A brief analysis of certain numerical methods used to solve stochastic differential equations

by

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DECLARATION

I, the undersigned, hereby declare that the dissertation submitted herewith for the degree Magister Scientiae to the University of Pretoria contains my own work, independent work and has not been submitted for any degree at any other university.

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DEDICATION AND THANKS

I would like to dedicate my thesis to my parents for their unwavering support and who instilled in us the need for education.

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ABSTRACT

Stochastic differential equations (SDE's) are used to describe systems which are influenced by randomness. Here, randomness is modelled as some external source interacting with the system, thus ensuring that the stochastic differential equation provides a more realistic mathematical model of the system under investigation than deterministic differential equations.

The behaviour of the physical system can often be described by probability and thus understanding the theory of SDE's requires the familiarity of advanced probability theory and stochastic processes.

SDE's have found applications in chemistry, physical and engineering sciences, microelectronics and economics. But recently, there has been an increase in the use of SDE's in other areas like social sciences, computational biology and finance. In modern financial practice, asset prices are modelled by means of stochastic processes. Thus, continuous-time stochastic calculus plays a central role in financial modelling.

The theory and application of interest rate modelling is one of the most important areas of modern finance. For example, SDE's are used to price bonds and to explain the term structure of interest rates. Commonly used models include the Cox-Ingersoll-Ross model; the Hull-White model; and Heath-Jarrow-Morton model.

Since there has been an expansion in the range and volume of interest rate related products being traded in the international financial markets in the past decade, it has become important for investment banks, other financial institutions, government and corporate treasury offices to require ever more accurate, objective and scientific forms for the pricing, hedging and general risk management of the resulting positions.

Similar to ordinary differential equations, many SDE's that appear in practical applications cannot be solved explicitly and therefore require the use of numerical methods. For example, to price an American put option, one requires the numerical solution of a free-boundary partial differential equation.

There are various approaches to solving SDE's numerically. Monte Carlo methods could be used whereby the physical system is simulated directly using a sequence of random numbers. Another method involves the discretisation of both the time and space variables. However, the most efficient and widely applicable approach to solving SDE's involves the discretisation of the time variable only and thus generating approximate values of the sample paths at the discretisation times.

This paper highlights some of the various numerical methods that can be used to solve stochastic differential equations. These numerical methods are based on the simulation of sample paths of time discrete approximations. It also highlights how these methods can be derived from the Taylor expansion of the SDE, thus providing opportunities to derive more advanced numerical schemes.

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BASIC NOTATION

t	time
T	maturity date
W_t	Wiener process
S_t	stochastic process
X_t	stochastic process
Var (X)	variance of the random variable *X*
E(X)	the expectation of the random variable *X*
$\sim N(\mu, t)$	Normal distributed with expectation μ and variance *t*
$\sim$	with distribution
$\mathcal{F}_t$	filtration
Δt	small increment in time *t*
Δn	small increment in *n*
$\mathcal{L}^2$	set of square-integrable functions
Ω	sample space
[a,b]	closed interval $\{x \varepsilon \mathcal{R} : a \leq x \leq b\}$
$\mathcal{R}$	set of real numbers
Cov (X)	covariance of the random variable *X*
$a \varepsilon A$	a is an element of the set A
$\lvert x \rvert$	the Euclidean norm of a vector $x \varepsilon \mathcal{R}^d$
$f : R \to R$	a function f from R to R
a. s.	almost surely
w. p. 1	with probability 1
P	probability

CHAPTER I - INTRODUCTION

Differential equations are used to explain and model the behaviour of systems over time. They have been applied to many areas from chemistry and physics to engineering to medicine and finance. For example, the growth of a population can be modelled over short periods of time by assuming that the population grows continuously with time at a rate proportional to the number of individuals present at that time:

$$\frac{dN(t)}{dt} = \lambda N(t),$$

where $N(t)$ is the number of individuals at time t and λ is the constant birth rate of the population. The motion of a swinging pendulum can be described by the following second-order differential equation (Burden and Faires, 1997):

$$\frac{\partial^2 \theta}{\partial t^2} - \frac{g}{L}\sin\theta = 0,$$

where L is the length of the pendulum, g is the gravitational constant of the earth and θ is the angle the pendulum makes with the vertical equilibrium position. In the theory of the spread of contagious diseases, the following elementary differential equation can be used to predict the number of infected individuals in a population at any time:

$$\frac{dy(t)}{dt} = k\, x(t)\, y(t),$$

where $x(t)$ is the number of susceptible individuals at time t and $y(t)$ is the number of infected individuals (Burden and Faires, 1997).

Many ordinary differential equations can be solved analytically, and when this is not possible, numerical methods are used. The use of numerical methods has increased in the recent years with the increase in computing power of personal computers.

Ordinary differential equations have proved useful in explaining the behaviour of systems over time. However, when there are uncertainties and when randomness enters the system, the differential equations need to be adapted in order to model the random fluctuations. Stochastic differential equations (SDE's) arise in a natural manner in the description of these systems which are influenced by randomness. The behaviour of the physical system can be described by probability and thus understanding the theory of SDE's requires the familiarity of advanced probability theory and stochastic processes.

Traditionally, SDE's have found applications in chemistry, physical and engineering sciences, microelectronics and economics. But recently, there has been an increase in the use of SDE's in other areas like social sciences, computational biology and finance. Cobb (1998) provides some examples where stochastic differential equations are applied to social sciences. In Brown, *et al*, (2006), stochastic differential equations are used to model response times for simple decision making. Saarinen, *et al*, (2006) models the intrinsic dynamic behaviour of neurons using stochastic differential equations. Manninen, *et al,* (2006) use stochastic differential equations to model the behaviour of neuronal signal transduction networks. Carletti (2006) considers the application of SDE's in biosciences where the Euler – Maruyama and Milstein methods are used to determine the solution of the SDE's that model virus - bacteria interaction.

SDE's are also prominent in finance where they are used to price bonds and to explain the term structure of interest rates. The short-term interest rate can be modelled by a stochastic differential equation, suggested by Cox, *et al*, (1985), which is known as the Cox-Ingersoll-Ross model. The Hull-White model, which is

an example of an Ornstein-Uhlenbeck process, is used to model future interest rates using the current term structure of interest rates. The model was proposed by Hull and White (1987) as a solution to the pricing of a European call option on an asset that has stochastic volatility. Another model by Heath, *et al,* (1992) shows that under no-arbitrage conditions, the instantaneous forward rate can be expressed by a stochastic differential equation. This model is commonly known as the Heath-Jarrow-Morton model.

It has generally been accepted that modelling asset price movements is based on the idea that asset prices can be represented by stochastic processes (Hughston, 1996). The standard continuous dynamic model for an asset price is obtained by assuming that the stochastic process S_t for the price of the asset at time *t* is an Itô process, characterised by the stochastic equation:

$$\frac{dS_t}{S_t} = \mu_t\, dt + \sigma_t\, dW_t\,,$$

which can be written in differential form as:

$$dS_t = \mu_t\, S_t\, dt + \sigma_t\, S_t\, dW_t\,.$$

This model says that the infinitesimal price movement dS_t at time *t*, which is expressed as a percentage of the price, is given by the sum of a drift $\mu_t\, dt$ and a rapidly fluctuating term $\sigma_t\, dW_t$, where W_t is a Wiener process (Hughston, 1996).

There is a well-developed theory surrounding asset pricing and numerous models have been suggested. Models range from the famous Black-Scholes model, binomial and trinomial models, and dynamic arbitrage to the Heath-Jarrow-Morton family of models. The continuous time models require the use of stochastic processes and most of them involve stochastic differential equations.

The theory and application of interest rate modelling, which forms part of the general theory of dynamic asset pricing, is one of the most important areas of modern finance. In the past decade, there has been an expansion in the range and volume of interest rate related products being traded in the international financial markets. According to the International Swaps and Derivatives Association, transactions outstanding in interest rate swaps, currency swaps and interest rate options at the close of 1995 stood at $17.713 trillion in notional principal, while this figure stood at $183.6 trillion at the end of 2004 (International Swaps and Derivatives Association).

Thus it is important for investment banks, other financial institutions, government and corporate treasury offices to require ever more accurate, objective and scientific forms for the pricing, hedging and general risk management of the resulting positions. The short term interest rate is important in determining companies' exposure to market movements. Thus, models (like the Hull-White and Heath-Jarrow-Morton models) would be used to explain the evolution of interest rates and to provide forward estimates of interest rate movements. However, since many of these models that arise in practice cannot be solved analytically, numerical methods are employed.

During the past twenty years, there has been an accelerating interest in developing numerical methods for stochastic differential equations, especially in engineering and physical sciences (Burrage and Burrage, 1996). This has been supported by continuous improvements in computing capability and the equivalent decrease in costs of personal computers.

In the light of the volume of interest rate related derivatives trade worldwide, there is a need to highlight and to understand the available numerical methods that could be used to solve the stochastic differential equations, thus providing a more accurate and efficient way for the pricing and hedging of derivatives

products. Further, these numerical methods aide in bridging the gap between the well advanced theory of SDE's and its application to specific examples.

There are various methods that have been proposed to solve SDE's numerically. Monte Carlo methods can be used to simulate the behaviour of the system. Under this method, the physical process is simulated directly using a sequence of random numbers and there is no need to specify the differential equation that describes the behaviour of the system. The physical system is described by probability density functions and then the Monte Carlo simulation can begin by random sampling from the probability density function (Casella and Robert, 2005). Many simulations are performed and the desired result is taken as an average over the number of observations.

However, Kloeden & Platen (1992) claim that this method is to some extent inefficient because it does not use the special structure of the drift and diffusion coefficients. Another method to solve SDE's is to make use of the discretisation of both time and space variables, so that the solution is approximated as finite state Markov chains. This method is plausible for simple problems, but for high dimension problems, this method can involve a considerable amount of computing time because the transition matrices contain a lot of unnecessary information which must be repeatedly reprocessed during computations.

Another method involves the finite discretisation of the time interval [0, T] only and not the state variable. This time discrete approximation can be used to generate approximate values of the sample paths at each step of the discretisation times. The simulated sample paths can then be analysed using statistical methods to determine how good the approximation is to the exact solution. This method is efficient and can be easily implemented on a digital computer. Consequently, it has been used widely and preferred to other methods because it has lower computational costs.

Having realised the importance and the recent increased use of SDE's, the main aim of this thesis is to present a brief analysis of the various numerical methods that have been developed for solving SDE's, focusing on strong and weak schemes. In addition, this thesis shows how these numerical schemes can be derived from Taylor expansions of the stochastic differential equations, thus providing opportunities for the derivation of more advanced numerical schemes and the application of existing schemes by other researchers that do not have a solid background in modern probability theory.

Analogous to deterministic ordinary differential equations where the Taylor expansion is used to derive various numerical methods, the Itô-Taylor expansion for stochastic differential equations is used to derive various numerical methods. The Euler and Milstein schemes provide a good starting point to introduce numerical methods for SDE's. This is then extended to higher order Taylor schemes. These schemes involve simulating the derivatives of the coefficients of the drift and diffusion terms. Further schemes are presented which replace the derivatives with finite differences. Implicit schemes are then presented, which takes into account previous simulated values and thus involves less computing. These schemes are presented for both the strong and weak convergence criteria.

The numerical methods are based on time discrete approximations. Time discrete approximations for both the strong and weak convergence criteria will be presented. Whereas time discrete approximations which satisfy the strong convergence criterion involves the simulation of sample paths at each step of the discretisation time, approximations that satisfy the weak convergence criterion involve the approximation of some function of the Itô process such as the first and second moments at a given final time T. Further, the thesis also contrasts the different numerical schemes by providing some analytical results of the scheme and comparing it with the known solution. This is done by using *Matlab* software. The effect of varying the step size is also considered.

CHAPTER II – LITERATURE REVIEW

The understanding of stochastic differential equations (SDE's) forms part of stochastic calculus. A clear understanding thus requires the knowledge of modern probability theory.

The theory of stochastic differential equations is well established. It was originally developed by mathematicians as a tool for explicit construction of paths of diffusion processes for given coefficients of drift and diffusion. According to Arnold (1974: xi) SDE's "were first treated in 1908 by Langevin in the study of Brownian motion of a particle in fluid". Itô (1942) first introduced stochastic integrals "to formulate the stochastic differential equation that determines Kolmogorov's diffusion process" (Mao, 1991: 1). Since then the theory of stochastic differential equations within the theory of stochastic processes has expanded.

There are several books on stochastic processes and stochastic differential equations. Gihman and Skorohod (1972) provide one of the first extensive accounts of the calculus of random differential equations defined in terms of the Wiener process. Relevant results from probability theory and stochastic processes are covered. The first part of the book presents the theory of one-dimensional stochastic equations whose solutions are Markov diffusion processes while the second part provides a general definition of a stochastic differential equation based on the idea of a line integral along a random curve.

While these sources provide a general theory of stochastic differential equations, they are generally inaccessible to persons intending to apply them. Arnold (1974) provides a moderately advanced level of the subject, covering areas from the fundamentals of probability theory to Markov and diffusion processes. Examples of both stochastic integrals and stochastic differential equations are also given while at the same time providing the solution of the SDE as Markov and diffusion

processes. Arnold (1974) goes further by considering the stability of stochastic dynamic systems as well as optimal control of dynamic systems.

A more theoretical approach to stochastic processes is provided in Gihman and Skorohod (1979). The book covers the theory of martingales, stochastic integrals, stochastic differential equations, diffusion and continuous Markov processes. The understanding of this book requires knowledge of advanced probability theory and as such is not suited for those looking to apply SDE's in their research, especially for researchers in non-traditional fields like computational biology and social sciences; areas in which there is an increase in the use of SDE's.

Unlike Arnold (1974) who provides some applications, Schuss (1980) presents the theory but, more importantly, the book provides a wide range of applications of stochastic differential equations of the Itô type. The material is also presented so that applied mathematicians, physicists and engineers may be able to use it, without the necessary deeper understanding of modern probability and measure theory. The book thus bridges the gap between the mathematical theory and wide range of areas in which stochastic differential equations may arise; areas from statistical mechanics and transport theory to mathematical genetics.

Understanding stochastic differential equations require the understanding of modern probability theory and stochastic calculus. The book by Karatzas and Shreve (1988) explores stochastic processes in their continuous time context and is thus suitable for readers who are acquainted with the Markov and martingale properties in discrete time. Although, measure-theoretic probability is kept at a minimum, the book covers areas from martingales and stopping times to the construction of Brownian motion. Stochastic integration and stochastic differential equations are presented, and the book also provides theoretical underpinnings of strong and weak solutions of stochastic differential equations, which are utilised in deriving numerical methods for SDE's.

These early publications provide a concise theoretical framework for the study of stochastic differential equations. The problem of finding solutions of SDE's, whether numerically or analytically, is kept at a minimum since numerical analysis is a branch of mathematics on its own and in order to implement and derive numerical methods, the theory has to be well developed and understood. These publications thus focused on introducing the theory as well as trying to close the gap between the well-developed theory and the various applications of SDE's.

Early attempts were made in the area of numerical methods for stochastic differential equations. Milstein (1974) provides an early account for constructing a numerical method for solving stochastic differential equations. This method is known as the Milstein method. Hovanessian and Chang (1977) provide an application of the central difference and predictor methods for finding a solution of differential equations with stochastic inputs.

Numerical methods for SDE's can be constructed by translating a deterministic numerical method (like the Euler method or Runge-Kutta method) and applying it to a stochastic ordinary differential equation. However, merely translating a deterministic numerical method and applying it to an SDE will generally not provide accurate methods (Burrage and Burrage, 1996). Suitably appropriate numerical methods for SDE's should take into account a detailed analysis of the order of convergence as well as stability of the numerical scheme and the behaviour of the errors. The Euler-Maruyama method for SDE's is the simplest method which is a direct translation of the deterministic Euler method, but according to Burrage and Burrage (1996), this method is not very accurate. However, this method is useful in that it provides a starting point for more advanced numerical methods for SDE's.

A very concise publication by Kloeden and Platen (1992) provides a comprehensive and systematic presentation of numerical methods available for SDE's. The book focuses on time discretisation methods for initial value problems of SDE's with Itô diffusion as their solutions. Numerical methods for both the strong and weak order of convergence are presented.

The preliminary part covers introductory areas of probability and stochastic processes that are required for understanding the remaining parts of the book. It is thus recommended for those without much theoretical knowledge in measure and probability theory but nevertheless would like to apply SDE's in their research areas. This book can also be used by those with advanced mathematical background and who are interested in the theoretical developments and underlying mathematical issues. For a more theoretical treatment of stochastic differential equations, the reader is referred, in addition to those mentioned earlier, to the monograph by Ikeda and Watanabe (1981).

While the book by Kloeden and Platen (1992) provides the numerical schemes, the complementary book by Kloeden, *et al*, (1994) presents the computations associated with these numerical methods. The computation of the moments or sample paths of a given SDE is important for the effective practical application of SDE's and thus the book also focuses on the algorithms associated with these computations. Both of these books (by Kloeden, *et al*, (1994); and Kloeden and Platen (1992)) are highly recommended for those with limited technical background in mathematics but nevertheless are interested in modelling and applying standard numerical methods in their research areas. A familiarity with basic programming skills is essential in order to understand the material and to effectively carry out the practical exercises.

Since the publication of these books, others have also focused on the numerical schemes while emphasising algorithms based on typical software. This is mainly due to the advances in computer technology and decrease in costs of computing

power. Cyganowski, *et al,* (2002) provide basic results of stochastic differential equations and modern probability theory with the help of the computer software package Maple®, although SDE's are not covered in any great detail. This book is not a conventional mathematics book and it is intended to provide an intuitive background for those not necessarily in mathematics but who are nevertheless interested in modern probability theory. The mathematical proofs are omitted but the book uses Maple® to help the reader understand intuitively the ideas under discussion.

There has been an increase in the number of literary articles written, with most at an advanced level. Apart from the theory (which has been adequately developed and presented in the publications listed above) many of these articles focus on advanced topics related to numerical methods for SDE's. Others take a step forward to derive new numerical schemes and to adapt existing ones for specific purposes, problems that arise frequently in the physical and engineering sciences.

Higham (2001) provides an accessible introduction to numerical methods for stochastic differential equations. Topics covered include the Euler-Maruyama method, Milstein method and Monte Carlo methods and Matlab is used to simulate numerical solutions of the SDE's. Mean-square stability and asymptotic stability, as well as strong and weak convergence, are also discussed from a practical viewpoint. This paper is suitable for readers with little or no knowledge of advanced probability theory or stochastic processes.

Naess (2001) focuses on path integration methods for calculating the probability law of the solution of stochastic differential equations. The article focuses on calculating the joint probability density function of the phase space Markov vector process that solves the nonlinear SDE. Mannella (2002) concentrates on the simulations of stochastic processes on a computer, but focuses on algorithms to simulate rare fluctuations because this is a topic of great interest in the study of

optimal paths. Problems connected to the treatment of boundaries and correlated noise are also discussed. “Rare fluctuations are fluctuations which bring the stochastic system very far from the phase space which the system explores most of the time” Mannella (2002: 1). The algorithms are amended to simulate rare fluctuations when the simulation hits a prescribed boundary in the phase space.

This thesis uses the strong and weak convergence criteria to assess the efficiency of the numerical scheme. However, Mannella (2002) also studies the long-time dynamics of the SDE, unlike the convergence schemes which use short-time dynamics. This is motivated by the fact that good behaviour of the system at short times does not imply any good behaviour at long times. While this thesis studies SDE’s driven by white noise, Mannella (2002) provides an introduction to SDE’s that are driven by non-white noise. The simplest non-white noise is the exponentially correlated white noise. This is important as it provides insight and opportunities to apply numerical methods to SDE’s that are not driven by white noise for some real systems.

Most of the literature on SDE’s focus on the engineering and physical science systems, but there is an increase in the application of SDE’s to other fields. Cobb (1998) provides examples of the use of SDE’s in the social sciences. Carletti (2003) applies Runge-Kutta type methods for stochastic ordinary differential equations and the Euler-Maruyama method for stochastic delay differential equations arising in biosciences. Frutos (2005) applies implicit-explicit Runge-Kutta methods as an alternative method for pricing financial derivatives, especially to value American type contracts. The methods provided in these articles assume little knowledge of advanced probability theory and thus the article is practical in that these powerful methods can be applied to support theories in other areas.

Recently, there has been an increase in numerical methods used to solve SDE’s as the current existing methods are being expanded. This increase is partly due

to the increase in computing powers and therefore much more complex numerical schemes can be solved for many applications. High strong order explicit Runge-Kutta methods for stochastic ordinary differential equations are presented in Burrage and Burrage (1996) which are a more efficient class of explicit Runge-Kutta methods than the current existing methods.

