \frac{1}{\lambda}p(\lambda), && (2.12)\\ S(s)R(\lambda) &= \mathcal{L}_\lambda\{S_s\} - p(\lambda) \\ &= e^{\lambda s}p(\lambda) - (e^{\lambda\cdot} * S)(s) - p(\lambda). && (2.13) \end{aligned}$$

When taking transforms again, this time at μ with respect to s,

$$\begin{aligned} p(\mu)R(\lambda) &= \frac{1}{\mu-\lambda}p(\lambda) - \frac{1}{\mu-\lambda}p(\mu) - \frac{1}{\mu}p(\lambda), \text{ or} \\ P(\mu)R(\lambda) &= \frac{\mu}{\mu-\lambda}[p(\lambda) - p(\mu)] - p(\lambda), \end{aligned}$$

and when simplifying we obtain

$$P(\lambda) - P(\mu) = (\mu - \lambda)P(\mu)R(\lambda).$$

By interchanging the role of λ and μ and then rewriting we have

$$P(\mu)R(\lambda) = P(\lambda)R(\mu). \tag{2.14}$$

When retracing our steps and employing the uniqueness theorem, Section 6.1, we obtain the partial converse, with (2.3) holding for almost every s and t. Strong continuity ensures that the relation holds for all s, $t \in R^+$. □

Remark 2.3 *It would be tempting to derive (2.14) from (2.10) as can be done formally in the same manner as in Theorem 2.2, but this can only be justified if $P(\lambda)$ is a bounded operator and we do not assume this here.*

Corollary 2.1 *The domain $\mathcal{D}_S := P(\lambda)[Y] \subset X$ does not depend on the choice of λ.*

Proof. Let $x = P(\mu)y$, then $x = P(\lambda)\big[1 - (\lambda - \mu)R(\mu)\big]y$. □

When we add the invertibility assumption more results follow.

An essential property for the development of a theory in the case of an empathy is that under the invertibility condition E is a semigroup. This has the analogy which one would expect:

Theorem 2.4 *Under the invertibility condition E has the integrated semigroup property (1.4).*

Proof.

$$\begin{aligned}
[S(t)E(r)]E(s) &= \int_0^t \big[S(\rho + r)E(s) - S(\rho)E(s)\big]\, d\rho \\
&= \int_0^t \big[S(s)E(\rho + r) - S(s)E(\rho)\big]\, d\rho
\end{aligned}$$

$$
\begin{aligned}
&= S(s)\int_0^t \big[E(\rho+r)-E(\rho)\big]\,d\rho \\
&= S(t)\int_0^r \big[E(\rho+s)-E(\rho)\big]\,d\rho
\end{aligned}
$$

where we used the permutation identity (2.9) in the last step. If we take the Laplace transform at ξ with respect to t,

$$P(\xi)E(r)E(s) = P(\xi)\int_0^r \big[E(\rho+s)-E(\rho)\big]\,d\rho$$

and the invertibility of $P(\xi)$ yields (1.5). □

Remark 2.4 *From here onwards we always assume invertibility of $P(\xi)$ for at least one ξ.*

Results obtained for S so far all have analogies for E, if we think of E as a special integrated empathy. While keeping in mind Theorem 2.4 it has been known [A] that R satisfies the pseudo-resolvent equation

$$R(\lambda)-R(\mu) = (\mu-\lambda)R(\lambda)R(\mu) = (\mu-\lambda)R(\mu)R(\lambda), \tag{2.15}$$

but it may also be seen from Theorem 2.3 by letting $X = Y$ and $S = E$, for then $P = R$. Corresponding to (2.13) we have for integrated semigroups

$$E(s)R(\lambda) = e^{\lambda s}r(\lambda) - (e^{\lambda\cdot} * E)(s) - p(\lambda). \tag{2.16}$$

Similar to (2.10) we have that

$$E(t)R(\lambda) = R(\lambda)E(t), \tag{2.17}$$

and from the pseudo-resolvent equation it follows as before that

$$R(\mu)R(\lambda) = R(\lambda)R(\mu). \tag{2.18}$$

It is also true that $\mathcal{D}_E := R(\lambda)[Y] \subset Y$ is independent of λ (as per Corollary 2.1). We also see, from $R(\mu)y = 0$ implies $R(\lambda)y = (\mu-\lambda)R(\lambda)R(\mu)y = 0$, that the null space $N_E := \mathrm{Ker}R(\lambda)$ does not depend on the choice of λ. These two facts are quite well known, see e.g. ([Y2], Chapter 8, Proposition, p.215). For $P(\lambda)$ there is a more intricate property which is also a direct consequence of the pseudo-resolvent equations. The proofs for the following two results were first given in [S2].

Lemma 2.1 *Under the minimal and invertibility assumptions,* $N_E \cap KerP(\lambda) = N_E \cap KerP(\mu)$ *for every positive* λ *and* μ.

Proof. If $y \in N_E \cap \text{Ker}P(\mu)$ then by (2.11) $P(\lambda)y = (\mu - \lambda)P(\lambda)R(\mu)y = 0$. Hence $y \in \text{Ker}P(\lambda)$. □

The invertibility assumption has strong implications.

Theorem 2.5 *The invertibility of* $P(\xi)$ *for one* ξ *implies that the linear operators* $P(\lambda)$ *are all invertible.*

Proof. Suppose that $P(\lambda)y = 0$. By (2.11) we have $P(\xi)y = (\lambda - \xi)P(\xi)R(\lambda)y$. The invertibility of $P(\xi)$ now gives $y = (\lambda - \xi)R(\lambda)y$ and hence $R(\xi)y = (\lambda - \xi)R(\xi)R(\lambda)y$. Comparison with (2.15) shows that $R(\lambda)y = 0$. By Lemma 2.1, $y \in N_E \cap \text{Ker}P(\lambda) = N_E \cap \text{Ker}P(\xi)$ and $y = 0$. This means that $P(\lambda)$ is invertible. □

In the same manner as in the original empathy theory, we have the corresponding *representation equations* which facilitate many of the proofs in the sequel (note that (2.19) is not dependent on invertibility while (2.20) is).

Theorem 2.6 (Representation equations) *Let* $y = R(\lambda)y_\lambda \in \mathcal{D}_E$. *Then*

$$S(t)y = e^{\lambda t}\left[\frac{1}{\lambda}(1 - e^{-\lambda t})P(\lambda)y_\lambda - \int_0^t e^{-\lambda s}S(s)y_\lambda \, ds\right]; \qquad (2.19)$$

$$E(t)y = e^{\lambda t}\left[\frac{1}{\lambda}(1 - e^{-\lambda t})y - \int_0^t e^{-\lambda s}E(s)y_\lambda \, ds\right]. \qquad (2.20)$$

Proof. Note that (2.13) may be rewritten to obtain the expressions (2.19) and (2.20) respectively. □

Remark 2.5 *The equation (2.19) does not depend on the invertibility assumption. Moreover we do not necessarily have that y_λ is unique, which would be the case if $R(\lambda)$ were invertible.*

Theorem 2.7 *$S(t)[\mathcal{D}_E] \subset \mathcal{D}_S$ and $E(t)[\mathcal{D}_E] \subset \mathcal{D}_E$, and for $y \in \mathcal{D}_E$ the mappings $t \mapsto S(t)y$ and $t \mapsto E(t)y$ are continuous.*

Proof. Suppose $y \in \mathcal{D}_E$ then $S(t)y = P(\lambda)E(t)y_\lambda$ by (2.10). The inclusion $E(t)\mathcal{D}_E \subset \mathcal{D}_E$ follows from (2.17). The continuity follows from the continuity in t of the Bochner integral and Theorem 2.6 where S and E map from $\mathcal{D}_E$. □

Remark 2.6 *The trajectory of S, defined by $y \in Y$, is the 'curve' $T_{S,y} := \{S(t)y \mid t > 0\} \subset X$ and the trajectories of E are defined analogously. Another way of stating Theorem 2.7 would then be to say that $\{T_{S,y} \mid y \in \mathcal{D}_E\} \subset \mathcal{D}_E$ and that these trajectories are all continuous curves.*

2.4 Initial continuity and invertibility

The assumption of initial continuity (of S) added to the minimal and invertibility conditions has far reaching consequences. It will be understood that this condition is assumed from here onwards.

Firstly it enables us to have a "right-hand derivative" of $S(t)y$ for all $y \in \mathcal{D}_E$ at $t = 0$, and secondly the surprising fact that initial continuity for S only, gives the invertibility of every $R(\lambda)$. This demonstrates a subtle interdependence of the properties of E and that of S, which is related to the integrated empathy relation.

The initial continuity condition for E is a separate assumption, and will always be explicitly stated. The consequences of initial continuity for S has analogies for E, when initial continuity of E is assumed instead.

Theorem 2.8 *If S satisfies the initial continuity condition the following is true:*

(a) $C_0 y := \lim_{t\to 0+} t^{-1} S(t)y$ *exists for* $y \in \mathcal{D}_E$*;*

(b) The operators $R(\lambda)$ *are all invertible;*

(c) $C_0 = P(\lambda)R^{-1}(\lambda)$ *is invertible and* $C_0[\mathcal{D}_E] = \mathcal{D}_S$*;*

(d) For $y \in \mathcal{D}_E$ $C_0 y = \lim_{\lambda\to\infty} \lambda P(\lambda)y$*.*

Proof. Statement (a) follows from (2.19) after division by t and letting $t \to 0+$. The assumption that $S(t) \to 0$ as $t \to 0+$ lets the integral term fall away. So $C_0 y = P(\lambda)y_\lambda$, if $y = R(\lambda)y_\lambda$. For (b), suppose $y \in \mathcal{D}_E$ has the two representations $y = R(\lambda)y_\lambda = R(\lambda)z_\lambda$, then $C_0 y = P(\lambda)y_\lambda = P(\lambda)z_\lambda$ hence by Theorem 2.5, $y_\lambda = z_\lambda$ which gives the result. The validity of (c) follows because of the invertibility of $P(\lambda)$ and $R(\lambda)$.

In order to prove, (d) we first note that the condition of initial continuity implies that $\lim_{\lambda\to\infty} P(\lambda) = 0$ after the dominated convergence theorem is applied in the definition of P. From (2.11), (acting on y_μ)

$$\lambda P(\lambda)R(\mu)y_\mu - P(\mu)y_\mu = \mu P(\lambda)R(\mu)y_\mu - P(\lambda)y_\mu,$$

then we set $y = R(\mu)y_\mu$ to obtain

$$\lambda P(\lambda)y - C_0 y = P(\lambda)\left[\mu y - y_\mu\right]$$

and the statement follows when $\lambda \to \infty$. □

The analogies for E are seen to be

Theorem 2.9 *If E satisfies an initial continuity condition the following is true for all* $y \in \mathcal{D}_E$*,*

(a) $\lim_{t\to 0+} t^{-1} E(t)y = y$*;*

(b) $\lim_{\lambda\to\infty} \lambda R(\lambda)y = y$*.*

2.5 The generator and implicit Cauchy problem

The assumptions we have made thus far, (the invertibility assumption and the initial continuity assumption), enable us, similar to the theory of integrated semigroups and using identical formulae and arguments to [S2], to define the concept of the *generator* of an integrated empathy $\langle S, E\rangle$. This will be a pair of operators $\langle A, B\rangle$ with common domain $\mathcal{D}_S$ each mapping into Y, chosen in such a manner that

$$P(\lambda) = (\lambda B - A)^{-1}. \tag{2.21}$$

The operators A and B are defined as follows

$$B \quad := \quad B_\lambda = {C_0}^{-1} = R(\lambda)P^{-1}(\lambda); \tag{2.22}$$

$$A \quad := \quad A_\lambda = \big[\lambda R(\lambda) - I_Y\big]P^{-1}(\lambda), \tag{2.23}$$

with I_Y the identity on Y.

It is straightforward to check that (2.21) is satisfied. Indeed,

$$(\lambda B - A)^{-1} = \Big(\lambda R(\lambda)P^{-1}(\lambda) - \big[\lambda R(\lambda)P^{-1}(\lambda) - P^{-1}(\lambda)\big]\Big)^{-1} = P(\lambda).$$

However we still need to know that A and B do not depend on λ for this definition to be meaningful.

