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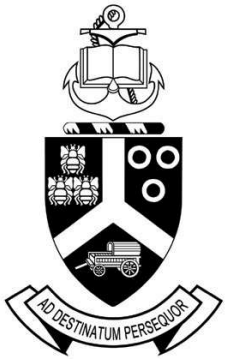
# Time dependent entanglement features, and other quantum information aspects, of two-qubits systems interacting with an environment

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I declare that the thesis, which I hereby submit for the degree of Magister Scientiae in Physics at the University of Pretoria, is my own work and has not been previously submitted by me at this or any other tertiary institution.

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# Time dependent entanglement properties, and other quantum-information aspects, of two-qubit systems interacting with an environment

by

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## Summary

Quantum systems usually suffer from unavoidable interactions with the environment. In most experiments, it is virtually impossible to isolate the system under study from the environment. It is thus imperative to study and understand how quantum systems interact with their surroundings.

In the present study, I consider the evolution of quantum entanglement in a two-qubit system interacting with an environment in a regime where the non-Markovian effects are important. The present thesis is organized as follows:

Chapter one is a general introduction to the thesis.

In chapter two, some preliminary concepts that will be used in the subsequent chapters will be introduced and defined. These concepts include *qubits*, *density matrix*, *Quantum entanglement* and *Entropic measures* in quantum information. Ways to detect or quantify entanglement in a quantum system will also be discussed in the section on quantum entanglement.

Chapter three will be about open quantum systems in general: concepts like *master equations*, *Markovian quantum systems* and *non-Markovian quantum systems* will be briefly reviewed.

Chapter four will deal with time dependent entanglement features of two-qubit and multi-qubit systems interacting with an environment, basically all the results obtained in my study will be presented there.

Some general conclusions will be drawn in chapter five.

# Contents

Chapter 1

# Introduction

Craftsmen and engineers have been developing devices to manipulate and transmit information by classical means from the very beginnings of civilizations. The formal theoretical study of these processes was initiated by pioneers of computer science and information theory in the 1930's and 1940's. Only recently, however, scientists discovered that quantum entities can be used to process information in novel, non-classical and highly counter-intuitive ways. Quantum mechanics [1, 2] arose in the beginning of the twentieth century from the need to understand thermal properties of radiation and the discrete spectral features of atoms. Physical theory at that time could not explain certain phenomena such as black body radiation, where it predicted the existence of an "ultraviolet catastrophe" involving infinite energy. Quantum mechanics is to date the most successful theory of physics as it describes with fantastic accuracy the structure of atoms and molecules, the properties of materials, fusion reactions in stars, and the list goes on. In its field theoretical version, quantum mechanics also describes the physics of elementary particles. Excepting gravitation, the laws of quantum mechanics apply to all known physical phenomena.

Quantum information theory studies the use of quantum mechanical systems to transmit and process information. In the past few years, this emerging, multidisciplinary field [3] received a lot of attention and grew at a remarkable speed.

This is because the laws of quantum physics opened doors to new opportunities of manipulating information in ways that were not possible classically. Quantum information differs from classical information in many respects, for example, quantum entities may be in a superposition of states unlike classical entities. Another difference is that an arbitrary quantum state cannot be *cloned*, that is, it is impossible to construct a copy of a quantum state without destroying the original whereas this is easily done in classical information.
Quantum information theory is a vast field, but there are few fundamental goals uniting work in the filed: [3]

- The identification of elementary classes of static resources in quantum mechanics (this is identified as type of information). Examples are qubits and Bell states (section 2.7) shared between two distant particles
- The identification of elementary classes of dynamical processes in quantum mechanics (this is identified as types of information processing). Examples are the ability to store quantum information over a period of time and the process of protecting quantum information from the effects of noise.
- The quantification of resource tradeoffs incurred performing elementary dynamical processes, for example, what minimal resources are required to transfer quantum information reliably between two parties using a noisy communications channel.

One of the most fundamental aspects behind quantum information is quantum entanglement. It allows one to perform tasks that are impossible or very difficult otherwise [3]. Examples of these tasks include quantum teleportation [4], superdense coding [5], quantum computing [6] and quantum cryptography [7]. Quantum entanglement is a physical resource proper to quantum mechanics that is associated with non classical correlations between subsystems of quantum composite systems. Quantum entanglement was initially regarded as a problematic and paradoxical aspect of quantum physics, when Einstein Podolski and Rosen pointed it out in

an attempt to show that quantum mechanics was an incomplete theory [8]. Nowadays, entanglement is seen as a fundamental phenomenon that lies at the heart of our understanding of quantum physics. It also constitutes a valuable resource for implementing information-related tasks.

Building quantum information processing devices is a great scientific and engineering challenge. One major limitation is noise. This is mainly because quantum entanglement is a very fragile resource. Real quantum systems suffer from unavoidable couplings with the environment [12, 13]. These interactions with the environment are regarded as noise in quantum information theory. Consequently, in many practical situations, quantum systems must be treated as open systems since it is not possible to get complete isolation from the surrounding. In order to build useful quantum information processing systems, it is crucial to understand these interactions with the environment.

The theory of open quantum systems plays a major role in quantum physics since a complete isolation of the system from the environment is not feasible, and a complete microscopic description or control of the environment degrees of freedom is not possible (or only partially so). The study of open quantum systems is a vast topic and there are many ways to approach it. In chapter 3, some basic aspects of this subject will be reviewed.

Open quantum systems may be subdivided into two major groups, depending on the restrictions we put on the system or the assumptions we make. These two groups are *Markovian quantum systems* and *non-Markovian quantum systems*.
In the case of Markovian systems (section 3.2.6) it is assumed that "memory effects" in the system-environment interaction are not important. In other words, it is assumed that any self-correlations within the environment created by the coupling to the system decay rapidly compared to the characteristic time over which the state of the system varies noticeably.

In the non-Markovain regime, on the other hand (section 3.3), the environment correlation time is greater than, or of the same order as, the relaxation time over which the state of the system changes. "Memory effects" are thus considered important and are taken into account.

The present study is mainly about the evolution of quantum entanglement of two-qubit systems interaction with a reservoir in a regime where the non-Markovian effects are important [12, 44, 45]. The dynamics in this case is quite different to the one corresponding to single qubit cases.

It was interesting to note that the entanglement disappears completely at finite times for some initial states. This is known as "Entanglement sudden death" (ESD) and has been observed experimentally (see Almeida et al. [43]). It is very important to study and understand ESD because the realization of quantum information and computation depends on the longevity of entanglement in multi-qubits states.

The System studied here was previously presented by B. Bellomo et al. [44, 45] but they considered only particular initial states. The aims of the present study are the following:

- My first goal is to determine a global picture of the average, typical behaviour of the system for important families of initial states. I will do this by recourse to a numerical approach based upon the generation of random initial states and the evaluation of the concomitant time dependent average properties.

- The second goal is to consider entanglement features that were not considered by the previous researchers. I will study the relationship between the time evolutions of the two-qubits entanglement, on the one hand, and their mixedness on the other. I will study the behavior of entanglement indicators based on $q$-entropies and on appropriate uncertainty relations.

- I will also consider the entanglement dynamics of systems consisting of more than two qubits.

Chapter 2

# Multi-partite quantum systems, Entanglement and entanglement indicators

It was stated in the introduction that quantum mechanics opens novel ways to process and transmit information. This is possible, to a large extent, because of quantum entanglement, as it is the fundamental resource that makes many quantum information tasks feasible. Quantifying quantum entanglement [10, 11, 26] is an important subject in quantum information theory. It is also important to develop practical entanglement indicators to estimate the amount of entanglement associated with a given quantum state.

The aim of this introductory chapter is to define some preliminary concepts that are important in quantum information theory and will be use extensively later. The first preliminary concept that will be presented is the *qubit* or quantum bit.

## 2.1 The Qubit

A qubit (quantum bit) is a 2-level quantum mechanical system (that is, a system described by a 2-dimensional Hilbert space). Similarly to the role played by the bit in classical computation and information, the qubit is the fundamental unit of quantum mechanical devices that process information.

Examples of physical realizations of qubits are the spin degrees of freedom of a spin-1/2 particle, the polarization state of a photon, or an atom where we focus our attention on quantum superpositions of only two relevant eigenstates.

Just like a classical bit, a qubit also has a state. Two possible states of a qubit are $|0\rangle$ and $|1\rangle$ which correspond to the state 0 and 1 of a bit. Unlike a classical bit, which can only have either of the two states 0 and 1, a qubit can also be in a superposition of state. That is, a qubit can be in a state like

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \tag{2.1}$$

where $\alpha$ and $\beta$ are complex numbers satisfying $|\alpha|^2 + |\beta|^2 = 1$. It is thus clear that a qubit can be in an infinity of possible states since one can find infinitely many complex numbers satisfying $|\alpha|^2 + |\beta|^2 = 1$. So one may think that it is, in principle, possible to store a great deal of information in a qubit. But this is not true because when the state of a qubit given by Eq. 2.1 is measured, one obtains either the state 0 or 1 with probability $|\alpha|^2$ and $|\beta|^2$ respectively, and, after the measurement, the state of the qubit is collapsed from the superposition state to the state resulting from the measurement. As an illustration, consider a spin-1/2 particle, an electron for instance. When the $z$-component of the spin is measured, one gets either $+$ (for spin up) or $-$ (for spin down) and the post measurement state of the particle is the state that is consistent with the result. Or, more explicitly, the post measurement state of the particle will be spin up if the value $+$ was obtained and spin down if the value $-$ was obtained.

A useful geometrical representation to picture the state of a qubit is as follows: since $|\alpha|^2 + |\beta|^2 = 1$, one may write Eq. 2.1 as

$$|\psi\rangle = \mathrm{e}^{i\gamma}\left(\cos\frac{\theta}{2}|0\rangle + \mathrm{e}^{i\varphi}\sin\frac{\theta}{2}|1\rangle\right), \tag{2.2}$$

where $\theta$, $\varphi$ and $\gamma$ are real numbers. As the overall phase of a quantum state does not have any physical significance (only the phase difference between $\alpha$ and $\beta$ is important), we can rewrite the state as

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + \mathrm{e}^{i\varphi}\sin\frac{\theta}{2}|1\rangle. \tag{2.3}$$

All possible states may be obtained by restricting the parameters to $0 \leq \theta \leq \pi$ and $0 \leq \varphi \leq 2\pi$. Thus any state is a point on the unit sphere with azimuthal angle $\varphi$ and polar angle $\theta$. This sphere is often referred to as the *Bloch sphere*.

### 2.1.1 Bell states

Bell states are two-qubits states of particular importance in quantum information. They are sometimes called *EPR pairs*, after Einstein, Podolsky and Rosen who were the first to point out the strange properties of entangled quantum states [8]. Bell states are

$$|\beta_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \tag{2.4}$$

$$|\beta_{01}\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \tag{2.5}$$

$$|\beta_{10}\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \tag{2.6}$$

$$|\beta_{11}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \tag{2.7}$$

The notation above may be generalized as [3]

$$|\beta_{xy}\rangle = \frac{1}{\sqrt{2}}(|0y\rangle + (-1)^x|1\bar{y}\rangle) \tag{2.8}$$

where $\bar{y}$ is the negation of $y$.

Bell states are maximally entangled two-qubit states and they play an important role in quantum protocols like quantum teleportation (section 2.4.2) and super dense coding. An interesting property of Bell states is the following: suppose one has a pair of qubits in the state $|\beta_{00}\rangle$. If one measures the state of the first qubit, the result is either 0 with probability $\frac{1}{2}$ (and the post measurement state is $|00\rangle$) or 1, also with probability $\frac{1}{2}$ (and the resulting state is $|11\rangle$). Therefore a measurement of the state of the second qubit will definitely be 0 in the first case or 1 in the second one. In other words, the results are correlated. John Bell proved that these correlations are beyond any classical correlation [9].

## 2.2 Density matrix and mixed states

The density matrix (or statistical operator) formalism was introduced independently by J. Von Neumann and L. Landau in 1927 to describe physical situations involving either

- incompletely specified quantum states or
- subsystems of composite quantum systems.

This formalism is very useful to describe a system whose state is not completely known and it is used extensively in quantum statistical mechanics. The formalism is also necessary for describing quantum mechanical experiments where noise is inevitable. In virtually all quantum mechanical experiments it is impossible to conpletely isolate the quantum system under study. The system becomes entangled with the environment through unwanted, but unavoidable, interactions. Nevertheless, it is sometimes necessary to describe the system without taking the environment explicitly into account. The density operator formalism constitutes an indispensable tool in situations where one needs to describe subsystems of a composite quantum system.

Consider a quantum system in one of the states of an ensemble of pure states $\{p_i, |\psi_i\rangle\}$ where $p_i$ is the probability associated with the state $|\psi_i\rangle$. The states $|\psi_i\rangle$ do not need to be orthogonal to each other. The density operator for the system is then defined as

$$\rho \equiv \sum_i p_i |\psi_i\rangle\langle\psi_i|. \tag{2.9}$$

In the density operator formalism one represents quantum states by hermitian operators acting on the system's Hilbert space, instead of unit vectors on this space. For any quantum state vector $|\psi\rangle$, the corresponding density operator is the projection operator $|\psi\rangle\langle\psi|$. That is, the system is in the state $|\psi\rangle$ with probability 1 in this case, and the system is said to be in a pure state.

The properties of a density matrix are:

1. **Hermiticity**: $\rho$ is a hermitian operator, that is, $\rho = \rho^\dagger$
2. **Trace condition**: $\mathrm{Tr}\rho = 1$, the density operator has trace equal to one. In other words, $\rho$ is normalized.
3. **Positivity condition**: For any vector $v$, $\langle v|\rho|v\rangle \geq 0$, that is, the density operator is a positive semidefinite operator. Alternatively, one may say that $\rho$ is Hermitian with nonnegative eigenvalues.

The trace property is easily proved as follows:
consider the density matrix given by Eq. 2.9, then

$$\mathrm{Tr}(\rho) = \sum_i p_i \mathrm{Tr}(|\psi_i\rangle\langle\psi_i|) = \sum_i p_i = 1$$

since the $p_i$ are probabilities.
For the last condition, suppose that $|\varphi\rangle$ is an arbitrary vector in the state space, then

$$\begin{aligned}\langle\varphi|\rho|\varphi\rangle &= \sum_i p_i\langle\varphi|\psi_i\rangle\langle\psi_i|\varphi\rangle \\ &= \sum_i p_i|\langle\varphi|\psi_i\rangle|^2 \\ &\geq 0.\end{aligned}$$

An important fact to mention here is that it is possible to find two different ensembles of pure states leading to the same density operator [3], For instance consider an equal mixture of the states $|0\rangle$ and $|1\rangle$; the density operator obtained from that mixed state is $\rho = \frac{1}{2}\mathbb{I}$ where $\mathbb{I}$ is the identity operator. The same density operator can be obtained by mixing equally the states $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$, as it is easy to verify that

$$\frac{1}{2}|+\rangle\langle+| + \frac{1}{2}|-\rangle\langle-| = \frac{1}{2}\mathbb{I}.$$

As stated above, a state that can be represented by a state vector is pure and a state that cannot be represented by a state vector is called a mixed state. We use the density operator formalism to describe mixed states. Now that pure and mixed states are defined, one may ask the following question: given a density operator, how can one tell wether it represents a pure or a mixed state? One simple criterion to check if a density operator represents a pure state or not is to compute the trace of that density operator squared, because for every density operator $\rho$,

$$\mathrm{Tr}\rho^2 \leq 1, \tag{2.10}$$

with equality if and only if $\rho$ represents a pure state. That is, if and only if $\rho = |\psi\rangle\langle\psi|$ for some unit vector $|\psi\rangle$.

One of the most important applications of density matrices in quantum information theory is the description of subsystems of composite quantum systems through the *reduced density matrix*. Consider for example a composite system made of two subsystems $A$ and $B$, and let $\rho^{AB}$ be a density matrix describing a state of the composite system $AB$. Then the reduced density matrix of the subsystem $A$ is given by

$$\rho^A = \mathrm{Tr}_B \rho^{AB}, \tag{2.11}$$

that is, $\rho^A$ is obtained by taking the partial trace over subsystem $B$. Likewise, the reduced density matrix of subsystem $B$ is obtained by performing a partial trace over subsystem $A$.

The partial trace is defined as follows: let $|\varphi_1^A\rangle$ and $|\varphi_2^A\rangle$ be two vectors on the state space $A$ and $|\varphi_1^B\rangle$ and $|\varphi_2^B\rangle$ be two vectors on the state space of $B$, then

$$\mathrm{Tr}_B(|\varphi_1^A\rangle\langle\varphi_2^A| \otimes |\varphi_1^B\rangle\langle\varphi_2^B|) = |\varphi_1^A\rangle\langle\varphi_2^A|\mathrm{Tr}(|\varphi_1^B\rangle\langle\varphi_2^B|). \tag{2.12}$$

### 2.2.1 Schmidt decomposition

The Schmidt decomposition [3, 27] is a very useful tool for the study of composite quantum systems and it is stated as follows:

Suppose $|\psi\rangle$ is a pure state of a composite quantum system, $AB$. Then there exist orthonormal basis $\{|i_A\rangle\}$ for system $A$ and $\{|i_B\rangle\}$ for system $B$ such that

$$|\psi\rangle = \sum_i \sqrt{\lambda_i}|i_A\rangle|i_B\rangle; \tag{2.13}$$

where $\lambda_i$ are non-negative real numbers satisfying $\sum_i \lambda_i = 1$. These are known as *Schmidt coefficients*. The basis $|i_A\rangle$ and $|i_B\rangle$ are called *Schmidt basis*, the number of non-zero valued $\lambda_i$ is known as the *Schmidt number* for the state $|\psi\rangle$, and it is at most $min(d_A, d_B)$ where $d_A$ and $d_B$ are the dimensions of subsystems $A$ and $B$ respectively. The *Schmidt coefficients* (and consequently also the Schmidt numbers) of a given state are invariant under local unitary transformations $U = U_A \otimes U_B$. This invariance makes the Schmidt number a useful tool. As an example, the *Schmidt number* can be use as way to check if a given pure bipartite state is separable.

An interesting consequence of the Schmidt decomposition is the following: consider the state $|\psi\rangle$ given above, this state is a pure state of the composite system $AB$. The reduced density matrices of the subsystems $A$ and $B$ are then given by $\rho^A = \sum_i \lambda_i |i_A\rangle\langle i_A|$ and $\rho^B = \sum_i \lambda_i |i_B\rangle\langle i_B|$ respectively. Thus the reduced density matrices of the two subsystems have the same eigenvalues. This implies that any function of the density matrix that depends only on the eigenvalues will have the same value for both reduced density matrices.

## 2.3 Entropic Measures in Quantum information theory.

The concept of entropy first arose in the theories of thermodynamics and statistical physics, but after Shannon's work [28], it turned out to also play a fundamental role in information theory. Entropy is a very important concept in classical and quantum information theory. One of its useful applications is as a measure of the degree of "mixedness" of a given quantum state. Entropy can be defined as a measure of the uncertainty in the state of a physical system. In this section, the definitions and basic properties of few relevant quantum entropic measures will be

reviewed.

### 2.3.1 Von Neumann Entropy

The von Neumann entropy is important because of its relation with the thermodynamics entropy. For a quantum state $\rho$ it is defined as

$$S(\rho) = -k\mathrm{Tr}(\rho \ln \rho), \tag{2.14}$$

where $k$ is the Bolzmann's constant. In quantum information theory, it is usually defined it as

$$S(\rho) \equiv -\mathrm{Tr}(\rho \log_2 \rho), \tag{2.15}$$

where the logarithm is taken in base two. If the eigenvalues of $\rho$ are known to be $\lambda_i$, the von Neumann entropy may be written as

$$S(\rho) = -\sum_i \lambda_i \log_2 \lambda_i. \tag{2.16}$$

In these expressions, it is understood that $0 \cdot \log 0 \equiv 0$.
Some of the properties of the von Neumann entropy are [3, 29]:

1. **Purity**. The entropy is non-negative and for a pure state $\rho = |\varphi\rangle\langle\varphi|$, $S(\rho) = 0$. This is straightforward because if $\rho$ represents a pure state, all the eigenvalues of $\rho$ are zero, except one, which has the value one. Thus the von Neumann entropy of the pure state is $S(\rho) = 1 \times \log_2 1 = 0$

2. **Invariance under unitary transformation.** That is,

$$S(U\rho U^\dagger) = S(\rho) \tag{2.17}$$

   for any unitary transformation $U$.

3. In a $d$ dimensional Hilbert space, the von Neumann entropy is at most $\log_2 d$, that is,

$$S(\rho) \leq \log_2 d, \tag{2.18}$$

with equality when all the eigenvalues are nonzero and equal, in other words, $S(\rho) = \log_2 d$ if and only if the system is in the completely mixed state $\mathbb{I}/d$, where $\mathbb{I}$ is the $d \times d$ identity matrix.

4. Suppose that the composite system $AB$ is in a pure state, then the two marginal density matrices $\rho_A$ and $\rho_B$ have the same entropy,

$$S[\rho_A] = S[\rho_B]. \tag{2.19}$$

This can simply be proven by using the Schmidt decomposition. As stated in section 2.2.1, one consequence of the Schmidt decomposition is that if the composite system is in a pure state, the reduced density matrices of the subsystems have the same eigenvalues.