The Runge-Kutta methods based on the increments of the Wiener process have, at most, strong order 1.5 convergence. In order to increase the order of convergence, Burrage and Burrage (1996) construct a new class of Runge-Kutta methods by adding multiple stochastic integral terms from the stochastic Taylor series. An explicit four-stage method of strong order 2.0 convergence is then constructed.

While most of the methods considered focus on the strong convergence class, Platen (1995) provides a short survey on weak schemes for SDE's and discuss several implicit and predictor-corrector type methods. Weak schemes are important if the analysis is focussed on evaluating some functional of the Itô stochastic differential equation. The advantage of using implicit and predictor corrector methods is that these schemes overcome most of the numerical stability problems that occur, especially in systems where there are extremely different time scales involved (i.e. stiff systems) (Platen, 1995). The explicit numerical methods are not able to control the propagation of errors in stiff systems. Thus, implicit methods are used. Therefore, weak approximations are employed as these provide more freedom in constructing implicit schemes. Stability of the stochastic numerical scheme depends more on the "approximations own dynamical behaviour which characterizes its ability to control the propagation of errors" (Platen, 1995: 69).

When using a particular numerical scheme, not only is the nature and order of convergence of numerical schemes important, but the stability of a numerical scheme is vital. Stability of a numerical scheme is essential to avoid the possible

explosion of the numerical solution. Abukhaled (2004) discusses mean square stability of second-order weak numerical methods. The closed form of the second moment is used to create a mean square stability criterion for the second-order weak numerical schemes. Tocino (2005) studies the mean square stability of the second-order two-stage explicit Runge-Kutta methods which have been proposed by Tocino and Vigo-Aguiar (2002). The similarity between Abukhaled (2004) and Tocino (2005) is that both papers focus on stability of weak schemes with respect to the second moment. Tocino (2005) goes further by studying stability of linear SDE's with multiplicative noise.

Tocino and Ardanuy (2002) develop a class of explicit Runge-Kutta schemes of second order in the weak sense for systems of stochastic differential equations with multiplicative noise. Two Runge-Kutta schemes of the third order have also been developed for scalar SDE's with constant diffusion terms.

In all of the numerical schemes considered, a constant step size was used. Lehn, *et al*, (2002) reviews adaptive schemes which use variable step sizes in the numerical scheme. A re-examination of the two main step size control algorithms is conducted and their efficiency is compared in a simulation study. Adaptive schemes may be one possible way to reduce computational costs. These scheme were proposed as an alternative to the higher order schemes that require simulating correlated multiple Itô-integrals of the stochastic Taylor expansion and evaluating functions (for each time step) which can be a very difficult and time-consuming task. In higher order schemes, multiple stochastic integrals have to be approximated. This becomes difficult because the multiple stochastic integrals do not depend continuously on the trajectories of the Wiener process (Lehn, *et al*, 2002).

The SDE's that are considered in this thesis have local Lipschtiz coefficients and the numerical schemes were based on this assumption. Zhang (2006) proves that the Euler-Maruyama approximation for SDE's for non-Lipschitz coefficients

converges uniformly to the solution in the L^p – space with respect to the time and starting points.

Convergence of schemes for stochastic differential equations is shown in Fleury (2006). Almost sure convergence is established for explicit and implicit Euler schemes, explicit Milstein schemes, stochastic Newmark scheme and the implicit Itô-Milstein scheme. The family of stopping times is used to ensure the almost sure convergence of the schemes.

Burrage and Tian (2001) present a composite Euler method for the strong solution of stochastic differential equations driven by *d*-dimensional Wiener processes. This method is a combination of the semi-implicit Euler method and the implicit Euler methods and is used to obtain improved stability properties than the Euler methods.

Tian and Burrage (2001) introduce implicit Taylor methods for stiff Itô stochastic differential equations. These implicit Taylor methods are based on the relationship between Itô stochastic integrals and backward stochastic integrals. The following methods are considered: the implicit Euler – Taylor method with strong order 0.5; the implicit Milstein – Taylor method with strong order 1.0; and the implicit Taylor method with strong order 1.5. The authors claim that the stability properties of the implicit Euler – Taylor and implicit Milstein – Taylor are much better than those of the corresponding semi – implicit Euler and Milstein methods. The implicit Euler – Taylor and implicit Milstein – Taylor methods can be used to solve stochastic differential equations which are stiff in both the deterministic and stochastic components. According to the authors, the numerical results for the convergence and stability properties suggest that these implicit methods are very promising for stiff stochastic differential equations.

The basic theory of SDE's and numerical techniques for solving them is well developed. This provides the foundation for further research and analysis.

According to Higham (2001), current research is being pursued in a number of directions. There is the construction of methods with high order of strong or weak convergence or improved stability; the design of variable time step algorithms; and the analysis of long-term properties such as ergodicity for nonlinear problems.

Thus the area of stochastic differential equations, its solutions and wide range of applications (from biology to engineering, and from chemistry to finance) makes it an interesting area for research, as it encompasses many fields of mathematics; from stochastic analysis to applied mathematics to numerical analysis. Researchers would thus require the indispensable tools of modern probability theory in order to understand this rich and diverse branch of mathematics.

This chapter highlights that the study SDE's requires a thorough understanding of modern probability theory. Numerous authors and texts on modern probability theory and stochastic calculus have been mentioned. The excellent text by Kloeden and Platen (1992) provides an excellent introduction to numerical methods for solving SDE's. This provides a background for understanding and expanding these numerical methods. Higher order schemes that have been developed and reviewed recently are highlighted. Recent advances in stability issues and convergence of numerical schemes have also been discussed. The following chapter provides some preliminary concepts starting with motivating the use of an SDE.

CHAPTER III – MATHEMATICAL PRELIMINARY

3.1 Wiener process and white noise

A standard Wiener process $W = \{W(t), t \geq 0\}$ is defined as a Gaussian process with independent increments such that:

$$W(0) = 0, \qquad \text{with probability 1}$$
$$E(W(t)) = 0, \qquad \text{and}$$
$$Var(W(t) - W(s)) = t - s,$$

for all $0 \leq s \leq t$. This process was proposed by Norbert Wiener as a mathematical description of Brownian motion; which describes the erratic motion of a grain of pollen on a water surface due to its being continually bombarded by water molecules.

Thus, $W(t) - W(s) \sim N(0, t-s)$ for $0 \leq s < t$ and the increments $W(t_2) - W(t_1)$ and $W(t_4) - W(t_3)$ are independent for all $0 \leq t_1 < t_2 \leq t_3 < t_4$.

The Wiener process has sample paths that are almost surely continuous functions of time, but the process W_t is nowhere differentiable (Dana and Jeanblanc, 2003).

3.2 Appearance of the stochastic differential equation

An ordinary differential equation

$$\frac{dx}{dt} = a(t, x)$$

may be written in symbolic differential form as:

$$dx = a\,(t,x)\,dt\,.$$

The above equation can also be written in integral form as:

$$x(t) = x_0 + \int_{t_0}^{t} a(s, x(s))\,ds\,,$$

where $x(t) = x(t; x_0, t_0)$ is a solution satisfying the initial condition $x(t_0) = x_0$.

Ordinary differential equations are used to model the time varying behaviour of systems. Consider the following equation that models population growth[1]:

$$\frac{dx}{dt} = ax(t)\,,$$

where $x(t)$ is the earth's population at time t and a is the constant growth rate. This model however shows that the growth is unlimited. The model was revised later to eliminate the effect of unlimited growth[2]. The model suggests that as the population gets high, there is a tendency for individuals to fight over scarce resources. These effects are incorporated in the model as:

$$\frac{dx}{dt} = ax - mx^2\,. \qquad (3.1)$$

This model assumes a constant growth rate a. However, when the system undergoes disturbances, stochastic noise is introduced in the model. The growth

[1] This model was first suggested by Thomas R Malthus in 1798 to model the population growth rate

[2] The model was revised by Pierre-Francois Verhulst in 1838.

rate a need not be constant and in many applications, it is assumed that the growth rate varies. Assume that the growth rate a varies randomly as $a = \lambda + \sigma \xi_t$, where λ is some constant rate and $\sigma \xi_t$ is a rapidly varying component and ξ_t a stochastic process with zero mean. Substitute $a = \lambda + \sigma \xi_t$ in equation (3.1) to obtain:

$$\frac{dx}{dt} = (\lambda + \sigma \xi_t) x - mx^2$$

$$\Rightarrow \quad \frac{dx}{dt} = \left(\lambda x - mx^2\right) + \sigma x \xi_t . \tag{3.2}$$

Equation (3.2) can be written in symbolic differential form as:

$$dx = \left(\lambda x - mx^2\right) dt + \sigma x \xi_t dt$$

or

$$dx = a\, dt + \sigma x \xi_t dt , \tag{3.3}$$

where $a = \left(\lambda x - mx^2\right)$.

Equation (3.3) thus provides a more realistic model of the state of the population at time t. Therefore, randomness is modelled as some external sources interacting with the system. Equation (3.3) can be interpreted as the integral equation:

$$x(t) = x_0 + \int_{t_0}^{t} a\left(s, x(s)\right) ds + \int_{t_0}^{t} b\left(s, x(s)\right) \xi_s \, ds , \tag{3.4}$$

where $b = \sigma x$.

Equation (3.4) is written symbolically as a stochastic differential equation of the form:

$$dX_t = a\left(t, X_t\right)dt + b\left(t, X_t\right)\xi_t dt\,. \tag{3.5}$$

Here, $a\left(t, X_t\right)dt$ represents the deterministic or average drift term while $b\left(t, X_t\right)\xi_t dt$ is a rapidly varying continuous random component called the diffusion. The term ξ_t is a standard Gaussian random variable for each t and $b\left(t, X_t\right)$ is a space-time dependent intensity factor. For the special case when $a = 0$ and $b = 1$, notice that ξ_t should be the derivative of a Wiener process;

$$\begin{aligned} & d\,X_t = \xi_t\, dt \\ \Rightarrow \quad & d\,X_t = dW_t\,. \end{aligned}$$

Thus the SDE in equation (3.5) could be written as:

$$dX_t = a\left(t, X_t\right)dt + b\left(t, X_t\right)dW_t\,,$$

or in integral form as:

$$X_t\left(\omega\right) = X_{t_0}\left(\omega\right) + \int_{t_0}^{t} a\left(s, X_s\left(\omega\right)\right)ds + \int_{t_0}^{t} b\left(s, X_s\left(\omega\right)\right)dW_s\left(\omega\right), \tag{3.6}$$

where W_t is a Wiener process.

The second integral of equation (3.6) cannot be interpreted as the Riemann (or Lebesgue) integral since ξ_t in equation 3.5 is a stochastic process and is not known. Therefore equation (3.6) cannot be solved in the usual sense. This is due to the fact that the Wiener process W_t is nowhere differentiable and the stochastic process ξ_t does not exist as a conventional function of t. Further, the continuous sample paths of a Wiener process do not have bounded variation on any bounded time interval (Kloeden and Platen, 1992).

3.3 Itô Integral

As mentioned, the Wiener process is nowhere differentiable so the white noise process does not exist as a conventional function of t**.** Thus the second integral of equation (3.6) cannot be an ordinary (Riemann or Lebesgue) integral. Further, the continuous sample paths of a Wiener process are not of bounded variation on any bounded time interval, so the second integral cannot be interpreted as a Riemann-Stieltjes integral either (Kloeden and Platen, 1992).

Itô (1951) overcame this problem by defining the integral by using mean square convergence. Consider the second integral in equation (3.6) above and let $b(s, x(s)) = b$ be a constant:

$$I(b) = \int_{t_0}^{t} b \; dW_s \; .$$

One would expect that

$$I(b) = b\,\{W_t(\omega) - W(t_0)\},$$

no matter how it is defined. Consider an integral of a random function f over the time interval $t = [0,1]$:

$$I(f)(\varpi) = \int_{t_0}^{t} f(s,\omega)\, dW_s(\omega). \qquad (3.7)$$

The integral is defined in two steps. Firstly, consider the case when the process f is simple and secondly, when the process f is generalised. In order to proceed with defining the Itô integral, the following ancillary definitions are required.

Definition

1. A process *X* is **adapted** to the filtration $\{\mathcal{F}_t : t \geq 0\}$ if $\{X_t \in \mathcal{F}_t,\ \forall t \geq 0\}$, where $\{\mathcal{F}_t : t \geq 0\}$ is an increasing family of σ – algebras.

2. A process $f \in \mathcal{L}^2$ if:

   - The process f is adapted, and
   - The process f satisfies

$$\int_a^b E\left[f^2(s)\right] ds \ < \ \infty .$$

3. The process f is simple if:

   - $f \in \mathcal{L}^2$, and
   - There exists fixed points $t_1,...,t_{n+1}$ with $0 = t_1 < t_2 < ... < t_{n+1} = 1$, such that f is piecewise constant on any $t_j \leq t < t_{j+1}$, i.e.

$$f(t) = f_j, \qquad t_j \ \leq \ t \ < \ t_{j+1} .$$

Return to the definition of the Itô – integral.

i. The process f is simple

The interval [0, 1] is sub-divided into n sub-intervals, with $0 = t_1 < t_2 < ... < t_{n+1} = 1$. In order to define this integral, Itô commenced by first defining f as a **non-random** step function (piecewise constant) $f(t,\omega) = f_j$ on the interval $t_j \le t < t_{j+1}$ for $j = 1, 2, ...n$. Thus, equation (3.7) becomes:

$$I(f)(\omega) = \sum_{j=1}^{n} f_j \left\{ W_{t_{j+1}}(\omega) - W_{t_j}(\omega) \right\}. \qquad (3.8)$$

Since equation (3.8) is a sum of random variables with zero mean, it implies that $I(f)(\omega)$ is a random variable with zero mean. Thus, equation (3.8) defines the stochastic integral for the case when f is simple.

ii. The process f is generalised

Suppose that $\{\mathcal{F}_t : t \ge 0\}$ is an increasing family of σ – algebras such that W_t is $\mathcal{F}_{t_j}$ – measurable for each $t \ge 0$. The integral in equation (3.7) is then extended to consider random step functions $f(t,\omega) = f_j(\omega)$ on the interval $t_j \le t < t_{j+1}$ for $j = 1, 2, ...n$. Since $f(t,\omega) = f_j(\omega)$ are **random** step functions, it is assumed that each f_j is $\mathcal{F}_{t_j}$ – measurable and that f_j is also mean square integrable over the sample space. This measurability condition for the random step functions ensures the non-anticipativeness of the integrand. It is also assumed that f is continuous in t for all $\omega \in \Omega$.

Consider:

$$E\left(W_{t_{j+1}} - W_{t_j} \middle| \mathcal{F}_{t_j}\right). \tag{3.9}$$

Equation (3.9) is zero with probability one, since we assumed that each W_t is $\mathcal{F}_{t_j}$ - measurable for each $t \geq 0$. Therefore it follows that

$$f_j \left\{W_{t_{j+1}} - W_{t_j}\right\}$$

is $\mathcal{F}_{t_{j+1}}$ - measurable and integrable. Therefore, the expectation is zero for each $j = 1, 2, ...n$. That is:

$$E\left(f_j \left\{W_{t_{j+1}} - W_{t_j}\right\}\right) = E\left(f_j \, E\left(W_{t_{j+1}} - W_{t_j} \middle| \mathcal{F}_{t_j}\right)\right) = 0.$$

Therefore, the integral $I(f)$ is defined as:

$$I(f)(\omega) = \sum_{j=1}^{n} f_j(\omega)\left\{W_{t_{j+1}}(\omega) - W_{t_j}(\omega)\right\} \tag{3.10}$$

with probability one. $I(f)$ is $\mathcal{F}_1$ - measurable because the j^{th} term is $\mathcal{F}_{t_{j+1}}$ - measurable and hence $\mathcal{F}_1$ - measurable. Further, $I(f)$ is integrable over Ω and has zero mean. Equation (3.10) is defined when it was assumed that $f(t, \omega) = f_j(\omega)$ are **random** step functions.

The random step functions are then extended to general functions. Consider the case for a general integrand $f : [0, 1] \times \Omega \to \mathcal{R}$. It is assumed that there are

random step functions $f^{(n)}$ converging to f. The integral $I(f)$ is then defined as the limit of integrals $I(f^{(n)})$ of random step functions $f^{(n)}$ converging to f.

It is assumed that f is continuous in *t* for all $\omega \in \Omega$ and $f(t,.)$ is $\mathcal{F}_t$ - measurable for t ϵ [0, 1]. Form a partition $0 = t_1^{(n)} < t_2^{(n)} < ..... < t_{n+1}^{(n)} = 1$ such that as $n \to \infty$, we have:

$$\delta^{(n)} = \max_{1 \le j \le n} \left\{ t_{j+1}^{(n)} - t_j^{(n)} \right\} \to 0 .$$

Define a step function $f^{(n)}$ by $f^{(n)}(t,\omega) = f(t_j^{(n)}, \omega)$. In order to define the integral for general functions, the appropriate mode of convergence needs to be defined so that the step functions $f^{(n)}$ converge to the integrand f. Thus equation (3.10) becomes:

$$I(f^{(n)})(\omega) = \sum_{j=1}^{n} f^{(n)}(t_j^{(n)}, \omega) \left\{ W_{t_{j+1}^{(n)}}(\omega) - W_{t_j^{(n)}}(\omega) \right\}. \qquad (3.11)$$

The problem is to characterise the limit of the finite sums with respect to an appropriate mode of convergence. To solve this problem, Itô used the mean square convergence. Assume that

$$E\left(\left| f^{(n)}(t,.) - f(t,.) \right|^2 \right) \to 0 \text{ as } n \to \infty \text{ for t ϵ [0, 1].}$$

Then the mean square limit of $I(f^{(n)})$ exists and is unique, with probability one. This mean square limit is denoted by $I(f)$ and is called the Itô stochastic integral.

