Order convergence on Archimedean vector lattices and applications

by

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Contents

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"*Mathematics, rightly viewed, possesses not only truth, but supreme beauty- a beauty cold and austere, like that of sculpture.*"

-Bertrand Russel

1. INTRODUCTION

1.1 Vector Lattices

The theory of ordered vector spaces, and in particular vector lattices, also known as Riesz spaces, has its origins in the work of three mathematicians around the year 1935 namely F. Riesz in Hungary, L. V. Kantorovitch in the former Soviet Union and H. Freudenthal in the Netherlands. Riesz was interested in what is in the modern literature called the order dual of a partially ordered vector space and presented his findings at the 1928 International Mathematics conference at Bologna [68] as well as in a 1940 paper [69], a translation of a 1937 paper in Hungarian. There he proved a result on the nature of the ordered vector space of all bounded linear functionals on a vector lattice. Kantorovitch studied convergence and algebraic properties of ordered vector spaces in some detail, see for instance [40], [41] and [42]. He found, amongst others, a more general version of Riesz's theorem. He was soon joined by more mathematician in Leningrad, among them A. G. Pinsker and A. I. Judin. Freudenthal contributed a powerful spectral theorem for vector lattices in his 1936 paper [32]. The importance of this result can be seen in the fact that both the Radon-Nikodym Theorem in measure theory and the spectral theorem for Hermitian operators on a Hilbert space follow as corollaries to it. Not many years later, in the period 1940 to 1944, important contributions were published in Japan, notably those of H. Nakano ([58], [59], [60] and [61]), T. Ogasawara and K. Yosida, and in the United States by H. F. Bohnenblust and S. Kakutani. These papers dealt with, among other things, the concrete representation of a vector lattice as spaces of 'nearly finite' continuous functions. With this three major schools of research in ordered linear spaces were established, in the Soviet Union, in Japan and in the United States.

It was only after 1971 that the various notations and terminology used by the different schools were united and brought under a common banner in the book [52] by W. A. J. Luxemburg and A. C. Zaanen and the second volume [81] of that same work. It is interesting, though, to note the differences in approach and nature of the work of the different research communities, see for instance [83] and [51]. We will submit to the notation and terminology of the American school as it is by far the most common one in use today.

Definition 1.1 *A pair $(E, \leq)$ consisting of a real vector space E and a partial order $\leq$ defined on E is called a vector lattice if the following conditions are satisfied for all $f, g, h \in E$ and all real numbers $\alpha > 0$.*
(i) If $f \leq g$ then $f + h \leq f + h$;

(ii) *If* $f \leq g$ *then* $\alpha f \leq \alpha g$*;*
(iii) E *is a lattice.*

Of particular interest here are those vector lattices that satisfy the following addition property.

Definition 1.2 *A vector lattice* E *is Archimedean whenever the relation*

$$0 \leq nf \leq g, n \in \mathsf{N}$$

implies that $f = 0$*.*

In the Soviet terminology a vector lattice is known as a K-lineal while Nakano and the Japanese school call it a semi-ordered linear space. A well known example of a vector lattice is the space $\mathcal{C}(X)$ of all continuous functions on a topological space X. Another is the space $M(X, \mu)$ of all μ-almost everywhere finite valued μ-measurable functions on the nonempty point set X, where μ is a countably additive non-negative measure on the σ-field Λ of subsets of X, ordered pointwise μ-almost everywhere. These examples also motivate the choice of notation for elements of a vector lattice commonly used in the literature. Since we will also apply our results to function spaces we conform to the standard, that is, we denote such elements always by lower case roman characters, mostly f, g, h and so forth.

The notation $f \vee g = \sup\{f, g\}$ and $f \wedge g = \inf\{f, g\}$ is dominant in the literature, and we use it as well. For a vector lattice E and $f \in E$ we make use of the following notation: The positive part f^+ and the negative part f^- of f are given respectively by

$$f^+ = f \vee 0, f^- = (-f) \vee 0.$$

The modulus $|f|$ of f is defined to be

$$|f| = (-f) \vee f. \tag{1.1}$$

It is obvious that $-f^- = f \wedge 0$ and for any $f \in E$ we have

$$f = f^+ - f^-,$$

$$f^+ \wedge f^- = 0,$$

$$|f| = f^+ + f^-.$$

The positive cone of a vector lattice E is denoted E^+, that is,

$$E^+ = \{f \in E : 0 \leq f\}.$$

For a vector lattice the positive cone is generating, that is, it determines the partial order on the space through

$$f \leq g \Leftrightarrow g - f \in E^+.$$

The theory of vector lattices is considered by most to be more or less complete. Most of the recent contributions to the field form part of a program initiated by A. C. Zaanen, see [82]. The aim of this program is to reprove the classical results using elementary methods without the use of cumbersome representation theorems, see for instance [38]. The motivation for such an investigation is that elementary methods reveal more of the underlying structure than proofs by representation. Other recent contributions include the characterization of those vector lattices that can be represented as a space of finite continuous functions on some topological space, [1].

1.2 Order Convergence

Our interest lies chiefly in the 'convergence properties' of a vector lattice, in particular we are interested in order convergence of sequences. Our work can therefore be viewed as a continuation, be it some seventy years later, of the work of Kantorovitch. In particular, we are interested in the following notion of convergence.

Definition 1.3 *(i) The sequence* (f_n) *on a (vector) lattice* E *is said to decrease to the element* $f \in E$ *if* $f_{n+1} \leq f_n$ *for every* $n \in \mathsf{N}$ *and* $f = \inf\{f_n : n \in \mathsf{N}\}$*. We denote this by* $f_n \downarrow f$*.*
(ii) The sequence (f_n) *on a (vector) lattice* E *is said to increase to the element* $f \in E$ *if* $f_n \leq f_{n+1}$ *for every* $n \in \mathsf{N}$ *and* $f = \sup\{f_n : n \in \mathsf{N}\}$*. We denote this by* $f_n \uparrow f$*.*
(iii) A sequence (f_n) *on the (vector) lattice* E *order converges to* $f \in E$ *if there exists a sequence* (λ_n) *increasing to* f *and a sequence* (μ_n) *decreasing to* f *such that*

$$\lambda_n \leq f_n \leq \mu_n, n \in \mathsf{N}.$$

We denote this by $f_n \to f$*.*

It should be noted that some authors use the term 'order convergence' to mean some other type of convergence, see for instance [19]. The relation between these different types of convergence is a subject of interest in its own right and has been studied by several authors, including Kent [49] and May and McArthur [56]. One can, for instance, define a notion of 'order convergent filter', see [47], [48], [49] or [27]. The concept of order convergene given in Definition 1.3 can be generalized in a straight forward way to nets, [19]. In non-complete posets there are some disadvantages to using this generalization. For instance, there are residually constant nets which do not converge. Rennie introduced a modification which avoids such pathologies in [67]. Also in [67] a definition of order convergent nets that is equivalent to order convergence of filters is given. This definition is in general not equivalent to any of the other notions of order convergence on a poset. If, however, the poset is a complete lattice, then they are all equivalent, see [27]. In the present work we make no assumption on the completeness of the lattice structure and restrict ourselves to Definition 1.3.

Definition 1.3 is mostly used in cases where no algebraic structure other than the order relation is considered, for instance in [19]. On a vector lattice Definition 1.3 is equivalent to the following, which is standard in vector lattice theory and can be found in [52] or [31].

Definition 1.4 *The sequence (f_n) on a vector lattice E is said to order converge to the element $f \in E$ if there exists a sequence (μ_n) on E that decreases to 0 such that*

$$|f - f_n| \leq \mu_n, n \in \mathsf{N}. \tag{1.2}$$

We denote this by $f_n \to f$.

Note that the convergence defined above is determined only by the elements of the space itself and the order on the space. No other set is involved, nor does the definition rely on some mapping from the set into the reals, or any other set for that matter, as is the case for convergence in a metric space, for instance. However, the basic notion of some eventual minimization involving the terms of the sequence and its limit that is encountered in most forms of convergence used in functional analysis is preserved. Indeed, for a metric space (X, ρ) convergence in the metric is defined through

$$(x_n) \to_\rho x \Longleftrightarrow \left\{ \begin{array}{l} \text{for each } \varepsilon > 0 \text{ there is } N_\varepsilon \in \mathsf{N} \\ \text{such that } \rho(x, x_n) < \varepsilon, n \geq N_\varepsilon \end{array} \right. . \tag{1.3}$$

The eventual minimization in (1.3) takes place in R where as the sequence is minimized (eventually) in the space E itself in (1.2).

1.3 Topological Type Structures

Our motivation for the investigation of convergence properties of vector lattices is that convergence of topological type processes is one of the most basic concepts connected with the theory of function spaces, and it is there that we will also seek to apply our theory. The specific topological type processes that are generally considered include sequences [30], nets (generalized sequences) [46] and filters [39]. Within the classical functional analysis the convergence is generally induced by a topology on the function space. The topology can be defined through a norm, as is the case with the well known L^p-spaces where the norm is given by

$$\|f\| := \left(\int (f(x))^p \, dx \right)^{1/p}.$$

Alternatively, one can induce a locally convex topology through a family of semi-norms. Here the weak*-topology on the dual $\mathcal{L}E$ of a Banach space E is a classical example, where the semi-norms are defined by

$$\rho_f(\varphi) := |\varphi(f)|, \varphi \in \mathcal{L}E$$

and f runs over all nonzero elements of E. It is worth noting that the majority of the topologies usually considered on function spaces are locally convex. Of particular interest to us here is the space $\mathcal{C}(X)$ of all continuous functions defined on a topological space X. The most widely studied of these spaces is the particular case when X is compact. It is usual, in this instance, to consider the space $\mathcal{C}(X)$ equipped with the supremum-norm given by

$$\|f\| = \sup\{|f(x)| : x \in X\}.$$

However, when the space X is not necessarily compact more general topologies are considered on $\mathcal{C}(X)$. There are two important groups of topologies on $\mathcal{C}(X)$: The set-open topologies, and the uniform topologies. Set-open topologies are defined in terms of networks [54] of subsets of X. In particular, for a closed network α on X the set-open topology on $\mathcal{C}(X)$ with respect to α is generated by the subbase

$$\{[A, V] : A \in \alpha \text{ and } V \text{ is open in } \mathsf{R}\}$$

where $[A, V] = \{f \in \mathcal{C}(X) : f(A) \subseteq V\}$. Particularly common in application are the compact-open topology and the point-open topology. The point-open topology corresponds to pointwise convergence of sequences (or nets) in $\mathcal{C}(X)$. The uniform topologies are defined similarly. All the topologies defined above induce convergence classes on $\mathcal{C}(X)$ that are at least as restrictive as the pointwise convergence. Hence the study of these spaces is called by some authors [8] C_p-theory, where the 'p' indicates the connection with pointwise convergence of sequences (or nets). This theory has been used as a powerful tool in studying certain equivalences between topological spaces. A detailed survey of the subject can be found in [8] and [54]. Since $\mathcal{C}(X)$ is a vector lattice we can consider order convergence of sequences on it. This convergence, unlike those described above, is more general than the pointwise convergence.

If (K, τ) is a topological space we denote the convergence of a sequence (f_n) on K to an element $f \in K$ by $f_n \rightarrow_\tau f$. With every topological space (K, τ) one can associate a mapping σ_τ from K into the powerset of the set of all sequences on K by

$$(f_n) \in \sigma_\tau(f) \Leftrightarrow f_n \rightarrow_\tau f.$$

The mapping σ_τ satisfies the following properties, known as the Moore-Smith Axioms, that were introduced in [57], see [46] or [74] for a more recent presentation:

MS1 (Constants) If (f_n) is a sequence such that $f_n = f$ for every n, then $(f_n) \in \sigma_\tau(f)$;

MS2 (Subsequences) If a sequence (f_n) belongs to $\sigma_\tau(f)$, then so does each subsequence of (f_n);

MS3 (Divergence) If a sequence (f_n) does belong to $\sigma_\tau(f)$, then there exists a subsequence of (f_n), no subsequence of which belongs to $\sigma_\tau(f)$;

MS4 (Iterated limits) If for every $n \in \mathsf{N}$ the sequence $(f_{n\,m})$ belongs to $\sigma_\tau(f_n)$ and the sequence (f_n) belongs to $\sigma_\tau(f)$, then there exists a strictly increasing mapping $k : \mathsf{N} \to \mathsf{N}$ such that the sequence $(f_{n\,k(n)})$ belongs to $\sigma_\tau(f)$.

Condition (MS3) is also known as the Urysohn Property while condition (MS4) is sometimes called the Diagonal Property.

Inversely, it may happen that a set S together with a mapping σ from S into the powerset of the set of all sequence on S is given. The question here is whether or not there exists a topology τ on S such that $\sigma = \sigma_\tau$. This need not be the case, and a characterization of those mappings σ that can be identified with a topology was obtained by Moore and Smith [57].

Theorem 1.1 *Let σ be a mapping from the set S into the powerset of all sequences on S. Then there exists a topology τ on S such that $\sigma = \sigma_\tau$ if and only if the mapping σ satisfies the Moore-Smith Axioms (MS1) to (MS4).*

The characterization above was originally stated in terms of nets, but is applies equally well to sequences. With this result the issue might seem to be settled. There are, however, a large variety of examples, many of practical importance, that do not satisfy all of the axioms. These include convergence almost everywhere on the space of measurable functions on a measure space (X, Λ, μ) and continuous convergence of sequences of continuous functions. Within the classical topology [46] it seems that such structures are too weak to allow for a general treatment by topological methods. If, however, one considers a more general notion of a Topological Type Structure, namely that of a convergence space, it is possible to develop quite a strong theory for many of the Topological Type Structures that do not satisfy all the axioms (MS1) to (MS4), as is demonstrated in [15].

The present work is largely concerned with order convergence on vector lattices which in general is also not topological, see [19] and [78]. The order convergence generally only satisfies the Axioms (MS1) and (MS2). These two axioms define the rather weak concept of a sequential convergence structure, see [15]. We show that order convergence on certain partially ordered sets does, however, satisfy some additional properties which enables us to employ methods from the theory of convergence spaces.

1.4 Convergence Spaces

Let a set K be given. A filter on K is a nonempty collection $\mathcal{F}$ of nonempty subsets of K such that $\mathcal{F}$ is closed under finite intersections and the formation of supersets. A filter $\mathcal{F}$ is said to be finer than a filter $\mathcal{G}$ if the inclusion $\mathcal{G} \subseteq \mathcal{F}$ holds. We also say that $\mathcal{G}$ is coarser than $\mathcal{F}$. A filter is said to be an ultrafilter if it is not properly contained in any other filter.

A subset $\mathcal{B}$ of a filter $\mathcal{F}$ is a filter base for $\mathcal{F}$ if every element of $\mathcal{F}$ contains an element of $\mathcal{B}$. We call $\mathcal{F}$ the filter generated by $\mathcal{B}$ and write $\mathcal{F} = [\mathcal{B}]$. Every subset B of K generates a filter called the filter generated by B and and it is denotes $[\{B\}]$ or $[B]$ for short. In case $B = \{f\}$ we write $[f]$ and call it the filter generated by f.

Definition 1.5 *A mapping λ from the set K into the power set of all filters on K is called a convergence structure and the pair (K, λ) a convergence space if the following conditions are satisfied:*
(i) $[f] \in \lambda(f)$ *for every* $f \in K$*;*
(ii) $\mathcal{F} \cap \mathcal{G} \in \lambda(f)$ *for every* $\mathcal{F}, \mathcal{G} \in \lambda(f)$*;*
(iii) if $\mathcal{G}$ belongs to $\lambda(f)$ then every finer filter $\mathcal{F}$ belongs to $\lambda(f)$.

In order not to be entangled in cumbersome notation a convergence space (K, λ) is often denoted simply by K.

The simplest example of a convergence structure is the one induced by a topology.

Example 1.1 *Let (K, τ) be a topological space. Denote by $\mathcal{U}_\tau(f)$ the neighbourhood filter at $f \in K$ with respect to τ. Define the mapping λ_τ from K into the powerset of all filters on K as*

$$\mathcal{F} \in \lambda_\tau(f) \Leftrightarrow \mathcal{F} \supseteq \mathcal{U}_\tau(f).$$

It is easily verified that λ_τ does indeed define a convergence structure on K, [15].

However, convergence structure is a more general concept than topology. In fact, most convergence structures of practical importance cannot be defined in terms of a topology as in Example 1.1. Consider the following example.

Example 1.2 *Let M be the set of all real-valued measurable functions on a measure space $(\Omega, \mathcal{A}, \mu)$. Define a convergence structure λ on M as follows: a filter $\mathcal{F}$ converges to f in (M, λ) if $\mathcal{F}$ converges to f almost everywhere in Ω. There is no topology on M that induces the convergence structure λ, see [15] or [63].*

From the above it is clear that the theory of convergence spaces contains that of topological spaces as a special case. These spaces have been studied for over fifty years, and as in Definition 1.5, are defined in terms of filter convergence. It should be noted that some authors, [35] for instance, refer to a convergence structure in the sense of [15] as a pseudo-topology while others use this term to refer to a specific class of convergence space, the Choquet spaces [34]. The term pseudo-topology is used in [70], [71] and [72] to indicate the most general notion of a Topological Type Structure of wich convergence spaces is a highly important special case. We will consider convergence structures only in the sense of [18] and [15] and adopt the terminology of [15].

The classical topological methods have generally been available since the appearance of Hausdorff's book in 1914, but the notion of a convergent filter was defined only in 1937, see [23]. Filter convergence spaces were first considered by Choquet in 1948, see [24], and initially studied by Fischer [29] and Kowalsky [50], although their definitions do not coincide exactly with the modern one. At first these spaces were mainly used in applications to analysis and topology, notably in [11], [50], [79], [43], [44], [45] and in particular in [18] during the 1960's and 1970's. During this time the field of convergence spaces developed into an abstract theory. The concepts from classical topology were generalized to the wider setting of convergence spaces, and by the time of the second conference on convergence spaces and its applications [36], a fully fledged theory had developed. Such problems as the validity of the closed graph theorem [14], the extremal compactification of convergence spaces [21] and generalizations of the Hahn-Banach extension theorem to convergence vector spaces were considered, see [12],[13] and [55].

It is generally accepted that functional analysis was initiated by Banach in 1932. In [10] such fundamental results as the Hahn-Banach Theorem, the Open Mapping Theorem and the Principal of Uniform Boundedness are proved. When mathematicians realized the power of these results, a way was sought in which to generalize them. This generalization was obtained through the use of topological spaces and, in particular, topological vector spaces. There are, however, some difficulties when working exclusively with classical topological structures of which we mention a few.

A very important concept in functional analysis is that of an inductive limit. Within the framework of topological spaces, the limit is far removed from its component spaces, as indicated in [17]. Hence there are some difficulties when lifting properties to the components and properties of the components are not well preserved by the limit. Convergence space inductive limits, on the other hand, show remarkable permanence properties and properties of the limit are easily lifted to the component spaces.

Another problem when dealing with topological vector spaces is that there is no natural topology for the dual of a locally convex topological vector space. Although there are many different possible topologies that can be defined on the dual, for example the strong, the Mackey, the weak and the weak* topologies to name but a few, each of which has its advantages and disadvantages.

Convergence structures, through the continuous convergence structure, provide a beautiful dual structure for locally convex topological vector spaces. In fact, it was shown [20] that every complete locally convex topological vector space is continuously reflexive, that is, every complete locally convex topological vector space is isomorphic to its bidual, equipped with the continuous convergence structure. Moreover, if the space is not complete, it can be embedded as a dense subspace of its bidual which is isomorphic to the completion of the original space. This was the first major result for convergence spaces in functional analysis.

One of the most powerful tools available in normed or metrizable spaces is the use of sequential arguments. In the more general setting of a topological vector space, one must impose strong countability conditions on the space in order to make use of these methods. But these conditions usually imply that the space is a Fréchet space. Therefore sequential methods are rarely accessible to us there. The countability conditions in convergence vector spaces are much more lax so that sequences come into play quite naturally in many instances where they do not suffice in the topological case. In particular, Beattie and Butzmann showed in [15] that the space $\mathcal{D}_c(\Omega)$ of all test functions with compact support on an open subset Ω of $\mathbb{R}^n$ is second countable, as is the space of test functions $\mathcal{E}_c(\Omega)$, considered with the continuous convergence structure. They also showed that the spaces $\mathcal{L}_c\mathcal{E}_c(\Omega)$ and $\mathcal{L}_c\mathcal{D}_c(\Omega)$ of distributions and distributions with compact support are second countable when considered with the continuous convergence structure.

If, for a given set K, a mapping σ from K into the powerset of the set of all sequences on K that does not satisfy (MS3) and (MS4) is given, it is natural to ask whether or not there exists a convergence structure λ that induces σ The characterization of such spaces is due to Butzmann, Beattie and Herrlich [16]. Since this result will be applied in the current work we will discuss it in some detail.

Definition 1.6 *A mapping σ from a set K into the powerset of all sequences on K is called a sequential convergence structure and the pair (K, σ) a sequential convergence space if the following conditions are satisfied:*
(i) The constant sequence with value f belongs to $\sigma(f)$.
(ii) If a sequence belongs to $\sigma(f)$ then so does each of its subsequences.

Every convergence structure λ on a set K induces a sequential convergence structure σ_λ on K in the following way. A sequence (f_n) on a convergence space (K, λ) converges to $f \in K$ if and only the Fréchet filter generated by (f_n), that is, the filter

$$\langle (f_n) \rangle = [\{\{f_n : n \geq k\} : k \in \mathbb{N}\}] \tag{1.4}$$

converges to f. The sequence (f_n) is said to belong to $\sigma_\lambda(f)$ whenever it converges to f.

For a given sequential convergence space (K, σ) the solution to the question posed above, that is, the existence or not of a convergence structure λ on K that induces σ, was solved in [16]. The solution is stated in terms of category theory, but an alternative version of the result can be found in [15]. We give the statement in the latter form.

Theorem 1.2 *Let (K, σ) be a sequential convergence space. Then there exists a convergence structure λ on K such that $\sigma = \sigma_\lambda$ if and only if the following is true for all $f \in K$ and all sequences $(f_n), (g_n)$ on K:*
(i) If (f_n) belongs to $\sigma(f)$ and $\langle (f_n) \rangle = \langle (g_n) \rangle$ then (g_n) belongs to $\sigma(f)$;
(ii) If (f_n) belongs to $\sigma(f)$ and (g_n) belongs to $\sigma(f)$ then $(f_n) \lozenge (g_n)$ belongs to $\sigma(f)$.
Here $(f_n) \lozenge (g_n)$ denotes the trivial mixing of (f_n) and (g_n), i.e. $((f_n) \lozenge (g_n))_{2n-1} = f_n$ and $((f_n) \lozenge (g_n))_{2n} = g_n$ for all $n \in \mathsf{N}$.

A sequential convergence space that satisfies the conditions of Theorem 1.2 is called an FS-space. For an arbitrary FS-space (K, σ) the convergence structure λ on K that induces σ, that is, the convergence structure λ such that $\sigma_\lambda = \sigma$ need not be unique. In fact, even if σ is topological, that is, it satisfies the Moore-Smith Axioms (MS1) through (MS4), the topology that induces σ is not uniquely determined. The plurality of convergence structures associated with σ is due to the failure of convergence spaces, in general, to be completely described by sequential convergence. However, if suitable countability conditions are imposed on the required convergence structure λ the uniqueness of such a structure follows quite naturally.

1.5 The Order Convergence Structure

As it was stated earlier, we are interested in order convergence on a lattice, and in particular on a vector lattice. On a lattice $(L, \leq)$ or a vector lattice $(E, \leq)$ there are various ways in which to define a notion of convergence of sequences arising from the order on L ,respectively E.

It is well known, see [52] or [78], that order convergence is a Hausdorff (separated) sequential convergence structure in the sense of Definition 1.6, which we denote by σ_o. In general order convergence is non-topological in that it fails to satisfy axioms (MS3) and (MS4) of the Moore-Smith Axioms. In fact, to require that order convergence on an Archimedean vector lattice E satisfy (MS4) is so strong as to force the convergence to be 'uniform', see [52] and [80].

Example 1.3 *Consider the space $\mathcal{C}(\mathsf{R})$ of all continuous real valued functions on R. It is well known that $\mathcal{C}(\mathsf{R})$ is an Archimedean vector lattice when considered with the pointwise operations and order. Define the sequence (f_n) on*

$\mathcal{C}(\mathsf{R})$ in the following way. Divide the interval $[0,1]$ in half. Then divide each of the intervals $\left[0,\frac{1}{2}\right]$ and $\left[\frac{1}{2},1\right]$ in half and continue this process indefinitely to obtain a sequence of intervals $(I_n)=[a_n,b_n]$. Let, for each $n\in\mathsf{N}$, the function f_n be given by

$$f_n(x)=\begin{cases}1-n(a_n-x) & \text{if } a_n-\frac{1}{n}\leq x<a_n\\ 1 & \text{if } x\in I_n\\ 1-n(x-b_n) & \text{if } b_n<x\leq b_n+\frac{1}{n}\\ 0 & \text{otherwise}\end{cases}.$$

Clearly, for every $x\in[0,1]$, there are an infinite number of terms in the sequence $(f_n(x))$ such that $f_n(x)=1$. This implies that the sequence (f_n) can not order converge to 0. Now take any subsequence (f_{n_k}) of (f_n) and let (I_{n_k}) be the sequence of intervals associated with (f_{n_k}) as above. At least one of the intervals I_1 or I_2 must contain infinitely many of the intervals I_{n_k}. Assume that this is the case for I_1. Again, at least one of the intervals I_3 or I_4 must contain infinitely many of the intervals I_{n_k}. Assume that this is the case for I_3. In this way we obtain a sequence $\left(I_{n_j}\right)$ such that

$$I_{n_{j+1}}\subset I_{n_j},\, j\in\mathsf{N}. \tag{1.5}$$

Let $f_{n_{k_j}}$ be the first term in (f_{n_k}) such that $I_{n_{k_j}}\subset I_{n_j}$. Define the sequence $(\mu_{n_{k_j}})$ by

$$\mu_{n_{k_j}}(x)=f_{n_j}(x).$$

Using (1.5) it can easily be seen that the sequence $(\mu_{n_{k_j}})$ decreases to 0 and satisfies $f_{n_{k_j}}\leq\mu_{n_{k_j}}$ so that $(f_{n_{k_j}})$ order converges to 0. Therefore order convergence on $\mathcal{C}(\mathsf{R})$ does not satisfy Axiom (MS3) of the Moore-Smith Axioms.

The 'topological properties' of order convergence of filters, and in particular its relation to certain topologies induced by the order relation, has been studied, amongst others, in [27] for arbitrary posets and in [26] for complete lattices.

Of interest to us is the paper [65] where the topological completion of commutative lattice groups under order convergence (in our sense) of nets and sequences were discussed. The term 'topological completion' is very loosely used there. Firstly, order convergence is in general non-topological. Secondly, he constructs a 'completion' of a sequential convergence group. This is, in general, of little interest as there are examples of sequential convergence groups having no completion, see [33].

There have been some attempts to define convergence structures on a vector lattice, see [28] and [37]. However, none of these structures induce sequential convergence identical with the order convergence of sequences. The problem of finding a convergence structure, if it exists, that induces order convergence of sequences on a vector lattice remained an open problem. To formulate it precisely, the question can be stated as follows: Given a vector lattice E, is it possible to define a convergence structure λ on E such that $\sigma_o=\sigma_\lambda$.

We will answer this question in the affirmative. In fact, we will prove a stronger result than this. We show that if L is a σ-distributive lattice, that is,

$$g \wedge \sup \{f_n : n \in \mathsf{N}\} = \sup \{g \wedge f_n : n \in \mathsf{N}\} \tag{1.6}$$

whenever the first supremum exists, then the sequential convergence space (L, σ_o) is an FS-space. As a result of the strong distributive properties of vector lattices, we will obtain the corresponding result for vector lattices as a corollary to the above.

Having shown that for any σ-distributive lattice L there exists a convergence structure λ that induces σ_o, we are interested in obtaining an explicit description of λ. As we discussed in Section 1.4, there is, however, no unique convergence structure that induces σ_o. We define a mapping λ_o from L into the powerset of the set of all filters on L by requiring that $\mathcal{F} \in \lambda_o(f)$ whenever there is a coarser filter with a countable basis consisting of appropriate order intervals

$$[\lambda, \mu] = \{g \in E : \lambda \leq g \leq \mu\}$$

We will call the mapping λ_o 'order convergence structure' and deduce the following fundamental properties:

- The mapping λ_o defines a convergence structure on L.
- The convergence structure λ_o induces the sequential convergence structure σ_o.
- The convergence structure λ_o is first countable.
- The convergence structure λ_o is regular and Hausdorff.

Definition 1.7 *Let K be a convergence space.*
(i) K is said to be first countable if, for each filter converging to a point f, there is a coarser one with a countable basis that still converges to f.
(ii) K is said to be first countable if there exists a countable basis at each $f \in E$, that is, for every $f \in E$ there exists a countable collection $\mathcal{B}$ of subsets of K such that for every filter $\mathcal{F}$ that converges to f there is a coarser filter with a basis from $\mathcal{B}$.

Definition 1.8 *A convergence space K is said to be*
(i) T_1 if every finite point set is closed;
(ii) Hausdorff if $\mathcal{F} \rightarrow f$ and $\mathcal{F} \rightarrow g$ imply that $f = g$;
(iii) regular if $a(\mathcal{F})$, the filter generated by $\{a(F) : F \in \mathcal{F}\}$ converges to a point $x \in X$ whenever $\mathcal{F}$ converges to x.

As is the case for topological spaces the implications

$$T_1 + \text{Regular} \Rightarrow \text{Hausdorff} \Rightarrow T_1 \tag{1.7}$$

With every convergence structure one can associate a topology that is in some sense 'closest' to it. This topology is introduced through the concept of an open set, an extension of the familiar topological notion.

Definition 1.9 *Let (K, λ) be a convergence space.*
(i) For every $f \in K$ the filter

$$\mathcal{V}_f = \bigcap \{\mathcal{F} : \mathcal{F} \in \lambda(f)\}$$

is called the neighbourhood filter at f and its elements are called the neighbourhoods of f.
(ii) A set $U \subseteq K$ is said to be open if it is a neighbourhood of each of its points.
(iii) For each subset A of K the set

$$a(A) = \{f \in K : \exists \mathcal{F} \in \lambda(f) \text{ on } K \text{ such that} A \in \mathcal{F}\}$$

is called the adherence of A.
(iv) A set $A \subseteq K$ is called closed if $a(A) = A$.

Although the concepts defined above coincide with the topological notions of 'open set', 'closed set' and 'closure' when the convergence structure is induced by a topology as in Example 1.1, there are, in general, significant differences. The adherence operator is in general not indempotent so that the adherence of a set is not necessarily closed while the neighbourhood filter of a point need not converge. These phenomena are not at all pathological and quite characteristic of convergence spaces.

With every convergence structure we now associate the following topology.

Definition 1.10 *Let K be a convergence space. A set U is open in the associated topology of K if it is an open set in the sense of Definition 1.9 (ii). We call the set K equipped with the associated topology the topological modification of K and denote it by $o(K)$.*

It is well known [19] that the following topology, called the order topology, is the finest topology on a poset that preserves order convergence.

Definition 1.11 *Let P be a poset. We say that a subset U of P is open in the order topology if for every $f \in U$ and every sequence (f_n) on P such that (f_n) order converges to f there exists $N \in \mathbb{N}$ such that $f_n \in U$ for every $n \geq N$. We denote the order topology by τ_o.*

It is clear that there exists a strong connection between order convergence structure λ_o on a σ-distributive lattice L and the order topology. We will show in Section 2.2 that the order topology is in fact the associated topology of λ_o. This result is representative of a number of statements that are 'new' in terms of the convergence structure λ_o, but are simply known results that are restated in terms of convergence structures. The associated topology is of interest mainly because it has exactly the same continuous functions (into a topological space) than the convergence structure with which it is associated. Recall the following definitions.

Definition 1.12 *Let K and L be convergence spaces and $\varphi : K \to L$ a mapping.*
(i) φ is continuous at $f \in K$ if $\varphi(\mathcal{F})$ converges to $\varphi(f)$ in L whenever $\mathcal{F}$ converges to f in K. It is continuous on K if it is continuous at every $f \in K$.
(ii) φ is sequentially continuous at $f \in K$ if $(\varphi(f_n))$ converges to $\varphi(f)$ in L whenever (f_n) converges to f in K. It is sequentially continuous on K if it is continuous at every $f \in K$.
(iii) φ is an embedding if it is continuous, injective and has a continuous inverse.
(iv) φ is a homeomorphism if it is a surjective embedding.

We now focus our attention on order convergence on a vector lattice. As a consequence of the sequential continuity of the group structure of a vector lattice with respect to order convergence, a fact that is well known [52], it is also continuous with respect to order convergence structure. In general, scalar multiplication is not continuous and hence some further assumption on the vector lattice is required for order convergence structure to be compatible with the linear structure. Indeed, we show that this holds true if and only if the space is Archimedean.

In the case of a vector lattice, the associated topology of order convergence structure, the order topology, is particularly well behaved as it is compatible with the group structure of the space, see Section 2.2. Moreover, if the space is Archimedean, we show that scalar multiplication is also continuous. This is not simply a consequence of a general result for convergence vector spaces, as there are examples of convergence vector spaces where the associated topology is not compatible with the linear structure, see [15][Remark 4.3.31]. This result implies very strong separation properties of order convergence structure. In particular, we will show that, for a not necessarily Archimedean vector lattice E the convergence space (E, λ_o) is functionally regular and functionally Hausdorff.

Definition 1.13 *Let K be a convergence space. Then K is said to be:*
(i) functionally regular if, for each filter $\mathcal{F}$ that converges to $f \in K$, the filter

$$\overline{\mathcal{F}}^{\sigma} = \left[\left\{\overline{F}^{\sigma} : F \in \mathcal{F}\right\}\right],$$

where $\overline{F}^{\sigma}$ denotes the closure of F in the initial topology with respect to $\mathcal{C}(K)$ on K, also converges to $f \in K$;
(ii) functionally Hausdorff if the initial topology on K with respect to $\mathcal{C}(K)$ is Hausdorff.

Definition 1.14 *Let K be a convergence space. A filter $\mathcal{F}$ on K converges to $f \in K$ in the initial topology σ with respect to $\mathcal{C}(K)$ on K whenever $\varphi(\mathcal{F})$ converges to $\varphi(f)$ for every $\varphi \in \mathcal{C}(K)$.*

The importances of the concepts defined above can be seen in their application to the characterization of an important class of convergence spaces, the class of c-embedded convergence spaces, see [15]. Order convergence structure in general does not belong to this class as it requires quite a strong restriction on the nature of convergence in the space, namely, the space must also be Choquet, see [24].

As mentioned above, if the vector lattice E is not Archimedean, then order convergence structure is not compatible with the underlying linear structure of the vector lattice E. The concept of a convergence vector space is a generalization of the concept of a topological vector space, the latter being a generalization of normed spaces. Normed spaces have been in use since Banach's 1928 book, see [10]. The powerful results established there served to motivate mathematicians to seek a means to generalize the concept of a normed space. It was not until some twenty years later that topological vector spaces were considered as a means to such a generalization. It is perhaps surprising that such a long period of time elapsed before this generalization was made as the machinery of general topology had been available since the appearance of Hausdorff's 1914 book. The further extension of the idea of a topological vector space came about in the late 1950s and the 1960s as a convergence vector space.

Definition 1.15 *A convergence structure λ on a real vector space E is called a vector space convergence structure and the pair (E, λ) a convergence vector space whenever the mappings*

$$+ : E \times E \rightarrow E$$

and

$$\cdot : \mathsf{R} \times E \rightarrow E$$

such that $+ : (f, g) \longmapsto f + g$ and $\cdot : (\alpha, f) \longmapsto \alpha f$ are continuous.

In Definition 1.15 above, the products $E \times E$ and $\mathsf{R} \times E$ are equipped with the product convergence structure which is defined as follows.

Definition 1.16 *Let (K_i) be a family of convergence spaces. A filter $\mathcal{F}$ on $\prod_{i \in I} K_i$ converges to $f \in \prod_{i \in I} K_i$ in the product convergence structure if, for all $i \in I$, there are filters $\mathcal{F}_i \rightarrow f(i)$ in K_i such that $\mathcal{F} \supseteq \prod_{i \in I} \mathcal{F}_i$. Here $\prod_{i \in I} \mathcal{F}_i$ denotes the Tychonoff product of the filters $\mathcal{F}_i$, that is, the filter based on*

$$\left\{ \prod_{i \in I} F_i : F_i \in \mathcal{F}_i \text{ for all } i \in I,\ F_i \neq K_i \text{ for only finitely many } i \in I \right\}. \quad (1.8)$$

The motivation for the generalization of topological vector spaces came by and large from functional analysis and its applications, in particular form the theory of PDEs, see for instance [34] and [11]. Indeed, it came to light that the classical topology fails to describe certain very interesting notions of convergence. The most prominent example to date is the continuous convergence structure [18] that has been shown to be very useful in applications, see [20] and [15].

As far as the order convergence structure on a vector lattice is concerned it has already been stated that topolocigal spaces, and hence topological vector spaces, fails give an adequite desciption of order convergence. In Section 2.1 we show that if the vector lattice E is Archimedean, then order the convergence structure is indeed a vector space convergence structure. Convergence vector spaces therefore provides an appropriate context for the study of order convergence on Archimedean vector lattices where topological vector spaces fail to do so. Convergence vector spaces and order convergence have never been associated in this way before and in this lies the novelty of our approach. Moreover, it is shown that the convergence vector space (E, λ_o) is both locally convex and locally bounded.

Definition 1.17 *Let E be a convergence vector space.*
(i) A subset B of E is bounded whenever the filter $\mathcal{N}B$ converges to 0.
(ii) A filter $\mathcal{F}$ on E is called bounded whenever the filter

$$\mathcal{NF} = [\{NF : N \in \mathcal{N}, F \in \mathcal{F}\}]$$

converges to 0.
(iii) E is locally bounded if and only if every filter that converges in E contains a bounded set.

Definition 1.18 *A convergence vector space E is locally convex if for every filter $\mathcal{F}$ which converges to 0 in E the filter $co(\mathcal{F})$, the filter generated by $\{co(F) : F \in \mathcal{F}\}$, also converges to 0. Here $co(F)$ denotes the convex hull of F.*

It should be noted that local convexity is not nearly as strong a property in the convergence vector space setting as it is for the topological case.

The properties associated with bounded sets, on the other hand, are far more significant as it is related to the problem of completing a given convergence vector space. The key to solving this problem is that every Cauchy filter must be bounded. In fact, in our definition of the order convergence structure λ_o this is specifically kept in mind. Hence the use of order bounded sets in the definition of a convergent filter.

Definition 1.19 *Let E be a convergence vector space.*
(i) A filter $\mathcal{F}$ on E is a Cauchy filter if the filter $\mathcal{F} - \mathcal{F}$ converges to 0 in E.
(ii) A sequence (f_n) on E is a Cauchy sequence if $\langle (f_n) \rangle$ is a Cauchy filter.
(iii) E is a complete convergence vector space if every Cauchy filter on E converges to some $f \in E$.

For a given convergence vector space E, the objective is to construct a convergence vector space $\widetilde{E}$ with the following properties:

C1 $\widetilde{E}$ is a complete Hausdorff convergence vector space.

C2 There exists a linear embedding $i : E \to \widetilde{E}$ such that $i(E)$ is dense in $\widetilde{E}$ in the sense that $a(i(E)) = \widetilde{E}$ where the adherence is taken in $\widetilde{E}$.

C3 If F is a complete Hausdorff convergence vector space and $T : E \to F$ a continuous linear mapping, then there exists a continuous linear mapping $\widetilde{T} : \widetilde{E} \to F$ such that $T(f) = \widetilde{T}(i(f))$ for every $f \in E$.

In general, such a construction is not possible, see [35] where a characterization of those space that can be completed in this way is given. Here the properties of the convergence vector space E as related to boundedness are crucial. In fact, if E is a Hausdorff convergence vector space, then there exists a convergence vector space $\widetilde{E}$, called the completion of E, that satisfies conditions (C1) through (C3) if and only if every Cauchy filter on E is bounded, see [35].

Indeed, let $\mathcal{F}$ be a Cauchy filter on E that is not bounded and suppose that there exists a completion $\widetilde{E}$ of E. Since $\widetilde{E}$ is complete, the image filter $i(\mathcal{F})$ must converge to some $\widetilde{f} \in \widetilde{E}$. But every convergent filter is bounded so that $\mathcal{N}i(\mathcal{F}) = i(\mathcal{N}\mathcal{F})$ must converge to 0 in $\widetilde{E}$. But since i is an embedding, it follows that $\mathcal{N}\mathcal{F}$ converges to 0 in E, a contradiction.

In the case of an Archimedean vector lattice E equipped with the order convergence structure, we prove the following results in Section 2.3. First we show that there exists a complete convergence vector space $\widetilde{E}$ and a mapping $i : (E, \lambda_o) \to \widetilde{E}$ that satisfies the conditions (C1) through (C3). As there are examples of Hausdorff convergence vector spaces that can not be completed, this result is far from being trivial. Having established the existence of the completion of (E, λ_o) it is of interest to find a concrete representation of the completion $\widetilde{E}$. The abstract construction of the space $\widetilde{E}$ follows closely the procedure for completing a topological vector space, see for instance [46]. The second result settles this question in a rather elegant way. We relate the concepts of completeness and completion in terms of the order on E to the convergence space completeness of (E, λ_o).