5. **Concavity:** The von Neumann entropy is a concave functional on the space of density matrices. That is, given n density matrices $\rho_i$, for $\lambda_1, \lambda_2, \cdots, \lambda_n \geq 0$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_n = 1$,

$$S(\lambda_1\rho_1 + \cdots + \lambda_n\rho_n) \geq \lambda_1 S(\rho_1) + \cdots + \lambda_n S(\rho_n). \tag{2.20}$$

6. **Additivity:** If $\rho = \rho_A \otimes \rho_B$, then $S(\rho) = S(\rho_A) + S(\rho_B)$.

### 2.3.2 The Linear Entropy

The linear entropy is an entropic measure that is sometimes preferred to the von Neumann entropy due to its many computational advantages, both from numerical and analytical points of view. The linear entropy of a density matrix $\rho$ is given by

$$S_L(\rho) = 1 - \mathrm{Tr}(\rho^2). \tag{2.21}$$

One clearly sees that to compute $S_L$, there is no need to diagonalize the density matrix like in the case of the von Neumann entropy.
The usefulness of the linear entropy as a measure of the degree of mixedness is based on an important property of the density matrix: for a given density matrix $\rho$, $\mathrm{Tr}(\rho^2) \leq 1$ with equality if and only if $\rho$ represents a pure state. From that, it

is clear that $S_L(\rho)$ is zero if and only if $\rho$ represents a pure state. On the other hand, $S_L$ adopts its maximum value for the completely mixed density matrix.

### 2.3.3 $q$-entropies

The $q$-entropies are entropic measures involving the quantity $\mathrm{Tr}(\rho^q)$. The most important $q$-entropies are the Tsallis [30, 36, 33, 34, 35] and the Rényi [30] entropies. A noticeable fact about the $q$-entropies is their dependence on the parameter $q$. Each value of $q$ corresponds to a specific entropic measure. In other words, the $q$-entropies provide one with a family of entropic measures. It is interesting to note that the $q$-entropies reduce to the von Neumann entropy in the limit $q \to 1$.

#### Rényi entropy

The Rényi entropy is defined as follows:

$$S_q^{(R)} = \frac{1}{1-q} \ln(\mathrm{Tr}(\hat{\rho}^q)) \tag{2.22}$$

and in the limit $q \to \infty$ the Rényi entropy becomes

$$S_\infty^{(R)}(\rho) = -\ln(\lambda_m) \tag{2.23}$$

where $\lambda_m$ is the maximum eigenvalue of the density matrix $\rho$.

#### Tsallis entropy

The Tsallis entropy on the other hand is defined as

$$S_q^{(T)} = \frac{1}{q-1}(1 - \mathrm{Tr}(\hat{\rho}^q)) \tag{2.24}$$

In the limit $q \to 1$, the von-Neumann entropy is recovered. The linear entropy $S_L$ corresponds to $S_q^{(T)}$ with $q = 2$.

## 2.4 Quantum entanglement

The word entanglement comes from the translation of the German word "Verschränkung "used by Erwin Schödinger to describe a correlation of a quantum

nature [37]. Quantum entanglement is a physical resource proper to quantum mechanics that is associated with non classical correlations between subsystems of quantum composite systems. A pure state of a composite quantum system is entangled if it cannot be factorized in terms of pure states of each of the subsystems. Otherwise the state is said to be separable. Consider a bipartite system consisting of two subsystems $A$ and $B$. A pure state of this system is entangled if it cannot be factorized as $|a\rangle \otimes |b\rangle$, where $|a\rangle$ is a pure state of subsystem $A$ and $|b\rangle$ is a pure state of subsystem $B$. As an example, consider two qubits in the Bell state $|\beta_{00}\rangle = \frac{1}{\sqrt{2}}\{|00\rangle + |11\rangle\}$. It is impossible to attribute to either qubit a definite pure state. In other words, it is impossible to write that Bell state as $|a\rangle \otimes |b\rangle$, where $|a\rangle$ is a state of the first qubit and $|b\rangle$ is a state of the second qubit. The state $|00\rangle$ on the other hand is separable and it is clear that each of the qubits are in the state $|0\rangle$.

A mixed state of the bipartite system described above is entangled if it cannot be represented as a mixture of factorizable pure states of the system. That is, if its density matrix cannot be written as [38]

$$\rho = \sum_i p_i |a_i\rangle\langle a_i| \otimes |b_i\rangle\langle b_i|, \tag{2.25}$$

where the coefficients $p_i$ are probabilities ($0 \leq p_i \leq 1, \sum_i p_i = 1$) and the $|a_i\rangle$ and $|b_i\rangle$ are pure states of subsystems $A$ and $B$ respectively.
An example of a mixed separable state is

$$\rho = \frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|) \tag{2.26}$$

and an example of a mixed entangle state is the Werner state

$$\rho_W = \frac{1-p}{4}\mathbb{I}_4 + p|\beta_{00}\rangle\langle\beta_{00}|. \tag{2.27}$$

That is, a mixture of a maximal entanglement via the Bell state $|\beta_{00}\rangle$ and a maximal incoherence via the two-particles identity. This state is entangled for $p > 1/3$. Entangled states cannot be prepared locally by acting on each of the subsystem individually. This is related to the fact that the amount of entanglement does

not change under local unitary transformation. This last statement can be proved using the Schmidt decomposition [27] (section 2.2.1).

### 2.4.1 Entanglement measures and entanglement indicators

It is clear by now that quantum entanglement is a very important resource in quantum information theory, but to be able to use it, one needs to find a way to measure or to quantify it. In this section, entanglement measures for bipartite systems will be defined. The focus is on bipartite entanglement due to the intrinsic importance of bipartite systems, and also because the most useful multi-partite entanglement measures are based on the bipartite case.
An entanglement measure $E(\cdot)$ may be seen as a functional that takes a quantum state of a bi-partite system to a nonnegative real number, that is,

$$E : D(\mathcal{H}) \rightarrow \mathbb{R}^{+} \tag{2.28}$$

where $D(\mathcal{H})$ is the set of density operators on the Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$.
Quantifying entanglement is an important topic in quantum information theory, but determining the amount of entanglement associated with general mixed states of multi-partite systems is a very difficult task.
Various measures of entanglement appear in the literature, some of these measures will be presented in details in the following sections. These measures are subdivided into two groups: on the one hand we have *operational measures* like the entanglement cost $E_C$ and the entanglement of distillation $E_D$ and on the other hand we have the *abstract measures* that quantify entanglement mathematically but are not operationally defined in terms of entanglement manipulation. Examples of abstract entanglement measures include the *reduced von Neumann entropy* and the *entanglement of formation*. Some of these abstract measures coincide with the operational ones.
A good entanglement measure has to fulfill certain requirements that will be listed below, but it is still debatable whether all of these requirements are necessary. In

fact, some of the entanglement measures that will be presented below don't satisfy all of them. The fundamental requirement that everyone agrees with is that an entanglement measure must be monotonic under LOCC [25]. In other words, the amount of entanglement should not increase by local quantum operation and classical communications [39].
The requirements for an entanglement measure are:

1. **If $\rho$ is separable then** $E(\rho) = 0$.

2. **Normalization**: the entanglement of a maximally entangled state of two $d$-dimensional systems $A$ and $B$ is given by

$$E(P_+^d) = \log_2 d \tag{2.29}$$

where $P_+^d = |\Phi_+^d\rangle\langle\Phi_+^d|$ and $|\Phi_+^d\rangle = \sum_{i=1}^{d} \frac{1}{\sqrt{d}}|i_A\rangle\otimes|i_B\rangle$ is a maximally entangle state in $d \times d$ dimensions and $\{|i_A\rangle\}$ and $\{|i_B\rangle\}$ are orthonormal basis of systems $A$ and $B$ respectively.

3. **No increase under LOCC**: the entanglement of $\rho$ cannot be increased by applying local operations to $\rho$ and classically communicating. That is,

$$E(\Lambda_{LOCC}(\rho)) \leq E(\rho). \tag{2.30}$$

4. **Continuity**: In the limit of vanishing distance between two density matrices, the difference between their entanglement should tend to zero. That is,

$$E(\rho) - E(\sigma) \to 0 \quad for \quad ||\rho - \sigma|| \to 0. \tag{2.31}$$

5. **Additivity**: The entanglement of a number $n$ of identical copies of a given state $\rho$ should be equal to $n$ times the entanglement of one copy,

$$E(\rho^{\otimes n}) = nE(\rho). \tag{2.32}$$

6. **Subadditivity**: The entanglement of a tensor product of two states $\rho$ and $\sigma$ must be less than or equal to the sum of the entanglement of each of the states, i.e.

$$E(\rho \otimes \sigma) \leq E(\rho) + E(\sigma). \tag{2.33}$$

7. **Convexity**: The entanglement measure should be a convex function, that is,

$$E(\lambda\rho + (1-\lambda)\sigma) \leq \lambda E(\rho) + (1-\lambda)E(\sigma) \tag{2.34}$$

for $0 < \lambda < 1$.

In an attempt to quantify entanglement for bi-partite quantum systems, one should distinguish between the simpler case of pure states entanglement and the more complicated case of mixed states entanglement. It should be stressed that there are entanglement indicators that are very useful for estimating the amount of entanglement, even if they do not comply with all the above properties.

#### Entanglement measures for pure bi-partite states

##### Entropy of entanglement

For a pure bipartite state each of the subsystems is described by a mixed state. The more mixed the subsystems are, the more entangled the overall state is. Thus, one can use the degree of mixture of the subsystems as a measure of the entanglement present in the global state. Since quantum entropies measure the degree of mixture of a given quantum state, one can use the von Neumann entropy of the reduced density matrices (sometimes called *the reduced von Neumann entropy*) as a measure of the amount of the entanglement of the global state. This measure of entanglement is also known as the **entropy of entanglement** ($E_E$).

#### Entanglement of distillation and the entanglement cost

As stated above, these entanglement measures have a direct operational meaning in terms of the manipulation of entanglement.

**The entanglement cost** is defined as the minimum number of Bell states needed to create a given state by means of LOCC. In other words, it tells us how expensive it is to create an entangled state, i.e. what is the ratio of the number of maximally entangled $|Bell\rangle$ states over the number of produced output states $\rho$, minimized over all LOCC operation. In the limit of infinitely many states, $E_C$ is given by

$$E_C(\rho) = \inf_{\{\Lambda_{LOCC}\}} \lim_{n_\rho \to \infty} \frac{n^{in}_{|Bell\rangle}}{n^{out}_{\rho}}. \tag{2.35}$$

**The Entanglement of distillation** on the other hand is the maximum yield of Bell states that can be obtained, optimized over all possible LOCC protocols. In other words, it tells one how much entanglement can be extracted from a given entangled state $\rho$, i.e. what the ratio is of the number of maximally entangled output states $|Bell\rangle$ over the needed input states $\rho$, maximized over all LOCC operations. In the limit of infinitely many states, $E_D$ is given by

$$E_D(\rho) = \sup_{\{\Lambda_{LOCC}\}} \lim_{n_\rho \to \infty} \frac{n^{out}_{|Bell\rangle}}{n^{in}_{\rho}}. \tag{2.36}$$

It was shown in [39] that these three entanglement measures coincide, that is, $E_D(|\psi\rangle) = E_C(|\psi\rangle) = E_E(|\psi\rangle)$ for a given pure state $|\psi\rangle$. This is a consequence of *the uniqueness theorem for entanglement measures* [25, 40, 26, 41, 42]. This theorem states that any pure state measure of entanglement coincides with the entropy of entanglement under appropriate conditions. The minimal conditions for the theorem to hold are given in [40].

#### Mixed bi-partite states entanglement measures

It is much more complicated to quantify entanglement in the case of mixed states than it is for pure states. A different approach has to be used. The reduced von Neumann entropy fails to quantify mixed states' entanglement because it doesn't distinguish between quantum and classical correlation.

The *entanglement cost* ($E_C$) and the *entanglement of distillation* ($E_D$) discussed above may also be used to quantify mixed state entanglement. It was shown in

[40] that under appropriate conditions, the *entanglement of distillation* and the *entanglement cost* provide upper and lower bounds for mixed states entanglement measures. That is, for some entanglement measures $E$, we have $E_D \leq E \leq E_C$. $E_C$ and $E_D$ are entanglement measures with operational meaning, but there also exist abstract mixed states entanglement measures. These include the *entanglement of formation* and *the Negativity*. These two entanglement measures will be the subject of the following sections.

#### The entanglement of formation

The *entanglement of formation*, ocasionally called *the entanglement of creation*, may be seen as the generalization of the entropy of entanglement to mixed states. The entanglement of formation is defined as follows:
Let $\rho$ be the density matrix jointly describing the pair of quantum systems $A$ and $B$. Consider all possible decompositions $\{p_i, |\psi_i\rangle\}$ of $\rho$ as a mixture of pure states $|\psi_i\rangle$. In other words, all the mixtures $\{p_i, |\psi_i\rangle\}$ such that

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|. \tag{2.37}$$

For each of the pure states $|\psi_i\rangle$, the entanglement $E(\psi_i)$ is quantified by the entropy of either subsystems A and B, as discussed above. The entanglement of formation is then the minimum over all the aforementioned decompositions of the corresponding average entanglement,

$$E(\rho) = \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i E(\psi_i). \tag{2.38}$$

This minimization problem constitutes in general a formidable one that can only be approached numerically. However, an analytical expression for the entanglement of formation has been obtained by Wooters in the two-qubits case. Wooters'

celebrated formula is given by [47]:

$$E = h\left(\frac{1+\sqrt{1-C^2}}{2}\right) \tag{2.39}$$

where $h(x) = -x\log_2(x) - (1-x)\log_2(1-x)$ is Shannon's entropy function and $C$ is the concurrence defined by

$$C(\rho) = max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}. \tag{2.40}$$

The $\lambda_i$ are the eigenvalues, in decreasing order, of the hermitian matrix

$$R = \sqrt{\sqrt{\rho}\tilde{\rho}\sqrt{\rho}} \tag{2.41}$$

and

$$\tilde{\rho} = (\sigma_2 \otimes \sigma_2)\rho^*(\sigma_2 \otimes \sigma_2) \tag{2.42}$$

is known as the spin-flip state with

$$\sigma_2 = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{2.43}$$

**The Negativity**

Another abstract measure of entanglement is the negativity [50, 54, 55, 56, 57]. The negativity is defined as follows: Consider a density matrix $\rho$ describing a state of a bipartite system composed by two subsystems A and B. Let us assume that this composite system has a finite-dimensional Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$. Let $\rho^{T_A}$ denote the partial transpose of $\rho$ with respect to subsystem A. That is, $\rho^{T_A}$ is the Hermitian, trace-normalized operator having matrix elements

$$\langle i_A, j_B|\rho^{T_A}|k_A, l_B\rangle \equiv \langle k_A, j_B|\rho|i_A, l_B\rangle. \tag{2.44}$$

Then the negativity is defined as

$$\mathcal{N}(\rho) \equiv \frac{\left\|\rho^{T_A}\right\|_1 - 1}{2} \tag{2.45}$$

where $\|\cdot\|_1$ is the trace norm, which is defined for any Hermitian operator $A$ as $\|A\|_1 = tr\sqrt{A^\dagger A}$. Alternatively, the negativity can be defined as

$$\mathcal{N}(\rho) \equiv \sum_i \frac{|\lambda_i| - \lambda_i}{2} \tag{2.46}$$

where $\lambda_i$ are the eigenvalues of $\rho^{T_A}$. In other words, the negativity is the absolute value of the sum of the negative eigenvalues of $\rho^{T_A}$.

The Negativity measure is closely related to the Peres criterion for separability [58], also known as the *positivity of the partial transpose* ( PPT) criterion. This criterion states that for $\rho$ to be separable, all the eigenvalues of $\rho^{T_A}$ must be positive or zero. The reasoning behind Peres criterion is the following:

Since any separable bi-partite state can be written as in Eq. 2.25, its partial transpose is given by

$$\rho_{sep}^{T_A} = \sum_i p_i (|a_i\rangle\langle a_i|)^T \otimes |b_i\rangle\langle b_i|. \tag{2.47}$$

The $(|a_i\rangle\langle a_i|)^T$ are again valid density matrices for the subsystem $A$ (the $(|a_i\rangle\langle a_i|)^T$ are Hermitian, non-negative matrices with unit trace). Therefore $\rho_{sep}^{T_A} \geq 0$, in other words, $\rho_{sep}^{T_A}$ is a positive operator.

The PPT criterion is a necessary separability criterion, but not a sufficient one because some entangled states pass the PPT criterion. Entangled states verifying the PPT criterion are called PPT entangled states. In the special case of two-qubits states, the PPT separability criterion is both necessary and sufficient.

The fact that the Negativity relies on the PPT criterion is also its weakness because it fails to distinguish between separable states and entangled PPT states. That is, the Negativity is zero for both separable states and PPT entangled states. However, from the practical point of view, the negativity is still the best available entanglement measure for mixed states of composite systems of dimensions larger than $2 \times 2$. Indeed, it is the measure normally used by researchers when studying the entanglement of mixed states in these systems.

**Entanglement indicators based on the $q$-entropies.**

Given a composite quantum system, $AB$, consisting of two subsystems $A$ and $B$, the entropies associated with the composite system as a whole, $S[AB]$, and the entropies associated with the subsystems, $S[A]$ and $S[B]$, allow for a direct, information-theoretical way of characterizing the entanglement exhibited by certain quantum states. A very intuitive property of classical composite systems is that the global entropy of the complete system is always larger than or equal to the individual entropies associated with each of the subsystems. However, this is not always true in the case of composite quantum systems. For instance, when one has such a composite system in a pure quantum state, the entropies of its subsystems are in general not zero, in spite of the fact that the entropy of the global system vanishes. Indeed, for pure states of bi-partite quantum systems the entropy of one of the subsystems constitutes a valuable quantitative measure for the amount of entanglement exhibited by the state.

In the case of mixed states of bi-partite quantum systems it is also the case that, unlike what happens with classical systems, the entropy of a sub-system may be greater than the entropy of the global system if the composite system is in an entangled state. However, the situation for mixed states is more complicated than it is for pure states. All non-entangled states comply with the classical entropic inequalities, but some entangled states also verify those inequalities. This means that, if for a given state the entropy of a subsystem is larger than the entropy of the complete system, then we know for sure that the state under consideration is entangled. However, if the state complies with the classical entropic inequalities, we cannot be sure that the state is separable.

On the basis of the above considerations, it is interesting to consider entropic

differences of the form

$$D_q = S_q[A] - S_q[AB] \tag{2.48}$$

as indicators of entanglement. If $D_q$ is positive, then one can conclude that the state is entangled, but in the case when $D_q$ is negative or zero, one cannot conclude that the state is separable, for there are entangled states for which the value of $D_q$ is negative. Thus, in the case of mixed states, the quantities $D_q$ lead to sufficient (but not necessary) criteria for entanglement.

An entanglement measure was defined in section 2.4.1 as a functional that takes quantum states to positive real numbers, but the quantity $D_q$ defined here can be negative for certain entangle states. So for the sake of conforming with the definition, only the positive values of $D_q$ will be considered. The quantity $D_q$ will be set equal to zero whenever $S_q[A] - S_q[AB] < 0$.
The entropic entanglement indicators $D_q$ are closely related to the other entropic indicators or entropic measures:

- For pure states and $q = 1$, $D_q$ coincides with the entropy of entanglement.
- For pure states and $q = 2$, $D_q$ coincides with the linear entropy.
- In the case $q = 2$ (for general states, pure or mixed), $D_q$ is closely related to the Mintert-Buchleitner lower bound $E_{MB}$ for the squared concurrence (section 2.4.1).

#### Entanglement indicators based on uncertainty relations

Uncertainty relations lie at the heart of some of the most basic aspects of quantum physics. It turns out that it is possible to formulate uncertainty relations that will only be violated by entangled states. In other words, all separable states comply with these relations, but some entangled states violate them. However, not all entangled states violate these relations. Consequently, these relations provide suf-

ficient (but not necessary) conditions for entanglement [61].