The Itô integral is defined similarly on any bounded interval $[t_0, t]$ resulting in a random variable

$$X_t(\omega) = \int_{t_0}^{t} f(s,\omega)\, dW_s(\omega),$$

which is $\mathcal{F}_t$ - measurable and mean square integrable with zero mean and

$$E\left(X_t^2\right) = \int_{t_0}^{t} E\left(f(s,.)^2\right) ds\,.$$

Since mean square limits are used, then for any $0 < t_1 < t_2 < \ldots < t_{n+1}$:

$$\begin{aligned}\sum_{j=1}^{n} W_{t_j}\left\{ W_{t_{j+1}} - W_{t_j} \right\} &= \frac{1}{2}W_t^2 - \frac{1}{2}\sum_{j=1}^{n}\left\{ W_{t_{j+1}} - W_{t_j} \right\}^2 \\ &= \frac{1}{2}W_t^2 - \frac{1}{2}t\,,\end{aligned}$$

since $\sum_{j=1}^{n}\left\{ W_{t_{j+1}} - W_{t_j} \right\}^2$, the mean square limit of the sum of squares, is equal to t. Therefore

$$\int_0^t W_s(\omega)\, dW_s(\omega) = \frac{1}{2}W_t^2(\omega) - \frac{1}{2}t\,, \qquad (3.12)$$

which is in contrast to the following from classical calculus for a differentiable function $w(t)$ with $w(0) = 0$:

$$\int_0^t w(s)\, dw(s) = \int_0^{w(t)} d\left(\frac{1}{2}w^2\right) = \frac{1}{2}w^2(t)\,.$$

3.4 Itô formula

For each $t \geq t_0$, define a stochastic process Y_t by:

$$Y_t(\omega) = U\left(t, X_t(\omega)\right),$$

where $U(t,x)$ has continuous second order partial derivatives and X_t is given by:

$$dX_t(\omega) = a(t,\omega)dt + b(t,\omega)dW_t(\omega).$$

For brevity, the above is written as:

$$dx = a\,dt + b\,dW\,. \tag{3.13}$$

Consider:

$$dY_t = U\left(t+dt, X_t + dX_t\right) - U\left(t, X_t\right).$$

The Taylor expansion for U gives:

$$\begin{aligned} dY_t \quad &= U\left(t+dt, X_t + dX_t\right) - U\left(t, X_t\right) \\ &= \left\{ \frac{\partial U}{\partial t}\,dt + \frac{\partial U}{\partial x}\,dx \right\} \\ &\quad + \frac{1}{2}\left\{ \frac{\partial^2 U}{\partial t^2}(dt)^2 + 2\frac{\partial^2 U}{\partial t\,\partial x}\,dt\,dx + \frac{\partial^2 U}{\partial x^2}(dx)^2 \right\} + \ldots \end{aligned} \tag{3.14}$$

Substitute equation (3.13) into equation (3.14) to obtain:

$$dY_t = \left\{ \frac{\partial U}{\partial t}\,dt + \frac{\partial U}{\partial x}\left(a\,dt + b\,dW\right) \right\}$$

$$+\frac{1}{2}\left\{ \frac{\partial^2 U}{\partial t^2}\left(dt\right)^2 + 2\frac{\partial^2 U}{\partial t\,\partial x}dt\left(a\,dt + b\,dW\right) + \frac{\partial^2 U}{\partial x^2}\left(a\,dt + b\,dW\right)^2 \right\}$$

$$= \left\{ \frac{\partial U}{\partial t}\,dt + a\frac{\partial U}{\partial x}dt + b\frac{\partial U}{\partial x}dW \right\}$$

$$+\frac{1}{2}\left\{ \frac{\partial^2 U}{\partial t^2}\left(dt\right)^2 + 2a\frac{\partial^2 U}{\partial t\,\partial x}dt\,dt + 2b\frac{\partial^2 U}{\partial t\,\partial x}dW\,dt \right\}$$

$$+\frac{1}{2}\left\{ \frac{\partial^2 U}{\partial x^2}\left(a^2\left(dt\right)^2 + 2a\,b\,dt\,dW + b^2\left(dW\right)^2\right) \right\}.$$

Taking expectations and noting that (Chang, 2004)

$$E\left(\left(dW\right)^2\right) = dt,$$

$$E\left(dWdt\right) = 0, \text{ and}$$

$$E\left(\left(dt\right)^2\right) = 0,$$

equation (3.14) simplifies to:

$$dY_t = \left\{ \frac{\partial U}{\partial t}\left(t, X_t\right) + a\frac{\partial U}{\partial x}\left(t, X_t\right) + \frac{1}{2}b^2\frac{\partial^2 U}{\partial x^2}\left(t, X_t\right) \right\} dt + b\frac{\partial U}{\partial x}\left(t, X_t\right)dW_t. \quad (3.15)$$

3.5 Stochastic Taylor expansions

As with the determinist Taylor expansion, the stochastic Taylor formula for the expansion of smooth functions of an Itô process is used to construct numerical methods for stochastic differential equations. The stochastic Taylor formula, which is called the Itô – Taylor expansion, is derived by repeatedly applying the Itô formula (3.15). For any twice continuously differentiable function $f : R \to R$, apply the Itô formula to obtain:

$$f(X_t) = f(X_{t_0}) + \int_{t_0}^{t} \left(a(X_s) f'(X_s) + \frac{1}{2} b^2(X_s) f''(X_s) \right) ds + \int_{t_0}^{t} b(X_s) f'(X_s) dW_s .$$

Introduce the following operators,

$$L^0 f = a f' + \frac{1}{2} b^2 f'' ,$$

$$L^1 f = b f' , \tag{3.16}$$

to obtain:

$$f(X_t) = f(X_{t_0}) + \int_{t_0}^{t} L^0 f(X_s) ds + \int_{t_0}^{t} L^1 f(X_s) dW_s , \tag{3.17}$$

for any $t \in [t_0, T]$. If $f(x) = x$, then $L^0 f = a$ and $L^1 f = b$. Thus the above is just the original Itô equation for X_t:

$$X_t = X_{t_0} + \int_{t_0}^{t} a(X_s) ds + \int_{t_0}^{t} b(X_s) dW_s . \tag{3.18}$$

If formula (3.17) is again applied to the functions $f = a$ and $f = b$ in equation (3.18), the following is obtained:

$$X_t = X_{t_0} + \int_{t_0}^{t} \left(a\left(X_{t_0}\right) + \int_{t_0}^{s} L^0 a\left(X_z\right) dz + \int_{t_0}^{s} L^1 a\left(X_z\right) dW_z \right) ds$$

$$+ \int_{t_0}^{t} \left(b\left(X_{t_0}\right) + \int_{t_0}^{s} L^0 b\left(X_z\right) dz + \int_{t_0}^{s} L^1 b\left(X_z\right) dW_z \right) dW_s$$

$$= X_{t_0} + a\left(X_{t_0}\right) \int_{t_0}^{t} ds + b\left(X_{t_0}\right) \int_{t_0}^{t} dW_s + R\,, \qquad (3.19)$$

where:

$$R = \int_{t_0}^{t} \int_{t_0}^{s} L^0 a\left(X_z\right) dz\, ds + \int_{t_0}^{t} \int_{t_0}^{s} L^1 a\left(X_z\right) dW_z\, ds$$

$$+ \int_{t_0}^{t} \int_{t_0}^{s} L^0 b\left(X_z\right) dz\, dW_s + \int_{t_0}^{t} \int_{t_0}^{s} L^1 b\left(X_z\right) dW_z\, dW_s\,.$$

Repeat this procedure by applying the formula (3.17) to $f = L^1 b$ in equation (3.19) to obtain:

$$X_t = X_{t_0} + a\left(X_{t_0}\right) \int_{t_0}^{t} ds + b\left(X_{t_0}\right) \int_{t_0}^{t} dW_s$$

$$+ L^1 b\left(X_{t_0}\right) \int_{t_0}^{t} \int_{t_0}^{s} dW_z\, dW_s + \overline{R}\,, \qquad (3.20)$$

where:

$$\overline{R} \quad = \int_{t_0}^{t} \int_{t_0}^{s} L^0 a\left(X_z\right) dz\, ds + \int_{t_0}^{t} \int_{t_0}^{s} L^1 a\left(X_z\right) dW_z\, ds$$

$$+ \int_{t_0}^{t} \int_{t_0}^{s} L^0 b\left(X_z\right) dz\, dW_s + \int_{t_0}^{t} \int_{t_0}^{s} \int_{t_0}^{z} L^0 L^1 b\left(X_u\right) du\, dW_z\, dW_s$$

$$+ \int_{t_0}^{t} \int_{t_0}^{s} \int_{t_0}^{z} L^1 L^1 b\left(X_u\right) dW_u\, dW_z\, dW_s \,.$$

The Itô – Taylor expansion can thus be considered as a generalisation of both the Itô formula and the deterministic Taylor formula.

3.6 Convergence Criteria

There are five commonly used concepts for the convergence of random sequences (Kloeden and Platen, 1992). These are:

i. Convergence with probability one:

$$P\left(\left\{\omega \in \Omega : \lim_{n\to\infty} \left| X_n(\omega) - X(\omega) \right| = 0\right\}\right) = 1 .$$

ii. Mean-square convergence:

$$E\left(X_n^2\right) < \infty \,, \quad \text{for n = 1, 2, ….,}$$

$$E\left(X^2\right) < \infty \,, \quad \text{and}$$

$$\lim_{n\to\infty} E\left(\left|X_n - X\right|^2\right) = 0 .$$

iii. Convergence in probability:

$$\lim_{n\to\infty} P\left(\left\{\omega \in \Omega; \left|X_n(\omega) - X(\omega)\right| \geq \varepsilon\right\}\right) = 0,$$

for all $\varepsilon > 0$.

iv. Convergence in distribution:

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x),$$

for all continuity points of F_X.

v. Weak convergence

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f(x)\, d\, F_{X_n}(x) = \int_{-\infty}^{\infty} f(x) d\, F_X(x),$$

for all test functions $f : R \to R$.

In these definitions, all the random variables are defined on a common probability space (Ω, F, P).

Convergence of random sequences is classified into two classes, namely, strong convergence and weak convergence. Convergence with probability one, mean-square convergence and convergence in probability are the most commonly used convergences in the strong class while convergence in distribution and weak convergence are classified from the weak class. For the weak class, only the distribution function is required and not the actual random variables or the underlying probability space.

Since many SDE's cannot be solved explicitly, numerical schemes are employed. There are various numerical schemes (see, Kloeden and Platen, 1992) and in order to access their usefulness and practicality, certain criteria are required in which to access the various schemes. The convergence criterion is just one of many other criteria, like mean square stability, asymptotic stability and cost of computation, which can be used when assessing the usefulness of different numerical schemes.

The numerical schemes for SDE's employed in this paper are recursive in that the trajectories of the solution are computed at discrete time steps. Thus, use is made of the convergence criterion that involves calculating some function of the error at a specific time step, usually at the end of the time interval. Global errors can also be computed. Hofmann and Müller-Gronbach (2004) provides an analysis of $L^2([0,1])$ – error of general numerical methods based on multiple Itô – integrals for pathwise approximation of scalar stochastic differential equations on the interval [0,1].

Fleury (2006) proves almost sure convergence for the explicit Euler, implicit Euler scheme and the Milstein schemes assuming that the coefficients of the equation are only locally Lipschitz and the solution process is unique, continuous and does not explode. Bernard and Fleury (2001) consider convergence in probability for numerical schemes for stochastic differential equations. This thesis uses the strong and weak convergence criteria defined below.

3.6.1 Strong convergence criterion

In many practical areas, like direct simulations, filtering or testing statistical estimators, a good pathwise approximation is usually required and for these instances, the absolute error criterion is appropriate. This criterion gives a measure of the pathwise closeness at the end of the time interval $[0,T]$ (Kloeden and Platen, 1992).

Consider a particular sample path of the Wiener process i.e. W_T is given (and hence known) therefore there is no randomness in the SDE and hence no randomness in X_T (Cao and Pope, 2003). The increments in the given Wiener process are then used to obtain the numerical approximation $Y(T)$. The absolute error criterion is defined as:

$$\varepsilon = E\left(\left| X_T - Y(T) \right| \right).$$

Here, the Euclidean norm is used. X_T is the Itô process at time T while $Y(T)$ is the approximation obtained by approximately integrating the SDE in a sequence of time steps i.e. from the numerical scheme. Therefore, the error is the expectation of the absolute value of the difference between the approximation $Y(T)$ and the Itô process X_T at time T.

The numerical scheme is consistent if the approximation $Y(T)$ converges to X_T as Δt tends to zero. Therefore, a discrete time approximation $Y(T)$ with maximum time step size δ converges strongly to X at time T if (Kloeden and Platen, 1992):

$$\lim_{\delta \to 0} E\left(\left| X_T - Y(T) \right| \right) = 0. \qquad (3.21)$$

There are various discrete time approximations that can be derived from the Itô – Taylor expansion and in order to compare different discrete time approximations, the order of convergence of the numerical scheme is used.

A discrete time approximation Y^{δ} converges strongly with order $\gamma > 0$ at time T if there exists a positive constant C, which does not depend on δ, and a $\delta_0 > 0$ such that:

$$\varepsilon(\delta) = E\left(\left| X_T - Y(T) \right| \right) \leq C\delta^{\gamma},$$

for each $\delta \in (0, \delta_0)$. Thus the numerical scheme *is strong pth order accurate* if the error is of order Δt^{p} (Cao and Pope, 2003).

3.6.2 Weak convergence criterion

In many practical problems, approximating some functional of the Itô process is of interest, such as the probability distribution, its mean and variance. Thus, the weak convergence criterion is used since the requirements for their simulation are not as demanding as for pathwise approximations (Kloeden and Platen, 1992). Here the sample path W_T is not known but is drawn from the distribution of Wiener processes.

Since W_T is a random variable, X_T is a random variable. The numerical approximation $Y(T)$ is also a random variable because $Y(T)$ is obtained using samples of Wiener-process increments. Therefore, convergence of X_T and $Y(T)$ is only considered in distribution. The convergence in distribution is analysed in terms of means $g(X(T))$ of test functions $g(x)$ (Cao and Pope, 2003).

The test functions $g(x)$ are bounded, infinitely differentiable and the means exist as $|x|$ tends to infinity. The numerical scheme is *weak p^{th} order accurate* if the error

$$\varepsilon = \left| \left(g\left(X\left(T\right)\right)\right) - \left(g\left(Y\left(T\right)\right)\right) \right|$$

is of order Δt^p. Thus,

$$\left| \left(g\left(X\left(T\right)\right)\right) - \left(g\left(Y\left(T\right)\right)\right) \right| \leq C\Delta t^p .$$

A general discrete time approximation Y with maximum time step size δ converges weakly to X at time T as $\delta \to 0$ with respect to a class C of test functions $g : \mathbb{R}^d \to \mathbb{R}$ if we have:

$$\lim_{\delta \to 0} \left| E\left(g\left(X_T\right)\right) - E\left(g\left(Y\left(T\right)\right)\right) \right| = 0, \tag{3.22}$$

for g $\in$ C.

A time discrete approximation Y converges weakly with order $\beta > 0$ to X at time T as $\delta \to 0$, if for each polynomial g there exists a positive constant C, which does not depend on δ, and a finite δ_0 such that:

$$\left| E\left(g\left(X_T\right)\right) - E\left(g\left(Y\left(T\right)\right)\right) \right| \leq C\delta^{\beta},$$

for each $\delta \in \left(0, \delta_0\right)$.

Whereas, the strong convergence criterion gives a measure of the closeness of the pathwise approximation to the Itô process, the weak convergence criterion

gives an approximation of the probability distribution of X_T. Carletti (2006: 425) states that:

> *"the strong order of convergence measures the rate at which the 'mean of the error' decays as $\Delta t \to 0$. The weak order of convergence measures the rate of decay of the 'error of the means' [as $\Delta t \to 0$]".*

This chapter provides the mathematical preliminary required to understand the following chapters. The ordinary differential equation that models population growth was used to motivate the appearance of a stochastic differential equation. The Wiener process was defined; the Itô-integral was motivated; and the Itô formula for calculating solutions of SDE's was derived. Since many SDE's do not have explicit solutions, numerical methods are used. Analogous to the numerical methods obtained from Taylor expansions to solve deterministic ordinary differential equations, the methods used in this thesis are derived from stochastic Taylor expansions. Hence, there was the need to introduce the stochastic Taylor expansion.

As with all numerical methods, in order to determine the efficiency of the numerical scheme, a method to determine the error is introduced. The different convergences are classified according to the strong and weak class of convergence. Convergence criteria are then used. This is used to determine the accuracy and efficiency of a particular numerical scheme. Having completed this preliminary, the next chapter focuses on some common numerical schemes that satisfy the strong convergence criteria are studied. These schemes are provided such that they can be expanded to provide other numerical schemes.

CHAPTER IV – NUMERICAL METHODS: STRONG APPROXIMATIONS

As mentioned in the first chapter, there are various methods to solve SDE's numerically. These include Monte Carlo methods which can be used to simulate the behaviour of the random system. Another method makes use of the discretisation of both time and space variables. However, this thesis considers discrete time approximations which involve the finite discretisation of the time interval $[0,T]$ only and not the state variable.

Strong approximations involve simulating the solution of SDE's when a good pathwise approximation is required. This occurs in direct simulations of the solution, or when filtering or testing statistical estimators. The absolute error criterion defined in Chapter III is used as a measure of the pathwise closeness at the end of the time interval $[0,T]$. There are various strong approximations and higher order numerical schemes (Runge-Kutta methods) analogous to the schemes used to solve deterministic differential equations, but the focus in this thesis is on strong Taylor schemes, explicit strong schemes and implicit strong approximations.

4.1 Strong Taylor approximations

As is the case of using deterministic Taylor expansions to derive numerical methods for ordinary differential equations, the stochastic Taylor expansion (3.20) is used to derive time discrete approximations with respect to the strong convergence criterion (3.21). The number of terms to be included in the stochastic Taylor expansion in the approximating numerical scheme will depend on the desired order of convergence. These strong Taylor schemes provide a pathwise approximation of the exact solution.

(i) Euler-Maruyama Method

The Euler – Maruyama method applied to an SDE is similar to the Euler method used to solve an ordinary differential equation. Consider the following scalar SDE:

$$dX_t = a\left(t, X_t\right)dt + b\left(t, X_t\right)dW_t,$$

or in integral form:

$$X_t(\omega) = X_{t_0}(\omega) + \int_{t_0}^{t} a\left(s, X_s(\omega)\right)ds + \int_{t_0}^{t} b\left(s, X_s(\omega)\right)dW_s(\omega), \qquad (4.1.1)$$

where $X = \{X_t, t_0 \le t \le T\}$ is an Itô process with initial value $X_{t_0} = X_0$. Subdivide the time interval $[0,T]$ into N subintervals according to the following discretisation:

$$t_0 = \tau_0 < \tau_1 < \cdots < \tau_n < \cdots < \tau_N = T.$$

The Euler approximation is defined as a continuous time stochastic process $Y = \{Y(T), t_0 \le t \le T\}$ satisfying the iterative scheme:

$$Y_{n+1} = Y(\tau_n) + a\left(\tau_n, Y(\tau_n)\right)\left(\tau_{n+1} - \tau_n\right) + b\left(\tau_n, Y(\tau_n)\right)\left(W_{\tau_{n+1}} - W_{\tau_n}\right), \qquad (4.1.2)$$

for $n = 0,1,2,...,N-1$ with initial value $Y_0 = X_0$. The Euler scheme is obtained by considering the first three terms of the Itô – Taylor expansion (3.20).

$$X_t = X_{t_0} + a\left(X_{t_0}\right)\int_{t_0}^{t} ds + b\left(X_{t_0}\right)\int_{t_0}^{t} dW_s$$

$$+L^1 b\left(X_{t_0}\right)\int_{t_0}^{t}\int_{t_0}^{s} dW_z\, dW_s + \overline{R}, \qquad (4.1.3)$$

where $\overline{R}$ is the remainder and is defined in equation (3.20). Equation (4.1.3) is the Itô – Taylor expansion of $X_t(\omega)$ in equation (4.1.1). The Itô – Taylor expansion is useful in approximating a sufficiently smooth function in a neighbourhood of a given point to a desired order of accuracy. Thus, considering the first three terms of equation (4.1.3) provides the Euler scheme in (4.1.2), where each term on the right hand side of equation (4.1.2) approximates the corresponding term on the right hand side of equation (4.1.1).

For brevity, equation (4.1.2) is written as:

$$Y_{n+1} = Y_n + a\,\Delta n + b\,\Delta W_n,$$

where

$$\Delta n = \tau_{n+1} - \tau_n = \int_{\tau_n}^{\tau_{n+1}} ds,$$

$$\Delta W_n = W_{\tau_{n+1}} - W_{\tau_n} = \int_{\tau_n}^{\tau_{n+1}} dW_s,$$

$$a = a\left(\tau_n, Y\left(\tau_n\right)\right),$$

$$b = b\left(\tau_n, Y\left(\tau_n\right)\right),$$

$$Y_n = Y\left(\tau_n\right).$$

The Euler scheme for a deterministic ordinary differential equation is obtained if $b = 0$ in equation (4.1.2). Thus, the main difference between the Euler scheme for deterministic ordinary differential equations and the Euler – Maruyama scheme

for SDE's is that the following random increments need to be generated for the SDE:

$$\Delta W_n = W_{\tau_{n+1}} - W_{\tau_n},$$

for $n = 0,1,2,...,N-1$ of the Wiener process $W = \{W_t, t \geq 0\}$, as defined in Chapter II.

The Euler scheme determines values of the approximating process at the discretisation times only. The values at the intermediate instances can be calculated by using either the piecewise constant interpolation method or the linear interpolation method. An overview of these methods is provided in Kloeden, *et al*, (1994: 307).

The Euler scheme is an example of a time discrete approximation (or difference method) in which the continuous time differential equation is replaced by a discrete time difference equation generating values $Y_1, Y_2, ..., Y_n$ to approximate $X_{t_1}, X_{t_2}, ..., X_{t_n}$ at given discretisation times $t_0 < t_1 < ... < t_n$. The Euler scheme is the simplest strong Taylor approximation and attains an order of convergence $\gamma = 0.5$. The proof is given in Kloeden & Platen (1992: 341 - 343).

(ii) Milstein Scheme

The Milstein scheme is obtained by considering the first four terms of Taylor expansion given in equation (3.20). It is given as:

$$X_t = X_{t_0} + a\left(X_{t_0}\right)\int_{t_0}^{t} ds + b\left(X_{t_0}\right)\int_{t_0}^{t} dW_s \; + L^1 b\left(X_{t_0}\right)\int_{t_0}^{t}\int_{t_0}^{s} dW_z \, dW_s \, .$$

Use formula (3.12) to obtain:

$$\int_{t_n}^{t_{n+1}} \int_{t_n}^{t} dW_s \, dW_t = \frac{1}{2}\left\{\left(\Delta W_n\right)^2 - \Delta n\right\}.$$

From equation (3.16), $L^1 b = bb'$, thus:

$$L^1 b\left(X_{t_0}\right) \int_{t_0}^{t} \int_{t_0}^{s} dW_z \, dW_s = \frac{1}{2} bb' \left\{\left(\Delta W_n\right)^2 - \Delta n\right\}.$$

Therefore, the Milstein scheme is defined as:

$$Y_{n+1} = Y\left(\tau_n\right) + a\left(\tau_n, Y\left(\tau_n\right)\right)\left(\tau_{n+1} - \tau_n\right) + b\left(\tau_n, Y\left(\tau_n\right)\right)\left(W_{\tau_{n+1}} - W_{\tau_n}\right) +$$

$$\frac{1}{2} b\left(\tau_n, Y\left(\tau_n\right)\right) b'\left(\tau_n, Y\left(\tau_n\right)\right)\left\{\left(W_{\tau_{n+1}} - W_{\tau_n}\right)^2 - \left(\tau_{n+1} - \tau_n\right)\right\}.$$

For brevity, this is written as:

$$Y_{n+1} = Y_n + a\,\Delta n + b\,\Delta W_n + \frac{1}{2} b\,b'\left\{\left(\Delta W_n\right)^2 - \left(\Delta n\right)\right\}.$$

The term b' is the partial derivative of b with respect to x. i.e. $b' = db/dx$.