Proposition 2.1 *The operators A and B as defined in (2.22–2.23) do not depend on the choice of λ.*

Proof. It can be seen from Theorem 2.8(a) and (c), and the definition of the operator that the claim holds for B. Formal inversion of the resolvent equation gives

$$P^{-1}(\lambda) - P^{-1}(\mu) = (\lambda - \mu)R(\lambda)P^{-1}(\lambda) = (\lambda - \mu)B.$$

Now

$$\begin{aligned} A_\lambda - A_\mu &= \big[\lambda R(\lambda)P^{-1}(\lambda) - P^{-1}(\lambda)\big] - \big[\mu R(\mu)P^{-1}(\mu) + P^{-1}(\mu)\big] \\ &= (\lambda - \mu)B - \big(P^{-1}(\lambda) - P^{-1}(\mu)\big) = 0 \end{aligned}$$

which implies that A_λ is independent of λ and A is well-defined. $\square$

We will need some identities and results which involve the generator $\langle A, B\rangle$. To obtain these we apply (2.10) and (2.17) to the definition. The subdomains $\mathcal{D}_E^k; k = 1, 2, \ldots$ are defined recursively by $\mathcal{D}_E^1 := \mathcal{D}_E$, $\mathcal{D}_E^k := R(\lambda)\left[\mathcal{D}_E^{k-1}\right]$. For consistency we take $\mathcal{D}_E^0 := Y$. It is seen that the sequence $\{\mathcal{D}_E^k\}$ of subspaces of Y are decreasing.

Theorem 2.10 *Let $x \in \mathcal{D}_S$ and $y = Bx = R(\lambda)y_\lambda \in \mathcal{D}_E$. Then*

(a) $Ax = AC_0y = (\lambda y - y_\lambda)$;

(b) $E(t)y = BS(t)y$;

(c) $AS(t)y = E(t)AC_0y = E(t)Ax$;

For $y \in \mathcal{D}_E^{k+1}$, $k \geq 1$,

(d) $AC_0y \in \mathcal{D}_E^k$ and $S(t)AC_0y = C_0AS(t)y$.

Proof. To obtain these identities is a matter of applying the definitions of A and B and using simple algebra. $\square$

Remark 2.7 *The present assumptions allow us to write (2.19) in a more familiar form, cf. ([ABHN], Lemma 3.2.2(d), p.125), as follows: Let $y = Bx$; $x \in \mathcal{D}_S$ then*

$$\begin{aligned}
S(t)y &= \frac{1}{\lambda}(e^{\lambda t} - 1)C_0y - e^{\lambda t}\int_0^t e^{\lambda s}S(s)y_\lambda\, ds \\
&= \frac{1}{\lambda}(e^{\lambda t} - 1)x - e^{\lambda t}\int_0^t e^{\lambda s}S(s)P^{-1}(\lambda)x\, ds \\
&= \frac{1}{\lambda}(e^{\lambda t} - 1)x - e^{\lambda t}\int_0^t e^{\lambda s}S(s)(\lambda B - A)x\, ds \\
&= \frac{1}{\lambda}(e^{\lambda t} - 1)x - \lambda e^{\lambda t}\int_0^t e^{\lambda s}S(s)Bx - e^{\lambda t}\int_0^t e^{\lambda s}S(s)Ax\, ds
\end{aligned}$$

by letting $\lambda \to 0+$,

$$S(t)y = tx + \int_0^t S(s)Ax\,ds. \tag{2.24}$$

Theorem 2.11 *If* $y \in \mathcal{D}_E$, *the functions* $t \mapsto v(t) := S(t)y$ *and* $t \mapsto w(t) := E(t)y$ *are differentiable in* X *and* Y *respectively, and*

$$u(t) := v'(t) = S(t)AC_0y + C_0y = C_0\big[AS(t)y + y\big]; \tag{2.25}$$

$$w'(t) = E(t)AC_0y + y. \tag{2.26}$$

Moreover

$$\lim_{t\to 0+} Bu(t) = y. \tag{2.27}$$

Proof. The differentiability can be seen from Theorem 2.6. The expressions for v' and w' are obtained from (2.19) and (2.20), as well as Theorem 2.10(d). To prove (2.27) we notice that(2.25) combined with Theorem 2.10(b) gives the identity

$$Bu(t) = E(t)AC_0y + y \tag{2.28}$$

and since $AC_0y \in \mathcal{D}_E$ it follows from (2.20) that $E(t)AC_0y \to 0$ as $t \to 0+$. $\square$

Remark 2.8 *Since* $\mathcal{D}_E^{k+1}$ *is a nested sequence, Theorem 2.11 also holds if* $y \in \mathcal{D}_E^{k+1}$, $k \geq 0$.

We introduce the notion of a tangent trajectory whereby Theorem 2.11 may be interpreted in a dynamic systems framework.

Remark 2.9 *If the function* $t \mapsto S(t)y$ *is differentiable in* X *the curve* $T'_{S,y} = \big\{\frac{d}{dt}S(t)y \mid t > 0\big\}$ *is called the* **tangent trajectory** *of* S *determined by* y. *If* $x = \lim_{t\to 0+}\frac{d}{dt}S(t)y$ *exists, then we say* $T'_{S,y}$ **emanates** *from* x.

The subdomains of $\mathcal{D}_S$ are defined recursively by $\mathcal{D}_S^{k+1} := P(\lambda)\big[\mathcal{D}_E^k\big] \subset \mathcal{D}_S$ for $k = 1, 2, \ldots$ and $\mathcal{D}_S^0 := X$.

The equation (2.29) is the integrated equivalent of (1.6). Further differentiability of $E(t)$ will ensure that (1.6) is satisfied.

Theorem 2.12 *If $y \in \mathcal{D}_E^{k+1}$ for $k \geq 1$, then $T'_{S,y}$ exists, $T'_{S,y} \subset \mathcal{D}_S^k$ and emanates from $C_0 y$. The tangent trajectory is an affine transformation of the trajectory, (see Remark 2.6 and Remark 2.9), and is generated by the solution of the implicit integral equation*

$$Bu(t)y = y + A \int_0^t u(s)\, ds. \tag{2.29}$$

If $k \geq 2$ the tangent trajectory is generated by the solution of the implicit Cauchy problem (1.6–1.7).

Proof. The first statement is an interpretation of Theorem 2.11 on noticing that $T'_{S,y} = \{u(t) \mid t > 0\}$. It is also seen from Theorem 2.7 that the function $t \mapsto v'(t)$ is continuous and hence $v(t) = \int_0^t v'(s)\, ds = \int_0^t u(s)\, ds$. The identity (2.29) now follows from (2.25).

Finally, if $k \geq 2$ let $z = AC_0 y$ and $w(t) := E(t)z$. Theorem 2.10(d) implies that $z \in \mathcal{D}_E^{k+1}$ for $k \geq 1$ and hence that w is differentiable. According to (2.26) we have $w'(t) = E(t)AC_0 z + z = BS(t)AC_0 z + z = AS(t)z + z = A\big[S(t)AC_0 y + C_0 y\big] = Au(t)$, by (2.28). But by (2.25) and Theorem 2.10(d) $Bu(t) = AS(t)y + y = E(t)AC_0 y + y = w(t) + y$, so that $\frac{d}{dt}Bu(t)y = w'(t) = Au(t)$. □

2.6 Strong continuity and implications of the Post-Widder inversion theorem

We will add the strong continuity assumption in this section, with the surprising result that it ensures that $E(t)$ maps Y into the closure of $\mathcal{D}_E$ and $S(t)$ maps into the closure of $\mathcal{D}_S$ for each t. Initial continuity of S will be assumed throughout.

Theorem 2.13 *Suppose that S is strongly continuous. If $y \in \mathcal{D}_E^k$ for $k \geq 1$, then $T'_{S,y}$ exists, $T'_{S,y} \subset \mathcal{D}_S^{k-1}$ and emanates from $C_0 y$. If $y \in \mathcal{D}_E^{k+1}$ for $k \geq 1$ the tangent trajectory is an affine transformation of the trajectory and is generated by a solution of the implicit integral equation (2.29). If E is also strongly continuous, the tangent trajectory is generated by a solution of the implicit Cauchy problem (1.6–1.7).*

Proof. When the proof of Theorem 2.12 is retraced it is seen that less regularity of the initial value y is required under strong continuity. □

The condition of strong continuity also has the consequences mentioned in the first paragraph of this section, i.e. $E(t)Y \subset \text{Cl}[\mathcal{D}_E]$ and $S(t)Y \subset \text{Cl}[\mathcal{D}_S]$ for every $t > 0$. For the description we define the closed subspaces $Y_E := \text{Cl}[\mathcal{D}_E] \subset Y$ and $Y_S := \text{Cl}[\mathcal{D}_S] \subset X$. We will need expressions for the *Widder operators* associated with $r(\lambda)$ and $p(\lambda)$, in terms of the operators $R(\lambda)$ and $P(\lambda)$. For $f \in C^\infty\{(0,\infty)\colon X \text{ or } Y\}$ these operators are defined by

$$(L_k f)(\lambda) := \left[\frac{(-1)^k}{k!}\right] \lambda^{k+1} f^{(k)}(\lambda) \text{ for } k = 1, 2, \ldots$$

with $f^{(k)}$ the k'th derivative of f. The Post Widder inversion Theorem, (Appendix B, Section 6.3), states that if f is the Laplace transform of a function g then $(L_k f)(k/t) \to g(t)$ as $k \to \infty$ for all t in the Lebesgue set of g. If g is continuous the convergence is uniform over compact subintervals of $(0,\infty)$.

In the evaluation of the Widder operators the expressions

$$R^{(n)}(\lambda) = (-1)^n n! R^{n+1}(\lambda); \tag{2.30}$$

$$P^{(n)}(\lambda) = (-1)^n n! P(\lambda) R^n(\lambda) \tag{2.31}$$

are employed. These expressions are obtained by using the resolvent equations (2.11) and (2.15) to differentiate R and P.

Lemma 2.2

$$(L_k r)(\lambda) = R(\lambda) \sum_{n=0}^{k} \lambda^n R^n(\lambda); \tag{2.32}$$

$$(L_k p)(\lambda) \quad = \quad P(\lambda)\sum_{n=0}^{k}\lambda^n R^n(\lambda). \tag{2.33}$$

Proof. We prove (2.33) with the aid of the Leibnitz rule and (2.31). The proof of (2.32) is similar.

$$\begin{aligned}
(L_k p)(\lambda) &= \frac{(-1)^k}{k!}\lambda^{k+1}p^{(k)}(\lambda)\\
&= \frac{(-1)^k}{k!}\lambda^{k+1}\left[\frac{1}{\lambda}P(\lambda)\right]^{(k)}\\
&= \frac{(-1)^k}{k!}\lambda^{k+1}\sum_{n=0}^{k}\left[\binom{k}{n}(-1)^{k-n}\frac{(k-n)!}{\lambda^{k-n+1}}\,(-1)^n n! P(\lambda)R^n(\lambda)\right]\\
&= P(\lambda)\sum_{n=0}^{k}\lambda^n R^n(\lambda).
\end{aligned}$$

□

Note that $(L_k r)(\lambda)\colon Y \to \mathcal{D}_E$ and $(L_k p)(\lambda)\colon Y \to \mathcal{D}_S$. Therefore by an application of the Post-Widder theorem we obtain

Theorem 2.14 *If S and E are strongly continuous, then $E(t)[Y] \subset Y_E$ and $S(t)[Y] \subset X_S$. In addition $(L_k p)(k/t) \to S(t)$ and $(L_k r)(k/t) \to E(t)$ in the strong operator topology. The convergence is uniform on compact subintervals of* $(0, \infty)$.

2.7 Bounded resolvents

The boundedness of the resolvent operators $P(\lambda)$ or $R(\lambda)$ has not been a prerequisite for any of the results obtained thus far, and no conditions have been imposed on S or E implying the boundedness of these resolvents. Indeed theorems with far-reaching consequences may be proved in making such assumptions.

In this section we investigate consequences of adding boundedness of some or all of the resolvent operators. These results involve:

(a) The behaviour of the family E at $t = 0$;

(b) The properties of the operator C_0;

(c) Closedness of the generator $\langle A, B\rangle$.