Let us consider a family of observables of a quantum mechanical system represented by a set of hermitian operators $\{\hat{A}_i\}$. The uncertainty of $A_i$ is then given by

$$\delta A_i^2 = \langle \hat{A}_i^2 \rangle - \langle \hat{A}_i \rangle^2. \tag{2.49}$$

This positive quantity can only be zero in the case of perfect predictability of the measurement outcome, that is, only if the quantum state is an eigenstate of $\hat{A}_i$. This implies that a quantum state with zero uncertainty in all the properties $\hat{A}_i$ must be a simultaneous eigenstate of all the operators in $\{\hat{A}_i\}$. In the case where these uncertainties cannot be all zero (because the $\hat{A}_i$ do not all share a common eigenstate), there must be a lower bound $U > 0$ for the sum of uncertainties

$$\sum_i \delta A_i^2 \geq U. \tag{2.50}$$

The value of $U$ may be difficult to find in cases where the operators $\hat{A}_i$ have a difficult form. There are, however, cases where this limit is relatively easy to compute. For $N$-level systems, one can obtain such a limit by using the spin algebra of the corresponding spin $l = (N-1)/2$ system with

$$(\hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2)|\psi\rangle = l(l+1)|\psi\rangle \tag{2.51}$$

for any state $|\psi\rangle$. In this case one obtains the uncertainty limit [61]

$$\delta L_x^2 + \delta L_y^2 + \delta L_z^2 = \langle \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 \rangle - (\langle \hat{L}_x^2 \rangle + \langle \hat{L}_y^2 \rangle + \langle \hat{L}_z^2 \rangle) \geq l. \tag{2.52}$$

In the case of two-level systems, the spin are often expressed in terms of the normalized Pauli matrices. The uncertainty relation above (Eq. 2.52) becomes

$$\delta\sigma_1^2 + \delta\sigma_2^2 + \delta\sigma_3^2 \geq 2. \tag{2.53}$$

Now, given a bi-partite system $AB$, let us consider two sets of observables $\{A_i\}$ and $\{B_i\}$ referring, respectively, to the subsystems $A$ and $B$. These two sets of operators satisfy the sum uncertainty relations given by

$$\begin{aligned} \sum_i \delta A_i^2 &\geq U_A, \\ \sum_i \delta B_i^2 &\geq U_B. \end{aligned} \tag{2.54}$$

Here, the Hilbert spaces of systems $A$ and $B$ do not need to have the same dimension. The operator properties $\hat{A}_i + \hat{B}_i$ then define a set of joint properties of the two systems that can be determined by local measurement on $\hat{A}_i$ and $\hat{B}_i$, respectively. It follows that the uncertainties of $\hat{A}_i + \hat{B}_i$ are equal to the sum of the local uncertainties for product states $\rho = \rho(A) \otimes \rho(B)$. That is,

$$\delta(A_i + B_i)^2 = \delta A_i^2 + \delta B_i^2 \tag{2.55}$$

Therefore, the measurement statistics of the product states are limited by the uncertainty relation

$$\sum_i \delta(A_i + B_i)^2 \geq U_A + U_B. \tag{2.56}$$

This uncertainty limit is also valid for mixture of product states

$$\rho = \sum_m p_m \hat{\rho}_m(A) \otimes \hat{\rho}_m(B), \tag{2.57}$$

since the uncertainties of a mixture are always greater than or equal to the average uncertainties of the components. Therefore, any separable state satisfies the local uncertainty relation (Eq. 2.56). Any state violating Eq. 2.56 must be entangled. One can define the relative violation of local uncertainty relation for a pair of quantum systems $A$ and $B$ as

$$C_{LUR} = 1 - \frac{\sum_i \delta(A_i + B_i)^2}{2U} \tag{2.58}$$

where $U$ is the uncertainty limit for a single system (here, we assume that $U_A = U_B = U$).

**Mintert-Buchleitner observable concurrence lower bound**

All the entanglement measures presented so far are indirect ways to quantify entanglement in a given quantum state. In this section, we review a method of obtaining the lower bound of the entanglement in a mixed state $\varrho$ by making few experimental measurement on a twofold copy $\varrho \otimes \varrho$ [62, 63]. This lower bound is given by

$$[c(\varrho)]^2 \geq \mathrm{Tr}(\varrho \otimes \varrho V_i), \tag{2.59}$$

$(i = 1, 2)$ with $V_1 = 4(P_- - P_+) \otimes P_-$ and $V_2 = 4P_- \otimes (P_- - P_+)$

$P_-$ is the projector on the antisymmetric subspace of the two copies of either subsystem and $P_+$ is the symmetric counterpart of $P_-$. The lower bound on the right-hand side can be expressed in terms of the purities of $\varrho$, $\varrho_r^{(1)}$ and $\varrho_r^{(2)}$ that is,

$$\mathrm{Tr}(\varrho \otimes \varrho V_i) = 2[\mathrm{Tr}\varrho^2 - \mathrm{Tr}(\varrho_r^{(i)})^2], \tag{2.60}$$

where $\varrho_r^{(i)}$ is the reduced density matrix of either subsystem.

In the case $q = 2$ the quantity $E_{MB} = S_q[A] + S_q[B] - 2S_q[AB]$ (evaluated using Tsallis $q$-entropies) is the lower bound for the squared concurrence of the state. The quantity $E_{MB}$ is *an experimentally measurable quantity* [62]. For quantum states verifying the equality $S_2[A] = S_2[B]$ the alluded measurable quantity coincides with $D_2$ introduced in section 2.4.1. This is particularly important because the amount of entanglement (as measured, for instance, by the squared concurrence) is not a directly measurable quantity. It is therefore important, both from the practical and the theoretical points of view, to investigate in detail the properties of experimentally measurable indicators of entanglement, especially if they also establish lower bounds for the amount of entanglement.

**Entanglement measure for multi-partite (more than two parts) systems**

It is important to understand multi-partite systems' entanglement since this is crutial to the realization of quantum information processing devices. For a multi-partite system in a pure state, one way to quantify entanglement is to compute

the average value of the entropic entanglement measures for all the possible bi-partitions [53, 52, 57].

A practical entanglement measure for n-qubit pure states $|\psi\rangle$ was introduced by Meyer and Wallach [51]. Brennen [53] showed that this measure is equivalent to the average of all the single-qubit linear entropies,

$$Q(|\psi\rangle) = 2\left(1 - \frac{1}{n}\sum_{k=1}^{N} \mathrm{Tr}\rho_k^2\right), \tag{2.61}$$

where $\rho_k$ is the density operator for the $k$th qubit after tracing out the rest. This expression was generalized by Scot [52] to the multiqubit states $|\psi\rangle$. By considering all the possible bipartite divisions one obtains

$$\begin{aligned} Q_m(|\psi\rangle) &= \frac{2^m}{2^m - 1}\left(1 - \frac{m!(n-m)!}{n!}\sum_s \mathrm{Tr}\rho_s^2\right), \\ m &= 1, \ldots, \lfloor n/2 \rfloor, \end{aligned} \tag{2.62}$$

where the sum is taken over all the subsystems $s$ constituted by $m$ qubits, $\rho_s$ are the corresponding marginal density matrices, and $\lfloor x \rfloor$ is the integer part of $x$.
In the case of a multi-partite system in mixed state, one can used the average of the negativity of all the possible partitions [56, 54].

Let us consider an n-qubit state in the form

$$|\psi\rangle = \sum_{k=0}^{2^n-1} c_k |k\rangle \tag{2.63}$$

where the $c_k \in \mathbb{C}$ obey the normalization condition. Each $k$ is a basis state $|a_1 a_2 \cdots a_n\rangle$, where $a_1 a_2, \cdots, a_n$ is the binary representation of the integer $k$, with $a_i \in \{0, 1\}$. Now we construct a density operator of a mixed state $\rho$ using $N$ pure

states of the same form as $|\psi\rangle$:

$$\begin{aligned}
\rho &= \sum_{j=1}^{N} p_j |\psi_j\rangle\langle\psi_j| \\
&= \sum_{j=1}^{N} p_j \sum_{a_1\ldots a_n=0}^{1} c^j_{a_1\ldots a_n}|a_1\ldots a_n\rangle \sum_{a'_1\ldots a'_n=0}^{1} c^{j*}_{a'_1\ldots a'_n}\langle a'_1\ldots a'_n| \\
&= \sum_{j=1}^{N} p_j \sum_{\substack{a_1\ldots a_n=0\\ a'_1\ldots a'_n=0}}^{1} d^j_{a_1\ldots a_n a'_1\ldots a'_n}|a_1\ldots a_n\rangle\langle a'_1\ldots a'_n|
\end{aligned} \tag{2.64}$$

where $d^j_{a_1\ldots a_n a'_1\ldots a'_n=0} = c^j_{a_1\ldots a_n} c^{j*}_{a'_1\ldots a'_n=0}$. The partial transpose $\rho^{T_{\{i\}}}$ of $\rho$ with respect to the index $i$ (also known as the cut set $\{i\}$) is obtained by transposing the bits $a_i$ and $a'_i$ in the basis states. That is,

$$\begin{aligned}
\rho^{T_{\{i\}}} &= \sum_{j=1}^{N} p_j \sum_{\substack{a_1\ldots a_n=0\\ a'_1\ldots a'_n=0}}^{1} d^j_{a_1\ldots a_i\ldots a_n a'_1\ldots a'_i\ldots a'_n}|a_1\ldots a'_i\ldots a_n\rangle\langle a'_1\ldots a_i\ldots a'_n| \\
&= \sum_{j=1}^{N} p_j \sum_{\substack{a_1\ldots a_n=0\\ a'_1\ldots a'_n=0}}^{1} d^j_{a_1\ldots a'_i\ldots a_n a'_1\ldots a_i\ldots a'_n}|a_1\ldots a_i\ldots a_n\rangle\langle a'_1\ldots a'_i\ldots a'_n|
\end{aligned} \tag{2.65}$$

The partial transpose of larger set of indices (larger *cut set*) is constructed similarly, by transposing the bits corresponding to each of the index in the set. Each density matrix has $2^n$ different partial transposes. To get a good indication of the amount of entanglement, one must investigate all the possible cuts. In practice, however, only $2^{n-1}-1$ of the $2^n$ partial transposes need to be calculated. This is because certain cuts are equivalent and the trivial partial transpose with respect to the empty cut $\rho^{T_{\{\emptyset\}}}$ is the original density matrix, which is known to have no negative eigenvalues.

### 2.4.2 Example of an application of Entanglement

Entanglement is important because it is a fundamental resource that can be exploited to perform tasks that are impossible or very difficult otherwise. Quan-

tum entanglement is one of the pillars of the emerging, new quantum information technologies. Entanglement is at the core of information-related tasks such as quantum computation, quantum cryptography, quantum teleportation and super dense coding. For an example of application of quantum entanglement, quantum teleportation, proposed by Bennett et al in 1993, will be discussed below.

#### Quantum teleportation

Quantum teleportation [3, 4] may be seen as a technique for moving quantum states around even in the absence of a quantum communication channel between the sender and the recipient.

Consider two observers "Alice" and "Bob" that live far apart, and suppose that they met long time ago and shared an EPR pair (section 2.1.1). Now Alice wishes to send a qubit in a state $|\psi\rangle$ to Bob, but the state of the qubit and the location of Bob are unknown to Alice and there is no quantum communication channel between her and Bob (that is, she can only send classical information to Bob). One way to achieve this task happens to be quantum teleportation.

Let $\{|0\rangle_A, |1\rangle_A\}$ and $\{|0\rangle_B, |1\rangle_B\}$ be basis for Alice's and Bob's EPR particles respectively, and let $\{|0\rangle_Q, |1\rangle_Q\}$ be a basis for the qubit that Alice wishes to send to Bob. The state of that qubit is $|\psi\rangle = a|0\rangle_Q + b|1\rangle_Q$ with $a$ and $b$ unknown amplitudes satisfying $|a^2| + |b^2| = 1$. The EPR particles are prepared in the Bell state $|\beta_{00}\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B)$. The entire system is thus in a pure state $|\psi_{QAB}\rangle = |\psi\rangle|\beta_{00}\rangle$. To teleport the qubit, Alice must couple it with the EPR pair. She does so by performing a complete measurement on the system consisting of the qubit and her EPR particle. The measurement is performed in the Bell basis given by

$$|\beta_{00}\rangle = \frac{1}{\sqrt{2}}(|0\rangle_Q|0\rangle_A + |1\rangle_Q|1\rangle_A)$$
$$|\beta_{01}\rangle = \frac{1}{\sqrt{2}}(|0\rangle_Q|1\rangle_A + |1\rangle_Q|0\rangle_A)$$
$$|\beta_{10}\rangle = \frac{1}{\sqrt{2}}(|0\rangle_Q|0\rangle_A - |1\rangle_Q|1\rangle_A)$$
$$|\beta_{11}\rangle = \frac{1}{\sqrt{2}}(|0\rangle_Q|1\rangle_A - |1\rangle_Q|0\rangle_A).$$

These four states constitute a complete orthonormal basis for the particle A and the qubit Q.

The complete state of the three particles before Alice's measurement is

$$\begin{aligned} |\psi_{QAB}\rangle &= (a|0\rangle_Q + b|1\rangle_Q)[\frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B)] \\ &= \frac{a}{\sqrt{2}}(|0\rangle_Q|0\rangle_A|0\rangle_B + |0\rangle_Q|1\rangle_A|1\rangle_B) \\ &\quad + \frac{b}{\sqrt{2}}(|1\rangle_Q|0\rangle_A|0\rangle_B + |1\rangle_Q|1\rangle_A|1\rangle_B). \end{aligned} \tag{2.66}$$

From the Bell basis, we get the following relations:

$$|0\rangle_Q|0\rangle_A = \frac{\sqrt{2}}{2}(|\beta_{00}\rangle + |\beta_{10}\rangle) \tag{2.67}$$
$$|0\rangle_Q|1\rangle_A = \frac{\sqrt{2}}{2}(|\beta_{01}\rangle + |\beta_{11}\rangle) \tag{2.68}$$
$$|1\rangle_Q|0\rangle_A = \frac{\sqrt{2}}{2}(|\beta_{01}\rangle - |\beta_{11}\rangle) \tag{2.69}$$
$$|1\rangle_Q|1\rangle_A = \frac{\sqrt{2}}{2}(|\beta_{00}\rangle - |\beta_{10}\rangle). \tag{2.70}$$

By substituting these into $|\psi_{QAB}\rangle$, one gets

$$\begin{aligned} |\psi_{QAB}\rangle &= \frac{a}{2}[(|\beta_{00}\rangle + |\beta_{10}\rangle)|0\rangle_B + (|\beta_{01}\rangle + |\beta_{11}\rangle)|1\rangle_B \\ &\quad + \frac{b}{2}[(|\beta_{01}\rangle - |\beta_{11}\rangle)|0\rangle_B + (|\beta_{00}\rangle - |\beta_{10}\rangle)|1\rangle_B] \\ &= \frac{1}{2}[|\beta_{00}\rangle(a|0\rangle_B + b|1\rangle_B) + |\beta_{01}\rangle(a|1\rangle_B + b|0\rangle_B) \\ &\quad + |\beta_{10}\rangle(a|0\rangle_B - b|1\rangle_B) + |\beta_{11}\rangle(a|1\rangle_B - b|0\rangle_B)] \end{aligned} \tag{2.71}$$

So depending on Alice's measurement outcome, Bob's particle will be in one of the four pure states

$$\begin{aligned} &a|0\rangle_B + b|1\rangle_B \\ &a|1\rangle_B + b|0\rangle_B \\ &a|0\rangle_B - b|1\rangle_B \\ &a|1\rangle_B - b|0\rangle_B. \end{aligned} \tag{2.72}$$

Thus to complete the process, Alice has to communicate the result of the measurement to Bob. After that Bob can recover the state $|\psi\rangle = a|0\rangle_B + b|1\rangle_B$ by applying an appropriate operator to his particle. If the measurement outcome is $|\beta_{00}\rangle$, Bob doesn't need to do anything since his particle will be in the state $|\psi\rangle$. If the outcome is $|\beta_{01}\rangle$, then Bob's particle is in the state $a|1\rangle_B + b|0\rangle_B$ so he has to apply the operator

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.73}$$

to recover $|\psi\rangle = a|0\rangle_B + b|1\rangle_B$.
If the outcome is $|\beta_{10}\rangle$, Bob's particle will be in the state $a|0\rangle_B - b|1\rangle_B$. Consequently, he has to apply the operator

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.74}$$

to recover the state $|\psi\rangle$ of the qubit.
And finally if the outcome is $|\beta_{11}\rangle$, Bob's particle will be in the state $a|1\rangle_B - b|0\rangle_B$. So in order to recover $|\psi\rangle = a|0\rangle_B + b|1\rangle_B$, he will have to apply first the operator $X$ and then the $Z$ operator.

Chapter 3

# Entanglement and open quantum systems

## 3.1 Introduction

An open quantum system may be defined as a system $S$ which is coupled to another system $B$, called the environment. In other words, it is a subsystem of the combined system $(S + B)$. A simple illustration is given in Fig. 3.1. The study of open systems is a very important subject in quantum physics in general and in quantum information theory in particular. This is because real quantum systems often suffer from unwanted and unavoidable coupling with the environment [12]. Quantum systems must then be regarded as open systems since it is not possible to get complete isolation from the surrounding. In order to build useful quantum information processing systems, it is crucial to understand these interactions with the environment and how they affect the entanglement of quantum systems.

To study how quantum systems interact with their environment is also important from the point of view of fundamental physics. This is because these interactions, leading to a set of effects generally referred to as decoherence, justify the framework and intuition of classical physics as an acceptable approximation, even though the underlying processes are quantum mechanical. In other words, decoherence is the mechanism by which the classical picture emerges out of the quantum mechanical background. Consequently, it determines the location of the quantum-classical

boundary.

The study of open quantum systems is a vast topic, and there are many ways to approach it. The aim of this chapter is not to explore open quantum systems in depth, for this is beyond the scope of the present study. Here, I introduce and explain basic concepts on open systems that are crucial to the understanding of the system that I will study.

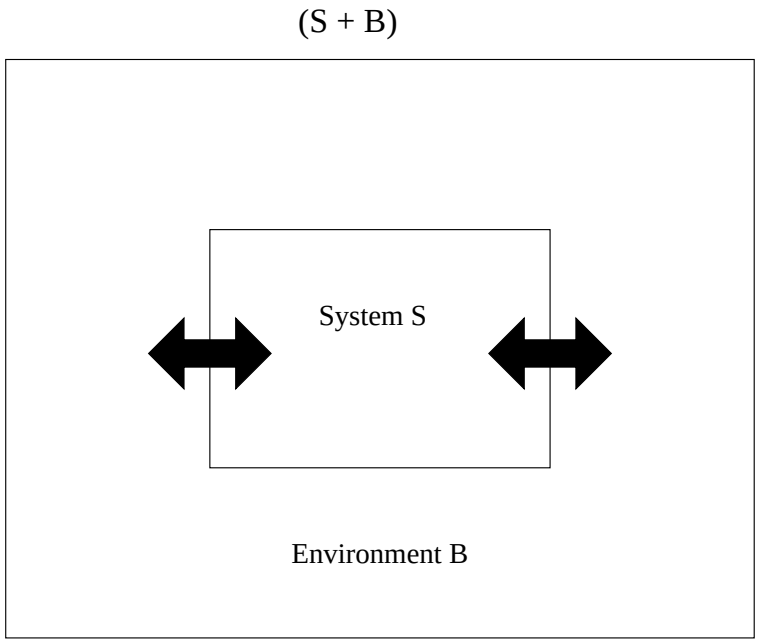


Figure 3.1: Simple Schematic picture of an open quantum system

When dealing with a closed quantum system one represents its dynamics by a unitary evolution operator. In general, this is not possible in the case of open systems. Consider for example the illustration given by Fig 3.1. The state of the subsystem $S$ will change due to the interaction with the environment $B$ and due to its internal dynamics. After some time, it will no longer be possible to represent its resultant state in term of unitary Hamiltonian dynamics. It is therefore necessary to formulate the dynamics of open system by means of appropriate equations of motion. These equations of motion are usually called *master equations*.

In the study of open quantum systems, one tries to develop a simpler description in a reduced state space formed by a restricted set of physically relevant variables. This is because in many important situations, a complete mathematical model of the combined system dynamics is too complex. As an example, one can mention the case where the environment consists of infinitely many degrees of freedom (a reservoir), or the case where the modes of the environment are neither known exactly nor controllable. This simpler description is achieved by means of various analytical methods and approximation techniques.

It was stated earlier that the evolution of open quantum systems is described by *master equations*. These master equations relieve one from the need of having to first determine the dynamics of the total system-environment combination and to then trace out the degrees of freedom of the environment as we usually do with close quantum systems. The next section is about Quantum Markov processes in general and the Born-Markov master equation in particular. After that, I will discuss the Non-Markovian regime.

## 3.2 Master equations formalism

Before discussing Master equations, it is crucial to define few preliminary concepts from probability theory. These concepts are *Random variables*, *Stochastic process*, and *Markov processes*. I shall first briefly explain from the point of view of classical probability the main difference between Markovian and non-Markovian processes. Afterwards, I shall review the basics of quantum master equations.

### 3.2.1 Random variables

A random variable is basically an unknown quantity whose value is determined by the randomness of some experiment. This concept is very important in probability theory because in many situations one is not interested in an experiment itself, but rather in some consequence of its random outcome. Such consequences, when real valued, may be thought of as functions called random variables which map

the sample space $\Omega$ into $\mathbb{R}$. The sample space $\Omega$ is the set of all possible outcomes of an experiment. [12, 14].Thus a *random variable* $X$ is a map

$$X : \Omega \mapsto \mathbb{R} \tag{3.1}$$

which assigns to each elementary event $\omega \in \Omega$ a real number $X(\omega)$.
As an illustrative example, consider an experiment where a fair coin is tossed twice: $\Omega = \{HH, HT, TH, TT\}$. For $\omega \in \Omega$, let $X(\omega)$ be the number of heads, then $X(HH) = 2$, $X(HT) = X(TH) = 1$, $X(TT) = 0$.

### 3.2.2 Stochastic processes

A *stochastic process* is a random variable whose statistical properties change in time [12, 14]. Stochastic processes may be seen as a generalization of the idea of deterministic time evolution. The latter is usually given in terms of differential equations describing the deterministic change in time of some variable while in a stochastic process, such a deterministic evolution is replaced by a probabilistic law for the time development of the variable [12].

A more mathematically rigorous definition of stochastic processes is as follows: a stochastic process is a family of random variables $X(t)$ on a common probability space depending on a parameter $t \in T$. The parameter $t$ may represent the time variable as it is the case in most physical applications. In that case, the parameter space $T$ is an interval of the real time axis. According to this definition, for each fixed $t$, the quantity $X(t)$ is a map from the sample space $\Omega$ into $\mathbb{R}$. That is, $X$ is a map,

$$X : \Omega \times T \to \mathbb{R} \tag{3.2}$$

which associates with each $\omega \in \Omega$ and with each $t \in T$ a real number $X(\omega, t)$. When $\omega$ is kept fixed, the mapping

$$t \mapsto X(\omega, t), t \in T, \tag{3.3}$$

is called a realization, trajectory, or sample path of the stochastic process.