Definition 1.20 *Let E be a vector lattice.*
(i) E is said to be Dedekind σ-complete if every non-empty countable set that is bounded from above has a least upper bound.
(ii) E is Dedekind complete if every non-empty subset of E that is bounded from above has a least upper bound in E.
(iii) E is called order separable if every non-empty subset D of E possessing a supremum contains an at most countable subset possessing the same supremum as D.
(iv) E is called super Dedekind complete if it is Dedekind complete and order separable.

Indeed, it is shown that the space $\widetilde{E}$ is isomorphic, in the sense of convergence vector spaces, to the space $(E^{\#}, \lambda_o)$, where $E^{\#}$ denotes the Dedekind σ-completion of E, that is, the smallest Dedekind σ-complete vector lattice that contains E as a vector lattice subspace. Incidentally, in the coarse of studying the completion problem for (E, λ_o) we also solve a problem from the theory of vector lattices. Zaanen [83] defines a notion of an order Cauchy sequence on a vector lattice as follows.

Definition 1.21 *Let E be a vector lattice.*
(i) A sequence (f_n) on E is order Cauchy whenever there exists a sequence (μ_n) that decreases to 0 such that for every $n \in \mathbb{N}$ there exists $N_n \in \mathbb{N}$ such that

$$|f_m - f_k| \leq \mu_n \tag{1.9}$$

for every $k, m \geq N_n$.
(ii) E is order complete if every order Cauchy sequence order converges in E.

In Section 2.3 we show that an Archimedean vector lattice E is order complete if and only if E is Dedekind σ-complete.

Given a convergence space (K, λ) and a subset A of K, the convergence structure λ on the larger set K induces a convergence structure on A in a natural way.

Definition 1.22 *Let K be a convergence space and A a subset of K. The subspace convergence structure on A is the initial convergence structure with respect to the inclusion mapping $e : A \rightarrow K$. A filter $\mathcal{F}$ on A converges to f in A if and only if $[\mathcal{F}_K]$ converges to f in K.*

When the convergence structure on K is defined through some special structure on K, the subset A usually inherits this structure. Hence we can define a convergence structure on A with respect to the additional structure. However, the convergence structure so defined on A does not necessarily coincide with the subspace convergence structure induced on A by (K, λ). Generally one must make some additional assumptions on A to insure that the subspace structure is identical with the convergence structure defined on A independent of K.

Indeed, for the order convergence structure on an Archimedean vector lattice such problems also arise. We study this problem through the concepts of ideals, σ-ideals and bands.

On an algebraic ring it is of great interest to study ring ideals. For a vector lattice one can define a notion of ideal analogous to that encountered in ring theory, see [83]. Recall that a subring of a ring is an ideal if it absorbs products. An (order) ideal in a vector lattice is defined similarly in requiring that the vector sublattice 'absorbs' elements under the order relation.

Definition 1.23 *Let E be a vector lattice and A a linear subspace of E.*
(i) A is called a vector sublattice of E if it is closed under the formation of finite suprema.
(ii) A is called an ideal in E if it is a vector sublattice and it is solid, that is, for any $f \in A$ and any $g \in E$ such that $|g| \leq |f|$ it follows that $g \in A$.
(iii) A is called a σ-ideal in E if it is an ideal and if it is closed under the formation of countable suprema.
(iv) A is called a band in E if it is an ideal and if it is closed under formation of arbitrary suprema.

The structure of ideals and bands of a vector lattice is much studied in the literature, see for instance [52] and [83]. The motivation for such a study stems from the close relationship that exist between ideals and Riesz homomorphisms.

Definition 1.24 *Let E and F be vector lattices and $\pi : E \to F$ monotone linear mapping, that is, $\pi f \leq \pi g$ in F whenever $f \leq g$ in E.*
(i) The mapping π is a Riesz homomorphism if

$$\pi(f \vee g) = (\pi f) \vee (\pi g)$$

for all $f, g \in E$.
(ii) The mapping π is a Riesz isomorphism if it is a bijective Riesz homomorphism and its inverse T^{-1} is monotone.

Indeed, the kernel

$$\ker \pi = \{f \in E : \pi f = 0\}$$

of a Riesz homomorphism is an ideal in E and the ideal-properties of $\ker \pi$ serve to characterize some further properties that π might satisfy.

Another motivation for the use of bands is that, for a large class of vector lattices, it is possible to represent the space as the direct sum of two bands A and A^d that are 'orthogonal' in the sense that

$$|f| \wedge |g| = 0$$

for every $f \in A$ and $g \in A^d$, and $A \cap A^d = \{0\}$. Hence every $f \in E$ can be decomposed in a unique way as a sum

$$f = f_1 + f_2$$

where $f_1 \in A$ and $f_2 \in A^d$. For the details the reader is referred to [52][Section 2.4] or any text on vector lattices.

As is the case in ring theory, one can define for an ideal A of a vector lattice E the quotient vector lattice E/A. This is the (algebraic) quotient space of all equivalence classes modulo A. The element of E/A containing $f \in E$ will be denoted $[f]$. Hence we have $[f_1] = [f_2]$ if and only if $f_1 - f_2 \in A$. The space E/A is made into a vector lattice in the following way.

Definition 1.25 *Let E be a vector lattice and A an ideal of E. Then $[f] \leq [g]$ in E/A whenever there exists $f_1 \in [f]$ and $g_1 \in [g]$ such that $f_1 \leq g_1$.*

The interest in the quotient vector lattice is due to the fact that if $\pi : E \rightarrow F$ is a Riesz homomorphism, then $E/\ker\pi$ is Riesz isomorphic to F.

We are interested in the behavior of ideals of the Archimedean vector lattice E when considered as subspaces of (E, λ_o), as well as the relationship between order convergence structure on E/A and the quotient convergence structure induced on E/A by (E, λ_o).

Definition 1.26 *Let K be a convergence space, A a set and a surjection $q : K \rightarrow A$. A filter $\mathcal{F}$ converges in the quotient convergence structure on A to a point $g \in A$ if and only if there are points $f_1, ..., f_n \in K$ and for each k a filter $\mathcal{F}_k$ which converges to f_k such that $q(f_k) = g$ for all k and $\mathcal{F} \supseteq q(\mathcal{F}_1) \cap ... \cap q(\mathcal{F}_n)$.*

In general, order convergence on a subspace A, and even on an ideal, does not coincide with the convergence induced on A by order convergence on E. If, however, A is a band and E is Dedekind complete, the situation is much improved. It is shown in Section 2.4 that if E is Dedekind σ-complete the ideal A of E is a closed subspace of (E, λ_o) if and only if A is a σ-ideal. Moreover, if A is a band and E is Dedekind complete, then A is a closed subspace of (E, λ_o) and the order convergence structure on A, when considered as a vector lattice on its own, coincides with the subspace structure it inherits from (E, λ_o). We also show that if E is Dedekind σ-complete and A is a σ-ideal, then sequential convergence induced from E coincides with order convergence of sequences on A.

Let us now fix, once and for all, the following notation. If E is a vector lattice and G is a vector sublattice of E, then by (G, λ_o) we mean the set F equipped with order convergence structure as defined on it as a vector lattice in its own right. If we refer to the subspace convergence structure inherited from E, we will explicitly mention it.

Regarding the quotient space E/A, we will show in Section 2.4 that order convergence of sequences on E/A coincides with sequential convergence with respect to the quotient convergence structure on E/A, as induced from E by the quotient mapping $q : E \rightarrow E/A$ defined by $q(f) = [f]$, whenever E is Dedekind σ-complete and A is a σ-ideal. The restriction on the completeness of E and A is necessary for two reasons. Firstly, if A is not ru-complete, then the quotient E/A need not be Archimedean. Therefore the order convergence structure will not be a vector space convergence structure, while the convergence space quotient is a convergence vector space. Secondly, if A is not a σ-ideal, then the projection mapping $\pi_A : E \rightarrow E/A$, mapping f into $[f]$, is not necessarily continuous.

As stated earlier, it is shown that the bounded subsets of (E, λ_o) are exactly the order bounded subset of E. This fact becomes particularly relevant when applied to the study of bounded operators between (E, λ_o) and (F, λ_o). In order to study bounded operators on a convergence vector space one considers the following convergence structure, called the Mackey modification.

Definition 1.27 *Let E be a convergence vector space and denote by $\mathcal{B}$ the set of all bounded subsets of E. (i) A filter $\mathcal{F}$ converges to $f \in E$ in the Mackey convergence structure μ on E whenever there exists $B \in \mathcal{B}$ such that*

$$\mathcal{F} - f \supseteq \mathcal{N}B.$$

The vector space E equipped with the Mackey convergence structure is called the Mackey modification of E and is denoted $\mu(E)$.
(ii) Way say that E is a Mackey space if $\mu(E) = E$.

Clearly $\mu(E)$ and E share the same bounded sets, $\mu(E)$ is first countable and locally bounded. In Section 2.5 we use the fact that $\mu(E)$ is first countable to characterize those Archimedean vector lattices for which $\mu(E)$ is complete in terms of relatively uniform convergence.

Definition 1.28 *Let E be a vector lattice. A sequence (f_n) on E converges relatively uniformly (ru) to $f \in E$ whenever there exists $\mu \in E^+$ such that for every $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ so that*

$$|f - f_n| < \varepsilon\mu$$

for every $n \geq N_\varepsilon$.

Note that (ru) convergence of a sequence implies order convergence to the same limit, but the converse of this does not necessarily hold. Indeed, consider the following simple example.

Example 1.4 *Consider the set $\mathcal{C}(\mathbb{R})$ of all continuous functions on the real line equipped with pointwise operations and order. This defines on $\mathcal{C}(\mathbb{R})$ the structure of an Archimedean vector lattice. Define the sequence (f_n) as*

$$f_n(x) = \begin{cases} 1 - n|x| & \text{if } -\frac{1}{n} \leq x \leq \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}.$$

The sequence (f_n) is positive and decreasing. Moreover, for every $x \neq 0$ the sequence $(f_n(x))$ decreases to 0. Therefore the sequence (f_n) decreases to 0, and hence it order converges to 0.
On the other hand, $f_n(0) = 1$ for every $n \in \mathbb{N}$ so that (f_n) can not converge relatively uniformly to 0.

Also, if the space E is not Archimedean, then the limit of a (ru) convergent sequence need not be unique. For Archimedean spaces the situation is much better and limits are unique.

Definition 1.29 *Let E be an Archimedean vector lattice.*
(i) A sequence (f_n) on E is called (ru) Cauchy if there exists $\mu \in E^+$ so that for every $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that

$$|f_n - f_m| < \varepsilon\mu$$

for every $n \geq N_\varepsilon$.
(ii) E is said to be (ru) complete if every (ru) Cauchy sequence converge (ru) to some $f \in E$.

We show that the convergent sequences with respect to the Mackey convergence structure are exactly the relatively uniformly convergent sequences on E. Moreover, we show that $\mu(E)$ is complete if and only if E is (ru) complete.

In general the Mackey modification of a convergence vector space has a relatively simple structure compared with the origenal convergence structure. In the case of the order convergence structure this is also the case. In particular, we show in Section 2.5 that the Mackey modification of $\mu(E)$ of (E, λ_o) is strongly first countable whenever E posseses a strong order unit.

Definition 1.30 *Let E be a vector lattice. An element $e \in E^+$ is called a strong order if there exists for every $f \in E^+$ a positive integer N_f such that*

$$f \leq N_f e.$$

In general the Mackey convergence structure does not satisfy such a strong countability condition and hence our result is by no means a triviality.

1.6 Continuous and Bounded Operators

Since we are essentially performing a functional analysis on linear spaces it is appropriate to consider the linear mappings between such spaces. Various classes of linear mappings between vector lattices have been studied extensively, particularly when the spaces are Banach lattices, that is, there is a norm on the space that is compatible with the order structure, and the space is norm complete, or the codomain is Dedekind complete We will consider the general case of two Archimedean vector lattices E and F equipped with the order convergence structure. We are specifically interested in the following classes of operators, see [83].

Definition 1.31 *Let E and F be vector lattices and $T : E \to F$ a linear mapping.*
(i) We call T order bounded if the image $T(A)$ of every order bounded subset A of E under T is an order bounded subset of F. The set of all order bounded

operators from E into F is denoted $L_b(E,F)$.
(ii) We call T σ-order continuous if, for every sequence (f_n) on E that satisfies $f_n \downarrow 0$,

$$\inf\{|Tf_n| : n \in \mathbf{N}\} = 0$$

in F. The set of all σ-order continuous operators from E into F is denoted $L_c(E,F)$.

Note that an operator $T : E \to F$ is σ-order continuous whenever T maps order convergent sequences into order convergent sequences, thus motivating the terminology. The converse is true if T is positive or if F is Dedekind complete.

There is a natural way in which to order the space $L_b(E,F)$. Define the relation "$\leq$" on $L_b(E,F)$ by saying that

$$T \leq S \Longleftrightarrow \left(f \in E^+ \Longrightarrow (S-T)(f) \in F^+\right) \tag{1.10}$$

The order (1.10) makes $L_B(E,F)$ into a partially ordered vector space. The inclusion

$$L_c(E,F) \subseteq L_b(E,F)$$

therefore implies the same order structure on $L_c(E,F)$. In general, however, without further conditions on F, neither $L_b(E,F)$ nor $L_c(E,F)$ is a vector lattice and hence order convergence need not be induced by a convergence structure. In this general case we can therefore not consider the spaces $L_b(E,F)$ and $L_c(E,F)$ with the order convergence structure. On the other hand, if F is Dedekind complete, then $L_b(E,F)$ becomes a Dedekind complete vector lattice and $L_c(E,F)$ is a band in $L_b(E,F)$, and hence also a Dedekind complete vector lattice. In terms of the order convergence structure, we then have the following interpretation of the above.

If E and F are vector lattices with F Dedekind complete, then $(L_b(E,F), \lambda_o)$ is a complete convergence vector space and $(L_c(E,F), \lambda_o)$ is a closed subspace of $(L_b(E,F), \lambda_o)$. In particular, the spaces $(L_b(E), \lambda_o) = (L_b(E,\mathbf{R}), \lambda_o)$ and $(L_c(E), \lambda_o) = (L_c(E,\mathbf{R}), \lambda_o)$ are both complete convergence vector spaces.

We are interested in operators $T : E \to F$ that are either bounded or continuous with respect to the order convergence structure on the Archimedean vector lattices E and F.

Definition 1.32 *Let E and F be convergence vector spaces. The operator $T : E \to F$ is said to be bounded if the image $T(B)$ of any bounded subset B of E under T is a bounded subset of F.*

Unlike in the case of a normed space, boundedness of an operator does not imply continuity. Indeed, even for locally convex topological vector spaces E and F the implication

$$T \text{ bounded} \Longrightarrow T \text{ continuous}$$

does not generally hold. The inverse implication, however, still remains true for convergence vector spaces, that is, if $T : E \to F$ is an operator, then

$$T \text{ continuous} \Longrightarrow T \text{ bounded.}$$

We denote the set of all continuous operators from E into F by $\mathcal{L}(E, F)$, and if the space is equipped with the continuous convergence structure, it is denoted $\mathcal{L}_c(E, F)$. In particular, we write $\mathcal{L}(E)$ for $\mathcal{L}(E, \mathsf{K})$ and $\mathcal{L}_c(E)$ to mean $\mathcal{L}_c(E, \mathsf{K})$. If instead $\mathcal{L}(E)$ is considered with the weak* topology, that is, the topology induced by the semi norms

$$\rho_f : \varphi \to |\varphi(f)|$$

where the parameter f runs through all of $E \setminus \{0\}$, we denote the space $\mathcal{L}_s(E)$.

As far as the set $\mathcal{L}(E, F)$ is concerned, where both E and F are Archimedean vector lattices equipped with the order convergence structure, the main result that we obtain is that $\mathcal{L}(E, F) = L_c(E, F)$ whenever F is Dedekind complete, that is, the operator $T : E \to F$ is continuous if and only if it is sequentially continuous. In particular, it follows, therefore, that $\mathcal{L}(E)$ consists of all the σ-order continuous linear functionals. We will see an application of this fact when we consider the problem of embedding E into $\mathcal{L}(E)$.

This is by no means a triviality. In general first countability of a convergence space is not sufficient to ensure that the equivalence

$$T \text{ continuous} \Longleftrightarrow T \text{ sequentially continuous}$$

holds. A stronger countability condition on the codomain space is usually necessary.

If we consider $\mathcal{L}_c(E, F)$, the set $\mathcal{L}(E, F)$ equipped with the continuous convergence structure as a subspace of $\mathcal{C}_c(E, F)$, then we obtain a surprising result. If F is Dedekind complete, so that (F, λ_o) is complete, then the space $\mathcal{L}_c(E, F)$ is a complete convergence vector space.

Definition 1.33 *Let K and L be convergence spaces. Then a filter $\mathcal{F}$ on $\mathcal{C}(K, L)$ converges to $\varphi_0 \in \mathcal{C}(K, L)$ in the continuous convergence structure if and only if $\omega_{K,L}(\mathcal{F} \times \Phi)$ converges to $\varphi_0(f_0)$ whenever Φ converges to f_0 in K. Here*

$$\omega_{K,L} : \mathcal{C}(K, L) \times K \to L$$

denotes the evaluation mapping defined by $\omega_{K,L}(\varphi, f) = \varphi(f)$.

This is not the case for arbitrary convergence vector spaces E and F. For an example of complete, Hausdorff, regular convergence vector spaces E and F such that $\mathcal{L}_c(E, F)$ is not complete, see [22].

As mentioned earlier, the Mackey modification $\mu(E)$ of a convergence vector space E provides us with a useful tool with which to study bounded operators. In fact, the novelty of the Mackey convergence structure is that an operator

$T : \mu(E) \to \mu(F)$ is continuous if and only if it is bounded. Since E and $\mu(E)$ share the same bounded sets, it follows that the continuous operators between $\mu(E)$ and $\mu(F)$ are exactly the bounded operators from E into F. Furthermore, if E and F are Mackey spaces, then an operator $T : E \to F$ is continuous if and only if it is bounded.

Since, for an Archimedean vector lattice E the bounded subsets of (E, λ_o) is exactly the order bounded subsets of E, it follows that the continuous mappings from $\mu(E)$ into $\mu(F)$, where F is another Archimedean vector lattice, are exactly the order bounded operators from E into F, that is, $\mathcal{L}(\mu(E), \mu(F)) = L_b(E, F)$.

One of the cornerstone theorems of functional analysis is the Banach-Steinhaus theorem, also called the principle of uniform boundedness. This result has been known to hold for Banach spaces since the 1920s, see [10], and since then many variations of for more general spaces have appeared. The most common variant is the following: If E and F are locally convex topological vector spaces with E is barrelled, then every pointwise bounded subset of $\mathcal{L}(E, F)$ is equicontinuous.

Definition 1.34 *Let E and F be convergence vector spaces. Then a subset $H \subseteq \mathcal{L}(E, F)$ is equicontinuous if the filter*

$$H(\mathcal{F}) = [\{\{Tf : T \in H\} : F \in \mathcal{F}\}]$$

converges to 0 in F whenever the filter $\mathcal{F}$ converges to 0 in E.

Definition 1.35 *A convergence vector space E is called barrelled if every bounded subset of $\mathcal{L}_s(E)$ is equicontinuous.*

In [15] the scope of this result is greatly enlarged to include many convergence vector spaces. We apply the results obtained there to order convergence on an Archimedean vector lattice E and its Mackey modification $\mu(E)$. The first step in this direction is to show that both (E, λ_o) and $\mu(E)$ are barrelled. We then apply this result, together with those obtained in [15], to find sufficient conditions on a convergence vector space F such that $((E, \lambda_o), F)$ and $(\mu(E), F)$ are Banach-Steinhaus pairs.

Definition 1.36 *Let E and F be convergence vector spaces. The pair (E, F) is called a Banach-Steinhaus pair whenever every bounded subset of $\mathcal{L}_s(E, F)$ is equicontinuous.*

The results described above then give rise to a Banach-Steinhaus type theorem for σ-order continuous operators between Archimedean vector lattices. In particular, we obtain the following results:

BS Let E and F be Archimedean vector lattices, with F Dedekind complete such that L_cF separates the points of F. If the sequence (T_n) of σ-order continuous operators $T_n : E \to F$ converges pointwise, with respect to order convergence structure on E and F, to a linear mapping $T : E \to F$, then T is σ-order continuous.

Beside the Banach-Steinhaus theorems for convergence vector spaces, the results described above also rely on a certain relationship between the convergence vector space (E, λ_o) and its second dual space $\mathcal{L}(\mathcal{L}(E, \lambda_o), \lambda_o)$. We show that whenever the first dual separates the points of E, then there exists a vector sublattice of the second dual that is isomorphic, as a convergence vector space when equipped with order convergence structure to (E, λ_o). This is an application of the duality theorems for vector lattices that can be found in [81].

There have been previous attempts at proving Banach-Steinhaus type theorems for vector lattices, see for instance [73] where a result similar to (BS) is proved for order continuous operators. There, however, the further condition that the space E is of 'Grothendick type' is imposed. Furthermore, it is required that the limit operator already be order bounded. Our result neither implies Schaefer's result, nor does it follow from it. If, however, the space F is super Dedekind complete and of Grothendick type the two results are equivalent.

1.7 Hausdorff Continuous Functions

We consider functions on a topological space X with values extended closed real intervals, that is, functions $f : X \to \mathbb{I}\overline{\mathsf{R}}$. Here $\mathbb{I}\overline{\mathsf{R}}$ denotes the set

$$\mathbb{I}\overline{\mathsf{R}} = \left\{[\underline{a}, \overline{a}] : \underline{a} \leq \overline{a} \in \overline{\mathsf{R}}\right\} \tag{1.11}$$

where $\overline{\mathsf{R}} = \mathsf{R} \cup \{\pm\infty\}$. A partial order was defined on $\mathbb{I}\overline{\mathsf{R}}$ by Markov in [53] as

$$[\underline{a}, \overline{a}] \leq [\underline{c}, \overline{c}] \Leftrightarrow \underline{a} \leq \underline{c}, \overline{a} \leq \overline{c}. \tag{1.12}$$

With every interval $a \in \mathbb{I}\overline{\mathsf{R}}$ one can associate a nonnegative extended real number $\omega(a)$, the width of a, which is defined as

$$\omega(a) = \begin{cases} \overline{a} - \underline{a} & \text{if } \overline{a}, \underline{a} \in \mathsf{R} \\ \infty & \text{if } \overline{a} = +\infty \text{ or } \underline{a} = -\infty \\ 0 & \text{if } \overline{a} = \underline{a} = \pm\infty \end{cases}. \tag{1.13}$$

By the absolute value of an interval $a = [\underline{a}, \overline{a}]$ we mean the extended positive real number defined by

$$|a| = \max\{|\underline{a}|, |\overline{a}|\}. \tag{1.14}$$

We denote by $\mathsf{A}(X)$ the set of extended closed real interval valued functions on X, that is,

$$\mathsf{A}(X) = \left\{f : X \to \mathbb{I}\overline{\mathsf{R}}\right\}. \tag{1.15}$$

By viewing every extended real number as an interval $a = [a, a]$, $\overline{\mathsf{R}}$ can be treated as a subset of $\mathbb{I}\overline{\mathsf{R}}$ so that the set of extended real valued functions

$$\mathcal{A}(X) = \left\{f : X \to \overline{\mathsf{R}}\right\} \tag{1.16}$$

is contained in the set $\mathsf{A}(X)$. For any $f \in \mathsf{A}(X)$ and $\varepsilon > 0$ we denote by W_f^ε

$$W_f^\varepsilon = \{x \in X : \omega(f(x)) > \varepsilon\}.$$

The set

$$W_f = \{x \in X : \omega(f(x)) > 0\} \tag{1.17}$$

on which f assumes proper interval values can be represented as

$$W_f = \bigcup_{\varepsilon > 0} W_f^\varepsilon. \tag{1.18}$$

Using the partial order defined in (1.12) on $\mathbb{I}\overline{\mathbb{R}}$, a partial order that extends the usual one on $\mathcal{A}(X)$ may be defined on $\mathsf{A}(X)$ in a pointwise way as

$$f \leq g \Leftrightarrow f(x) \leq g(x), x \in X. \tag{1.19}$$

Note that since each $f \in \mathsf{A}(X)$ takes extended interval values, we may write these functions in an interval way as $f = [\underline{f}, \overline{f}]$ where $\underline{f} \leq \overline{f} \in \mathcal{A}(X)$. In [9] Baire defined the operators I and S, known respectively as the lower and upper Baire operators, for real valued functions of a real variable, but recently the definitions were generalized to extended closed interval valued functions on an arbitrary topological space, see [3]. The definitions may be written in the following way:

$$I(f)(x) = \sup_{V \in \mathcal{V}_x} \inf\{z \in f(y) : y \in V\}, x \in X \tag{1.20}$$

$$S(f)(x) = \inf_{V \in \mathcal{V}_x} \sup\{z \in f(y) : y \in V\}, x \in X \tag{1.21}$$

Here $\mathcal{V}_x$ denotes the set of neighbourhoods at $x \in X$. The mappings I and S are closely connected with the concept of semi-continuous functions as introduced by Baire [9], for real valued functions of a real variable, and subsequently generalized to more arbitrary spaces by several authors.

Definition 1.37 *A function $f \in \mathcal{A}(X)$ is called lower semi-continuous at $x \in X$ if for every $m < f(x)$ there exists $V \in \mathcal{V}_x$ such that $m < f(y)$ for all $y \in V$. If $f(x) = -\infty$, then f is assumed lower semi-continuous at x.*

Definition 1.38 *A function $f \in \mathcal{A}(X)$ is called upper semi-continuous at $x \in X$ if for every $m > f(x)$ there exists $V \in \mathcal{V}_x$ such that $m > f(y)$ for all $y \in V$. If $f(x) = +\infty$, then f is assumed upper semi-continuous at x.*

Semi-continuous functions are characterized as the fixed points of the operators I and S, that is,

$$f \text{ is lower semi-continuous on } X \Leftrightarrow I(f) = f \tag{1.22}$$

$$f \text{ is upper semi-continuous on } X \Leftrightarrow S(f) = f \tag{1.23}$$

From (1.20) through (1.21) it is clear that, for any $f \in \mathsf{A}(X)$ it holds that

$$I(f) \leq S(f). \tag{1.24}$$

From (1.24) it is evident that the mapping $F : \mathsf{A}(X) \to \mathsf{A}(X)$ defined by

$$F(f) = [I(f), S(f)] \tag{1.25}$$

is well defined. This mapping, called the Graph completion operator, was first defined by Sendov in [75] for the special case where X is a closed real interval, but the definition was again generalized to arbitrary topological spaces in [3]. Through the Graph completion operator a notion of continuity of interval valued functions was defined [75]. We recount this definition here as it will come into play on several occasions.

Definition 1.39 *A function $f \in \mathsf{A}(X)$ is said to be Sendov continuous, or S-continuous for short, if it is a fixed point of the graph completion operator, that is, if $F(f) = f$. We denote the set of S-continuous functions by $\mathsf{F}(X)$.*

Remark 1.1 *In the literature the S-continuous functions are often called segment continuous, or s-continuous for short. We adopt the current notation in homage of B. Sendov who first defined this notion of continuity for interval valued functions.*

Also in [75], Sendov defined the concept of a Hausdorff continuous closed extended interval valued functions of a real variable. Anguelov extended this definition to arbitrary topological spaces in [3]. The definition is in terms of a minimality condition imposed on the inclusion of interval functions. Alternative characterizations were obtained in [3], [78] and will be recounted in the appendix.

Definition 1.40 *A function $f \in \mathsf{A}(X)$ is called Hausdorff continuous, or H-continuous for short, if for every $g \in \mathsf{A}(X)$ satisfying the inclusion $g(x) \subseteq f(x), x \in X$, we have that $F(g)(x) = f(x), x \in X$. We denote the set of all H-continuous functions on X by $\mathsf{H}(X)$.*

Remark 1.2 *It is important to note that this concept of H-continuity differs greatly form the H-continuous functions that is encountered in set valued analysis.*

Note that at first glance the definition does not involve neighbourhoods of the points of X. However, it does involve the graph completion operator, the definition of which does involve these neighbourhoods. This ensures that a number of the properties of continuous functions are preserved by the H-continuous functions. For instance, two H-continuous functions are equal if and only if equality holds pointwise on a dense set.

The Hausdorff continuous interval valued functions are mostly used in situations where certain discontinuities occur. Initially they were applied mainly to problems in approximation theory, see [75] and [76]. The renewed interest in the H-continuous functions is a result of several recent applications to other areas in mathematics. It was shown in [3] that the Dedekind completion of the space $\mathcal{C}(X)$ of continuous functions can be obtained though certain spaces of H-continuous functions. In particular, if X is a metric space then the space of all finite H-continuous functions

$$\mathsf{H}_{ft}(X) = \{f \in \mathsf{H}(X) : |f(x)| < \infty, x \in X\} \tag{1.26}$$

is exactly the Dedekind completion of $\mathcal{C}(X)$. The Dedekind completion of $\mathcal{C}_b(X)$, the set of bounded continuous functions, was previously characterized as a space of 'normal semi-continuous functions' in [25]. The novelty of the result obtained in [3] is that the entire space $\mathcal{C}(X)$, and not only its subspace $\mathcal{C}_b(X)$,is completed as a space of functions on the same topological space X. The completion of $\mathcal{C}_b(X)$ is also characterized as the set of bounded H-continuous functions. Another application of H-continuous functions is the use of the space of nearly finite H-continuous functions, that is, the space

$$\mathsf{H}_{nf}(X) = \{f \in \mathsf{H}(X) : |f(x)| < \infty, x \in D \text{ - closed nowhere dense}\}, \tag{1.27}$$

in the order completion method [62]. The order completion method is a powerful theory for solving arbitrary continuous nonlinear PDEs using only the usual measurable functions. The regularity of the solutions obtained in this way has been improved significantly, see [6], as the solutions can in fact be assimilated using the nearly finite H-continuous functions. The applications mentioned above all have at their centre the strong properties that the respective sets of H-continuous functions exhibit when viewed as ordered spaces. Indeed, the spaces $\mathsf{H}_{ft}(X)$ and $\mathsf{H}_{nf}(X)$ are both Dedekind complete. It is for this reason that the we are interested in the order convergence on these spaces. In particular, we investigate order convergence on $\mathsf{H}_{ft}(X)$. In [78] we considered order convergence on the set $\mathsf{H}_{ft}(X)$ of finite H-continuous functions where X is an open subset of R^n. It was shown that order convergence satisfies neither Axiom (MS3) nor Axiom (MS4) of the Moore-Smith Axioms. There, however, the space $\mathsf{H}_{ft}(X)$ is considered only as a lattice so that the results on order convergence outlined above does not apply. In order to utilize the results on order convergence on vector lattices in this work some further structure is required. In particular, it must be shown that $\mathsf{H}_{ft}(X)$ is an Archimedean vector lattice.

When defining the algebraic operations of a linear space on $\mathsf{H}_{ft}(X)$ it is desirable that they extend the pointwise operations on $\mathcal{C}(X)$. It is therefore tempting to define the operations on $\mathsf{H}_{ft}(X)$ in a pointwise way. For scalar multiplication this does not pose significant problems. For addition, however, one runs into trouble immediately. The function

$$(f+g)(x) = \left[\underline{f}(x) + \underline{g}(x), \overline{f}(x) + \overline{g}(x)\right], x \in X \tag{1.28}$$

need not be H-continuous.

Example 1.5 *Let $X = [0,1]$. Consider the functions f and g on X defined by*

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ [0,1] & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

and

$$g(x) = \begin{cases} 0 & \text{if } x < 0 \\ [-1,0] & \text{if } x = 0 \\ -1 & \text{if } x > 0 \end{cases} .$$

Adding f and g pointwise as in (1.28) yields

$$(f+g)(x) = \begin{cases} 0 & \text{if } x < 0 \\ [-1,1] & \text{if } x = 0 \\ 0 & \text{if } x > 0 \end{cases}$$

which is clearly not H-continuous.

Example 1.5 indicates that the difficulties in defining pointwise operations on $\mathsf{H}_{ft}(X)$ lies in the points where the functions assume proper interval values. The obstruction is that the established operations of interval analysis does not introduce a linear structure on the set IR of closed finite real intervals, see [5].

It would therefore be appropriate to somehow 'remove' the sets W_f and W_g when defining the sum '$f+g$'. Indeed, following this approach it was shown that if X is an open subset of R^n, the space $\mathsf{H}_{ft}(X)$ can be made into a linear space such that the operations extend the pointwise operations on $\mathcal{C}(X)$, see [77]. In fact, it was shown that, under certain mild assumptions, $\mathsf{H}_{ft}(X)$ is the largest real linear space that extends the pointwise operations on $\mathcal{C}(X)$. That result is essentially based on the fact that any open subset of R^n is a Baire space, that is, the complement of any set of first Baire category is dense. The result can therefore be generalized to any Baire space X. However, when X is not a Baire space the Baire category argument used in [77] and in this work fails.

As we stated above, one of the most striking and important properties of the set $\mathsf{H}_{ft}(X)$ when compared with the function spaces usually studied in functional analysis is that it is Dedekind complete with respect to the order (1.19). It is therefore of considerable importance to study the relationship between the order and the algebraic structure on $\mathsf{H}_{ft}(X)$. We will show that the desired connection exists in that $\mathsf{H}_{ft}(X)$ possesses the structure of a vector lattice. In particular, since $\mathsf{H}_{ft}(X)$ is Dedekind complete it follows by the Main Inclusion Theorem, see Appendix A, that it is an Archimedean vector lattice. The power of this result is twofold. Firstly, it allows for application of the established theory of vector lattices to the H-continuous functions. Moreover, it enables us to make use of the theory of order convergence as developed in Chapter 2 of this work.

In addition to the general results on order convergence obtained in Chapter 2, this mode of convergence on the vector lattice $\mathsf{H}_{ft}(X)$ satisfies a further property when X is a compact metric space. One can characterize the order convergent sequences on $\mathsf{H}_{ft}(X)$ through the Hausdorff distance on $\mathsf{H}_{ft}(X)$. This characterization was obtained in [78] for X a compact subset of R^n. Here we will extend it to arbitrary compact metric spaces.

The Hausdorff distance, as the name indicates, was introduced by Felix Hausdorff in his famous 1914 book on topology, as a metric on the set of closed subsets of a metric space (M, ρ). In particular, for any two closed subsets A and B of M, the Hausdorff distance $r(A, B)$ between them is defined as

$$r(A,B) = \max\left\{\sup_{a\in A}\inf_{b\in B}\rho(a,b), \sup_{b\in B}\inf_{a\in A}\rho(a,b)\right\}. \tag{1.29}$$

Consider the set $M = X \times \mathsf{R}$ endowed with the metric

$$\rho\left((x_1, y_1), (x_2, y_2)\right) = \max\left\{d(x_1, x_2), |y_1 - y_2|\right\} \tag{1.30}$$

where d is the distance function on $X \times X$. Then any $f \in \mathsf{A}(X)$ that assumes only finite values satisfies

$$\{(x, f(x)) : x \in X\} \subset M,$$

that is, the graph of f is a subset of M. If, therefore, $f, g \in \mathsf{A}(X)$ are finite and their graphs are closed subsets of M, the Hausdorff distance

$$r(f,g) = \max\left\{\sup_{x\in X}\inf_{y\in X}\rho((x, f(x)), (y, g(y))), \sup_{Y\in x}\inf_{x\in X}\rho((x, f(x)), (y, g(y)))\right\} \tag{1.31}$$

is well defined. In particular, if we consider the set $\mathsf{F}_{ft}(X)$ of all finite valued S-continuous functions on X, we show that (1.31) defines a metric on that space. It is call the Hausdorff distance, or H-distance for short. The definition of this distance on $\mathsf{F}_{ft}(X)$ is due to Sendov [76] for the particular case when X is a compact interval in the real line. There it is shown that $(\mathsf{F}_{ft}(X), r)$ is a complete metric space with the surprising property that the unit ball is compact. Recall that in a normed space the compactness of the unit ball is equivalent to the space being of finite dimension.

In [78] the work of Sendov was in part extended to the case when X is a subset of R^n. Moreover, some relations between the Hausdorff distance and order convergence on $\mathsf{H}_{ft}(X)$ were established. Our goal here is the further generalization of the results in [78] to arbitrary metric spaces.

A first step towards the generalization obtained in [78] was an alternative expression for the H-distance. This is obtained through the so called δ-Baire operators I_δ and S_δ. Although they were at first defined only for X a subset of R^n, the expression remains unchanged for more general metric spaces. The δ-lower Baire and δ-upper Baire operators I_δ and S_δ are defined as

$$I_\delta(f)(x) = \inf\{z \in f(y) : y \in B_\delta(x)\}, \tag{1.32}$$

$$S_\delta(f)(x) = \sup\{z \in f(y) : y \in B_\delta(x)\}, \tag{1.33}$$

for any $\delta > 0$ and $f \in \mathsf{A}(X)$. Note that since the open balls form a basis for the topology induced on X by the metric, it holds that

$$I(f)(x) = \sup_{\delta>0} I_\delta(f)(x), x \in X \tag{1.34}$$

and

$$S(f)(x) = \inf_{\delta>0} S_\delta(f)(x), x \in X \tag{1.35}$$

for every $f \in \mathsf{A}(X)$. For any $f, g \in \mathsf{F}_{ft}(X)$ define the one-sided H-distance $\eta(f, g)$ as

$$\eta(f, g) = \inf\{\delta > 0 : I_\delta(g)(x) - \delta \leq f(x) \leq S_\delta(g)(x) + \delta, x \in X\}. \tag{1.36}$$

We show that the H-distance $r(f, g)$ is given by

$$r(f, g) = \max\{\eta(f, g), \eta(g, f)\} \tag{1.37}$$

For all $f, g \in \mathsf{F}_{ft}(X)$. For functions f and g that are not S-continuous the H-distance is defined as the H-distance between their completed graphs. That is,

$$r(f, g) = r(F(f), F(g)). \tag{1.38}$$

Note that the one-sided H-distance is not a metric (distance function) on $\mathsf{F}_{ft}(X)$, since there exist functions $f, g \in \mathsf{F}_{ft}(X)$ such that

$$\eta(f, g) = 0, f \neq g.$$

We generalize a characterization of order convergent sequence on $\mathsf{H}_{ft}(X)$ in terms of the one-sided H-distance that was obtained in [78]. Furthermore, we show that under certain conditions of equi-H-continuity, order convergence and convergence in the H-distance are equivalent.

1.8 Order Convergence on $\mathcal{C}(X)$

Consider the set $\mathcal{C}(X)$ of all continuous real functions defined on a given topological space X with a point-wise defined partial order, that is, for $f, g \in \mathcal{C}(X)$

$$f \leq g \Longleftrightarrow f(x) \leq g(x),\ x \in X. \tag{1.39}$$

It is well known, see for instance [52], that the pointwise operations and the pointwise order (1.39) make $\mathcal{C}(X)$ into an Archimedean vector lattice. In fact, a large number of vector lattices are Riesz isomorphic to a subspace of $\mathcal{C}(X)$ for some topological space X, see [1]. Therefore, when we consider $\mathcal{C}(X)$ we are actually working with quite a large class of vector lattices.

Since $\mathcal{C}(X)$ is an Archimedean vector lattice, the theory of order convergence developed in this work applies. Firstly, it follows that $(\mathcal{C}(X), \sigma_o)$, where σ_o is the sequential convergence structure on $\mathcal{C}(X)$ given by the order convergence with respect to the partial order (1.39), is a sequential convergence space. As for Archimedean vector lattices in general, there is no topology on $\mathcal{C}(X)$ that induces the sequential convergence structure σ_o, except in some special cases. Indeed, in Example 1.3 we showed that σ_o violates the third Moore-Smith Axiom (MS3) and in [78] it is shown that the fourth Moore-Smith Axiom (MS4) also fails.

Let us note that since $\mathcal{C}(X)$ is a lattice, any finite subset of $\mathcal{C}(X)$ has both supremum and infimum which are respectively the point-wise supremum and infimum. However, the existence of supremum and infimum of infinite sets cannot be guaranteed. In particular the supremum and infimum in the Definition 1.4 might not exist as the space $\mathcal{C}(X)$ is in general neither Dedekind complete nor Dedekind σ-complete. In fact, a necessary and sufficient condition on the topological space X for $\mathcal{C}(X)$ to be Dedekind complete is that X is completely regular and extremely disconnected, see [52] [Section 43]. Furthermore, when the supremum and/or infimum exist they are not necessarily equal to the point-wise supremum and/or infimum of the respective sequences of functions as the later ones might not be continuous functions at all. This is demonstrated in the following example which also shows that the order convergence on $\mathcal{C}(X)$ is not point-wise.

Example 1.6 *Take $X = \mathsf{R}$ with the usual topology on R and consider the sequence of functions (f_n) given by*

$$f_n(x) = \begin{cases} 1 - n|x| & \text{if } x \in \left(-\frac{1}{n}, \frac{1}{n}\right) \\ 0 & \text{otherwise} \end{cases} \tag{1.40}$$

Let f denote the constant zero function, that is, $f(x) = 0$, $x \in \mathsf{R}$. Then f is the largest lower bound of the set $\{f_n : n \in \mathsf{N}\}$ in $\mathcal{C}(\mathsf{R})$ with respect to the partial order (1.39), that is, $f = \inf\{f_n : n \in \mathsf{N}\}$. Using also that (f_n) is a decreasing sequence and taking $\lambda_n = f$ and $\mu_n = f_n$, $n \in \mathsf{N}$, we obtain from Definition 1.4 that the sequence $(f_n)_{n\in\mathsf{N}}$ order converges to f. Note that f is not a point-wise limit of (f_n) and that the point-wise limit is actually not a continuous function.