Whereas, Euler – Maruyama scheme has order γ = 0.5, the Milstein scheme has order γ = 1 (Kloeden and Platen, 1992).

The orders of strong and weak convergence of the stochastic Euler and Milstein schemes are low. In order to improve the order of convergence, multiple stochastic integrals of W_t are included in the numerical scheme. This is because

the simple increments ΔW_t do not provide enough information about the sample paths of Wiener process W_t inside the discretisation subinterval $[\tau_{n+1} - \tau_n]$ to ensure higher order of approximation. Generally, the numerical scheme implicitly uses a linear interpolation in the subinterval. Multiple stochastic integrals of W_t that occur in the stochastic Taylor expansion provide additional information about the sample paths of the driving Wiener process within the discretisation interval.

(iii) Order 1.5 Strong Taylor Scheme

A more accurate strong Taylor scheme can be obtained by including further multiple stochastic integrals from the stochastic Taylor expansion in the scheme. The order γ = 1.5 strong Taylor scheme is derived by adding more terms from the Itô – Taylor expansion to the Milstein scheme.

The order 1.5 strong Taylor scheme is given as (see Kloeden & Platen, 1992: 351):

$$X_{n+1} = X_n + a\,\Delta n + b\,\Delta W_n + \frac{1}{2} b\,b' \left\{ (\Delta W_n)^2 - \Delta n \right\} + b\,\frac{\partial a}{\partial x}\,\Delta Z_n$$

$$+ \left\{ \frac{\partial b}{\partial t} + \frac{\partial b}{\partial x} + \frac{1}{2} b^2 \frac{\partial^2 b}{\partial x^2} \right\} \left\{ \Delta W_n\,\Delta n - \Delta Z_n \right\}$$

$$+ \frac{1}{2} \left\{ \frac{\partial a}{\partial t} + a\frac{\partial a}{\partial x} + \frac{1}{2} b^2 \frac{\partial^2 a}{\partial x^2} \right\} (\Delta n)^2$$

$$+ \frac{1}{2} b \left\{ b\,\frac{\partial^2 b}{\partial x^2} + \left(\frac{\partial b}{\partial x} \right)^2 \right\} \left\{ \frac{1}{3} (\Delta W_n)^2 - \Delta n \right\} \Delta W_n ,$$

where:

$$\Delta Z_n = \int_{\tau_n}^{\tau_{n+1}} \int_{\tau_n}^{s_2} dW_{s_1}\, ds_2 \tag{4.1.4}$$

is a normally distributed random variable with the following properties:

$$E\left(\Delta Z_n\right) = 0\,,$$

$$Var\left(\Delta Z_n\right) = \frac{1}{3}\left(\Delta n\right)^3\,,$$

$$Cov\left(\Delta Z_n\, \Delta W_n\right) = \frac{1}{2}\left(\Delta n\right)^2\,. \tag{4.1.5}$$

The properties of ΔZ_n are important from a numerical perspective. The multiple stochastic integral ΔZ_n cannot be generated directly during the numerical approximation process with these properties. In order to generate ΔZ_n, two independent $N\left(0,1\right)$ distributed random variables U_1 and U_2 are generated in the numerical approximation and then the following transformation is made:

$$\Delta W = U_1\sqrt{\Delta n}\,,$$

$$\Delta Z = \frac{1}{2}\sqrt{\left(\Delta_n\right)^3}\left(U_1 + \frac{1}{\sqrt{3}}U_2\right). \tag{4.1.6}$$

It has been shown in Kloeden and Platen (1992: 351) that ΔZ_n defined in equation (4.1.6) has the desired properties (4.1.5). Thus, in order to generate ΔZ_n defined in equation (4.1.4) during the numerical process, two independent $N\left(0,1\right)$ distributed random variables U_1 and U_2 are generated and the

transformation in (4.1.6) is made. This transformation is essential in order to generate the stochastic multiple integral in the numerical approximation.

Higher order strong Taylor schemes can be obtained by including even more terms from the stochastic Taylor expansions. These scheme, however, become increasingly more complex, but in some cases simple formulae can be obtained by using the special structure of the SDE. Using this approach, the order 2.0 strong Taylor scheme can be derived (Kloeden & Platen, 1992: 356).

4.2 Explicit strong schemes

(i) Explicit order strong schemes

Using strong Taylor approximations involves determining and evaluating the derivatives of the various orders of the drift and diffusion coefficient as well as the coefficients themselves for each step. Implementing this procedure can be time consuming. Therefore, to avoid the use of derivatives, the derivatives in the strong schemes are replaced by their corresponding finite differences.

Using the Milstein scheme, one can derive the explicit order 1 scheme by replacing the derivatives by the corresponding difference ratios. However, these differences require the use of supporting values of the coefficients at additional points.

Use the following Milstein scheme:

$$Y_{n+1} = Y_n + a\,\Delta n + b\,\Delta W_n + \frac{1}{2} b\,b' \left\{ \left(\Delta W_n \right)^2 - \left(\Delta n \right) \right\}$$

and replace the derivative b' with finite differences, to obtain the explicit order 1 strong scheme which has the following form (Kloeden & Platen (1992: 374)):

$$Y_{n+1} = Y_n + a\ \Delta n + b\ \Delta W_n + \frac{1}{2\sqrt{\Delta n}}\left\{ b\left(\tau_n, \overline{Y_n} \right) - b\left(\tau_n, Y_n\right) \right\} \left\{ \left(\Delta W_n \right)^2 - \Delta n \right\},$$

where:

$$\overline{Y_n} = Y_n + a\ \Delta n + b\ \sqrt{\Delta n}\ .$$

An explicit order 1.5 strong scheme can also be derived by replacing the derivatives in the order 1.5 strong Taylor scheme by corresponding finite differences.

In principle, the derivatives of the strong Taylor schemes can be replaced to obtain corresponding explicit schemes. This procedure may work well for low order explicit schemes but as the order is increased the formulae become more complicated. Sometimes, the special structure of the equation under consideration can be used to derive relatively simple higher order explicit schemes which do not involve the derivatives of the drift and diffusion coefficients. Kloeden & Platen (1992) provide an explicit order 2 strong scheme for additive noise, using the Stratonovich notation, to simplify the notation. This is possible, since an SDE which is written using Itô integrals can be equivalently written using Stratonovich integrals (see Kloeden & Platen (1992: 154 – 160).

(ii) Multi step schemes

Numerical methods using information from previous discretisation sub-intervals when calculating y_{n+1} on the interval $t_n < t < t_{n+1}$ achieve higher accuracy than those involving just a single time step. In these multi-step methods, y_{n+1} depends on the previous *k* values $y_n, y_{n-1}, \ldots y_{n-k}$ for some *k > 1*. The initial *k* values are generated using an appropriate one-step method.

Another advantage of using multi-step over single step methods is that they are more computationally efficient because they require only one new evaluation of the right hand side of the SDE for each iteration. Sometimes, these multi-step schemes are more stable for larger time steps, although there are multi-step schemes that are unstable (Kloeden and Platen, 1992). Because stochastic simulations require the calculation of many different realisations of the approximating process, efficiency and stability are crucial factors to be taken into account when using stochastic multi-step schemes.

Kloeden & Platen (1992: 385 - 389) provide a general explanation of how one would obtain a two-step order 1 strong scheme and a two-step order 1.5 strong scheme using a 2-dimensional Itô system.

4.3 Implicit strong approximations

Implicit strong schemes are those schemes in which the unknown quantity y_{n+1} appears on both sides of the SDE and in general y_{n+1} cannot be isolated algebraically. These schemes are used to simulate the solution of *stiff* stochastic differential equations.

(i) Implicit Euler Scheme

Generating one of the simplest implicit schemes is rewriting the Euler scheme to obtain the *implicit* Euler scheme:

$$Y_{n+1} = Y_n + a\left(\tau_{n+1}, Y_{n+1}\right)\Delta n + b\,\Delta W\,, \qquad (4.3.1)$$

where only the drift term is implicit, in order to ensure a solution, and $b = b\left(\tau_n, Y_n\right)$. A family of implicit Euler schemes can be generated:

$$Y_{n+1} = Y_n + \{\alpha a(\tau_{n+1}, Y_{n+1}) + (1-\alpha)a\}\Delta n + b\Delta W ,$$

where $\alpha \in [0,1]$ characterizes the degree of implicitness. When α = 0, the explicit Euler scheme is obtained; the implicit scheme when α = 1 and for α = 0.5 the generalisation of the deterministic trapezoidal method is obtained.

Implementing the implicit scheme will require solving an additional algebraic equation at each time step. This is usually accomplished with standard numerical methods such as the Newton-Raphson method. Further, using implicit schemes improves the stability of simulations considerably without too much additional computational effort (Kloeden and Platen, 1992).

(ii) The implicit Milstein scheme

The implicit Milstein scheme is derived analogously:

$$Y_{n+1} = Y_n + a(\tau_{n+1}, Y_{n+1})\Delta n + b\Delta W + \frac{1}{2}bb'\{(\Delta W)^2 - \Delta n\}, \qquad (4.3.2)$$

where again only the drift term contains the unknown y_{n+1}.

The family of implicit schemes:

$$Y_{n+1} = Y_n + \{\alpha a(\tau_{n+1}, Y_{n+1}) + (1-\alpha)a\}\Delta n + b\Delta W + \frac{1}{2}bb'\{(\Delta W)^2 - \Delta n\},$$

where again, $\alpha \in [0,1]$ characterizes the degree of implicitness. When α = 0, the explicit Milstein scheme is obtained; the implicit scheme when α = 1 and for α = 0.5 the generalisation of the deterministic trapezoidal method is obtained.

Implicit schemes for the order 1.5 and order 2 strong Taylor schemes can also be obtained, as well as derivative free schemes and multi-step schemes. Kloeden & Platen (1992) provide the implicit versions of these schemes for the 1-dimensional as well as multi-dimensional case. In addition, they also consider the special cases when there is additive and commutative noise, sometimes using the equivalent Stratonovich representation to ease the notation.

CHAPTER V – NUMERICAL METHODS: WEAK APPROXIMATIONS

The numerical methods in chapter IV are based on the strong convergence criterion where a good pathwise approximation is required, like in areas where direct simulations are required, filtering or testing statistical estimators. However, in many practical problems, a good pathwise approximation may not be of interest, but rather in some functional of the Itô process, like its probability distribution and its moments because it may not possible to determine these analytically. Thus, the weak convergence criterion is used, defined in chapter II, in cases where we need to approximate expectations of the functionals of the Itô process, which governs the stochastic differential equation.

5.1 Weak Taylor schemes

Weak Taylor approximations are obtained from using the stochastic Taylor expansion, introduced in chapter II, with a specified number of terms, depending on the order of weak convergence that is required.

(i) Weak Euler Scheme

Consider the following Euler scheme introduced in chapter IV, which only has the ordinary time integral and simple Itô integral:

$$Y_{n+1} = Y_n + a\,\Delta n + b\,\Delta W_n\,,$$

where:

$$\Delta n = \tau_{n+1} - \tau_n = \int_{\tau_n}^{\tau_{n+1}} ds\ ,$$

$$\Delta W_n = W_{\tau_{n+1}} - W_{\tau_n} = \int_{\tau_n}^{\tau_{n+1}} dW_s\ ,$$

$$a = a\left(\tau_n, Y\left(\tau_n\right)\right),$$

$$b = b\left(\tau_n, Y\left(\tau_n\right)\right),$$

$$Y_n = Y\left(\tau_n\right).$$

Since, in the weak approximation, estimating some functional of the Itô process is of interest, the Gaussian increments ΔW_n can be replaced by other random variables, say $\Delta\widehat{W}_n$, which has similar moment properties to ΔW_n. Thus the weak Euler scheme has the form:

$$Y_{n+1} = Y_n + a\,\Delta n + b\,\Delta\widehat{W}_n, \tag{5.1.1}$$

where the $\Delta\widehat{W}_n$ must be independent $A_{\tau_{n+1}}$- measurable random variables. For example, $\Delta\widehat{W}_n$ could be a two-point distributed random variable with:

$$P\left(\Delta\widehat{W}_n = \pm\sqrt{\Delta n}\right) = \frac{1}{2}.$$

If a and b are sufficiently smooth, and four times continuously differentiable, then the Euler approximation has order of weak convergence β = 1.0 (Kloeden & Platen (1992: 457)). This is the order 1.0 weak Taylor scheme.

(ii) The order 2.0 weak Taylor scheme

More accurate weak Taylor schemes can be derived by adding further multiple stochastic integrals from the stochastic Taylor expansion, similar to the strong convergence schemes. The order β = 2.0 weak scheme is obtained by adding all the double stochastic integrals from the Itô – Taylor expansion (3.20) to the Euler scheme. This would provide more information about the probability measure of

the underlying Itô process. Since only the probability measure is of interest and not the sample paths itself, the multiple stochastic integrals could be replaced with simpler random variables. These random variables have only to coincide in their lower order moments with those of the Wiener process to provide sufficient accurate approximation of the probability law of the Itô diffusion.

The order 2.0 weak Taylor scheme is:

$$Y_{n+1} = Y_n + a\,\Delta n + b\,\Delta W_n + \frac{1}{2}bb'\left\{\left(\Delta W_n\right)^2 - \Delta n\right\}$$

$$+ a'b\,\Delta Z_n + \frac{1}{2}\left(aa' + \frac{1}{2}a''b^2\right)\left(\Delta n\right)^2$$

$$+ \left(ab' + \frac{1}{2}b''b^2\right)\left\{\Delta W_n\,\Delta n - \Delta Z_n\right\},$$

where, as previously, ΔZ_n represents the double Itô integral:

$$\Delta Z_n = \int_{\tau_n}^{\tau_{n+1}} \int_{\tau_n}^{s_2} dW_{s_1} ds_2 .$$

In a similar way, order 3.0 and order 4.0 weak Taylor schemes can be derived by adding all of the multiple Itô integrals of multiplicity three (for order β = 3.0) from the Itô –Taylor expansion and all of the multiple Itô integrals of multiplicity four for order β = 4.0, respectively. Kloeden & Platen (1992) derive the order 3.0 weak scheme and order 4.0 scheme for the general multi-dimensional case. However, it is difficult to generate multiple integrals of higher multiplicity and this is exacerbated when the corresponding coefficient functions become complicated. Thus, for weak convergence, simpler random variables with analogous moment properties to the Wiener process are substituted for the multiple Itô integrals.

These schemes become simpler depending on the structure of the SDE, for example, if there is *scalar* or *additive* noise.

5.2 Explicit weak approximations

Weak Taylor schemes require the determination and evaluation of derivatives of various orders of the drift and diffusion coefficients. Similar to what was done for the strong schemes, schemes that avoid the use of derivatives can also be derived.

(i) Explicit order 2.0 weak schemes

The explicit order 2.0 weak scheme is given as:

$$\begin{aligned} Y_{n+1} &= Y_n + \frac{1}{2}\left(a\left(\bar{Y} \right) + a \right)\Delta n \\ &+ \frac{1}{4}\left(b\left(\bar{Y}^{+} \right) + b\left(\bar{Y}^{-} \right) + 2b \right)\Delta \widehat{W}_n \\ &+ \frac{1}{4}\left(b\left(\bar{Y}^{+} \right) - b\left(\bar{Y}^{-} \right) \right)\left\{ \left(\Delta \widehat{W}_n \right)^2 - \Delta n \right\}(\Delta n)^{-0.5} \end{aligned}$$

with supporting values:

$$\bar{Y} = Y_n + a\Delta n + b\Delta \widehat{W}_n ,$$

$$\bar{Y}^{\pm} = Y_n + a\Delta n \pm b\sqrt{\Delta n} ,$$

with $\Delta \widehat{W}_n$ being $A_{\tau_{n+1}}$ – measurable. This measurability condition holds if $\Delta \widehat{W}_n$ is chosen at each step to be independent. For example, $\Delta \widehat{W}_n$ could be Gaussian or it could be three-point distributed with:

$$P\left(\Delta\widehat{W}_n = \pm\sqrt{3\Delta n}\right) = \frac{1}{6},$$

$$P\left(\Delta\widehat{W}_n = 0\right) = \frac{2}{3}.$$

The explicit scheme can be extended in principle to the order 3.0 and order 4.0 weak Taylor schemes by replacing the derivative of the drift and diffusion coefficients by their corresponding finite differences. These formulae become more complicated as the order is increased. However, one could take into account the special structure of the SDE, like whether there is additive noise or not. See Kloeden & Platen (1992: 488) for further reference.

5.3 Implicit weak approximations

Implicit weak approximations involve substituting ΔW_n in the implicit strong schemes by other random variables $\Delta\widehat{W}_n$, with similar moment properties as the Wiener process ΔW_n.

(i) Implicit weak Euler scheme

Thus for example, ΔW_n can be replaced by $\Delta\widehat{W}_n$ in equation (4.3.1) to obtain the implicit weak Euler scheme:

$$Y_{n+1} = Y_n + a\left(\tau_{n+1}, Y_{n+1}\right)\Delta n + b\,\Delta\widehat{W}_n\,,$$

where only the drift term is implicit. The following two-point distributed random variable can be used:

$$P\left(\Delta\widehat{W}_n = \pm\sqrt{\Delta n}\right) = \frac{1}{2},$$

for $\Delta\widehat{W}_n$. A family of implicit Euler schemes can also be formed, as in Chapter IV:

$$Y_{n+1} = Y_n + \left\{(1-\alpha)\,a\left(\tau_n, Y_n\right) + \alpha\, a\left(\tau_{n+1}, Y_{n+1}\right)\right\}\Delta n + b\left(\tau_n, Y_n\right)\Delta\widehat{W}_n,$$

where, as previously, α can be interpreted as the degree of implicitness. With α = 0.0 the above scheme reduces to the simplified weak Euler scheme (5.1.1) while with α = 0.5, the scheme represents a stochastic generalisation of the trapezoidal method.

(ii) The implicit order 2.0 weak Taylor scheme

The order 2.0 weak Taylor scheme is adapted to obtain the implicit scheme. It has the following form:

$$\begin{aligned} Y_{n+1} &= Y_n + a\left(Y_{n+1}\right)\Delta n + b\Delta\widehat{W} \\ &\quad -\frac{1}{2}\left\{a\left(Y_{n+1}\right)a'\left(Y_{n+1}\right) + \frac{1}{2}b^2\left(Y_{n+1}\right)a''\left(Y_{n+1}\right)\right\}\left(\Delta n\right)^2 \\ &\quad +\frac{1}{2}bb'\left\{\left(\Delta\widehat{W}\right)^2 - \Delta n\right\} \\ &\quad +\frac{1}{2}\left\{-a'b + ab' + \frac{1}{2}b''b^2\right\}\Delta\widehat{W}\Delta n \end{aligned}$$

where $\Delta\widehat{W}$ is $N\left(0, \Delta n\right)$. $\Delta\widehat{W}$ could also be three-point distributed with:

$$P\left(\Delta\widehat{W}_n = \pm\sqrt{3\Delta}\right) = \frac{1}{6},$$

$$P\left(\Delta\widehat{W}_n = 0\right) = \frac{2}{3}.$$

(iii) The implicit order 2.0 weak scheme

This scheme can be derived from the order 2.0 strong scheme by ensuring that the drift term only is implicit and ΔW_n is replaced by $\Delta\widehat{W}_n$. It has the following form,

$$\begin{aligned} Y_{n+1} &= Y_n + \frac{1}{2}\left(a + a\left(Y_{n+1}\right)\right)\Delta n \\ &+ \frac{1}{4}\left(b\left(\overline{Y}^{+}\right) + b\left(\overline{Y}^{-}\right) + 2b\right)\Delta\widehat{W} \\ &+ \frac{1}{4}\left(b\left(\overline{Y}^{+}\right) - b\left(\overline{Y}^{-}\right)\right)\left\{\left(\Delta\widehat{W}\right)^2 - \Delta n\right\}(\Delta n)^{-0.5} \end{aligned}$$

with supporting value

$$\overline{Y}^{\pm} = Y_n + a\Delta n \ \pm\ b\sqrt{\Delta n}\,.$$

5.4 Predictor – Corrector methods

Kloeden & Platen (1992) provide other weak approximating schemes: the weak order $\beta = 1$ and weak order $\beta = 2.0$ predictor-corrector methods. These methods are used mainly because of their numerical stability. Further, the difference

between the predicted and corrected values at each time step provides an indication of the local error.