Note that we do not make the strong continuity assumption in this section nor is initial continuity of E assumed. The *ancestral space* of the family E is defined as

$$\mathrm{Anc}(E) := \Big\{ y \in Y \mid \lim_{t\to 0+} \frac{1}{t} E(t)y \text{ exists} \Big\}.$$

From (2.20) it is seen that $\mathcal{D}_E^2 \subset \mathrm{Anc}(E)$ and for $y \in \mathcal{D}_E^2$, $\lim_{t\to 0+} \frac{1}{t}E(t)y = y$. In order to extend this result we need representations similar to (2.19–2.20) which will now be obtained, (note that we have initial continuity of S which gave the invertibility of the respective resolvents).

Proposition 2.2 *For arbitrary $y \in Y$, $t > 0$ and $\lambda > 0$, $\int_0^t e^{-\lambda s} S(s)y\, ds \in \mathcal{D}_s$. Also,*

$$E(t)y = e^{\lambda t}\left[\frac{1}{\lambda}(1 - e^{-\lambda t})y - P^{-1}(\lambda)\int_0^t e^{-\lambda s} S(s)y\, ds\right]. \tag{2.34}$$

Proof. The equation follows from:

$$P(\lambda)E(t)y = e^{\lambda t}\left[\frac{1}{\lambda}(1 - e^{-\lambda t})P(\lambda)y - \int_0^t e^{-\lambda s} S(s)y\, ds\right], \tag{2.35}$$

after noting that (2.35) implies that the integral on the right is in $\mathcal{D}_S$.

To derive (2.35) the calculation is similar to that of obtaining (2.19) from the integrated empathy relation but now we set $S(t)R(\lambda) = P(\lambda)E(t)$ in (2.13). □

Theorem 2.15 *Suppose that the operator $P(\lambda)$ is bounded for at least one value of λ. If $y \in Anc(E)$ then $\lim_{t\to 0+} \frac{1}{t}E(t)y = y$.*

Proof. The boundedness of $P(\lambda)$ means that $P^{-1}(\lambda)$ is closed. The claim then follows from the initial continuity of S and (2.34). □

Notice that properties of S once again have consequences for E. When E has continuity properties of its own there is much more that can be said:

Theorem 2.16 *If E is initially continuous, then*

(a) $\mathcal{D}_E \subset Anc(E)$;

(b) If $R(\lambda)$ is bounded for a single λ then $\lim_{t\to 0+} \frac{1}{t}E(t)y = y$ for all $y \in Anc(E)$.

When the strong continuity of E is assumed as well, $Anc(E) \subset Y_E$.

Proof. Part (a) is implied by Theorem 2.9(a). In the proof of (b) the formula

$$E(t)y = e^{\lambda t}\left[\frac{1}{\lambda}(1-e^{-\lambda t})y - R^{-1}(\lambda)\int_0^t e^{-\lambda s}E(s)y\,ds\right] \tag{2.36}$$

may be used, or it may be observed that both this formula and (b) is true if we consider that E is an integrated empathy with $S = E$. For then (b) is the same as Theorem 2.15. The last statement follows from Theorem 2.14. □

The boundedness of $P(\lambda)$ for each λ has even more implications, specifically regarding the operator C_0.

Theorem 2.17 *If every $P(\lambda)$, $\lambda > 0$ is bounded, and there exists a real number K such that $\lambda\|P(\lambda)\| < K$ for all $\lambda > 0$, then the operator C_0 is bounded and the continuous extension of C_0 to Y_E, which we call C, has the property that $Cy = \lim_{\lambda\to\infty} \lambda P(\lambda)y$ for all $y \in Y_E$. If both S and E are strongly continuous, $S(t) = CE(t)$.*

Proof. The result follows by applying the Uniform Boundedness Principle to Theorem 2.8(d). The second statement follows from Theorem 2.14 and the identity (2.10). □

In the theory of semigroups and integrated semigroups a pertinent feature is that the generator is closed (or closable). The situation in empathy theory is not different, but the corresponding property that the pair $\langle A, B\rangle$ mapping into the product space is closed, is nevertheless surprising.

A number of results may be proved regarding the closedness of the generator, all of which involve boundedness conditions on either or both $P(\lambda)$ and $R(\lambda)$.

Theorem 2.18 *The operator $\langle A, B\rangle \colon x \in \mathcal{D}_S \subset X \mapsto Y \times Y$ is closed provided that $P(\lambda)$ is bounded for at least two values of λ.*

Proof. Let $P(\lambda)$ and $P(\mu)$ be bounded for $\lambda \neq \mu$. Suppose $x_n \to x$ and $\langle Ax_n, Bx_n\rangle \to \langle y_1, y_2\rangle$ as $n \to \infty$ where $x_n \in \mathcal{D}_S$. Then

$$\lambda Bx_n + Ax_n \to \lambda y_1 + y_2,$$

$$\mu Bx_n + Ax_n \to \mu y_1 + y_2, \text{ as } n \to \infty.$$

The closedness of $P^{-1}(\lambda)$ and $P^{-1}(\mu)$ now implies that $x \in \mathcal{D}_S$ and that

$$\lambda Bx + Ax = \lambda y_1 + y_2,$$

$$\mu Bx + Ax = \mu y_1 + y_2.$$

On solving for Ax and Bx we obtain $\langle Ax, Bx\rangle = \langle y_1, y_2\rangle$. $\square$

This result is in contrast to other works in implicit Cauchy problems where it is often assumed that both of the operators A and B are closed (e.g. [AF], [CS], [FY]). Much of the present significance of our work regarding applications depends on Theorem 2.18. This is because applications in boundary value problems do exist where the operator B is not closable. The case where B is closable is summarized in the following theorem. This theorem is quoted from [S1] and [S2].

Theorem 2.19 *The operator B is closable if and only if C is invertible.*

Proof. If C is invertible then the inverse is closed and $B = C_0^{-1}$ is the restriction of a closed operator and is therefore closable.

For the converse suppose that C_0^{-1} is closable and $Cy = 0$. Let $\{y_n\} \subset \mathcal{D}_E$ be such that $y_n \to y$. Then $x_n = Cy_n = C_0 y_n \to 0$ and $y_n = C_0 x_n \to y$ as $n \to \infty$ and hence $y = 0$. □

When the generator $\langle A, B\rangle$ is closed we may say more about the implicit Cauchy problem, indeed, of the solutions to some implicit integral equations.

Proposition 2.3 *Suppose the generator $\langle A, B\rangle$ of an integrated empathy is closed. Then, for $y \in \mathcal{D}_E$, $\int_0^t S(s)y\,ds \in \mathcal{D}_S$ and $A\int_0^t S(s)y\,ds = \int_0^t AS(s)y\,ds$. Furthermore*

$$E(t)y = ty + \int_0^t AS(s)y\,ds. \tag{2.37}$$

Proof. From Theorem 2.10(b) and (c) we see that $AS(s)y = E(s)Ax$ and $BS(s)y = E(s)y$ are integrable (with $y = Bx$). Since $\langle A, B\rangle$: $\mathcal{D}_S \subset X \to Y \times Y$ is closed, it follows that $\int_0^t S(s)yds \in \mathcal{D}_S$ and that both A and B commute with the integral. Therefore $P^{-1}(\lambda) = \lambda B - A$ will also commute. Thus (2.34) may in this case be written in the form

$$E(t)y = te^{\lambda t}\left[\left(\frac{1-e^{-\lambda t}}{\lambda t}\right)y - \lambda\int_0^t E(s)yds + \int_0^t AS(s)yds\right].$$

By letting $\lambda \to 0+$ in this expression we obtain the final statement. □

Corollary 2.2 *Suppose the generator $\langle A, B\rangle$ of an integrated empathy is closed and S is strongly continuous. For $y \in \mathcal{D}_E$ let $v(t) := S(t)y$. Then v satisfies the implicit integral equation*

$$Bv(t) = ty + A\int_0^t v(s)\,ds. \tag{2.38}$$

Proof. The last statement in Theorem 2.17 is applied to (2.37). □

Remark 2.10 *The form of equation (2.38) is apparently unavoidable. It would be preferable to have a direct analogy of (2.29) similar to that for integrated semigroups ([ABHN], p.130) but the commutation relation Theorem 2.10(d)* *does not make sense for* $y \in \mathcal{D}_E$. *Also, operator* B *does not have to possess all the agreeable properties of the identity operator.*

2.8 Lipschitz continuity

In [S2] uniformly bounded empathies are considered and appropriate necessary and sufficient conditions are imposed on the resolvents. The corresponding property for an integrated empathy turns out to be stronger, the families should be Lipschitz continuous. The conditions in the form of inequalities are very similar to the Hille-Yosida condition. Note that in the proof of this result the form of the convolution theorem given in Section 6.5 is crucial.

Theorem 2.20 *A necessary and sufficient condition for an integrated empathy to be Lipschitz continuous with Lipschitz constants* M *and* N *is that for all* $\lambda > 0$ *and* $k = 0, 1, 2, \ldots$

$$\|\lambda P(\lambda)[\lambda R(\lambda)]^k\| \leq M; \tag{2.39}$$

$$\|[\lambda R(\lambda)]^{k+1}\| \leq N. \tag{2.40}$$

Lipschitz continuous empathies are also initially continuous.

Proof. Assume the uniform Lipschitz continuity of S. Denote by E^{k*} the k-fold convolution of E with itself.

To find an estimate for $\|(S*E^{k*})(t)\|$ we evaluate the convolutions $(S*E^{k*})(t) = \int_0^t \int_0^{s_1} \ldots \int_0^{s_k} S(t-s_1)E(s_1-s_2)E(s_2-s_3)\ldots E(s_{k-1}-s_k)E(s_k)ds_k \ldots ds_1$. By using (2.5) we may evaluate the integrand

$$\begin{aligned} &S(t-s_1)E(s_1-s_2)\ldots E(s_{k-1}-s_k)E(s_k) \\ &= \int_0^{s_1-s_2} \ldots \int_0^{s_{k-1}-s_k} \int_0^{s_k} \Big[S(t-s_1+\xi) - S(\xi)\Big] d\rho_k \ldots d\rho_1 \end{aligned} \tag{2.41}$$

where $\xi = \rho_1 + \ldots + \rho_k$. Because of the Lipschitz condition

$$\Big\|[S(t-s_1+\xi) - S(\xi)]\Big\| \leq M(t-s_1).$$

When calculating the integral we obtain

$$\|S{*}E^{k*}\| \leq M \int_0^t \int_0^{s_1} \ldots \int_0^{s_k} (t{-}s_1)(s_1{-}s_2)(s_2{-}s_3)\ldots(s_{k-1}{-}s_k)(s_k)ds_k\ldots ds_1$$

and after inspection $\|S * E^{(k*)}\| \leq M[t]^{(k+1)*}$, where $[t]$: $t \to t$.

It also follows from the convolution theorem (Appendix B, Section 6.6) that $p(\lambda)r^k(\lambda) = \mathcal{L}_\lambda\{S * E^{k*}\}$, and therefore we have

$$\left\| \frac{1}{\lambda}P(\lambda)\left[\frac{1}{\lambda}R(\lambda)\right]^k \right\| \leq \int_0^\infty e^{-\lambda s} M[t]^{(k+1)*} ds = \frac{1}{\lambda^{2k+2}}M.$$

Inequality (2.39) follows. The proof of (2.39) is essentially the same.