The above example shows that order convergence does not imply point-wise convergence. The converse is also true, point-wise convergence does not in general imply order convergence. However, under some assumptions on X, e.g. X compact, and for certain classes of sequences, e.g. bounded sequences, point-wise convergence implies order convergence.

Since $\mathcal{C}(X)$ is an Archimedean vector lattice we can define the order convergence structure λ_o on it. However, since $\mathcal{C}(X)$ is in general not Dedekind σ-complete, it follows that the order convergence structure is not complete. We show that, under the additional assumption that X is a metric space, the set of all finite H-continuous functions considered with order convergence structure satisfies conditions (C1) through (C3). Hence every Cauchy sequence on $\mathcal{C}(X)$, with respect to the order convergence structure, order converges to some finite H-continuous function.

1.9 Summary of the Main Results

This work deals with three previously unrelated topics namely convergence spaces, order convergence on vector lattices and Hausdorff continuous functions. A consequence of this is that we obtain a large number of results, some of greater interest than others. Therefore we highlight the more important contributions.

MR1 Order convergence on a σ-distributive lattice, and hence on a vector lattice, is an FS-convergence structure, that is, there exists a convergence structure that induces the order convergence of sequences. One such structure is the order convergence structure λ_o. Moreover, if the vector lattice is Archimedean, then λ_o is a vector space convergence structure.

MR2 For an Archimedean vector lattice E there exists a complete convergence vector space $\widetilde{E}$ and a mapping $i : E \rightarrow \widetilde{E}$ that satisfy the conditions (C1) through (C3). Indeed, we show that the space $\widetilde{E}$ is isomorphic to the Dedekind σ-completion of E equipped with the order convergence structure.

MR3 If E and F are Archimedean vector lattices with F Dedekind complete, then the linear operators from E into F continuous with respect to order convergence structure are exactly the σ-order continuous operators. Furthermore, the set $\mathcal{L}_c(E, F)$ of all continuous linear operators equipped with the continuous convergence structure is complete. We also obtain the result that if the dual of (F, λ_o) separates the points of F, then the pointwise limit of a sequence of σ-order continuous operators is a σ-order continuous operator.

MR4 We extend the linear structure on the set of all finite H-continuous functions, as defined in [77] for open subsets of R^n, to the more general case where the functions are defined on a Baire space. We also show that the linear structure is compatible with the order relation so that $\mathsf{H}_{ft}(X)$ is a Dedekind complete vector lattice.

MR5 Order convergence structure on $\mathcal{C}(X)$ is in general not a complete vector space convergence structure. We show that for a Baire space X, the completion can be obtained as a set of finite H-continuous functions on

X. If X is a metric space we show that the space $(\mathsf{H}_{ft}(X), \lambda_o)$ is the completion of $(\mathcal{C}(X), \lambda_o)$.

2. ORDER CONVERGENCE STRUCTURE ON A VECTOR LATTICE

2.1 The Order Convergence Structure

In this section we first consider order convergence on a σ-distributive lattice L. One of the difficulties when dealing with order convergence, and non topological convergence in general, is the absence of the diagonal property (MS4). We therefore obtain a diagonal theorem for monotone sequences on a lattice that serves as a powerful tool in obtaining further results for order convergence in general.

Theorem 2.1 *Let L be a lattice with respect to a given partial ordering $\leq$.*
(i) For every $n \in \mathsf{N}$ let the sequence (f_{mn}) be bounded and increasing and let

$$\widehat{f}_n = \sup\{f_{mn} : m \in \mathsf{N}\}, n \in \mathsf{N},$$

$$\widetilde{f}_n = \sup\{f_{mn} : m = 1, ..., n\}, n \in \mathsf{N}.$$

If the sequence $\left(\widehat{f}_n\right)$ is bounded and increasing and $\sup\left\{\widehat{f}_n : n \in \mathsf{N}\right\}$ exists, then the sequence $\left(\widetilde{f}_n\right)$ is bounded and increasing and

$$\sup\left\{\widehat{f}_n : n \in \mathsf{N}\right\} = \sup\left\{\widetilde{f}_n : n \in \mathsf{N}\right\}.$$

(ii) For every $n \in \mathsf{N}$ let the sequence (f_{nm}) be bounded and decreasing and let

$$\widehat{f}_n = \inf\{f_{mn} : m \in \mathsf{N}\}, n \in \mathsf{N},$$

$$\widetilde{f}_n = \inf\{f_{mn} : m = 1, ..., n\}, n \in \mathsf{N}.$$

If the sequence $\left(\widehat{f}_n\right)$ is bounded and decreasing and $\inf\left\{\widehat{f}_n : n \in \mathsf{N}\right\}$ exists, then the sequence $\left(\widetilde{f}_n\right)$ is decreasing and bounded and

$$\inf\left\{\widehat{f}_n : n \in \mathsf{N}\right\} = \inf\left\{\widetilde{f}_n : n \in \mathsf{N}\right\}.$$

Proof. Let the sequence $\left(\widehat{f}_n\right)$ be increasing and bounded such that $f = \sup\left\{\widehat{f}_n : n \in \mathsf{N}\right\}$ exists. Using the monotonicity of the sequences $(f_{mn})_{m\in\mathsf{N}}$,

$n \in \mathbb{N}$, we have

$$\begin{aligned}\widetilde{f}_n &= \sup\{f_{mn} : m = 1, ..., n\} \\ &\leq \sup\{f_{mn+1} : m = 1, ..., n\} \\ &\leq \sup\{f_{mn+1} : m = 1, ..., n+1\} \\ &= \widetilde{f}_{n+1}\end{aligned}$$

which implies that the sequence is increasing. We will show next that the sequence $\left(\widetilde{f}_n\right)$ is bounded from above and that $\sup\left\{\widetilde{f}_n : n \in \mathbb{N}\right\} = f$ exists. Since for each $k \leq n$ we have $f_{kn} \leq \widehat{f}_k \leq \widehat{f}_n$ it follows that

$$\widetilde{f}_n \leq \widehat{f}_n \leq f, n \in \mathbb{N}.$$

Let $\widetilde{f}$ be an upper bound for the sequence $\left(\widetilde{f}_n\right)$. Then it is easy to see that $f_{mk} \leq \widetilde{f}$ for every $k, m \in \mathbb{N}$. Indeed,

$$k \leq m \Longrightarrow f_{km} \leq \widetilde{f}_m \leq \widetilde{f}$$

and

$$k > m \Longrightarrow f_{km} \leq f_{kk} \leq \widetilde{f}_k \leq \widetilde{f}.$$

Therefore

$$\widehat{f}_n = \sup\{f_{mn} : m \in \mathbb{N}\} \leq \widetilde{f}.$$

Hence every upper bound of $\left(\widetilde{f}_n\right)$ is an upper bound for $\left(\widehat{f}_n\right)$ so that every upper bound $\widetilde{f}$ of $\left(\widetilde{f}_n\right)$ satisfies $f \leq \widetilde{f}$. Therefore $\sup\left\{\widetilde{f}_n : n \in \mathbb{N}\right\} = f$.
(ii) This is proved in a similar way as (i) above. ■

Although the diagonal property (MS4) generally fails for order convergence, Theorem 2.1 above allows us to obtain many interesting results. This is due to the fact, as can be seen from Definition 1.3 (iii), that the monotone sequence essentially determine the order convergent sequences.

This next result can be found for the specific case of a vector lattice in [52]. The proof employed there relies not so much on the distributive properties of the lattice structure but follows a more algebraic approach. It is of interest to us here as it permits a construction that is essential to the main result of this section: Order convergence on a σ-distributive lattice defines an FS-convergence structure.

Lemma 2.1 *Let L be a σ-distributive lattice with (f_n) and (g_n) sequences on L.*
(i) If $f_n \uparrow f$ and $g_n \uparrow g$ then the sequences $(h_n^1) = (f_n \wedge g_n)$ and $(h_n^2) = (f_n \vee g_n)$ increase to $h^1 = f \wedge g$ and $h^2 = f \vee g$ respectively.
(ii) If $f_n \downarrow f$ and $g_n \downarrow g$ then the sequences $(h_n^1) = (f_n \wedge g_n)$ and $(h_n^2) = (f_n \vee g_n)$ decrease to $h^1 = f \wedge g$ and $h^2 = f \vee g$ respectively.

Proof. (i) It is clear that $(h_n^2) = (f_n \vee g_n)$ increases to $h_n^2 = f \vee g$, so we prove only that $(h_n^1) = (f_n \wedge g_n)$ increase to $h^1 = f \wedge g$. Since $f_n \leq f_{n+1}$ and $g_n \leq g_{n+1}$ for every $n \in \mathsf{N}$ it follows that $f_n \wedge g_n \leq f_{n+1}$ and $f_n \wedge g_n \leq g_{n+1}$ for every $n \in \mathsf{N}$. Therefore $h_n^1 \leq h_{n+1}^1 = f_{n+1} \wedge g_{n+1}$ for every $n \in \mathsf{N}$. In the same way $h_n^1 \leq f \wedge g, n \in \mathsf{N}$. Since L is σ-distributive, it follows from

$$\sup\{g_n : n \in \mathsf{N}\} = g$$

that $f_n \wedge g_m \uparrow f_n \wedge g$ for every $n \in \mathsf{N}$. But since

$$\sup\{f_n : n \in \mathsf{N}\} = f$$

we obtain $f_n \wedge g \uparrow f \wedge g$, again by the σ-distributivity of L. It now follows by Theorem 2.1 (i) that the sequence (h_n) defined by

$$h_n = \sup\{f_n \wedge g_m : m \leq n\}$$

increases to $f \wedge g$. But

$$h_n = \sup\{f_n \wedge g_m : m \leq n\} = f_n \wedge g_n = h_n^1, n \in \mathsf{N}$$

by the monotonicity of the sequence (g_n). This yields the desired convergence.
(ii) This follows by similar arguments as (i) above. ■

Theorem 2.2 *Let L be a σ-distributive lattice. Then order convergence on L satisfies conditions (i) and (ii) of Theorem 1.2.*

Proof. (i) Let (f_n) and (g_n) be sequences on L where $f_n \to f$ and $\langle(f_n)\rangle = \langle(g_n)\rangle$. Since the sequence (f_n) converges to f there exists by Definition 1.3 (iii) sequences (λ_n) and (μ_n) such that $\lambda_n \uparrow f$ and $\mu_n \downarrow f$ and

$$\lambda_n \leq f_n \leq \mu_n, n \in \mathsf{N}.$$

Firstly, note that for every $m \in \mathsf{N}$ the elements λ_m and μ_m are respectively lower and upper bounds for the set $\{f_n : n \geq m\}$. Indeed, we have

$$\lambda_m \leq \lambda_n \leq f_n \leq \mu_n \leq \mu_m, n \geq m.$$

Furthermore, since $\{f_n : n \geq m\} \in \langle(f_n)\rangle = \langle(g_n)\rangle$ there exists $k_m \in \mathsf{N}$ such that

$$\{g_n : n \geq k_m\} \subseteq \{f_n : n \geq m\}.$$

Then λ_m and μ_m are respectively lower and upper bounds of the set $\{g_n : n \geq k_m\}$. Hence we can construct inductively an increasing sequence of naturals $k_1, k_2, k_3, ...$ such that

$$\lambda_m \leq g_n \leq \mu_m, n \geq k_m, m \in \mathsf{N}. \tag{2.1}$$

We can now define two new sequences (l_n) and (u_n) as follows:

$$l_n = \inf\{g_1, ..., g_{k_1-1}, \lambda_1\}, n = 1, 2, ..., k_1 - 1$$

$$l_n = \lambda_n, n = k_n, k_n + 1, ..., k_{n+1} - 1, n = 1, 2, ...$$
$$u_n = \sup \{g_1, ..., g_{k_1-1}, \mu_1\}, n = 1, 2, ..., k_1 - 1$$
$$u_n = \mu_n, n = k_n, k_n + 1, ..., k_{n+1} - 1, n = 1, 2, ...$$

Clearly, (l_n) is increasing and (u_n) is decreasing. From the inequality (2.1) it follows that

$$l_n \leq g_n \leq u_n, n \geq m, m \in \mathsf{N}.$$

We also have

$$\sup \{l_n : n \in \mathsf{N}\} = \sup \{\lambda_n : n \in \mathsf{N}\} = f$$

and

$$\inf \{u_n : n \in \mathsf{N}\} = \inf \{\mu_n : n \in \mathsf{N}\} = f$$

so that the sequence (g_n) order converges to f by Definition 1.3 (iii).
(ii) Let (f_n) and (g_n) be sequence on L that order converges to $f \in L$. Now consider the trivial mixing $(h_n) = (f_n \Diamond g_n)$ defined by

$$h_{2n-1} = f_n, h_{2n} = g_n. \tag{2.2}$$

According to Definition 1.3 (iii) there exists sequences (λ_n^1) and (μ_n^1), and (λ_n^2) and (μ_n^2) such that $\lambda_n^1, \lambda_n^2 \uparrow f$ and $\mu_n^1, \mu_n^2 \downarrow f$ and

$$\begin{aligned} \lambda_n^1 &\leq f_n \leq \mu_n^1, n \in \mathsf{N}, \\ \lambda_n^2 &\leq g_n \leq \mu_n^2, n \in \mathsf{N}. \end{aligned} \tag{2.3}$$

Combining (2.2) through (2.3) we obtain

$$\begin{aligned} \lambda_n^1 &\leq h_{2n-1} \leq \mu_n^1, \\ \lambda_n^2 &\leq h_{2n} \leq \mu_n^2. \end{aligned} \tag{2.4}$$

Define the sequence (μ_n^*) as

$$\mu_n^* = \sup \{\mu_n^1, \mu_n^2\}. \tag{2.5}$$

By Lemma 2.1 (ii) this sequence decreases to $\sup \{f, f\} = f$. If we now consider the trivial mixing $(\mu_n) = (\mu_n^* \Diamond \mu_n^*)$ of μ_n^* with itself, then (2.4) and (2.5) imply that

$$h_n \leq \mu_n, n \in \mathsf{N}$$

and it is obvious that $\mu_n \downarrow f$. In the same way an increasing sequence (λ_n) can be constructed so that Definition 1.3 (iii) is satisfied. This completes the proof. ■

We now obtain, as a Corollary to Theorem 2.2 above, the following result for vector lattices.

Corollary 2.1 *For any vector lattice E the sequential convergence space (E, σ_o) is an FS-space.*

Proof. This follows on application of Theorems 2.2 and A.1. ■

In the introduction to this work it is mentioned that a given FS-convergence structure is not induced by a unique convergence structure. There is, however, a unique sequentially determined convergence structure that induces the FS-convergence structure. For order convergence we will consider rather the following definition as it exhibits some strong properties related to boundedness.

Definition 2.1 *Let L be a σ-distributive lattice. Define the mapping λ_o from E into the powerset of all filters on E as follows. A filter $\mathcal{F}$ belongs to $\lambda_o(f)$ if and only if there exists a coarser filter $\mathcal{G}$ with a countable basis of the form*

$$\{[\lambda_n, \mu_n] : n \in \mathsf{N}\}$$

where $\lambda_n \uparrow f$ and $\mu_n \downarrow f$.

We proceed to show that the mapping λ_o does indeed define a convergence structure and induces the sequential convergence structure σ_o.

Lemma 2.2 *Let L be a σ-distributive lattice. If a filter $\mathcal{F} \in \lambda_o(f)$ has a countable basis $\{F_1, F_2, ...\}$ where F_1 is order bounded and $F_1 \supseteq F_2 \supseteq ...$, then there exists a decreasing sequence (λ_n) and an increasing sequence (μ_n) such that*

$$\lambda_n \leq g \leq \mu_n, g \in F_n, n \in \mathsf{N}$$

and

$$f = \sup\{\lambda_n : n \in \mathsf{N}\} = \inf\{\mu_n : n \in \mathsf{N}\}.$$

Proof. Let the filter $\mathcal{F}$ on L satisfy the conditions listed above. By Definition 2.1 there exists a coarser filter $\mathcal{G}$ with a countable basis of the form $\left\{\left[\widetilde{\lambda}_n, \widetilde{\mu}_n\right] : n \in \mathsf{N}\right\}$ such that $\widetilde{\lambda}_n \uparrow f$ and $\widetilde{\mu}_n \downarrow f$. As $\mathcal{G}$ is coarser than $\mathcal{F}$ there exists for every $n \in \mathsf{N}$ a natural number k_n such that $F_{k_n} \subseteq \left[\widetilde{\lambda}_n, \widetilde{\mu}_n\right]$. The required sequence can now be constructed as follows:

$$\lambda_j = \begin{cases} \inf\left\{\widetilde{\lambda}_1, g_1\right\} & , j = 1, ..., k_1 - 1 \\ \widetilde{\lambda}_n & , j = k_n, k_n + 1, ..., k_{n+1} - 1 \end{cases},$$

$$\mu_j = \begin{cases} \sup\{\widetilde{\mu}_1, g_2\} & , j = 1, ..., k_1 - 1 \\ \widetilde{\mu}_n & , j = k_n, k_n + 1, ..., k_{n+1} - 1 \end{cases},$$

where $g_1 \leq g \leq g_2$ for every $g \in F_1$. ■

Theorem 2.3 *The mapping λ_o from a σ-distributive lattice L into the powerset of all filters on L defines a first countable convergence structure on L such that $\sigma_{\lambda_o} = \sigma_o$.*

Proof. Conditions (i) and (iii) of Definition 1.5 are obvious so we will prove only conditions (ii).
Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be filters on E that converge to $f \in L$ and let $\left(\lambda_n^{(1)}\right), \left(\mu_n^{(1)}\right)$ and $\left(\lambda_n^{(2)}\right), \left(\mu_n^{(2)}\right)$ be the sequences associated with the filters $\mathcal{F}_1$ and $\mathcal{F}_2$ according to Definition 2.1. Define the sequences (λ_n) and (μ_n) as follows:

$$\begin{aligned} \lambda_n &= \inf\left\{\lambda_n^{(1)}, \lambda_n^{(2)}\right\}, \\ \mu_n &= \sup\left\{\mu_n^{(1)}, \mu_n^{(2)}\right\}. \end{aligned}$$

By Theorem A.3 (i) and (ii) respectively $\lambda_n \uparrow f$ and $\mu_n \downarrow f$. Clearly the filter generated by the base

$$\{[\lambda_n : \mu_n] : n \in \mathsf{N}\}$$

is coarser than the filter $\mathcal{F} = \mathcal{F}_1 \cap \mathcal{F}_2$ so it follows by Definition 2.1 that $\mathcal{F}$ converges to f.
The convergence structure λ_o is first countable by definition. It remains to prove that λ_o induces σ_o.
The implication '(f_n) order converges to f implies (f_n) converges to f in σ_o' is obvious. For the reverse implication, consider a sequence (f_n) that converges to f in σ_o. Let $(\widetilde{\lambda}_n), (\widetilde{\mu}_n)$ be the sequences associated with the filter

$$\langle f_n\rangle = [\{\{f_n : n \geq m\} : m \in \mathsf{N}\}]$$

in terms of Definition 2.1, that is, the filter $\mathcal{G}$ given by

$$\mathcal{G} = \left[\left\{\left[\widetilde{\lambda}_n, \widetilde{\mu}_n\right] : n \in \mathsf{N}\right\}\right]$$

is coarser than $\langle f_n\rangle$ and $\widetilde{\lambda}_n \uparrow f$ and $\widetilde{\mu}_n \downarrow f$. Since $\langle f_n\rangle$ is finer than $\mathcal{G}$ it follows that there exists $m \in \mathsf{N}$ such that $\{f_n : n \geq m\} \subseteq \left[\widetilde{\lambda}_1, \widetilde{\mu}_1\right]$. Since there are only a finite number of terms of (f_n) that is not in $\left[\widetilde{\lambda}_1, \widetilde{\mu}_1\right]$ it follows that (f_n) is bounded. Now we apply Lemma 2.2 to the filter $\langle (f_n)\rangle$. Accordingly, there exists a increasing sequence (λ_n) and a decreasing sequence (μ_n) such that

$$\lambda_n \leq f_m \leq \mu_n, m \geq n, n \in \mathsf{N}$$

and

$$f = \sup\{\lambda_n : n \in \mathsf{N}\} = \inf\{\mu_n : n \in \mathsf{N}\}.$$

It follows that (f_n) order converges to f. ∎

Again as a corollary to the above we obtain the corresponding result for vector lattices.

Corollary 2.2 *For a vector lattice E the mapping λ_o from the E into the powerset of all filters on E defines a first countable convergence structure on E such that $\sigma_{\lambda_o} = \sigma_o$.*

Proof. By Theorem A.1 E is distributive, and hence σ-distributive. Theorem 2.3 now implies the desired result. ■

Having established that the convergence structure λ_o induces order convergence of sequences we commence an investigation of the convergence space properties of this structure. We begin this investigation with studying the separation properties of order convergence structure. This investigation will be continued in the next section in the particular case of a vector lattice.

Theorem 2.4 *Let L be a σ-distributive lattice. Then (L, λ_o) is a regular convergence space.*

Proof. Let the filter $\mathcal{F}$ converge to f in (L, λ_o). By Definition 2.1 there exist sequences $\lambda_n \uparrow f$ and $\mu_n \downarrow f$ such that the filter generated by

$$\{[\lambda_n, \mu_n] : n \in \mathsf{N}\}$$

is coarser than $\mathcal{F}$. Since (L, λ_o) is first countable (Theorem 2.3) it follows by Proposition B.1 that for any subset A of L

$$a(A) = \{f \in L : \exists (f_n) \subseteq A, f_n \rightarrow f\}.$$

We first show that every interval $[g_1, g_2]$ is closed. Let (f_n) be a sequence in $[g_1, g_2]$ that converges to some $f \in L$. By Definition 2.1 (iii) there exists sequences (λ'_n) and (μ'_n) such that $\lambda'_n \uparrow f$ and $\mu'_n \downarrow f$ and

$$\lambda'_n \leq f_n \leq \mu'_n, n \in \mathsf{N}. \tag{2.6}$$

Suppose that $g_1 \not\leq f$. Since the sequence (μ'_n) decreases to f it follows by (2.6) and the inclusion $(f_n) \subset [g_1, g_2]$ that $f < f \vee g_1 \leq \mu'_n$ for every $n \in \mathsf{N}$. But $f = \inf\{\mu'_n : n \in \mathsf{N}\}$, a contradiction. Therefore our assumption that $g_1 \not\leq f$ is false so that $g_1 \leq f$. In the same way it follows that $f \leq g_2$ so that $[g_1, g_2]$ is closed.

Since the filter generated by

$$\{[\lambda_n, \mu_n] : n \in \mathsf{N}\}$$

is coarser than $\mathcal{F}$, there exists for every $n \in \mathsf{N}$ a set $F \in \mathcal{F}$ such that $[\lambda_n, \mu_n] \subseteq F$. But

$$a([\lambda_n, \mu_n]) = [\lambda_n, \mu_n] \subseteq F \subseteq a(F).$$

Therefore the filter $[\{[\lambda_n, \mu_n] : n \in \mathsf{N}\}]$ is coarser than $a(\mathcal{F})$ so by Definition 2.1 $a(\mathcal{F})$ converges to f. ■

Corollary 2.3 *For a σ-distributive lattice L the convergence space (L, λ_o) is Hausdorff.*

Proof. Clearly the space (L, λ_o) is T_1 and by Theorem 2.4 above it is regular. By (1.7) (L, λ_o) is Hausdorff. ■

The corresponding result for vector lattices now follow as a straight forward corollary to the above.

Corollary 2.4 *For a vector lattice E the convergence space (E, λ_o) is regular and Hausdorff.*

Proof. This follows from Theorem 2.4 and Corollary 2.3. ■

We now turn our attention to the special case at hand, that is, order convergence on a vector lattice and, in particular, an Archimedean vector lattice. We proceed to show that, if a vector lattice is Archimedean, then the convergence structure λ_o is compatible with the algebraic structure. This result fails when the assumption that the vector lattice is Archimedean is relaxed as this is equivalent to the sequential continuity of scalar multiplication.

Theorem 2.5 *For an Archimedean vector lattice E the order convergence structure λ_o is a vector space convergence structure so that the pair (E, λ_o) is a convergence vector space.*

Proof. Denote by $d : E \times E \to E$ the addition mapping, that is, $d(f_1, f_2) = f_1 + f_2$ for all $f_1, f_2 \in E$. Let $\pi_1, \pi_2 : E \times E \to E$ be the projection mappings around the first and second coordinate respectively. Assume that $\mathcal{F}$ is a filter on $E \times E$ that converges to (f_1, f_2) in the product convergence structure on $E \times E$. Since the product convergence structure is the initial convergence structure on $E \times E$ with respect to the projections π_1 and π_2 it follows that $\pi_i(\mathcal{F}) \in \lambda_o(f_i)$, $i = 1, 2$. By Definition 2.1 there exists filters $\mathcal{G}_1$, $\mathcal{G}_2$ respectively generated by the bases

$$\left\{ \left[\lambda_n^{(i)}, \mu_n^{(i)} \right] : n \in \mathbb{N} \right\}, i = 1, 2$$

where the sequences $\left(\lambda_n^{(i)} \right)$, $i = 1, 2$, increase to f_i, $i = 1, 2$, and the sequences $\left(\mu_n^{(i)} \right)$, $i = 1, 2$, decrease to f_i, $i = 1, 2$. It is easy to see that the filter $\mathcal{H}$ on $E \times E$ generated by a basis

$$\left\{ \left[\lambda_n^{(1)}, \mu_n^{(1)} \right] \times \left[\lambda_n^{(2)}, \mu_n^{(2)} \right] : n \in \mathbb{N} \right\}$$

is coarser than $\mathcal{F}$. Furthermore, since $\pi_i(\mathcal{H}) = \mathcal{G}_i$, $i = 1, 2$, the filter $\mathcal{H}$ converges to (f_1, f_2). The image filter $d(\mathcal{H})$ is generated by the basis

$$\left\{ \left[\lambda_n^{(1)} + \lambda_n^{(2)}, \mu_n^{(1)} + \mu_n^{(2)} \right] : n \in \mathbb{N} \right\}.$$

The sequential continuity of the addition mapping d implies that the increasing sequence $\left(\lambda_n^{(1)} + \lambda_n^{(2)} \right)$ and the decreasing sequence $\left(\mu_n^{(1)} + \mu_n^{(2)} \right)$ both converge to $f_1 + f_2$. Therefore the filter $d(\mathcal{H})$ converges to $f_1 + f_2$ and since $d(\mathcal{A})$ is finer than $d(\mathcal{H})$ it follows that $d(\mathcal{A})$ also converges to $f_1 + f_2$ so that addition is continuous.

The continuity of scalar multiplication follows by similar arguments. ■

The following result gives a more general criterion for a filter to be convergent with respect to the order convergence structure. Its main use will be in characterizing the filters that converge to 0 in such a way as to ease certain manipulations.

Proposition 2.1 *Let E be a vector lattice. If the sequences (f_n) and (g_n) both order converge to $f \in E$ and satisfy*

$$f_n \leq g_n, n \in \mathsf{N},$$

then the filter $[\{[f_n, g_n] : n \in \mathsf{N}\}]$ converges to f in (E, λ_o).

Proof. By assumption the sequences (f_n) and (g_n) both converge to $f \in E$ so that there exists sequences (λ_n^1) and (λ_n^2) and (μ_n^1) and (μ_n^2) such that $\lambda_n^1, \lambda_n^2 \uparrow f$ and $\mu_n^1, \mu_n^2 \downarrow f$ and

$$\lambda_n^1 \leq f_n \leq \mu_n^1, n \in \mathsf{N} \tag{2.7}$$

and

$$\lambda_n^2 \leq g_n \leq \mu_n^2, n \in \mathsf{N}. \tag{2.8}$$

Define the sequences (λ_n) and (μ_n) as

$$\begin{aligned} \lambda_n &= \inf\left\{\lambda_n^1, \lambda_n^2\right\}, n \in \mathsf{N}, \\ \mu_n &= \sup\left\{\mu_n^1, \mu_n^2\right\}, n \in \mathsf{N}. \end{aligned}$$

By Theorem A.3 the sequence (λ_n) increases to f and the sequence (μ_n) decreases to f. It follows from (2.7) through (2.8) that

$$[f_n, g_n] \subseteq [\lambda_n, \mu_n], n \in \mathsf{N}$$

and hence

$$[\{[\lambda_n, \mu_n] : n \in \mathsf{N}\}] \subseteq [\{[f_n, g_n] : n \in \mathsf{N}\}].$$

Therefore $[\{[f_n, g_n] : n \in \mathsf{N}\}]$ converges to f by Definition 2.1. ■

Proposition 2.2 *Let E be an Archimedean vector lattice. Then a filter $\mathcal{F}$ on E converges to 0 in (E, λ_o) if and only if there exists a sequence $(\widetilde{\mu}_n)$ such that $\widetilde{\mu}_n \downarrow 0$ and*

$$[\{[0, \widetilde{\mu}_n] : n \in \mathsf{N}\}] \subseteq |\mathcal{F}|$$

where $|\mathcal{F}| = [\{|F| : F \in \mathcal{F}\}]$.

Proof. Let the filter $\mathcal{F}$ on E converges to 0 in (E, λ_o). By Definition 2.1 there exists sequences (λ_n) and (μ_n) such that $\lambda_n \uparrow 0$ and $\mu_n \downarrow 0$ and

$$[\{[\lambda_n, \mu_n] : n \in \mathsf{N}\}] \subseteq \mathcal{F}. \tag{2.9}$$

By (2.9) above there exists for every $n \in \mathsf{N}$ an $F_n \in \mathcal{F}$ such that $F_n \subseteq [\lambda_n, \mu_n]$. Hence the inclusion

$$|F_n| \subseteq [0, \widetilde{\mu}_n], n \in \mathsf{N} \tag{2.10}$$

where $\widetilde{\mu}_n = \sup\{|\lambda_n|, |\mu_n|\}$ follows. By Theorem A.2 (iv) and Theorem A.3 (ii) the sequence $(\widetilde{\mu}_n)$ decreases to 0. By Proposition 2.1 above it follows that the filter $[\{[0, \widetilde{\mu}_n] : n \in \mathsf{N}\}]$ converges to 0 and by (2.10)

$$[\{[0, \widetilde{\mu}_n] : n \in \mathsf{N}\}] \subseteq |\mathcal{F}|.$$

For the converse, suppose that there exists a sequence $(\widetilde{\mu}_n)$ such that $\widetilde{\mu}_n \downarrow 0$ and

$$[\{[0, \widetilde{\mu}_n] : n \in \mathsf{N}\}] \subseteq |\mathcal{F}| \tag{2.11}$$

Define the sequences (λ_n) and (μ_n) by $\lambda_n = -\mu_n$ and $\mu_n = \widetilde{\mu}_n$ for every $n \in \mathsf{N}$. By Theorem A.2 (ii) the sequence (λ_n) increases to 0. But the inclusion (2.11) implies that there exists, for every $n \in \mathsf{N}$, an $F_n \in \mathcal{F}$ such that $|F_n| \subseteq [0, \mu_n]$. But this implies that $F_n \subseteq [\lambda_n, \mu_n]$ and hence

$$[\{[\lambda_n, \mu_n] : n \in \mathsf{N}\}] \subseteq \mathcal{F}.$$

This completes the proof. ∎

Local convexity is strong property for a topological vector space to poses. In fact, it is within the setting of locally convex spaces that functional analysis is usually performed. For convergence vector spaces it is not as strong a property. It is, however, useful when studying equicontinuity of sets of linear mappings. In particular, in some cases it plays a role in obtaining a Banach-Steinhauss theorem. It follows by an easy calculation that order convergence is a locally convex vector space convergence structure.

Theorem 2.6 *For any Archimedean vector lattice E the convergence vector space (E, λ_o) is locally convex.*

Proof. Let $\mathcal{F}$ converge to 0 in (E, λ_o). By Definition 2.1 there exists sequences $\lambda_n \uparrow 0$ and $\mu_n \downarrow 0$ such that the filter generated by

$$\{[\lambda_n, \mu_n] : n \in \mathsf{N}\}$$

is coarser than $\mathcal{F}$. Since each interval $[\lambda_n, \mu_n]$ is convex it follows by Definition 1.18 that $[\lambda_n, \mu_n] \in co(\mathcal{F})$ for every $n \in \mathsf{N}$ so that $[\{[\lambda_n, \mu_n] : n \in \mathsf{N}\}]$ is coarser than $co(\mathcal{F})$. Therefore $co(\mathcal{F})$ converges to 0 in (E, λ_o) by Definition 2.1. ∎

We proceed to characterize the bounded subsets with respect to order convergence structure. The implication of this result is not immediately evident, but its significance will becomes clear when we study the Mackey modification $\mu(E)$ of (E, λ_o) and the completion of (E, λ_o).

Theorem 2.7 *Let E be an Archimedean vector lattice. A subset B of E is bounded in the order convergence structure if and only if it is order bounded, that is, there exists $f_1, f_2 \in E$ such that*

$$f_1 \leq f \leq f_2, f \in B.$$

Proof. Suppose that the subset B of E is bounded with respect to λ_o. According to Definition 1.3 (i) the filter $\mathcal{N}B$ converges to 0. By Definition 2.1 there exists sequence $\lambda_n \uparrow 0$ and $\mu_n \downarrow 0$ such that the filter generated by

$$\{[\lambda_n, \mu_n] : n \in \mathsf{N}\}$$

is coarser than $\mathcal{N}B$. A basis for $\mathcal{N}B$ is given by the collection

$$\left\{\left(-\frac{1}{n},\frac{1}{n}\right)B : n\in\mathsf{N}\right\}$$

and since $[\{[\lambda_n,\mu_n] : n\in\mathsf{N}\}]$ is coarser than $\mathcal{N}B$ it follows that there exists $m\in\mathsf{N}$ such that

$$\left(-\frac{1}{m},\frac{1}{m}\right)B\subseteq[\lambda_1,\mu_1].$$

Therefore

$$\frac{1}{m+1}B\subseteq[\lambda_1,\mu_1]$$

so that $f\leq(m+1)\mu_1$ and $(m+1)\lambda_1\leq f$ for every $f\in B$.
Conversely, assume that the set B is order bounded, that is, there exists $f_1,f_2\in E$ such that

$$f_1\leq f\leq f_2, f\in B.$$

Let $g=\sup\{|f_1|,|f_2|\}$. Then

$$-g\leq f\leq g, f\in B. \tag{2.12}$$

Since E is Archimedean the sequence $\left(\widetilde{\lambda}_n\right)=\left(-\frac{g}{n}\right)$ increases to 0 and the sequence $(\widetilde{\mu}_n)=\left(\frac{g}{n}\right)$ decreases to 0. But by (2.12)

$$\left(-\frac{1}{n},\frac{1}{n}\right)B\subseteq\left[-\frac{1}{n}g,\frac{1}{n}g\right], n\in\mathsf{N}$$

so $\mathcal{N}B$ converges to 0 by Definition 2.1. This completes the proof. ■

Corollary 2.5 *For an Archimedean vector lattice E the convergence vector space (E,λ_o) is locally bounded.*

Proof. By Definition 2.1, if a filter $\mathcal{F}$ converges to, say f, then there exist sequences $\lambda_n\uparrow f$ and $\mu_n\downarrow f$ such that the filter generated by

$$\{[\lambda_n,\mu_n] : n\in\mathsf{N}\}$$

is coarser than $\mathcal{F}$. Therefore $[\lambda_1,\mu_1]\in\mathcal{F}$ and by Theorem 2.7 $[\lambda_1,\mu_1]$ is bounded in (E,λ_o). ■

2.2 Continuous Functions on (E,λ_o)

We begin our investigation of the space $\mathcal{C}((E,\lambda_o))$ of all continuous real valued functions defined on (E,λ_o) by determining the associated topology of the order convergence. In light of Proposition B.2 (iii) and Definition 1.11 it is not at all surprising that it turns out to be the order topology. Our interest in the associated topology stems from Proposition B.2, that is, the continuous functions from a convergence space into the reals is exactly those real valued mappings that are continuous with respect to the associated topology.

Theorem 2.8 *Let E be a vector lattice. Then the associated topology of (E, λ_o) is the order topology.*

Proof. Suppose that U is an open set with respect to the associated topology of (E, λ_o). Let $f \in U$ and suppose that (f_n) order converges to f. By Theorem 2.3 the filter

$$\langle f_n \rangle = [\{f_n : n \leq k\} : k \in \mathsf{N}]$$

converges to f in (E, λ_o). By Definitions 1.9 and 1.10 U belongs to every filter that converges to f so that $U \in \langle f_n \rangle$. Therefore there exists $k \in \mathsf{N}$ such that $\{f_n : n \leq k\} \subseteq U$. Since $f \in U$ and the sequence (f_n) were arbitrary it follows by Theorem 1.11 that U is τ_o-open so that $o((E, \lambda_o))$ is coarser than the order topology.
Now let U be a τ_o-open subset of E and let $f \in U$. Let the filter $\mathcal{F}$ converge to f. By Definition 2.1 there exists sequence $\lambda_n \uparrow f$ and $\mu_n \downarrow f$ such that the filter

$$[\{[\lambda_n, \mu_n] : n \in \mathsf{N}\}]$$

is coarser than $\mathcal{F}$. If we can show that there exists $n \in \mathsf{N}$ such that $[\lambda_n, \mu_n] \subseteq U$ then $U \in \mathcal{F}$. Since f and $\mathcal{F}$ were chosen in an arbitrary way, this implies that U is open in $o((E, \lambda_o))$ which completes the proof. So assume, for the sake of obtaining a contradiction, that this is not the case, that is, for every $n \in \mathsf{N}$ there exists $f_n \in [\lambda_n, \mu_n]$ such that $f_n \notin U$. The sequence (f_n) thus obtained clearly order converges to f, but lies entirely outside of U. Therefore U can not be τ_o-open, contrary to our assumption. Therefore $[\lambda_n, \mu_n] \subseteq U$ for some $n \in \mathsf{N}$. This completes the proof. ■

This next result is quite special in that it is not true for convergence vector spaces in general. We will show that the topological modification $o(E)$ of (E, λ_o) is a topological vector space. For an example of a convergence vector space for which this does not hold, see [15][Remark 4.3.31]. This result has significant consequences for the space $\mathcal{C}((E, \lambda_o))$ and its relation with (E, λ_o).

Theorem 2.9 *Let E be an Archimedean vector lattice. Then (E, τ_o) is a topological vector space.*

Proof. We must show that the mappings

$$+ : (E, \tau_o) \times (E, \tau_o) \to (E, \tau_o)$$

and

$$\cdot : (E, \tau_o) \times \mathsf{R} \to (E, \tau_o)$$

are continuous. So let U be an open subset of (E, τ_o).and consider its inverse image under addition, that is, the set

$$V = \{(f, g) \in E \times E : f + g \in U\}. \tag{2.13}$$

We must show that the set V is open in the product space $(E, \tau_o) \times (E, \tau_o)$. This will hold if and only if there exists for every $(f, g) \in V$ open subsets V_1 and V_2 of (E, τ_o) such that $f \in V_1$, $g \in V_2$ and

$$V_1 \times V_2 \subseteq V.$$

The set V_1 is open if and only if for every $f \in V_1$ and every sequence (f_n) that order converges to f there exists a natural number N such that $f_n \in V_1$ for every $n \geq N$ and similarly for V_2. Therefore the set V is open if and only if for every $(f, g) \in V$ and every sequence $(\widetilde{f_n}) = (f_n, g_n)$ on $E \times E$ such that (f_n) order converges to f and (g_n) order converges to g there exists a natural number N such that $(\widetilde{f_n}) \in V$ for every $n \geq N$.
Let $(\widetilde{f_n}) = (f_n, g_n)$ be such a sequence. By Theorem A.4 the sequence $(h_n) = (f_n + g_n)$ order converges to $f + g$. But $f + g \in U$ by (2.13) and since U is open by assumption, it follows by Definition 1.11 that there exists $N_1 \in \mathbb{N}$ such that

$$f_n + g_n = h_n \in U, n \geq N_1. \tag{2.14}$$

By (2.13) and (2.14) it follows that $(f_n, g_n) \in V$ for every $n \geq N_1$ so that V is open. Therefore 'addition' is continuous with respect to the order topology.
The proof that scalar multiplication is a continuous mapping from $(E, \tau_o) \times \mathbb{R}$ into (E, τ_o) follows in the same way as above. ∎

If we now recall Pontryagin's theorem on the complete regularity of topological groups, that is, every topological group is completely regular, we obtain the following corollary to Theorem 2.9 above.

Corollary 2.6 *Let E be an Archimedean vector lattice. Then (E, τ_o) is a completely regular topological space.*

Proof. This follows immediately from Pontryagin's Theorem and Theorem 2.9. ∎

We proceed to state a useful characterization of continuous functions on (E, λ_o) into an arbitrary convergence space K.

Theorem 2.10 *Let E be a vector lattice. Then a mapping $\varphi : E \to K$, where K is a convergence space, is continuous if and only if φ is countably continuous, that is, if the filter $\mathcal{F}$ on E converges to $f \in E$ in (E, λ_o) and has a countable basis, then $\varphi(\mathcal{F})$ converges to $\varphi(f)$ in K.*
If K is topological, then φ is continuous if and only if it is sequentially continuous.