The predictor terms are given by the weak Taylor or explicit weak schemes and the corrector terms are the corresponding implicit schemes made explicit by the predicted value $\overline{Y}_{n+1}$ instead of the Y_{n+1} on the right hand side of the implicit scheme.

Consider the following modified trapezoidal method of weak order $\beta = 1$ with corrector:

$$Y_{n+1} = Y_n + \frac{1}{2}\left\{ a\left(\overline{Y}_{n+1}\right) + a \right\} \Delta n + b\,\Delta \widehat{W}_n$$

The predictor term is the weak Euler scheme:

$$\overline{Y}_{n+1} = Y_n + a\,\Delta n \; + \; b\,\Delta \widehat{W}_n \,.$$

Since, in the weak schemes, ΔW_n can be replaced by other random variables with similar properties, $\Delta \widehat{W}_n$ could be chosen as a $N\left(0, \Delta n\right)$ normally distributed random variable, or as a two-point distributed random variable with:

$$P\left(\Delta \widehat{W}_n = \pm\sqrt{\Delta n}\right) = \frac{1}{2}\,.$$

CHAPTER VI – NUMERICAL RESULTS

This chapter of the thesis deals with the numerical tests, which have been programmed in Matlab. The strong and weak convergence criteria lead to different discrete time approximations, which are only efficient with respect to one of these two criteria. It is therefore important to clarify the aim of the simulation before choosing an appropriate scheme, deciding on whether a good pathwise approximation of the Itô process is required or whether an approximation of some functional of the Itô process is the real objective.

Initially, Brownian motion is generated. Then various numerical schemes, which approximate the solution of the stochastic differential equation, are compared to the exact solution of a linear SDE with multiplicative noise. This can be easily accomplished since we have a closed solution for the linear SDE. Numerical estimates are provided for the strong convergence schemes of Chapter IV as well as estimates for the absolute error using the absolute error criterion. Numerical estimates are then provided for the order 2.0 weak Taylor scheme of chapter V. Further, the numerical schemes are also applied to the case when the SDE has additive noise. The Ornstein-Uhlenbeck process is used as an example of a linear SDE with additive noise.

6.1 Generating Brownian motion in Matlab

The underlying difference between deterministic and stochastic differential equations is the need to generate the following random increments of the Brownian motion for the SDE:

$$\Delta W_n = W_{t_n} - W_{t_{n-1}}. \tag{6.1}$$

For computational purposes, it is necessary to discretise the Brownian motion, where W_t is specified at discrete t values. Therefore, let $\Delta t = T/N$ for some

positive integer N and for T on interval $[0,T]$. From the definition of Brownian motion:

$$\Delta W_n = W_{t_n} - W_{t_{n-1}} \sim N\left(0, t_n - t_{n-1}\right),$$

or equivalently:

$$\Delta W_n = W_{t_n} - W_{t_{n-1}} \sim \sqrt{t_n - t_{n-1}}\ N(0,1)$$

where $N(0,1)$ denotes a standard normally distributed random variable with zero mean and variance equal to one. Here $(t_n - t_{n-1}) = \Delta t$ is the variance of the Brownian motion random variable. In *Matlab,* the function **randn (1,N)** will generate N random variables from the standard normal distribution. In order to generate a random variable with variance equal to $\Delta t,$ random variables from the standard normal distribution are generated using the *Matlab* function **randn (1,N)** and each of these variables are then multiplied by $\sqrt{\Delta t},$ resulting in the random increments in equation (6.1).

From equation (6.1),

$$\Delta W_1 = W_{t_1} - W_{t_0},$$
$$\Delta W_2 = W_{t_2} - W_{t_1},$$

implying that:

$$\begin{aligned} \Delta W_1 + \Delta W_2 &= W_{t_2} - W_{t_1} + W_{t_1} - W_{t_0} \\ &= W_{t_2}, \end{aligned}$$

since $t_0 = 0$ and $W_0 = 0$. Therefore:

$$W_{t_n} = \sum_{j=1}^{n} \Delta W_j$$

will generate a discretised Brownian motion. The *Matlab* function **cumsum (dW)**, where $dW = \Delta W$, will generate a discretised Brownian motion during the simulations.

Figure 6.1 shows a single simulation of a discretised Brownian path, the *Matlab* code for which is provided in Appendix A1. The Brownian path is conducted on the interval $T = [0,1]$ and $N = 500$, so that $\Delta t = 1/500 = 0.002$.

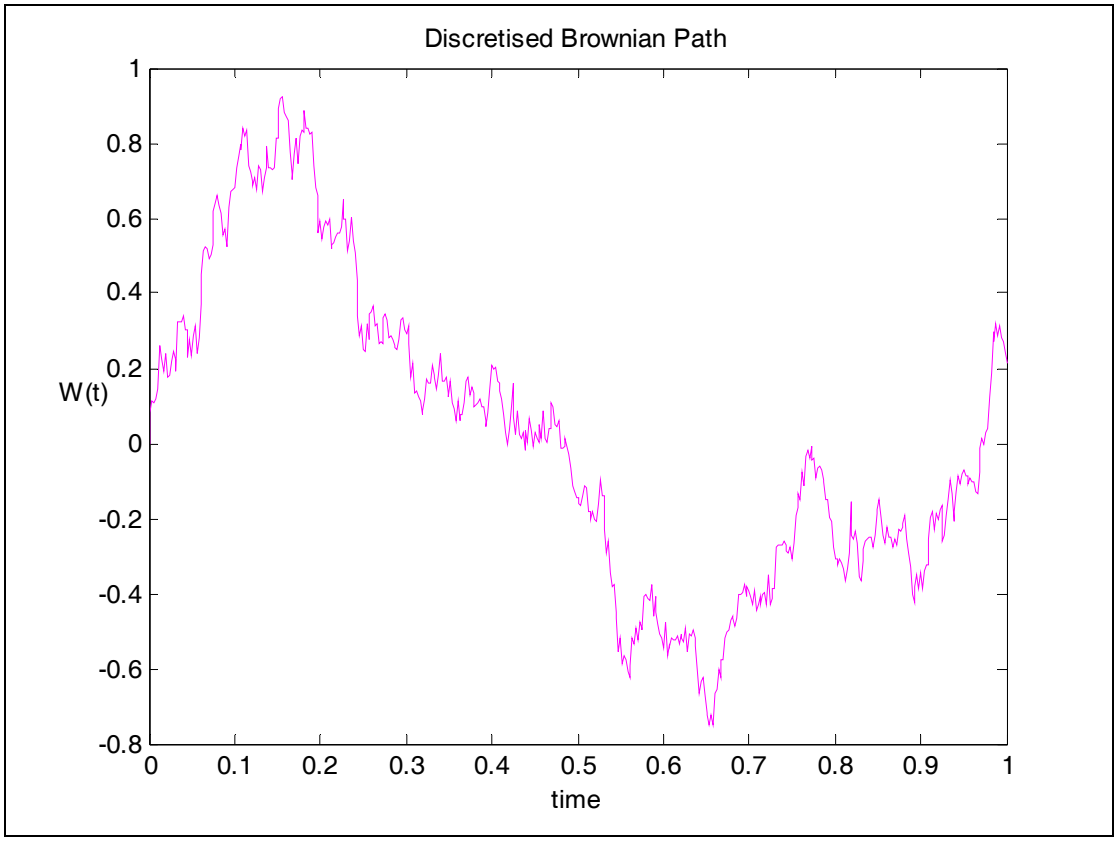

Figure 6.1: Discretised Brownian path

6.2 Strong Taylor approximations – SDE with multiplicative noise

Consider the Itô process $X = \{X_t, t \geq 0\}$ satisfying the following stochastic differential equation:

$$dX_t = a(t, X_t)dt + b(t, X_t)dW_t. \tag{6.2}$$

Let

$$a(t, X_t) = \alpha X_t,$$
$$b(t, X_t) = \beta X_t,$$

so that equation (6.2) becomes a linear stochastic differential equation with multiplicative noise:

$$dX_t = \alpha X_t dt + \beta X_t dW_t. \tag{6.3}$$

In integral form, the stochastic differential equation is written as:

$$X_t = X_0 + \int_0^t \alpha X_s ds + \int_0^t \beta X_s dW_s,$$

where α and β are constants.

To derive the closed form solution, Itô's formula (3.15) is used.

Let $Y(t) = \log X(t)$ in equation (3.15).

Applying Itô's formula (3.15):

$$\begin{aligned} d\left[\log X(t)\right] &= \frac{dX(t)}{X(t)} - \frac{1}{2}\frac{\left[dX(t)\right]^2}{X(t)^2} \\ &= \left(\alpha\, dt + \beta\, dW\right) - \frac{1}{2}\beta^2\, dt \\ \Rightarrow \quad d\left[\log X(t)\right] &= \left(\alpha - \frac{\beta^2}{2}\right)dt + \beta\, dW\,. \end{aligned}$$

Integrate both sides to obtain:

$$\begin{aligned} \log X(t) - \log X(0) &= \int_0^t \left(\alpha - \beta^2/2\right)\, ds \;+\; \int_0^t \beta\, dW \\ \Rightarrow \quad \log\left[X(t)/X(0)\right] &= \left(\alpha - \beta^2/2\right)t + \beta W(t) \\ \Rightarrow \quad X(t)/X(0) &= \exp\left\{\left(\alpha - \beta^2/2\right)t + \beta W(t)\right\} \\ \Rightarrow \quad X(t) &= X(0)\exp\left\{\left(\alpha - \beta^2/2\right)t + \beta W(t)\right\}. \end{aligned}$$

Therefore, the solution for the above linear SDE in equation (6.3) is:

$$X_t = X_0 \exp\left(\left(\alpha - \frac{1}{2}\beta^2\right)t + \beta W_t\right), \tag{6.4}$$

where:

$$\begin{aligned} X(0) &= X_0\,, \\ X(t) &= X_t\,. \end{aligned}$$

(i) The Euler scheme

From Chapter II, the Euler approximation is given by:

$$Y_{n+1} = Y(\tau_n) + a\left(\tau_n, Y(\tau_n)\right)\left(\tau_{n+1} - \tau_n\right) + b\left(\tau_n, Y(\tau_n)\right)\left(W_{\tau_{n+1}} - W_{\tau_n}\right). \qquad (6.5)$$

To simulate the Euler approximation (6.5), for example (6.3), start with the initial value $Y_0 = X_0$ and proceed recursively to generate the value:

$$Y_{n+1} = Y_n + \alpha Y_n \Delta t + \beta Y_n \Delta W_n .$$

Noting that:

$$a\left(\tau_n, Y(\tau_n)\right) = \alpha Y_n ,$$
$$b\left(\tau_n, Y(\tau_n)\right) = \beta Y_n ,$$
$$\left(\tau_{n+1} - \tau_n\right) = \Delta t ,$$
$$\left(W_{\tau_{n+1}} - W_{\tau_n}\right) = \Delta W_t .$$

For $Y_0 = X_0 = 1$, $\alpha = 1.5$ and $\beta = 1$, the following plot is generated (Figure 6.2), using step size $\Delta t = 2^{-5}$, (of which the Matlab routine is presented in Appendix A.2):

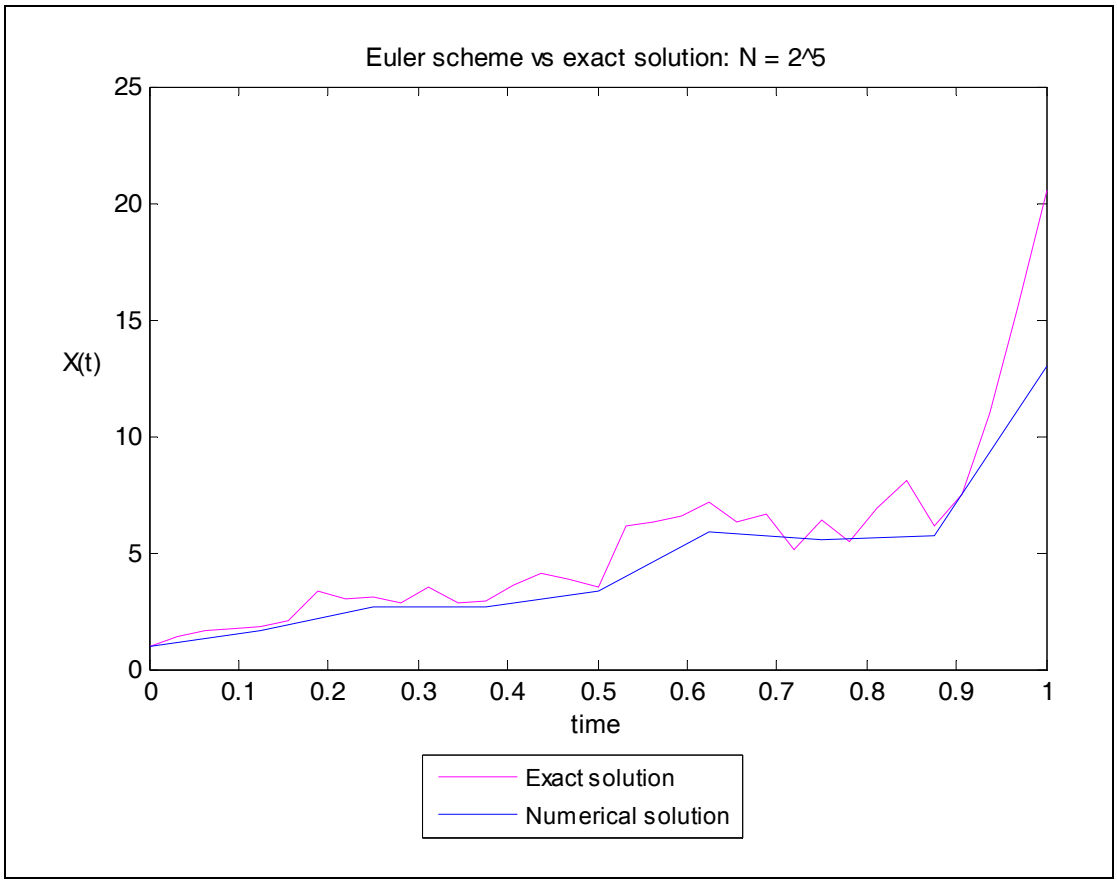

Figure 6.2: Exact solution and the Euler approximation with N = 2^5

The error is 7.5709 when the step size is $\Delta t = 2^{-5}$.

The linearly interpolated Euler approximation differs from the exact solution when using a step size $\Delta t = 2^{-5}$. However, as the step size increases, the Euler approximation gets closer to resembling the true solution. Figure 6.3 plots the exact solution and the Euler approximation with $N = 2^{10}$ and the associated error is 0.0195.

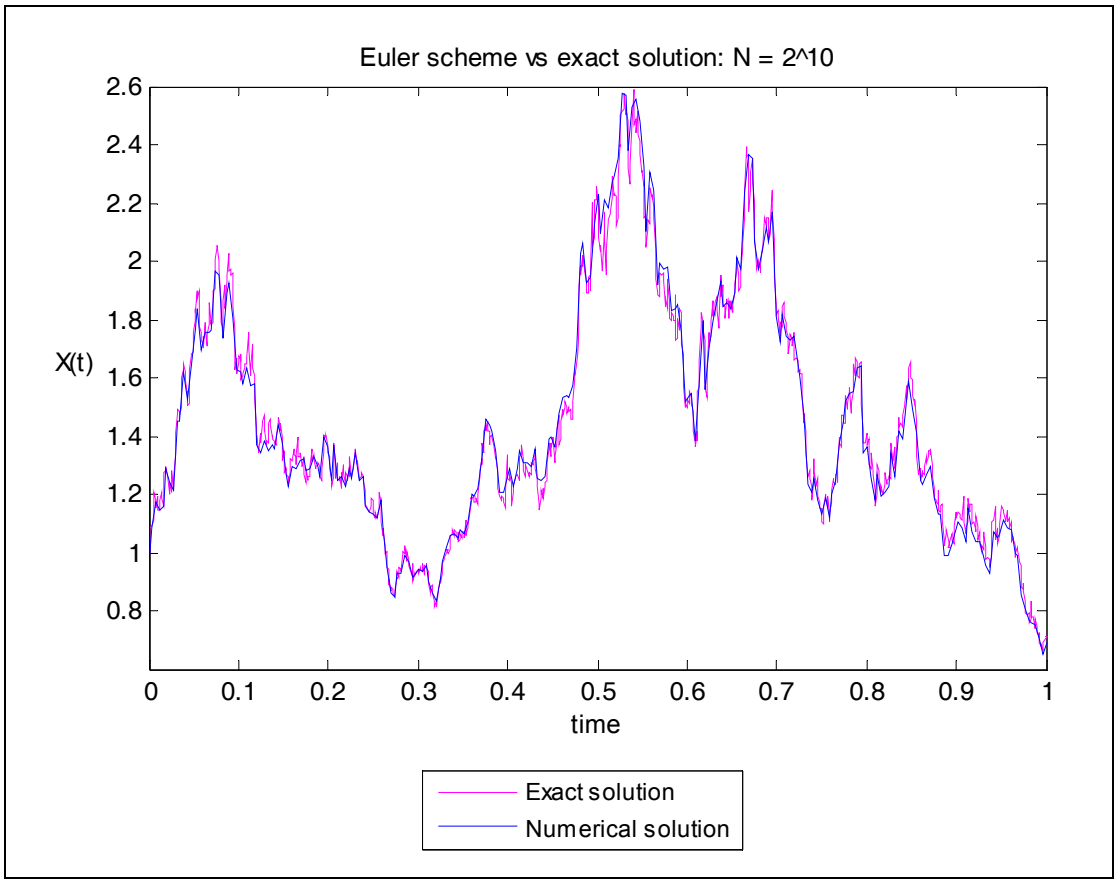

Figure 6.3: Exact solution and the Euler approximation with N = 2^10

Varying the size of the number of time steps from $N = 2^5$ to $N = 2^{14}$ produces errors that generally decrease as N increases. The errors are recorded in Table 6.2 when the number of time steps is increased from $N = 2^5$ to $N = 2^{14}$.

Euler Scheme	
N	**Error**
2 ^ 5	7.5709
2 ^ 6	5.4325
2 ^ 7	0.7487
2 ^ 8	0.0094
2 ^ 9	0.1677
2 ^ 10	0.0195
2 ^ 11	0.0087
2 ^ 12	0.0582
2 ^ 13	0.0457
2 ^ 14	0.0670

Table 6.1: Errors generated by the Euler scheme

Figure 6.4 shows that the numerical scheme is a much better approximation as the number of time steps is increased to $N = 2^{14}$.

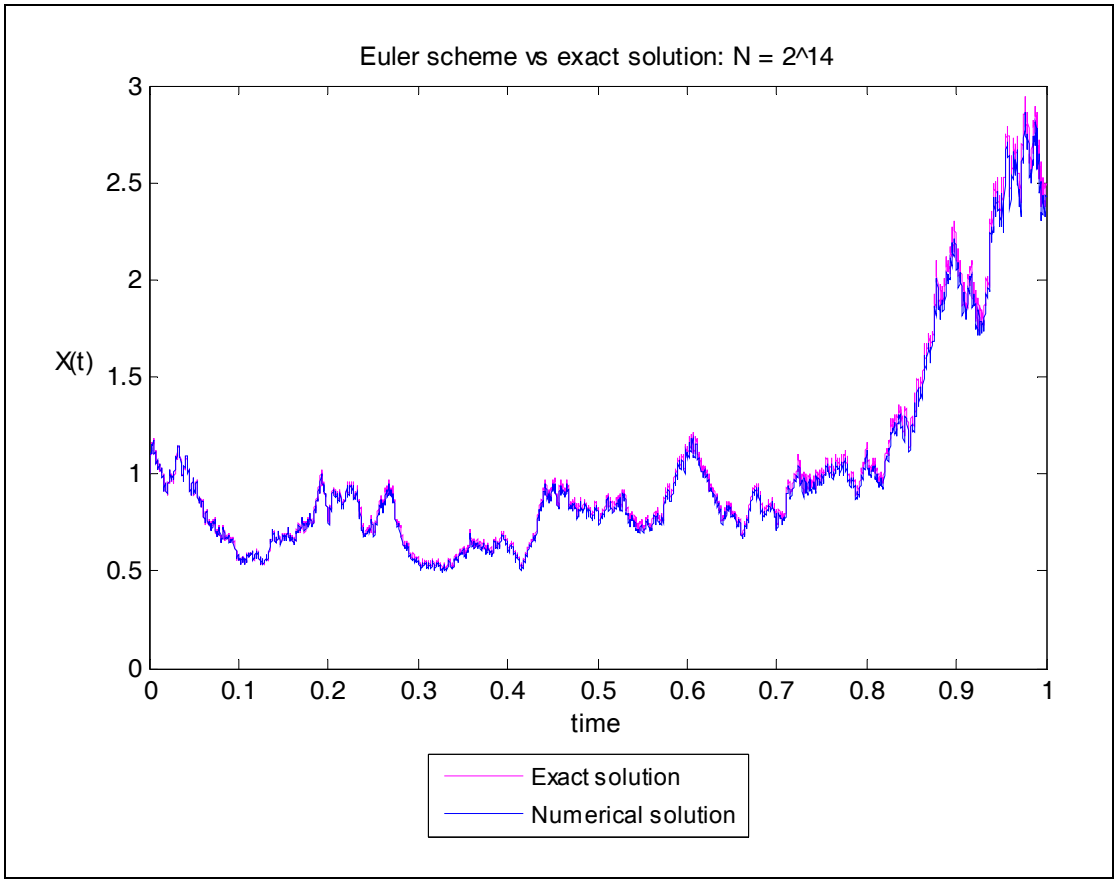

Figure 6.4: Exact solution and Euler approximation with N = 2^14

(ii) The Milstein scheme

The Milstein Scheme is given by:

$$Y_{n+1} = Y(\tau_n) + a(\tau_n, Y(\tau_n))(\tau_{n+1} - \tau_n) + b(\tau_n, Y(\tau_n))(W_{\tau_{n+1}} - W_{\tau_n}) +$$

$$\frac{1}{2} b(\tau_n, Y(\tau_n)) b'(\tau_n, Y(\tau_n)) \{ (W_{\tau_{n+1}} - W_{\tau_n})^2 - (\tau_{n+1} - \tau_n) \}.$$

For the example (6.3), $dX_t = \alpha X_t\, dt + \beta X_t\, dW_t$, the Milstein scheme is:

$$Y_{n+1} = Y_n + \alpha Y_n \Delta t + \beta Y_n \Delta W_t + \frac{1}{2} \beta^2 Y_n \{(\Delta W_t)^2 - \Delta t\},$$

because for $b(t, X_t) = \beta X_t$, the derivative of b with respect to x is $b'(t, X_t) = \beta$, therefore $b'(\tau_n, Y(\tau_n)) = \beta$.