### 3.2.3 Markov processes

Markov processes are very important in natural sciences in general and in physics in particular. Many important processes arising in equilibrium statistical mechanics may be classified as Markovian if an appropriate set of variables are chosen.
A Markov Process is a Stochastic Process with a short memory. That is, a process which rapidly forgets its past history. Consequently, Markov Processes are relatively easy to deal with. One can formulate the condition of the fast decrease of memory effects in terms of conditional probability as follows:

$$P(X(t) \in B|X(t_m) = x_m, ..., X(t_1) = x_1) = P(X(t) \in B|X(t_m) = x_m). \quad (3.4)$$

This equation is known as the Markov condition. It states that the probability of the event $X \in B$ conditioned on $m$ previous events $X(t_m) = x_m, ..., X(t_1) = x_1$ depends only on the latest event $X(t_m) = x_m$. Random processes that do not comply with the above condition are called non-Markovian.

### 3.2.4 Dynamical maps

Let us consider the illustration of the open system given in Fig. 3.1. It is clear that the total system $(S + B)$ is a closed system. Its evolution is therefore unitary. The reduced density matrix of the subsystem $S$ is obtained by using the ordinary formalism of closed systems, that is, by taking the partial trace over the degrees of freedom of the environment

$$\rho_S(t) = \mathrm{Tr}_B\{U(t, t_0)\rho(t_0)U^\dagger(t, t_0)\}; \quad (3.5)$$

where $U(t, t_0)$ correspond to the time evolution of the total system. The equation of motion for the marginal density matrix is obtained by taking the partial trace over the environment on both sides of the Liouville-von Neumann equation for the total system

$$\frac{d}{dt}\rho_S(t) = \frac{1}{i\hbar}\mathrm{Tr}_B[H(t), \rho(t)]. \quad (3.6)$$

In general, the dynamics of the reduced system defined by the exact equations 3.5 and 3.6 will be quite involved, if not intractable.

In the master-equation formalism, the marginal density matrix $\rho_S(t)$ is directly calculated from an expression of the form

$$\hat{\rho}_S(t) = \hat{V}(t)\hat{\rho}_S(0), \tag{3.7}$$

where the operator $\hat{V}(t)$ is the *dynamical map* that generates the evolution of $\hat{\rho}_S(t)$. $\hat{V}(t)$ is sometimes referred to as a "superoperator" because it represents an operator that in turn acts on another operator. Equation 3.7 is called a master equation for $\hat{\rho}_S(t)$, and it represents the most general form that such a master equation may take.

To discuss the concept of dynamical map further, suppose that at the initial time $t = 0$, the state of the total system $S + B$ is an uncorelated product state $\rho(0) = \rho_S(0)\otimes\rho_B$ where $\rho_S(0)$ is the initial state of the reduced system $S$ and $\rho_B$ represents some reference state of the environment. The transformation describing the time evolution of the subsystem $S$ may be written in the form

$$\hat{\rho}_S(0) \mapsto \hat{\rho}_S(t) = \hat{V}(t)\hat{\rho}_S(0) \equiv \mathrm{Tr}_B\{\hat{U}(t,0)[\hat{\rho}_S(0)\otimes\hat{\rho}_B]\hat{U}^\dagger(t,0)\}. \tag{3.8}$$

If the reference state $\hat{\rho}_B$ and the final time are fixed, this relation (Eq. 3.8) defines a map from the space $S(\mathcal{H}_S)$ ($\mathcal{H}_S$ is the Hilbert space of the subsystem $S$) of the density matrices of the reduced system into itself,

$$\hat{V}(t) : S(\mathcal{H}_S) \to S(\mathcal{H}_S). \tag{3.9}$$

This map, describing the time evolution of the open system, is a dynamical map. To discuss the dynamical map $\hat{V}(t)$ above, the assumption that the time $t \geq 0$ is fixed was made. If one allows the time to vary, one gets a one parameter family $\hat{V}(t)|t \geq 0$ of dynamical maps, where $\hat{V}(0)$ is the identity map. Such a family describes the whole future time evolution of the open system. This time evolution is very involved in general. However, if the characteristic time scales over which the environment correlation function decay are much smaller than the characteristic time scale of the system's evolution, memory effects in the reduced

system dynamics can be neglected. One thus expects Markovian-type behavior as in the classical theory.

### 3.2.5 Operator-sum representation

The *operator-sum representation* or *Kraus representation* [3, 15] is a useful tool in the study of one part of a composite quantum system. Consider a composite system made of two subsystems $A$ and $B$, and suppose that its initial density matrix $\rho_{AB}$ is a tensor product state of the form

$$\rho_{AB}(0) = \rho_A(0) \otimes |b_0\rangle\langle b_0|, \tag{3.10}$$

that is, subsystem $A$ has the the density matrix $\rho_A(0)$ initially and subsystem $B$ is in the pure state $|b_0\rangle$. The time evolution of the bipartite system is governed by the unitary time evolution operator, that is,

$$\rho_{AB}(t) = \hat{U}_{AB}[\rho_A(0) \otimes |b_0\rangle\langle b_0|]\hat{U}_{AB}^{\dagger}. \tag{3.11}$$

The density matrix of the subsystem $A$ is obtained by taking the partial trace of the total density matrix,

$$\rho_A(t) = \mathrm{Tr}_B(\hat{U}_{AB}[\rho_A(0) \otimes |b_0\rangle\langle b_0|]\hat{U}_{AB}^{\dagger}). \tag{3.12}$$

Now let $\{|b_k\rangle\}$ be an orthonormal basis for the state space of the subsystem $B$, then the previous equation may be written as

$$\begin{aligned} \rho_A(t) &= \sum_k \langle b_k|\hat{U}_{AB}[\rho_A(0) \otimes |b_0\rangle\langle b_0|]\hat{U}_{AB}^{\dagger})|b_k\rangle. \\ &= \sum_k \langle b_k|\hat{U}_{AB}|b_0\rangle\rho_A(0)\langle b_0|\hat{U}_{AB}^{\dagger}|b_k\rangle \\ &= \sum_k E_k\rho_A(0)E_k^{\dagger}. \end{aligned} \tag{3.13}$$

where $E_k \equiv \langle b_k|\hat{U}_{AB}|b_0\rangle$. The operators $E_k$ are known as the *operation elements* or the *Kraus operators*.

It follows from the unitarity of $\hat{U}_{AB}$ that the $E_k's$ satisfy the property

$$\begin{aligned}\sum_k E_k^\dagger E_k &= \sum_k \langle b_0|\hat{U}_{AB}^\dagger|b_k\rangle\langle b_k|\hat{U}_{AB}|b_0\rangle \\ &= \langle b_0|\hat{U}_{AB}^\dagger \hat{U}_{AB}|b_0\rangle \\ &= I_A. \end{aligned} \tag{3.14}$$

This property is generally known as the *completeness relation*.
Here, it is assumed that the subsystem $B$ is initially in the pure state $|b_0\rangle$. The above argument will still hold even if this assumption is not correct. This is because if $B$ starts in a mixed state, one can always introduce a second system to purify it through a process known as *purification*.

### 3.2.6 The Born-Markov Master equation

Master equations are very useful in the description of the dynamics of open quantum systems, but their power is only unlocked once certain assumptions about the system-environment states and dynamics are imposed. In the case of the Born-Markov master equation, two main approximations are involved [12, 17, 18]:

1. *The Born approximation*: here it is assumed that the environment is large and practically unaffected by the interactions with the system. As a consequence, changes of the density operator of the environment are negligible and the system-environment state remains in an approximate product state at all times. That is,
$$\hat{\rho}(t) \approx \hat{\rho}_S(t) \otimes \hat{\rho}_B \tag{3.15}$$
with $\hat{\rho}_B$ approximately constant at all times.

2. *The Markov approximation*: here, memory effects of the environment are neglected. In other words, it is assumed that any self-correlations within the environment created by the coupling to the system decay rapidly compared to the characteristic time over which the state of the system varies noticeably.

### Derivation of the Born-Markov Master equation

In order to derive the Born-Markov Master equation, consider a system $S$ interacting with an environment $B$ (see Fig. 3.1). The Hamiltonian describing the combination $S + B$ is of the form

$$H = H_S \otimes I_B + I_S \otimes H_B + H_{SB} = H_0 + H_{SB} \tag{3.16}$$

where $H_S$ describe the system, $H_B$ the environment and $H_{SB}$ the interaction. Let $\rho$ be the density matrix describing the composite system $S + B$. Then the density matrix describing the system is $\rho_S = \mathrm{Tr}_B \rho$. The evolution of $\rho$ is governed by the Liouville-von Neumann equation

$$i\hbar\dot{\rho} = [H, \rho] \tag{3.17}$$

If one assumes weak interaction, then it is possible to separate the fast motions due to $H_0 = H_S + H_B$ and the slow motions due to the interaction $H_{SB}$. Now let make use of the interaction picture by defining [17]

$$\begin{aligned} i\hbar\dot{U}_S &= H_S U_S, \qquad U_S(0) = I_S, \\ i\hbar\dot{U}_B &= H_B U_B, \qquad U_B(0) = I_B, \\ U &= U_S \otimes U_B \end{aligned} \tag{3.18}$$

$$\tilde{\rho} = U^\dagger \rho U, \qquad \tilde{H}_{SB} = U^\dagger H_{SB} U \tag{3.19}$$

After substituting Eq. 3.18 and Eq. 3.19 into Eq. 3.17, we get

$$i\hbar\dot{\tilde{\rho}} = [\tilde{H}_{SB}, \tilde{\rho}] \tag{3.20}$$

which is equivalent to

$$\tilde{\rho}(t) = \tilde{\rho}(0) + \frac{1}{i\hbar}\int_0^t d\tau [\tilde{H}_{SB}(\tau), \tilde{\rho}(\tau)] \tag{3.21}$$

where $\tilde{\rho}(0) = \rho(0)$.

By inserting Eq. 3.21 into the right hand side of Eq. 3.20 we get

$$\dot{\tilde{\rho}}(t) = \frac{1}{i\hbar}[\tilde{H}_{SB}(t), \rho(0)] - \frac{1}{\hbar^2}\int_0^t d\tau[\tilde{H}_{SB}(t), [\tilde{H}_{SB}(\tau), \tilde{\rho}(\tau)]]. \tag{3.22}$$

The density matrix of the system $S$ in the interaction picture is obtained by taking the partial trace of $\tilde{\rho}$ over the environment degrees of freedom, that is,

$$\begin{aligned} \mathrm{Tr}_B(\tilde{\rho}) &= \mathrm{Tr}_B(U^\dagger \rho U) \\ &= U_S^\dagger \mathrm{Tr}_B(U_B^\dagger \rho U_B) U_S \\ &= U_S^\dagger \mathrm{Tr}_B(\rho U_B^\dagger U_B) U_S \\ &= U_S^\dagger \mathrm{Tr}_B \rho U_S \\ &= U_S^\dagger \rho_S U_S \\ &= \tilde{\rho}_S. \end{aligned} \tag{3.23}$$

Now if one takes the trace of Eq. 3.22 over the environment degrees of freedom, one gets

$$\dot{\tilde{\rho}}_S(t) = \frac{1}{i\hbar}\mathrm{Tr}_B\{[\tilde{H}_{SB}(t), \rho(0)]\} - \frac{1}{\hbar^2}\int_0^t d\tau \mathrm{Tr}_B\{[\tilde{H}_{SB}(t), [\tilde{H}_{SB}(\tau), \tilde{\rho}(\tau)]]\}. \tag{3.24}$$

To continue from this point, one needs to make few assumptions.

(i) The first assumption is that the system and the environment are not entangled initially, that is, at $t = 0$,

$$\rho(0) = \rho_S(0) \otimes \rho_B(0) \tag{3.25}$$

where $\rho_s$ and $\rho_B$ are the system and environment density matrices respectively.

(ii) The second assumption is that

$$\mathrm{Tr}_B\{[\tilde{H}_{SB}, \rho(0)]\} = 0. \tag{3.26}$$

Even if this is not the case, it is always possible to redefine $H_S$ and $H_{SR}$ to fulfill that assumption since $\mathrm{Tr}_B\{[\tilde{H}_{SB}, \rho(0)]\}$ is an operator acting on the system alone.

(iii) The third assumption is the Born approximation, in other words, one assumes that the coupling is so weak and the environment so large that its state is unaffected by the interaction. Consequently, the density operator of the system-environment combination remains at all times in an approximate product form. That is,

$$\tilde{\rho}(\tau) \approx \tilde{\rho}_S(\tau) \otimes \tilde{\rho}_B \tag{3.27}$$

so Eq. 3.24 becomes

$$\dot{\tilde{\rho}}_S(t) = -\frac{1}{\hbar^2}\int_0^t d\tau \mathrm{Tr}_B\{[\tilde{H}_{SB}(t), [\tilde{H}_{SB}(\tau), \tilde{\rho}_S(\tau)\tilde{\rho}_B]]\}. \tag{3.28}$$

(iv) The fourth assumption one makes is the Markov approximation. In Eq. 3.28, computing the change of $\tilde{\rho}_S$ at time $t$ requires knowledge of $\tilde{\rho}_S$ at all previous times $\tau < t$. The point of using the Markov approximation here is thus to transform Eq. 3.28 into a time-local master equation. By replacing $\tilde{\rho}_S(\tau)$ with $\tilde{\rho}_S(t)$ in Eq. 3.28, one gets

$$\dot{\tilde{\rho}}_S(t) = -\frac{1}{\hbar^2}\int_0^t d\tau \mathrm{Tr}_B\{[\tilde{H}_{SB}(t), [\tilde{H}_{SB}(\tau), \tilde{\rho}_S(t)\tilde{\rho}_B]]\}. \tag{3.29}$$

Now let $\sigma_i$ be a basis of Hermitian operator acting on the system, then one can expand the $H_{SB}$ as

$$H_{SB} = \sum_{i=0}^{M-1} \sigma_i B_i \tag{3.30}$$

where the operator $B_i$ acts on the environment and $M = N^2$, and $N$ is the dimension of the system's Hilbert space. We also have the transformation equations

$$\tilde{H}_{SB}(t) = U^\dagger(t) H_{SB} U(t) = \sum_i \tilde{\sigma}_i(t)\tilde{B}_i(t), \tag{3.31}$$

where

$$\tilde{\sigma}_i(t) \equiv U_S^\dagger(t)\sigma_i U_S(t), \qquad \tilde{B}_i(t) \equiv U_B^\dagger(t) B_i U_B(t). \tag{3.32}$$

Inserting Eq. 3.31 and 3.32 into Eq. 3.29 yields

$$\dot{\tilde{\rho}}_S(t) = -\frac{1}{\hbar^2}\sum_{i,j}\int_0^t d\tau \mathrm{Tr}_B\{[\tilde{\sigma}_i(t)\tilde{B}_i(t),[\tilde{\sigma}_j(\tau)\tilde{B}_j(\tau),\tilde{\rho}_S(t)\tilde{\rho}_B]]\}. \tag{3.33}$$

Expanding the commutator on the right-hand side of this equation gives

$$\begin{aligned}\dot{\tilde{\rho}}_S(t) &= -\frac{1}{\hbar^2}\sum_{i,j}\int_0^t d\tau \mathrm{Tr}_B\{\tilde{\sigma}_i(t)\tilde{B}_i(t)\tilde{\sigma}_j(\tau)\tilde{B}_j(\tau)\tilde{\rho}_S(t)\tilde{\rho}_B \\ &\quad -\tilde{\sigma}_i(t)\tilde{B}_i(t)\tilde{\rho}_S(t)\tilde{\rho}_B\tilde{\sigma}_j(\tau)\tilde{B}_j(\tau) - \tilde{\sigma}_j(\tau)\tilde{B}_j(\tau)\tilde{\rho}_S(t)\tilde{\rho}_B\tilde{\sigma}_i(t)\tilde{B}_i(t) \\ &\quad +\tilde{\rho}_S(t)\tilde{\rho}_B\tilde{\sigma}_j(\tau)\tilde{B}_j(\tau)\tilde{\sigma}_i(t)\tilde{B}_i(t)\}. \end{aligned}\tag{3.34}$$

Using the cyclic property of the trace, one gets

$$\begin{aligned}\dot{\tilde{\rho}}_S(t) &= -\frac{1}{\hbar^2}\sum_{i,j}\int_0^t d\tau \mathrm{Tr}_B\{\tilde{\sigma}_i(t)\tilde{\sigma}_j(\tau)\tilde{\rho}_S(t)\tilde{B}_i(t)\tilde{B}_j(\tau)\tilde{\rho}_B \\ &\quad -\tilde{\sigma}_i(t)\tilde{\rho}_S(t)\tilde{\sigma}_j(\tau)\tilde{B}_j(\tau)\tilde{B}_i(t)\tilde{\rho}_B - \tilde{\sigma}_j(\tau)\tilde{\rho}_S(t)\tilde{\sigma}_i(t)\tilde{B}_i(t)\tilde{B}_j(\tau)\tilde{\rho}_B \\ &\quad +\tilde{\rho}_S(t)\tilde{\sigma}_j(\tau)\tilde{\sigma}_i(t)\tilde{B}_j(\tau)\tilde{B}_i(t)\tilde{\rho}_B\}. \end{aligned}\tag{3.35}$$

Let the environment correlation functions be defined as

$$\tilde{\Gamma}_{ij}(t,\tau) \equiv \mathrm{Tr}_B\tilde{B}_i(t)\tilde{B}_j(\tau)\tilde{\rho}_B. \tag{3.36}$$

After inserting this relation in the previous equation, one gets

$$\begin{aligned}\dot{\tilde{\rho}}_S(t) &= -\frac{1}{\hbar^2}\sum_{i,j}\int_0^t d\tau \mathrm{Tr}_B\{\tilde{\sigma}_i(t)\tilde{\sigma}_j(\tau)\tilde{\rho}_S(t)\tilde{\Gamma}_{ij}(t,\tau) \\ &\quad -\tilde{\sigma}_i(t)\tilde{\rho}_S(t)\tilde{\sigma}_j(\tau)\tilde{\Gamma}_{ij}(t,\tau) - \tilde{\sigma}_j(\tau)\tilde{\rho}_S(t)\tilde{\sigma}_i(t)\tilde{\Gamma}_{ij}(t,\tau) \\ &\quad +\tilde{\rho}_S(t)\tilde{\sigma}_j(\tau)\tilde{\sigma}_i(t)\tilde{\Gamma}_{ij}(t,\tau)\}. \end{aligned}\tag{3.37}$$

If one now assumes that the environment correlation functions are memoryless, that is,

$$\tilde{\Gamma}_{ij}(t,\tau) = \tilde{\gamma}_{ij}\delta(t-\tau) \tag{3.38}$$

substituting that in the previous equation, one gets

$$\dot{\tilde{\rho}}_S(t) = -\frac{1}{\hbar^2}\sum_{i,j}\frac{\tilde{\gamma}_{ij}}{2}\int_0^t d\tau \mathrm{Tr}_B\{\tilde{\sigma}_i(t)\tilde{\sigma}_j(t)\tilde{\rho}_S(t) - \tilde{\sigma}_i(t)\tilde{\rho}_S(t)\tilde{\sigma}_j(t) - \tilde{\sigma}_j(t)\tilde{\rho}_S(t)\tilde{\sigma}_i(t) + \tilde{\rho}_S(t)\tilde{\sigma}_j(t)\tilde{\sigma}_i(t)\}. \quad (3.39)$$

### Master equation in the Lindblad form

The Lindblad equation or master equation in the Lindblad form is the most general type of Markovian master equation describing non-unitary evolution of the density matrix $\rho$ that is trace preserving and completely positive for any initial condition. It was first shown by Lindblad [16] and Gorini et al. [19] that the most general master equation ensuring the positivity of the density matrix is of the form

$$\dot{\rho} = -\frac{i}{\hbar}[H,\rho] + \sum_j [2L_j\rho L_j^\dagger - \{L_j^\dagger L_j,\rho\}], \quad (3.40)$$

where $\{x,y\} = xy + yx$ is an anticommutator, $H$ is the system Hamiltonian representing the coherent part of the system and $L_j$ are the *Lindblad operators*, representing the coupling of the system with the environment [3].
To derive the master equation in the Lindblad form, one needs to use the operator-sum representation or the Kraus representation discussed in section 3.2.5.

As stated earlier, the evolution of the density matrix of a system evolving unitarily is governed by the Liouville-von-Neumann equation (Eq. 3.17), that is,

$$\dot{\rho} = -\frac{i}{\hbar}[H,\rho].$$

One can solve this equation and obtain

$$\rho(t) = \mathrm{e}^{-iHt}\rho(0)\mathrm{e}^{iHt} \quad (3.41)$$

if the Hamiltonian $H$ is time independent. The aim here is to generalize this equation to the case of Markovian but non-unitary evolution. That is, one wishes

to get an evolution equation of the form

$$\dot{\rho}(t) = \mathcal{L}[\rho]. \tag{3.42}$$

The linear operator $\mathcal{L}$, also sometimes referred to as the generator, is called the *Lindbladian*. It generates a finite superoperator in the same sense that a Hamiltonian $H$ generates unitary time evolution. The formal solution of equation 3.42 is

$$\rho(t) = \mathrm{e}^{\mathcal{L}t}[\rho(0)] \tag{3.43}$$

if $\mathcal{L}$ is time independent.