To prove the converse observe that the Widder operators defined in (6.1) satisfy the following:

$$\frac{d}{d\lambda}(L_k p)(\lambda) = \frac{k+1}{\lambda}\big[(L_k p) - (L_{k+1}p)\big] = -\frac{k+1}{\lambda^2}\lambda P(\lambda)[\lambda R(\lambda)]^k, \qquad (2.42)$$

where (2.31) is used. When integrating (2.42) we find that

$$(L_k p)(\lambda) - (L_k p)(\mu) = -(k+1)\int_\mu^\lambda \rho P(\rho)[\rho R(\rho)]^k \frac{d\rho}{\rho^2}. \qquad (2.43)$$

From (2.39) and (2.43) it now follows that

$$\big\|(L_k p)(\lambda) - (L_k p)(\mu)\big\| \leq M(k+1)\int_\mu^\lambda \frac{d\rho}{\rho^2} = M(k+1)\left[\frac{1}{\mu} - \frac{1}{\lambda}\right]. \qquad (2.44)$$

With $\lambda = k/t$ and $\mu = k/(t+h)$ this becomes

$$\big\|(L_k p)\big(k/t\big) - (L_k p)\big(k/(t+h)\big)\big\| \leq M\left(1 + \frac{1}{k}\right)h. \qquad (2.45)$$

It follows from this and the Post-Widder theorem that

$$\|S(t+h) - S(t)\| \leq \lim_{k\to\infty} \big\|(L_k p)\big(k/t\big) - (L_k p)\big(k/(t+h)\big)\big\| \leq Mh.$$

□

Remark 2.11 *Note that this theorem gives exact bounds M and N, namely the constants in the Lipschitz continuity inequalities and the characterizing inequalities (2.39–2.40) are identical.*

It is fortunate that a much larger class of integrated empathies and corresponding Cauchy problems are covered by this theory. An integrated empathy is called *exponentially Lipschitz continuous* if there are constants M and a such that

$$\begin{aligned} \|S(t+h)-S(t)\| &\leq Me^{at}h \text{ and} & (2.46)\\ \|E(t+h)-E(t)\| &\leq Me^{at}h. & (2.47) \end{aligned}$$

The transformations

$$\begin{aligned} S_{\{a\}}(t) &:= e^{-at}S(t)+a\int_0^t e^{-as}S(s)\,ds; & (2.48)\\ E_{\{a\}}(t) &= e^{-at}E(t)+a\int_0^t e^{-as}E(s)\,ds & (2.49) \end{aligned}$$

gives an integrated empathy $\langle S_{\{a\}}, E_{\{a\}}\rangle$ that is (uniformly) Lipshitz continuous with Lipschitz constants M and N as may be confirmed with a direct calculation.

The transformations are valid for both positive and negative a and are consequently invertible.

Furthermore $P_{\{a\}}(\lambda) = P(\lambda - a)$ and $R_{\{a\}}(\lambda) = R(\lambda - a)$, which means that an exponentially Lipschitz continuous integrated empathy may always be transformed into an integrated empathy with the resolvents defined for any positive argument λ. It also follows that the corresponding generator is $\langle A + aB, B\rangle$ with the implicit differential equation

$$\frac{d}{dt}[Bu(t)] = [A + aB]u(t).$$

The simplest way to confirm all the above claims is to take Laplace transforms on both sides of (2.48–2.49), cf. ([ABHN], Proposition 3.2.6, p.128). The notion discussed here corresponds to that of an *exponentially bounded* empathy covered in [S3].

2.9 Characterization of the generator

In line with Theorem 6.4 the generator of a *Lipschitz continuous* integrated empathy is characterized. This is in contrast to the case of an empathy where only *exponential boundedness* is assumed. However, the space Y must then have the Radon-Nikodým property [S2] and in the present setting this is not necessary. The development is parallel to that of Arendt for integrated semigroups, ([A], Theorem 4.1; [ABHN], Theorem 3.3.1, Chapter 3, p.135–136).

Let $A, B\colon \mathcal{D} \subset X \to Y$ be given linear operators. We formally define for $\lambda > 0$,

$$P(\lambda, \langle A, B\rangle) := (\lambda B - A)^{-1}; \tag{2.50}$$

$$R(\lambda, \langle A, B\rangle) := BP(\lambda). \tag{2.51}$$

The following is the promised characterization of the generator in terms of the Hille-Yosida type inequalities given in the previous section. With the theory we have developed it is a straightforward consequence of Arendt's theorem.

Theorem 2.21 *The pair $\langle A, B\rangle$ is the generator of a Lipschitz continuous integrated empathy $\langle S, E\rangle$ with $P(\lambda) = P(\lambda, \langle A, B\rangle)$ and $R(\lambda) = R(\lambda, \langle A, B\rangle)$ if and only if $\lambda B - A\colon \mathcal{D} \to Y$ is bijective, the operators $P(\lambda, \langle A, B\rangle)$ and $R(\lambda, \langle A, B\rangle)$ are bounded for all $\lambda > 0$ and satisfy the inequalities (2.39–2.40) for $k = 0, 1, 2, \ldots$.*

Proof.

$$\begin{aligned} P(\lambda, \langle A, B\rangle) - P(\mu, \langle A, B\rangle) &= (\lambda B - A)^{-1} - (\mu B - A)^{-1} \\ &= (\lambda B - A)^{-1}\big((\mu B - A) - (\lambda B - A)\big)(\mu B - A)^{-1} \\ &= (\mu - \lambda)P(\lambda, \langle A, B\rangle)R(\lambda, \langle A, B\rangle), \end{aligned} \tag{2.52}$$

and hence, by letting B act on both sides of (2.52),

$$R(\lambda, \langle A, B\rangle) - R(\mu, \langle A, B\rangle) = (\mu - \lambda)R(\lambda, \langle A, B\rangle)R(\lambda, \langle A, B\rangle). \tag{2.53}$$

Since $P(\lambda, \langle A, B\rangle)$ and $R(\lambda, \langle A, B\rangle)$ satisfy the resolvent equations (2.52–2.53) it may be verified that both are infinitely differentiable when considered as functions of the parameter λ and in fact,

$$P^{(k)}(\lambda, \langle A, B\rangle) = (-1)^k k! P(\lambda, \langle A, B\rangle) R^k(\lambda, \langle A, B\rangle); \tag{2.54}$$

$$R^{(k)}(\lambda, \langle A, B\rangle) = (-1)^k k! R^{k+1}(\lambda, \langle A, B\rangle). \tag{2.55}$$

By using the formulae (2.54) and (2.55) and the inequalities (2.39–2.40) it is seen that $(\lambda^{n+1}/n!)P^{(k)}(\lambda, \langle A, B\rangle)$ and $(\lambda^{n+1}/n!)R^{(k)}(\lambda, \langle A, B\rangle)$ are bounded uniformly in λ and n. Arendt's theorem, (Appendix B, Theorem 6.4), implies that there exists a Lipschitz continuous double family $\langle S, E\rangle$ such that $P(\lambda, \langle A, B\rangle) = \lambda\mathcal{L}_\lambda\{S\}$ and $R(\lambda, \langle A, B\rangle) = \lambda\mathcal{L}_\lambda\{E\}$. Because P and R satisfy the resolvent equations (2.52–2.53) and $\langle S, E\rangle$ is strongly continuous, it follows from Theorem 2.3 that $\langle S, E\rangle$ is an integrated empathy. □

Remark 2.12 *The condition of Lipschitz continuity falls just short of assuming that $\langle S, E\rangle$ is differentiable, and if Y has the Radon-Nikodým property the notions coincide, as we shall see in the next section.*

2.10 Differentiability and the Radon-Nikodým property

Corresponding to the property of (uniform) Lipschitz continuity of an integrated empathy is the uniform boundedness of an empathy, i.e. $\|S(t)\| \leq M$ and $\|E(t)\| \leq N$. In fact Theorem 2.20 holds verbatim with this change and "integrated empathy" substituted by "empathy".

The characterization result for an empathy follows much easier in this context than the original of Sauer, ([S2], Theorem 8.2). It is well known that a Lipschitz continuous function on a space that has the Radon-Nikodým property is differentiable almost everywhere. This is extended to the whole of R^+ for an integrated semigroup by using the semigroup property, by Arendt ([A], Theorem 6.2). He makes use of a set-theoretic argument to show that the set of points where E is not differentiable is empty. By using this result and some of our previous theorems, we are able to prove (in analogy to Arendt):

Theorem 2.22 *Suppose that the space Y has the Radon-Nikodým property and let $\langle S, E\rangle$ be a Lipschitz continuous integrated empathy with generator $\langle A, B\rangle$. Then there exists a uniformly bounded empathy $\langle S', E'\rangle$ with generator $\langle A, B\rangle$ such that $P(\lambda)$ and $R(\lambda)$ are the Laplace transforms of S and E respectively, and*

$$S(t) := \int_0^t S'(\sigma)\, d\sigma; \tag{2.56}$$

$$E(t) := \int_0^t E'(\sigma)\, d\sigma. \tag{2.57}$$

Proof. Of course E is an integrated semigroup. As proved by Arendt, the resolvent equation (2.11) and the Hille-Yosida inequality (2.40) implies that $R(\lambda)$ is the Laplace transform of a semigroup E' on Y (on recalling that Y has the Radon-Nikodým property), see Section 6.2. By using Theorem 2.17 with (2.39) for the case $k = 0$ it is clear that the operator $C\colon Y_E \to X$ is a bounded extension of the operator C_0. Let $S'(t) = CE'(t)$. One may easily see that $\langle S', E'\rangle$ is indeed an empathy, (see Example 2.1) with the desired properties:

The uniform boundedness is established by the analogy of Theorem 2.20 for empathies ([S2], Theorem 8.2). Observe that $\left\langle \int_0^t S'(\sigma)\, d\sigma,\, \int_0^t E'(\sigma)\, d\sigma \right\rangle$ is an integrated empathy (Example 2.2). We see that this is exactly $\langle S, E\rangle$ as follows: The resolvents $P(\lambda)$ and $R(\lambda)$ are the same for both. The uniqueness theorem for the Laplace transform (Section 6.1) and the strong continuity of S and E now implies the equality. □

Chapter 3

The n-times integrated empathy

In the same spirit as [A] a general theory of *more-than-once integrated empathies* may be developed, and we give an outline here. The results obtained are often identical to those of Chapter 2 and clues as to where the results differ may be obtained from Arendt.

3.1 More-than-once integrated semigroups

For a general positive integer n Arendt [A] gives the *n-times integrated* semigroup relation as

$$E(t)E(s) = \frac{1}{(n-1)!}\left[\int_t^{s+t}(s+t-\rho)^{n-1}E(\rho)\,d\rho - \int_0^s (s+t-\rho)^{n-1}E(\rho)\,d\rho\right]. \tag{3.1}$$

Clearly the case $n = 1$ corresponds to (1.4). It is assumed in [A] that $\lim_{t\to 0+} E(t)x = 0$ for every $x \in X$.

Suppose that T is a C_0-semigroup. With

$$E(t) = \int_0^t \dots \int_0^t T(t)(dt)^n = \int_0^t (t-s)^{n-1} T(s)\, ds$$

it may be verified with a calculation that (3.1) holds. Thus we see that this notion is indeed a generalization of an n-times integrated semigroup. The reason behind the form of R that follows, also becomes clear.

With R given by $R(\lambda) = \lambda^n r(\lambda) = \lambda^n \mathcal{L}_\lambda\{E\}$ the pseudo-resolvent equation

$$R(\lambda) - R(\mu) = (\mu - \lambda) R(\lambda) R(\mu)$$

is equivalent to the relation (3.1), provided E is strongly continuous ([A], Theorem 3.1). If R is the resolvent of the linear operator A, then A is called the generator of E, as for semigroups.

The characterizing inequalities corresponding to (1.3) have a different form in the more-than-once integrated case, ([A], Theorem 4.1):

Theorem 3.1 *A linear operator A is the generator of an $(n+1)$-times integrated semigroup E satisfying*

$$\|E(t+h) - E(t)\| \leq Mh \text{ for all } h > 0$$

if and only if

$$\left\| \frac{\lambda^{k+1}}{k!} \frac{d^k}{d\lambda^k} \left[\frac{1}{\lambda^n} R(\lambda) \right] \right\| \leq M \text{ for all } \lambda > 0, \;\; k = 0, 1, 2, \dots, \tag{3.2}$$

(assuming that $R(\lambda)$ exists for all $\lambda > 0$).

([ABHN], Proposition 3.2.4, Theorem 3.3.1) may also be referred to regarding these results.

Furthermore the Cauchy problem has the integrated form:

Theorem 3.2 *If A is the generator of an n-times integrated semigroup then for all $x \in \mathcal{D}_{\mathcal{A}}$,*

$$S(t)x = (t^n/n!)x + \int_0^t S(\rho) Ax \, d\rho \tag{3.3}$$

and

$$A\int_0^t S(\rho)x\,d\rho = S(t)x - (t^n/n!)x. \tag{3.4}$$

3.2 Corresponding definitions and assumptions

The correct definition of an n-times integrated empathy is not unexpected. Let $\langle S, E\rangle$ be a family of bounded linear operators with $S(t)\colon X \to Y$, $E(t)\colon Y \to Y$ where X, Y are Banach spaces and the operators are defined for all $t > 0$.