Proof. If the mapping φ is continuous, then it is countably continuous. Conversely, suppose φ is countably continuous and let the filter $\mathcal{F}$ converge to f in (E, λ_o). By Theorem 2.3 (E, λ_o) is first countable so that there exists a coarser filter $\mathcal{G}$ with a countable basis that converges to f. Obviously

$$\varphi(\mathcal{G}) \subseteq \varphi(\mathcal{F}),$$

and since $\varphi(\mathcal{G})$ converges to $\varphi(f)$ by assumption, it follows that by Definition 1.5 (iii) that $\varphi(\mathcal{F})$ converges to $\varphi(f)$ in K. Hence φ is continuous.
Now suppose that K is topological. It is sufficient to show that sequential continuity implies continuity as the converse is true by default. Let $\varphi : (E, \lambda_o) \to K$ be a sequentially continuous mapping. By Definition 1.12 (ii) the sequence $(\varphi(f_n))$ converges to $\varphi(f)$ in K whenever (f_n) converges to f in (E, λ_o). By Corollary 2.2 (f_n) converges to f in (E, λ_o) if and only if (f_n)order converges to f so that $\varphi : (E, \tau_o) \to K$ is continuous by Proposition A.3. But by Theorem 2.10 $(E, \tau_o) = o((E, \lambda_o))$ so that Proposition B.2 (ii) implies that $\varphi : (E, \lambda_o) \to K$ is continuous. ∎

As particular cases of the above we now have the following.

Corollary 2.7 *Let E be an Archimedean vector lattice. Then a mapping $\varphi : E \to \mathbb{R}$ is continuous in the order convergence structure on E if and only if $\varphi(f_n)$ converges to $\varphi(f)$ in $\mathbb{R}$ whenever (f_n) order converges to f in E.*

Proof. This is a direct consequence of Theorem 2.10 above. ∎

As a result of Corollary 2.6 above we obtain the following separation result for the space (E, λ_o). The significance of such a result is clear as it is a necessary condition for a convergence space to be c-embeddeble. Although (E, λ_o) in in general not c-embeddable this result would still be of some use when studying nonlinear phenomena.

Theorem 2.11 *For an Archimedean vector lattice E the convergence vector space (E, λ_o) is functionally regular and functionally Hausdorff.*

Proof. We start by showing that, for any $f \leq g \in E$ the interval $[f_1, f_2]$ is closed in (E, σ). By Theorem A.2 (v) and Proposition A.2 $[f_1, f_2]$ is τ_o-closed. By Corollary 2.6 and Proposition B.2 (ii) there exists for every $h \notin [f_1, f_2]$ some $\varphi_0 \in \mathcal{C}((E, \lambda_o))$ such that $\varphi_0(h) = 0$, $\varphi_0([f_1, f_2]) = \{1\}$ and $0 \leq \varphi_0(g) \leq 1$ for every $g \in E$. Therefore

$$\left\{ g \in E : |\varphi_0(g)| < \frac{1}{2} \right\} \subset E \setminus [f_1, f_2]$$

so that $E \setminus [f_1, f_2]$ is σ-open by Definition 1.14 (ii) so that $[f_1, f_2]$ is σ-closed. Now let the filter $\mathcal{F}$ converge to f in (E, λ_o). By Definition 2.1 there exists sequences $\lambda_n \uparrow f$ and $\mu_n \downarrow f$ such that

$$[\{[\lambda_n, \mu_n] : n \in \mathbb{N}\}] \subseteq \mathcal{F}.$$

But each interval $[\lambda_n, \mu_n]$ is closed in (E, σ) so it follows by Definition 1.13 that

$$[\{[\lambda_n, \mu_n] : n \in \mathsf{N}\}] \subseteq \overline{\mathcal{F}}^{\sigma}.$$

Again by Definition 2.1 the filter $\overline{\mathcal{F}}^{\sigma}$ converges to f in (E, λ_o) so that (E, λ_o) is functionally regular.
We proceed to show that (E, σ) is Hausdorff. By Corollary 2.7 a function $\mathcal{C}((E, \lambda_o)) = \mathcal{C}((E, \tau_o))$ and by Corollary 2.6 (E, τ_o) is completely regular. Therefore, for every $f \neq g \in (E, \lambda_o)$ there exists $\varphi_1 \in \mathcal{C}((E, \lambda_o))$ such that $\varphi_1(f) = 0$, $\varphi_1(g) = 1$ and $0 \leq \varphi_1(h) \leq 1$ for every $g \in E$. Therefore the σneighborhoods

$$U = \left\{ h \in E : |\varphi_1(h)| < \frac{1}{3} \right\}$$

and

$$V = \left\{ h \in E : |\varphi_1(h)| > \frac{1}{3} \right\}$$

of f and g respectively are disjoint. This completes the proof. ■

2.3 Convergence Space Completion of (E, λ_o)

The aim of this section is to give a concrete description of the convergence vector space (E, λ_o). We will achieve this goal in four steps. First we characterize the Cauchy sequences on (E, λ_o) through order conditions. We proceed to determine exactly those Archimedean vector lattices for which (E, λ_o) is complete. In the third step we establish the existence of a convergence vector space G that satisfies conditions (C1) through (C3) as stated in Section 1.6. Finally we present the concrete description of this completion G.

Theorem 2.12 *Let E be an Archimedean vector lattice. Then a sequence (f_n) is Cauchy in (E, λ_o) if and only if it is order Cauchy.*

Proof. First note that (1.9) of Definition 1.21 is equivalent to

$$f_m - f_k \leq \mu_n, m, k \geq n. \tag{2.15}$$

Indeed, if (2.15) holds then

$$f_k - f_m \leq \mu_n, m, k \geq n.$$

By (1.1) we then have

$$|f_m - f_k| = \sup \{(f_m - f_k), -(f_m - f_k)\} \leq \mu_n, m, k \geq n.$$

The inverse implication is obvious.
It is clear that an order Cauchy sequence is Cauchy with respect to the convergence structure λ_o.
Now suppose that the sequence (f_n) is Cauchy in (E, λ_o), that is, the filter $\langle (f_n) \rangle$ is Cauchy. By Definition 1.19 (i) the filter $\langle (f_n) \rangle - \langle (f_n) \rangle$ converges to 0 in (E, λ_o). Therefore there exists a coarser filter $\mathcal{G}$ with a countable basis of the form $\{[\widetilde{\lambda}_n, \widetilde{\mu}_n] : n \in \mathsf{N}\}$ where the sequence $(\widetilde{\lambda}_n)$ increases to 0 and the sequence $(\widetilde{\mu}_n)$ decreases to 0. The filter $\langle (f_n) \rangle - \langle (f_n) \rangle$ has as base the collection

$$\{\{f_m - f_k : m, k \geq l\} : l \in \mathsf{N}\}.$$

Since $\mathcal{G}$ is coarser than $\langle (f_n) \rangle - \langle (f_n) \rangle$ there exists for every $n \in \mathsf{N}$ an $l_n \in \mathsf{N}$ such that

$$\{f_m - f_k : m, k \geq l_n\} \subseteq [\widetilde{\lambda}_n, \widetilde{\mu}_n].$$

Therefore there exists $l_1 \in \mathsf{N}$ such that

$$\{f_m - f_k : m, k \geq l_1\} \subseteq [\widetilde{\lambda}_1, \widetilde{\mu}_1].$$

Therefore only a finite number of terms of the form $f_m - f_k$ are not in the interval $\left[\widetilde{\lambda}_1, \widetilde{\mu}_1\right]$. Since E is a lattice it follows that the set $\{f_m - f_k : m, k \in \mathsf{N}\}$ is order bounded. Lemma 2.2 therefore implies the existence of the desired sequence. ∎

Theorem 2.13 *Let E be an Archimedean vector lattice. Then the convergence vector space (E, λ_o) is complete if and only if E is Dedekind σ-complete.*

Proof. Suppose that E is Dedekind σ-complete. Now let (f_n) be an increasing Cauchy sequence on E. As in the proof of Theorem 2.12 it follows that (f_n) is bounded. But E is Dedekind σ-complete so that (f_n) has a supremum f, and hence (f_n) converges to f. By Theorem A.6 E is order complete and hence by Theorem 2.12 every Cauchy sequence in (E, λ_o) converges in E. By Proposition B.5 (E, λ_o) is a complete convergence vector space since λ_o is a first countable vector space convergence structure by Theorems 2.2 and 2.5.
Conversely, suppose that (E, λ_o) is complete but not Dedekind σ-complete. Let $E^\#$ be the Dedekind σ-completion of E. We first show that (E, λ_o) is a subspace of $\left(E^\#, \lambda_o\right)$. For this purpose it is sufficient to show that if a filter $\mathcal{F}$ converges to f in $\left(E^\#, \lambda_o\right)$, then the restriction $\mathcal{F}|E$ of $\mathcal{F}$ to E converges to f in (E, λ_o). By Definition 2.1 there exists sequences (λ_n) and (μ_n) on $E^\#$ such that $\lambda_n \uparrow f$ and $\mu_n \downarrow f$ and

$$[\{[\lambda_n, \mu_n] : n \in \mathsf{N}\}] \subseteq \mathcal{F}. \tag{2.16}$$

Since $E^\#$ is the Dedekind σ-completion of E there exists, for every $n \in \mathsf{N}$, a sequence $(\lambda'_{n\,m})$ on E such that $\lambda'_{n\,m} \uparrow \lambda_n$. By Theorem 2.1 (ii) we can construct a sequence $\left(\widetilde{\lambda}_n\right)$ on E in such a way that $\widetilde{\lambda}_n \leq \lambda_n, n \in \mathsf{N}$ and

$\widetilde{\lambda}_n \uparrow f$. Similarly we can define a sequence $(\widetilde{\mu}_n)$ on E in such a way that $\mu_n \leq \widetilde{\mu}_n, n \in \mathbb{N}$ and $\widetilde{\mu}_n \downarrow f$. By (2.16) there exists for every $n \in \mathbb{N}$ a set $F_n \in \mathcal{F}$ such that $F_n \subseteq [\lambda_n, \mu_n]$. But

$$[\lambda_n, \mu_n] \, |E \subseteq \left[\widetilde{\lambda}_n, \widetilde{\mu}_n\right], n \in \mathbb{N} \tag{2.17}$$

where the second interval is taken in E. Combining (2.17) with the inclusion $F_n \subseteq [\lambda_n, \mu_n]$ yields the desired convergence of $\mathcal{F}\,|E$ so that (E, λ_o) is a subspace of $\left(E^{\#}, \lambda_o\right)$.
Since (E, λ_o) is complete it follows by Proposition and the above that it is a closed subspace of $\left(E^{\#}, \lambda_o\right)$. But since $E \neq E^{\#}$ there exists an increasing sequence (f_n) on E and $f \in E^{\#} \setminus E$ such that $f_n \uparrow f$. This contradicts the fact that (E, λ_o) is closed. Therefore $E = E^{\#}$ which completes the proof. ■

Lemma 2.3 *Let E be an Archimedean vector lattice. A filter $\mathcal{F}$ on E is bounded with respect to λ_o if and only if there exists $g_1 \leq g_2 \in E$ and $F \in \mathcal{F}$ such that $F \subseteq [g_1, g_2]$.*

Proof. Suppose that the filter $\mathcal{F}$ is bounded. Then by Definition 1.3 (ii) and Proposition 2.2 there exists a sequences (μ_n) such that $\mu_n \downarrow 0$ and

$$[\{[0, \mu_n] : n \in \mathbb{N}\}] \subseteq |\mathcal{N}\mathcal{F}| \, .$$

Hence there exists for every $n \in \mathbb{N}$ two sets $N_n \in \mathcal{N}$ and $F_n \in \mathcal{F}$ such that

$$|N_n F_n| = |N_n| \, |F_n| \subseteq [0, \mu_n] \, .$$

But there exists $\varepsilon_n > 0$ such that $[0, \varepsilon_n] \subseteq |N_n|$ so that

$$[0, \varepsilon_n] \, |F_n| \subseteq [0, \mu_n] \, .$$

Therefore

$$0 \leq \varepsilon_n \, |f| \leq \mu_n, f \in F_n$$

so that

$$0 \leq |f| \leq \frac{1}{\varepsilon_n} \mu_n, f \in F_n.$$

The desired inclusion is obtained when setting

$$g_1 = -\frac{1}{\varepsilon_n} \mu_n, g_2 = \frac{1}{\varepsilon_n} \mu_n.$$

Conversely, suppose that there exists $F \in \mathcal{F}$ and $g_1 \leq g_2 \in E$ such that $F \subseteq [g_1, g_2]$. By Theorem 2.7 and Definition 1.3 (i) the filter $\mathcal{N}F$ converges to 0. But $\mathcal{N}F$ is coarser than $\mathcal{N}\mathcal{F}$ so that $\mathcal{N}\mathcal{F}$ converges to 0. This completes the proof. ■

Theorem 2.14 *Let E be an Archimedean vector lattice. Then there exists a convergence vector space $\widetilde{E}$ that satisfies (C1) through (C3).*

Proof. We must show that (E, λ_o) satisfies the condition of Theorem B.1. By Corollary 2.3 it suffices to show that every Cauchy filter on (E, λ_o) is bounded. So let $\mathcal{F}$ be a Cauchy filter on (E, λ_o) and consider the filter

$$\mathcal{N}\mathcal{F} = [\{NF : N \in \mathcal{N}, F \in \mathcal{F}\}]$$

where $\mathcal{N}$ denotes the zero neighbourhood filter of R. Since $\mathcal{F}$ is Cauchy the filter

$$\mathcal{F} - \mathcal{F} = [\{F_1 - F_2 : F_1, F_2 \in \mathcal{F}\}]$$

converges to 0. By Definition 2.1 there exists a sequences (μ_n) and (λ_n) such that (λ_n) increases to 0 and (μ_n) decreases to 0 and

$$[\{[\lambda_n, \mu_n] : n \in \mathsf{N}\}] \subseteq \mathcal{F} - \mathcal{F}.$$

Therefore, for every $n \in \mathsf{N}$, there exists $F_1^n, F_2^n \in \mathcal{F}$ such that $F_1^n - F_2^n \subseteq [\lambda_n, \mu_n]$. Let $g \in F_2^n$ be given. Then for every $f \in F_1^n$

$$\lambda_n \leq f - g \leq \mu_n$$

so that $\lambda_n + g \leq f \leq \mu_n + g$. By Lemma 2.3 every Cauchy filter on (E, λ_o) is bounded so that the desired completion $\widetilde{E}$ exists. ■

Theorem 2.15 *Let E be an Archimedean vector lattice and denote by $E^\#$ its Dedekind σ-completion. Then $\left(E^\#, \lambda_o\right)$ is the convergence vector space completion of (E, λ_o). That is, $\left(E^\#, \lambda_o\right)$ satisfies (C1) through (C3).*

Proof. We first show that (E, λ_o) is a dense subspace of $\left(E^\#, \lambda_o\right)$. It must be shown that the convergence structure induced on E as a subspace of $\left(E^\#, \lambda_o\right)$ is the order convergence structure on E. If a filter $\mathcal{F}$ converges to $f \in E$ in (E, λ_o) then it obviously converges to f in $\left(E^\#, \lambda_o\right)$. Conversely, suppose that the filter $\mathcal{F}$ converges to $f \in E$ with respect to the subspace structure induced on E by $\left(E^\#, \lambda_o\right)$. By Definitions 1.22 and 2.1 there exists sequences (λ_n) and (μ_n) on $\left(E^\#, \lambda_o\right)$ such that $\lambda_n \uparrow f$ and $\mu_n \downarrow f$ and

$$[\{[\lambda_n, \mu_n] : n \in \mathsf{N}\}] \subseteq [\mathcal{F}]_{E^\#}.$$

Since $E^\#$ is the Dedekind σ-completion of E there exists for every $n \in \mathsf{N}$ sequence (λ_{nm}) and (μ_{nm}) on E such that (λ_{nm}) increases to λ_n and (μ_{nm}) decreases to μ_n. Theorem 2.1 guarantees the existence of sequences $\left(\widetilde{\lambda}_n\right)$ and $(\widetilde{\mu}_n)$ on (E, λ_o) such that $\widetilde{\lambda}_n \uparrow f$ and $\widetilde{\mu}_n \downarrow f$ and $[\lambda_n, \mu_n] \subseteq \left[\widetilde{\lambda}_n, \widetilde{\mu}_n\right]$ when the second interval is considered as a subset of $E^\#$. Therefore the following inclusions hold:

$$\left[\left\{\left[\widetilde{\lambda}_n, \widetilde{\mu}_n\right] : n \in \mathsf{N}\right\}\right] \subseteq [\{[\lambda_n, \mu_n] : n \in \mathsf{N}\}] \subseteq [\mathcal{F}]_{E^\#}.$$

But this implies that

$$\left[\left\{\left[\widetilde{\lambda}_n, \widetilde{\mu}_n\right] : n \in \mathsf{N}\right\}\right] \subseteq \mathcal{F}$$

when the intervals are taken in E. Therefore (E, λ_o) is a subspace of $\left(E^{\#}, \lambda_o\right)$. The denseness follows by Theorem 2.2 and Proposition B.1.
Now let F be a complete Hausdorff convergence vector space and $T : E \to F$ linear and continuous. First note that if the sequence (f_n) is Cauchy on (E, λ_o) then its image under T, that is, the sequence (Tf_n), is Cauchy in F. Since F is Hausdorff and complete by assumption, the sequence (Tf_n) converges to a unique $f_0 \in F$. For every $f \in E^{\#}$ let (λ_n) and (μ_n) be sequences on E such that $\lambda_n \uparrow f$ and $\mu_n \downarrow f$. Obliviously the filter

$$\mathcal{F} = [\{[\lambda_n, \mu_n] : n \in \mathsf{N}\}]$$

is a Cauchy filter on E and hence the filter $T(\mathcal{F})$ is a Cauchy filter on F, and since F is complete $T(\mathcal{F})$ converges to some $g^f \in F$. For every $f \in E^{\#}$ define the mapping $T^{\#} : E^{\#} \to F$ by

$$T^{\#} f = g^f.$$

The definition is independent of the particular choice of sequences (λ_n) and (μ_n). To see this, let (λ_n') and (μ_n') be different from (λ_n) and (μ_n) such that $\lambda_n' \uparrow f$ and $\mu_n' \downarrow f$, and define the filter $\mathcal{F}'$ in the same way as $\mathcal{F}$. Theorems A.2 (i), (ii) and A.4 together with the inclusion

$$\mathcal{F} - \mathcal{F}' \supseteq [\{[\lambda_n - \mu_n', \mu_n - \lambda_n'] : n \in \mathsf{N}\}]$$

implies that the filter $\mathcal{F} - \mathcal{F}'$ converges to 0 in E so that $T(\mathcal{F} - \mathcal{F}')$ converges to 0 in F. But

$$\begin{aligned} T(\mathcal{F} - \mathcal{F}') &= [\{T(F - F') : F \in \mathcal{F} \in, F' \in \mathcal{F}'\}] \\ &= T(\mathcal{F}) - T(\mathcal{F}') \end{aligned}$$

and since $T(\mathcal{F})$ converges to g^f in F, it follows that $T(\mathcal{F}')$ also converges to g^f in F.
To see that T is linear, let the sequences (λ_n) and (μ_n) be as before and let (λ_n') and (μ_n') be sequences on E such that $\lambda_n' \uparrow f'$ and $\mu_n' \downarrow f'$. By Theorems A.2 (i), (ii) and A.4 $(\lambda_n + \lambda_n')$ increases to $f + f'$ and $(\mu_n + \mu_n')$ decreases to $f + f'$. Define the filters $\mathcal{F}$ and $\mathcal{F}'$ on E as before. Clearly

$$T(\mathcal{F} + \mathcal{F}') = T(\mathcal{F}) + T(\mathcal{F}').$$

But $T(\mathcal{F})$ converges to $T^{\#} f$ and $T(\mathcal{F}')$ converges to $T^{\#} f'$ and $T(\mathcal{F} + \mathcal{F}')$ converges to $T^{\#}(f + f')$ since the filter

$$\mathcal{F} + \mathcal{F}' \supseteq [\{[\lambda_n + \lambda_n', \mu_n + \mu_n'] : n \in \mathsf{N}\}].$$

In the same way it can be shown that $T(\alpha f) = \alpha Tf$ for every $f \in E$ and any scalar α.
It is clear that $Tf = T^{\#}f$ for every $f \in E$, so it remains to show that $T^{\#}$ is continuous. By Definition 2.1 we need only consider filters on $E^{\#}$ of the form

$$\mathcal{F} = [\{[\lambda_n, \mu_n] : n \in \mathsf{N}\}]$$

where (λ_n) and (μ_n) are sequences on $E^{\#}$ that increase to $f \in E^{\#}$ and decrease to $f \in E^{\#}$ respectively. Since, for each $n \in \mathsf{N}$, we can approximate λ_n by an increasing sequence and μ_n by an decreasing sequence on E, by Theorem 2.1 we can construct sequences $\left(\lambda_n^{\#}\right)$ and $\left(\mu_n^{\#}\right)$ on E such that

$$\lambda_n^{\#} \leq \lambda_n \leq \mu_n \leq \mu_n^{\#}, n \in \mathsf{N}$$

and $\left(\lambda_n^{\#}\right)$ increases to f and $\left(\mu_n^{\#}\right)$ decreases to f. Define the filter $\mathcal{F}^{\#}$ on E by

$$\mathcal{F}^{\#} = \left[\left\{\left[\lambda_n^{\#}, \mu_n^{\#}\right] : n \in \mathsf{N}\right\}\right].$$

Then $T\left(\mathcal{F}^{\#}\right)$ converges to $T^{\#}f$ by definition. But

$$T\left(\mathcal{F}^{\#}\right) \subseteq T^{\#}\left(\mathcal{F}^{\#}\right) \subseteq T^{\#}(\mathcal{F})$$

so that $T^{\#}(\mathcal{F})$ converges to $T^{\#}f$. This completes the proof. ■

2.4 Subspaces and Quotient Spaces

In this section we discuss the permanence properties of the order convergence structure related to the formation of subspaces and quotients. For the general case of filter convergence strong results can only be obtained under rather restrictive conditions. The situation is much better when we consider sequential convergence, which seems to indicate that the sequentially determined convergence structure λ_{σ_o} may be better suited to a study of subspaces and quotients.

Theorem 2.16 *Let E be an Archimedean vector lattice, A an ideal in E and (f_n) a sequence on A. Then the following holds.*
(i) If E is Dedekind σ-complete, then A is a closed subspace of (E, λ_o) if and only if A is a σ-ideal in E.
(ii) If E is Dedekind σ-complete and A is a σ-ideal in E, then (f_n) order converges in E if and only if (f_n) order converges in A.
(iii) If E is Dedekind complete and A is a band in E, then order convergence on structure on A coincides with the subspace convergence structure on A induced by (E, λ_o).

Proof. (i) Let A be a σ-ideal in E and consider a sequence (f_n) on A such that converges to some $f \in E$. By Proposition B.1 we must show that $f \in A$. By Definition 1.23 (iii)

$$g_1 = \inf \{f_n : n \in \mathsf{N}\} \in A$$

and

$$g_2 = \sup \{f_n : n \in \mathsf{N}\} \in A$$

so that $g = \sup \{|g_1|, |g_2|\}$ is in A^+. The above supremum and infimum exist since (f_n) is bounded and E is Dedekind σ-complete. But $g_1 \leq f \leq g_2$ so that $|f| \leq g$. By Definition 1.23 (ii) f belongs to A.
Conversely, suppose that the ideal A is a closed subspace of (E, λ_o). By Proposition B.1 A contains all the limits of convergent sequences contained in A. Now consider the subset $\{f_n : n \in \mathsf{N}\}$ of A with the property that $f = \sup \{f_n : n \in \mathsf{N}\}$ exists. The sequence $\left(\widetilde{f}_n\right)$ on A defined by

$$\widetilde{f}_n = \sup \{f_k : k \leq n\}, n \in \mathsf{N}$$

is increasing a bounded from above by f. In fact, $\widetilde{f}_n \uparrow f$. To see this, suppose the opposite, that is, there exists an upper bound $g \in E$ of $\left(\widetilde{f}_n\right)$ such that $f \cdot g$. Then it is clear that g is an upper bound of $\{f_n : n \in \mathsf{N}\}$ so that $\inf \{f, g\} < f$ is also an upper bound for $\{f_n : n \in \mathsf{N}\}$, contrary to the assumption that f was the least upper bound of $\{f_n : n \in \mathsf{N}\}$. The sequence $\left(\widetilde{f}_n\right)$ therefore converges to f so that $f \in A$.
(ii) Let (f_n) order converge to $f \in A$ in E. Since E is Dedekind σ-complete Theorem applies. Therefore

$$f_n \geq \lambda_n = \inf \{f_k : k \geq n\} \uparrow f$$

and

$$f_n \leq \mu_n = \sup \{f_k : k \geq n\} \downarrow f.$$

But since A is a σ-ideal, $\lambda_n, \mu_n \in A$ for every $n \in \mathsf{N}$. Therefore Definition 1.3 (iii) implies that (f_n) order converges to f in A. The inverse implication is trivial.
(iii) Let A be a band in E and let the filter $\mathcal{F}$ converge to 0 in the subspace convergence structure induced on A from E, that is, the filter $[\mathcal{F}]_E$ converges to 0 in E. By Proposition 2.2 there exists a sequence (μ_n) on E^+ such that $\mu_n \downarrow 0$ and

$$[\{[0, \mu_n] : n \in \mathsf{N}\}] \subseteq |[\mathcal{F}]_E| = [|\mathcal{F}|]_E .$$

Therefore there exists, for every $n \in \mathsf{N}$ an element $F_n \in \mathcal{F}$ such that

$$|F_n| \subseteq [0, \mu_n] . \tag{2.18}$$

Since A is a band in E (2.18) implies that $\widetilde{\mu}_n = \sup |F_n| \in A$. But $0 \leq \widetilde{\mu}_n \leq \mu_n$ and $\mu_n \downarrow 0$ so that $\widetilde{\mu}_n \downarrow 0$ and

$$|F_n| \subseteq [0, \widetilde{\mu}_n], n \in \mathsf{N}.$$

Hence $[\{[0, \widetilde{\mu}_n] : n \in \mathsf{N}\}] \subseteq |\mathcal{F}|$ and by Proposition 2.2 $\mathcal{F}$ converge to 0 in (A, λ_o). For the general case where $\mathcal{F}$ converge to $f \in A$ we consider the filter $\mathcal{F} - f$ which converges to 0. The inverse implication is obvious. ■

We proceed by considering the quotient space associated with an ideal as described in Section 1.5. In general, the quotient vector lattice so obtained need not even be Archimedean. Moreover, some assumptions on the vector lattice E and the ideal A of E are necessary for the quotient mapping to be continuous with respect to order convergence. The result we obtain therefore does not apply to general Archimedean vector lattices.

Theorem 2.17 *Let E be a Dedekind σ-complete vector lattice and let A be a σ-ideal in E. Then a sequence converges in the quotient convergence structure on $E \setminus A$ with respect to the order convergence structure on E whenever it order converges on $E \setminus A$.*

Proof. Let the sequence $([f_n])$ order converge to 0 on $E \setminus A$. By Definition 1.3 (iii) there exists sequences $([\lambda_n])$ and $([\mu_n])$ such that $([\lambda_n])$ increases to $[0]$ and $([\mu_n])$ decreases to $[0]$ and

$$[\lambda_n] \leq [f_n] \leq [\mu_n], n \in \mathsf{N}.$$

Since the projection π_A is a surjection and E is Dedekind σ-complete, Proposition A.4 implies the existence of sequences (λ_n') and (μ_n') on E such that, for every $n \in \mathsf{N}$, $\pi_A(\lambda_n') = [\lambda_n]$ and $\pi_A(\mu_n') = [\mu_n]$ and (λ_n') increases to 0 and (μ_n') increases to 0. Because π_A is a surjection there exists a sequence (f_n') on E such that $\pi_A(f_n') = [f_n]$ for every $n \in \mathsf{N}$. Define the sequence (f_n) through

$$f_n = (f_n' \vee \lambda_n') \wedge \mu_n', n \in \mathsf{N}.$$

Clearly (f_n) satisfies

$$\lambda_n' \leq f_n \leq \mu_n', n \in \mathsf{N} \tag{2.19}$$

and by Definition 1.24 (i)

$$\begin{aligned} \pi_A(f_n) &= \pi_A((f_n' \vee \lambda_n') \wedge \mu_n') \\ &= \pi_A(f_n' \vee \lambda_n') \wedge \pi_A(\mu_n') \\ &= \pi_A(f_n') \vee \pi_A(\lambda_n') \wedge \pi_A(\mu_n') \\ &= [f_n] \vee [\lambda_n] \wedge [\mu_n] \\ &= [f_n] \end{aligned} \tag{2.20}$$

for every $n \in \mathsf{N}$. By (2.19) the sequence (f_n) order converges to 0. But from (2.20) it follows that

$$\begin{aligned}\pi_A(\langle(f_n)\rangle) &= [[\{\{\pi_A(f_n) : n \geq k\} : k \in \mathsf{N}\}]] \\ &= [[\{\{[f_n] : n \geq k\} : k \in \mathsf{N}\}]] \\ &= \langle([f_n])\rangle\end{aligned}$$

so that Proposition B.4 implies that $([f_n])$ converges to $[0]$ in the quotient convergence structure on $E \setminus A$.
Conversely, suppose that the sequence $([f_n])$ converges to $[0]$ in the quotient convergence structure on $E \setminus A$. By Proposition B.4 there exists a filter $\mathcal{F}$ on E such that $\mathcal{F}$ converges to 0 and $\pi_A(\mathcal{F}) \subseteq \langle([f_n])\rangle$. By Definition 2.1 there exists sequences (λ_n) and (μ_n) on E such that $\lambda_n \uparrow 0$ and $\mu_n \downarrow 0$ and

$$\{[\lambda_n, \mu_n] : n \in \mathsf{N}\} \subseteq \mathcal{F}.$$

Therefore, for every $n \in \mathsf{N}$, there exists $k_n \in \mathsf{N}$ such that

$$\{[f_n] : n \geq k_n\} \subseteq \pi_A([\lambda_n, \mu_n]) = [\pi_A(\lambda_n), \pi_A(\mu_n)].$$

where the equality above is a result of the surjectivety of π_A. Hence we obtain inductively an increasing sequence of naturals $k_1, k_2, k_3, \ldots$ such that

$$\pi_A(\lambda_n) \leq [f_n] \leq \pi_A(\mu_n), n \geq k_n. \tag{2.21}$$

Now define the sequence $([\mu'_n])$ on $E \setminus A$ by

$$[\mu'_n] = \sup\{[f_1], \ldots, [f_{k_1-1}], \pi_A(\mu_1)\}, n = 1, 2, \ldots, k_1 - 1$$

$$[\mu'_n] = \pi_A(\mu_n), n = k_n, k_n + 1, \ldots, k_{n+1} - 1, n = 1, 2, \ldots$$

Since A is a σ-ideal, it follows that π_A is a Riesz σ-homomorphism so that the sequence $([\mu'_n])$ decreases to $[0]$, and by the construction and (2.21) $[f_n] \leq [\mu'_n], n \in \mathsf{N}$. In the same way we can construct a sequence $([\lambda'_n])$ such that $([\lambda'_n])$ increases to $[0]$ and $[\lambda'_n] \leq [f_n], n \in \mathsf{N}$ so that $[f_n]$ order converges to $[0]$. Since A is a σ-ideal Theorem A.9 implies that $E \setminus A$ is Archimedean so that order convergence structure is compatible with the linear structure. But by Proposition B.4 the quotient convergence structure is a vector space convergence structure so that the result follows by linearity and the above. ∎

2.5 The Mackey Modification of (E, λ_o)

As we discussed in Section 1.5, for every convergence vector space F we can define a vector space convergence structure, called the Mackey modification of F, that has exactly the same bounded subsets as F. In the context of order convergence on an Archimedean vector lattice this has a particularly interesting application.

The next result is analogue to Proposition 2.2 and proves equally useful in simplifying the technical nature of many arguments.

Proposition 2.3 *Let E be an Archimedean vector lattice. Then a filter $\mathcal{F}$ on E converges to 0 in $\mu(E)$ if and only if there exists $f \in E^+$ such that*

$$\left[\left\{\left\{\alpha_n g : 0 \leq g \leq f, 0 < \alpha_n < \frac{1}{n}\right\} : n \in \mathsf{N}\right\}\right] \subseteq |\mathcal{F}|$$

where $|\mathcal{F}| = [\{|F| : F \in \mathcal{F}\}]$.

Proof. Let the filter $\mathcal{F}$ converge to 0 in $\mu(E)$. By Definition 1.17 (iii) there exists a bounded subset B of (E, λ_o) such that

$$\mathcal{N}B \subseteq \mathcal{F}.$$

But by Theorem 2.7 there exists $f \in E^+$ such that $B \subseteq [-f, f]$. But this implies that

$$\left[\left\{\left\{\alpha_n g : 0 \leq g \leq f, 0 < \alpha_n < \frac{1}{n}\right\} : n \in \mathsf{N}\right\}\right] \subseteq \mathcal{N}|B|$$

and since

$$\mathcal{N}|B| = \left[\left\{\left\{\alpha_n h : h \in B, 0 < \alpha_n < \frac{1}{n}\right\} : n \in \mathsf{N}\right\}\right] \subseteq |\mathcal{F}|$$

the result follows.
Conversely, suppose that there exists $f \in E^+$ such that

$$\left[\left\{\left\{\alpha_n g : 0 \leq g \leq f, 0 < \alpha_n < \frac{1}{n}\right\} : n \in \mathsf{N}\right\}\right] \subseteq |\mathcal{F}|.$$

Then for every $n \in \mathsf{N}$ there exists $F_n \in \mathcal{F}$ such that

$$F_n \subseteq \left\{\alpha_n g : -f \leq g \leq f, 0 < \alpha_n < \frac{1}{n}\right\}$$

and hence $\mathcal{N}[-f, f] \subseteq \mathcal{F}$. But by Theorem 2.7 $[-f, f]$ is bounded in (E, λ_o) so that $\mathcal{F}$ converges to 0 in $\mu(E)$. This completes the proof. ■

Corollary 2.8 *Let E be an Archimedean vector lattice. Then a filter $\mathcal{F}$ on E converges to $f \in E$ in $\mu(E)$ if and only if there exists $\lambda \leq f \leq \mu \in E$ such that*

$$\left[\left\{\left\{\alpha_n h : h \in [\lambda, \mu], 0 < \alpha_n < \frac{1}{n}\right\} : n \in \mathsf{N}\right\}\right] \subseteq \mathcal{F}.$$

Proof. Let the filter $\mathcal{F}$ on E converges to $f \in E$ in $\mu(E)$. Then the filter $\mathcal{F} - f$ converges to 0 so that Proposition 2.3 implies that there exists $g \in E^+$ such that

$$\left[\left\{\left\{\alpha_n h : 0 \leq h \leq g, 0 < \alpha_n < \frac{1}{n}\right\} : n \in \mathsf{N}\right\}\right] \subseteq |\mathcal{F} - f|. \tag{2.22}$$

If we now set $\lambda = f - g$ and $\mu = f + g$ it follows from (2.22) that, for every $n \in \mathsf{N}$, there exists $F_n \in \mathcal{F}$ such that

$$F_n \subseteq \left\{\alpha_n h : 0 \leq h \leq g, 0 < \alpha_n < \frac{1}{n}\right\}$$

so that the result follows.
The inverse direction follows immediately from Definition 1.27 (i) and Theorem 2.7. ■

By Proposition B.6 the Mackey modification $\mu(F)$ of a convergence vector space F is first countable. Hence sequential convergence suffices to determine adherences and completeness. The characterization of the convergent sequences in $\mu(F)$ is therefore a matter deserving of investigation.

Proposition 2.4 *Let E be an Archimedean vector lattice. A sequence (f_n) on E converges to $f \in E$ in $\mu(E)$ if and only if it converges relatively uniformly to f.*

Proof. Suppose that the sequence (f_n) converges to $f \in E$ in $\mu(E)$. Therefore the Fréchet filter

$$\langle (f_n) \rangle = [\{\{f_n : n \geq k\} : k \in \mathsf{N}\}] \tag{2.23}$$

converges to f in $\mu(E)$ so that the filter $\langle (f_n) \rangle - f$ converges to 0. By Proposition 2.3 there exists $g \in E^+$ such that

$$\left[\left\{\left\{\alpha_n h : 0 \leq h \leq g, 0 < \alpha_n < \frac{1}{n}\right\} : n \in \mathsf{N}\right\}\right] \subseteq |\langle (f_n) \rangle - f|.$$

Then for each $n \in \mathsf{N}$ there exists $F_n \in \langle (f_n) \rangle$ such that

$$|F_n - f| \subseteq \left\{\alpha_n h : 0 \leq h \leq g, 0 < \alpha_n < \frac{1}{n}\right\}. \tag{2.24}$$

From (2.23) and (2.24) it follows that there exists for each $n \in \mathsf{N}$ a natural number K_n such that

$$\{|f - f_n| : n \geq K_n\} \subseteq |F_n - f| \subseteq \left\{\alpha_n h : 0 \leq h \leq g, 0 < \alpha_n < \frac{1}{n}\right\}. \tag{2.25}$$

Now take any $\varepsilon > 0$ and select $N_\varepsilon \in \mathsf{N}$ such that $\frac{1}{N_\varepsilon} < \varepsilon$. Set $N'_\varepsilon = K_{N_\varepsilon}$. Then (2.25) implies that

$$|f - f_n| \leq \frac{1}{N_\varepsilon} g < \varepsilon g, n \geq N'_\varepsilon$$

so that (f_n) converges relatively uniformly to f.
Conversely, assume that the sequence (f_n) converges relatively uniformly to $f \in E$. By Definition 1.28 there exists $g \in E^+$ with the property that for every $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that

$$|f - f_n| < \varepsilon g, n \geq N_\varepsilon.$$

Setting $\varepsilon = \frac{1}{n}$ it is clear that

$$|F_n - f| \subseteq \left\{ \alpha_n h : 0 \leq h \leq g, 0 < \alpha_n < \frac{1}{n} \right\}$$

where

$$F_n = \left\{ f_k : k \leq N_{\frac{1}{n}} \right\}$$

so that $|\langle (f_n) \rangle - f|$ converges to 0 in $\mu(E)$ and hence $\langle (f_n) \rangle$ converges to f in $\mu(E)$. This completes the proof. ■

We now apply Proposition 2.4 above to form a characterization of those Archimedean vector lattices E for which $\mu(E)$ is complete.

Theorem 2.18 *Let E be an Archimedean vector lattice. Then $\mu(E)$ is a complete convergence vector space if and only if E is relatively uniformly complete.*

Proof. Since $\mu(E)$ is first countable by Proposition B.6, it suffices by Proposition B.5 and Definition 1.29 (i) to show that the Cauchy sequences of $\mu(E)$ are exactly the relatively uniformly Cauchy sequences.
Let (f_n) be a Cauchy sequence on $\mu(E)$. By Definition 1.19 (ii) the filter $\langle (f_n) \rangle - \langle (f_n) \rangle$ converges to 0. By Proposition 2.3 there exists $g \in E^+$ such that there exists for every $n \in \mathbb{N}$ a natural number K_n such that

$$\{|f_m - f_k| : m, k \geq K_n\} \subseteq \left\{ \alpha_n h : 0 \leq h \leq g, 0 < \alpha_n < \frac{1}{n} \right\}. \tag{2.26}$$

Now take any $\varepsilon > 0$ and select $N_\varepsilon \in \mathbb{N}$ such that $\frac{1}{N_\varepsilon} < \varepsilon$ and set $N'_\varepsilon = K_{N_\varepsilon}$. Then it follows by (2.26) that $|f_m - f_k| \leq \varepsilon g$ for all $m, k \geq N'_\varepsilon$. Therefore the sequence (f_n) is relatively uniformly Cauchy.
Now assume that the sequence (f_n) is relatively uniformly Cauchy. By Definition 1.29 (i) there exists $g \in E^+$ such that for every $\varepsilon > 0$ there is a natural number N_ε such that

$$|f_k - f_m| < \varepsilon g$$

for all $m, k \geq N_\varepsilon$. Since the filter $\langle (f_n) \rangle - \langle (f_n) \rangle$ is based on the collection of sets

$$\{\{f_k - f_m : k \geq k_1, m \geq k_2\} : k_1, k_2 \in \mathbb{N}\}$$

it follows upon setting $\varepsilon = \frac{1}{n}$ that

$$\{|f_k - f_m| : k, m \geq N_\varepsilon\} \subseteq \left\{ \alpha_n h : 0 \leq h \leq g, 0 < \alpha_n < \frac{1}{n} \right\}$$

for every $n \in \mathsf{N}$ so that

$$\left[\left\{\alpha_n h : 0 \leq h \leq g, 0 < \alpha_n < \frac{1}{n} : n \in \mathsf{N}\right\}\right] \subseteq |\langle (f_n) \rangle - \langle (f_n) \rangle| .$$

By Proposition 2.3 the filter $\langle (f_n) \rangle - \langle (f_n) \rangle$ converges to 0 so that $\langle (f_n) \rangle$ is a Cauchy filter. This completes the proof. ■

The Mackey modification $\mu(E)$ has a relatively simple structure when compared with the order convergence structure. Moreover, it exhibits unusually strong countability properties under some mild assumptions on E. In particular, the following result holds.