If we consider a system $S$ interacting with the environment $B$ as illustrated in figure 3.1, the density matrix of the system $S$ is given by

$$\dot{\rho}_S = \mathrm{Tr}_B(\dot{\rho}_{SB}) = \mathrm{Tr}_B(-\frac{i}{\hbar}[H, \rho_{SB}]) \tag{3.44}$$

But one cannot expect this formula to be expressed in terms of $\rho_S$ alone. For the following derivation, it is assumed that Markovian approximation applies. A general superoperator $\mathcal{S}$ (an operator acting on another operator) has a Kraus representation

$$\rho(t; t+dt) = \mathcal{S}(t; t+dt)(\rho(t)) = \sum_k E_k(t)\rho(0)E_k^\dagger(t) \tag{3.45}$$

and $\mathcal{S}(t; t+dt)$ is a superoperator mapping $\rho(t)$ into $\rho(t+dt)$. If one assumes that

$$\rho(t+dt) = \rho(t) + O(dt), \tag{3.46}$$

then one of the Kraus operators will be $E_0 = I + O(dt)$ and the other will be of the order $\sqrt{dt}$. one can thus write

$$\begin{aligned} E_0 &= I + \frac{1}{\hbar}(-iH + K)dt, \\ E_k &= L_k\sqrt{dt}, k = 1, 2, 3 \ldots \end{aligned} \tag{3.47}$$

where $H$ and $K$ are hermitian operators, and the $L_k$ are the Lindblad operators. The operator $K$ can be obtained with the normalization condition,

$$\begin{aligned} I &= \sum_k E^\dagger E_k \\ &= [I + \frac{1}{\hbar}(iH + K)dt][I + \frac{1}{\hbar}(-iH + K)dt] + \sum_{k>0} L_k^\dagger L_k dt \end{aligned} \tag{3.48}$$

After a little algebra, the previous equation becomes

$$\begin{aligned} \frac{2}{\hbar}Kdt &+ \sum_{k>0} L_k^\dagger L_k dt + O((dt)^2) = 0 \\ \therefore K &= -\frac{\hbar}{2}\sum_{k>0} L_k^\dagger L_k. \end{aligned} \tag{3.49}$$

Substituting Eq. 3.49 into 3.45 and making use of the assumption

$$\rho(t + dt) = \rho(t) + \dot{\rho}(t)dt, \tag{3.50}$$

(which is possible under the Markovian approximation) one gets

$$\begin{aligned} \rho(t) + \dot{\rho(t)}dt &= \rho(t) - \frac{i}{\hbar}[H, \rho(t)]dt \\ &= \sum_{k>0}(L_k\rho(t)L_k^\dagger - \frac{1}{2}L_k^\dagger L_k\rho(t) - \frac{1}{2}\rho(t)L_k^\dagger L_k)dt. \end{aligned} \tag{3.51}$$

Now equating the terms of order $dt$, we obtain Lindblad's equation

$$\dot{\rho} \equiv \mathcal{L}[\rho] = -\frac{i}{\hbar}[H, \rho] + \sum_{k>0}(L_k\rho L_k^\dagger - \frac{1}{2}L_k^\dagger L_k\rho - \frac{1}{2}\rho L_k^\dagger L_k) \tag{3.52}$$

This master equation is usually referred to as the GKSL (Gorini, Kossakowski, Sudarshan and Lindblad) master equation. The first term in $\mathcal{L}[\rho]$ is the usual von-Neumann terms that generates unitary evolutions, the other terms describe possible transitions that the system may undergo due to interactions with the environment.

## 3.3 Non-Markovian processes

In order to derive the Born-Markov master equation above, certain assumptions were made. For instance it was assumed that system-environment coupling is weak

and that the memory effects of the environment are negligible. In many situations, however, these assumptions are not fulfilled. The system in that case is said to be non-Markovian.

It is important to understand non-Markovian approximation because studies showed the limits of the Markovian description of quantum computation and quantum error correction [20, 21]. Moreover, in order to describe decoherence in many solid state systems, one often needs to use non-Markovian approaches [22, 23]

Equations of motion for the density matrix of systems evolving under non-Markovian conditions are usually obtained by using *projection operator techniques* [12]. The basic idea behind projection operator techniques is to regard the operation of tracing over the environment degrees of freedom as a formal projector $\mathcal{P}$ in the state space of the total system. The super-operator has the property of a projector operator, that is $\mathcal{P}^2 = \mathcal{P}$. The density matrix $\mathcal{P}\rho$ is referred to as the *relevant* part of the density matrix $\rho$ of the total system. In the same way, one defines another projector $\mathcal{Q}$ onto the *irrelevant* part $\mathcal{Q}\rho$ where $\mathcal{P} + \mathcal{Q}$ is the identity map. The aim here is to derive an equation of motion for the relevant part $\mathcal{P}\rho$. The two projector operator techniques mostly used are the Nakajima-Zwanzig (NZ) and the time convolutionless (TCL) techniques. Both these techniques lead to an exact equation of motion for the relevant part $\mathcal{P}\rho$.

The Nakajima-Zwanzig technique leads to an integrodifferencial equation that involves a retarded time integration over the history of the reduced system while the time convolutionless technique yields a first order differential equation local in time.

The main system that will be studied in the next chapter is a two-level system interacting with an environment. It might be helpful to complete this chapter by introducing an example of a two-dimensional system interacting with the environment.

**Spontaneous decay of a two-level system [12, 24]**

In this example, a two-level system $S$ decaying spontaneously into the field vacuum is considered. This is an exactly solvable model.

The Hamiltonian of the total system is

$$H = H_S + H_B + H_I = H_0 + H_I, \tag{3.53}$$

where

$$H_0 = \omega_0 \sigma_+ \sigma_- + \sum_k \omega_k b_k^\dagger b_k, \tag{3.54}$$

and

$$H_I = \sigma_+ \otimes B + \sigma_- \otimes B^\dagger \tag{3.55}$$

with $B = \sum_k g_k b_k$. $\omega_0$ is the transition frequency of the two-level system $S$ and $\sigma_\pm$ are the raising and lowering operators. The index $k$ labels the different field modes of the reservoir with frequencies $\omega_k$, $b_k^\dagger$ and $b_k$ are the modes' creation and annihilation operators, and $g_k$ are the coupling constants.
Now let introduce the states [24]

$$\psi_0 = |0\rangle_S \otimes |0\rangle_B, \tag{3.56}$$

$$\psi_1 = |1\rangle_S \otimes |0\rangle_B, \tag{3.57}$$

$$\psi_k = |0\rangle_S \otimes |k\rangle_B, \tag{3.58}$$

where $|0\rangle_S = \sigma_- |1\rangle_S$ and $|1\rangle_S = \sigma_+ |0\rangle_S$ are the ground and exited states of the system respectively, $|0\rangle_B$ denotes the vacuum state of the reservoir, and $|k\rangle_B = b_k^\dagger |0\rangle_B$. In the interaction picture, the state $\phi(t)$ of the total system obeys the Schrödinger equation

$$\frac{d}{dt}\phi(t) = -iH_I(t)\phi(t), \tag{3.59}$$

where

$$H_I(t) = \sigma_+(t)B(t) + \sigma_-(t)B^\dagger(t) \tag{3.60}$$

is the interaction picture Hamiltonian with

$$\sigma_{\pm}(t) = \sigma_{\pm} \mathrm{e}^{\pm i\omega_0 t}, \tag{3.61}$$

and

$$B(t) = \sum_k g_k b_k \mathrm{e}^{-i\omega_k t}. \tag{3.62}$$

The 'particle number' operator is

$$N = \sigma_+\sigma_- + \sum_k b_k^\dagger b_k, \tag{3.63}$$

and one can easily show that it commutes with the total Hamiltonian given by Eq. 3.53. That is,

$$[H, N] = 0. \tag{3.64}$$

This means that $N$ is a conserved quantity. As a consequence, any initial state of the form

$$\phi(0) = c_0\psi_0 + ci(0)\psi_1 + \sum_k c_k(0)\psi_k \tag{3.65}$$

evolves after a time $t$ into a state

$$\phi(t) = c_0\psi_0 + c_1(t)\psi_1 + \sum_k c_k(t)\psi_k. \tag{3.66}$$

The amplitude $c_0$ is constant since $H_I(t)\psi_0 = 0$, while the amplitudes $c_1(t)$ and $c_k(t)$ are time dependent.

After substituting Eq. 3.66 into the Schrödinger equation (Eq. 3.59), one obtains the following differential equations for the time evolution of the coefficients:

$$\dot{c}_1(t) = -i\sum_k g_k \mathrm{e}^{i(\omega_0-\omega_k)t} c_k(t), \tag{3.67}$$

$$\dot{c}_k(t) = -ig_k^* \mathrm{e}^{-i(\omega_0-\omega_k)t} c_1(t). \tag{3.68}$$

If one assumes that there are no photons in the initial state, one gets $c_k(0) = 0$. A close equation for $c_1(t)$ can then be obtained by solving the second equation and inserting the solution into the first.

$$\dot{c}_1(t) = -\int_0^t dt_1 f(t-t_1)c_1(t_1), \tag{3.69}$$

where the kernel $f(t-t_1)$ is given by the correlation function

$$f(t-t_1) = \mathrm{Tr}_B B(t) B^\dagger(t_1) \rho_B \mathrm{e}^{i\omega_0(t-t_1)}, \tag{3.70}$$

where $\rho_B = (|0\rangle\langle 0|)_B$ is the vacuum state of the reservoir. This kernel may be expressed in terms of the spectral density $J(\omega)$ of the reservoir,

$$f(t-t_1) = \int d\omega J(\omega) \mathrm{e}^{i(\omega_o-\omega)(t-t_1)}. \tag{3.71}$$

Equation 3.69 can now be solved using Laplace transformation if one knows the form of the spectral density. The density matrix of the two-level system can thus be expressed as

$$\rho_S(t) = \mathrm{Tr}_B |\phi(t)\rangle\langle\psi(t)| = \begin{pmatrix} |c_1(t)|^2 & c_0^* c_1(t) \\ c_0 c_1^*(t) & 1-|c_1(t)|^2 \end{pmatrix}. \tag{3.72}$$

# Chapter 4

# Time dependent entanglement features of two-qubit and multi-qubit systems interacting with an environment.

## 4.1 System studied

First I will consider a system consisting of two qubits each interacting with a reservoir, then I will attempt to generalize the results to N qubits. As stated in [44], the dynamics of the two-qubit density matrix elements can be obtained from that of the single qubit, provided that the different parts are independent. Consider a system composed by $N$ independent parts $\tilde{S} = \tilde{1}, \tilde{2}, ..., \tilde{N}$. Each part consists of a qubit $(S = 1, 2, ..., N)$ interacting with a reservoir $R_s = R_1, R_2, ..., R_N$ respectively. Each qubit $S$ and the corresponding reservoir $R_S$ are initially considered statistically independent or uncorrelated. The time dependent reduced density matrix of the single qubit $S$ is given by

$$\hat{\rho}^S(t) = \mathrm{Tr}_{R_S}\{\hat{U}^{\tilde{S}}(t)\hat{\rho}^S(0) \otimes \hat{\rho}^{R_S}(0)\hat{U}^{\tilde{S}\dagger}\} \tag{4.1}$$

where the trace is over the reservoir $R_S$ degrees of freedom and $\hat{U}^{\tilde{S}}(t)$ is the time evolution operator for the part $\tilde{S}$. The density matrix of the reservoir can be written in its spectral decomposition as

$$\hat{\rho}^{R_S}(0) = \sum_{\alpha_S} \lambda_{\alpha_S} |\varphi_{\alpha_S}\rangle\langle\varphi_{\alpha_S}|. \tag{4.2}$$

Let $|\varphi_{\beta_S}\rangle$ be an orthonormal basis for the state space of the environment. Using the operator sum representation (section 3.2.5), Eq. 4.1 becomes

$$\begin{aligned}\hat{\rho}^S(t) &= \sum_{\alpha_S\beta_S} \lambda_{\alpha_S}\langle\varphi_{\beta_S}|\hat{U}^{\tilde{S}}(t)[\hat{\rho}^S(0)\otimes|\varphi_{\alpha_S}\rangle\langle\varphi_{\alpha_S}|]\hat{U}^{\tilde{S}\dagger}|\varphi_{\beta_S}\rangle \\ &= \sum_{\alpha_S\beta_S} \lambda_{\alpha_S}\langle\varphi_{\beta_S}|\hat{U}^{\tilde{S}}(t)|\varphi_{\alpha_S}\rangle\hat{\rho}^S(0)\langle\varphi_{\alpha_S}|\hat{U}^{\tilde{S}\dagger}|\varphi_{\beta_S}\rangle \\ &= \sum_{\alpha_S\beta_S} \hat{W}^S_{\alpha_S\beta_S}(t)\hat{\rho}^S(0)\hat{W}^{S\dagger}_{\alpha_S\beta_S}(t) \qquad (4.3)\end{aligned}$$

where the operators $\hat{W}^S_{\alpha_S\beta_S}(t)$ are given by

$$\hat{W}^S_{\alpha_S\beta_S}(t) = \sqrt{\lambda_{\alpha_S}}\langle\varphi_{\beta_S}|\hat{U}^{\tilde{S}}(t)|\varphi_{\alpha_S}\rangle. \qquad (4.4)$$

Given the basis $|l_S\rangle, l_S = 0_S, 1_S$ for the qubit $S$. Inserting the identity operators $I_S = \sum|l_S\rangle\langle l_S|$ between the $\hat{W}^{\tilde{S}}$'s operators and the density matrix in Eq. 4.3, it follows that the reduced density matrices of the qubits $S = 1, ..., N$ have the form

$$\langle i_S|\hat{\rho}^S(t)|i'_S\rangle \equiv \rho^S_{i_Si'_S}(t) = \sum_{l_Sl'_S}\sum_{\alpha_S\beta_S}\langle i_S|\hat{W}^S_{\alpha_S\beta_S}(t)|l_S\rangle\langle l_S|\hat{\rho}^S(0)I_S|l'_S\rangle\langle l'_S|\hat{W}^{\dagger S}_{\alpha_S\beta_S}(t)|i'_S\rangle. \qquad (4.5)$$

Now by setting

$$A^{l_Sl'_S}_{i_Si'_S}(t) \equiv \sum_{\alpha_S\beta_S}\langle i_S|\hat{W}^S_{\alpha_S\beta_S}(t)|l_S\rangle\langle l'_S|\hat{W}^{\dagger S}_{\alpha_S\beta_S}(t)|i'_S\rangle, \qquad (4.6)$$

the density matrix elements of each qubit are given by

$$\begin{aligned}\rho^1_{i_1i'_1}(t) &= \sum_{l_1l'_1} A^{l_1l'_1}_{i_1i'_1}(t)\rho^j_{l_1l'_1}(0) \\ \rho^2_{i_2i'_2}(t) &= \sum_{l_2l'_2} A^{l_2l'_2}_{i_2i'_2}(t)\rho^j_{l_2l'_2}(0) \\ &\vdots \\ \rho^N_{i_Ni'_N}(t) &= \sum_{l_Nl'_N} A^{l_Nl'_N}_{i_Ni'_N}(t)\rho^j_{l_Nl'_N}(0). \qquad (4.7)\end{aligned}$$

And the reduced density matrix elements of the N-qubit system are given by

$$\begin{aligned}\langle i_1, i_2, ..., i_N|\hat{\rho}(t)|i'_1, i'_2, ....i'_N\rangle &= \rho_{i_1 i'_1, i_2 i'_2, ..., i_N i'_N}(t) \\ &= \sum_{l_1 l'_1}\sum_{l_2 l'_2}\cdots\sum_{l_N l'_N} A^{l_1 l'_1}_{i_1 i'_1}(t) A^{l_2 l'_2}_{i_2 i'_2}(t) \times \cdots \\ &\quad \times A^{l_N l'_N}_{i_N i'_N}(t)\rho_{l_1 l'_1, l_2 l'_2, ..., l_N l'_N}(0). \qquad (4.8)\end{aligned}$$

where the coefficients $A^{l_S l'_S}_{i_S i'_S}(t)$ are the same as those in Eq. 4.7. It is thus clear from Eq. 4.7 and Eq. 4.8 that the dynamics of N-qubits can can be obtained from that of a single qubit.

For the evolution of the single qubit, let consider the "qubit + reservoir" Hamiltonian already discussed in section 3.3

$$H = \omega_0 \sigma_+ \sigma_- + \sum_k \omega_k b^\dagger_k b_k + (\sigma_+ B + \sigma_- B^\dagger) \qquad (4.9)$$

where $B = \sum_k g_k b_k$, $\omega_0$ denotes the transition frequency of the two-level system (that is, a qubit) and $\sigma_\mp$ stands for the system's raising and lowering operators. The reservoir is represented as a set of field modes, $b^\dagger_k$ and $b_k$ being the concomitant creation and annihilation operators associated with the $k$-mode. These field modes are characterized by frequencies $\omega_k$ and coupling constants $g_k$ with the two-level system. The Hamiltonian (4.9) may describe, for instance, a qubit consisting of the excited and ground electronic states of a two-level atom that interacts with the quantized electromagnetic modes associated with a high-Q cavity [44, 12]. The effective spectral density of the reservoir is assumed to be of the form

$$J(\omega) = \frac{1}{2\pi}\frac{\gamma_0 \lambda^2}{(\omega - \omega_0)^2 + \lambda^2}, \qquad (4.10)$$

where $\gamma_0$ and $\lambda$ are positive parameters with dimensions of inverse time [44].
The dynamics of the single qubit is then described by the density matrix [12]

$$\rho(t) = \begin{pmatrix} \rho_{11}(0)P_t & \rho_{10}(0)\sqrt{P_t} \\ \rho_{01}(0)\sqrt{P_t} & \rho_{00}(0) + \rho_{11}(0)(1 - P_t) \end{pmatrix}. \qquad (4.11)$$

From Eq. 4.8 and Eq. 4.11, the elements of density matrix of the two-qubit system in the basis $B = \{|1\rangle \equiv |11\rangle, |2\rangle \equiv |10\rangle, |3\rangle \equiv |01\rangle, |4\rangle \equiv |00\rangle\}$ are:

$$\begin{aligned}
\rho_{11}^T(t) &= \rho_{11}^T(0)P_t^2, \\
\rho_{22}^T(t) &= \rho_{22}^T(0)P_t + \rho_{11}^T(0)P_t(1-P_t), \\
\rho_{33}^T(t) &= \rho_{33}^T(0)P_t + \rho_{11}^T(0)P_t(1-P_t), \\
\rho_{44}^T(t) &= 1 - [\rho_{11}^T + \rho_{22}^T + \rho_{33}^T], \\
\rho_{12}^T(t) &= \rho_{12}^T(0)P_t^{3/2}, \qquad \rho_{13}^T(0) = \rho_{13}^T(0)P_t^{3/2}, \\
\rho_{14}^T(t) &= \rho_{14}^T(0)P_t, \qquad \rho_{23}^T(0) = \rho_{23}^T(0)P_t, \\
\rho_{24}^T(t) &= \sqrt{P_t}[\rho_{24}^T(0) + \rho_{13}^T(0)P_t(1-P_t)], \\
\rho_{34}^T(t) &= \sqrt{P_t}[\rho_{34}^T(0) + \rho_{12}^T(0)P_t(1-P_t)]
\end{aligned} \tag{4.12}$$

and $\rho_{ij}^T(t) = \rho_{ji}^{T*}(t)$, that is, the matrix $\rho^T(t)$ is Hermitian. The function $P_t$ is given in the non-Markovian regime by [12, 44]

$$P_t = \mathrm{e}^{-\lambda t}\left[\cos\left(\frac{dt}{2}\right) + \frac{\lambda}{d}\sin\left(\frac{dt}{2}\right)\right]^2 \tag{4.13}$$

where $d = \sqrt{2\gamma_0\lambda - \lambda^2}$. By using a similar procedure, one can obtain the matrix elements of the dynamics of a n-qubit system from those corresponding to a single qubit.

In the present study, only the non Markovian regime is considered. To that end, $\lambda = 0.01\gamma_0$ will be used [44].

## 4.2 Time dependent features of two-qubit systems interacting with the environment

In the two-qubit case, it was observed that for some initial states, entanglement disappears for finite time, the two qubits remain disentangled for some time and then the entanglement get revived. For example it occurs with the state $|\Phi\rangle = \alpha|00\rangle + \beta|11\rangle$ for certain vales of $\alpha$. This is illustrated by Fig. 4.1

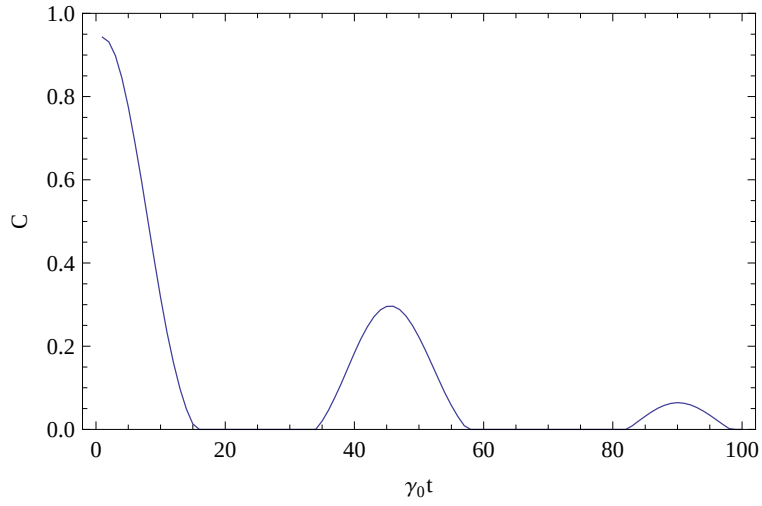


Figure 4.1: Plot of the concurrence, C against the quantity $\gamma_0 t$ for the initial state $|\Psi\rangle = \alpha|00\rangle + \beta|11\rangle$ with $\alpha^2 = 1/3$ and $\lambda = 0.01\gamma_0$.