$\langle S, E\rangle$ is an *n-times integrated empathy* if it satisfies

$$S(t)E(s) = \frac{1}{(n-1)!}\left[\int_t^{s+t}(s+t-\rho)^{n-1}S(\rho)\,d\rho - \int_0^s (s+t-\rho)^{n-1}S(\rho)\,d\rho\right]. \tag{3.5}$$

However we have found the following notation which we introduce for this chapter very helpful.

If $f\colon t \to f(t)$; $t > 0$ then the *shifted function* is defined as $f_{[s]}(t) = f(t+s)$ for $s > 0$. In addition let $f_n(t) = t^{n-1}/(n-1)!$ for $n \geq 1$.

It may be verified with a routine calculation that the definition (3.5) is equivalent to

$$S(t)E(s) = (f_n * S_{[s]})(t) - (f_{n,[s]} * S)(t). \tag{3.6}$$

It should be noted that we use the same notation but definitions of the resolvent operators are extended,

$$P(\lambda) = \lambda^n p(\lambda) = \lambda^n \mathcal{L}_\lambda\{S\}$$

and

$$R(\lambda) = \lambda^n r(\lambda) = \lambda^n \mathcal{L}_\lambda\{E\}.$$

The *initial continuity condition* is an extension of that of Section 2.2. It is given by

$$\lim_{t\to 0}\frac{1}{t^{n-1}}S(t)y = 0 \text{ for every } y \in Y. \tag{3.7}$$

Note that this gives $\lim_{t\to 0}\frac{1}{t^r}S(t)y = 0$ for each $r = 0, 1, \ldots, n-1$.

The other assumptions are identical to those given in Section 2.2 except for the minimal assumption (1.9) which is extended to (3.5).

3.3 Resolvent relations and consequences

The general form of Theorem 2.3 may be proved in essentially the same way (with $P(\lambda)$ and $R(\lambda)$ as in the above).

Theorem 3.3 *If $\langle S, E\rangle$ is an n-times integrated empathy, then the pseudo-resolvent equation*

$$P(\lambda) - P(\mu) = (\mu - \lambda)P(\lambda)R(\mu) = (\mu - \lambda)P(\mu)R(\lambda) \tag{3.8}$$

holds under the minimal assumptions. As a partial converse, if the strong continuity assumption is added for $\langle S, E\rangle$ then (3.8) implies that $\langle S, E\rangle$ satisfies the n-times integrated empathy relation (3.6).

Proof. If we take Laplace transforms on both sides of (3.6) at λ with respect to t we obtain

$$S(s)r(\lambda) = \mathcal{L}_\lambda\big\{(f_n * S_{[s]})(t)\big\} - \mathcal{L}_\lambda\big\{(f_{n,[s]} * S)(t)\big\}.$$

The convolution theorem gives

$$\begin{aligned} S(s)r(\lambda) &= \frac{1}{\lambda^n}\mathcal{L}_\lambda\{S_{[s]}\} - \Big(e^{\lambda s}\frac{1}{\lambda^n} - (e^{\lambda\cdot} * f_n)(s)\Big)p(\lambda) \\ S(s)R(\lambda) &= e^{\lambda s}p(\lambda) - (e^{\lambda\cdot} * S)(s) - \Big(\frac{1}{\lambda^n}e^{\lambda s} - (e^{\lambda\cdot} * f_n)(s)\Big)P(\lambda). \end{aligned} \tag{3.9}$$

When taking transforms again, this time at μ with respect to s,

$$p(\mu)R(\lambda) = \frac{1}{\mu-\lambda}p(\lambda) - \frac{1}{\mu-\lambda}p(\mu) - \frac{1}{\mu-\lambda}\Big(\frac{1}{\lambda^n} - \frac{1}{\mu^n}\Big)P(\lambda)$$

which simplifies to

$$P(\mu)R(\lambda) = \frac{1}{\mu-\lambda}P(\lambda) - \frac{1}{\mu-\lambda}P(\mu). \tag{3.10}$$

The converse is obtained in the same way as in Theorem 2.3. □

The following theorem corresponds to Theorem 2.4. To prove it in the same way seems to be a daunting task, so we shall use the resolvent characterization of the n-times integrated empathy and semigroup instead.

Theorem 3.4 *If $\langle S, E\rangle$ is an n-times integrated empathy and $P(\xi)$ is invertible for some ξ then R satisfies the pseudo-resolvent equation*

$$R(\mu) - R(\lambda) = (\lambda-\mu)R(\lambda)R(\mu). \tag{3.11}$$

If in addition E is strongly continuous, then it is an n-times integrated semigroup.

Proof. Through applying (3.8),

$$\begin{aligned} &P(\xi)R(\mu) - P(\xi)R(\lambda) \\ &\quad= \frac{1}{\mu-\xi}\big(P(\xi)-P(\mu)\big) - \frac{1}{\lambda-\xi}\big(P(\xi)-P(\lambda)\big) \\ &\quad= \frac{1}{\lambda-\xi}\cdot\frac{1}{\mu-\xi}\Big[(\lambda-\mu)P(\xi) + (\mu-\xi)P(\lambda) + (\xi-\lambda)P(\mu)\Big], \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} &(\lambda-\mu)P(\xi)R(\lambda)R(\mu) \\ &\quad= \frac{\lambda-\mu}{\lambda-\xi}\Big[P(\xi)R(\mu) - P(\lambda)R(\mu)\Big] \\ &\quad= \frac{\lambda-\mu}{\lambda-\xi}\Big[\frac{1}{\mu-\xi}\big(P(\xi)-P(\lambda)\big) - \frac{1}{\lambda-\mu}\big(P(\mu)-P(\lambda)\big)\Big] \\ &\quad= \frac{1}{\lambda-\xi}\cdot\frac{1}{\mu-\xi}\Big[(\lambda-\mu)\big(P(\xi)-P(\lambda)\big) - (\mu-\xi)\big(P(\mu)-P(\lambda)\big)\Big] \end{aligned}$$

which simplifies to (3.12). Therefore

$$P(\xi)R(\mu) - P(\xi)R(\lambda) = (\lambda - \mu)P(\xi)R(\lambda)R(\mu).$$

By using the invertibility assumption (3.11) is obtained. The rest of the statement of the theorem is clear from Theorem 3.3, (see also the first section of this chapter). □

As before all $P(\lambda)$ are invertible under the invertibility condition and further the correct form of the representation equation (2.19) may be found from (3.9) and it is, with $y \in \mathcal{D}_E$, $y = R(\lambda)y_\lambda$,

$$S(t)y = \frac{1}{\lambda^n}\left[e^{\lambda t} - \sum_{k=0}^{n-1}\frac{(\lambda t)^k}{k!}\right]P(\lambda)y_\lambda - e^{\lambda t}\int_0^t e^{-\lambda s}S(s)y_\lambda\,ds. \tag{3.13}$$

Given initial continuity it is possible to define the operator C_0 that plays the same role as in the once integrated empathy. It is seen from (3.7) and (3.13) that the limit exists and we may define C_0: $\mathcal{D}_E \to X$ as

$$C_0 y := \lim_{t\to 0+}\left[\frac{n!}{t^n}\right]S(t)y.$$

3.4 The Cauchy problem

The generator $\langle A, B\rangle$ is defined with the exact formulae of Section 2.5:

$$\begin{aligned} B &= {C_0}^{-1} = R(\lambda)P^{-1}(\lambda); \\ A &= \left[\lambda R(\lambda) - I_Y\right]P^{-1}(\lambda). \end{aligned}$$

The "unintegrated" form of the implicit Cauchy problem, i.e.

$$\begin{aligned} \frac{d}{dt}\left[Bu(t)\right] &= Au(t) \\ \lim_{t\to 0+} Bu(t) &= y \end{aligned}$$

(with $\langle A, B\rangle$ the generator of the n-times integrated empathy $\langle S, E\rangle$) is solved for $y \in \mathcal{D}_E^{n+2}$. For this we let $v(t) := S(t)y$ and $u(t) = v^{(n)}(t)$ after careful inspection of (3.13) implies that the n-th derivative of v exists.

3.5 Lipschitz continuity and characterization

The characterizing inequalities corresponding to (1.3) have a different form in the more-than-once integrated case.

Theorem 3.5 *The pair $\langle A, B\rangle$ is the generator of an n-times integrated empathy $\langle S, E\rangle$ satisfying the Lipschitz continuity condition*

$$\|S(t+h)-S(t)\| \leq Mh \text{ and } \|E(t+h)-E(t)\| \leq Nh \text{ for all } h>0$$

if and only if

$$\left\| \frac{\lambda^{k+1}}{k!} \frac{d^k}{d\lambda^k}\left[\frac{1}{\lambda^{n-1}}P(\lambda)\right]\right\| \leq M \text{ for all } \lambda > 0, \tag{3.14}$$

and

$$\left\| \frac{\lambda^{k+1}}{k!} \frac{d^k}{d\lambda^k}\left[\frac{1}{\lambda^{n-1}}R(\lambda)\right]\right\| \leq N \text{ for all } \lambda > 0, k = 0,1,2,\dots . \tag{3.15}$$

Proof. The proof is almost identical to that of Theorem 2.21. Note that for S and E to be Lipschitz continuous $\left\|\frac{\lambda^{k+1}}{k!}\left[\lambda p(\lambda)\right]^{(k)}(\lambda)\right\|$ and $\left\|\frac{\lambda^{k+1}}{k!}\left[\lambda r(\lambda)\right]^{(k)}(\lambda)\right\|$ must be uniformly bounded by Theorem 6.4, and furthermore $p(\lambda) = \frac{1}{\lambda^n}P(\lambda)$, $r(\lambda) = \frac{1}{\lambda^n}R(\lambda)$. □

Chapter 4

Applications and possible extensions

4.1 Examples in physics: dynamic boundary conditions

Two worked out examples are given, both of which involve partial differential equations with dynamic boundary conditions, on the space $C(0,1)$ with the supremum norm. Some of the significance of the examples lie in the following remarks:

We shall discover that in both the examples neither is the domain of the generator pair dense, nor does X or Y have the Radon-Nikodým property. This means that the theory in [S2] is not applicable. Furthermore it is not necessary to verify the closedness of A or B, since we have that $\langle A, B\rangle$ is closed once we have established the uniform boundedness of $P(\lambda)$.

The first example consists of the heat/diffusion equation and the second the wave equation rewritten as a symmetric hyperbolic system of first order equations. In both cases all constants have been removed, so that the results are simply mathematical illustrations, which can be adapted for situations such as described in Appendix A.

4.1.1 The heat equation in one spatial dimension

We consider the following system of equations:

$$v_t(x,t) - v_{xx}(x,t) = 0; \quad 0 < x < 1; \ t > 0 \tag{4.1}$$

$$\frac{d}{dt}v(1,t) + v_x(1,t) = 0 \tag{4.2}$$

subject to the boundary and initial conditions

$$v(0,t) = 0; \tag{4.3}$$

$$v(x,0) = v_0(x); \tag{4.4}$$

$$v(1,0) = v_0. \tag{4.5}$$

This system is now written as an implicit evolution equation in the space $X = C[0,1]$ of continuous functions. Let $\mathcal{D} = \{v \in X \mid v \in C^2(0,1);\ v(0) = 0\}$ and let $Y = X \times R^1$. The operators $A, B\colon \mathcal{D} \to Y$ are defined as follows: $Av = \langle v_{xx}, -v_x(1)\rangle$; $Bv = \langle v, v(1)\rangle$. The implicit equation is then $\frac{d}{dt}[Bv(t)] = Av(t)$ with initial condition $\lim_{t\to 0+} Bv(t) = y \in Y$. To apply the theory in Chapter 2 it is first necessary to find estimates for $\|\lambda P(\lambda)\|$ and $\|\lambda^n R^n(\lambda)\|$. This in turn involves the explicit solution of $(\lambda B - A)v = \langle f, g\rangle \in Y$ for $\lambda = \omega^2 > 0$, which leads to the boundary value problem:

$$\omega^2 v(x) - v''(x) = f(x); \quad 0 < x < 1; \tag{4.6}$$

$$v(0) = 0; \tag{4.7}$$

$$\omega^2 v(1) + v'(1) = g. \tag{4.8}$$

Demonstrating the existence and uniqueness of a solution to this problem can be done with standard techniques. See e.g. [Y1].