The Milstein scheme with the same step size $\Delta t = 2^{-5}$ gives a much better approximation to the true solution than does the Euler scheme with the same step size. The error for the Milstein scheme is 5.8242 while the error is 7.5709 for the Euler scheme when $N = 2^5$. Figure 6.5 shows the Milstein scheme approximation to the exact solution when $N = 2^5$.

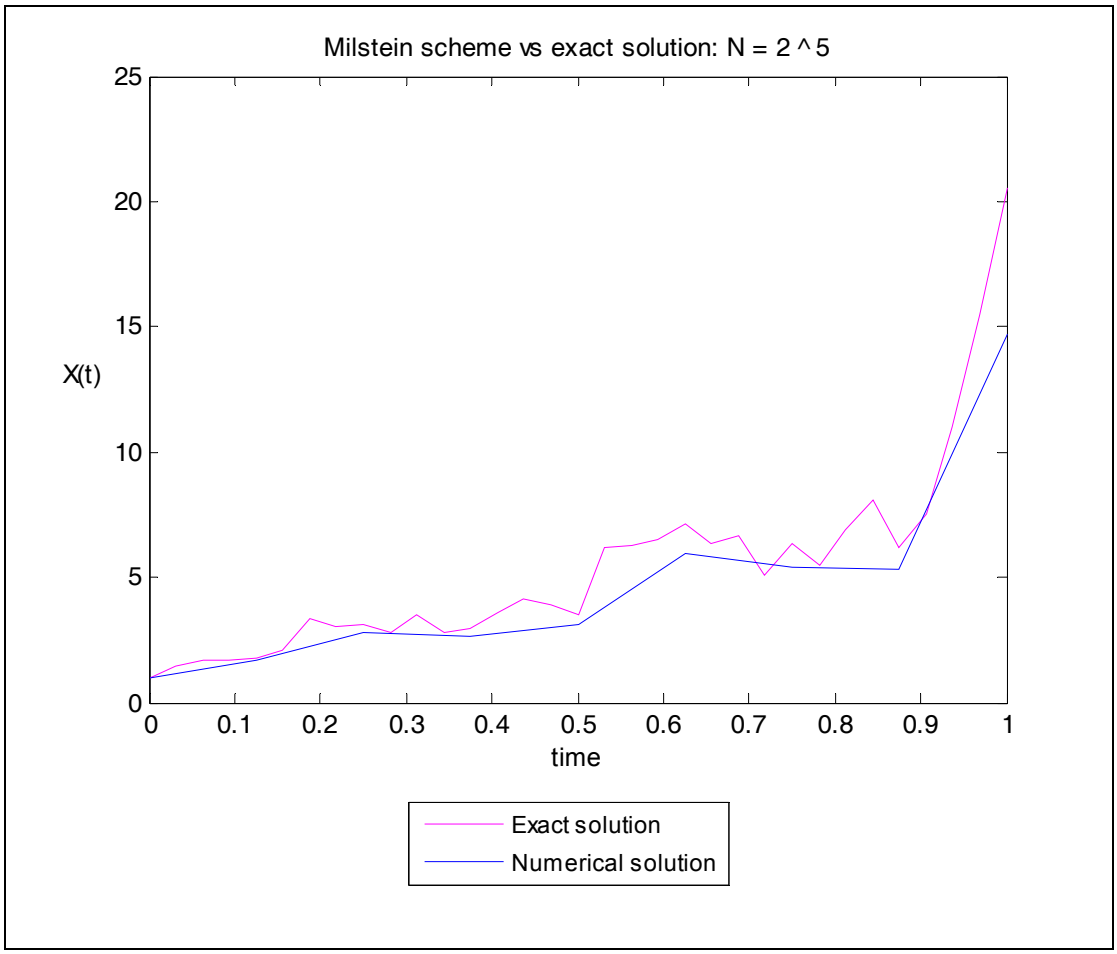

Figure 6.5: Exact solution and Milstein scheme with N = 2^5

A much better result is obtain when the number of time steps is increased to $N = 2^{10}$. This is depicted in Figure 6.6. The error using the Milstein approximation is 0.0068 when $\Delta t = 2^{-10}$. The Milstein scheme is therefore an improved method over the Euler method, decreasing the error of the Euler scheme by approximately 65% when $N = 2^{10}$.

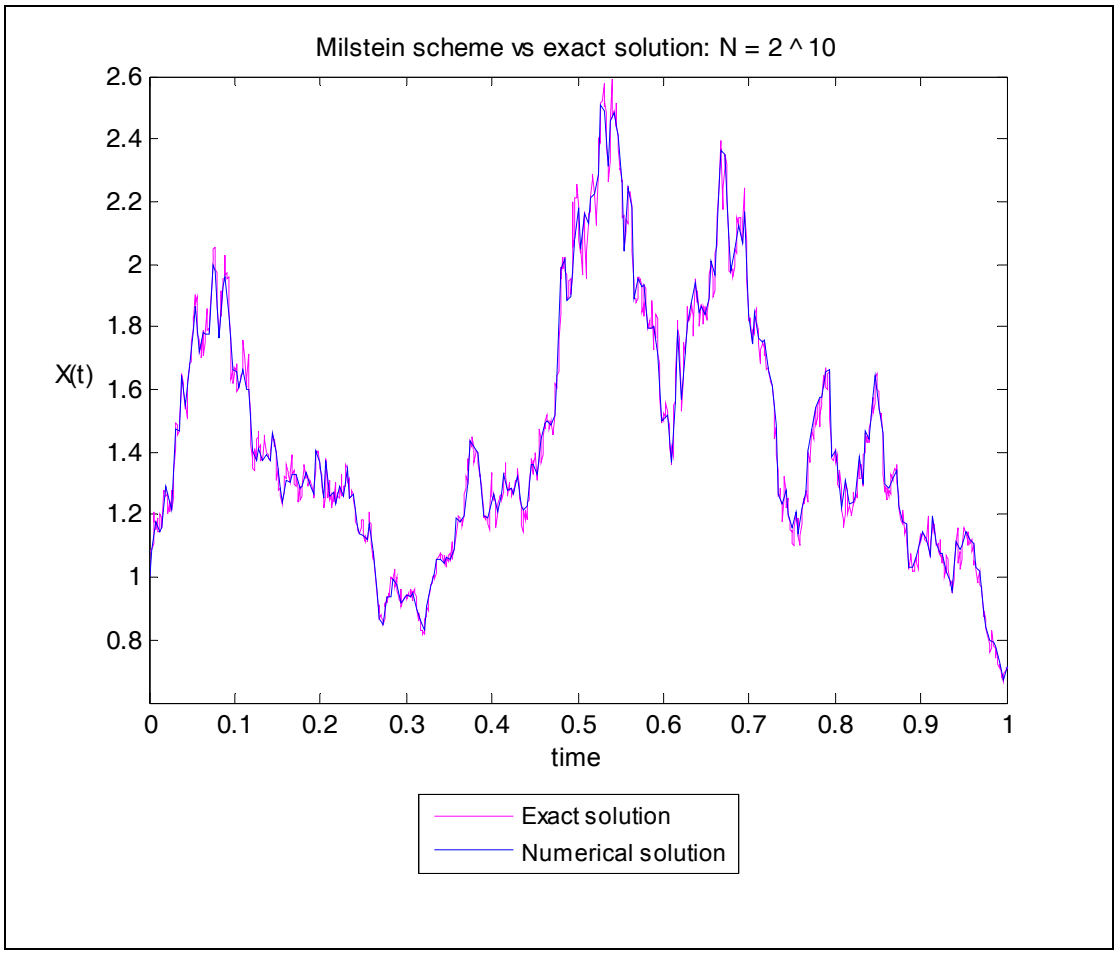

Figure 6.6: Exact solution and Milstein scheme with N = 2^10

The size of N is varied from $N = 2^5$ to $N = 2^{14}$ and again, the corresponding errors decrease as N increases (Table 6.2).

Milstein Scheme	
N	**Error**
2 ^ 5	5.8242
2 ^ 6	3.4856
2 ^ 7	0.0786
2 ^ 8	0.0064
2 ^ 9	0.0241
2 ^ 10	0.0068
2 ^ 11	0.0025
2 ^ 12	0.0023
2 ^ 13	0.0011
2 ^ 14	7.3684E-04

Table 6.2: Errors generated by the Milstein scheme

When $N = 2^{14}$, the numerical scheme traces the exact solution very closely, which is clearly visible in Figure 6.7.

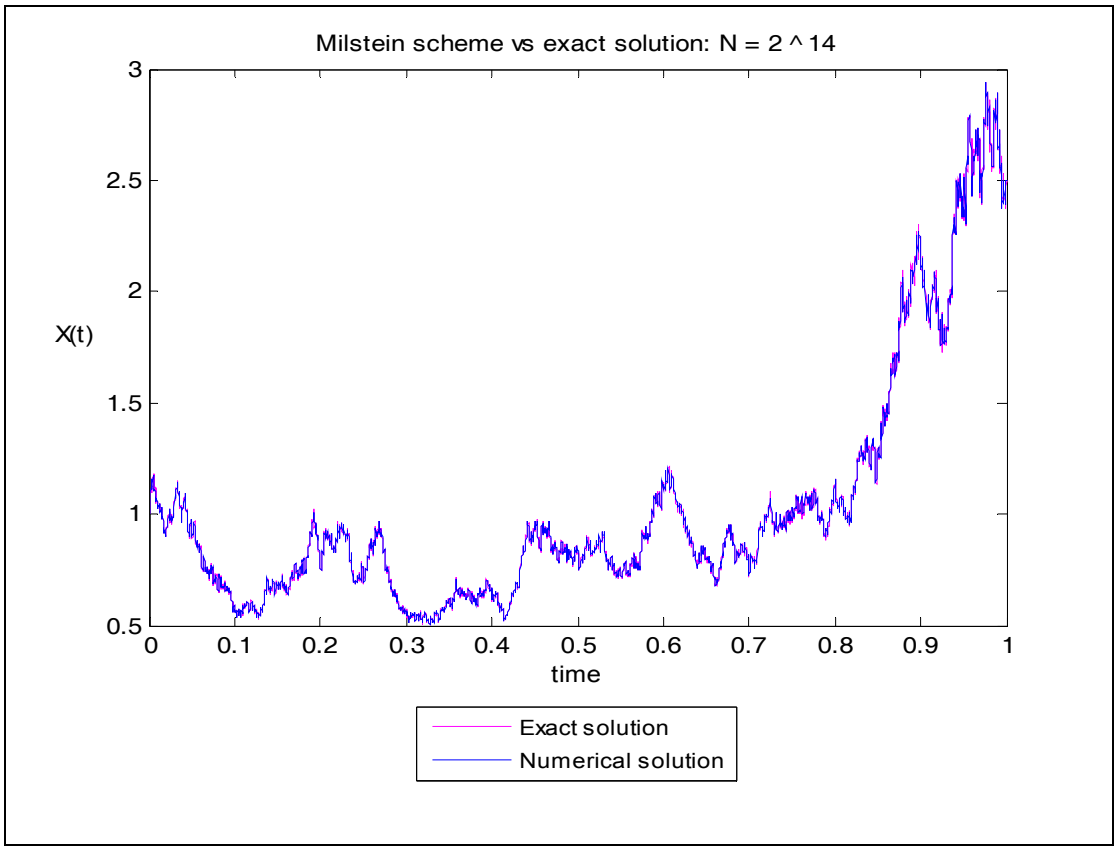

Figure 6.7: Milstein scheme and exact solution when N = 2^14

6.3 Explicit and implicit strong convergence schemes

The following integration schemes satisfying the strong convergence criterion are presented:

- Explicit order 1 strong scheme
- Implicit Euler scheme
- Implicit Milstein schemes

(i) Explicit order 1 strong schemes

$$Y_{n+1} = Y_n + a(\tau_n, Y_n)\,\Delta n + b(\tau_n, Y_n)\,\Delta W_n + \frac{1}{2\sqrt{\Delta n}}\left\{ b\left(\tau_n, \overline{Y_n}\right) - b(\tau_n, Y_n)\right\}\left\{(\Delta W_n)^2 - \Delta n\right\}, \qquad (6.6)$$

where:

$$\overline{Y_n} = Y_n + a(\tau_n, Y_n)\;\Delta n + b(\tau_n, Y_n)\sqrt{\Delta n}\,.$$

For example (6.3),

$$(\tau_{n+1} - \tau_n) = \Delta n\,,$$

$$\left(W_{\tau_{n+1}} - W_{\tau_n}\right) = \Delta W_n\,,$$

$$a(\tau_n, Y_n) = \alpha Y_n\,,$$

$$b(\tau_n, Y_n) = \beta Y_n\,,$$

$$b\left(\tau_n, \overline{Y_n}\right) = \beta \overline{Y_n}\,,$$

$$\overline{Y_n} = Y_n + \alpha Y_n\,\Delta n + \beta Y_n\,\sqrt{\Delta n}\,,$$

and

$$\begin{aligned} b\left(\tau_n, \overline{Y_n}\right) - b\left(\tau_n, Y_n\right) &= \beta \overline{Y_n} - \beta Y_n \\ &= \beta\left(\overline{Y_n} - Y_n\right) \\ &= \beta\left(Y_n + \alpha Y_n \Delta n + \beta Y_n \sqrt{\Delta n} - Y_n\right) \\ &= \beta Y_n\left(\alpha \Delta n + \beta \sqrt{\Delta n}\right). \end{aligned}$$

Therefore, equation (6.6) becomes:

$$Y_{n+1} = Y_n + \alpha Y_n \Delta_n + \beta Y_n \Delta W_n + \frac{1}{2\sqrt{\Delta_n}}\left\{\beta Y_n\left(\alpha \Delta_n + \beta\sqrt{\Delta_n}\right)\right\}\left\{\left(\Delta W_n\right)^2 - \Delta_n\right\}.$$

Figure 6.8 shows the explicit order 1 strong scheme approximation to the exact solution, when $N = 2^5$ and the error is 5.1262.

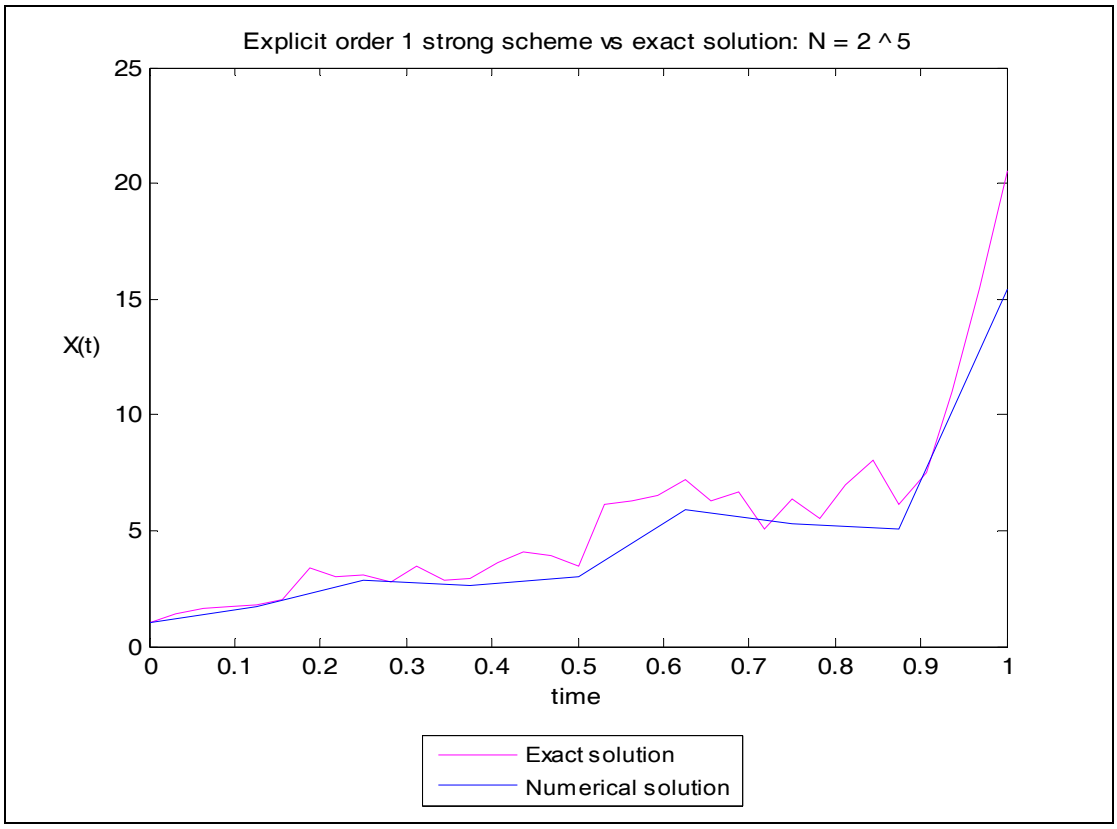

Figure 6.8: Explicit order 1.0 strong scheme and exact solution when N = 2^5

The error is 0.0041 when $N = 2^8$; a marked improvement over both the Euler and Milstein Schemes.

When $N = 2^{14}$, the error is 0.0023 and the numerical scheme is a very close approximation to the exact solution. This is clearly visible in Figure 6.9.