One clearly sees that the concurrence vanishes and then reappears after a finite time.

### 4.2.1 General behaviour of two-qubit systems interacting with the environment.

In order to investigate the average features characterising the entanglement dynamics associated with a given family of initial states, the average properties of the concomitant evolutions are computed. To determine these averages I generate random initial states (within the alluded family) uniformly distributed according to the Haar measure [30, 59]. I shall consider a family of maximally entangled initial states, a family of partially entangled pure initial states all sharing the same amount of entanglement, and a family of Werner states.

To study the typical, average behavior of the entanglement dynamics of a pair of qubits evolving from an initial maximally entangled state, one can represent the

initial states $|\Psi_e\rangle$ as [59]

$$|\Psi_e\rangle = (I_2 \otimes U_1)\,|\Psi_0\rangle \tag{4.14}$$

where

$$|\Psi_0\rangle \;=\; \frac{1}{\sqrt{2}}\,(|01\rangle + |10\rangle), \tag{4.15}$$

$I_2$ denotes the two-dimensional identity matrix and $U_1$ is a unitary matrix on $SU(2)$. This unitary matrix can be conveniently parameterized as

$$U = \begin{pmatrix} -\sin\vartheta e^{-i\theta_2} & \cos\vartheta e^{i\theta_1} \\ \cos\vartheta e^{-i\theta_1} & \sin\vartheta e^{i\theta_2} \end{pmatrix} \tag{4.16}$$

where $\theta_{1,2} \in [0, 2\pi]$ and $\vartheta \in [0, 2\pi]$. In terms of the three parameters $\theta_1$, $\theta_2$, and $\vartheta$, the maximally entangled state reads,

$$|\Psi\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} \cos\vartheta e^{i\theta_1} \\ \sin\vartheta e^{i\theta_2} \\ -\sin\vartheta e^{-i\theta_2} \\ \cos\vartheta e^{-i\theta_1} \end{bmatrix}, \tag{4.17}$$

where $|\Psi\rangle$ is represented as a column vector in terms of its coefficients with respect to the computational basis. To generate the initial states, random unitary matrices $U$ are generated uniformly according to the Haar measure. The angles $\theta_i$ have to be generated uniformly distributed in $[0, 2\pi]$. $\vartheta$ on the other hand, is obtained by first generating a random variable $\epsilon$, uniformly distributed in $[0, 1]$ and then by computing $\vartheta = \arcsin[\epsilon^{1/2}]$.

More generally, random pure states exhibiting a fixed, prescribed amount of entanglement can be generated using the representation

$$|\Psi_\alpha\rangle \;=\; (I_2 \otimes U_1)\Big(\sqrt{1-\alpha^2}|01\rangle + \alpha|10\rangle\Big), \tag{4.18}$$

which leads to the parameterization

$$|\Psi_\alpha\rangle = \begin{bmatrix} \sqrt{1-\alpha^2}\cos\vartheta e^{i\theta_1} \\ \sqrt{1-\alpha^2}\sin\vartheta e^{i\theta_2} \\ -\alpha\sin\vartheta e^{-i\theta_2} \\ \alpha\cos\vartheta e^{-i\theta_1} \end{bmatrix} \tag{4.19}$$

where one can change the degree of entanglement by using different values of $\alpha$, for instance the value $\alpha = \frac{1}{\sqrt{2}}$ will give the maximally entangled states above. The parameters $\theta_{1,2}$ and $\vartheta$ appearing in (4.19) have to be generated in the same way as in the case of the maximally entangled states.

#### Maximally Entangled Initial States

In this section, the typical, average entanglement dynamics corresponding to maximally entangled initial states will be explored. To this end, I generate random maximally entangled initial states according to the procedure described in the previous section and compute, for different times, the averages of the concurrence $C$ and the linear entropy $S_L$. The average values of the concurrency (left) and that of the linear entropy (right) are depicted in Fig. 4.2 as a function of the dimensionless quantity $\gamma_0 t$. It transpires from this figure that, even though on average, the concurrence does vanish at certain times, it does not stay equal to zero during finite time intervals. In other words, the finite time interval of vanishing entanglement before the entanglement revivals (that is observed for certain initial states) is not a feature characterising the average entanglement dynamics. This observation will be of relevance later when comparing the entanglement dynamics of the two-qubit system with the entanglement behaviours corresponding to three-qubit or four-qubit systems.

It is a well-known trend that the amount of entanglement exhibited by a quantum state of a bipartite system tends to decrease as the degree of mixedness of the state increase (see [30] and references therein). In fact, all two-qubits states with

a linear entropy $S_L = 1 - \mathrm{Tr}(\rho^2)$ larger than 2/3 have zero entanglement (that is, are separable). The abovementioned general trend relating entanglement and mixedness is consistent with the average behaviours of the concurrence and the linear entropy during the first half of the initial period of entanglement decrease observed in Fig. 4.2. During this first part of the two qubits evolution, the concurrence (and, consequently, the amount of entanglement) decreases while the degree of mixedness increases. However, after this first phase of the evolution the pattern changes: the concurrence and the mixedness increase or decreases simultaneously. In particular, during the entanglement revivals, the entanglement and the degree of mixedness of the two-qubit system tend to adopt their maximum values at the same time.

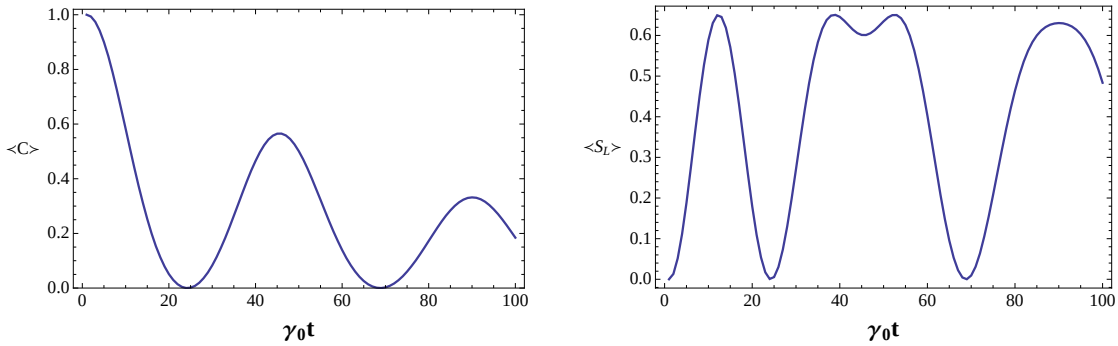


Figure 4.2: Plot of the average value of the concurrence against the quantity $\gamma_0 t$ (left), and the average value of the linear entropy against the quantity $\gamma_0 t$ (right) for the maximally entangled states.

Fig. 4.3 shows the plot of the average value of the concurrence, $\langle C \rangle$, (continuous line) and the dispersion,

$$\Delta C = \sqrt{\langle C^2 \rangle - \langle C \rangle^2}, \tag{4.20}$$

(dotted line) against the quantity $\gamma_0 t$ for the two-qubit maximally entangled state. The plot of the dispersion remains low which indicates that the typical values of the concurrence of the states generated are not very far from the mean value.

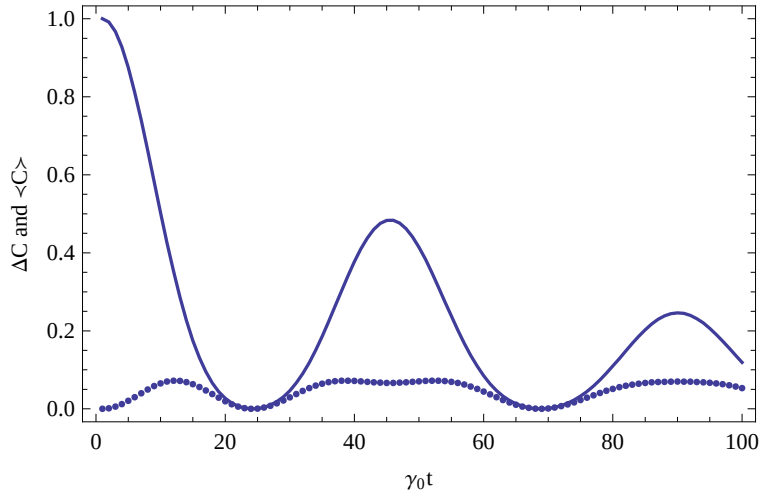


Figure 4.3: Plot of the average value of the concurrence, $\langle C\rangle$,(continuous line) and the dispersion, $\Delta C$,(dotted line) against the quantity $\gamma_0 t$ for the two qubits state $|\Psi\rangle$.

When considering the relationship between the amount of entanglement and the degree of mixedness of two qubits states, the maximally entangled mixed states (MEMS) play an important role. The MEMS [60] states are two qubits states that have the maximum possible value of the concurrence for a given degree of mixture. Their density matrix is given by

$$\rho_{MEMS} = \begin{pmatrix} g(\gamma) & 0 & 0 & \gamma/2 \\ 0 & 1-2g(\gamma) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \gamma/2 & 0 & 0 & g(\gamma) \end{pmatrix} \tag{4.21}$$

where

$$g(\gamma) = \begin{cases} \gamma/2, & \gamma \geq 2/3 \\ 1/3, & x = \gamma < 2/3 \end{cases} \tag{4.22}$$

Some aspects of the entanglement dynamics of the two-qubit system can be illuminated if one considers the trajectory followed by this system in the mixedness-concurrence plane, and compare this trajectory with the curve corresponding to the MEMS states. Fig. 4.4 shows a plot of the average value of the concurrence against the average value of the linear entropy (continuous line) for maximally entangled initial states. The curve in the $(S_L - C)$-plane corresponding to the concurrence $C_{mems}$ associated with maximally entangled mixed states (MEMS) of linear entropy $S_L$ is also depicted (dotted line).

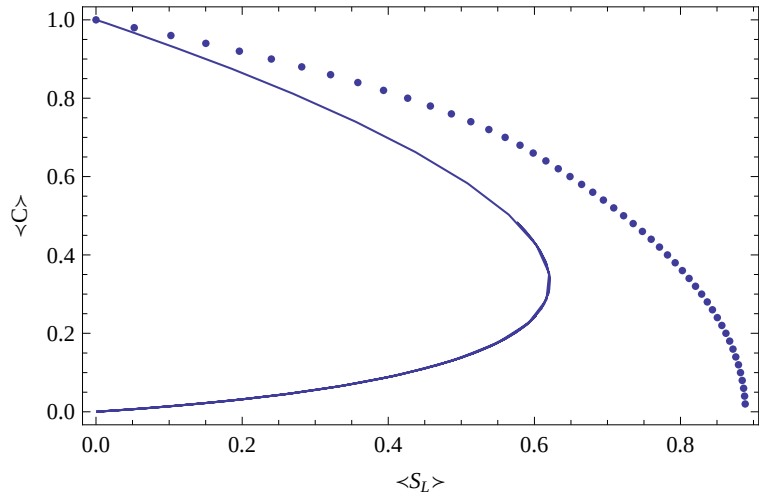


Figure 4.4: Plot of the average value of the concurrence, $\langle C \rangle$, (continuous line) and the concurrence of the maximally entangled mixed states, $C_{mems}$,(dotted line) against the linear entropy.

It can be appreciated in Fig. 4.4 that the average trajectory in the $(C - S_L)$-plane associated with maximally entangled initial states has two branches: an upper branch that stays relatively close to the MEMS curve and a lower branch that departs drastically from the MEMS. During the first phase of entanglement decrease, the average evolution associated with maximally entangled initial states describes the complete trajectory depicted in Fig. 4.4, starting with states of maximum entanglement and zero mixedness, and ending with states of zero entanglement and zero mixedness. During the periods of entanglement revival, the average evo-

lution follows the lower branch, first in the direction corresponding to an increase of entanglement and mixedness, and then in the opposite direction. During the first entanglement revival the two-qubit states reach the point of maximum $\langle S_L \rangle$ in the $\langle S_L \rangle - \langle C \rangle$ curve, and retrace part of the upper branch. In the second and later entanglement revivals, the two-qubit states remain in the lower branch.

### Partially Entangled Pure Initial States

The average behavior corresponding to pure, partially entangled initial states is qualitatively similar to the one corresponding to maximally entangled initial states, but with a $\langle S_L \rangle - \langle C \rangle$ obviously starting with states of less than maximum entanglement and zero mixedness (that is $S_L = 0$). The average behavior, as a function of $\gamma_0 t$, of the concurrence and the linear entropy is depicted in Fig. 4.5 for initial states of the form

$$|\Phi\rangle = \alpha|00\rangle + \beta|11\rangle, \tag{4.23}$$

for $\alpha^2 = 1/3$. It can be seen in Fig. 4.5 that the finite time intervals of zero entan-

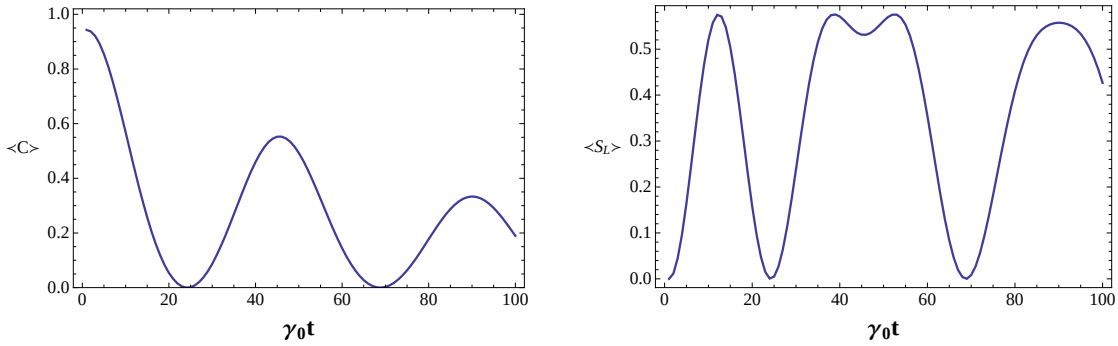


Figure 4.5: Plot of the average value of the concurrence against the quantity $\gamma_0 t$ (left), and the average value of the linear entropy against the quantity $\gamma_0 t$ (right) for the partially entangled states with $\alpha^2 = 1/3$.

glement disappear when we consider the average behavior of the above-mentioned

states. This is in clear contrast with the behaviour of some particular initial states with $\alpha^2 = 1/3$, whose associated trajectories show rather long intervals with zero entanglement like in Fig. 4.1.

An analytical expression linking $S_L$ and $C$ during the evolution associated with individual partially entangled initial states of the form $\alpha|00\rangle + \beta|11\rangle$ is obtained. The trajectory on the $(S_L - C)$-plane corresponding to these initial states is given by

$$S_L = \left( \frac{C^2}{4\alpha^2(1-\alpha^2)} - \frac{C}{2\alpha\sqrt{1-\alpha^2}} \right). \tag{4.24}$$

The average trajectories corresponding to a family of initial pure states with the same entanglement (concurrence) as the state (4.23) are depicted in Fig. 4.6 for different values of $\alpha$.

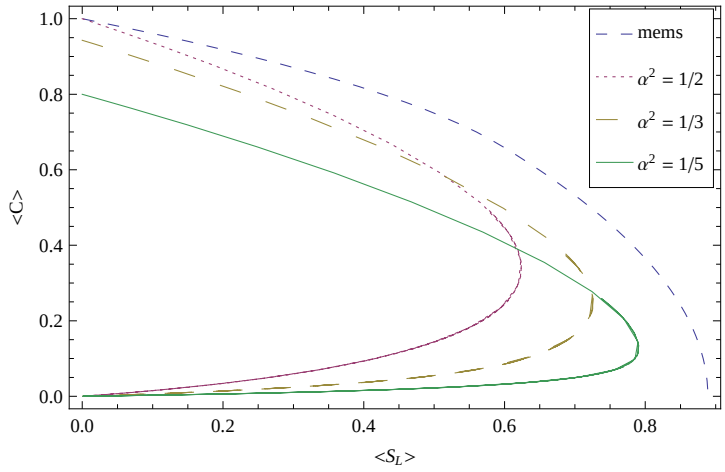


Figure 4.6: Plot of the average value of the concurrence, $\langle C \rangle$, and the concurrence of the maximally entangled mixed states, $C_{mems}$, against the linear entropy for various values of $\alpha$ for the initial state $\alpha|00\ + \beta|11\rangle$.

It transpires from Fig. 4.6 that the smaller the initial entanglement, the larger the maximum degree of mixedness achieved by the two-qubit system during its

evolution. Another trend that can be observed in Fig. 4.6 is that the smaller the initial entanglement, the closer the second branch of the average trajectory is to the $\langle C\rangle = 0$ line. This, of course, is related to the increasing length of the time intervals of zero entanglement corresponding to initial pure states of decreasing entanglement.

The time averaged amount of entanglement exhibited by an evolving composite system is also an interesting quantity to consider. The time average of the concurrence of the two-qubit system,

$$\langle C\rangle_t = \frac{1}{\tau}\int_0^\tau C(t)\,dt, \tag{4.25}$$

was computed numerically. Here, $\tau$ is the time when the concurrence vanishes for the second time (that is, $\tau$ corresponds to the end of the first revival event). In particular, $\langle C\rangle_t$ was computed by using the Bell states

$$|\beta_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \tag{4.26}$$

$$|\beta_{01}\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \tag{4.27}$$

$$|\beta_{10}\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \tag{4.28}$$

$$|\beta_{11}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \tag{4.29}$$

as initial states, obtaining the values $\langle C\rangle_t = 0.225336$ for $|\beta_{00}\rangle$ and $|\beta_{10}\rangle$ and $\langle C\rangle_t = 0.376867$ for $|\beta_{01}\rangle$ and $|\beta_{11}\rangle$. A numerical search for the maximum value of $\langle C\rangle_t$ among evolutions starting with a maximally entangled initial state yielded a maximum value $\langle C\rangle_t^{(\text{max.})} = 0.376867$. This maximum value is achieved by the states $|\beta_{01}\rangle$ and $|\beta_{11}\rangle$.

### Initial Mixed States of Two Qubits

This section is about the entanglement dynamics associated with mixed initial states of the form

$$\rho = \gamma|\Psi\rangle\langle\Psi| + \frac{1-\gamma}{4}I \tag{4.30}$$

where $0 < \gamma < 1$, $I$ is the $4 \times 4$ identity matrix and $|\Psi\rangle$ is a maximally entangled pure state of the form of Eq. 4.14. The state $\rho$ represents a mixture of a maximally entangled pure state and the completely mixed state $\frac{I}{4}$. The state $\rho$ is entangled for $\gamma > 1/3$.

In order to study the typical, average behavior of initial mixed states of the form of Eq. 4.30, maximally entangled states $|\Psi\rangle$ were generated (according to the procedure explained in Section 4.2.1). Then the average properties associated with the evolutions corresponding to the family of states (4.30) were computed.

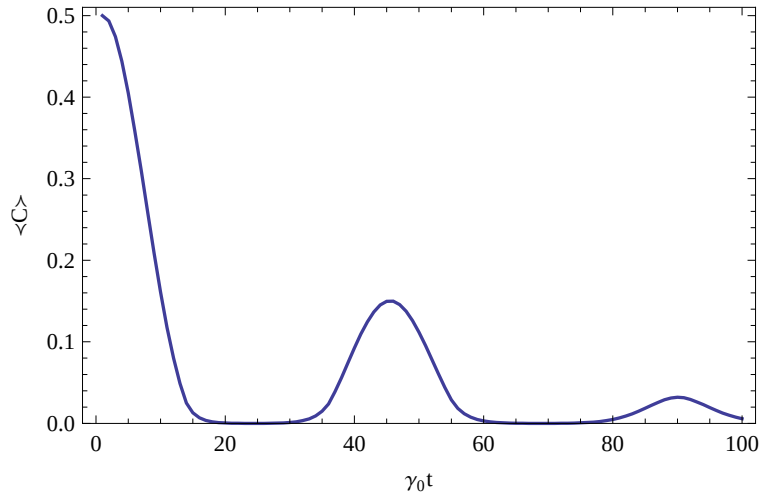


Figure 4.7: Average value of the concurrence, $\langle C\rangle$ as a function of $\gamma_0 t$ for the initial mixed states $\gamma|\Psi\rangle\langle\Psi| + \frac{1-\gamma}{4}I$ with $\gamma = \frac{2}{3}$.