In order to find suitable norm estimates for the solution it is expedient to represent it in terms of Green's function. One may solve for v by standard methods, using an integrating factor, but the form of solution is such that it is difficult to find suitable bounds.

For this purpose the solution is represented in the form $v = u + w$ by adding the solution of the homogeneous problem ($f = 0$ in (4.6–4.8)), u, and the solution with homogeneous boundary conditions ($g = 0$), w.

Setting

$$\Gamma(\omega) := \cosh \omega + \omega \sinh \omega \tag{4.9}$$

we may verify by direct substitution that

$$u(x) = \left[\frac{g}{\omega \Gamma(\omega)} \right] \sinh \omega x \tag{4.10}$$

solves the homogeneous problem (including boundary conditions).

The component w will be represented by means of Green's function. For the construction of Green's function we firstly need two linearly independent solutions to the differential equation (4.6), one satisfying the left boundary condition, (4.7), and the other the right (4.8). We identify suitable candidates, namely

$$p(x) = \sinh \omega x; \tag{4.11}$$

$$q(x) = \cosh \omega(1 - x) + \omega \sinh \omega(1 - x). \tag{4.12}$$

It is readily verified that both satisfy (4.6) and that $p(0) = 0$ and $q'(1) + \omega^2 q(1) = 0$. To find Green's function it is necessary to calculate the Wronskian determinant of p and q. This is a tedious process, but after many cancellations the simple form

$$\mathrm{Wr} := pq' - p'q = -\omega\Gamma(\omega) < 0$$

is obtained, which shows that p and q are linearly independent for each $\omega > 0$.

Green's function, G ([Y1], Chapter 2, p.64) may now be defined as

$$G(x,\xi) = -\frac{1}{\mathrm{Wr}} \begin{cases} p(\xi)q(x) & \text{for } 0 \le \xi \le x, \\ p(x)q(\xi) & \text{for } x < \xi \le 1. \end{cases} \tag{4.13}$$

It follows that

$$w(x) = \int_0^1 G(x,\xi) f(\xi)\, d\xi. \tag{4.14}$$

To verify that w indeed solves the given system we may note that it is given by

$$w(x) = \frac{1}{\mathrm{Wr}} \left(q(x) \int_0^x p(\xi) f(\xi)\, d\xi + p(x) \int_x^1 q(\xi) f(\xi)\, d\xi \right)$$

and that p and q solve (4.6) and satisfy their respective boundary conditions.

Substituting w and using the product rule for differentiation, (4.6) is verified.

In order to find a bound for $|w|$ in terms of the maximum value of $|f|$, the integral in (4.14) is calculated for the constant function $f(x) = 1$. Again, after some rather tedious calculations and bearing in mind the definition of Wr we obtain

$$W(x) = \int_0^1 G(x,\xi)\, d\xi = \frac{1}{\omega^2} - \frac{1}{\omega^2 \Gamma(\omega)} [q(x) + \omega p(x)].$$

Next the solution $v = P(\lambda)(f,g)$ is estimated in the obvious norm, namely $\|f\|_X = \sup_{0 \le x \le 1} |f(x)|$. We do this by handling the terms u and w separately. It is readily seen that $\frac{\sinh(\omega)}{\Gamma(\omega)} = \frac{1}{\coth(\omega)+\omega} < \frac{1}{\omega}$, and it follows from (4.10) that

$$|u(x)| \le \frac{|g|}{\omega \Gamma(\omega)} \sinh \omega \le \frac{|g|}{\omega^2}\,. \tag{4.15}$$

We observe from the formulas for p, q and the Green's function G that G is non-negative. By (4.14) we see $W(x) \ge 0$. Therefore

$$|w(x)| \le \int_0^1 G(x,\xi)\, d\xi \|f\|_X \le W(x) \|f\|_X < \frac{\|f\|_X}{\omega^2}\,. \tag{4.16}$$

Combining (4.15) and (4.16) gives the desired estimate:

$$\|\lambda P(\lambda)\langle f,g\rangle\|_X < \|f\|_X + |g|. \tag{4.17}$$

The final task is to obtain the bounds for $\|\lambda^n R^n\|$. This will be done by choosing a norm on Y which makes $\lambda R(\lambda) = BP(\lambda)$ a contraction. For the space Y

let $\|\langle f,g\rangle\|_Y := 2(\|f\|_X + |g|)$. We have that $BP(\lambda)\langle f,g\rangle = Bv = \langle v, v(1)\rangle$. Thus, $\|R(\lambda)\langle f,g\rangle\|_Y = \|Bv\|_Y \leq 2\|v\|_X$. This combined with (4.17) gives the estimates:

$$\|\lambda P(\lambda)\| \leq \frac{1}{2} \tag{4.18}$$

$$\|\lambda R(\lambda)\| \leq 1. \tag{4.19}$$

Consequently, the inequalities (2.39–2.40) are satisfied with $M = \frac{1}{2}$ and $N = 1$. By Theorem 2.21 the operator pair $\langle A, B\rangle$ is the generator of a Lipschitz continuous integrated empathy $\langle S, E\rangle$. Thus Theorem 2.13 is applicable, and the implicit Cauchy problem

$$\begin{aligned} \frac{d}{dt}\big[Bv(t)\big] &= Av(t); \\ \lim_{t\to 0+} Bv(t) &= y \end{aligned}$$

associated with the dynamic boundary value problem (4.1–4.5) can be solved for $y \in \mathcal{D}_E^2 \subset Y$.

In retracing the construction of the function w above, it is found that $\mathcal{D}_E^2 = B\big[\mathcal{D} \cap C^4(0,1)\big]$. The initial states for the system (4.6–4.8) seem to be very restricted. Indeed in an L^2 setting (where the Radon-Nikodým property holds), the semigroup E is holomorphic even in higher spatial dimensions ([S1], [vdM]). It is suspected that the holomorhic property can also be found in the background in the present setting, cf. ([ABHN], Section 6.1, p.395 *ff.*).

In this example we could obtain the estimates because we could construct explicit solutions and the underlying Green's function was explicitly known. The extension to higher spatial dimensions in which the dynamic boundary conditions can be much more complex ([S3], [vdM]), presents a greater challenge.

4.1.2 The wave equation

Let $u(x,t) := (u_1(x,t), u_2(x,t))$ for $0 \le x \le 1$ and $t > 0$. Let $L := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. We consider the evolution equation

$$u_t(x,t) - Lu_x(x,t) = 0; \quad x \in (0,1),\ t > 0 \tag{4.20}$$

subject to the boundary conditions

$$\begin{aligned} u_1(0) &= 0; & (4.21) \\ \frac{d}{dt}u_1(1,t) + u_2(1,t) &= 0. & (4.22) \end{aligned}$$

The system (4.21–4.22), represents the wave equation written as a symmetric positive system, see Appendix B, (Friedrichs, [F2]), with a dynamic boundary condition. The form (4.20) is closer to the conservation law and constitutive equation from which the wave equation is derived. In the present formulation u_1 represents the velocity field and u_2 the internal force. The dynamic boundary condition (4.22) represents the interaction between the medium and a mass attached to the endpoint $x = 1$ [S4].

To write this as an implicit evolution equation, let $X := C[0,1] \times C[0,1]$, $\mathcal{D}_0 := \{v \in C[0,1] \cap C^1(0,1) \mid v(0) = 0\}$, $\mathcal{D}_1 := C[0,1] \cap C_1(0,1)$, $\mathcal{D} := \mathcal{D}_0 \times \mathcal{D}_1 \subset X$ and $Y := X \times R^1$.

The norm for $C[0,1]$ is $\|h\| := \sup_{x\in[0,1]} |h(x)|$ and the norms in X and Y are defined as $\|v\|_X := \max\{\|v_1\|, \|v_2\|\}$ and $\|\langle f, f_3\rangle\|_Y := 2\big[\|f\|_X + |f_3|\big]$ respectively.

The operators $A, B\colon \mathcal{D} \to Y$ are $Au(\cdot\,,t) := \big\langle Lu(\cdot\,,t), u_2(1,t)\big\rangle$ and $Bu(\cdot\,,t) := \big\langle u(\cdot\,,t), u_1(1,t)\big\rangle$. The equation (4.20) and the boundary conditions (4.21–4.22) with "natural" initial conditions may then be written in the form

$$\begin{aligned} \frac{d}{dt}[Bu(t)] &= Au(t); & (4.23) \\ \lim_{t\to 0^+} Bu(t) &= y \in Y. & (4.24) \end{aligned}$$

Resolvents are once again investigated by considering the system of first order differential equations $\lambda Bu - Au = \langle f, f_3 \rangle$ written in the form

$$\lambda u(x) - Lu'(x) = f(x) = \big\langle f_1(x), f_2(x) \big\rangle;\ x \in (0,1) \tag{4.25}$$

$$u_1(0) = 0; \tag{4.26}$$

$$\lambda u_1(1) + u_2(1) = f_3, \tag{4.27}$$

for $\lambda > 0$. We shall represent the solution of (4.25 – 4.27) in the form $u = v + w$ with v the solution of the homogeneous equation ($f = 0$) and w the solution of (4.25 – 4.27) with $f_3 = 0$. A direct verification shows that

$$v(x) = \left[\frac{f_3}{\Gamma(\lambda)}\right] \big\langle \sinh \lambda x, \cosh \lambda x \big\rangle$$

with $\Gamma(\lambda)$ as defined in (4.9).

The component w of the solution will be represented by a Green's matrix function, which we shall construct with the aid of two solutions of the homogeneous equation, namely

$$\varphi(x) = \big\langle \sinh \lambda x, \cosh \lambda x \big\rangle, \tag{4.28}$$

$$\text{and } \psi(x) = \Big\langle -\big[\cosh \lambda(1-x) + \lambda \sinh \lambda(1-x)\big], \sinh \lambda(1-x) + \lambda \cosh \lambda(1-x) \Big\rangle. \tag{4.29}$$

It is seen that φ satisfies the boundary condition (4.26) and ψ satisfies (4.27) with $f_3 = 0$. The two solutions (4.28) and (4.29) are linearly independent vector functions. Indeed,

$$\psi_2(x)\varphi_1(x) - \varphi_2(x)\psi_1(x) = \Gamma(\lambda),$$

with $\Gamma(\lambda)$ as is defined in the previous section.

The Green's matrix function is defined by

$$G(x,\xi) = \frac{1}{\Gamma(\lambda)} \begin{cases} \psi(x) \otimes J\varphi(\xi) & \text{for } 0 \le \xi \le x, \\ \varphi(x) \otimes J\psi(\xi) & \text{for } x < \xi \le 1 \end{cases} \tag{4.30}$$

where the symbol $\otimes$ denotes the tensor product and J the diagonal matrix $\mathrm{diag}(-1,1)$. The solution w is then represented in the form

$$\begin{aligned} w(x) &= \int_0^1 G(x,\xi) f(\xi)\, d\xi \\ &= \left[\int_0^x J\varphi(\xi)\cdot f(\xi)\, d\xi\right]\psi(x) + \left[\int_x^1 J\psi(\xi)\cdot f(\xi)\, d\xi\right]\varphi(x) \quad (4.31) \end{aligned}$$

as is seen from (4.30). This can be verified by substitution in (4.25–4.27) with $f_3 = 0$.

We proceed from here to estimate the solution $u = P(\lambda)\langle f, f_3\rangle$. In the first place — after some elementary, but nontrivial considerations — it is seen from (4.1.2) that

$$\lambda\|v\|_X \leq \left[\frac{\lambda\cosh\lambda}{\Gamma(\lambda)}\right]|f_3| \leq |f_3|. \quad (4.32)$$

The second step is to estimate $w(x)$. This is done by estimating the terms which constitute the representation (4.31) in a manner similar to the estimates obtained in Example 4.1.1. The estimates are more delicate, but we finally obtain

$$\lambda\|w\|_X \leq \|f\|_X. \quad (4.33)$$

Combining (4.32) and (4.33) yields

$$\|\lambda P(\lambda)\| \leq \frac{1}{2}. \quad (4.34)$$

Once again, as in Example 4.1.1, we obtain (by letting $R(\lambda) = BP(\lambda)$)

$$\|\lambda R(\lambda)\| \leq 1. \quad (4.35)$$

Combination of the estimates (4.34) and (4.35) leads to conclusions similar to that of Example 4.1.1 with the exception that $\mathcal{D}_E^2 = B\big[\mathcal{D}_0 \cap C^2(0,1) \times \mathcal{D}_1 \cap C^2(0,1)\big]$.