Theorem 2.19 *Let E be an Archimedean vector lattice with a strong order unit e. Then the Mackey modification $\mu(E)$ of (E, λ_o) is strongly first countable.*

Proof. Consider the collection

$$\mathcal{B}_0 = \left\{\left\{\alpha_n f : f \in [-m'e, m'e], 0 < \alpha_n < \frac{1}{n}\right\} : n, m \in \mathsf{N}\right\}$$

of subsets of E. We will show that $\mathcal{B}$ is a basis for $\mu(E)$ at 0. Let the filter $\mathcal{F}$ converge to 0 in $\mu(E)$. By Corollary 2.8 there exists $\lambda \leq 0 \leq \mu \in E$ such that

$$\left[\left\{\left\{\alpha_n f : f \in [\lambda, \mu], 0 < \alpha_n < \frac{1}{n}\right\} : n \in \mathsf{N}\right\}\right] \subseteq \mathcal{F}.$$

Since e is a strong order unit there exists $m' \in \mathsf{N}$ such that

$$-m'e \leq \lambda \leq \mu \leq m'e. \tag{2.27}$$

From (2.27) it now follows that

$$\left\{\alpha_n f : f \in [\lambda, \mu], 0 < \alpha_n < \frac{1}{n}\right\} \subseteq \left\{\alpha_n f : f \in [-m'e, m'e], 0 < \alpha_n < \frac{1}{n}\right\}$$

so that the filter

$$\left[\left\{\left\{\alpha_n f : f \in [-m'e, m'e], 0 < \alpha_n < \frac{1}{n}\right\} : n \in \mathsf{N}\right\}\right] \tag{2.28}$$

is coarser than $\mathcal{F}$. But by Corollary 2.8 the filter (2.28) converges to 0 so that $\mathcal{B}$ is a basis for $\mu(E)$ at 0. It now follows by linearity that

$$\mathcal{B}_g = \left\{\left\{\alpha_n f + g : f \in [-m'e, m'e], 0 < \alpha_n < \frac{1}{n}\right\} : n, m \in \mathsf{N}\right\}$$

is a basis for $\mu(E)$ at every $g \in E$. ■

It is therefore of interest to determine the conditions on E that ensure that $\mu(E)$ and (E, λ_o) are identical.

Theorem 2.20 *Let E be an Archimedean vector lattice. Then (E, λ_o) is a Mackey space if and only if order convergence on E is stable.*

Proof. Suppose that order convergence is stable. Let the filter $\mathcal{F}$ converges to 0 in (E, λ_o). By Definition 2.1 there exist sequences $\lambda_n \uparrow 0$ and $\mu_n \downarrow 0$ such that the filter generated by

$$\{[\lambda_n, \mu_n] : n \in \mathsf{N}\}$$

is coarser than $\mathcal{F}$. Without loss of generality we can choose $\lambda_n = -\mu_n$. By assumption, there exists a sequence of real numbers $0 < \alpha_n \uparrow \infty$ such that $\alpha_n \mu_n \rightarrow 0$. Therefore there exists a sequence $\widetilde{\mu}_n \downarrow 0$ such that $\alpha_n \mu_n \leq \widetilde{\mu}_n$, or equivalently,

$$\mu_n \leq \frac{1}{\alpha_n}\widetilde{\mu}_n \leq \frac{1}{\alpha_n}\widetilde{\mu}_1, n \in \mathsf{N}.$$

Similarly,

$$-\frac{1}{\alpha_n}\widetilde{\mu}_1 \leq -\frac{1}{\alpha_n}\widetilde{\mu}_n \leq \lambda_n, n \in \mathsf{N}.$$

The collection

$$\left\{\left[-\frac{1}{\alpha_n}\widetilde{\mu}_1, \frac{1}{\alpha_n}\widetilde{\mu}_1\right] : n \in \mathsf{N}\right\}$$

forms a base for the filter $\mathcal{N}[-\widetilde{\mu}_1, \widetilde{\mu}_1]$. Since

$$[\lambda_n, \mu_n] \subseteq \left[-\frac{1}{\alpha_n}\widetilde{\mu}_1, \frac{1}{\alpha_n}\widetilde{\mu}_1\right], n \in \mathsf{N}$$

it follows that $\mathcal{N}[-\widetilde{\mu}_1, \widetilde{\mu}_1] \subseteq \mathcal{F}$. On the other hand, if a filter $\mathcal{F}$ is finer than $\mathcal{N}B$ for some bounded set B, it converges to 0 since $\mathcal{N}B$ converges to 0 by Definition 1.3 (i).
Conversely, suppose that the convergence vector space (E, λ_o) is a Mackey space and let the sequence (f_n) order converge to f. Then the filter $\langle (f_n) \rangle$ based on the collection

$$\{\{f_n : n \geq k\} : k \in \mathsf{N}\}$$

converges to 0. Hence there exists a bounded set B such that $\mathcal{N}B$ is coarser than $\langle (f_n) \rangle$. Since the filter $\mathcal{N}B$ is based on the collection

$$\left\{\left(-\frac{1}{m}, \frac{1}{m}\right) B : m \in \mathsf{N}\right\}$$

it follows that for every $m \in \mathsf{N}$ there exists $k_m \in \mathsf{N}$ such that

$$\{f_n : n \geq k_m\} \subseteq \left(-\frac{1}{m^2}, \frac{1}{m^2}\right) B \subseteq \left[-\frac{1}{m^2}g, \frac{1}{m^2}g\right]$$

where $g > 0$ and $-g \leq f \leq g, f \in B$. If there exists m_0 such that $k_m = k_{m+1}$ for every $m \geq m_0$ then $f_n = 0$ for every $n \geq k_{m_0}$ and we are done. So assume the opposite, that is, $k_m \uparrow \infty$. Now construct the sequence (α_n) as follows. For $n \geq k_1$ let

$$\alpha_n = m, k_m \leq n < k_{m+1}$$

and for $n < k_1$ let $\alpha_n = 1$. Then $\alpha_n \uparrow \infty$ and

$$-\frac{1}{m}g \leq \alpha_n f_n \leq \frac{1}{m}g, n \geq k_m.$$

Because E is Archimedean, it follows that $\alpha_n f_n \to 0$. This completes the proof. ■

3. A BANACH-STEINHAUS THEOREM

3.1 Linear Mappings on (E, λ_o)

The purpose of this section is to characterize the continuous linear operators mapping (E, λ_o) into (F, λ_o), where both E and F are Archimedean vector lattices, in terms of order-theoretic conditions. Indeed, we show that the continuous operators are σ-order continuous. The converse of this is generally not true. However, for positive operators σ-order continuity is equivalent to continuity in the order convergence structure. Moreover, if the codomain space is Dedekind complete, we show that the continuous operators are exactly the σ-order continuous operators. This is equivalent to the statement that every sequentailly continuous operator is continuous. In general, first countability is not a strong enough condtion to ensure this equivalence between continuity and sequential continuity.

Proposition 3.1 *Let E be an Archimedean vector lattice and F a convergence vector space. A linear mapping $T : E \to F$ is continuous with respect to the order convergence structure if and only if the filter*

$$[\{T([0, \mu_n]) : n \in \mathsf{N}\}]$$

converges to 0 in F whenever the sequence (μ_n) on E^+ decreases to 0.

Proof. Suppose that the filter $\mathcal{F}$ converges to 0 in E. By Definition 2.1 there exists sequences (λ'_n) and (μ'_n) such that (λ'_n) increases to 0 and (μ'_n) decreases to 0 and

$$[\{[\lambda'_n, \mu'_n] : n \in \mathsf{N}\}] \subseteq \mathcal{F}.$$

By assumption the filters

$$[\{T([0, \mu'_n]) : n \in \mathsf{N}\}]$$

and

$$[\{T([0, -\lambda'_n]) : n \in \mathsf{N}\}]$$

both converge to 0 in F. But by the linearity of T

$$[\{T([0, -\lambda'_n]) : n \in \mathsf{N}\}] = -[\{T([\lambda'_n, 0]) : n \in \mathsf{N}\}]$$

so that the filter

$$[\{T([\lambda'_n, 0]) : n \in \mathsf{N}\}]$$

also converges to 0. Therefore the filter

$$\mathcal{G} = [\{T([\lambda'_n, 0]) : n \in \mathsf{N}\}] \cap [\{T([0, \mu'_n]) : n \in \mathsf{N}\}]$$

converges to 0 by Definition 1.5. For each $G \in \mathcal{G}$ there exists $m, n \in \mathsf{N}$ such that

$$T([\lambda_m, \mu_n]) \subseteq G.$$

Without loss of generality we may assume that $m \leq n$. Then by virtue of the monotonicity of the sequences (λ'_n) and (μ'_n) it follows that

$$T([\lambda'_n, \mu'_n]) \subseteq G.$$

It follows that

$$\mathcal{G} \subseteq [\{T([\lambda'_n, \mu'_n]) : n \in \mathsf{N}\}] \subseteq T(\mathcal{F})$$

so that $T(\mathcal{F})$ converges to 0. Since the filter $\mathcal{F}$ was arbitrary, it follows that T is continuous at 0 and hence on E.
The inverse implication is trivial. ■

The above proposition provides us with a tool to analise the relationship between the σ-order continuous operators between two vector lattices and the operators continuous with respect to the order convergence structure. In general, the set of sequentially continuous mappings between two first countable convergence vector spaces is properly larger than the set of continuous linear mappings. Part (iii) of the result which we state below is therefore nontrivial.

Theorem 3.1 *Let E and F be Archimedean vector lattices and $T : E \to F$ a linear mapping.*
(i) If T is continuous with respect to the order convergence structure on E and F then T is σ-order continuous.
(ii) If T is positive and σ-order continuous, then T is continuous with respect to the order convergence structure on E and F.
(iii) If F is Dedekind complete, then T is continuous if and only if it is σ-order continuous.

Proof. (i) Let (μ_n) decrease to 0 in E. By assumption $(T\mu_n)$ and hence $(|T\mu_n|)$ order converges to 0 in F. Therefore, by Definition 1.4, there exists a sequence (λ_n) on F that decreases to 0 such that

$$0 \leq |T\mu_n| = |0 - |T\mu_n|| \leq \lambda_n$$

Therefore $\inf\{|T\mu_n| : n \in \mathsf{N}\} = 0$ and hence T is σ-order continuous.
(ii) Let T be positive and σ-order continuous and let the sequence (μ_n) on E decrease to 0. Consider the filter

$$\mathcal{F} = [\{T([0, \mu_n]) : n \in \mathsf{N}\}]$$

on F. By the monotonicity of T it follows that

$$\mathcal{F} \supseteq [\{[0, T\mu_n] : n \in \mathsf{N}\}]$$

as $T([0,\mu_n]) \subseteq [0,T\mu_n]$ for every $n \in \mathsf{N}$. But $(T\mu_n)$ decreases to 0 since T is positive and σ-order continuous. Therefore $\mathcal{F}$ converges to 0 so that T is continuous by Propostition 3.1.
(iii) Asume that F is Dedekind complete and $T : E \rightarrow F$ is σ-order continuous. By Theorem A.10

$$T = T^{+} - T^{-} \tag{3.1}$$

where T^{+} and T^{-} are positive and σ-order continuous. Consider a sequence (μ_n) on E that decreases to 0. By Proposition 3.1 it is sufficient to show that

$$\mathcal{F} = [\{T([0,\mu_n]) : n \in \mathsf{N}\}]$$

converges to 0 in (F,λ_o). By the decompostion (3.1)

$$\mathcal{F} \supseteq \left[\left\{T^{+}([0,\mu_n]) : n \in \mathsf{N}\right\}\right] - \left[\left\{T^{-}([0,\mu_n]) : n \in \mathsf{N}\right\}\right]. \tag{3.2}$$

By the monotonicity of the operators T^{+} and T^{-} it follows that

$$T^{+}([0,\mu_n]) \subseteq \left[0,T^{+}\mu_n\right], T^{-}([0,\mu_n]) \subseteq \left[0,T^{-}\mu_n\right] \tag{3.3}$$

for every $n \in \mathsf{N}$. Combining (3.2) and (3.3) we obtain

$$\mathcal{F} \supseteq \left[\left\{\left[0,T^{+}\mu_n\right] : n \in \mathsf{N}\right\}\right] - \left[\left\{\left[0,T^{-}\mu_n\right] : n \in \mathsf{N}\right\}\right]. \tag{3.4}$$

But $(T^{+}\mu_n)$ and $(T^{-}\mu_n)$ both decrease to 0 so that (3.4) implies that T is continuous. ∎

A natural order exists on the set of σ-order continuous operators, but it does not make these spaces into vector lattices unless F is Dedekind complete. Hence the order convergence structure, which would seem to be a natural choice, can not be defined on them. In case F is Dedekind complete, one can define the order convergence structure on $\mathcal{L}(E,F)$. The resulting convergence space $(\mathcal{L}(E,F),\lambda_o)$ complete convergence vector space with all the properties discussed in Chapter 2.

Our interest in this chapter is to obtain a Banach-Steinhaus type theorem. A major tool in doing so will be the continuous convergence structure. As an application of Theorem 3.1 above we show that $\mathcal{L}_c(E,F)$, the set $\mathcal{L}(E,F)$ equipped with the continuous convergence structure is complete whenever F is Dedekind complete. In general, the space $\mathcal{L}_c(E,F)$ is not complete, even when F is. Our result is therefore highly non-trivail and its applicability is demonstrated by its use in proving our Banach-Steinhaus theorem. It is unknown at this time whether or not $\mathcal{L}_c(E,F)$ is complete for Dedekind σ-complete F.

Theorem 3.2 *Let E and F be vector lattices with F Dedekind complete. Then $\mathcal{L}_c(E,F)$ is a complete convergence vector space.*

Proof. Let $\mathit{\Psi}$ be a Cauchy filter on $\mathcal{L}_c(E,F)$ and let (f_n) converge to $f \in E$ in (E,λ_o). Then the filter

$$\mathit{\Psi}(\langle(f_n)\rangle) = [\{\{S(f_n) : n \geq k, S \in \mathit{\Psi}\} : k \in \mathsf{N}, \Psi \in \mathit{\Psi}\}]$$

is Cauchy in (F,λ_o). To see this, recall that by Definition 1.19 (ii) the filter $\mathit{\Psi}$ is Cauchy if and only if $\mathit{\Psi} - \mathit{\Psi}$ converges to 0. But by Definition 1.33 $\mathit{\Psi} - \mathit{\Psi}$ converges to 0 if and only if $\omega_{E,F}(\mathit{\Psi} - \mathit{\Psi}, \mathcal{F})$ converges to 0 in F for every filter $\mathcal{F}$ that converges to $f \in E$ in (E,λ_o). But

$$\omega_{E,F}(\mathit{\Psi} - \mathit{\Psi}, \langle(f_n)\rangle) \subseteq \mathit{\Psi}(\langle(f_n)\rangle) - \mathit{\Psi}(\langle(f_n)\rangle)$$

so that $\mathit{\Psi}(\langle(f_n)\rangle)$ is Cauchy in F. By assumption F is Dedekind complete and hence, by Theorem A.7 Dedekind σ-complete. It now follows by Theorem 2.13 that (F,λ_o) is complete so that $\mathit{\Psi}(\langle(f_n)\rangle)$ converges to some $f_{\mathit{\Psi}} \in F$. Define the mapping $T : E \to F$ by

$$T : f \to f_{\mathit{\Psi}}.$$

We show that Tf is independant of the specific choice of the sequence (f_n). Let (f_n) and (h_n) both converge to $f \in E$ in (E,λ_o). Then $(f_n - h_n)$ converges to 0 in (E,λ_o) so that $\mathit{\Psi}(\langle(f_n - h_n)\rangle)$ converges to 0 in F. But

$$\mathit{\Psi}(\langle(f_n - h_n)\rangle) \subseteq \mathit{\Psi}(\langle(f_n)\rangle) - \mathit{\Psi}(\langle(h_n)\rangle)$$

so that the latter filter also converges to 0. Since both $\mathit{\Psi}(\langle(f_n)\rangle)$ and $\mathit{\Psi}(\langle(h_n)\rangle)$ converge in F it follows that they converge to the same limit. The linearity of T follows in the same way. The mapping T is therefore a well-defined linear mapping from E into F. We will show that T is continuous and that the filter $\mathit{\Psi}$ converges to T.
Let the sequence (f_n) decrease to 0 in (E,λ_o) and suppose, for the sake of obtaining a contradiction, that $\inf\{|Tf_n| : n \in \mathsf{N}\} = h > 0$. The infimum exists on account of the Dedekind completeness of F. By the above it follows that the filter

$$|\mathit{\Psi}(f_n)| = [\{\{|Sf_n| : S \in \Psi\} : \Psi \in \mathit{\Psi}\}]$$

converges to $|Tf_n|$ in (F,λ_o) for every $n \in \mathsf{N}$. By Definition 2.1 there exists for every $n \in \mathsf{N}$ sequences (λ_{nm}) and (μ_{nm}) with the property that (λ_{nm}) increases to $|Tf_n|$ and (μ_{nm}) decreases to $|Tf_n|$ and there exixsts for every $m \in \mathsf{N}$ a $\Psi_m^n \in \mathit{\Psi}$ with the property that

$$\{|Sf_n| : S \in \Psi_m^n\} \subseteq [\lambda_{nm}, \mu_{nm}].$$

Note that we can choose the sequences (λ_{nm}) such that

$$0 \leq \lambda_{nm}, n, m \in \mathsf{N}. \tag{3.5}$$

Also, since $0 < h \leq |Tf_n|$ for every $n \in \mathsf{N}$, it follows that for every $n \in \mathsf{N}$ there exists $m_n \in \mathsf{N}$ such that

$$0 < \lambda_{nm}, m \geq m_n. \tag{3.6}$$

However, since (f_n) decreases to 0 the filter$|\mathit{\Psi}(\langle(f_n)\rangle)|$ converges to 0 in (F,λ_o) so that there exists a sequence (μ_n') that decreases to 0 with the property that for every $n\in\mathsf{N}$ there exists $\Psi_n\in\mathit{\Psi}$ and $k_n\in\mathsf{N}$ such that

$$\{|Sf_k|: S\in\Psi_n, k\geq k_n\}\subseteq[0,\mu_n']\,. \tag{3.7}$$

Define the sets $\Psi_n'\in\mathit{\Psi}$ as

$$\Psi_n'=\Psi_n\cap\Psi_{m_n}^n$$

for every $n\in\mathsf{N}$. Since $\Psi_n'\subseteq\Psi_n$ for every $n\in\mathsf{N}$ it follows by (3.7) that

$$\{|Sf_k|: S\in\Psi_n', k\geq k_n\}\subseteq[0,\mu_n']\,.$$

Therefore the filter

$$[\{\{|Sf_k|: S\in\Psi_n', k\geq k_n\}: n\in\mathsf{N}\}] \tag{3.8}$$

converges to 0 in (F,λ_o). Define the sequence (λ_n) by

$$\lambda_n=\sup\{\lambda_{km_k}: k\leq n\}\,. \tag{3.9}$$

Clearly the sequence (λ_n) is increasing and

$$\lambda_n\leq|Sf_n|\,, S\in\Psi_n'$$

for every $n\in\mathsf{N}$. It now follows by (3.5) and (3.9) that $0\leq\lambda_n$ for every $n\in\mathsf{N}$. But the convergence of the filter (3.8) now forces the equality $\lambda_n=0$. However, it follows from (3.6) and (3.9) that $0<\lambda_n$ for every $n\in\mathsf{N}$ which is clearly a contradiction. The assumption that $\inf\{|Tf_n|: n\in\mathsf{N}\}=h>0$ must therefore be untrue, and hence it must be true that $\inf\{|Tf_n|: n\in\mathsf{N}\}=0$. Since the sequence (f_n) that decreases to 0 was arbitrary the mapping T is σ-order continuous and hence by Theorem 3.1 (iii) it is continuous with respect to the order convergence structure on E and F.
Let the filter $\mathcal{F}$ converge to $f\in E$ in (E,λ_o). By Definition 2.1 there exists sequences (λ_n) and (μ_n) that respectively increase and decrease to f such that

$$\mathcal{G}=[\{[\lambda_n,\mu_n]: n\in\mathsf{N}\}]\subseteq\mathcal{F}.$$

We must show that $\omega_{E,F}(\mathit{\Psi},\mathcal{F})$ converges to Tf in (F,λ_o). Since $\omega_{E,F}(\mathit{\Psi},\mathcal{G})\subseteq\omega_{E,F}(\mathit{\Psi},\mathcal{F})$ it is sufficient to consider $\omega_{E,F}(\mathit{\Psi},\mathcal{G})$. It follows in the same way as above that the filter $\omega_{E,F}(\mathit{\Psi},\mathcal{G})$ converges to some $g\in F$. But $\mathit{\Psi}(\langle(\mu_n)\rangle)$ converges to Tf by our definition of the mapping T. It follows from the inclusion $\mathcal{G}\subseteq\langle(\mu_n)\rangle$ that

$$\omega_{E,F}(\mathit{\Psi},\mathcal{G})\subseteq\mathit{\Psi}(\langle(\mu_n)\rangle)\,.$$

Therefore it follows by Definition 1.5 (iii) that the $\mathit{\Psi}(\langle(\mu_n)\rangle)$ converges to some $g\in F$ in (F,λ_o). But by Corollary 2.4 (F,λ_o) is Hausdorff so that $g=Tf$. Since $f\in E$ and the filter $\mathcal{F}$ were arbitrary it follows that $\mathit{\Psi}$ converges to T in $\mathcal{L}_c(E,F)$ so that $\mathcal{L}_c(E,F)$ is complete. ■

3.2 A Duality Theorem for (E, λ_o)

Throughout this section we assume that E is an Archimedean vector lattice that also satisfies the following additional property.

SP1 $L_c E = \mathcal{L}E$ separates the points of E.

We show that under condition (SP1) the space (E, λ_o) is isomorphic to a space $(\sigma(E), \lambda_o)$ where $\sigma(E)$ is a vector lattice subspace of $(\mathcal{L}\mathcal{L}E, \lambda_o)$. We are interested in such a theorem only because it will prove to be a vital step in proving a Banach-Steinhaus theorem for σ-order continuous operators which is the main result of this chapter.

The result described above is a consequence of the classical embedding results in vector lattice theory, as described in Appendix A, and the following proposition on isomorphisms.

Proposition 3.2 *Let E and F be Archimedean vector lattices and $\pi : E \to F$ a Riesz isomorphism. Then $\pi : (E, \lambda_o) \to (F, \lambda_o)$ is a convergence space isomorphism.*

Proof. By Definition 1.24 (ii) the mapping π is a bijective Riesz homomorphism such that π^{-1} is monotone. It remains to show that π and π^{-1} are continuous. Let the filter $\mathcal{F}$ converge to 0 in (E, λ_o), that is, there exists sequences (λ_n) and (μ_n) such that (λ_n) increases to 0 and (μ_n) decrease to 0 in E and

$$\mathcal{G} = [\{[\lambda_n, \mu_n] : n \in \mathsf{N}\}] \subseteq \mathcal{F}.$$

Now, by the monotonicity and surjectiveness of π it follows that

$$\pi([\lambda_n, \mu_n]) = [\pi\lambda_n, \pi\mu_n], n \in \mathsf{N}.$$

It remains to show that $(\pi\lambda_n)$ increases to 0 and $(\pi\mu_n)$ decreases to 0. Since π is monotone, it follows that $(\pi\mu_n)$ is decreasing and positive in F. Supose that there exists $f > 0$ in F such that $f \leq \pi\mu_n$ for every $n \in \mathsf{N}$. Since π^{-1} is monotone, and surjective it follows that

$$0 < \pi^{-1} f \leq \pi^{-1}(\pi(\mu_n)) = \mu_n, n \in \mathsf{N}.$$

But then it is not true that (μ_n) decreases to 0 in E, a contradiction. Therefore $(\pi\mu_n)$ decreases to 0 in F. In a similar way $(\pi\lambda_n)$ increases to 0 so that π is continuous.
The continuity of π^{-1} follows in the same way. ■

With our isomorphism theorem in place, we now proceed to define a mapping $\sigma : E \to \mathcal{L}\mathcal{L}E$. Recall that for every $f \in E$ the the mapping $f^* : \mathcal{L}E \to \mathsf{R}$ defined by

$$f^*(\varphi) = \varphi(f), \varphi \in \mathcal{L}E$$

is linear. Since $\mathcal{L}E = L_c E$ is a band in $L_b E$, Theorems A.10 and A.12 apply so that $f^* \in \mathcal{L}\mathcal{L}E = L_c(L_c E)$. Therefore the mapping $\sigma : E \to \mathcal{L}(\mathcal{L}E, \lambda_o)$ given by

$$\sigma(f) = f^*$$

is well defined. The following is the main result of this section.

Theorem 3.3 *Let E be an Archimedean vector lattice. If E satisfies (SP1), then there exists a vector sublattice $\sigma(E)$ of $\mathcal{L}(\mathcal{L}E, \lambda_o)$ such that (E, λ_o) is isomorphic to $(\sigma(E), \lambda_o)$.*

Proof. According to Theorem 3.1 $\mathcal{L}E = L_c E$ and by Theorem A.10 $L_c E$ is a band in $L_b E$, and hence an ideal. By Theorem A.12 the mapping σ is a Riesz isomorphism into $L_n(L_c E)$. But by Theorem A.10 $L_n(L_c E)$ is a band in $L_c(L_c E)$, and by Theorem 3.1 $L_c(L_c E) = \mathcal{L}(\mathcal{L}E, \lambda_o)$. By Proposition 3.2 it follows that (E, λ_o) is isomorphic to $(\sigma(E), \lambda_o)$ where $\sigma(E)$ is the image of E under σ. ■

The above result will be applied in Section 3.4 to obtain a Banach-Steinhaus type theorem for σ-order continuous operators.

3.3 Some Banach-Steinhaus Pairs of the Form $(\mu(E), F)$

Before we prove our main result in the next section, we give some results on bounded operators. We will first characterize the bounded operators mapping (E, λ_o) into (F, λ_o), with both E and F Archimedean vector lattices, in terms the partial orders on E and F.

Theorem 3.4 *Let E and F be Archimedean vector lattices. Then the linear operator $T : E \to F$ is continuous with respect to the Mackey modification $\mu(E)$ and $\mu(F)$ if and only if it is order bounded.*

Proof. By Propostion B.7 T is continuous if and only if it maps bounded subsets of $\mu(E)$ into bounded subsets of $\mu(F)$. But by Propostion B.6 the bounded subsets of $\mu(E)$ and $\mu(F)$ are exactly the bounded subsets of E and F respectively. But by Theorem 2.7 the bounded subsets of (E, λ_o) and (F, λ_o) are exactly the order bounded subsets of E and F respectively. By Definition 1.31 (i) it follows that T is continuous if and only if it is order bounded. ■

The central result of this section is on the equicontinuity of pointwise bounded subsets of the dual of $\mu(E)$. In particular, we show that $\mu(E)$ is barrelled. This allows us to formulate sufficient conditions on a convergence vector space F for $(\mu(E), F)$ to be a Banach-Steinhaus pair.

Theorem 3.5 *Let E be an Archimedean vector lattice. Then the convergence vector space $\mu(E)$ is barreled.*

Proof. First note that according to Theorems 3.4 and A.10, $\mathcal{L}(\mu(E))$ is exactly the Dedekind complete vector lattice of all order bounded linear functionals on E. By Definition 1.35 we must show that every bounded subset of $\mathcal{L}_s(\mu(E))$ is equicontinuous. Let the subset H of $\mathcal{L}_s(\mu(E))$ be bounded. Then by Theorem A.11 there exists $\psi_1 \in \mathcal{L}_s(\mu(E))$ such that $\varphi \leq \psi_1$ for every $\varphi \in H$ and writing

$$\sup\{-\varphi : \varphi \in H\} = -\inf\{\varphi : \varphi \in H\}$$

if follows, again by Theorem A.11, that there exists $\psi_0 \in \mathcal{L}_s(\mu(E))$ such that $\psi_0 \leq \varphi$ for every $\varphi \in H$. Therefore

$$\varphi^+ = \varphi \vee 0 \leq \psi_1 \vee 0 = \psi_1^+ \tag{3.10}$$

and

$$\varphi^- = (-\varphi \vee 0) \leq (-\psi_0 \vee 0) = \psi_0^- \tag{3.11}$$

for every $\varphi \in H$. Now let the filter $\mathcal{F}$ on E converge to 0 in $\mu(E)$. It is easily checked that

$$\begin{aligned} H(\mathcal{F}) &= [\{\{\varphi(f) : f \in F, \varphi \in H\} : F \in \mathcal{F}\}] \\ &= [\{\{\varphi^+(f) - \varphi^-(f) : f \in F, \varphi \in H\} : F \in \mathcal{F}\}] \\ &\supseteq H^+(\mathcal{F}) - H^-(\mathcal{F}) \end{aligned}$$

where $H^+ = \{\varphi^+ : \varphi \in H\}$ and $H^- = \{\varphi^- : \varphi \in H\}$. It is therefore sufficient to prove that both $H^+(\mathcal{F})$ and $H^-(\mathcal{F})$ converge to 0. By Corollary 2.8 there exists $\lambda \leq 0 \leq \mu \in E$ such that

$$\mathcal{G} = \left[\left\{\left\{\alpha_n f : \lambda \leq f \leq \mu, |\alpha_n| < \frac{1}{n}\right\} : n \in \mathbb{N}\right\}\right] \subseteq \mathcal{F}.$$

By (3.10) and the monotonicity of ψ_1^+ and the φ^+ it follows that

$$\begin{aligned} H^+(\mathcal{G}) &= \left[\left\{\left\{\alpha_n \varphi^+(f) : \lambda \leq f \leq \mu, \varphi \in H, |\alpha_n| < \frac{1}{n}\right\} : n \in \mathbb{N}\right\}\right] \\ &\supseteq \left[\left\{\left\{\alpha_n \beta : \psi_1^+(\lambda) \leq \beta \leq \psi_1^+(\mu), |\alpha_n| < \frac{1}{n}\right\} : n \in \mathbb{N}\right\}\right] \end{aligned}$$

so that $H^+(\mathcal{G})$ converges to 0 in $\mathbb{R}$. In the same way, using (3.11), $H^-(\mathcal{G})$ converges to 0 in $\mathbb{R}$ so that $H(\mathcal{G})$ converges to 0 in $\mathbb{R}$. This completes the proof. ■

As a Corollary to the above, we now obtain the following sufficient conditions on a convergence vector space F in order for $(\mu(E), F)$ to be a Banach-Steinhaus pair.

Corollary 3.1 *Let E be an Archimedean vector lattice and F a convergence vector space. Then $(\mu(E), F)$ is a Banach-Steinhaus pair in each of the following cases:*
(i) F is a locally convex topological vector space
(ii) $F = \mathcal{L}_c G$ where G is either a first countable or locally bounded convergence vector space. In particular, G is an Archimedean vector lattice equiped with the Mackey modification of order convergence structure or the order convergence structure itself.

3.4 A Banach-Steinhaus Theorem for (E, λ_o)

We follow the standard approach in proving a Banach-Steinhaus type theorem and first consider the dual space. In particular, we show that (E, λ_o) is barrelled. This is a prerequisite for more general results, for if $(E, \mathbb{R})$ fails to be a Banach-Steinhaus pair, it is not to be expected that such a result should hold for a more general space than $\mathbb{R}$.

Theorem 3.6 *Let E be an Archimedean vector lattice. Then the convergence vector space (E, λ_o) is barreled.*

Proof. First note that according to Theorems 3.1 and A.10, $\mathcal{L}(E)$ is exactly the Dedekind complete vector lattice of all σ-order continuous linear functionals on E. By Definition 1.35 we must show that every bounded subset of $\mathcal{L}_s(E)$ is equicontinuous. Let the subset H of $\mathcal{L}_s(E)$ be bounded. Since $\mathcal{L}_s(E) \subseteq \mathcal{L}_s(\mu(E))$ it follows that H is bounded in $\mathcal{L}_s(\mu(E))$. Then by Theorems 3.4 and A.11 there exists $\psi_1 \in \mathcal{L}_s(\mu(E))$ such that $\varphi \leq \psi_1$ for every $\varphi \in H$ and writing

$$\sup\{-\varphi : \varphi \in H\} = -\inf\{\varphi : \varphi \in H\}$$

if follows, again by Theorem A.11, that there exists $\psi_0 \in \mathcal{L}_s(\mu(E))$ such that $\psi_0 \leq \varphi$ for every $\varphi \in H$. Therefore

$$0 \leq \varphi^+ = \varphi \vee 0 \leq \psi_1 \vee 0 = \psi_1^+ \tag{3.12}$$

and

$$0 \leq \varphi^- = (-\varphi \vee 0) \leq (-\psi_0 \vee 0) = \psi_0^-. \tag{3.13}$$

By Theorems 3.1 (iii), 3.4 and A.10 $\mathcal{L}(E)$ is a band in the Dedekind complete vector lattice $\mathcal{L}(\mu(E))$ so that ψ_1^+ and ψ_0^- can be taken to be σ-order continuous, and hence by Theorem 3.1 (iii) continuous on (E, λ_o). Now let the filter $\mathcal{F}$ on E converge to 0 in (E, λ_o). It is easily checked that

$$\begin{aligned} H(\mathcal{F}) &= [\{\{\varphi(f) : f \in F, \varphi \in H\} : F \in \mathcal{F}\}] \\ &= \left[\left\{\left\{\varphi^+(f) - \varphi^-(f) : f \in F, \varphi \in H\right\} : F \in \mathcal{F}\right\}\right] \\ &\supseteq H^+(\mathcal{F}) - H^-(\mathcal{F}) \end{aligned}$$

where $H^+ = \{\varphi^+ : \varphi \in H\}$ and $H^- = \{\varphi^- : \varphi \in H\}$. It is therefore sufficient to prove that both $H^+(\mathcal{F})$ and $H^-(\mathcal{F})$ converges to 0. By Definition 2.1 there exists sequences (λ_n) and (μ_n) that respectively increase and decrease to 0 such that

$$\mathcal{G} = [\{[\lambda_n, \mu_n] : n \in \mathsf{N}\}] \subseteq \mathcal{F}.$$

By (3.12) and the monotonicity of ψ_1^+ and the φ^+ it follows that

$$\begin{array}{rcl} H^+(\mathcal{G}) & = & \left[\left\{\left\{\varphi^+(f) : \lambda_n \leq f \leq \mu_n, \varphi \in H\right\} : n \in \mathsf{N}\right\}\right] \\ & \supseteq & \left[\left\{\left[\psi_1^+(\lambda_m), \psi_1^+(\mu_n)\right] : n \in \mathsf{N}\right\}\right] \end{array}$$

so that $H^+(\mathcal{G})$ converges to 0 in R. In the same way, using (3.13), $H^-(\mathcal{G})$ converges to 0 in R. so that $H(\mathcal{G})$ converges to 0 in R. The result follows. ■

Corollary 3.2 *Let E be an Archimedean vector lattice and F a convergence vector space. Then $((E, \lambda_o), F)$ is a Banach-Steinhaus pair in each of the following situations.*
(i) F is a locally convex topological vector space;
(ii) $F = \mathcal{L}_c G$ where G is either a first countable or locally bounded convergence vector space. In particular, G is an Archimedean vector lattice equiped with order convergence structure;

Corollary 3.2 states that if E and F are Archimedean vector lattices, equiped with order convergence structure, then $(E, \mathcal{L}_c F)$ is a Banach-Steinhuas pair. However, both the spaces $\mathcal{L}G$ and $\mathcal{L}(E, \mathcal{L}G)$ carry a natural order structure that makes them into Dedekind complete vector lattices. Hence we can equip them both with order convergence structure. We now utilize this fact, together with some results from vector lattice theorey, to obtain a Banach-Steinhaus type theorem for order convergence structure.

Proposition 3.3 *Let E be an Archimedean vector lattice. Then the order convergence structure on $\mathcal{L}E$ is finer than the continuous convergence structure.*

Proof. Let the filter Ψ on $\mathcal{L}E$ converge to $\varphi \in \mathcal{L}E$ in order convergence structure. By Definition 2.1 there exists sequences (ψ_n) and (ϕ_n) on $\mathcal{L}E$ such that (ψ_n) increases to φ and (ϕ_n) decreases to φ and

$$\Phi = [\{[\psi_n, \phi_n] : n \in \mathsf{N}\}] \subseteq \Psi.$$

Now let the filter $\mathcal{F}$ on E converge to $f \in E$ in the order convergence structure. Again by Definition 2.1 there exists sequences (λ_n) and (μ_n) such that (λ_n) increases to f and (μ_n) decreases to f and

$$\mathcal{G} = [\{[\lambda_n, \mu_n] : n \in \mathsf{N}\}] \subseteq \mathcal{F}.$$

It suffices to show that

$$\Phi(\mathcal{G}) = [\{\{\varsigma(g) : \psi_n \leq \varsigma \leq \phi_n, \lambda_k \leq g \leq \mu_k\} : n, k \in \mathsf{N}\}]$$

converges to $\varphi(f)$. Since $\mathcal{L}E$ is the Dedekind compelete vector lattice of all σ-continuous linear functionals on E by Theorems 3.1 (iii) and A.10 we have a decompostion

$$\phi = \phi^{+} - \phi^{-}$$

of any $\phi \in \mathcal{L}(E)$ into a difference of positive functionals. Hence we obtain the inclusion

$$\begin{aligned} \Phi(\mathcal{G}) \supseteq & \left[\left\{\left\{\varsigma^{+}(g) : \psi_n \leq \varsigma \leq \phi_n, \lambda_k \leq g \leq \mu_k\right\} : n,k \in \mathsf{N}\right\}\right] \\ & - \left[\left\{\left\{\varsigma^{-}(g) : \psi_n \leq \varsigma \leq \phi_n, \lambda_k \leq g \leq \mu_k\right\} : n,k \in \mathsf{N}\right\}\right]. \end{aligned}$$

But a similar decompostion $f = f^{+} - f^{-}$ holds in E so that

$$\begin{aligned} \Phi(\mathcal{G}) \supseteq & \left[\left\{\left\{\varsigma^{+}\left(g^{+}\right) : \psi_n \leq \varsigma \leq \phi_n, \lambda_k \leq g \leq \mu_k\right\} : n,k \in \mathsf{N}\right\}\right] \\ & - \left[\left\{\left\{\varsigma^{+}\left(g^{-}\right) : \psi_n \leq \varsigma \leq \phi_n, \lambda_k \leq g \leq \mu_k\right\} : n,k \in \mathsf{N}\right\}\right] \\ & - \left[\left\{\left\{\varsigma^{-}\left(g^{+}\right) : \psi_n \leq \varsigma \leq \phi_n, \lambda_k \leq g \leq \mu_k\right\} : n,k \in \mathsf{N}\right\}\right] \\ & + \left[\left\{\left\{\varsigma^{-}(g)^{-} : \psi_n \leq \varsigma \leq \phi_n, \lambda_k \leq g \leq \mu_k\right\} : n,k \in \mathsf{N}\right\}\right]. \end{aligned}$$

By Propositions A.1 the above reduces to

$$\begin{aligned} \Phi(\mathcal{G}) \supseteq & \left[\left\{\left[\psi_n^{+}\left(\lambda_k^{+}\right), \phi_n^{+}\left(\mu_k^{+}\right)\right] : n,k \in \mathsf{N}\right\}\right] \\ & - \left[\left\{\left[\psi_n^{+}\left(\mu_k^{-}\right), \phi_n^{+}\left(\lambda_k^{-}\right)\right] : n,k \in \mathsf{N}\right\}\right] \\ & - \left[\left\{\left[\phi_n^{-}\left(\lambda_k^{+}\right), \psi_n^{-}\left(\mu_k^{+}\right)\right] : n,k \in \mathsf{N}\right\}\right] \\ & + \left[\left\{\left[\phi_n^{-}\left(\lambda_k^{+}\right), \psi_n^{-}\left(\mu_k^{+}\right)\right] : n,k \in \mathsf{N}\right\}\right]. \end{aligned} \tag{3.14}$$

Since, by Proposition A.1 and Theorem A.2 (iii) and monotonicity, for every $k \in \mathsf{N}$ the sequence $\left(\psi_n^{+}\left(\lambda_k^{+}\right)\right)$ increases to $\varphi^{+}\left(\lambda_k^{+}\right)$, and the sequence $\left(\varphi^{+}\left(\lambda_k^{+}\right)\right)$ increases to $\varphi^{+}f$, there exists, according to Theorem 1.1 and (MS4), for every $n \in \mathsf{N}$ a natural number k_n such that $k_n \leq k_{n+1}$ and $\left(\psi_n^{+}\left(\lambda_{k_n}^{+}\right)\right)$ converges to $\varphi^{+}f^{+}$. By Theorem A.5 it follows that the sequence (g_m) defined by

$$g_m = \inf\left\{\psi_n^{+}\left(\lambda_{k_n}^{+}\right) : n \geq m\right\}$$

increases to $\varphi^{+}f^{+}$. But (g_m) also satisfies the inequality

$$g_m \leq \psi_m^{+}\left(\lambda_{k_m}^{+}\right), m \in \mathsf{N}. \tag{3.15}$$

In the same way we can construct a sequence (h_n) that decreases to $\varphi^{+}f^{+}$ and satisfies

$$\phi_m^{+}\left(\mu_{k_m}^{+}\right) \leq h_m, m \in \mathsf{N}. \tag{3.16}$$

By (3.15) and (3.16) the inclusion

$$\begin{aligned} \left[\left\{\left[\psi_n^{+}\left(\lambda_k^{+}\right), \phi_n^{+}\left(\mu_k^{+}\right)\right] : n,k \in \mathsf{N}\right\}\right] & \supseteq \left[\left\{\left[\psi_n^{+}\left(\lambda_{k_n}^{+}\right), \phi_n^{+}\left(\mu_{k_n}^{+}\right)\right] : n \in \mathsf{N}\right\}\right] \\ & \supseteq \left[\left\{\left[g_n, h_n\right] : n \in \mathsf{N}\right\}\right] \end{aligned}$$

holds. But $[\{[g_n, h_n] : n \in \mathsf{N}\}]$ converges to $\varphi^{+}f^{+}$ so that

$$\left[\left\{\left[\psi_n^{+}\left(\lambda_n^{+}\right), \phi_n^{+}\left(\mu_n^{+}\right)\right] : n \in \mathsf{N}\right\}\right] \to \varphi^{+}\left(f^{+}\right).$$

In the same way we also obtain

$$\left[\left\{\left[\psi_n^+\left(\mu_n^-\right),\phi_n^+\left(\lambda_n^-\right)\right]:n\in\mathbf{N}\right\}\right]\to\varphi^+\left(f^-\right),$$

$$\left[\left\{\left[\phi_n^-\left(\lambda_n^+\right),\psi_n^-\left(\mu_n^+\right)\right]:n\in\mathbf{N}\right\}\right]\to\varphi^-\left(f^+\right)$$

and

$$\left[\left\{\left[\phi_n^-\left(\mu_n^-\right),\psi_n^-\left(\lambda_n^-\right)\right]:n\in\mathbf{N}\right\}\right]\to\varphi^-\left(f^-\right)$$

where the convergence above takes place in $\mathbf{R}$. Hence the inclusion (3.14) implies that $\Phi(\mathcal{G})$ converges to

$$\varphi^+\left(f^+\right)-\varphi^+\left(f^-\right)-\varphi^-\left(f^+\right)+\varphi^-\left(f^-\right)=\varphi^+(f)-\varphi^-(f)=\varphi(f).$$

This completes the proof. ∎

By Proposition 3.3 above it follows that

$$\mathcal{L}\left(\left(E,\lambda_o\right),\left(\mathcal{L}F,\lambda_o\right)\right)\subseteq\mathcal{L}\left(\left(E,\lambda_o\right),\mathcal{L}_cF\right). \tag{3.17}$$

This is the first step in obtaining the desired Banach-Steinhuass theorem for σ-order continuous operators. The next step is to show that $\mathcal{L}\left(\left(E,\lambda_o\right),\left(\mathcal{L}F,\lambda_o\right)\right)$ is a closed subspace of $\mathcal{L}_c\left(\left(E,\lambda_o\right),\mathcal{L}_cF\right)$.