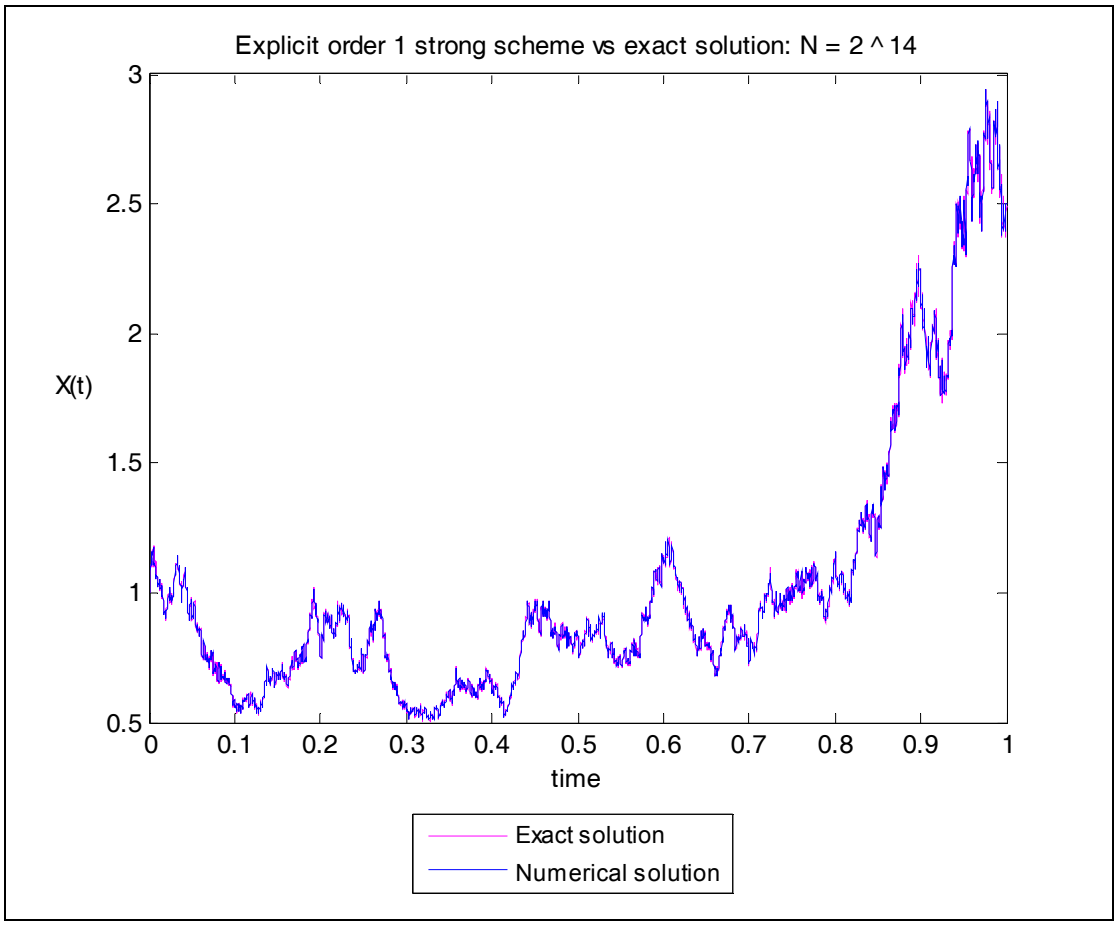

Figure 6.9: Explicit order 1.0 strong scheme and exact solution when N = 2^14

Table 6.3 provides errors by varying the number of time steps from $N = 2^5$ to $N = 2^{14}$. The errors that are produced generally decrease as N increases.

Explicit Order 1 Strong Scheme	
N	**Error**
2 ^ 5	5.1262
2 ^ 6	2.8150
2 ^ 7	0.0996
2 ^ 8	0.0041
2 ^ 9	0.0500
2 ^ 10	0.0092
2 ^ 11	0.0032
2 ^ 12	0.0052
2 ^ 13	0.0027
2 ^ 14	0.0023

Table 6.3: Errors generated by the explicit order 1.0 strong scheme

(ii) Implicit Euler scheme

In the implicit Euler scheme, only the drift term is written in implicit form:

$$Y_{n+1} = Y_n + a\left(\tau_{n+1}, Y_{n+1}\right)\Delta t + b\left(\tau_n, Y_n\right)\Delta W .$$

For example (6.3), $dX_t = \alpha X_t\, dt + \beta X_t\, dW_t$, the implicit Euler scheme is:

$$Y_{n+1} = Y_n + \alpha Y_{n+1}\Delta t + \beta Y_n \Delta W , \qquad (6.7)$$

since

$$a\left(\tau_{n+1}, Y_{n+1}\right) = \alpha Y_{n+1} .$$

Equation (6.7) implies that:

$$Y_{n+1} - \alpha Y_{n+1}\Delta t = Y_n + \beta Y_n \Delta W$$

$$\Rightarrow \qquad Y_{n+1} = \frac{Y_n\left(1+\beta\,\Delta W\right)}{\left(1-\alpha\,\Delta t\right)}.$$

Figure 6.10 shows the implicit Euler scheme approximation to the exact solution, when $N = 2^5$ and the error is 0.6587, which is lower than the errors produced by Euler scheme, Milstein scheme and the explicit order 1 strong scheme for $N = 2^5$.

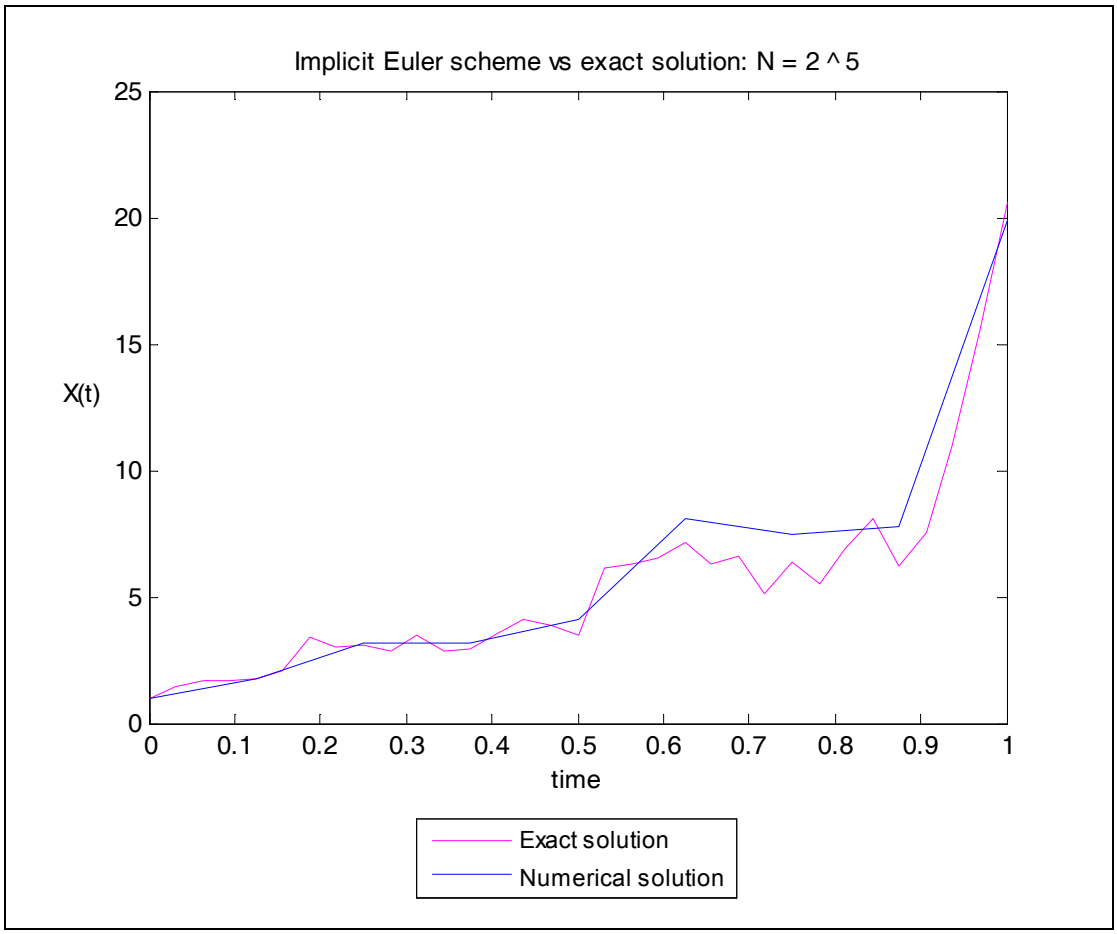

Figure 6.10: Implicit Euler scheme and exact solution when N = 2^5

Table 6.4 provides the errors when the number of time steps is increased from $N = 2^5$ to $N = 2^{14}$.

Implicit Euler Scheme	
N	**Error**
2 ^ 5	0.6587
2 ^ 6	1.4605
2 ^ 7	0.6119
2 ^ 8	0.0227
2 ^ 9	0.2174
2 ^ 10	0.0232
2 ^ 11	0.0103
2 ^ 12	0.0592
2 ^ 13	0.0461
2 ^ 14	0.0666

Table 6.4: Errors generated by the implicit Euler scheme

Figure 6.11 depicts the numerical approximation to the exact solution with an error of 0.0666, when $N = 2^{14}$. Again, the numerical scheme is a very close approximation to the exact solution.

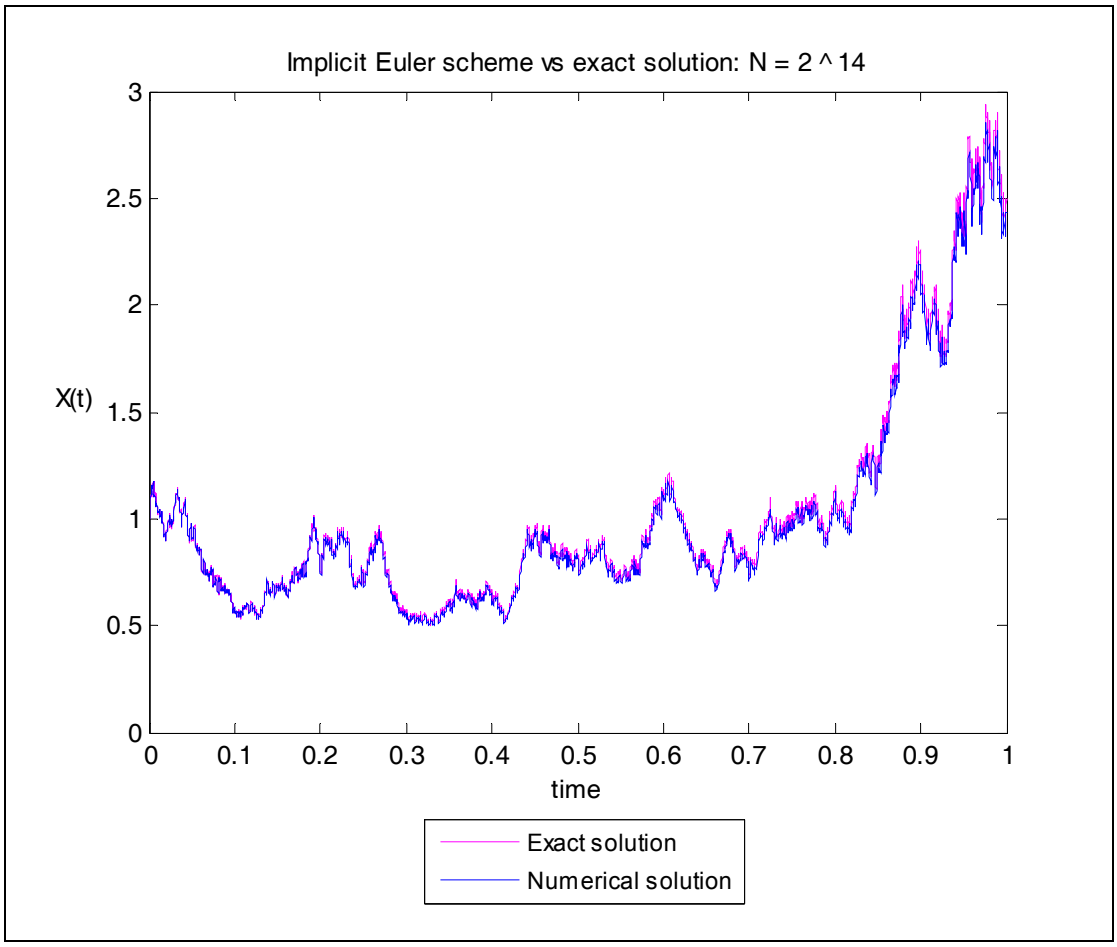

Figure 6.11: Implicit Euler scheme and exact solution when N = 2^14

(iii) Implicit Milstein scheme

The implicit Milstein scheme is defined as:

$$Y_{n+1} = Y(\tau_n) + a\left(\tau_{n+1}, Y(\tau_{n+1})\right)(\tau_{n+1} - \tau_n) + b\left(\tau_n, Y(\tau_n)\right)\left(W_{\tau_{n+1}} - W_{\tau_n}\right) +$$
$$\frac{1}{2} b\left(\tau_n, Y(\tau_n)\right) b'\left(\tau_n, Y(\tau_n)\right)\left\{\left(W_{\tau_{n+1}} - W_{\tau_n}\right)^2 - (\tau_{n+1} - \tau_n)\right\},$$

where again, only the drift term is depends on Y_{n+1}. For example (6.3), the implicit Milstein scheme is:

$$Y_{n+1} = Y_n + \alpha Y_{n+1}\Delta t + \beta Y_n \Delta W_t + \frac{1}{2}\beta^2 Y_n \left\{(\Delta W_t)^2 - \Delta t\right\}$$

$$\Rightarrow \quad Y_{n+1} - \alpha Y_{n+1}\Delta t = Y_n + \beta Y_n \Delta W_t + \frac{1}{2}\beta^2 Y_n \left\{(\Delta W_t)^2 - \Delta t\right\}$$

$$\Rightarrow \quad Y_{n+1}(1-\alpha\Delta t) = Y_n + \beta Y_n \Delta W_t + \frac{1}{2}\beta^2 Y_n \left\{(\Delta W_t)^2 - \Delta t\right\}$$

$$\Rightarrow \quad Y_{n+1} = \left(Y_n + \beta Y_n \Delta W_t + \frac{1}{2}\beta^2 Y_n \left\{(\Delta W_t)^2 - \Delta t\right\}\right)/(1-\alpha\Delta t).$$

Figure 6.12 shows the implicit Milstein scheme approximation to the exact solution, when $N = 2^5$ and the error is 1.7499, which is lower than the errors produced by Euler scheme, Milstein scheme and the explicit order 1 strong scheme for $N = 2^5$, but the error is not lower than the error of the implicit Euler scheme.

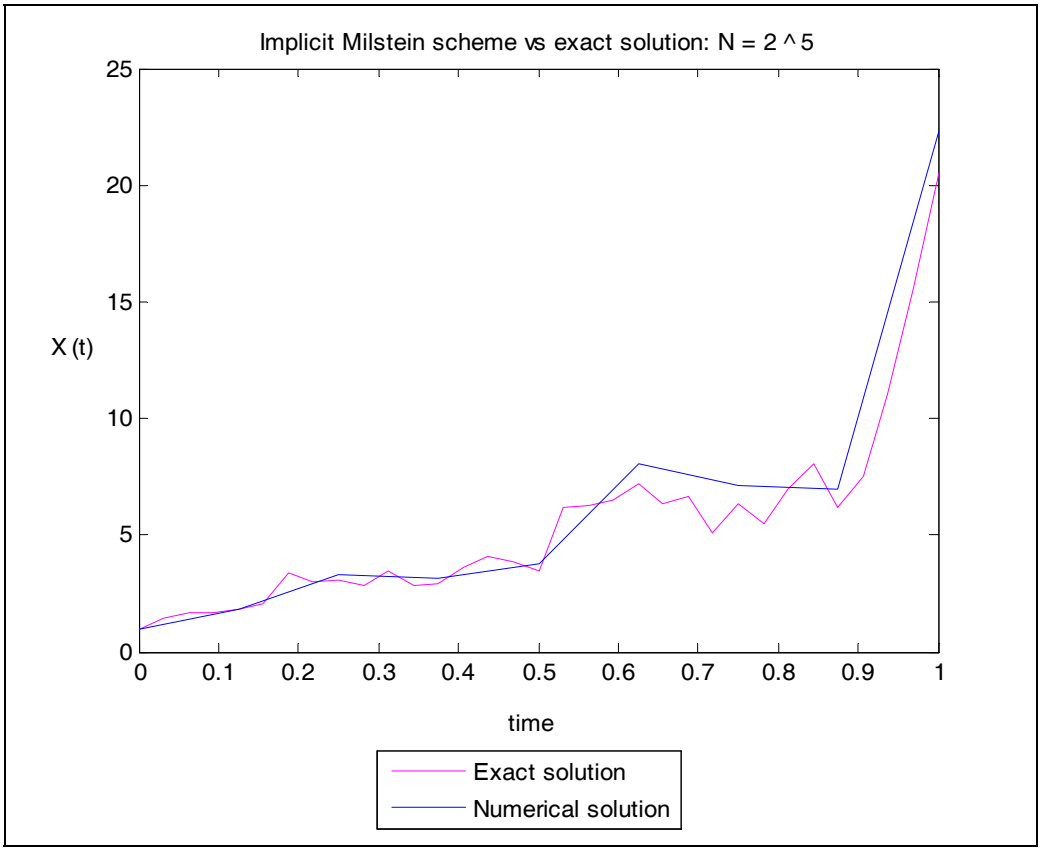

Figure 6.12: Implicit Milstein scheme and exact solution when N = 2^5

Table 6.5 provides the errors when the number of time steps is increased from $N = 2^5$ to $N = 2^{14}$.

Implicit Milstein Scheme	
N	**Error**
2 ^ 5	1.7499
2 ^ 6	1.0197
2 ^ 7	0.1247
2 ^ 8	0.0198
2 ^ 9	0.0210
2 ^ 10	0.0032
2 ^ 11	8.6286E-04
2 ^ 12	0.0014
2 ^ 13	7.2974E-04
2 ^ 14	0.0011

Table 6.5: Errors generated by the implicit Milstein scheme

The errors (for the associated time steps) are generally lower than the errors produced by the implicit Euler scheme.

The implicit Milstein approximation and the exact solution for $N = 2^{14}$ is provided in Figure 6.13.

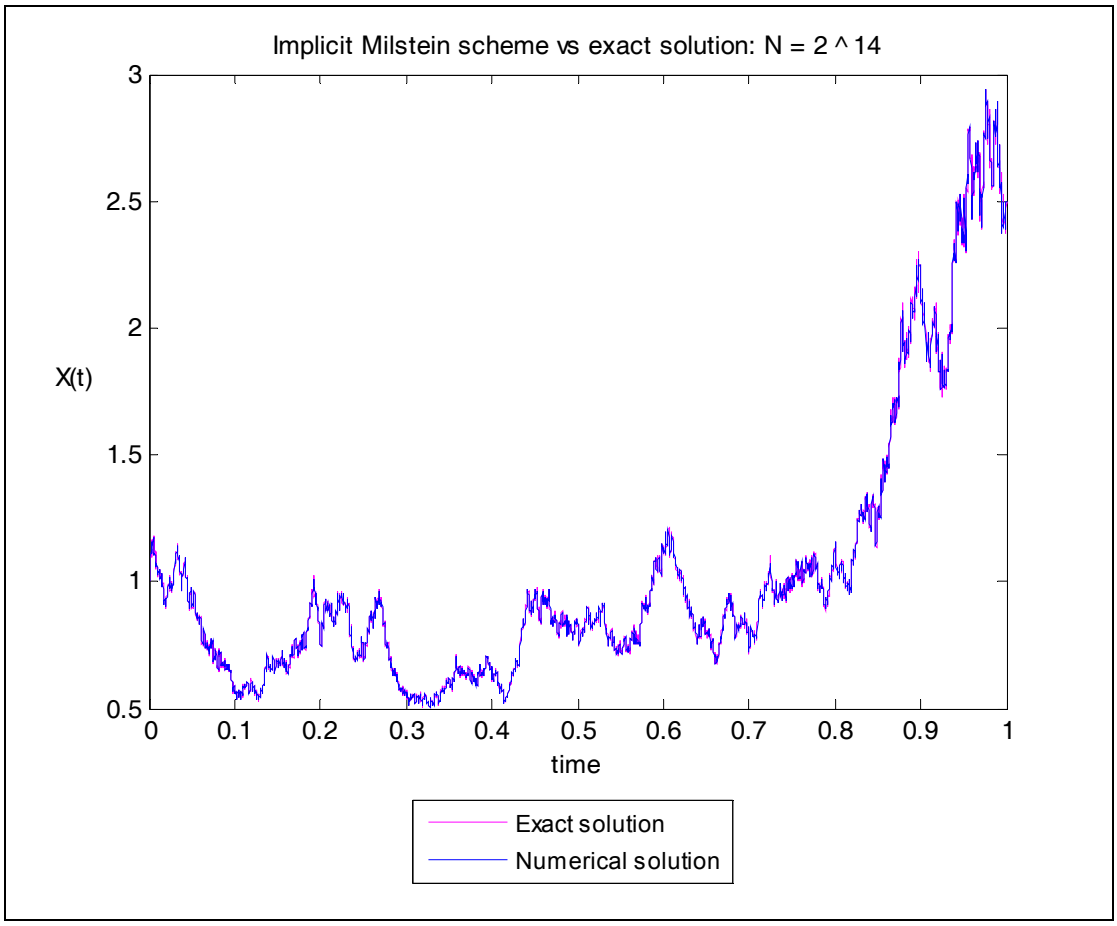

Figure 6.13: Implicit Milstein scheme and exact solution when N = 2^14

6.4 Weak convergence schemes

The order 2.0 weak Taylor scheme is defined as follows:

$$Y_{n+1} = Y_n + a\,\Delta n + b\,\Delta W_n + \frac{1}{2}bb'\left\{(\Delta W_n)^2 - \Delta n\right\}$$

$$+ a'b\,\Delta Z_n + \frac{1}{2}\left(aa' + \frac{1}{2}a''b^2\right)(\Delta n)^2$$

$$+ \left(ab' + \frac{1}{2}b''b^2\right)\left\{\Delta W_n\,\Delta n - \Delta Z_n\right\}, \qquad (6.8)$$

where ΔZ_n is the following double integral:

$$\Delta Z_n = \int_{\tau_n}^{\tau_{n+1}} \int_{\tau_n}^{s_2} dW_{s_1}\, ds_2 ,$$

with the following properties:

$$E\left(\Delta Z_n\right) = 0 , \tag{6.9}$$

$$Var\left(\Delta Z_n\right) = \frac{1}{3}\left(\Delta n\right)^3 , \tag{6.10}$$

$$Cov\left(\Delta Z_n\, \Delta W_n\right) = \frac{1}{2}\left(\Delta n\right)^2 . \tag{6.11}$$