4.1.3 Remarks concerning the examples

It was noted in Example 4.1.2 that the representation of the wave equation as a positive symmetric system of first order differential equations is, in a sense, closer to the mathematical modelling of the wave phenomenon. It is also true that the dynamic boundary condition in question is more naturally formulated in such a framework [S4].

For the heat equation the same is true. By not eliminating the flux from the model equations the system of differential equations is of the form

$$u_{1,t} - u_{2,x} = 0; \tag{4.36}$$

$$u_2 - u_{1,x} = 0. \tag{4.37}$$

The dynamic boundary condition is, in this formulation,

$$\frac{d}{dt}u_1(1,t) - u_2(1,t) = 0,$$

which once again, does not contain spatial derivatives at the boundary point $x = 1$. It would be interesting to study the heat transfer problem from this perspective. Example 4.1.2 gives us an indication that Green's functions for symmetric positive systems of first order equations may systematically be constructed with the aid of tensor products of linearly independent solutions of associated homogeneous problems.

4.2 Some areas for further research

Empathy theory is a relatively new development in mathematics ([S2], 1997; [S3] in preparation). In a way it runs along lines parallel to semigroup theory, which is a lot older ([HP], 1958). It seems to be promising for applications, and we believe the purely mathematical study is a worthwile pursuit in its own right.

There are many new areas of research that open up, including applications, only two of which are discussed below. Another possibility not discussed here is the study of the empathy relation without the requirement that S and E are linear operators, cf. [T]. More complete discussions of present applications are given by Sauer [S3].

4.2.1 Empathies involving multiple spaces

It would theoretically be possible to construct evolution operator relations like the empathy and integrated empathy relations between any number of spaces. The question one would ask here is whether there is any application for this, and indeed, what would the associated implicit Cauchy problems look like and what physical situation they might describe. It is even conceivable that such relations could be constructed and studied among an infinite sequence of spaces.

One could most probably use the methods we have developed to define generators and to find conditions similar to (2.39–2.40) for resolvents.

We illustrate with two examples involving three Banach spaces.

The following possibility readily springs to mind:

Let X, Y and Z Banach spaces, and $S_1(\cdot)\colon Z \to Y$, $S_2(\cdot)\colon Y \to X$ and $E(\cdot)\colon Z \to Z$ be families of linear operators.

Then $S_1(t+s) = S_1(t)E(s)$ and $S_2(t+s) = S_2(t)S_1(s)$ define "multiple empathy" relations between the three spaces. This could quite feasibly represent a physical situation with a compound, "double layer" boundary.

On the other hand consider three Banach spaces X, Y_1 and Y_2 with a bijection $B\colon Y_1 \to Y_2$ and families $S_1(\cdot)\colon Y_1 \to X$, $S_2(\cdot)\colon Y_2 \to X$ and $E_1(\cdot)\colon Y_1 \to Y_1$, $E_2(\cdot)\colon Y_2 \to Y_2$ satisfying the following:

$S_1(t+s) = S_1(t)E_1(s)$ and $S_2(t+s) = S_2(t)E_2(s)$ coupled by $S_1(t)y_1 = S_2(t)By_1$ for all $y_1 \in Y_1$, $y_2 \in Y_2$. This system could perhaps model a system with two distinct boundaries.

Note that any examples of the foregoing kind may also be formulated for integrated empathies. We hope that this kind of modelling could handle problems involving more than one dynamic boundary value condition.

4.2.2 Construction of more examples

Examples, especially purely theoretical examples are still relatively scarce. Consider our hierarchy of assumptions. Are there nontrivial examples of integrated semigroups or integrated empathies that do not satisfy strong continuity? Or that are strongly continuous without the Lipschitz condition? These questions beg attention.

Then, more specific to integrated empathies it should be mentioned that most examples at our disposal at this moment are based on those in Section 2.1. It would be good to have a few others as well.

Chapter 5

Appendix A: Models with dynamic boundary values

Two models from physics are discussed. Their mathematical treatment is given in Chapter 4.

5.1 The heat equation model

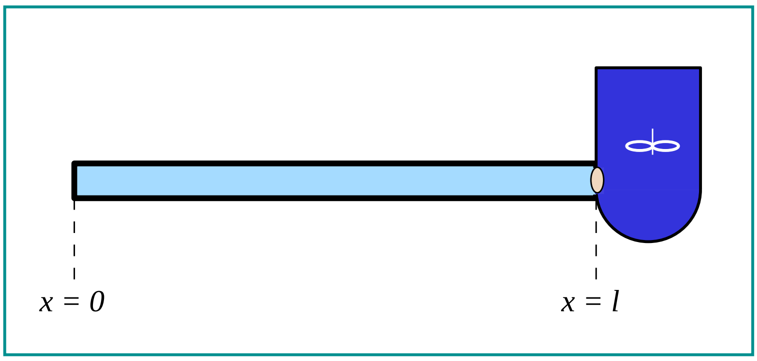

Figure 5.1: *A heat conduction problem*

The situation that the sketch illustrates is as follows:

A thin rod, which is thermally isolated along its length, is kept at a constant temperature of T_0 Kelvin at the endpoint $x = 0$, and an isolated container of fluid is attached to the other endpoint at $x = l$. The fluid is vigorously stirred so that the temperature may be assumed homogeneous and equal to the temperature of the rod at l. The temperature at position x and at time t is denoted by $v(x,t)$, and the temperature in the container at time t by $V(t)$, so that $v(0,t) = T_0$ and $v(l,t) = V(t)$.

We say the present situation is modelled with a dynamic boundary value condition, since at the boundary $x = l$ the boundary value is not a known function of time.

The partial differential equation in the rod will be discussed first, and then the equation that describes the dynamic boundary condition will be derived.

The following physical constants will be needed:

(a) ρ, the linear density of the rod in kgm^{-1};

(b) c, the specific heat of the rod in $Jkg^{-1}K^{-1}$;

(c) κ, the conductivity of the rod in WmK^{-1},

and the constant C, the amount of heat needed to raise the temperature of the fluid by 1 Kelvin, is the heat capacity of the mass of gas, in JK^{-1}.

The derivation of the heat equation (Fourier, 1822), may be found in standard texts on linear partial differential equations, see for example [S4], and also in physics texts.

The concept of the flux $\varphi(x,t)$, at position x and time t is essential in the derivation. The flux gives the rate of heat flow in *Watt* passing the point x in the positive direction at time t. The heat equation is obtained by combining the law of conservation of energy and the constitutive equation.

The equations simplify to the system

$$c\rho v_t(x,t) = -\varphi_x(x,t); \tag{5.1}$$

$$\varphi(x,t) = -\kappa v_x(x,t). \tag{5.2}$$

Provided $v(\cdot\,,t) \in C^2(0,l)$, the heat equation,

$$v_t(x,t) = \frac{\kappa}{c\rho} v_{xx}(x,t)$$

may be obtained, by the elimination of φ.

We turn to the derivation of the equation that describes the boundary condition at $x = l$.

The law of the conservation of energy implies that the rate of heat flow from the endpoint $x = l$ (the flux $\varphi(l,t)$) equals the rate of increase of the total thermal energy in the fluid. Thus

$$\frac{d}{dt}\left[CV(t)\right] = \varphi(l,t)$$

or, using (5.2) and the assumption that $V(t) = v(l,t)$,

$$\frac{d}{dt}\left[Cv(l,t)\right] = -\kappa v_x(l,t).$$

The system of equations with which we model the physical situation is therefore:

$$v_t(x,t) - \frac{\kappa}{c\rho} v_{xx}(x,t) = 0, \tag{5.3}$$

satisfying the boundary conditions

$$v(0,t) = T_0; \tag{5.4}$$

$$\frac{d}{dt}\left[v(l,t)\right] + \frac{\kappa}{C} v_x(l,t) = 0. \tag{5.5}$$

.

Remark 5.1 *The first term in (5.5) is an ordinary derivative, which follows from the fact that V is a function of t only. Note that $\frac{d}{dt}[v(l,t)]$ is not the same as $v_t(l,t)$.*

5.2 The wave equation model

The figure shows an elastic rod of length l which is kept fixed on one end, with a mass attached to the other end. The rod is free to vibrate only in the longitudinal direction. The linear density of the rod is ρ and Hooke's constant Λ. The attached mass will give rise to a dynamic boundary value condition as we shall show.

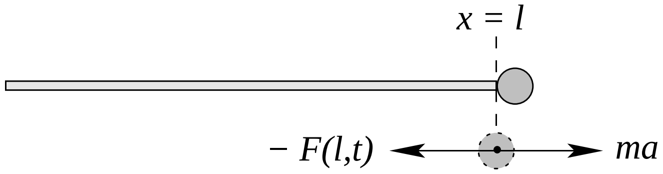

Figure 5.2: *A problem involving longitudinal vibrations*

The rod is modelled as a line segment of length l in the reference state (which we shall take as the initial state for convenience). The position of a point on the rod in the reference state is denoted by x. The mapping $y(\cdot, t): x \to X = y(x,t)$ signifies that a particle which is at position x in the reference state will be at position $X = y(x,t)$ at time t, see Figure 5.3. The function y is called the motion of the rod.

We discuss the evolution equation which describes the motion of the rod first, and then the boundary conditions.

The following variables are introduced:
$v(x,t) = \frac{\partial}{\partial t} y(x.t)$ is the velocity at $X(t) = y(x,t)$
and $F(x,t)$ is the internal force at position X at time t, with a tensile force taken as positive.

The derivation of the wave equation may now be found in a standard physics text. We only give an outline.

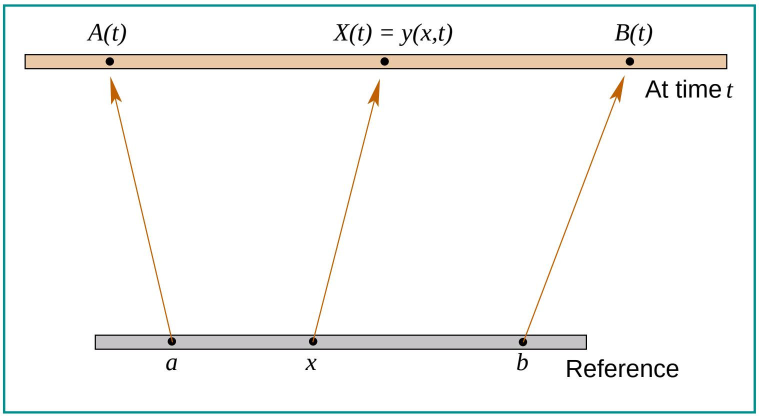

Figure 5.3: *Modelling a vibrating rod*

The law of the conservation of linear momentum reduces to

$$\rho v_t - F_x = 0, \tag{5.6}$$

while the constitutive equation for the rod, the infinitesimal version of Hooke's law, is

$$F = \Lambda[y_x - 1]. \tag{5.7}$$

If these are combined, the wave equation

$$\rho y_{tt} - \Lambda y_{xx} = 0 \tag{5.8}$$

is obtained. This can be written in the more familiar form by letting $c^2 = \Lambda/\rho$.

The wave equation may be written as a system of first order equations in some obvious ways, but writing it as *a symmetric hyperbolic system* in the sense of Friedrichs [F1], is natural and seems to be the most advantageous. This we do by differentiating (5.7) with respect to time, to obtain:

$$\begin{pmatrix} \rho & 0 \\ 0 & 1 \end{pmatrix} \partial_t \begin{pmatrix} v \\ F \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ -\Lambda & 0 \end{pmatrix} \partial_x \begin{pmatrix} v \\ F \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{5.9}$$

The boundary conditions are discussed next.

Suppose the object attached to the free end has mass m. The internal force F at $x = l$ has the effect of moving the body. Newton's second law of motion gives: $m\frac{d}{dt}[v(l,t] + F(l,t) = 0$.