Proposition 3.4 *Let E and F be Archimedean vector lattices. Then the continuous convergences structure on $\mathcal{L}\left(\left(E,\lambda_o\right),\left(\mathcal{L}F,\lambda_o\right)\right)$ is coarser than the subspace convergence structure inherrited from $\mathcal{L}_c\left(\left(E,\lambda_o\right),\mathcal{L}_cF\right)$ and*

$$a\left(\mathcal{L}\left(\left(E,\lambda_o\right),\left(\mathcal{L}F,\lambda_o\right)\right)\right)=\mathcal{L}\left(\left(E,\lambda_o\right),\left(\mathcal{L}F,\lambda_o\right)\right)$$

where the adherence is taken in $\mathcal{L}_c\left(\left(E,\lambda_o\right),\mathcal{L}_cF\right)$.

Proof. Let the filter $\mathit{\Psi}$ with a trace on $\mathcal{L}\left(\left(E,\lambda_o\right),\left(\mathcal{L}F,\lambda_o\right)\right)$ converge to $T\in\mathcal{L}_c\left(\left(E,\lambda_o\right),\mathcal{L}_cF\right)$. By Definition 1.33 the filter

$$\omega\left(\mathit{\Psi}\times\mathcal{F}\right)=\left[\left\{\left\{Sg:S\in\Psi,g\in F\right\}:\Psi\in\mathit{\Psi},F\in\mathcal{F}\right\}\right]$$

converges to Tf for every filter $\mathcal{F}$ on $\left(E,\lambda_o\right)$ that converges to f. Consider the trace filter $\mathit{\Psi}|A$ on $\mathcal{L}\left(\left(E,\lambda_o\right),\left(\mathcal{L}F,\lambda_o\right)\right)$ and the filter

$$\begin{aligned}\omega\left(\left(\mathit{\Psi}|A-\mathit{\Psi}|A\right)\times\mathcal{F}\right) &= [\{\{Sg-Ug:S\in\Psi_1,U\in\Psi_2,g\in F\}\\ &: \Psi_1,\Psi_2\in\mathit{\Psi}|A,F\in\mathcal{F}\}]\\ &\supseteq [\{\{Sg:S\in\Psi,g\in F\}:\Psi\in\mathit{\Psi}|A,F\in\mathcal{F}\}]\\ &\quad -[\{\{Sg:S\in\Psi,g\in F\}:\Psi\in\mathit{\Psi}|A,F\in\mathcal{F}\}]\\ &\supseteq \omega\left(\mathit{\Psi}\times\mathcal{F}\right)-\omega\left(\mathit{\Psi}\times\mathcal{F}\right).\end{aligned}$$

Therefore $\omega\left(\left(\mathit{\Psi}|A-\mathit{\Psi}|A\right)\times\mathcal{F}\right)$ converges to 0 and hence, since $\mathcal{F}$ is an arbitrary convergent filter, it follows that $\mathit{\Psi}|A-\mathit{\Psi}|A$ converges to 0 in $\mathcal{L}_c\left(\left(E,\lambda_o\right),\left(\mathcal{L}F,\lambda_o\right)\right)$ so that $\mathit{\Psi}|A$ is a Cauchy filter. But by Theorem 3.2 the

space $\mathcal{L}_c((E,\lambda_o),(\mathcal{L}F,\lambda_o))$ is complete since $\mathcal{L}F$ is Dedekind complete. Therefore $\Psi|A$ converges to some $S \in \mathcal{L}_c((E,\lambda_o),(\mathcal{L}F,\lambda_o))$. But $\omega(\Psi \times \mathcal{F}) \subseteq \omega(\Psi|A \times \mathcal{F})$ and hence $T = S$ so that $\mathcal{L}_c((E,\lambda_o),(\mathcal{L}F,\lambda_o))$ is coarser than $\mathcal{L}((E,\lambda_o),(\mathcal{L}F,\lambda_o))$ equiped with the subspace convergence structure inherrited from $\mathcal{L}_c((E,\lambda_o),\mathcal{L}_cF)$.
Since every filter that contains a set has a trace on it the above implies that $\mathcal{L}((E,\lambda_o),(\mathcal{L}F,\lambda_o))$ is closed. ∎

Propositions 3.3 and 3.4 above now enable us to proof the following result which is our main tool in studying the Banach-Steinhaus theorem for the order convergence structure.

Proposition 3.5 *Let E and F be Archimedean vector lattices. If (T_n) is a sequence in $\mathcal{L}((E,\lambda_o),(\mathcal{L}F,\lambda_o))$ that converges pointwise to a linear mapping $T : E \to \mathcal{L}F$, then $T \in \mathcal{L}((E,\lambda_o),(\mathcal{L}F,\lambda_o))$ and (T_n) converges continuously to T.*

Proof. By Corollary 3.2 (ii) $((E,\lambda_o),\mathcal{L}_cF)$ is a Banach-Steinhaus pair. But by (3.17) it follows that

$$(T_n) \subset \mathcal{L}((E,\lambda_o),\mathcal{L}_cF).$$

But by Proposition B.3 $\mathcal{L}_cF$ is regular and Choquet. Therefore Theorem B.3 implies that $T : E \to \mathcal{L}_cF$ is a continuous linear mapping and (T_n) converges continuously to T. Propostion 3.4 now implies the desired result. ∎

We now prove the main result of the chapter. We show that pointwise convergence of a sequence of positive operators implies that the limit is continuous. Moreover, if the space F is Dedekind complete, then this is true even when the operators involved are not positive.

Theorem 3.7 *Let E and G be Archimedean vector lattices such that $\mathcal{L}G$ separates the points of G.*
(i) If (T_n) is a sequence of postive operators in $\mathcal{L}((E,\lambda_o),(G,\lambda_o))$ that converges pointwise to a linear mapping $T : E \to G$, then $T \in \mathcal{L}((E,\lambda_o),(G,\lambda_o))$.
(ii) If (T_n) is a sequence in $\mathcal{L}((E,\lambda_o),(G,\lambda_o))$ that converges pointwise to a linear mapping $T : E \to G$, then T is σ-order continuous. If F is Dedekind complete, then $T \in \mathcal{L}((E,\lambda_o),(G,\lambda_o))$.

Proof. (i) By Theorem 3.3 (T_n) is contained in $\mathcal{L}((E,\lambda_o),(\mathcal{L}\mathcal{L}G,\lambda_o))$. In Proposition 3.5 above, set $F = \mathcal{L}G$. Then $T \in \mathcal{L}((E,\lambda_o),(\mathcal{L}\mathcal{L}G,\lambda_o))$ and (T_n) converges continuously to T. Now T takes all of its values in the subspace $\sigma(G)$ of $(\mathcal{L}\mathcal{L}G,\lambda_o)$, where $\sigma : G \to \mathcal{L}\mathcal{L}G$ is the Riesz isomorphism of Theorem 3.3. By Theorem 3.1 (i) $T : E \to \mathcal{L}\mathcal{L}G$ is σ-order continuous. Hence, for every sequence (μ_n) such that $\mu_n \downarrow 0$, we have

$$\inf\{|T\mu_n| : n \in \mathsf{N}\} = 0.$$

But $T(E) \subseteq \sigma(G)$ and since $\sigma(G)$ is a vector lattice subspace of $\mathcal{L}\mathcal{L}G$ it follows that $T : E \to G$ is σ-order continuous. Applying Theorem 3.1 (ii) completes the proof.

(ii) By Theorem 3.3 (T_n) is contained in $\mathcal{L}((E, \lambda_o), (\mathcal{L}\mathcal{L}G, \lambda_o))$. In Proposition 3.5 above, set $F = \mathcal{L}G$. Then $T \in \mathcal{L}((E, \lambda_o), (\mathcal{L}\mathcal{L}G, \lambda_o))$ and (T_n) converges continuously to T. Now T takes all of its values in the subspace $\sigma(G)$ of $(\mathcal{L}\mathcal{L}G, \lambda_o)$, where $\sigma : G \to \mathcal{L}\mathcal{L}G$ is the Riesz isomorphism of Theorem 3.3. By Theorem 3.1 (i) $T : E \to \mathcal{L}\mathcal{L}G$ is σ-order continuous. Hence, for every sequence (μ_n) such that $\mu_n \downarrow 0$, we have

$$\inf \{|T\mu_n| : n \in \mathsf{N}\} = 0.$$

But $T(E) \subseteq \sigma(G)$ and since $\sigma(G)$ is a vector lattice subspace of $\mathcal{L}\mathcal{L}G$ it follows that $T : E \to G$ is σ-order continuous. If F is Dedekind complete, the desired result follows upon application of Theorem 3.1 (iii). ■

4. HAUSDORFF CONTINUOUS FUNCTIONS AND THE CONVERGENCE SPACE COMPLETION OF $(\mathcal{C}(X), \lambda_O)$

4.1 Extension and Restriction of H-continuous Functions

Throughout this section we denote by X an arbitrary topological space. It is well known that if f is a continuous function on a dense subset D of X then there need not exists a continuous extension to all of X. On the other hand, the restriction of a continuous function g on X to a subset A of X is always continuous on A. The situation for H-continuous functions, however, is inverse to the above. As a motivation for the content of this section, consider the following examples.

Example 4.1 *Let X be the real line $\mathbb{R}$ and $D = \mathbb{R} \setminus \{0\}$. Define the function $f \in \mathcal{C}(D)$ as*

$$f(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}.$$

Suppose that f has an extension to X that is continuous at $x = 0$. Then for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that

$$|f(x) - f(y)| < \varepsilon$$

whenever $|x - y| < \delta_\varepsilon$. But $|f(-\delta/2) - f(\delta/2)| = |-1 - 1| = 2$ for every $\delta > 0$. So if we take $0 < \varepsilon < 2$ then there is no $\delta_\varepsilon > 0$ that satisfies the above condition, contrary to our assumption of continuity at 0.
On the other hand, consider the finite H-continuous function

$$\widetilde{f}(x) = \begin{cases} -1 & \text{if } x < 0 \\ [-1, 1] & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

which is clearly an extension of f to X.

Example 4.2 *Again let X be the real line $\mathbb{R}$. Let A be the subset $(0, +\infty]$ of X. Define the H-continuous function f on X as*

$$f(x) = \begin{cases} -1 & \text{if } x < 0 \\ [-1, 1] & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}.$$

We show that the restriction

$$f_{|A}(x) = \begin{cases} [-1,1] & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

of f to A is not H-continuous. Consider the interval valued function

$$g(x) = \begin{cases} [0,1] & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases} .$$

Clearly the inclusion $g(x) \subseteq f_{|A}(x)$ holds for every $x \in A$ and g is S-continuous on A. Therefore $F(g)(x) = g(x), x \in X$. But $g(x) \neq f_{|A}(0)$ so that $f_{|A}$ can not be H-continuous on A.

The definition of our extension operator makes use of a generalization of the graph completion operator. This generalization is obtained by only considering function values on a dense subset of X.

Let D be a dense subset of X. Then the generalized lower and upper Baire operators, as defined in [77], are given by

$$I(D,X,f)(x) = \sup_{V \in \mathcal{V}_x} \inf \{z \in f(y) : y \in V \cap D\}, x \in X, \tag{4.1}$$

$$S(D,X,f)(x) = \inf_{V \in \mathcal{V}_x} \sup \{z \in f(y) : y \in V \cap D\}, x \in X, \tag{4.2}$$

for very $f \in \mathsf{A}(D)$. Clearly,

$$I(D,X,f)(x) \leq S(D,X,f)(x), x \in X, \tag{4.3}$$

for each $f \in \mathsf{A}(D)$. It follows that the mapping $F : \mathsf{A}(D) \to \mathsf{A}(X)$, called the extended graph completion operator, defined by

$$F(D,X,f)(x) = [I(D,X,f)(x), S(D,X,f)(x)], x \in X, f \in \mathsf{A}(D) \tag{4.4}$$

is well defined and the inclusion

$$f(x) \subseteq F(D,X,f)(x), x \in D, \tag{4.5}$$

holds. When $D = X$ the mappings defined above reduce to the classical Baire Operators and Graph Completion Operator as defined by (1.20) through (1.25). In [77] the following properties of the operators introduced in (4.1) through (4.5) are listed:

GP1 The extended upper Baire, lower Baire and graph completion operators are all monotone increasing with respect to their functional argument, that is, if D is a dense subset of X, then for any two functions $f, g \in \mathsf{A}(D)$ we have

$$f(x) \leq g(x), x \in D \Rightarrow \begin{cases} I(D,X,f)(x) \leq I(D,X,f)(x), x \in X \\ S(D,X,f)(x) \leq S(D,X,f)(x), x \in X \\ F(D,X,f)(x) \leq F(D,X,f)(x), x \in X \end{cases} \tag{4.6}$$

GP2 The graph completion operator is inclusion isotone with respect to the functional argument, that is, if $f, g \in \mathsf{A}(D)$ where D is a dense subset of X, then

$$f(x) \subseteq g(x), x \in D \Rightarrow F(D, X, f)(x) \subseteq F(D, X, f)(x), x \in X. \quad (4.7)$$

GP3 The graph completion operator is inclusion isotone with respect to the set D in the sense that if D_1 and D_2 are dense subsets of X and $f \in \mathsf{A}(D_1 \cup D_2)$ then

$$D_1 \subseteq D_2 \Rightarrow \begin{cases} F(D_1, X, f)(x) \subseteq F(D_2, X, f)(x), x \in X \\ F(D_2, X, \cdot) \circ F(D_1, X, \cdot) = F(D_1, X, \cdot) \end{cases}. \quad (4.8)$$

In particular, this means that for any dense subset D of X and any $f \in \mathsf{A}(X)$ we have

$$F(D, X, f)(x) \subseteq F(f)(x), x \in X. \quad (4.9)$$

The next two theorems are proved in [4] and [77] for the special case where $X = \Omega$ is an open subset of R^p. The arguments differ little, if at all, from those used for the specific case considered there.

Theorem 4.1 *Let D be a dense subset of X and $f \in \mathsf{A}(D)$. Then $F(D, X, f)$ is S-continuous on X.*

Proof. By Definition 1.39 we must show that $F(F(D, X, f)) = F(D, X, f)$. But $F(f) = F(X, X, f)$ and $D \subseteq X$ so (4.8) implies

$$F(F(D, X, f)) = F(X, X, f) \circ F(D, X, f) = F(D, X, f)$$

which completes the proof. ■

As a result of the next result we will be able to easily establish certain extension theorems for H-continuous functions. It also provides a crucial link with the usual point-valued functions.

Theorem 4.2 *Let D be a dense subset of X and $f \in \mathcal{C}(D)$. Then the following statements hold:*
(i) $F(D, X, f)(x) = f(x), x \in D$;
(ii) $F(D, X, f) \in \mathsf{H}(X)$.

Proof. (i) Since f is continuous on D it follows that

$$F(D, X, f)(x) = F(D, D, f)(x) = f(x)$$

for every $x \in D$.
(ii) Theorem 4.1 states that $F(D, X, f)$ is S-continuous, so by Theorem C.7 we need only show that, for any $g \in \mathsf{F}(X)$ the inclusion

$$g(x) \subseteq F(D, X, f)(x), x \in X \quad (4.10)$$

implies that $f(x) = g(x), x \in X$. Since f is continuous on D it assumes degenerate interval values on D. Therefore the inclusion (4.10) and (i) imply that

$$g(x) = f(x), x \in D.$$

Hence we have

$$g(x) \subseteq F(D, X, f)(x) = F(D, X, g)(x) \subseteq F(g)(x) = g(x), x \in X$$

where the last inclusion follows by (4.9). It follows that $f(x) = g(x), x \in X$. This completes the proof. ■

Motivated by Example 1.6 we make the following definition and prove the resulting theorem.

Definition 4.1 *For any dense subset D of X define the mapping $E_D : \mathsf{H}(D) \rightarrow \mathsf{A}(X)$, which we shall call the extension mapping, as*

$$E_D(f)(x) = F(D, X, f)(x), x \in X.$$

Theorem 4.3 *Let D be a dense subset of X. The extension mapping $E_D : \mathsf{H}(D) \rightarrow \mathsf{A}(X)$ of Definition 4.1 maps $\mathsf{H}(D)$ into $\mathsf{H}(X)$ and*

$$E_D(f)(x) = f(x), x \in D.$$

Proof. By Theorem 4.1 the function $E_D(f)$ is S-continuous on X. Let $g \in \mathsf{F}(X)$ be such that

$$g(x) \subseteq E_D(f)(x), x \in X.$$

By Theorem C.7 we need only show that $g(x) = E_D(f)(x), x \in X$. Since f is H-continuous, and hence S-continuous, it follows that

$$E_D(f)(x) = F(D, X, f)(x) = F(D, D, f)(x) = f(x), x \in D.$$

Therefore the inclusion $g(x) \subseteq f(x)$ holds for every $x \in D$. But Proposition C.1 implies that g is S-continuous on D. By Theorem C.7 and the H-continuity of f it follows that $g(x) = f(x), x \in D$. But by (4.8) and the S-continuity of g

$$E_D(f)(x) = F(D, X, g)(x) \subseteq F(g)(x) = g(x) \subseteq E_D(f)(x).$$

Therefore $g(x) = E_D(f)(x), x \in X$ so that $E_D(f)$ is H-continuous. ■

Theorem 4.4 *For any $f \in \mathsf{H}(X)$ and any subset A of X such that $(A \setminus \text{int } A) \cap W_f = \emptyset$ the restriction $f_{|A}$ of f to A is H-continuous on A.*
In particular, the restriction of f to any open subset of X is H-continuous.

Proof. First note that if $\operatorname{int} A = \emptyset$ then our assumption on the set A implies that f is point valued at each point of A and hence by Theorem C.6 (ii) continuous on A. The restriction $f_{|A}$ is therefore continuous and hence H-continuous on A.
So assume that $\operatorname{int} A \neq \emptyset$. We proceed as follows: First we show that the restriction of f to $\operatorname{int} A$ is H-continuous. It then follows by Theorem 4.3 that we can extend this function to an H-continuous function on A. We then show that this function is the desired restriction.
By Proposition C.1 and Theorem C.2 we know that $f_{|\operatorname{int} A}$ is S-continuous on $\operatorname{int} A$. Therefore $I\left(f_{|\operatorname{int} A}\right)(x) = \underline{f}_{|\operatorname{int} A}(x), x \in \operatorname{int} A$ and $S\left(f_{|\operatorname{int} A}\right)(x) = \overline{f}_{|\operatorname{int} A}(x), x \in \operatorname{int} A$ so that

$$F\left(I\left(f_{|\operatorname{int} A}\right)\right)(x) = \left[\underline{f}_{|\operatorname{int} A}(x), S\left(\underline{f}_{|\operatorname{int} A}\right)(x)\right] \tag{4.11}$$

and

$$F\left(S\left(f_{|\operatorname{int} A}\right)\right)(x) = \left[I\left(\overline{f}_{|\operatorname{int} A}\right)(x), \overline{f}_{|\operatorname{int} A}(x)\right]. \tag{4.12}$$

Suppose that $S(\underline{f}_{|\operatorname{int} A})(x) \neq \overline{f}_{|\operatorname{int} A}(x)$ for some $x \in \operatorname{int} A$. By the monotonicity (Theorem C.1) of the operator S it follows that $S(\underline{f}_{|\operatorname{int} A})(x) \leq \overline{f}_{|\operatorname{int} A}(x) = \overline{f}(x)$. Therefore there must exists a neighbourhood U of x such that

$$\sup\left\{\underline{f}_{|\operatorname{int} A}(y) : y \in U \cap \operatorname{int} A\right\} < \overline{f}(x).$$

But then

$$\sup\left\{\underline{f}_{|\operatorname{int} A}(y) : y \in U \cap \operatorname{int} A\right\} < \sup\left\{\underline{f}(y) : y \in V\right\}, V \in \mathcal{V}_x \tag{4.13}$$

since $S\left(\underline{f}\right)(x) = \overline{f}(x)$. Since $\operatorname{int} A$ is open in X we have $U \cap \operatorname{int} A \in \mathcal{V}_x$. But then the inequality (4.13) can not hold, a contradiction. Therefore

$$S\left(\underline{f}_{|\operatorname{int} A}\right)(x) = \overline{f}_{|\operatorname{int} A}(x), x \in A.$$

In the same way we can show that

$$I\left(\overline{f}_{|\operatorname{int} A}\right)(x) = \underline{f}_{|\operatorname{int} A}(x), x \in \operatorname{int} A.$$

Therefore (4.11) and (4.12) imply that $F\left(I\left(f_{|\operatorname{int} A}\right)\right)(x) = F\left(S\left(f_{|\operatorname{int} A}\right)\right)(x), x \in \operatorname{int} A$. By Theorem C.4 the function $f_{|\operatorname{int} A}$ is H-continuous on $\operatorname{int} A$.
Now consider the H-continuous extension of $f_{|\operatorname{int} A}$ to A, call it $\widetilde{f}$. We need to show that $\widetilde{f}(x) = f(x), x \in A \setminus \operatorname{int} A$. By the S-continuity of $f_{|A}$ (Proposition C.1 and Theorem C.2) and (4.8)

$$\widetilde{f}(x) = F\left(\operatorname{int} A, A, f_{|\operatorname{int} A}\right)(x) \subseteq F\left(f_{|A}\right)(x) = f_{|A}(x), x \in A.$$

But by assumption $A \setminus \operatorname{int} A \cap W_f = \emptyset$ so that $f_{|A}$ is point valued on $A \setminus \operatorname{int} A$. The above inclusion then implies that $\widetilde{f}(x) = f(x), x \in A \setminus \operatorname{int} A$.

If A is open, then the condition $(A \setminus \operatorname{int} A) \cap W_f = \emptyset$ is automatically satisfied since $A = \operatorname{int} A$. This completes the proof. ■

We introduce a new class of H-continuous functions that appears in connection with the extension of finite H-continuous functions. If D a dense subset of X, then the boundary of D in X is defined to be the set

$$\partial D = X \setminus D. \tag{4.14}$$

A nearly finite H-continuous function f belongs to the set $\mathsf{H}_{ft}(X)^{(\partial D)}$ if it assumes finite values everywhere except possibly on the boundary ∂D of D.

Theorem 4.5 *Let D be an open and dense subset of X. The extension mapping E_D satisfies the following:*
(i) The mapping $E_D : \mathsf{H}(D) \to \mathsf{H}(X)$ is a bijection.
(ii) The restriction $E_D : \mathsf{H}_{nf}(D) \to \mathsf{H}_{nf}(X)$ is a bijection.
(iii) The restriction $E_D : \mathsf{H}_{ft}(D) \to \mathsf{H}_{ft}(X)^{(\partial D)}$ is a bijection.

Proof. (i) Suppose there are $f \neq g \in \mathsf{H}(D)$ such that $E_D(f) = E_D(g)$. By Theorem 4.3

$$f(x) = E_D(f)(x) = E_D(g)(x) = g(x), x \in D$$

which is clearly a contradiction.
For any $f \in \mathsf{H}(X)$ take the restriction $f_{|D}$ of f to D. By Theorem 4.4 $f_{|D}$ is H-continuous on D. By Theorem 4.3 its extension $E_D\left(f_{|D}\right)$ to X satisfies

$$E_D\left(f_{|D}\right)(x) = f_{|D}(x) = f(x) .x \in D.$$

But by assumption D is dense in X so that we have equality on X by Theorem C.10 (ii).
(ii) Since

$$D' = \{x \in D : \omega(f(x)) \neq \infty\}$$

is open and dense in D for every $f \in \mathsf{H}_{nf}(D)$ it follows by the denseness of D in X that D' is also dense in X. But $E_D(f)(x) = f(x), x \in D$ so that Theorem C.11 implies that $E_D(f)$ is nearly finite on X.
Conversely, suppose that f is nearly finite on X. Since D is open, and since f is finite on an open and dense subset of X, it follows that the same is true for $f_{|D}$.
(iii) Since

$$E_D(f)(x) = f(x), x \in D$$

for every $f \in \mathsf{H}_{ft}(X)$ it follows by the denseness of D in X and the fact that f is finite that $E_D(f)$ belongs to $\mathsf{H}_{ft}(X)^{(\partial D)}$. Conversely, if $f \in \mathsf{H}_{ft}(X)^{(\partial D)}$ then $f_{|D}$ is finite on D so the result follows by (i). ■

We conclude this section with the following example involving the Stone-Čhech compactificantion of a completely regular space.

Example 4.3 *Recall the classical Stone-Čech Theorem: If X is a completely regular space, then there exists a compact space βX such that:*
(i) There is a continuous mapping $\Delta : X \to \beta X$ with the property that $\Delta : X \to \Delta(X)$ is a homeomorphism.;
(ii) $\Delta(X)$ is dense in βX;
(iii) If $f \in \mathcal{C}_b(X)$, then there is a continuous map $f^\beta : \beta X \to \mathsf{R}$ such that $f^\beta \circ \Delta = f$.
Moreover, if Y is a compact space having these properties, then Y is homeomorphic to βX.
Let X be a completely regular space and let βX denote its Stone-Čech compactificantion. We will regard X as a subspace of βX. Every continuous function on X can be uniquely extended to a nearly finite H-continuous function $f^\#$ on βX such that $f^\beta \circ \Delta = f$. Moreover, if $f \in \mathcal{C}_b(X)$ then $f^\# = f^\beta$. Indeed, the first statement is evident from Theorem 4.5 (ii). The second statement follows by the denseness of X and Theorem C.10 (ii). Note that we can now express the function f^β as

$$f^\beta(x) = E_X(f)(x) = F(X, \beta X, f)(x), x \in \beta X.$$

4.2 The Vector Lattice $\mathsf{H}_{ft}(X)$

In [77] a linear structure was introduced on the space $\mathsf{H}_{ft}(X)$. This structure was only defined for the special case when $X = \Omega$ is an open subset of R^p. We extend it to the more general case where X is a Baire space. The definition of the operations as defined in [77] for $X = \Omega$ an open subset of R^p is extended in a straight forward way. In fact, the proof [77] that the operations there defined introduce on $\mathsf{H}_{ft}(X)$ the structure of a linear space can be applied with minimal modification.

The operations are defined in terms of the extended graph completion operator as introduced in Section 4.1 of this work and is not defined pointwise. However, if the pointwise sum yields an H-continuous function then it is the same as the sum.

Definition 4.2 *For any $f, g \in \mathsf{H}_{ft}(X)$ the sum $f \oplus g$ of f and g is defined as*

$$(f \oplus g)(x) = F(D, X, f + g)(x), x \in X$$

where $D_{fg} = X \setminus (W_f \cup W_g)$.
For $\alpha \in \mathsf{R}$ the scalar product αf is defined by

$$(\alpha f)(x) = \left[\min\left\{\alpha \underline{f}(x), \alpha \overline{f}(x)\right\}, \max\left\{\alpha \underline{f}(x), \alpha \overline{f}(x)\right\}\right].$$

Theorem 4.6 *Let X be a Baire space. Then the operations 'addition' and 'scalar multiplication' of Definition 4.2 introduce on $\mathsf{H}_{ft}(X)$ the structure of a vector space.*

Proof. By Theorem C.9 the sets W_f and W_g are both of first Baire category so that $W_f \cup W_g$ is also of first Baire category and hence $D = X \setminus (W_f \cup W_g)$ is dense in X. It therefore follows that the mapping

$$F(D, X, \cdot) : \mathsf{A}(D) \to \mathsf{F}(X)$$

is well defined. According to Theorem C.6 (ii) both f and g are continuous on D so that Theorem 4.2 implies that $f \oplus g$ is H-continuous on X. We now show that $f \oplus g$ is finite. Note that for every $x \in X$ there exists $V \in \mathcal{V}_x$ and $\varepsilon > 0$ such that

$$|f(y)| < \varepsilon, y \in V'. \tag{4.15}$$

Indeed, if this were not the case then for every $V \in \mathcal{V}_x$ and $\varepsilon > 0$ there would exists $y_\varepsilon \in V$ such that

$$|f(y_\varepsilon)| > \varepsilon.$$

But then either

$$\underline{f}(x) = I(f)(x) = \sup_{V \in \mathcal{V}_x} \inf \left\{ \underline{f}(y) : y \in V \right\} = -\infty$$

or

$$\overline{f}(x) = S(f)(x) = \inf_{V \in \mathcal{V}_x} \sup \left\{ \overline{f}(y) : y \in V \right\} = \infty$$

contrary to our assumption that f is finite. Similarly there exists $V'' \in \mathcal{V}_x$ and $\varepsilon' > 0$ such that

$$|g(y)| < \varepsilon', y \in V''. \tag{4.16}$$

Hence it follows by (4.15), (4.16) and Definition 4.2 that

$$\begin{aligned}
\overline{f \oplus g}(x) &= \inf_{V \in \mathcal{V}_x} \sup \left\{ \overline{f}(y) + \overline{g}(y) : y \in V \cap D \right\} \\
&= \inf_{V \in \mathcal{V}_x} \sup \left\{ \overline{f}(y) + \overline{g}(y) : y \in V \cap V' \cap V'' \cap D \right\} \\
&< \infty
\end{aligned}$$

and

$$\begin{aligned}
\underline{f \oplus g}(x) &= \sup_{V \in \mathcal{V}_x} \inf \left\{ \underline{f}(y) + \underline{g}(y) : y \in V \cap D \right\} \\
&= \inf_{V \in \mathcal{V}_x} \sup \left\{ \underline{f}(y) + \underline{g}(y) : y \in V \cap V' \cap V'' \cap D \right\} \\
&> -\infty.
\end{aligned}$$

We have shown that the mapping $\oplus : \mathsf{H}_{ft}(X) \times \mathsf{H}_{ft}(X) \to \mathsf{H}_{ft}(X)$ is well defined. To see that $\alpha f \in \mathsf{H}_{ft}(X)$ for every $f \in \mathsf{H}_{ft}(X)$ and $\alpha \in \mathbb{R}$, observe that

$$(\alpha f)(x) = \begin{cases} \left[\alpha \underline{f}(x), \alpha \overline{f}(x)\right] & \text{if } \alpha \geq 0 \\ \left[\alpha \overline{f}(x), \alpha \underline{f}(x)\right] & \text{if } \alpha < 0 \end{cases}, x \in X.$$

Consider the case $\alpha \geq 0$. Then

$$\begin{aligned} I(\alpha f)(x) &= \sup\left\{\inf\left\{\alpha \underline{f}(y) : y \in V\right\} : V \in \mathcal{V}_x\right\} \\ &= \sup\left\{\alpha \inf\left\{\underline{f}(y) : y \in V\right\} : V \in \mathcal{V}_x\right\} \\ &= \alpha \sup\left\{\inf\left\{\underline{f}(y) : y \in V\right\} : V \in \mathcal{V}_x\right\} \\ &= \alpha I(f)(x) \end{aligned}$$

for every $x \in X$. In the same way it follows that

$$F(I(\alpha f))(x) = \alpha F(I(f))(x), x \in X$$

and

$$F(S(\alpha f))(x) = \alpha F(S(f))(x), x \in X.$$

Applying Theorem C.4 twice yields $\alpha f \in \mathsf{H}_{ft}(X)$. The case $\alpha < 0$ follows in the same way.
We show that the associative law

$$(f \oplus g) \oplus h = f \oplus (g \oplus h) \tag{4.17}$$

is satisfied. It follows by Theorems C.6 (ii) and 4.2 (i) that

$$((f \oplus g) \oplus h)(x) = \overline{f}(x) + \overline{g}(x) + \overline{h}(x) = (f \oplus (g \oplus h))(x)$$

for every $x \in X \setminus (W_f \cup W_g \cup W_h)$. But since $W_f \cup W_g \cup W_h$ is of first Baire category by Theorem C.9 it follows that $X \setminus (W_f \cup W_g \cup W_h)$ is dense in X. The identity (4.17) now follows by Theorem C.10 (ii).
The other axioms of a linear space follow in the same way. ∎

Theorem 4.7 *The linear space $\mathsf{H}_{ft}(X)$ equipped with the partial order (1.19) is a Dedekind complete vector lattice.*

Proof. By Theorem C.12 any finite subset $\mathcal{B}$ of $\mathsf{H}_{ft}(X)$ has a supremum. Therefore $\mathsf{H}_{ft}(X)$ is a lattice.
Consider any $f, g, h \in \mathsf{H}_{ft}(X)$ such that $f \leq g$ and any $\alpha > 0$.
Since both the order relation and scalar multiplication are defined in a pointwise way, it follows easily that $\alpha f \leq \alpha g$.
By Definition 4.2 the sums $f \oplus h$ and $g \oplus h$ are given by

$$(f \oplus h)(x) = F(D_1, X, f + h)(x), x \in X$$

and

$$(g \oplus h)(x) = F(D_2, X, g + h)(x), x \in X$$

where $D_1 = X \setminus (W_f \cup W_h)$ and $D_2 = X \setminus (W_g \cup W_h)$. However, by Theorem C.9 and the fact that X is a Baire space the set

$$D = X \setminus (W_f \cup W_g \cup W_h) \subseteq D_1, D \tag{4.18}$$

is dense in X. By Theorem C.6 (ii) the functions f, g and h are continuous on D so that Theorem 4.2 (i) implies that

$$F(D, X, f+h)(x) = f(x)+h(x) \leq g(x)+h(x) = F(D, X, g+h)(x) \quad (4.19)$$

for every $x \in D$. But in the same way

$$(f \oplus h)(x) = f(x) + h(x), x \in D_1 \quad (4.20)$$

and

$$(g \oplus h)(x) = g(x) + h(x), x \in D_2. \quad (4.21)$$

It now follows by (4.18) through (4.21) that

$$(f \oplus h)(x) \leq (g \oplus h)(x), x \in D$$

and since D is dense in X Theorem C.10 (i) implies

$$(f \oplus h)(x) \leq (g \oplus h)(x), x \in X.$$

By Theorem C.12 $\mathsf{H}_{ft}(X)$ is Dedekind complete which completes the proof. ■

We are now in a position to apply the results of Chapter 2 to the set of finite H-continuous functions. In doing so we obtain the following comprehensive result.

Theorem 4.8 *Order convergence of sequences on $\mathsf{H}_{ft}(X)$ introduces the structure of an FS-convergence structure on the set. Moreover, the convergence space $(\mathsf{H}_{ft}(X), \lambda_o)$ is a locally bounded, locally convex, first countable, Hausdorff, regular, functionally regular and complete convergence vector space.*

When X is a metric space, one can define vector space operations on $\mathsf{H}_{ft}(X)$ in an order theoretic way.

It was shown in [3] that the $\mathsf{H}_{ft}(X)$ is the Dedekind completion of $\mathcal{C}(X)$ when X is a metric space. It is, however, well known that $\mathcal{C}(X)$ is a vector lattice when considered with the pointwise operations and partial ordering. It is therefore standard practice to extend the operations on $\mathcal{C}(X)$ to $\mathsf{H}_{ft}(X)$ in the following way. For every $f, g \in \mathsf{H}_{ft}(X)$ and real number $\alpha > 0$ we define the sum and scalar product as follows:

$$\begin{aligned} f \oplus' g &= \sup\{u+v : u, v \in \mathcal{C}(X), u \leq f, v \leq g\} \\ &= \inf\{u+v : u, v \in \mathcal{C}(X), f \leq u, g \leq v\} \end{aligned} \quad (4.22)$$

$$\begin{aligned} \alpha f &= \sup\{\alpha u : u \in \mathcal{C}(X), u \leq f\} \\ &= \inf\{\alpha u : u \in \mathcal{C}(X), f \leq u\} \end{aligned} \quad (4.23)$$

The above defines a vector space structure on $\mathsf{H}_{ft}(X)$. Note that the addition (4.22) should not be confused with the pointwise addition of H-continuous functions.

The linear structure of Definition 4.2 relates to pointwise operations and the operations (4.22) through (4.23) as follows.

Proposition 4.1 *For any* $f.g \in \mathsf{H}_{ft}(X)$ *the inclusion*

$$(f \oplus g)(x) \subseteq \left[\underline{f}(x) + \underline{g}(x), \overline{f}(x) + \overline{g}(x)\right] = (f+g)(x), x \in X$$

holds.

Proof. This is a direct consequence on (4.9) and Definition 4.2. ■

Theorem 4.9 *Let* X *be a complete metric space. Then (4.22) through (4.23) are equivalent to Definition 4.2.*

Proof. For any $u, v \in \mathcal{C}(X)$ such that $u \leq f$ and $v \leq g$ it follows by (1.12) and (1.19)

$$u(x) + v(x) \leq (f+g)(x), x \in X.$$

On the other hand, if $u', v' \in \mathcal{C}(X)$ is such that $f \leq u$ and $g \leq v$ then

$$(f+g)(x) \leq u'(x) + v'(x), x \in X.$$

By Proposition 4.1 it now follows that

$$u(x) + v(x) \leq (f \oplus g)(x) \leq u'(x) + v'(x), x \in X.$$

Therefore

$$f \oplus g \leq f \oplus' g = \sup\{u + v : u, v \in \mathcal{C}(X), u \leq f, v \leq g\}$$

and

$$f \oplus' g = \inf\{u' + v' : u', v' \in \mathcal{C}(X), f \leq u', g \leq v'\} \leq f \oplus g$$

so that $f \oplus g = f \oplus' g$.
The equivalence of the two definitions of scalar multiplication follows in the same way. ■

4.3 The Hausdorff Distance

Throughout this section we will assume X to be a metric space. In [76] it is shown that $\mathsf{F}_{ft}(X)$ equipped with the H-distance is a complete metric space when X is a compact interval of the real line. We generalize this result as follows: For an arbitrary metric space X we show that $\mathsf{F}_{ft}(X)$ is a metric space. Moreover, we show that $\mathsf{F}_{ft}(X)$ is a complete metric space whenever X is compact. The verification of the metric space axioms follows closely the argument employed in [76]. In order to prove completeness we make use of the characterization of the H-distance in terms of the one sided H-distance.