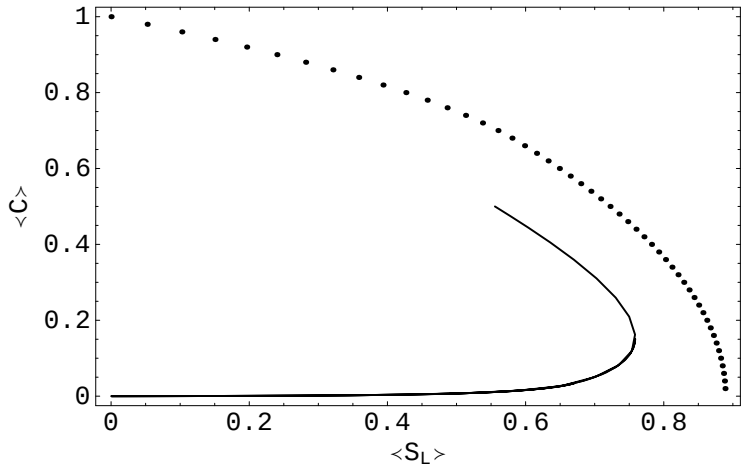


Figure 4.8: Average value of the concurrence $\langle C \rangle$ against the average linear entropy $\langle S_L \rangle$ for the same family of initial mixed states considered in Figure 4.7 (continuous line) and the concurrence of the maximally entangled mixed states $C_{mems}$ against $S_L$ (dotted line).

The results obtained, for $\gamma = \frac{2}{3}$, are summarized in Fig. 4.7 and Fig. 4.8. We can see in these figures that the behaviour of the initial mixed states (4.30) share some general features with the behaviour of the maximally entangled initial states considered previously. There is, however, one important difference (aside from the fact that the trajectory on the $\langle S_L \rangle - \langle C \rangle$ plane starts from an initial state of partial entanglement and finite linear entropy). The lower branch of the trajectory on the $\langle S_L \rangle - \langle C \rangle$ plane depicted in Fig. 4.8 has a long, almost horizontal part associated with states of very little, almost zero entanglement. This section of the lower branch corresponds to the time intervals between entanglement death and entanglement revivals in Fig. 4.7. This means that the existence of finite intervals of basically zero entanglement before entanglement revivals constitutes a typical, average property exhibited by the family of states (4.30).

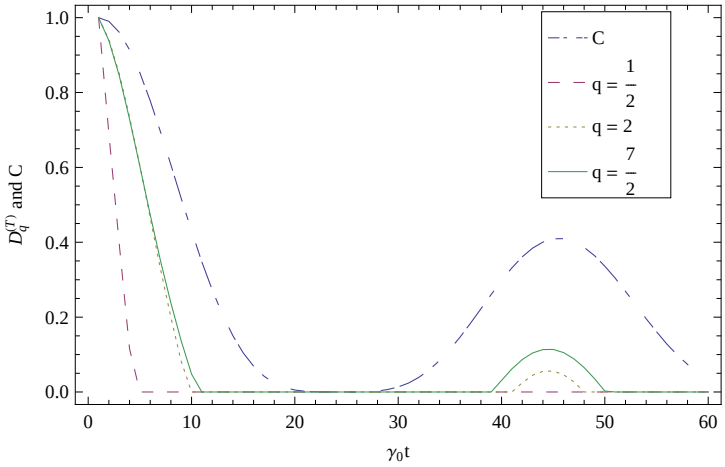


Figure 4.9: Plots of the concurrence $C$ and of the quantity $\frac{D_q^{(T)}}{D_q^{(T)}(t=0)}$ against $\gamma_0 t$ for different values of $q$ for the Tsallis' entropy. All depicted quantities are non-dimensional.

**Entanglement indicators based on the $q$-entropies**

In the model under consideration it has been observed that for some initial states, entanglement disappears at a finite time and afterwards the entanglement is revived [44]. Here, the $q$-entropic counterpart of this behaviour is investigated, the general results obtained are presented in [48]. Here, I compute for the initial Bell state

$$\frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle\Big) \tag{4.31}$$

and the strong non-Markovian regime corresponding to $\lambda = 0.01\gamma_0$, the time evolution of the entropic difference $D_q = S_q[A] - S_q[AB]$ (introduced in section 2.4.1), both for the Tsallis and the Rényi entropies.

The results are shown in Fig. 4.9 and 4.10, where the time evolutions of the concurrence $C$ and of the $D_q$ quantities are shown for the Tsallis and the Rényi entropies, respectively. In these figures the entropic differences $D_q$ are plotted against the non-dimensional variable $\gamma_0 t$. Since the quantities $D_q$ are used as en-

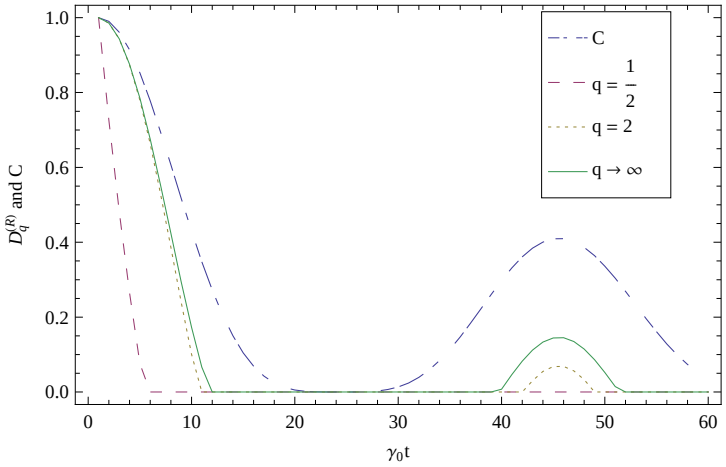


Figure 4.10: Plots of the concurrence $C$ and of the quantity $D_q^{(R)}$ against $\gamma_0 t$ for different values of $q$ for the Rényi entropy. All depicted quantities are non-dimensional.

tanglement indicators (and $D_q > 0$ is a sufficient but not necessary condition for entanglement) in Figures 4.9 and 4.10, $D_q$ is set equal to 0 (indicating that no entanglement is detected by this quantity) whenever $D_q$ becomes negative.

It is plain from Figures 4.9 and 4.10 that the first entanglement disappearance and its subsequent revival can be observed in the behavior of $D_q$. It also transpires from Figure 4.10 that the limit case $q \to \infty$ is the most favorable for these purposes. This is fully consistent with several previous studies by other researchers. Indeed, it is observed in Figure 4.10 that the Rényi based entropic difference $D_\infty^{(R)}$ is the one that detects the presence of entanglement for the largest time periods, both prior to the disappearance of entanglement and also during the first entanglement revival. During the initial decay of entanglement $D_\infty^{(R)}$ detects entanglement (that is, the concomitant classical entropic inequality does not hold) for approximately one half of the time interval where the two-qubits are entangled. During the first entanglement revival, the classical inequality is not verified for approximately one third of the time interval corresponding to non-vanishing entanglement. In

both cases the classical entropic inequality starts to be verified when the squared concurrence falls below a value around 0.2. Note that in the figures $C$ is plotted and not $C^2$, in order to make it easier to see the points where entanglement disappears. However, had $C^2$ been plotted instead of $C$, it would be clear that $2D_2^{(T)}$ does not constitute too bad a lower bound for $C^2$.

An analytical expression for the entropic difference $D_q^{(R)}$ associated with Rényi entropy in the limit $q \to \infty$ can be obtained for the initial Bell state (4.31), and it is given by

$$D_\infty^{(R)} = \ln\left[\frac{1 - P_t - P_t^2 + \sqrt{1 - 2P_t + P_t^2}}{2 - P_t}\right]. \tag{4.32}$$

For the same initial state, one can also obtain an analytical expression for the Tsallis entropy, and it is given by

$$\begin{aligned} D_q^{(T)} &= \frac{1}{q-1} \times \\ &\ln\left[\frac{2(1-P_t)^q P_t^q + (1 - P_t - P_t^2 - E_t)^q + (1 - P_t - P_t^2 + E_t)^q}{(2-P_t)^q + P_t^q}\right] \end{aligned} \tag{4.33}$$

where $E_t = \sqrt{1 - 2P_t + P_t^2}$.

The entropic differences corresponding to $q = 1/2$ are considerably less efficient as entanglement indicators during the aforementioned processes of entanglement decay and revival. On the other hand, it can also be seen in Figure 4.10 that the cases $q = 2$, even being not as good as the case $q = \infty$, is almost as efficient as this limit case. This feature of the $q = 2$ entropic differences is of particular interest because, in the present case, $D_2^{(T)}$ coincides with a recently discovered experimentally measurable entanglement indicator [62].

None of the entropic differences $D_q$ based on either the Tsallis or the Rényi entropies are able to detect the second entanglement revival occurring after its second

"death". However, these later events are less important that the first entanglement revival, because the actual amount of entanglement exhibited by the two-qubit system during the second entanglement "resurrection" is rather small and, consequently, of limited practical relevance.

### 4.2.2 Entanglement indicator based on the uncertainty relation

An entanglement measure based on the uncertainty principle was introduced in Section 2.4.1. All separable states (pure or mixed) of two-qubit systems satisfy

$$U = \delta[\sigma_1(A) + \sigma_1(B)]^2 + \delta[\sigma_2(A) + \sigma_2(B)]^2 + \delta[\sigma_3(A) + \sigma_3(B)]^2 \geq 4, \quad (4.34)$$

where $\sigma_i(A)$, $\sigma_i(B)$ $i = 1, 2, 3$, are the Pauli matrices corresponding to subsystems $A$ and $B$, repsectively, and $\delta O^2 = \langle O^2 \rangle - \langle O \rangle^2$ is the uncertainty of the observable $O$. On the basis of (4.34) we can regard the quantity

$$\frac{4 - U}{4} \quad (4.35)$$

as an entanglement indicator. Any state with $(4 - U)/4 > 0$ is necessarily entangled. On the other hand, if the above quantity is negative, the state may be entangled or separable. The entanglement indicator (4.35) is of interest because it is based on quantities that are in principle measurable.

The uncertainty limit U is plotted as a function of the quantity $\gamma_0 t$ for the initial state $\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$. As one can see in Figure 4.11, the value is less than four most of the time. There are some particular initial, maximally entangled states for which the entanglement of the time-dependent state $\rho$ is detected (at least part of the time) by the violation of the uncertainty relation (4.34). Therefore, for these states the quantity $(4 - U)/4 > 0$ exhibits positive values when the state $\rho$ has a large enough amount of entanglement. This behaviour can be seen in Figure 4.12 for the initial state $\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$.

The same quantities were also plotted for the initial state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ and the results are shown in Fig. 4.13 and 4.14. but in this case, it is interesting to note that the uncertainty relation is not violated, so the entanglement is not detected by the uncertainty limit in this case. One can therefore conclude that the uncertainty limit is a relatively good measure of entanglement but it does not detect entanglement in some states.

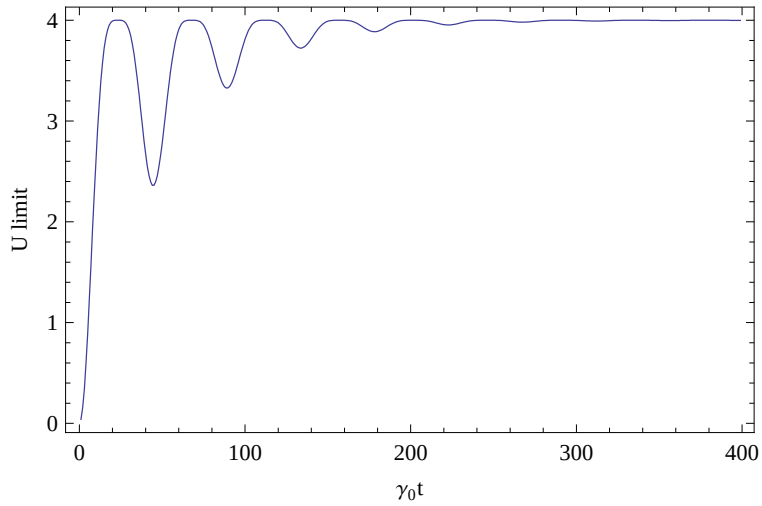


Figure 4.11: Plot of the Uncertainty relation as a function of the quantity $\gamma_0 t$ for the initial state $\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$.

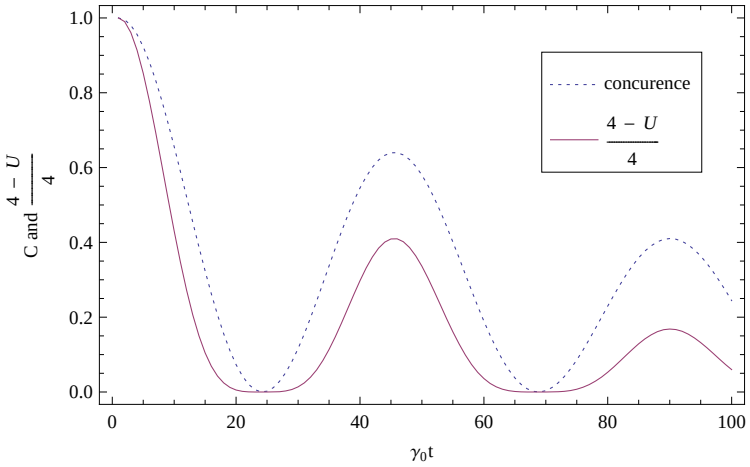


Figure 4.12: The uncertainty-based entanglement indicator and the concurrence $C$ against $\gamma_0 t$, for the initial state $\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$.

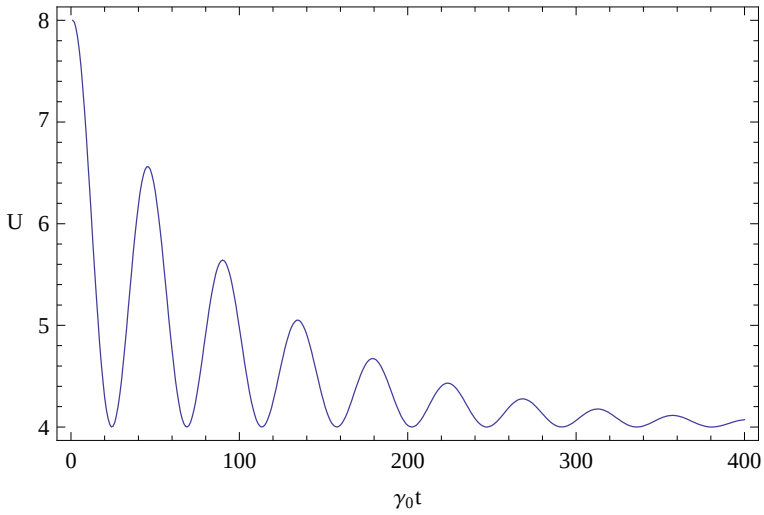


Figure 4.13: Plot of the Uncertainty relation as a function of the quantity $\gamma_0 t$ for the initial state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

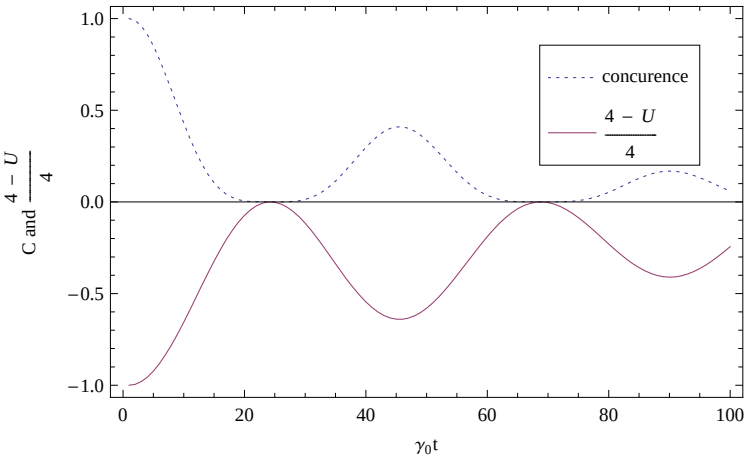


Figure 4.14: Plot of the concurrence (dotted line) and $\frac{4-U}{4}$ (solid line) as a function of the quantity $\gamma_0 t$ for the initial state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

The situation is different when one considers the average behavior of the uncertainty sum $U$ over the family of maximally entangled initial states. One can see in Fig. 4.15 that, on average, the time dependent states arising from maximally entangled initial states do not violate the uncertainty relation (4.34). Even though these states do exhibit on average (at certain times) a considerable amount of entanglement, this entanglement does not affect the local uncertainty relation (4.34).

Since the mean value of the uncertainty relation did not show any detection of entanglement, it would be interesting to randomly generate many initial states and obtain the mean values of the uncertainty relation, but only for the states that violate the uncertainty relation initially. The results are shown in Fig. 4.16 and Fig. 4.18. The same thing was done for the states that do not violate the uncertainty relation initially; the results are shown in Fig. 4.17 and Fig. 4.19.

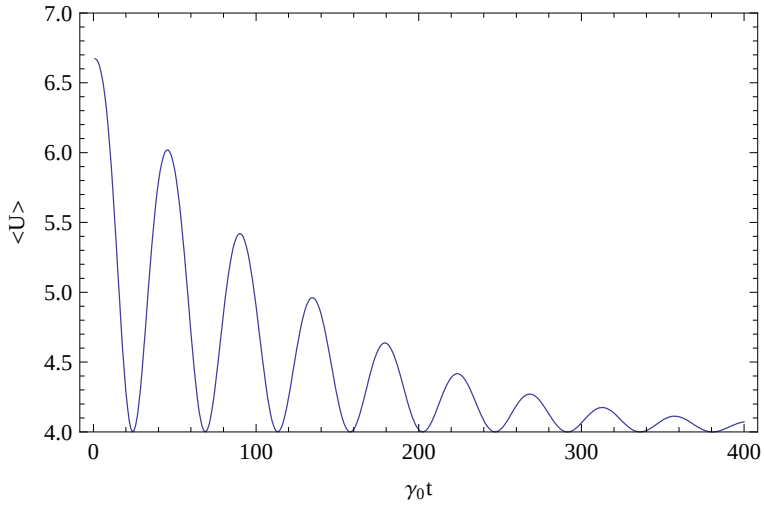


Figure 4.15: Plot of the mean of the uncertainty relation as a function of the quantity $\gamma_0 t$ for maximally entangled initial states generated randomly.

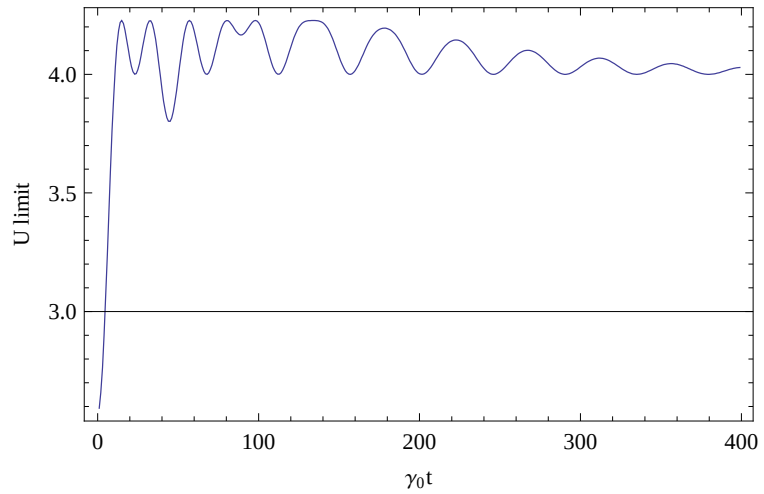


Figure 4.16: Plot of the mean of the uncertainty relation as a function of the quantity $\gamma_0 t$ for maximally entangled, randomly genarated, initial states that violate the uncertainty relation initially.

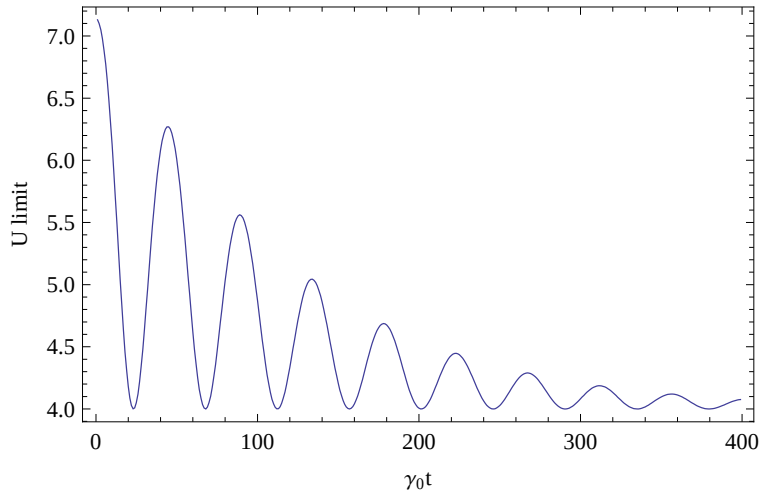


Figure 4.17: Plot of the mean of the uncertainty relation as a function of the quantity $\gamma_0 t$ for maximally entangled, randomly generated, initial states that do not violate the uncertainty relation initially.