Note, once again, that it is assumed that the velocity of the attached particle is the same as the velocity of the endpoint of the rod. Therefore $a = \frac{d}{dt}[v(l,t)]$.

The following boundary conditions are sufficient for the uniqueness of a solution to (5.9):

$$
\begin{aligned}
&v(0,t) = 0; \\
&m\frac{d}{dt}v(l,t) + F(x,l) = 0; \qquad (5.10) \\
&v(x,0) = a(x); \\
&F(x,0) = b(x),
\end{aligned}
$$

implying $v(l,0) = \alpha = a(l)$ and $F(l,0) = \beta = b(l)$. The dynamic boundary condition is (5.10).

Chapter 6

Appendix B: The Laplace transform in Banach spaces

6.1 The existence and uniqueness of the Laplace transform

As in Section 1.5, the *Laplace transform* of a measurable function $g\colon (0,\infty) \to X$ where X is a Banach space is defined by the Bochner integral

$$\mathcal{L}_\lambda\{g\} := \int_0^\infty e^{-\lambda t} g(t)\, dt.$$

The usual approach in the literature defines $\mathcal{L}_\lambda$ first as an *improper* integral and then works with absolutely convergent integrals. We do this from the outset.

If the Laplace transform of g exists for a given $\lambda = a$ then it also exists for all $\lambda > a$.

The following result is well known, see for instance ([ABHN], Chapter 1, p.41) where stronger forms are also given.

Theorem 6.1 (Uniqueness) *Suppose $\mathcal{L}_\lambda\{g\} = \mathcal{L}_\lambda\{h\}$ for all $\lambda > a$ where a is any real number. Then $g = h$ a.e.*

6.2 The transform as a pseudo-resolvent

The power of the Laplace transform in the study of the Cauchy problem originates in the fact that it transforms the evolution operator equations into resolvent relations, with the resolvent being the resolvent of the generator. For instance, for a semigroup we have the following ([A], Proposition 2.2):

Proposition 6.1 *E is an exponentially bounded semigroup if and only if the pseudo-resolvent equation*

$$R(\lambda) - R(\mu) = (\mu - \lambda)R(\mu)R(\lambda)$$

holds for $R(\lambda) := \mathcal{L}_\lambda\{E\}$ and all $\lambda,\ \mu > 0$.

Proof. By taking transforms we obtain (with the help of the operational rule, Proposition (6.2), and the convolution theorem):

$$\begin{aligned}
E(s)R(\lambda) &= \mathcal{L}_{\lambda,t}\{E(s)E(t)\} \\
&= \mathcal{L}_{\lambda,t}\{E(t+s)\} \\
&= e^{\lambda s}\Big(R(\lambda) - \int_0^s e^{-\lambda t}E(t)\,dt\Big) \\
&= e^{\lambda s}R(\lambda) - (e^{\lambda \cdot} * E)(s), \\
R(\mu)R(\lambda) &= \frac{1}{\mu - \lambda}\big(R(\lambda) - R(\mu)\big)
\end{aligned}$$

The converse follows from the uniqueness theorem as well as the strong continuity of E. □

Remark 6.1 *The details of the proof are very similar to that of Theorem 2.3. Strong continuity is in this case a consequence of the semigroup property and the measurability condition. This method of proof originates from [BS].*

6.3 The Post-Widder inversion theorem

Originally given by David Widder ([W], Chapter 7, Theorem 6a, p.289), the version of the theorem for Banach spaces we use is from Hille & Phillips ([HP], Chapter 6, Theorem 6.3.5, p.224). It gives an inversion formula for a vector-valued function f known to be a Laplace transform. From it the originating function g such that $f(\lambda) = \mathcal{L}_\lambda\{g\}$ may be recovered. The *Widder operators* associated with a function $f \in C^\infty\{(0,\infty) : X\}$ whith X a Banach space, are defined by

$$(L_k f)(\lambda) := \left[\frac{(-1)^k}{k!}\right] \lambda^{k+1} f^{(k)}(\lambda) \text{ for } k = 1, 2, \ldots \tag{6.1}$$

with $f^{(k)}$ the k'th derivative of f.

Theorem 6.2 *If f is the Laplace transform of a function g then $(L_k f)\left(\frac{k}{t}\right) \to g(t)$ as $k \to \infty$ for all t in the Lebesgue set of g. If g is continuous in some open interval the convergence is uniform over each compact subinterval.*

The theorem has been formulated in terms of the Laplace-Stieltjes transform ([ABHN], Chapter 2, p.75), where other results concerning inversion are also given.

6.4 Arendt's extension of Widder's theorem

Whereas the Post-Widder theorem gives an inversion formula when the function f is known to be a Laplace transform, Widder's theorem characterizes those functions that are Laplace transforms ([W], Theorem 16a, p.315; Theorem 16b, p.316). The theorem of Widder extended to a Banach space X holds only when the space has the Radon-Nikodým property. In fact it characterizes this property.

Theorem 6.3 *A Banach space X has the Radon-Nikodým property if and only if Widder's theorem holds for X, i.e. every $f \in C^\infty\{[0,\infty), X\}$ satisfying*

$$\sup_{n,\lambda}\left\{\left\|\frac{1}{n!}\lambda^{n+1}f^{(n)}(\lambda)\right\| : \lambda > 0, n = 1, 2, \ldots\right\} < \infty$$

is a Laplace transform, $f(\lambda) = \mathcal{L}_\lambda\{g\}$ where $g : [0,\infty) \to X$.

Remark 6.2 *The original proof is in ([A], Theorem 1.4) and makes use of the following characterization:*
A space X has the Radon-Nikodým property if and only if every Lipschitz continuous map $f\colon [0,1] \to X$ is differentiable a.e. See also ([ABHN], Chapter 2, p.74 and p.81).

The following *integrated version of Widder's theorem* which was also proved by Arendt, ([A], Theorem 1.1), holds in an arbitrary Banach space X.

Theorem 6.4 *Let $f\colon [0,\infty) \to X$ and $M \geq 0$. The following are equivalent:*

(a) $f \in C^\infty\{[0,\infty) : X\}$ and

$$\sup_{n,\lambda}\left\{\left\|\frac{1}{n!}\lambda^{n+1}f^{(n)}(\lambda)\right\| : \lambda > 0, n = 1, 2, \ldots\right\} \leq M;$$

(b) There exists a function $g\colon [0,\infty) \to X$ such that $f(\lambda) = \lambda\mathcal{L}_\lambda\{g\}$ for $\lambda > 0$ and g is Lipschitz continuous with Lipschitz constant M, i.e. $\|g(t+h) - g(t)\| \leq Mh$ for all non-negative t and h.

6.5 Operational properties

The following rules carry over from the real-valued transform, and the proofs are almost identical [ABHN].

Proposition 6.2 *Let $g \in L^1_{loc}\{(0,\infty), X\}$, $\mu \in C$ and $s \in R^+$ and let $g_s(t) := g(t+s)$, for $t \geq 0$ while $g_{-s}(t) := g(t-s)$ for $t \geq s$, and 0 otherwise.*

Let $\lambda \in C$ then

(a) $\mathcal{L}_\lambda\{e^{-\mu t}g(t)\}$ exists if and only if $\mathcal{L}_{\lambda+\mu}\{g(t)\}$ exists and then they are equal;

(b) $\mathcal{L}_\lambda\{g_{-s}\}$ exists if and only if $\mathcal{L}_\lambda\{g\}$ exists and then $\mathcal{L}_\lambda\{g_{-s}\} = e^{-\lambda s}\mathcal{L}_\lambda\{g\}$;

(c) $\mathcal{L}_\lambda\{g_s\}$ exists if and only if $\mathcal{L}_\lambda\{g\}$ exists and then

$$\mathcal{L}_\lambda\{g_s\} = e^{\lambda s}\Big(\mathcal{L}_\lambda\{g\} - \int_0^s e^{-\lambda t} g(t)\,dt\Big). \tag{6.2}$$

6.6 The convolution theorem

The well-known convolution theorem asserts that the Laplace transform of the convolution $(f * g)(t) := \int_0^t f(t-s)g(s)\ ds = \int_0^t f(s)g(t-s)\ ds$ equals the product of the Laplace transforms of f and g (provided that they exist). When we deal with the Laplace transform of vector-valued functions, the result makes sense if f is scalar and g is vector-valued. When we deal with families of linear operators, such as semi-groups, there is a meaningful analogue of the convolution theorem. The result Lemma 6.1 is given in [BS]. It was probably known but does not seem to have been recorded before. It has been used extensively in [S2].

Let X and Y be complex Banach spaces and let $F = \{F(t)\colon t > 0\}$ be a family of closed linear operators with a common domain $\mathcal{D}_F \subset Y$ which map into X. For $\lambda > 0$ we formally define the Laplace transform of F at λ as the operator

$$\mathcal{L}_\lambda\{F\}y := \int_0^\infty e^{-\lambda t}F(t)y\ dt;\ y \in \mathcal{D}_F.$$

We shall say that $\mathcal{L}_\lambda\{F\}$ *exists at* λ if the function $t \mapsto e^{-\lambda t}F(t)y$ is in $L^1((0,\infty), X)$.

Let us also consider a function $v\colon t \in (0,\infty) \mapsto v(t) \in \mathcal{D}_F$. We shall say that the *Laplace transform*

$$\mathcal{L}_\lambda\{v\} = \int_0^\infty e^{-\lambda t}v(t)\ dt$$

exists at $\lambda > 0$ if the function $t \mapsto e^{-\lambda t} v(t)$ is in $L^1((0,\infty), Y)$.

We formally define the convolution $F * v$ by

$$(F * v)(t) := \int_0^t F(t-s)v(s)\, ds = \int_0^t F(s)v(t-s)\, ds. \tag{6.3}$$

Lemma 6.1 *Suppose there is a* $\lambda > 0$ *such that the Laplace transforms* $\mathcal{L}_\lambda\{F\}$, $\mathcal{L}_\lambda\{v\}$ *and, for every* $t > 0$, *the Laplace transform of the function* $s \mapsto F(t)v(s)$, *exist at* λ.

Then $\mathcal{L}_\lambda\{v\} \in \mathcal{D}_F$. *If, in addition, the convolution* $F * v$ *exists, its Laplace transform exists at* λ *and*

$$\mathcal{L}_\lambda\{(F * v)\} = \mathcal{L}_\lambda\{F\} \cdot \mathcal{L}_\lambda\{v\}.$$

Proof. If the integral $\int_0^\infty e^{-\lambda s} F(t)v(s)\, ds$ exists for a $t > 0$, the existence of the Laplace transform $\mathcal{L}_\lambda\{v\}$ and the closedness of $F(t)$ implies that $\mathcal{L}_\lambda\{v\} \in \mathcal{D}_F$ and that

$$\int_0^\infty e^{-\lambda s} F(t)v(s)\, ds = F(t) \int_0^\infty e^{-\lambda s} v(s)\, ds = F(t)\, \mathcal{L}_\lambda\{v\}.$$

Hence,

$$\begin{aligned} \mathcal{L}_\lambda\{F\} \cdot \mathcal{L}_\lambda\{v\} &= \int_0^\infty e^{-\lambda t} F(t) \int_0^\infty e^{-\lambda s} v(s)\, ds\, dt \\ &= \int_0^\infty \int_0^\infty e^{-\lambda(t+s)} F(t)v(s)\, ds\, dt. \end{aligned}$$

The change of variables $s \mapsto \sigma = s + t$, $t \mapsto \tau = t$ and reversal of the order of integration yields

$$\mathcal{L}_\lambda\{F\} \cdot \mathcal{L}_\lambda\{v\} = \int_0^\infty e^{-\lambda\sigma} \int_0^\sigma F(\tau)v(\sigma - \tau)\, d\tau\, d\sigma \tag{6.4}$$

and the formula is proved. Reversal of the order of integration is possible because of the assumption that the convolution exists. □

Remark 6.3 *If* F *is a family of bounded linear operators defined on the whole space, the requirement that the Laplace transform of* $s \mapsto F(t)v(s)$ *exists, is superfluous.*

Remark 6.4 *When X and Y are the same, it is a simple matter to see that integer powers of the "transformed" operator correspond to multiple convolutions of the family of operators.*

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