In order to obtain the desired equivalence we assume that the space $X \times \mathsf{R}$ is equipped with the square metric

$$\rho\left(\left(x_1, y_1\right), \left(x_2, y_2\right)\right) = \max\left\{d\left(x_1, x_2\right), \left|y_1 - y_2\right|\right\} \tag{4.24}$$

where $d\left(\cdot,\cdot\right)$ denotes the metric on X. Recall also that for any subsets A and B of a metric space $\left(X, d\right)$ the 'distance' $d\left(A, B\right)$ is defined as

$$d\left(A, B\right) = \inf_{a \in A} \inf_{b \in B} d\left(a, b\right). \tag{4.25}$$

Theorem 4.10 *Let f and g be finite S-continuous functions on X. Then*

$$r\left(f, g\right) = \max\left\{\eta\left(f, g\right), \eta\left(g, f\right)\right\}.$$

Proof. It is sufficient to prove that

$$\eta\left(f, g\right) = \sup_{A \in g} \inf_{B \in f} \rho(A, B) \tag{4.26}$$

and

$$\eta\left(g, f\right) = \sup_{A \in f} \inf_{B \in g} \rho(A, B) \tag{4.27}$$

for any $f, g \in \mathsf{F}_{ft}\left(X\right)$. Let

$$\sup_{A \in g} \inf_{B \in f} \rho(A, B) = \delta.$$

We will show that

$$I_{\delta+\varepsilon}\left(g\right)\left(x\right) - \left(\delta + \varepsilon\right) \leq f\left(x\right) \leq S_{\delta+\varepsilon}\left(g\right)\left(x\right) + \left(\delta + \varepsilon\right), x \in X \tag{4.28}$$

for any $\varepsilon > 0$ and that

$$f\left(x\right) \ast \left[I_{\delta+\varepsilon}\left(g\right)\left(x\right) - \left(\delta - \varepsilon\right), S_{\delta+\varepsilon}\left(g\right)\left(x\right) + \left(\delta - \varepsilon\right)\right] \tag{4.29}$$

for every $0 < \varepsilon < \delta$ and some $x_0 \in X$.
We first establish the left inequality in (4.28). For the sake of obtaining a contradiction, suppose that there exists $x \in X$ such that

$$I_{\delta+\varepsilon}\left(g\right)\left(x\right) - \left(\delta + \varepsilon\right) > f\left(x\right).$$

By (1.32) it follows that

$$\inf\left\{z \in g\left(y\right) : d\left(x, y\right) < \delta + \varepsilon\right\} - \left(\delta + \varepsilon\right) > f\left(x\right)$$

so that

$$z - z' > \delta + \varepsilon \tag{4.30}$$

for all $z \in g\left(y\right)$ where $d\left(x, y\right) < \delta + \varepsilon$ and $z' \in f\left(x\right)$. Set

$$A = \left\{\left(x, z'\right) : z' \in f\left(x\right)\right\} \in f.$$

Then for any $B \in g$ (4.24) and (4.25) imply that

$$\begin{aligned}\rho(A,B) &= \inf_{a\in A}\inf_{b\in B}\rho(a,b)\\ &= \inf_{a\in A}\inf_{b\in B}\max\{d(x,y),|z-z'| : a=(x,z'), b=(y,z)\}.\end{aligned}$$

The implication

$$d(x,y) < \delta+\varepsilon \Rightarrow |z-z'| > \delta+\varepsilon$$

now follows by (4.30) so that $\rho(A,B) \geq \delta+\varepsilon$ for any $B \in g$. Therefore

$$\sup_{A\in g}\inf_{B\in f}\rho(A,B) \geq \delta+\varepsilon > \delta.$$

contrary to our assumption. The second inequality in (4.28) follows in the same way so that (4.28) must hold.
Now suppose that

$$f(x) \subseteq [I_{\delta+\varepsilon}(g)(x)-(\delta-\varepsilon), S_{\delta+\varepsilon}(g)(x)+(\delta-\varepsilon)], x \in X$$

for some $0<\varepsilon<\delta$. Then according to (1.32)

$$\inf\{z \in g(y) : d(x,y) < \delta-\varepsilon\} - (\delta-\varepsilon) \leq f(x), x \in X$$

so that, as above,

$$z - z' < \delta - \varepsilon \tag{4.31}$$

for all $z \in g(y)$ where $d(x,y) < \delta-\varepsilon$ and $z' \in f(x)$. Consider any

$$A = \{(x,z') : z' \in f(x)\} \in f$$

and choose

$$B = \{(y,z) : z \in g(y)\} \in g$$

such that $d(x,y) < \delta-\varepsilon$. As above

$$\begin{aligned}\rho(A,B) &= \inf_{a\in A}\inf_{b\in B}\rho(a,b)\\ &= \inf_{a\in A}\inf_{b\in B}\max\{d(x,y),|z-z'| : a=(x,z'), b=(y,z)\}\end{aligned}$$

so that (4.31) implies $\rho(A,B) < \delta-\varepsilon$. Hence for any $A \in f$

$$\sup_{A\in g}\inf_{B\in f}\rho(A,B) < \delta-\varepsilon.$$

contrary to our assumption. Therefore (4.29) must be true. Equation (4.26) is now a straight forward consequence of (4.28) through (4.29) and (1.36). The same arguments apply to $\eta(g,f)$ so that (4.27) also holds. This completes the proof. ■

Theorem 4.11 *The Hausdorff distance satisfies the axioms of a metric on* $\mathsf{F}_{ft}(X)$. *Moreover, if the metric space* X *is compact then* $(\mathsf{F}_{ft}(X), r)$ *is a complete.*

Proof. The Hausdorff distance satisfies the axioms of a metric on the set of closed subsets of $X \times \mathsf{R}$, and its restriction to any family of closed subsets will also be a metric. Therefore, it suffices to show that the graph of each S-continuous function f is a closed subset of $X \times \mathsf{R}$.
To this end, let (x_k) be a sequence on X and for each k let $y_k \in f(x_k)$. Suppose the sequence $((x_k, y_k))$ on $X \times \mathsf{R}$ converges to (x_0, y_0). We must show that $y_0 \in f(x_0)$. Since

$$f(x) = F(f)(x) = [I(f)(x), S(f)(x)],$$

we have for each k that

$$I(f)(x_k) \leq y_k \leq S(f)(x_k). \tag{4.32}$$

By the lower semi-continuity of $I(f)$, the upper semi-continuity of $S(f)$ and (4.32)

$$I(f)(x_0) \leq y_0 \leq S(f)(x_0),$$

or equivalently, $y_0 \in F(f)(x_0) = f(x_0)$. It remains to show that $(\mathsf{F}_{ft}(X), r)$ is complete.
Let X be compact and (f_n) be a Cauchy sequence on $\mathsf{F}_{ft}(X)$. Then for every $\varepsilon > 0$ there exists $N_\varepsilon \in \mathsf{N}$ such that

$$r(f_m, f_n) < \varepsilon \tag{4.33}$$

for all $m, n \geq N_\varepsilon$. Since X is compact every semi-continuous function on X is bounded, and hence by Theorem C.2 so is every S-continuous function. Upon applying Theorem 4.10 it follows by (1.37) and (4.33) that

$$\eta(f_m, f_n), \eta(f_n, f_m) < \varepsilon$$

By (1.36)

$$I_\varepsilon(f_m)(x) - \varepsilon \leq f_n(x) \leq S_\varepsilon(f_m)(x) + \varepsilon, x \in X \tag{4.34}$$

and

$$I_\varepsilon(f_n)(x) - \varepsilon \leq f_m(x) \leq S_\varepsilon(f_n)(x) + \varepsilon, x \in X \tag{4.35}$$

for every $m, n \geq N_\varepsilon$. Define $g, h \in \mathcal{A}(X)$ by

$$h(x) = \inf\left\{\sup\left\{\overline{f}_m(x) : m \geq n\right\} : n \in \mathsf{N}\right\} \tag{4.36}$$

and

$$g(x) = \sup\left\{\inf\left\{\underline{f}_m(x) : m \geq n\right\} : n \in \mathsf{N}\right\}. \tag{4.37}$$

Since the sequence (f_n) is bounded, both h and g are finite and bounded and

$$g(x) \leq h(x), x \in X.$$

Theorem C.1 (i) implies that

$$I(g)(x) \leq S(h)(x), x \in X$$

so that the function $f = [\underline{f}, \overline{f}]$ where,

$$\underline{f}(x) = I(g)(x), x \in X$$

and

$$\overline{f}(x) = S(h)(x), x \in X$$

is well defined and S-continuous by Theorems C.1 (iii) and C.2 and (C.1) through (C.2). It now follows by (4.34) and (4.36) through (4.37) that

$$I_\varepsilon(f)(x) - \varepsilon \leq f_n(x) \leq S_\varepsilon(f)(x) + \varepsilon, x \in X$$

for every $n \geq N_\varepsilon$. In a similar way it follows that

$$I_\varepsilon(f_n)(x) - \varepsilon \leq f(x) \leq S_\varepsilon(f_n)(x) + \varepsilon, x \in X$$

for every $n \geq N_\varepsilon$. This completes the proof. ∎

We proceed to characterize the H-continuous functions in terms of the H-distance, motivating the terminology. The characterization will be of some use later in this chapter. First, however, we determine the relationship between the order (1.19) on the set $\mathsf{A}_{ft}(X)$ and the H-distance.

Theorem 4.12 *Let $f_1, f_2, g_1, g_2 \in \mathsf{A}_{ft}(X)$ such that $f_1 \leq g_1 \leq f_2$ and $f_1 \leq g_2 \leq f_2$. Then $r(g_1, g_2) \leq r(f_1, f_2)$.*

Proof. By Theorem C.1 (i) $F(f_1) \leq F(g_1) \leq F(f_2)$ and $F(f_1) \leq F(g_2) \leq F(f_2)$, but $r(g_1, g_2) = r(F(g_1), F(g_2))$ and $r(f_1, f_2) = r(F(f_1), F(f_2))$. By Theorem C.1 (iii) $F(F(f_1)) = F(f_1)$ and similarly for the other functions so that $F(f_1), F(f_2), F(g_1), F(g_2) \in \mathsf{F}_{ft}(X)$. If we can prove the statement for arbitrary S-continuous functions, then it holds for all interval valued functions. So suppose that $f_1, f_2, g_1, g_2 \in \mathsf{F}_{ft}(X)$. Denote by Δ the set

$$\begin{aligned} \Delta = \{\delta > 0 : & I_\delta(f_2)(x) - \delta \leq f_1(x) \leq S_\delta(f_2)(x) + \delta, \\ & I_\delta(f_1)(x) - \delta \leq f_2(x) \leq S_\delta(f_1)(x) + \delta, x \in X\} \end{aligned}$$

By (1.37) it is clear that

$$r(f_1, f_2) = \inf \Delta. \tag{4.38}$$

For any $\delta \in \Delta$

$$I_\delta(g_1)(x) - \delta \leq I_\delta(f_2)(x) - \delta \leq f_1(x) \leq g_2(x), x \in X, \tag{4.39}$$

and

$$g_2(x) \leq f_2(x) \leq S_\delta(f_1)(x) + \delta \leq S_\delta(g_1)(x) + \delta, x \in X, \tag{4.40}$$

Inequalities (4.39) and (4.40) imply that

$$\Delta \subseteq \{\delta > 0 : I_\delta (g_1)(x) - \delta \leq g_2(x) \leq S_\delta (g_1)(x) + \delta\}. \tag{4.41}$$

From (4.38) and (4.41) we obtain

$$\begin{aligned} \eta(g_2, g_1) &= \inf \{\delta > 0 : I_\delta (g_1)(x) - \delta \leq g_2(x) \leq S_\delta (g_1)(x) + \delta\} \\ &\leq \inf \Delta \\ &= r(f_1, f_2). \end{aligned}$$

In the same way

$$\eta(g_1, g_2) \leq r(f_1, f_2)$$

so that

$$r(g_1, g_2) = \max\{\eta(g_1, g_2), \eta(g_2, g_1)\} \leq r(f_1, f_2).$$

■

Theorem 4.13 *Let $f = [\overline{f}, \underline{f}]$ be an S-continuous function. Then the following are equivalent:*
(i) The function f is H-continuous;
(ii) $r(\overline{f}, \underline{f}) = 0$.

Proof. (i)⇒(ii) We will show that if f is H-continuous, then for every $g \in \mathsf{A}(X)$ such that $g(x) \subseteq f(x), x \in X$,

$$r(f, g) = 0.$$

Assume that f is H-continuous. Then, by Definition 1.40, for every $g \in \mathsf{A}(X)$ such that $g(x) \subseteq f(x), x \in X$,

$$F(g)(x) = f(x).x \in X. \tag{4.42}$$

By (1.38) and the S-continuity of f

$$r(f, g) = r(F(f), F(g)) = r(f, F(g))$$

By Theorem 4.11 and (4.42)

$$r(f, g) = 0.$$

By (1.38) and Theorem 4.11

$$r(\overline{f}, \underline{f}) = r(F(\overline{f}), F(\underline{f})) \leq r(F(\overline{f}), f) + r(f, F(\underline{f})),$$

and by Theorem C.1 (ii) and the S-continuity of f $F(\overline{f})(x), F(\underline{f})(x) \subseteq f(x)$ for every x. Therefore $r(\overline{f}, \underline{f}) = 0$.
(ii)⟹(i) We will show that f is H-continuous if, for every $g \in \mathsf{A}(X)$ such that $g(x) \subseteq f(x), x \in X$,

$$r(f, g) = 0.$$

Assume that

$$r(f,g) = r(f,F(g)) = 0$$

for every $g \in \mathsf{A}(X)$ such that $g(x) \subseteq f(x), x \in X$. Then by Theorem 4.11

$$f(x) = F(f)(x) = F(g)(x), x \in X$$

for all such g. Therefore f is H-continuous.
Let $g \in \mathsf{A}(X)$ be any function satisfying the inclusion

$$g(x) \subseteq f(x), x \in X.$$

Then $\overline{f} \leq g \leq \underline{f}$, and also $\overline{f} \leq f \leq \underline{f}$, so it follows by Theorem 4.12 that

$$r(f,g) \leq r\left(\overline{f}, \underline{f}\right).$$

By assumption

$$r\left(\overline{f}, \underline{f}\right) = 0$$

so that $r(f,g) = 0$. Therefore f is H-continuous. ∎

Theorems 4.10, 4.11 and 4.13 above are the fundamental results, for the purpose of our study, concerning the H-distance on $\mathsf{F}_{ft}(X)$ and its subset $\mathsf{H}_{ft}(X)$. We will apply them in the next section to the order convergence of sequences of H-continuous functions.

4.4 Order Convergence and the Hausdorff Distance

This section has as its gaol the investigation of the relationship between order convergence and the H-distance. There are two paths that we will explore. The first seeks to characterize order convergence in terms of the one-sided H-distance. The other endeavours to find sufficient conditions on a sequence of H-continuous functions for convergence in the H-distance to coincide with order convergence.

For the purpose of attaining the former aim we generalize the idea of order convergence on $\mathsf{H}_{ft}(X)$ by considering the 'fat' order limit introduced in [4]. Let (f_n) be a sequence on $\mathsf{H}_{ft}(X)$ that is bounded in the sense that there exists $g, h \in \mathsf{H}_{ft}(X)$ such that

$$g \leq f_n \leq h, n \in \mathsf{N}. \tag{4.43}$$

For each $m \in \mathsf{N}$ define the functions $f_m^{(u)}$ and $f_m^{(l)}$ as

$$f_m^{(u)} = \left[\underline{f}_m^{(u)}, \overline{f}_m^{(u)}\right] = \sup\{f_n : n \leq m\}, m \in \mathsf{N}, \tag{4.44}$$

$$f_m^{(l)} = \left[\underline{f}_m^{(l)}, \overline{f}_m^{(l)}\right] = \inf\{f_n : n \leq m\}, m \in \mathbb{N}. \tag{4.45}$$

Each of these sequences are monotone. The sequence $(f_m^{(u)})$ is decreasing while the sequence $(f_m^{(l)})$ is increasing in the sense of Definition 1.3. It is clear that

$$\sup\{\inf\{f_n : n \geq m\} : m \in \mathbb{N}\} = \sup\left\{f_m^{(l)} : m \in \mathbb{N}\right\}, \tag{4.46}$$

$$\inf\{\sup\{f_n : n \geq m\} : m \in \mathbb{N}\} = \inf\left\{f_m^{(u)} : m \in \mathbb{N}\right\} \tag{4.47}$$

The sequences of real valued functions $(\underline{f}_m^{(l)})$ and $(\overline{f}_m^{(u)})$ are both monotone and bounded and hence pointwise convergent in the sense that

$$\sup\left\{\inf\left\{\underline{f}_n^{(l)}(x) : n \geq m\right\} : m \in \mathbb{N}\right\} = \inf\left\{\sup\left\{\underline{f}_n^{(l)}(x) : n \geq m\right\} : m \in \mathbb{N}\right\}$$

and

$$\sup\left\{\inf\left\{\overline{f}_m^{(u)}(x) : n \geq m\right\} : m \in \mathbb{N}\right\} = \inf\left\{\sup\left\{\overline{f}_m^{(u)}(x) : n \geq m\right\} : m \in \mathbb{N}\right\}$$

for every $x \in X$. We now come to the "fat" order limit mentioned above.

Definition 4.3 *Let (f_n) be a sequence on $\mathbb{H}_{ft}(X)$. Then the function $f^* = [\underline{f}^*, \overline{f}^*]$ where*

$$\underline{f}^*(x) = \lim_{n \in \mathbb{N}} \underline{f}_n^{(l)}(x), x \in X, \tag{4.48}$$

$$\overline{f}^*(x) = \lim_{n \in \mathbb{N}} \overline{f}_n^{(u)}(x), x \in X. \tag{4.49}$$

is called the "fat" order limit of the sequence (f_n)

The function $\underline{f}^*$ might as well be considered as the pointwise supremum of the set of lower semi-continuous functions $\left\{\underline{f}_n^{(l)}(x) : n \in \mathbb{N}\right\}$. By Theorem C.3 this implies that $\underline{f}^*$ is lower semi-continuous. Similarly the function $\overline{f}^*$ is upper semi-continuous. Theorem C.2 now implies that the function $f^* = [\underline{f}^*, \overline{f}^*]$ is S-continuous, but need not be H-continuous. The next two theorems were established in [4] for compact subsets Ω of $\mathbb{R}^n$. We consider the natural generalization to arbitrary compact metric spaces. The proof, however, changes little and makes use of the following lemma, again a generalization of a result obtained in [4].

Lemma 4.1 *Let (f_n) be a sequence of functions in $\mathcal{A}(X)$, X a compact metric space, that converges pointwise to a function $f \in \mathcal{A}(X)$.*
(i) If the sequence (f_n) is monotonically decreasing and the function $f_n, n \in \mathbb{N}$, are all upper semi-continuous then for every $\delta > 0$ and $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that

$$f(x) \leq f_n(x) \leq S_\delta(f)(x) + \varepsilon, x \in X$$

whenever $n \geq N_0$*.*
(ii) If the sequence (f_n) *is monotonically increasing and the function* $f_n, n \in \mathsf{N}$*, are all lower semi-continuous then for every* $\delta > 0$ *and* $\varepsilon > 0$ *there exists* $N_0 \in \mathsf{N}$ *such that*

$$I_\delta(f)(x) - \varepsilon \leq f_n(x) \leq f(x), x \in X$$

whenever $n \geq N_0$*.*

Proof. (i) Assume the opposite, that is, there exists $\delta > 0$ and $\varepsilon > 0$ such that, for every $N_0 \in \mathsf{N}$, there is $n \geq N_0$ and $x_n \in X$ such that

$$f_n(x_n) > S_\delta(f)(x_n) + \varepsilon.$$

We construct a subsequence (f_{n_k}) in the following way. Take any $N_0 \in \mathsf{N}$. There exists $n_1 \in \mathsf{N}$ and $x_1 \in X$ such that $f_{n_1}(x_1) > S_\delta(f)(x_n) + \varepsilon$. Now take $n_1 \in \mathsf{N}$. According to the assumption there exists $n_2 \in \mathsf{N}$ and $x_2 \in X$ such that $f_{n_2}(x_2) > S_\delta(f)(x_2) + \varepsilon$. In the same way if $n_k \in \mathsf{N}$ is already determined, we obtain from our assumption that there exists $n_{k+1} \in \mathsf{N}$ and $x_{k+1} \in X$ such that $f_{n_{k+1}}(x_{k+1}) > S_\delta(f)(x_{k+1}) + \varepsilon$. Thus the subsequence (f_{n_k}) is constructed inductively. Since X is compact there exists a subsequence (x_{k_m}) of the sequence (x_k) which converges to a point $x_0 \in X$. This implies that there exists $M_0 \in \mathsf{N}$ such that $x_{k_m} \in B_\delta(x_0)$ whenever $m > M_0$. Hence

$$f_{k_m}(x_{k_m}) > S_\delta(f)(x_{k_m}) + \varepsilon \geq f(x_0) + \varepsilon$$

Let $l > M_0$ be fixed. Using the monotonicity of the sequence (f_n), for $m > l$ we have

$$f_{n_{k_l}}(x_{k_m}) \geq f_{n_{k_m}}(x_{k_m}) > f(x_0) + \varepsilon.$$

Hence

$$S_\delta\left(f_{n_{k_l}}\right)(x_0) \geq f_{n_{k_l}}(x_{k_m}) > f(x_0) + \varepsilon.$$

Taking the infimum over all $\delta > 0$ the upper semi-continuity of $f_{n_{k_l}}$ we obtain

$$g_{n_{k_l}}(x_0) = S\left(g_{n_{k_l}}\right)(x_0) \geq f(x_0) + \varepsilon.$$

But $(g_{n_{k_l}}(x_0))$ converges to $f(x_0)$ so that we obtain a contradiction. This proves (i).
(ii) The proof follows in the same way as (i) above. ■

Theorem 4.14 *Consider a sequence* (f_n) *on* $\mathsf{H}_{ft}(X)$ *where* X *is a compact metric space. Let* $f^* = [\underline{f}^*, \overline{f}^*]$ *be the function defined by (4.44) through (4.49). Then for every* $\delta > 0$ *and* $\varepsilon > 0$ *there exists* $N_0 \in \mathsf{N}$ *such that*

$$I_\delta\left(\underline{f}^*\right)(x) - \varepsilon \leq f_n^{(l)}(x) \leq f_n(x) \leq f_n^{(u)}(x) \leq S_\delta\left(\overline{f}^*\right)(x) + \varepsilon, x \in X \quad (4.50)$$

whenever $n > N_0$*.*

Proof. We will prove the first and last inequalities in (4.50). Choose $\delta > 0$ and $\varepsilon > 0$. The sequence $(\overline{f}_n^{(u)})$ converges pointwise to the function $\overline{f}^*$ and satisfies the conditions of Lemma 4.1 (i). Hence there exists $N_0^{(u)} \in \mathsf{N}$ such that

$$\overline{f}_n^{(u)}(x) \leq S_\delta(f^*)(x) + \varepsilon, x \in X \tag{4.51}$$

whenever $n > N_0^{(u)}$. In the same way there exists $N_0^{(l)} \in \mathsf{N}$ such that

$$I_\delta(f^*)(x) - \varepsilon \leq \underline{f}^{(l)}(x), x \in X \tag{4.52}$$

whenever $n > N_0^{(l)}$. Setting $N_0 = \max\{N_0^{(u)}, N_0^{(l)}\}$ it follows by (4.51) through (4.52) that

$$I_\delta\left(\underline{f}^*\right)(x) - \varepsilon \leq \underline{f}_n^{(l)}(x) \leq \overline{f}_n^{(u)}(x) \leq S_\delta\left(\overline{f}^*\right)(x) + \varepsilon, x \in X.$$

Since the remaining inequalities in (4.50) are trivially true the result follows. ■

Theorem 4.15 *Consider a sequence (f_n) on $\mathsf{H}_{ft}(X)$ where X is a compact metric space. Let $f^* = [\underline{f}^*, \overline{f}^*]$ be the function defined by (4.44) through (4.49). Then*

$$\sup\{\inf\{f_n : n \geq m\} : m \in \mathsf{N}\}(x) \subseteq f^*(x), x \in X, \tag{4.53}$$

$$\inf\{\sup\{f_n : n \geq m\} : m \in \mathsf{N}\}(x) \subseteq f^*(x), x \in X. \tag{4.54}$$

Furthermore, if the sequence (f_n) order converges to the function $f \in \mathsf{H}_{ft}(X)$ then f is the unique H-continuous function satisfying the inclusion $f(x) \subseteq f^*(x), x \in X$.

Proof. Let δ and ε be arbitrary positive real numbers. It follows from Theorem 4.14 that there exists $N_0 \in \mathsf{N}$ such that the inequalities in (4.50) are satisfied whenever $n > N_0$. Let $n > N_0$ be fixed. From (4.50) we obtain

$$\begin{aligned} I_\delta\left(\underline{f}^*\right)(x) - \varepsilon &\leq \sup\{\inf\{f_n : n \geq m\} : m \in \mathsf{N}\}(x) \\ &\leq \inf\{\sup\{f_n : n \geq m\} : m \in \mathsf{N}\}(x) \\ &\leq S_\delta\left(\overline{f}^*\right)(x) + \varepsilon \end{aligned}$$

for every $x \in X$. Allowing δ and ε to approach 0 and using that the function f^* is S-continuous we have

$$\begin{aligned} \underline{f}^*(x) &\leq \sup\{\inf\{f_n : n \geq m\} : m \in \mathsf{N}\}(x) \\ &\leq \inf\{\sup\{f_n : n \geq m\} : m \in \mathsf{N}\}(x) \\ &\leq \overline{f}^*(x) \end{aligned}$$

which is equivalent to the inclusions (4.53) and (4.54).
Now let the sequence (f_n) order converge to $f \in \mathsf{H}_{ft}(X)$ and let

$$h \in \{g \in \mathsf{H}_{ft}(X) : f(x) \subseteq f^*(x), x \in X\}.$$

Then for every $n \in \mathsf{N}$ we have

$$f_n^{(l)}(x) \leq \underline{f}^*(x) \leq h(x) \leq \overline{f}^*(x) \leq f_n^{(u)}(x), x \in X.$$

It follows by (4.46) and (4.47) that

$$\sup\{\inf\{f_n : n \geq m\} : m \in \mathsf{N}\}(x) = \sup\left\{f_n^{(l)} : n \in \mathsf{N}\right\}(x) \leq h(x) \quad (4.55)$$

and

$$h(x) \leq \inf\left\{f_n^{(u)} : n \in \mathsf{N}\right\}(x) = \inf\{\sup\{f_n : n \geq m\} : m \in \mathsf{N}\}(x) \quad (4.56)$$

for every $x \in X$. Since $\mathsf{H}_{ft}(X)$ is Dedekind complete Theorem A.5 applies so that

$$\sup\{\inf\{f_n : n \geq m\} : m \in \mathsf{N}\} = f = \inf\{\sup\{f_n : n \geq m\} : m \in \mathsf{N}\}. \quad (4.57)$$

The result now follows upon combining (4.55) through (4.57). ∎

We have the following characterization of the order convergent sequences on $\mathsf{H}_{ft}(X)$ in terms of the one-sided Hausdorff distance (1.36).

Theorem 4.16 *Consider a sequence (f_n) on $\mathsf{H}_{ft}(X)$. The following are equivalent:*
(i) The sequence (f_n) order converges to $f \in \mathsf{H}_{ft}(X)$;
(ii) There exists an S-continuous function g such that the one-sided Hausdorff distance $\eta(f_n, g)$ tends to zero, that is, for every $\varepsilon > 0$ there exists an $N_\varepsilon \in \mathsf{N}$ such that, for every $n \geq N_\varepsilon$,

$$\eta(f_n, g) < \varepsilon,$$

and the set

$$\{\varphi \in \mathsf{H}_{ft}(X) : \varphi(x) \subseteq g(x), x \in X\} \quad (4.58)$$

has one and only one element.

Proof. (i)$\Longrightarrow$(ii) By Theorem 4.14, for every $\varepsilon > 0$ there exists an $N_\varepsilon \in \mathsf{N}$ such that

$$I_\varepsilon\left(\underline{f}^*\right)(x) - \varepsilon \leq f_n^{(l)}(x) \leq f_n(x) \leq f_n^{(u)}(x) \leq S_\varepsilon\left(\overline{f}^*\right)(x) + \varepsilon, x \in X. \quad (4.59)$$

whenever $n \geq N_\varepsilon$. By (4.59) and (1.36) it follows that for every $\varepsilon > 0$ there exists an $N_\varepsilon \in \mathsf{N}$ such that $\eta(f_\alpha, f) < \varepsilon$ for every $n \leq N_\varepsilon$. Furthermore, by Theorem 4.15 the set $\{\varphi \in \mathsf{H}_{ft}(X) : \varphi(x) \subseteq f(x), x \in X\}$ has exactly one element.
(ii)$\Longrightarrow$(i) By assumption, for every $\varepsilon > 0$ there exists an $N_\varepsilon \in \mathsf{N}$ such that for every $n > N_\varepsilon$

$$\eta(f_\alpha, g) < \frac{\varepsilon}{2}.$$

Hence by (1.36)

$$I_\varepsilon(g)(x) - \varepsilon \leq f_n(x) \leq S_\varepsilon(g)(x) + \varepsilon, x \in X.$$

Taking the supremum over $n > N_\varepsilon$ we find that

$$\sup\{f_n : n \leq N_\varepsilon\}(x) \leq S_\varepsilon(g)(x) + \varepsilon, x \in X.$$

If we now take the infimum over all $N_\varepsilon \in \mathsf{N}$, or equivalently, over all $\varepsilon > 0$, by (1.35)

$$h(x) = \inf\{\sup\{f_n : n \leq N_\varepsilon\} : N_\varepsilon \in \mathsf{N}\}(x) \leq S(g)(x), x \in X. \tag{4.60}$$

Taking the infimum over $n > N_\varepsilon$

$$\inf\{f_n : n \leq N_\varepsilon\}(x) \leq I_\varepsilon(g)(x) - \varepsilon, x \in X.$$

Now take the supremum over all $N_\varepsilon \in \mathsf{N}$, or equivalently, over all $\varepsilon > 0$, and again by (1.35)

$$h'(x) = \sup\{\inf\{f_n : n \leq N_\varepsilon\} : N_\varepsilon \in \mathsf{N}\} \geq I(g)(x), x \in X. \tag{4.61}$$

Clearly $h'(x) \leq h(x)$ for every $x \in X$ so that (4.60) through (4.61) implies

$$I(g)(x) \leq h'(x) \leq h(x) \leq S(g)(x), x \in X,$$

or equivalently, and since $g \in \mathsf{F}_{ft}(X)$,

$$h(x) \subseteq [I(g)(x), S(g)(x)] = F(g)(x) = g(x), x \in X,$$

$$h'(x) \subseteq [I(g)(x), S(g)(x)] = F(g)(x) = g(x), x \in X.$$

But by Theorem C.12, and because the set $\{\varphi \in \mathsf{H}_{ft}(X) : \varphi(x) \subseteq g(x), x \in X\}$ has exactly one element, the result follows immediately. ■

The above result shows that the order limit of a sequence of finite H-continuous functions is contained uniquely in a pointwise way in the 'fat' order limit. However, there is a stronger connection between the two limiting functions. Indeed, the 'fat' order limit equals the order limit on a dense set. In order to establish this result we state the following lemma. Although we only use it for the case when X is a compact metric space we prove it for the most general case currently known.

Lemma 4.2 *Let X any topological space and g finite and S-continuous on X.*
(i) If the set $D_g = X \setminus W_f$ is dense in X, then the set

$$\{\varphi \in \mathsf{H}_{ft}(X) : \varphi(x) \subseteq g(x), x \in X\} \tag{4.62}$$

has exactly one element.
(ii) If X is a Baire space the converse of (i) also holds.

Proof. (i) Suppose that f_1 and f_2 are distinct H-continuous functions that satisfy the inclusion (4.62). Then

$$\underline{g}(x) \leq f_1(x), f_2(x) \leq \overline{g}(x) = \underline{g}(x), x \in D_g$$

so that $f_1(x) = f_2(x) = g(x)$ for every $x \in D_g$. Since D_g is dense in X Theorem C.10 (ii) implies that $f_1 = f_2$. The set $\{\varphi \in \mathsf{H}_{ft}(X) : \varphi(x) \subseteq g(x), x \in X\}$ can therefore have no more than one element. Set

$$f(x) = F(I(S(\overline{g})))(x), x \in X.$$

By Theorem C.8 f is H-continuous and by Theorem C.1 (ii) f satisfies the desired inclusion.
(ii) By Theorem C.8 the functions $f_1 = F\left(S\left(I\left(\underline{g}\right)\right)\right)$ and $f_2 = F(I(S(\overline{g})))$ are both H-continuous and

$$F\left(S\left(I\left(\underline{g}\right)\right)\right) \leq F(I(S(\overline{g}))).$$

Since $\underline{g}$ and $\overline{g}$ are respectively lower and upper semi-continuous, this reduces to $f_1 = F\left(S\left(\underline{g}\right)\right)$ and $f_2 = F(I(\overline{g}))$. Since $\underline{g}(x) \subseteq g(x)$ and $\overline{g}(x) \subseteq g(x)$ for every $x \in X$, it follows by Theorem C.1 (i) and (ii) that

$$f_1(x), f_2(x) \subseteq g(x), x \in X.$$

Then by our assumption,

$$f_1(x) = F\left(S\left(\underline{g}\right)\right)(x) = F(I(\overline{g}))(x) = f_2(x), x \in X.$$

By (1.25) and Theorem C.1 (iii),

$$\left[I\left(S\left(\underline{g}\right)\right)(x), S\left(\underline{g}\right)(x)\right] = [I(\overline{g})(x), S(I(\overline{g}))(x)], x \in X.$$

Consider the set $D_\varepsilon = X \setminus W_g^\varepsilon$ for some $\varepsilon > 0$ and suppose that W_g^ε is not nowhere dense in X. Then, since there exists $a \in X$ and $V' \in \mathcal{V}_a$ that is contained in W_g^ε. By (4.42)

$$\omega(g(x)) = \overline{g}(x) - \underline{g}(x) \geq \varepsilon, x \in V',$$

or equivalently,

$$\overline{g}(x) \geq \underline{g}(x) + \varepsilon, x \in V'.$$

For every $x \in V$, there exists $V'' \in \mathcal{V}_x$ such that $V'' \subset V'$. Since $\overline{g}$ is upper semi-continuous (1.23) implies that

$$\begin{aligned}
\overline{g}(x) &= S(\overline{g})(x) = \inf\{\sup\{\overline{g}(y) : y \in V\} : V \in \mathcal{V}_x\} \\
&= \inf\{\sup\{\overline{g}(y) : y \in V\} : V \subset V''\} \\
&\geq \inf\left\{\sup\left\{\underline{g}(y) + \varepsilon : y \in V\right\} : V \subset V''\right\} \\
&= \inf\left\{\sup\left\{\underline{g}(y) : y \in V\right\} : V \subset V''\right\} + \varepsilon \\
&= S\left(\underline{g}\right)(x) + \varepsilon, x \in V'.
\end{aligned}$$

Also, for any $x \in V'$

$$\begin{aligned} I(\overline{g})(x) &= \sup\{\inf\{\overline{g}(y) : y \in V\} : V \in \mathcal{V}_x\} \\ &= \sup\{\inf\{\overline{g}(y) : y \in V\} : V \subset V''\} \\ &\geq \sup\left\{\inf\left\{S\left(\underline{g}\right)(y) + \varepsilon : y \in V\right\} : V \subset V''\right\} \\ &= \sup\left\{\inf\left\{S\left(\underline{g}\right)(y) : y \in V\right\} : V \subset V''\right\} + \varepsilon \\ &= I\left(S\left(\underline{g}(x)\right)\right) + \varepsilon, x \in V'. \end{aligned}$$

This is clearly a contradiction, so that W_g^{ε} is nowhere dense in X. Also, it is clear that

$$X \setminus W_g^{\varepsilon} = \{x \in X : \omega(g(x)) > \varepsilon\}$$

is open in X, so that W_g^{ε} is closed in X. Consequently, the set

$$W_g = \bigcup_{n=1}^{\infty} W_g^{\frac{1}{n}}$$

is of first Baire category in X. Since X is a Baire space it follows that $D_g = X \setminus W_g$ is dense in X. ■

Theorem 4.17 *Let X be a compact metric space. Let the sequence (f_n) on $\mathsf{H}_{ft}(X)$ order converges to $f \in \mathsf{H}_{ft}(X)$. Then the 'fat' order limit f^* associated with (f_n) satisfies*

$$f^*(x) = f(x), x \in D$$

where D is a dense subset of X.

Proof. By the Baire Category Theorem X is a Baire space. By Theorem 4.16 f is the unique H-continuous function that satisfies the inclusion

$$f(x) \subseteq f^*(x), x \in X. \tag{4.63}$$

By Lemma 4.2 the set D_{f^*} is dense in X so that the result follows by the inclusion (4.63). ■

It is possible, through the upper and lower δ-Baire operators, see (1.32) and (1.33), to define a modulus of continuity. This was done in the case where X is a compact interval of the real line in [76], but in the general case we define the modulus of continuity as

$$\omega_\delta(f) = \sup_{x \in X}\left[S_{\delta/2}(f)(x) - I_{\delta/2}(f)(x)\right]. \tag{4.64}$$

A function $f \in \mathcal{A}(X)$ is continuous if and only if $\omega_\delta(f) \to 0$ as $\delta \to 0^+$. The modulus of H-contnuity was defined similarly in [76] when X is a subset of the real line, and in general the modulus of H-continuity is

$$\tau_\delta(f) = r\left(S_{\delta/2}(f), I_{\delta/2}(f)\right). \tag{4.65}$$

As a first application of the modulus of H-continuity, we have the following characterization of H-continuous functions.

Theorem 4.18 *Let f be an S-continuous function defined on the metric space X. Then f is H-continuous whenever*

$$\tau_{\delta}(f) \to 0$$

as $\delta \to 0^{+}$ in the natural metric on $\mathbb{R}$. Moreover, if X is compact the converse of this also holds.

Proof. By (1.32) through (1.35) it is clear that

$$I_{\delta/2}(f) \leq I(f) \leq f \leq S(f) \leq S_{\delta/2}(f) \tag{4.66}$$

for every $\delta > 0$. By assumption, for every $\varepsilon > 0$ there exists a $\delta_{\varepsilon} > 0$ such that

$$\tau_{\delta}(f) = r\left(I_{\delta/2}(f), S_{\delta/2}(f)\right) < \varepsilon$$

for every $\delta < \delta_{\varepsilon}$. By (4.66) and the proof of Theorem 4.12

$$r(I(f), S(f)) = r(F(I(f)), F(S(f))) < \varepsilon$$

for every $\varepsilon > 0$. Therefore

$$r\left(\underline{f}, \overline{f}\right) = r(F(I(f)), F(S(f))) = 0,$$

and by Theorem 4.13 f is H-continuous.
Conversely, suppose that X is compact and f is finite and H-continuous. By (1.38) and (4.65)

$$\tau_{\delta}(f) = r\left(S_{\delta/2}(f), I_{\delta/2}(f)\right) = r\left(F\left(S_{\delta/2}(f)\right), F\left(I_{\delta/2}(f)\right)\right)$$

so that it follows by Theorem 4.11 that

$$\tau_{\delta}(f) \leq r\left(F\left(S_{\delta/2}(f)\right), f\right) + r\left(f, F\left(I_{\delta/2}(f)\right)\right)$$

Note that

$$S\left(S_{\delta/2}(f)\right)(x) \downarrow \overline{f}(x) \text{ as } \delta \downarrow 0, \tag{4.67}$$

$$I\left(S_{\delta/2}(f)\right)(x) \downarrow \underline{f}(x) \text{ as } \delta \downarrow 0, \tag{4.68}$$

$$S\left(I_{\delta/2}(f)\right)(x) \uparrow \overline{f}(x) \text{ as } \delta \downarrow 0 \tag{4.69}$$

and

$$I\left(I_{\delta/2}(f)\right)(x) \uparrow \underline{f}(x) \text{ as } \delta \downarrow 0. \tag{4.70}$$

Indeed, it follows from (1.33) and (1.35) that

$$\delta < \delta' \Rightarrow S_{\delta/2}(f)(x) \leq S_{\delta'/2}(f)(x) \leq S(f)(x), x \in X$$

so that Theorems C.1 (i) and C.4 imply

$$\delta < \delta' \Rightarrow \left\{ \begin{array}{c} \overline{f}(x) \leq S\left(S_{\delta/2}(f)\right)(x) \leq S\left(S_{\delta'/2}(f)\right)(x) \\ \underline{f}(x) = I(S(f))(x) \leq I\left(S_{\delta/2}(f)\right)(x) \leq I\left(S_{\delta'/2}(f)\right)(x) \end{array} \right., x \in X.$$

Now suppose there exists $x \in X$ and $c \in \mathsf{R}$ such that

$$\overline{f}(x) < c < S\left(S_{\delta/2}(f)\right)(x), \delta > 0. \tag{4.71}$$

By (1.33) and (1.35)

$$\begin{aligned} S\left(S_{\delta/2}(f)\right)(x) &= \inf\left\{\sup\left\{\sup\left\{\overline{f}(z) : z \in B_{\delta/2}(y)\right\} : y \in B_{\varepsilon}(x)\right\} : \varepsilon > 0\right\} \\ &= \inf\left\{\sup\left\{\overline{f}(y) : y \in B_{\delta/2+\varepsilon}(y)\right\} : \varepsilon > 0\right\} \end{aligned}$$

It now follows by (4.71) that

$$\overline{f}(x) < c \leq \inf\left\{S\left(S_{\delta/2}(f)\right)(x) : \delta > 0\right\} = S(f)(x) = \overline{f}(x)$$

which is clearly a contradiction so that (4.67) must hold. Similar arguments show that (4.68) through (4.70) hold.
Since X is compact the convergence (4.67) through (4.70) is uniform so that for every $\varepsilon > 0$ there exists $\delta_{\varepsilon} > 0$ such that

$$\delta < \delta_{\varepsilon} \Rightarrow \left\{ \begin{array}{l} S\left(S_{\delta/2}(f)\right)(x) - \overline{f}(x) < \frac{\varepsilon}{2} \\ \underline{f}(x) - I\left(I_{\delta/2}(f)\right)(x) < \frac{\varepsilon}{2} \end{array} \right., x \in X.$$

Therefore there exists for every $\varepsilon > 0$ a $\delta_{\varepsilon} > 0$ such that

$$S\left(S_{\delta/2}(f)\right)(x) < S_{\varepsilon}\left(\overline{f}\right)(x) + \varepsilon, x \in X \tag{4.72}$$

and

$$I_{\varepsilon}\left(\underline{f}\right)(x) - \varepsilon < I\left(I_{\delta/2}(f)\right)(x), x \in X \tag{4.73}$$

whenever $\delta < \delta_{\varepsilon}$. But

$$I\left(I_{\delta/2}(f)\right)(x) \leq S\left(I_{\delta/2}(f)\right)(x) \leq \overline{f}(x), x \in X \tag{4.74}$$

and

$$\underline{f}(x) \leq I\left(S_{\delta/2}(f)\right)(x) \leq S\left(S_{\delta/2}(f)\right)(x), x \in X. \tag{4.75}$$

Combining (4.72) through (4.75), we obtain

$$I_{\varepsilon}\left(\underline{f}\right)(x) - \varepsilon \leq I\left(S_{\delta/2}(f)\right)(x) \leq S\left(S_{\delta/2}(f)\right)(x) < S_{\varepsilon}\left(\overline{f}\right)(x) + \varepsilon, x \in X$$

and

$$I_{\varepsilon}\left(\underline{f}\right)(x) - \varepsilon < I\left(I_{\delta/2}(f)\right)(x) \leq S\left(I_{\delta/2}(f)\right)(x) \leq S_{\varepsilon}\left(\overline{f}\right)(x) + \varepsilon, x \in X.$$

It now follows by (1.36) that

$$\eta\left(F\left(S_{\delta/2}(f)\right), f\right) \to 0 \text{ and } \eta\left(F\left(I_{\delta/2}(f)\right), f\right) \to 0$$

as $\delta \to^{+} 0$. In the similar way as above it follows that

$$\eta\left(f, F\left(S_{\delta/2}(f)\right)\right) \to 0 \text{ and } \eta\left(f, F\left(I_{\delta/2}(f)\right)\right) \to 0$$

as $\delta \to^{+} 0$. This completes the proof. ■

Just as in the study of spaces of continuous functions in the Functional Analysis there is a concept of equi-continuity, we may introduce a similar concept, that of equi-H-continuity.