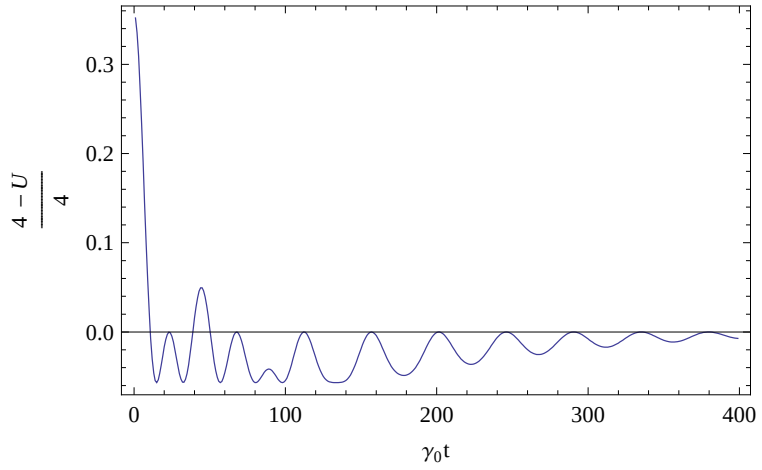


Figure 4.18: Plot of the $\frac{4-\langle U \rangle}{4}$ as a function of the quantity $\gamma_0 t$ for maximally entangled, randomly generated, initial states that violate the uncertainty relation initially.

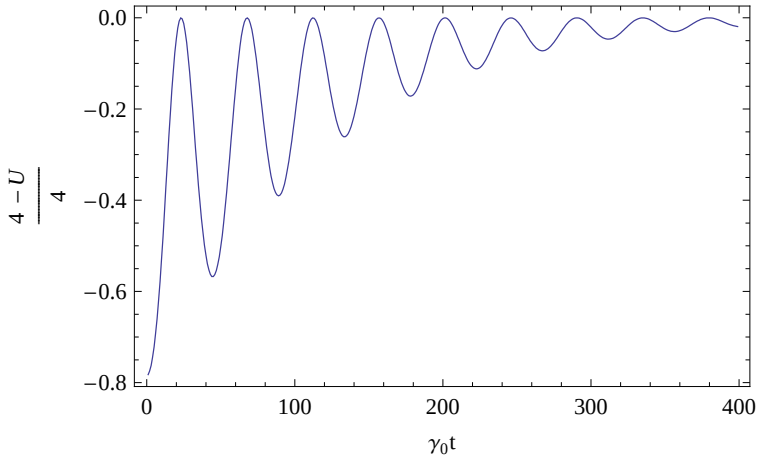


Figure 4.19: Plot of the $\frac{4-\langle U\rangle}{4}$ as a function of the quantity $\gamma_0 t$ for maximally entangled, randomly generated, initial states that donnot violate the uncertainty relation initially.

### 4.2.3 Minternt-Buchleitner (MB) Lower bound for the concurrence squared

Here, the measure $E_{MB}$ introduced in Section 2.4.1 is considered.

$$E_{MB}[\rho_{AB}] = 2\mathrm{Tr}[\rho_{AB}^2] - \mathrm{Tr}[\rho_A^2] - \mathrm{Tr}[\rho_B^2]. \tag{4.36}$$

This entanglement indicator is particularly interesting because, as was shown by Minternt and Buchleitner, it is an experimentally measurable quantity that provides a lower bound for the squared concurrence of $\rho_{AB}$,

$$C^2[\rho_{AB}] \geq E_{MB}[\rho_{AB}]. \tag{4.37}$$

Last, but certainly not least, the indicator $E_{MB}$ is a practical, mathematically simple to compute quantity.

It is interesting to examine the behaviour of $E_{MB}$ in a time dependent setting. The behaviour of $E_{MB}$ is compared with that of the squared concurrence $C^2$ in Figure 4.20 where both quantities are plotted against $\gamma_0 t$ for the initial state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. The average values of the concurrence squared $C^2$ and of the MB

lower bound were also computed for the evolutions corresponding to initially maximally entangled, randomly generated states. The results obtained are depicted in Figure 4.21.

The quantum states considered in Fig. 4.20 and Fig. 4.21 have $\mathrm{Tr}[\rho_A^2] = \mathrm{Tr}[\rho_B^2]$ and, consequently, for these states we can write $E_{MB}[\rho_{AB}] = 2\left(\mathrm{Tr}[\rho_{AB}^2] - \mathrm{Tr}[\rho_A^2]\right)$. One can verify in Fig. 4.20 and Fig. 4.21 that, indeed, the quantity $E_{MB}$ constitutes a lower bound for $C^2$. The results depicted in Fig. 4.20 indicate that, for the initial state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, the lower bound $E_{MB}$ provides a reasonably good estimate of the amount of entanglement exhibited by the two qubits during the first period of entanglement decrease. The quantity $E_{MB}$ is also able to detect the first entanglement revival, at least during the time interval around the peak value exhibited by $C^2$ in this revival (this interval corresponds, approximately, to one third of the duration of the first revival). On the contrary, $E_{MB}$ does not detect the second or later entanglement revivals.

As for the average behavior of the lower bound $E_{MB}$ corresponding to initial maximally entangled states, $E_{MB}$ provides a reasonable estimate for the squared concurrence during the first half of the first time interval of entanglement decrease. However, $E_{MB}$ does not detect the subsequent entanglement revivals.

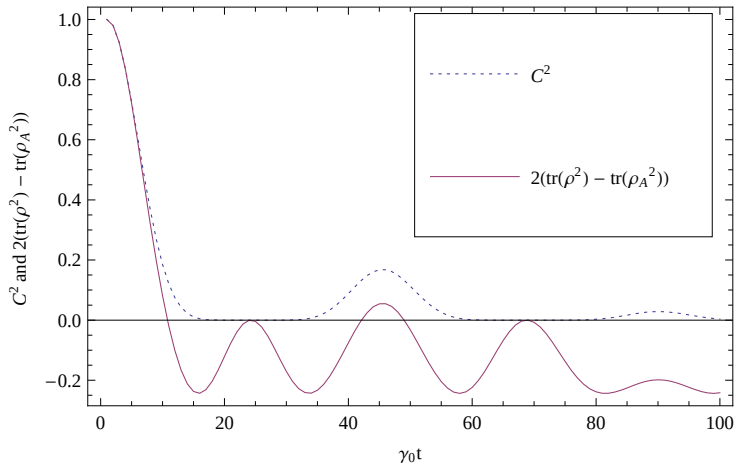


Figure 4.20: Plot of the concurrence squared and the MB lower bound $E_{Mb}$,as a function of $\gamma_0 t$, for the initial state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

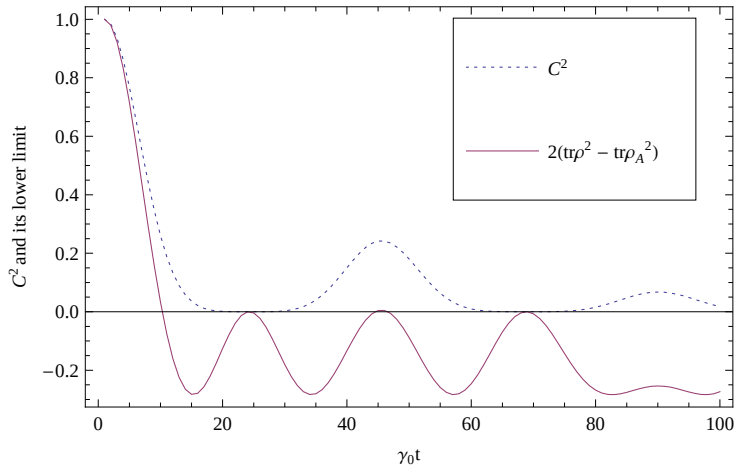


Figure 4.21: The averages of the concurrence squared and of the MB lower bound corresponding to initial maximally entangled states, as a function of $\gamma_0 t$.

## 4.3 N-qubits states ($N > 2$)

This section is about the entanglement dynamics of three-qubit and four-qubit systems interacting with an environment in non-Markovian regime. As in the two-qubit case, it is assumed that each qubit interacts with its own, independent

environment. In the case of three qubits or more, the GHZ (Greenberger-Horne-Zeilinger) state and the W states are studied. The general expression of the GHZ state is

$$|GHZ\rangle = \frac{|0\rangle^{\otimes n} + |1\rangle^{\otimes n}}{\sqrt{2}} \tag{4.38}$$

where $n$ is the number of qubits. For instance the GHZ state for three qubit will be

$$|GHZ\rangle = \frac{|000\rangle + |111\rangle}{\sqrt{2}} \tag{4.39}$$

The W state is generally given by

$$|W\rangle = \frac{1}{\sqrt{n}}(|100...0\rangle + |010...0\rangle + ... + |000...1\rangle) \tag{4.40}$$

For three qubits, the W state is

$$|W\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle) \tag{4.41}$$

The negativity (described in 2.4.1) is used here to quantify the entanglement in the case of more that two qubits. To determine the average behavior associated with initial states locally equivalent to the $n$-qubits $|GHZ\rangle$ state we generate random initial states of the form

$$(U_1 \otimes \cdots \otimes U_n)\ |GHZ\rangle, \tag{4.42}$$

where the unitary operators $U_i$ (acting on the $i$th single qubit) are generated randomly, independently and uniformly distributed according to the Haar measure, as described in Section 4.2.1. Then, the average, time dependent properties corresponding to the abovementioned random initial states are computed. A similar procedure was followed to study the average properties of evolutions corresponding to initial states locally equivalent to the $n$-qubits $|W\rangle$ state.

As in the two qubits case, the typical behavior of appropriate families of initial states was studied for three-qubit and and four-qubit systems. The most noticeable difference between the results obtained for two qubits and those obtained for

three or four qubits involves the finite time intervals of zero entanglement between entanglement revivals. For initial maximally entangled two-qubits states the finite time intervals of zero entanglement between entanglement revivals disappear when one computes the concomitant average behavior, as shown in Fig. 4.2 and Fig. 4.5. On the contrary, in the case of initial three-qubits states locally equivalent to the $|GHZ\rangle$ or the $|W\rangle$ states, the aforementioned intervals of zero entanglement survive after the averaging procedure, as depicted in Fig. 4.22 and Fig. 4.23). This means that the abovementioned finite time intervals of entanglement disappearance are robust features of the entanglement dynamics of three-qubit systems. This is consistent with the fact that the entanglement associated with $N$-qubit systems tends to become more fragile as the number of qubits increases.

Similar behaviour is observed in the case of four qubits. The general results in the four qubits case (Fig. 4.24 and Fig. 4.25) are similar to those in the three qubits case, with the difference that the interval of time over which ESD occurs is a bit wider in the four qubits case. As a general remark, one notices that as the number of qubits increases, ESD becomes more important.

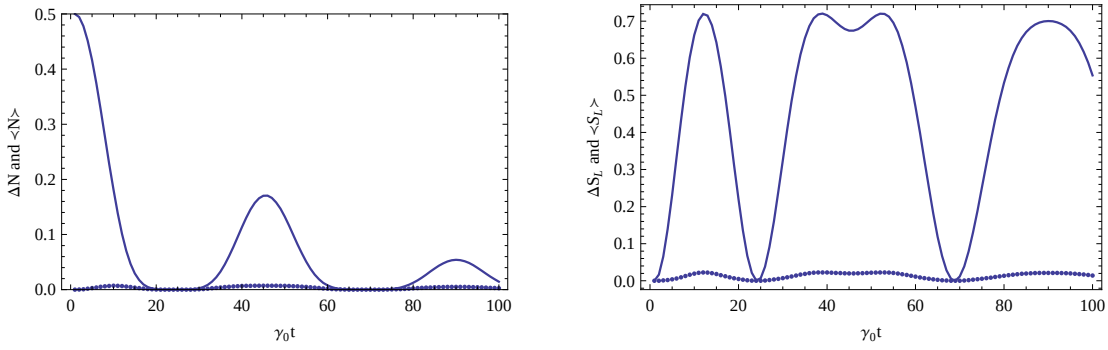


Figure 4.22: Plot of the average value of the negativity together with its dispersion against the quantity $\gamma_0 t$ (left), and the average value of the linear entropy together with its dispersion against the quantity $\gamma_0 t$ (right) for a family of the three qubits GHZ states.

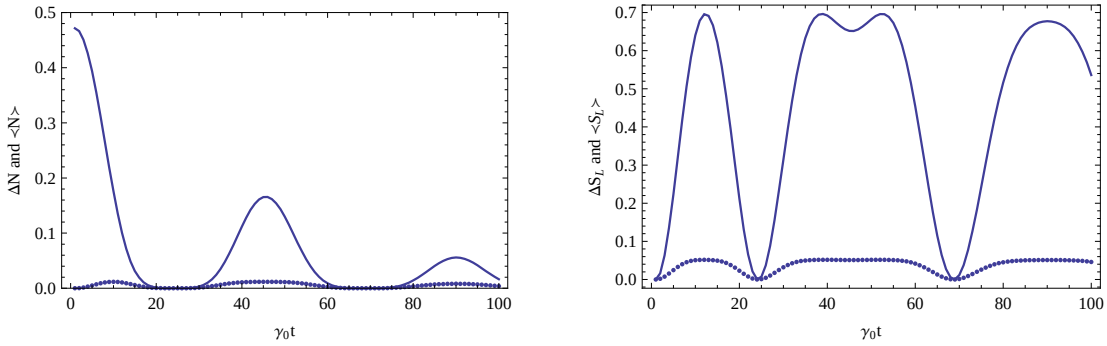


Figure 4.23: Plot of the average value of the negativity together with its dispersion against the quantity $\gamma_0 t$ (left), and the average value of the linear entropy together with its dispersion against the quantity $\gamma_0 t$ (right) for a family of the three qubits W states.

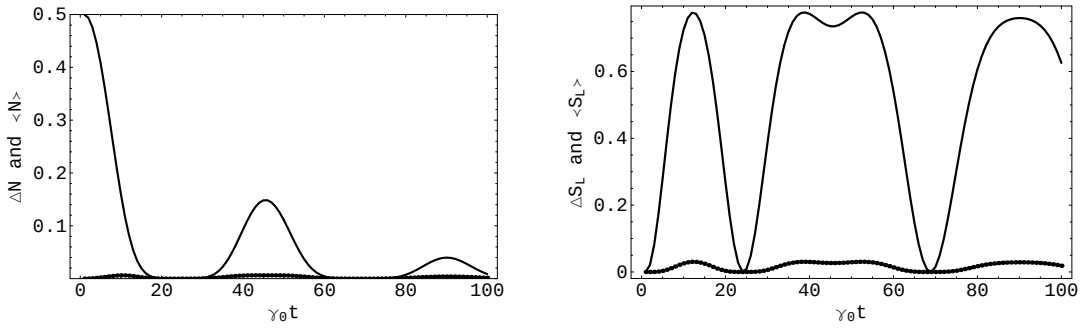


Figure 4.24: Plot of the average value of the negativity together with its dispersion against the quantity $\gamma_0 t$ (left), and the average value of the linear entropy together with its dispersion against the quantity $\gamma_0 t$ (right) for a family of the four qubits GHZ states.

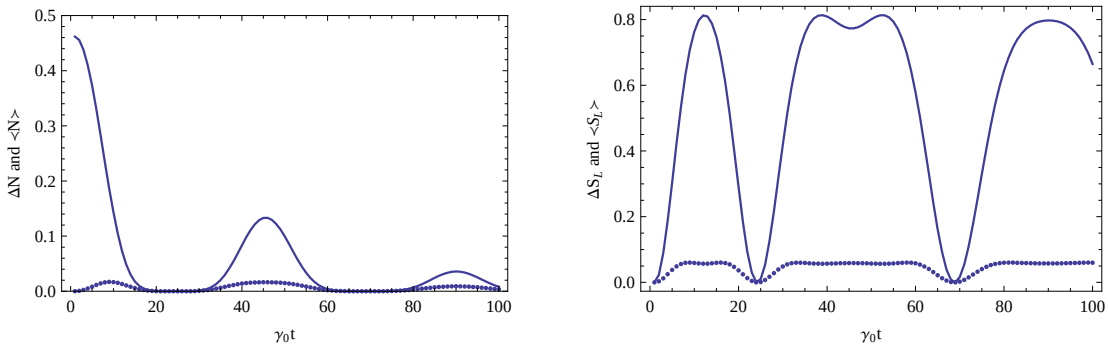


Figure 4.25: Plot of the average value of the negativity together with its dispersion against the quantity $\gamma_0 t$ (left), and the average value of the linear entropy together with its dispersion against the quantity $\gamma_0 t$ (right) for a family of the four qubits W states.

Chapter 5

# Conclusions

It was stated in the introduction that the current study has three main goals. The first goal is to get a global picture or a global typical behaviour of the system by randomly generating a sufficient number of initial states and observing how they behave in general. The second goal is to consider entanglement features that were not considered by previous researchers, and the third goal is to consider systems of more than two qubits.

In the case of two-qubits, the average, typical behavior corresponding to maximally entangled initial states, or that corresponding to pure, partially entangled states with a given amount of entanglement, does not have finite time intervals of zero entanglement between entanglement revivals, as is the case for some particular initial states belonging to the alluded families. On the contrary, when investigating the dynamics of entanglement associated with the families of initial states of three-qubits locally equivalent to the $|GHZ\rangle$ (Eq. 4.38) and to the $|W\rangle$ (Eq. 4.40) states, we found that the finite intervals of zero entanglement are still present in the average behavior. Consequently, the phenomena of entanglement sudden death and subsequent entanglement revival are robust properties of the evolutions associated with the abovementioned families of initial states of three-qubits. Similar results were obtained in the case of four-qubits. However, the phases of zero entanglement between entanglement revivals observed in connec-

tion with four-qubit initial states locally equivalent to the $|GHZ\rangle$ and to the $|W\rangle$ states are longer than the concomitant intervals for three-qubit systems. These features of the entanglement dynamics of three-qubit and four-qubit systems are consistent with the fact that, in general, entanglement becomes more fragile as the number of qubits of a system increases.

Entanglement measures based on the Tsallis and Rényi entropies were introduced in section 4.2.1. The time behaviour of entropic differences $D_q = S_q[A] - S_q[AB]$ between the $q$-entropy of one of the qubits and the $q$-entropy of the two qubits was investigated. The quantity $D_q$ is an entanglement indicator in the sense that $D_q > 0$ is a sufficient criterion for entanglement. The time evolution of $D_q$ was computed for various values of $q$, both for the Tsallis and for the Rényi entropies. Classical entropic inequalities are violated for (approximately) one half of the time interval corresponding to the initial entanglement decay, and for one third of the duration of the first entanglement revival. In both cases the classical inequalities are verified when the concurrence falls below a value of around 0.2. It is interesting to note that this behaviour shows some similarities with the behaviour exhibited for this system by the Bell inequalities. It was found in [46] that the Bell inequalities are satisfied (and, consequently, they don't detect entanglement) during an appreciable part of the first entanglement revival.

It was also observed that the limit case $q \to \infty$ constitutes the most favorable one for detecting the first decay of entanglement and the subsequent entanglement revival, in agreement with results obtained previously by other researchers in different contexts. However, the case $q = 2$ proved to be almost as good as the limit case $q \to \infty$ (particularly during the first phase of entanglement decay). This is specially relevant because, in the present case, $D_2^{(T)}$ constitutes an experimentally accessible indicator of entanglement.

The behaviour of two entanglement indicators based upon measurable quantities

was also investigated. The Minternt-Buchleitner lower bound $E_{MB}$ for the squared concurrence and an entanglement indicator based on the violation of a local uncertainty relation were considered. For the initial state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, the quantity $E_{MB}$ exhibited "sudden death" and one "revival". On the other hand, the average behavior of $E_{MB}$ corresponding to maximally entangled initial states has sudden death, but no revival. During the the period of entanglement decrease $E_{MB}$ provides a reasonable estimate for the squared concurrence.

The estimator based on the violation of local uncertainty relations does detect the entanglement of the evolving two-qubit state for some initial conditions. However, its average behavior for initial maximally entangled states corresponds to separable states. These findings are consistent with the results reported by Bellomo et al. in [46], where it was shown that even at times when the two-qubits system still has a considerable amount of entanglement it behaves "classically", as far as the Bell inequalities are concerned. Our present results show that the time dependent state of the two-qubits system, particularly during the entanglement revivals, also fails to exhibit other manifestations of entanglement, such as positive values of the Minternt and Buchleitner indicator $E_{MB}$, or the violation of local uncertainty relations.

In the case of initially mixed states of two-qubits, initial states of the form

$$\gamma|\Psi\rangle\langle\Psi| + \frac{1-\gamma}{4}I \tag{5.1}$$

are considered, where $\Psi$ is a maximally entangled state, $I$ is the $4 \times 4$ identity matrix and $0 < \gamma < 1$. This state is entangled for a finite if $\gamma > 1/3$. The general results obtained with $\gamma = 2/3$ show that entanglement disappears for a finite time and then gets revived.

I also investigated the connection between the time evolution of the amount of entanglement exhibited by the multi-qubit system on the one hand, and its global

degree of mixedness (as measured by the total linear entropy $S_L$) on the other. As a general trend, the entanglement exhibited by multi-partite quantum systems tends to decrease as the degree of mixedness increases. However, excepting the initial period of entanglement decrease, the systems considered here tend to exhibit the largest amount of entanglement simultaneously with the largest degrees of mixedness. Indeed, during the entanglement revivals entanglement and mixedness tend to increase and decrease together. The trajectory followed by the multi-qubit systems (for various families of initial states) in the $(\langle S_L \rangle - \langle \mathcal{E} \rangle)$-plane was determined. In all the cases studied, these trajectories exhibit the shape of an inverted "C" with two branches, one corresponding to the initial phase of entanglement decrease, and the second branch corresponding to the entanglement revivals. In the case of maximally entangled initial states of two-qubits, the first branch is relatively close to the MEMS curve, while the second branch departs drastically from it.

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