Definition 4.4 *A set* $\{f_\alpha\} \subset \mathsf{H}_{ft}(X)$ *is said to be equi-H-continuous if for every* $\varepsilon > 0$ *there exists a* $\delta_\varepsilon > 0$ *such that, for every* $\delta < \delta_\varepsilon$

$$\tau_\delta(f_\alpha) < \varepsilon$$

for every α. *Here* τ_δ *denotes the modulus of H-continuity defined in (4.65).*

Theorem 4.19 *Let* (f_n) *be a sequence on* $\mathsf{H}_{ft}(X)$ *and let* $g \in \mathsf{F}_{ft}(X)$ *be the Hausdorff limit of* (f_n)*, that is, for every* $\varepsilon > 0$ *there exists an* $N_\varepsilon \in \mathsf{N}$ *such that*

$$r(f_n, g) < \varepsilon$$

for every $N \geq N_\varepsilon$. *Also suppose that the set* $\{f_n : n \in \mathsf{N}\}$ *is equi-H-continuous. Then* $g \in \mathsf{H}_{ft}(X)$ *and* (f_n) *order converges to* g.

Proof. Because the set $\{f_n : n \in \mathsf{N}\}$ is equi-H-continuous, it follows by Definition 4.4 that for every $\varepsilon > 0$ there exists a $\delta_\varepsilon > 0$ such that

$$r\left(I_{\delta/2}(f_n), S_{\delta/2}(f_n)\right) < \varepsilon.$$

for every $\delta < \delta_\varepsilon$ and every $n \in \mathsf{N}$. Since g is the Hausdorff limit of the sequence (f_n), we have by (1.36) and (1.37) that there exists an $N_0 \in \mathsf{N}$ such that

$$I_{\delta/2}(f_n)(x) - \delta/2 \leq \underline{g}(x) \leq \overline{g}(x) \leq S_{\delta/2}(f_n)(x) + \delta/2, x \in X$$

whenever $n \leq N_0$. By Theorem 4.12 it follows that

$$r\left(\underline{g}, \overline{g}\right) < \varepsilon.$$

Since this holds for any $\varepsilon > 0$

$$r\left(\underline{g}, \overline{g}\right) = 0$$

so that g is H-continuous by Theorem 4.13. Furthermore, the set

$$\{\varphi \in \mathsf{H}_{ft}(X) : \varphi(x) \subseteq g(x), x \in X\}$$

contains only one element, namely g. Since the sequence (f_n) converges to g in the Hausdorff distance if and only if both the one-sided Hausdorff distances

$$h(f_\alpha, g),\ h(g, f_\alpha)$$

tends to zero, it follows by Theorem 4.16 that the sequence (f_n) order converges to some $\varphi \in \mathsf{H}(X)$ with the property that

$$\varphi(x) \subseteq g(x), x \in X.$$

Therefore $\varphi = g$ and we are done. ■

4.5 The Completion of $\mathcal{C}(X)$

As shown by Example 1.3 in the introduction to this work, it is not possible to have on $\mathcal{C}(X)$ a topology which induces the order convergence of sequences. However, using the results of Chapter 2 we can show that there exists a convergence structure on $\mathcal{C}(X)$ which induces the order convergence. Indeed, since $\mathcal{C}(X)$ is a real vector lattice with the point-wise defined addition, scalar multiplication and order, see [81][Example 4.2(6)], it follows from Corollary 2.1 that $(\mathcal{C}(X), \sigma_o)$ is an FS-space. Hence, there exists a convergence structure on $\mathcal{C}(X)$ which induces σ_o. In fact, it follows by Corollary 2.2 that the mapping λ_o, as defined in Definition 2.1, defines a first countable convergence structure on $\mathcal{C}(X)$ and induces σ_o. Furthermore, using that $\mathcal{C}(X)$ is an Archimedean vector lattice we have that $(\mathcal{C}(X), \lambda_o)$ is a convergence vector space.

It is shown in [15][Proposition 3.6.5] that for a first countable convergence vector space completeness and sequential completeness are equivalent. Since the convergence vector space $(\mathcal{C}(X), \lambda_o)$ is first countable it is sufficient to use sequential arguments with regard to its completeness. However we should recall that the Cauchy sequences are defined through λ_o not just σ_o. Consider the following example.

Example 4.4 *Let the sequence (g_n) on $\mathcal{C}(\mathbb{R})$ be given by*

$$g_n(x) = \begin{cases} -1 & \text{if } x \leq -\frac{1}{n} \\ nx & \text{if } -\frac{1}{n} < x < \frac{1}{n} \\ 1 & \text{if } x \geq \frac{1}{n} \end{cases}$$

The filter $\langle (g_n) \rangle - \langle (g_n) \rangle$ is generated by the basis $\{\{g_m - g_k : m, k \geq n\} : n \in \mathbb{N}\}$. It is easy to see that for any $n \in \mathbb{N}$ we have

$$-f_n \leq g_m - g_k \leq f_n, \ \ m \geq n, k \geq n,$$

where (f_n) is the sequence given in Example 1.6. Since (f_n) is a decreasing sequence with an infimum equal to the constant zero function, $(-f_n)$ and (f_n) are sequences that can be associated with the filter $\langle (g_n) \rangle - \langle (g_n) \rangle$ in terms of Definition 2.1. Hence the filter $\langle (g_n) \rangle - \langle (g_n) \rangle$ order converges to 0, which implies that (f_n) is a Cauchy sequence.

On the other hand it is quite clear that this sequence is not order convergent in $\mathcal{C}(X)$.

Example 4.4 shows that the convergence vector space $(\mathcal{C}(X), \lambda_o)$ is not complete.

The main aims of the this section is to construct a completion of the convergence vector space $(\mathcal{C}(X), \lambda_o)$ as a set of functions defined on the same domain X. Since the convergence structure λ_o is defined through the partial order on $\mathcal{C}(X)$ it is natural to consider the Dedekind order completion of $\mathcal{C}(X)$. In [3] the Dedekind order completion of $\mathcal{C}(X)$ was represented as a subset of the set of all finite Hausdorff continuous functions $\mathsf{H}_{ft}(X)$. It was also shown that in the special case when X is a metric space the Dedekind order completion of $\mathcal{C}(X)$ is exactly $\mathsf{H}_{ft}(X)$. Let us note that the Dedekind order completion of a poset does not give automatically a completion with respect to any uniform convergence structure defined through the order. In fact, it is shown in [64] that convergence with respect to the order topology on the Dedekind order completion of a poset does not imply convergence with respect to the order topology on the original poset. Hence the results given in the sequel with regard to the completion of $\mathcal{C}(X)$ through Hausdorff continuous functions are highly nontrivial.

It was shown in Section 4.2 that the set $\mathsf{H}_{ft}(X)$ can be made into a linear space whenever X is a Baire space by defining 'the sum' of two H-continuous functions f and g as

$$(f \oplus g)(x) = F(D, X, f+g)(x), x \in X \tag{4.76}$$

where $D = X \setminus (W_f \cup W_g)$. It is easily seen that the above is an extension of the pointwise operations on $\mathcal{C}(X)$. Indeed, since the continuous functions are the real valued H-continuous functions on X, it follows that $D = X$ in (4.76) for all continuous f and g so that the sum reduces to

$$(f \oplus g)(x) = F(f(x) + g(x)) = f(x) + g(x), x \in X.$$

Since we defined scalar multiplication on $\mathsf{H}_{ft}(X)$ in a pointwise way, it is also extends the operation on $\mathcal{C}(X)$ so that $\mathcal{C}(X)$ is a linear subspace of $\mathsf{H}_{ft}(X)$.

We also showed that $\mathsf{H}_{ft}(X)$ is a Dedekind complete vector lattice. But the order (1.19) is an extension of the pointwise order (1.39) on $\mathcal{C}(X)$. As $\mathcal{C}(X)$ is a lattice it now follows that $\mathcal{C}(X)$ is a vector sublattice of $\mathsf{H}_{ft}(X)$.

According to our results in Section 2.3, in particular Theorem , the completion of the convergence vector space $(\mathcal{C}(X), \lambda_o)$ is $\left(\mathcal{C}(X)^{\#}, \lambda_o\right)$, where $\mathcal{C}(X)^{\#}$ denotes the Dedekind σ-completion of $\mathcal{C}(X)$. By definition, $\mathcal{C}(X)^{\#}$ is the smallest Dedekind σ-complete vector lattice, with respect to inclusion, that contains $\mathcal{C}(X)$ as a vector lattice subspace. Since the Dedekind complete vector lattice $\mathsf{H}_{ft}(X)$ contains $\mathcal{C}(X)$ as a vector sublattice it follows that

$$\mathcal{C}(X)^{\#} \subseteq \mathsf{H}_{ft}(X) \tag{4.77}$$

whenever X is a Baire space. By (4.77) it follows that there exists a vector sublattice $\mathsf{H}^1_{ft}(X)$ of $\mathsf{H}_{ft}(X)$ such that $\left(\mathsf{H}^1_{ft}(X), \lambda_o\right)$ is the convergence vector space completion of $(\mathcal{C}(X), \lambda_o)$. Note that the convergence structure on $\mathsf{H}^1_{ft}(X)$ is not necessarily the subspace structure inherited from $(\mathsf{H}_{ft}(X), \lambda_o)$.

It is not known, however, wether or not equality always holds in (4.77). However, when X is a metric space, we have the following result.

Theorem 4.20 *Let X be a metric space. Then $\mathsf{H}_{ft}(X)$ is the Dedekind σ-completion of $\mathcal{C}(X)$ as a vector lattice.*

Proof. Let $f = [\underline{f}, \overline{f}] \in \mathbb{H}_{ft}(X)$. We will prove the existence of an increasing sequence (ψ_n) of continuous functions on X such that

$$f = \sup\{\psi_n : n \in \mathbb{N}\}.$$

Let ρ be the metric on X. We will use the function $h : \mathbb{R} \to (-1, 1) \subset R$ defined by

$$h(z) = \frac{z}{1 + |z|},\ z \in \mathbb{R}.$$

This real function is continuous and strictly increasing. The inverse function $h^{-1} : (-1, 1) \to \mathbb{R}$ is given by

$$h^{-1}(z) = \frac{z}{1 - |z|},\ z \in (-1, 1),$$

and is also continuous and strictly increasing.
Consider the functions $\varphi_n : X \times X \to \mathbb{R}$ defined by

$$\varphi_n(t, x) = h(\underline{f}(t)) + n\rho(t, x) = \frac{\underline{f}(t)}{1 + |\underline{f}(t)|} + n\rho(t, x),\ n \in \mathbb{N}. \tag{4.78}$$

It is easy to see that the function φ_n is bounded from below. Indeed, since the value of the metric ρ is always nonnegative and the fraction in (4.78) is greater than -1 we have $\varphi_n(t, x) > -1$. Then we can define

$$\psi_n(x) = \inf\{\varphi_n(t, x) : t \in X\},\ n \in \mathbb{N}.$$

First we will show that for every $n \in \mathbb{N}$ the function ψ_n is continuous on X. From the triangular inequality of the metric ρ for every $x, y, t \in X$ we have

$$\rho(t, y) - \rho(x, y) \le \rho(t, x) \le \rho(t, y) + \rho(x, y).$$

Therefore

$$\varphi_n(t, y) - n\rho(x, y) \le \varphi_n(t, x) \le \varphi_n(t, y) + n\rho(x, y).$$

Taking the infimum on $t \in X$ we obtain

$$\psi_n(y) - n\rho(x, y) \le \psi_n(x) \le \psi_n(y) + n\rho(x, y).$$

Hence we have the inequality

$$|\psi_n(x) - \psi_n(y)| \le n\rho(x, y)\ ,\quad x, y \in X\ ,$$

which implies that the function ψ_n is continuous on X.
Our second step is to prove that ψ_n satisfies the inequalities

$$-1 < \psi_n(x) \le \frac{\underline{f}(x)}{1 + |\underline{f}(x)|} < 1,\ x \in X. \tag{4.79}$$

For every $x \in X$ we have

$$\psi_n(x) = \inf\{\varphi(t,x) : t \in X\} \le \varphi(x,x) = h(\underline{f}(x)) = \frac{\underline{f}(x)}{1+|\underline{f}(x)|}. \qquad (4.80)$$

Furthermore, since -1 is a lower bound of $\varphi_n(t,x)$ the inequality

$$\psi_n(x) \ge -1$$

also holds. It remains to prove that $\psi_n(x) \ne -1$. Let us assume that there exists $x \in X$ such that $\psi_n(x) = -1$. Let the real number μ be such that $-1 < \mu < \frac{\underline{f}(x)}{1+|\underline{f}(x)|}$. Then we have

$$h(\underline{f}(x)) = \frac{\underline{f}(x)}{1+|\underline{f}(x)|} > \mu > -1.$$

Using standard techniques one can easily see that the function $h \circ \underline{f}$ is lower semi-continuous. Hence there exists $\varepsilon > 0$ such that

$$\frac{\underline{f}(t)}{1+|\underline{f}(t)|} = h(\underline{f}(t)) > \mu \quad \text{whenever} \quad \rho(t,x) < \varepsilon. \qquad (4.81)$$

Let now $\delta = \min\{n\varepsilon, \mu+1\}$. Since $\psi_n(x)$ is defined as an infimum on $t \in X$, there exists $t_\delta \in X$ such that

$$-1 = \psi_n(x) \le \varphi_n(t_\delta, x) \le \psi_n(x) + \delta = -1 + \delta$$

or, more precisely,

$$-1 \le \frac{\underline{f}(t_\delta)}{1+|\underline{f}(t_\delta)|} + n\rho(t_\delta, x) \le -1 + \delta.$$

Using simple manipulations we obtain

$$0 \le \rho(t_\delta, x) \;\le\; \frac{1}{n}\left(\delta - \left(1 + \frac{\underline{f}(t_\delta)}{1+|\underline{f}(t_\delta)|}\right)\right) \;<\; \frac{\delta}{n} \;\le\; \varepsilon \qquad (4.82)$$

$$-1 \le \frac{\underline{f}(t_\delta)}{1+|\underline{f}(t_\delta)|} \;\le\; -1+\delta \;\le\; \mu \qquad (4.83)$$

The contradiction between inequalities (4.82), (4.83) on the one side and the condition (4.81) on the other side show that the assumption that $\psi_n(x) = -1$ for some $x \in X$ is false. Therefore $f_n(x) > -1$, $x \in X$.
We will show that (f_n) where

$$f_n(x) = h^{-1}(\psi_n(x)) = \frac{\psi_n(x)}{1-|\psi_n(x)|}, \; x \in X, \; n \in \mathbb{N}, \qquad (4.84)$$

is the required sequence. Due to inequalities (4.79) the function f_n is well defined for every $x \in X$ and $n \in \mathbb{N}$. Moreover, f_n is continuous on X because ψ_n is continuous on X. Using the fact that the function h^{-1} is strictly increasing on the interval $(-1,1)$ and that the sequence $(\varphi_n)_{n\in\mathbb{N}}$ is increasing with n we obtain that (f_n) is an increasing sequence. Furthermore from the middle inequality in (4.79) we obtain

$$f_n(x) = h^{-1}(\psi_n(x)) \leq h^{-1}\left(\frac{\underline{f}(x)}{1+|\underline{f}(x)|}\right) = h^{-1}\left(h(\underline{f}(x)\right) = \underline{f}(x) \leq f(x),\ x \in X,\ n \in \mathbb{N}.$$

It remains to prove that $f = \sup\{f_n : n \in \mathbb{N}\}$. We will show first that $\underline{f}$ is the point-wise supremum of the sequence (f_n), that is,

$$\underline{f}(x) = \sup_{n\in\mathbb{N}}(f_n(x))\,,\ x \in X. \tag{4.85}$$

Let $x \in X$ and let $\varepsilon > 0$ be arbitrary. Using that the function $h \circ \underline{f}$ is lower semi-continuous there exists $\nu > 0$ such that $h(\underline{f}(t)) > h(\underline{f}(x)) - \varepsilon$ whenever $\rho(t,x) < \nu$. Let $m \in \mathbb{N}$ be such that $m \geq \dfrac{h(\underline{f}(x)) - \varepsilon + 1}{\nu}$. It is easy to see that

$$\varphi_n(t,x) \geq h(\underline{f}(x)) - \varepsilon\ ,\ t \in X,\ n \geq m. \tag{4.86}$$

Indeed, if

(a) $\rho(t,x) \geq \nu$ then $\varphi_n(t,x) > -1 + n\nu \geq -1 + \dfrac{h(\underline{f}(x)) - \varepsilon + 1}{\nu}\nu = h(\underline{f}(x)) - \varepsilon$;

(b) $\rho(t,x) < \nu$ then $\varphi_n(t,x) \geq h(\underline{f}(x)) - \varepsilon + n\rho(t,x) \geq h(\underline{f}(x)) - \varepsilon$.

Using (4.86) for $n \geq m$ we have

$$\psi_n(x) = \inf_{t\in X}\varphi_n(t,x) \geq h(\underline{f}(x)) - \varepsilon.$$

Therefore

$$\sup_{n\in\mathbb{N}}(\psi_n(x)) \geq h(\underline{f}(x)) - \varepsilon.$$

Since ε in the above inequality is arbitrary and using also (4.80) we obtain

$$\sup_{n\in\mathbb{N}}(\psi_n(x)) = h(\underline{f}(x)).$$

The function h^{-1} used in the definition of f_n, see (4.84), is continuous and strictly increasing. Then we have

$$\sup_{n\in\mathbb{N}}(f_n(x)) = \sup_{n\in\mathbb{N}}(h^{-1}(\psi_n(x))) = h^{-1}(\sup_{n\in\mathbb{N}}(\psi_n(x))) = h^{-1}(h(\underline{f}(x))) = \underline{f}(x),$$

which proves (4.85). Finally using Theorems C.12 and C.5 it follows from (4.85) that

$$\sup_{n\in\mathbb{N}} f_n = F(S(\underline{f})) = F(\overline{f}) = f.$$

A in a similar way we can construct a decreasing sequence (f_n') of continuous functions such that

$$f = \inf_{n \in \mathbb{N}} f_n'.$$

Now suppose that there exists $\mathsf{H}^1_{ft}(X) \subset \mathsf{H}_{ft}(X)$ that is Dedekind σ-complete and $\mathcal{C}(X) \subset \mathsf{H}^1_{ft}(X)$. Let $f \in \mathsf{H}_{ft}(X) \setminus \mathsf{H}^1_{ft}(X)$. By the above there exists an increasing sequence (f_n) on $\mathcal{C}(X)$ such that

$$f = \sup_{n \in \mathbb{N}} f_n.$$

But there also exists $f' \in \mathcal{C}(X)$ such that $f \leq f'$. Hence the sequence (f_n) is bounded from above in $\mathcal{C}(X)$ so that

$$f = \sup_{n \in \mathbb{N}} f_n \in \mathsf{H}^1_{ft}(X).$$

Since we choose $f \notin \mathsf{H}^1_{ft}(X)$ this is a contradiction so that $\mathsf{H}^1_{ft}(X) = \mathsf{H}_{ft}(X)$. The linear operations are extended to $\mathsf{H}_{ft}(X)$ as in (4.22) though (4.23). This completes the proof. ■

The completion of $(\mathcal{C}(X), \lambda_o)$ is now obtained as a straight forward corollary to the above.

Corollary 4.1 *Let X be a metric space. Then $(\mathsf{H}_{ft}(X), \lambda_o)$ is the convergence space completion of $(\mathcal{C}(X), \lambda_o)$ in the sense of (C1) through (C3).*

Proof. By Theorems 4.20 $\mathsf{H}_{ft}(X)$ is the Dedekind σ-completion of $\mathcal{C}(X)$. The result now follows upon application of Theorem 2.15. ■

5. CONCLUSION

The broad context of this work is a thorough treatment of order convergence of sequences on an Archimedean vector lattice within the framework of convergence structures. The motivation of this investigation is twofold. The first indication that such an investigation would be appropriate is to be found in [78] where we considered order convergence on sets of Hausdorff continuous functions. There, however, no further structure other than the order relation is assumed. With the introduction of a linear structure on the set of finite H-continuous functions on open subsets of R^n it became clear that the theory of vector lattices would supply a suitable abstract framework in which to study this phenomenon.

The second, far more specific problem that stirred our interest was that of finding a 'completion' with respect to order convergence on the set $\mathcal{C}(X)$ of all continuous functions. In order to define a satisfactory notion of 'Cauchyness' one must leave the realm of sequences and consider more arbitrary topological type processes. Moreover, sequential structures are ill-suited to the construction of 'completions'. In fact, there are examples of sequential convergence groups that have no completion at all, see [33]. A suitable theory for constructing such a completion was to be found in the theory of convergence spaces, see [15]. This second problem is solved here and in [7] where it is shown that for a metric space X the desired completion of $\mathcal{C}(X)$ is exactly the set $\mathsf{H}_{ft}(X)$ of all finite H-continuous functions on X. This result has been generalized slightly. It was shown that if a topological space X is a Baire space, then the completion of $(\mathcal{C}(X), \lambda_o)$ is a vector sublattice of the Dedekind complete vector lattice $\mathsf{H}_{ft}(X)$.

The obstacles to obtaining this result in its full generality is twofold. Firstly, it is unknown whether or not $\mathsf{H}_{ft}(X)$ is the Dedekind completion of $\mathcal{C}(X)$ for any space X more arbitrary than a metric space. Secondly, some difficulties are encountered when defining algebraic operations on $\mathsf{H}_{ft}(X)$ when X is neither a Baire space nor a metric space. Determining sufficient conditions on X to allow for a linear structure on $\mathsf{H}_{ft}(X)$ is an open problem.

We considered the problems related to order convergence in full generality. That is, we studied the order convergence of sequences on an Archimedean vector lattice E. The key result that we obtained in this respect is that order convergence is induced by a convergence structure. In fact, we constructed such a convergence structure λ_o and showed that it is compatible with the linear structure of the space E. This is the first time that the concept of order convergence on a vector lattice has been linked to that of a convergence structure. Generalizing the result obtained in [7], we showed that every Archimedean vector lattice E equipped with the order convergence structure λ_o can be completed in the sense of (C1) through (C3) by the Dedekind σ-completion of

E.

Closely connected with order convergence is the concept of relatively uniform (ru) convergence. As is the case with order convergence, (ru) convergence generally fails to be topological. However, as we showed, it is induced by a convergence structure. Moreover, this convergence structure is exactly the Mackey modification of the order convergence structure λ_o. The construction of the completion of of this convergence structure is still an open problem. We have only managed to show that completeness in the sense of convergence spaces is identical to the concept of (ru) completeness of the vector lattice. The concrete description of the completion has not been achieved, but it is our intention to pursue this in future research.

The convergence vector space (E, λ_o) and its Mackey modification $\mu(E)$, where E is an Archimedean vector lattice, provides an appropriate setting in which to study the order bounded and σ-order continuous operators. In particular, if F is a Dedekind compete vector lattice then the continuous operators between (E, λ_o) and (F, λ_o) are precisely the σ-order continuous operators, and the continuous operators between $\mu(E)$ and $\mu(F)$ are exactly the order bounded operators. Since the space $L_b(E, F)$ of order bounded operators and the space $L_c(E, F)$ of σ-order continuous operators are both Dedekind complete vector lattices, the order convergence structure can be defined on both, introducing the structure of a complete convergence vector space.

A particular novelty of this view of the σ-order continuous operators is that it allows for the use of the continuous convergence structure. It is shown that the space $L_c(E, F)$ is a complete convergence vector space when equipped with the continuous convergence structure. The interplay between these two convergence structures considered on $L_c(E, F)$ results in a Banach-Steinhaus type theorem. In particular, we show that if the dual of (F, λ_o) separates the points of F, then the pointwise limit of a sequence of σ-order continuous operators $T_n : E \to F$ is also σ-order continuous. Further research will seek to apply this result where appropriate.

In this work we only considered real vector lattices. However, complex vector lattices are of considerable interest in analysis as many important spaces are complex vector spaces. Moreover, the functional calculus is developed within the setting of complex vector lattices. It is therefore particularly relevant to investigate the possibilities for extending and applying the results obtained in this work for real spaces to complex vector lattices. In particular, one would consider the functional calculus mentioned above and how it relates to the convergence structures considered here.

The present work deals only with order convergence of sequences. In the Chapter 1, however, we mentioned other modes of convergence induced by an order relation. In particular, on a vector lattice one often considers directed sets, and specifically the upward and downward directed sets. This gives rise to a notion of order convergence of generalized sequences in much the same way as monotone sequences are employed in the definition of order convergent sequences, see Definition 1.3. We suggest that a convergence structure can be defined in the same spirit as the order convergence structure λ_o that will induce this convergence of generalized sequences. We believe that a theory similar to that developed here can be developed. This, with the proper completeness assumptions, would place the order continuous operators within the context of convergence vector spaces and perhaps the results developed for σ-order continuous operators in this work can be reworked to apply to the order continuous operators. Such an investigation is merited by the applications of the order continuous operators to functional analysis, for instance in integration theory and the theory of Banach lattices.

"A problem left to itself dries up or goes rotten. But fertilize a problem with a solution- you'll hatch out dozens."

-N. F. Simpson

A. VECTOR LATTICES

The following results are standard in vector lattice theory. Proofs can be found in [52] and [81]. A more recent presentation can be found in [83].

Theorem A.1 *Let E be a vector lattice. Then E is distributive as a lattice, and hence also σ-distributive.*

Proposition A.1 *Let E be a vector lattice and $f \leq g \in E$. Then $f^+ \leq g^+$ and $g^- \leq f^-$.*

Theorem A.2 *Let E be a vector lattice and let (f_n) and (g_n) be sequences on E.*
(i) The sequence (f_n) increases to f if and only if $(-f_n)$ decreases to $-f$.
(ii) The sequence (g_n) decreases to f if and only if $(-g_n)$ increases to $-f$.
(iii) If $f_n \rightarrow f$ and $g_n \rightarrow g$ then the sequence $(h_h) = (\sup\{f_n, g_n\})$ converges to $h = \sup\{f, g\}$. In particular, $f_n^+ \rightarrow f^+$ and $f_n^- \rightarrow f^-$.
(iv) If $f_n \rightarrow f$ then $|f_n| \rightarrow |f|$. In particular, if $f_n \uparrow 0$ then $|f_n| \downarrow 0$.
(v) If $f_n \rightarrow f$ and $f_n \geq g$ for every $n \in \mathbb{N}$ then $f \geq g$.
(vi) If $f_n \rightarrow f$ and $f_n \rightarrow g$ then $f = g$.

Theorem A.3 *Let E be a vector lattice.*
(i) If $f_n \uparrow f$ and $g_n \uparrow g$ then

$$\sup\{f_n, g_n\} \uparrow \inf\{f, g\},$$
$$\inf\{f_n, g_n\} \uparrow \inf\{f, g\}.$$

(ii) If $f_n \downarrow f$ and $g_n \downarrow g$ then

$$\sup\{f_n, g_n\} \downarrow \inf\{f, g\},$$
$$\inf\{f_n, g_n\} \downarrow \inf\{f, g\}.$$

Theorem A.4 *In a vector lattice E the operation addition is sequentially continuous with respect to order convergence, that is, if $f_n \rightarrow f$ and $g_n \rightarrow g$ then $f_n + g_n \rightarrow f + g$.*
Moreover, if E is Archimedean, then scalar multiplication is also sequentially continuous with respect to order convergence, that is, if $f_n \rightarrow f$ in E and $\alpha_n \rightarrow \alpha$ in $\mathbb{R}$ then the sequence $(\alpha_n f_n)$ order converges to αf.

Theorem A.5 *Let E be a Dedekind σ-complete lattice. Then the sequence (f_n) on E order converges to $f \in E$ if and only if*

$$\sup\{\inf\{f_n : n \geq k\} : k \in \mathbb{N}\} = \inf\{\sup f_n : n \geq k : k \in \mathbb{N}\}.$$

The sequence $(f_k') = (\inf\{f_n : n \geq k\})$ increases to f and the sequence $(f_k'') = (\sup\{f_n : n \geq k\})$ decreases to f.

Theorem A.6 *For a vector lattice E the following conditions are equivalent.*
(i) E is order complete;
(ii) Every increasing order Cauchy sequence has a supremum;
(iii) Every order Cauchy sequence has a supremum;

Proposition A.2 *An equivalent formulation of Definition 1.11 is the following: A subset A of P is closed in the order topology if and only if, for every sequence $(f_n) \subseteq A$ the convergence $f_n \to f$ implies that $f \in A$.*

Proposition A.3 *Let E be a vector lattice and K a topological space. A mapping $\varphi : E \to K$ is continuous in the order topology if and only if $(\varphi(f_n))$.converges to $\varphi(f)$ in K whenever (f_n) order converges to $f \in E$.*

Theorem A.7 *(Main Inclusion Theorem) Let E be a vector lattice. Then the following implications hold:*

$$\text{Super Ded. comp.} \Longrightarrow \text{Ded. comp.} \begin{array}{l} \Longrightarrow \text{De. } \sigma\text{-comp.} \\ \Longrightarrow \text{Proj. prop.} \end{array} \Longrightarrow \text{Princ. proj. prop.} \Longrightarrow \text{Arch.}$$

Theorem A.8 *Let E and F be vector lattices and $\pi : E \to F$ a Riesz homomorphism. Then π is a Riesz σ-homomorphism if and only if $\ker \pi$ is a σ-ideal.*

Proposition A.4 *If π is a Riesz homomorphism from the Dedekind σ-complete vector lattice L onto the vector lattice F and if (f_n) is a sequence in F satisfying $f_n \downarrow 0$, then thre exists a sequence (g_n) in E^+ such that $g_n \downarrow 0$ and $f_n = \pi g_n$ for every $n \in \mathbb{N}$.*

Theorem A.9 *Let E be an Archimedean vector lattice and A a σ-ideal of E. Then E/A is an Archimedean vector lattice.*

Theorem A.10 *Let E and F be vector lattices. Then $L_c(E,F)$ and $L_n(E,F)$ are linear subspaces of $L_b(E,F)$. If F is Dedekind complete $L_b(E,F)$ is a Dedekind complete vector lattice. The spaces $L_c(E,F)$ and $L_n(E,F)$ are both bands in $L_b(E,F)$.*

Theorem A.11 *Let E be a vector lattice and A a subset of L_bE. Then A is bounded by a bounded linear functional if and only A is pointwise bounded.*

Theorem A.12 *Let E be a vector lattice such that the ideal A of L_bE separates the points of E. Then the mapping $\sigma : E \to L_bB$ is a Riesz isomorphism. In fact, $\sigma(E)$ is contained in the set L_nB of all order continuous linear functionals on B.*

B. CONVERGENCE SPACES

This Appendix contains miscelenous results from the theory of convergence spaces that are applied in the main body of the present work. All results are taken from [15], except for Theorem B.1 which is taken from [35], where the proofs may also be located.

Proposition B.1 *For a subset A of a first countable convergence space K a point f belongs to $a\,(A)$ if and only if there exists a sequence in A which converges to f.*

Proposition B.2 *Let K and L be convergence spaces and φ a mapping from K into L.*
(i) If $\varphi : K \rightarrow L$ is continuous then so is $\varphi : o\,(K) \rightarrow o\,(L)$.
(ii) If L is topological, then $\varphi : K \rightarrow L$ is continuous if and only if $\varphi : o\,(K) \rightarrow o\,(L) = L$ is continuous.
(iii) $o\,(K)$ is the finest topological convergence structure on the same set that is coarser than K.

Proposition B.3 *Let K and L be convergence vector spaces. Then $\mathcal{C}_c\,(K, L)$ is Hausdorff, regular or Choquet whenever L is. In particular, $\mathcal{C}_c\,(K)$ is Hausdorff, regular and Choquet.*

Proposition B.4 *Let E be a convergence vector space, F a vector space and $p : E \rightarrow F$ a surjection. Then the quotient convergence structure on F is a vector space convergence structure and a fitler $\mathcal{Q}$ on F converges to $p(f) \in F$ in the quotient convergence structure if and only if there exists a filter $\mathcal{F}$ on E that converges to f such that $p\,(\mathcal{F}) \subseteq \mathcal{Q}$.*

Proposition B.5 *Let E be a first countable convergence vector space. Then E is complete if and only if E is sequentially complete, that is, every Cauchy sequence is convergent.*

Theorem B.1 *Let E be a Hausdorff convergence vector space. Then there exists a complete, Hausdorff convergence vector space $\widetilde{E}$ such that conditions (i) and (ii) above are satisfied. Furthermore, E is isomorphic to a dense subspace of $\widetilde{E}$ if and only if every Cauchy filter on E is bounded.*

Proposition B.6 *Let E be a convergence vector space. Then $\mu\,(E)$ is locally bounded and first countable. Moreover, $\mu\,(E)$ and E share the same bounded sets.*

Proposition B.7 *Let E and F be convergence vector spaces. Then a linear mapping $T : E \rightarrow F$ is bounded if and only if $T : \mu\,(E) \rightarrow \mu\,(F)$ is continuous.*

Theorem B.2 *Let E and F be convergence vector spaces with E barrelled. Then $(E, \mathcal{L}_c F)$ is a Banach-Steinhaus pair in each of the following two cases:*
(i) F is locally bounded.
(ii) E and F are first countable.

Corollary B.1 *Let E be a barrelled convergence vector space and F a locally convex topological vector space. Then (E, F) is a Banach-Steinhaus pair.*

Theorem B.3 *Let (E, F) be a Banach-Steinhaus pair and F regular and Choquet. If (T_n) is a sequence in $\mathcal{L}(E, F)$ that converges pointwise to a linear mapping $T : E \rightarrow F$, then T is continuous and (T_n) converges continuously to T.*

C. HAUSDORFF CONTINUOUS FUNCTIONS

The following is a collection of results on interval valued functions, and H-continuous functions in particular, that are used in this work. Proofs can be found in [3], [2] and [4].

Theorem C.1 *Concerning the mappings I, S and F, the following is true:*
(i) The operators I, S and F are all monotone with respect to the partial order (1.19), that is, for any two functions $f, g \in \mathbb{A}(X)$

$$f \leq g \Rightarrow I(f) \leq I(g), S(f) \leq S(g), F(f) \leq F(g);$$

(ii) The operator F is monotone with respect to the relation inclusion, that is, for any two functions $f, g \in \mathbb{A}(X)$

$$f(x) \subseteq g(x), x \in X \Rightarrow F(f)(x) \subseteq F(g)(x), x \in X;$$

(iii) The operators I, S and F are all indempotent, that is, for any $f \in \mathbb{A}(X)$

$$I(I(f)) = I(f), S(S(f)) = S(f), F(F(f)) = F(f).$$

There is a close connection between S-continuous functions and semi-continuous functions, and indeed H-continuous functions. For that reason we recall the definitions of semi-continuous functions, see [9].

The following fact is easily verified:

$$f \text{ is lower semi-continuous on } X \Leftrightarrow I(f) = f, \tag{C.1}$$

$$f \text{ is upper semi-continuous on } X \Leftrightarrow S(f) = f. \tag{C.2}$$

Theorem C.2 *Every pair consisting of a lower semi-continuous function $\underline{f}$ and an upper semi-continuous function $\overline{f}$ such that $\underline{f} \leq \overline{f}$ defines an S-continuous function $f(x) = [\underline{f}(x), \overline{f}(x)], x \in X$. Furthermore, if the set*

$$\{\varphi \in \mathcal{A}(X) : \underline{f} \leq \varphi \leq \overline{f}\}$$

does not contain any lower or upper semi-continuous functions, except for $\underline{f}$ and $\overline{f}$, respectively, then the function f is H-continuous.

Proposition C.1 *If f is an upper semi continuous (respectively lower semi continuous) function on X then its restriction to any subset A of X is upper semi continuous (respectively lower semi continuous) on A.*

Theorem C.3 *We have the following:*
(i) Let $L \subseteq \mathcal{A}(X)$ be a set of lower semi-continuous functions. Then the function l defined by

$$l(x) = \sup\{f(x) : f \in L\}$$

is lower semi-continuous.
(ii) Let $U \subseteq \mathcal{A}(X)$ be a set of upper semi-continuous functions. Then the function u defined by

$$u(x) = \inf\{f(x) : f \in U\}$$

is upper semi-continuous.

Theorem C.4 *Let $f \in \mathsf{A}(X)$. The following conditions are equivalent.*
(i) the function f is H-continuous;
(ii) $F(I(f)) = F(S(f)) = f$;
(iii) $S(I(f)) = S(f), I(S(f)) = I(f)$ and f is S-continuous.

Theorem C.5 *Let $f = [\underline{f}, \overline{f}] \in \mathsf{A}(X)$. Then f is H-continuous if and only if*

$$\overline{f} \text{ is upper semi-continuous}$$
$$\underline{f} \text{ is lower semi-continuous}$$
$$f = F(\underline{f}) = F(\overline{f})$$

Theorem C.6 *Let $f = [\underline{f}, \overline{f}]$ be a H-continuous function on X.*
(i) If $\underline{f}$ or $\overline{f}$ is continuous at a point $x \in X$ then $\underline{f}(x) = \overline{f}(x)$.
(ii) If $\underline{f}(x) = \overline{f}(x)$ for some $x \in X$ then both $\underline{f}$ and $\overline{f}$ is continuous at x.

Theorem C.7 *A function $f \in \mathsf{A}(X)$ is H-continuous if and only if the following two conditions are satisfied*
(i) f is S-continuous;
(ii) for any S-continuous function g the inclusion $g(x) \subseteq f(x), x \in X$, implies $g(x) = f(x), x \in X$.

Theorem C.8 *Let$f \in \mathsf{A}(X)$. Both the functions $F(S(I(f)))$ and $F(I(S(f)))$ are H-continuous and*

$$F(S(I(f))) \leq F(I(S(f))).$$

Theorem C.9 *Let $f \in \mathsf{H}(X)$. Then the set W_f^ε is closed nowhere dense in X. Consequently, the set*

$$W_f = \bigcup_{n \in \mathbb{N}} W_f^{\frac{1}{n}}$$

is of first Baire category.

Theorem C.10 *Let* $f, g \in \mathsf{H}(X)$ *and* D *a dense subset of* X. *If*
(i) $f(x) \leq g(x)$ *for each* $x \in D$, *then* $f(x) \leq g(x)$ *for each* $x \in X$;
(ii) $f(x) = g(x)$ *for each* $x \in D$, *then* $f(x) = g(x)$ *for each* $x \in X$;

Theorem C.11 *For an H-continuous function* f *to be finite it is suficient that* f *asumes finite values on a dense subset that need not be open.*

Theorem C.12 *Let* $\mathcal{B}$ *be a bounded subset of* $\mathsf{H}_{ft}(X)$ *and let the functions* $\varphi, \psi \in \mathcal{A}(X)$ *be defined by*

$$\varphi(x) = \inf\{\underline{f}(x) : f = [\underline{f}, \overline{f}] \in \mathcal{B}\}\ ,\ \ \psi(x) = \sup\{\overline{f}(x) : f = [\underline{f}, \overline{f}] \in \mathcal{B}\}\ ,\ x \in X.$$

Then both $\inf \mathcal{B}$ *and* $\sup \mathcal{B}$ *exists and*

$$\inf \mathcal{B} = F(I(\varphi)), \quad \sup \mathcal{B} = F(S(\psi)).$$

Furthermore if the set $\mathcal{B}$ *is finite then*

$$\inf \mathcal{B} = F(\varphi), \quad \sup \mathcal{B} = F(\psi).$$

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Order convergence on an Archimedean vector lattice and applications

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Summary

We consider here three as of yet unrelated concepts. That of order convergence on a vector lattice, convergence structures and Hausdorff continuous functions. It is shown that order convergence of sequences on a vector lattice is induced by a convergence structure. Such a convergence structure, called the order convergence structure, is defined and and we study its properties. In particular, it is shown that if the vector lattice is Archimedean then the order convergence structure is a vector space convergence structure and the completion the resulting convergence vector space is attained. This theory of order convergence is applied to obtain a Banach-Steinhaus theorem for σ-order continuous operators. We show that the set of all finite Hausdorff continuous functions defined on a Baire space constitutes a Dedekind complete vector lattice. Hence the theory developed for order convergence on Archimedean vector lattices is applicable. The completion of the order convergence structure on $\mathcal{C}(X)$ is obtained as a set of Hausdorff continuous functions.