For the example (6.3):

$$a\left(\tau_n, Y_n\right) = \alpha Y_n ,$$

$$b\left(\tau_n, Y_n\right) = \beta Y_n ,$$

$$a'\left(\tau_n, Y_n\right) = \alpha ,$$

$$b'\left(\tau_n, Y_n\right) = \beta ,$$

$$a''\left(\tau_n, Y_n\right) = 0 ,$$

$$b''\left(\tau_n, Y_n\right) = 0 .$$

Therefore the order 2.0 weak Taylor scheme (6.8) for example (6.3) is:

$$\begin{aligned} Y_{n+1} \quad &= Y_n + \alpha Y_n \Delta n + \beta Y_n \Delta W_n + \frac{1}{2}\beta Y_n \beta\left\{\left(\Delta W_n\right)^2 - \Delta n\right\} \\ &+ \alpha \beta Y_n \Delta Z_n + \frac{1}{2}\left(\alpha Y_n \alpha\right)\left(\Delta n\right)^2 + \left(\alpha Y_n \beta\right)\left\{\Delta W_n \Delta n - \Delta Z_n\right\} . \end{aligned}$$

In order to generate ΔZ_n with properties (6.9) – (6.11), let G_1 and G_2 be two independent standard normally distributed random variables which can be easily generated in Matlab with the command **randn**. Then the following definitions of ΔW_n and ΔZ_n will have the desired properties (6.9) – (6.11) (Kloeden and Platen, 1992):

$$\Delta W_n = \sqrt{\Delta n}\, G_1 \tag{6.12}$$

$$\Delta Z_n = \frac{1}{2}(\Delta n)^{3/2}\left(G_1 + \frac{1}{\sqrt{3}}\, G_2\right). \tag{6.13}$$

Therefore:

$$Var\left(\frac{1}{2}(\Delta n)^{3/2} G_1 + \frac{1}{2}(\Delta n)^{3/2}\frac{1}{\sqrt{3}}\, G_2\right) = \frac{1}{4}(\Delta n)^3 \operatorname{var}(G_1) + \frac{1}{4}(\Delta n)^3 \frac{1}{3}\operatorname{var}(G_2)$$

$$= \frac{1}{4}(\Delta n)^3 (1) + \frac{1}{4}(\Delta n)^3 \frac{1}{3}(1)$$

$$= \frac{1}{3}(\Delta n)^3,$$

since G_1 and G_2 are independent. Similarly:

$$Cov(\Delta Z_n\, \Delta W_n) = \frac{1}{2}(\Delta n)^2.$$

Thus ΔZ_n can be easily generated by specifying its functional form as equation (6.13) in Matlab. Note that equation (6.12) has been used previously to generate the Brownian path in the first section of this chapter.

Figure 6.14 depicts a comparison of the numerical approximation to the exact solution when $N = 2^5$. The associated error 2.1735. The order 2 weak Taylor scheme produces smaller errors than the Euler scheme, Milstein scheme and explicit order 1 strong scheme when $N = 2^5$.

However, the implicit Euler and implicit Milstein schemes perform much better than the order 2 weak Taylor scheme. This can be explained by the fact that the strong schemes provide a good pathwise approximation while the weak schemes, some functional of the Itô process is of interest.

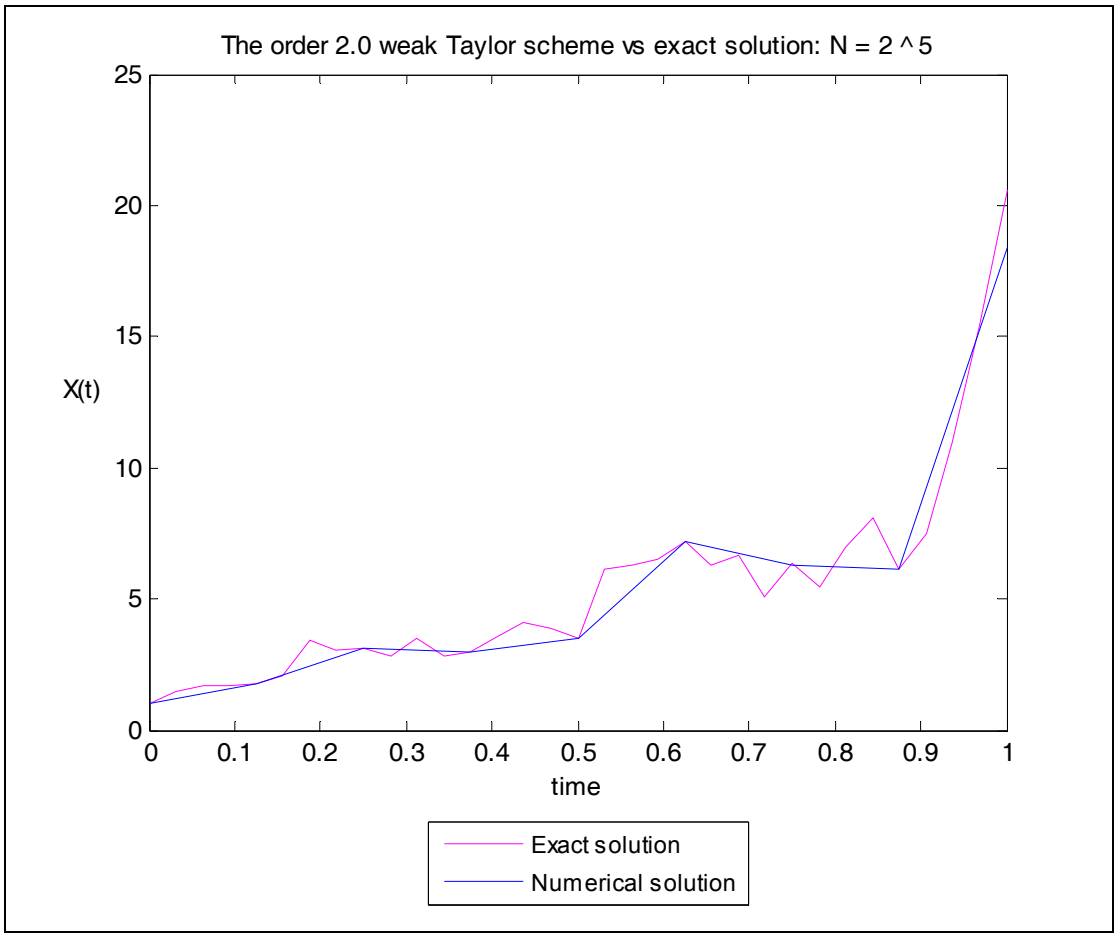

Figure 6.14: The order 2.0 weak Taylor scheme and exact solution when N = 2^5

When $N = 2^{10}$, the error is very small at 6.6027e-05. A comparison of the numerical scheme and exact solution when $N = 2^{10}$ is depicted in Figure 6.15. Just as in the case with the strong schemes, the order 2.0 weak Taylor scheme provides a very close approximation to the exact solution.

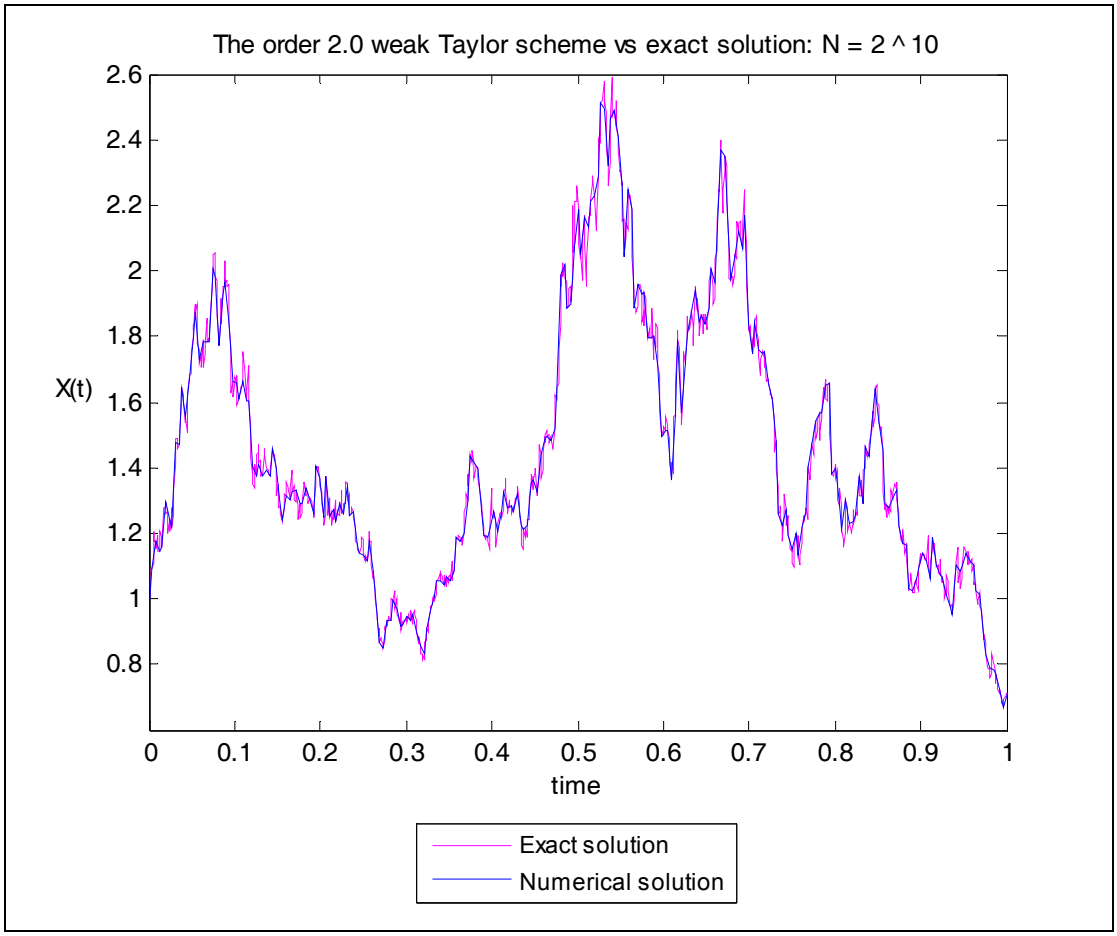

Figure 6.15: The order 2.0 weak Taylor scheme and exact solution when N = 2^10

The number of discretisation points was changed from $N = 2^5$ to $N = 2^{14}$, and the corresponding errors were obtained. The errors are provided in Table 6.6.

Order 2 Weak Taylor Scheme	
N	**Error**
2 ^ 5	2.1735
2 ^ 6	0.9191
2 ^ 7	0.0778
2 ^ 8	0.0173
2 ^ 9	0.0127
2 ^ 10	6.6027E-05
2 ^ 11	4.0001E-05
2 ^ 12	3.4032E-04
2 ^ 13	1.2792E-04
2 ^ 14	3.8124E-04

Table 6.6: Errors generated by the order 2.0 weak Taylor scheme

The order 2.0 weak Taylor scheme is a very close approximation to the exact solution for $N = 2^{14}$ (Figure 6.16).

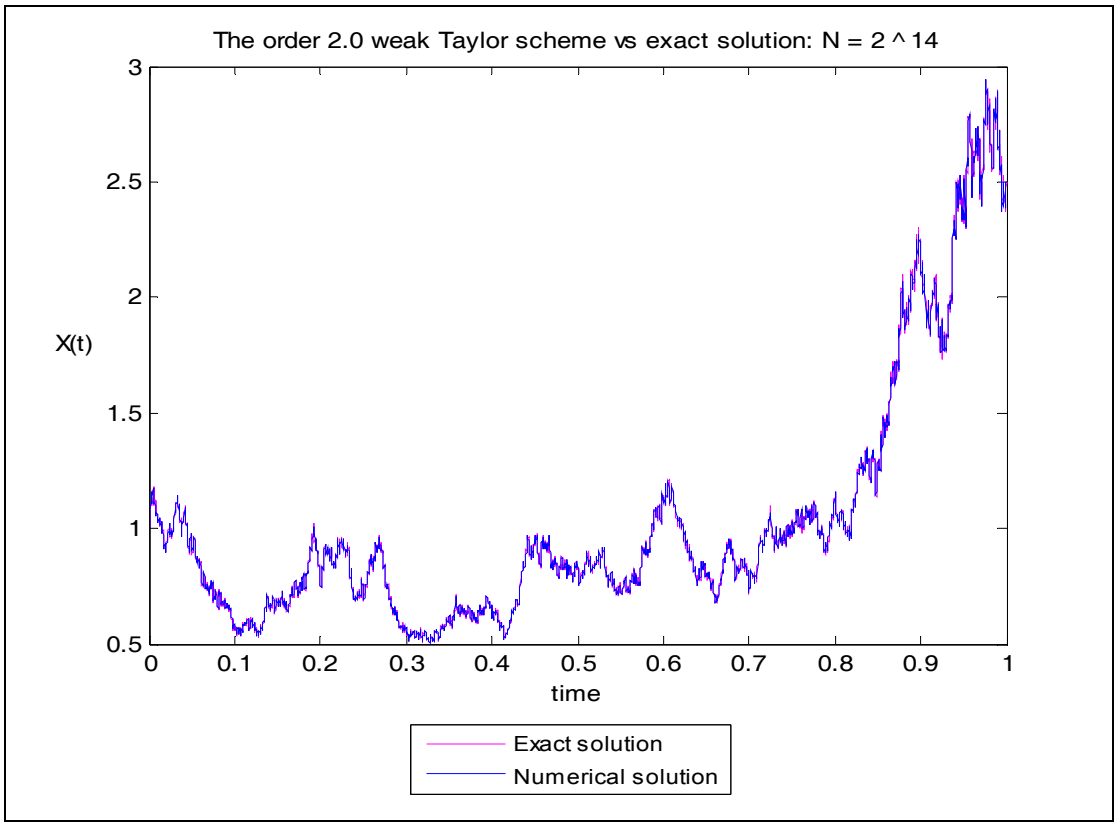

Figure 6.16: The order 2.0 weak Taylor scheme and exact solution when N = 2^14

Having specified the variable ΔZ_n with properties (6.9) – (6.11), it would then become fairly straightforward to implement numerical schemes for the explicit and implicit weak approximations.

6.5 Strong numerical schemes – SDE with additive noise

The Ornstein-Uhlenbeck process is an example of a stochastic differential equation with additive noise in that the co-efficient of the diffusion term is independent of the state variable. The Ornstein-Uhlenbeck process is a mean-reverting process (Hughston, 1996). The Vasicek model and the Cox-Ingersoll-Ross model are commonly used stochastic models for the term structure of interest rates and these models are based on the stochastic process being modelled by an Ornstein-Uhlenbeck process.

The Ornstein-Uhlenbeck process has the following form (Kloeden*, et al,* 1994):

$$X_t = X_0 - \int_0^t \alpha\left(X_s - \mu\right) ds + \int_0^t \beta\, dW_s \,.$$

Assume that the mean $\mu = 0$. The above stochastic integral equation is written symbolically as the following stochastic differential equation:

$$dXt = -\alpha\, X_t\, dt + \beta\, dW_t \,. \tag{6.14}$$

Using Itô's lemma, the Ornstein-Uhlenbeck process has the explicit solution (Kloeden*, et al,* 1994):

$$X_t = e^{-at} X_0 + e^{-at} \int_0^t e^{as} b\, dW_s \,.$$

The above solution is then compared to the numeric solution, using the different numerical schemes. This section provides results of the strong schemes when applied to the Ornstein-Uhlenbeck process.

i. Euler scheme

The Euler scheme is given as:

$$Y_{n+1} = Y(\tau_n) + a(\tau_n, Y(\tau_n))(\tau_{n+1} - \tau_n) + b(\tau_n, Y(\tau_n))(W_{\tau_{n+1}} - W_{\tau_n}).$$

For the Ornstein-Uhlenbeck process (6.14), we have:

$$a(\tau_n, Y(\tau_n)) = -\alpha Y_n,$$
$$b(\tau_n, Y(\tau_n)) = \beta.$$

Therefore, the Euler scheme for the Ornstein-Uhlenbeck process is:

$$Y_{n+1} = Y_n - \alpha Y_n \Delta n + \beta \Delta W,$$

where:

$$Y_n = Y(\tau_n),$$
$$(\tau_{n+1} - \tau_n) = \Delta n,$$
$$(W_{\tau_{n+1}} - W_{\tau_n}) = \Delta W.$$

Figure 6.17 shows the Euler scheme approximation to the exact solution for the Ornstein-Uhlenbeck process, when $N = 2^{10}$. The associated error is 0.2361.

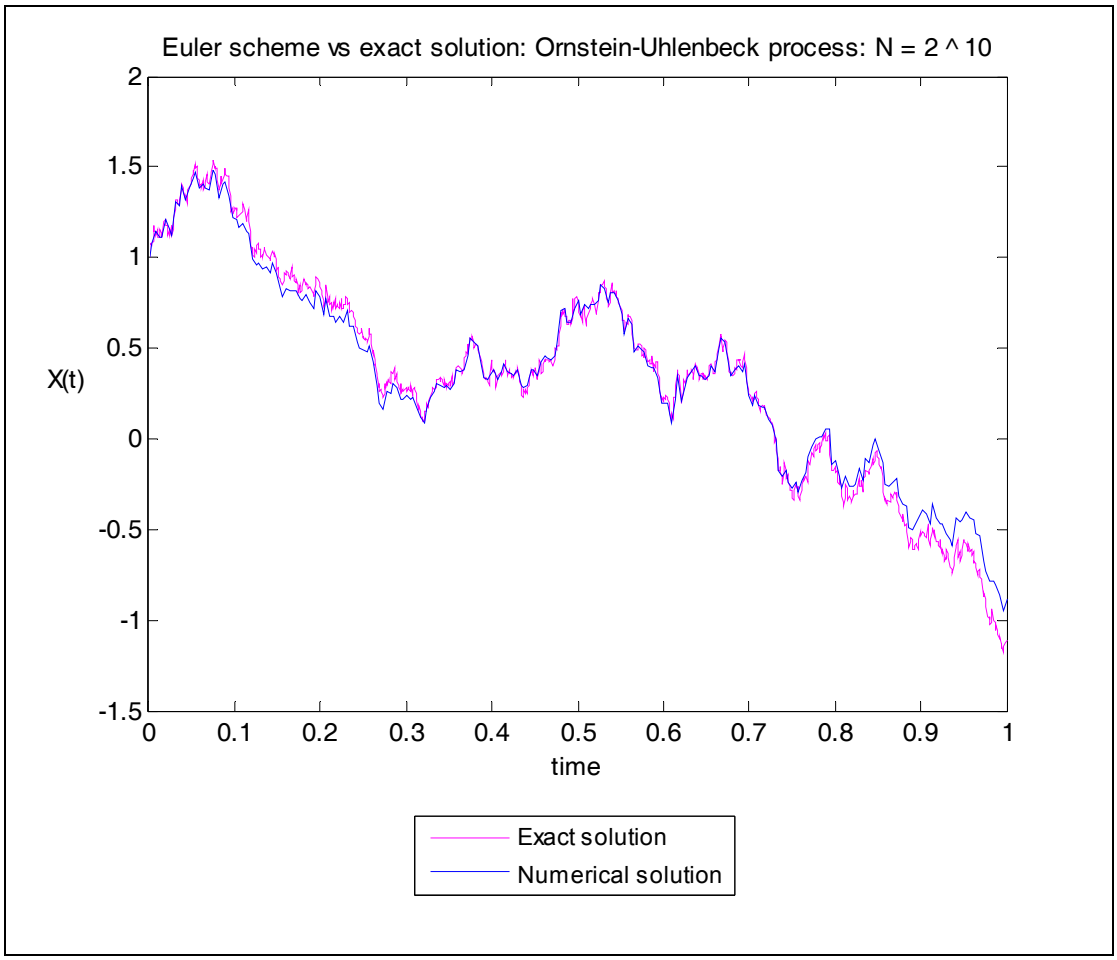

Figure 6.17: Euler scheme and exact solution: Ornstein-Uhlenbeck process when N = 2^10

The error produced by the Euler scheme for the Ornstein-Uhlenbeck process is much higher than the error produced for the SDE with multiplicative noise, for $N = 2^{10}$.

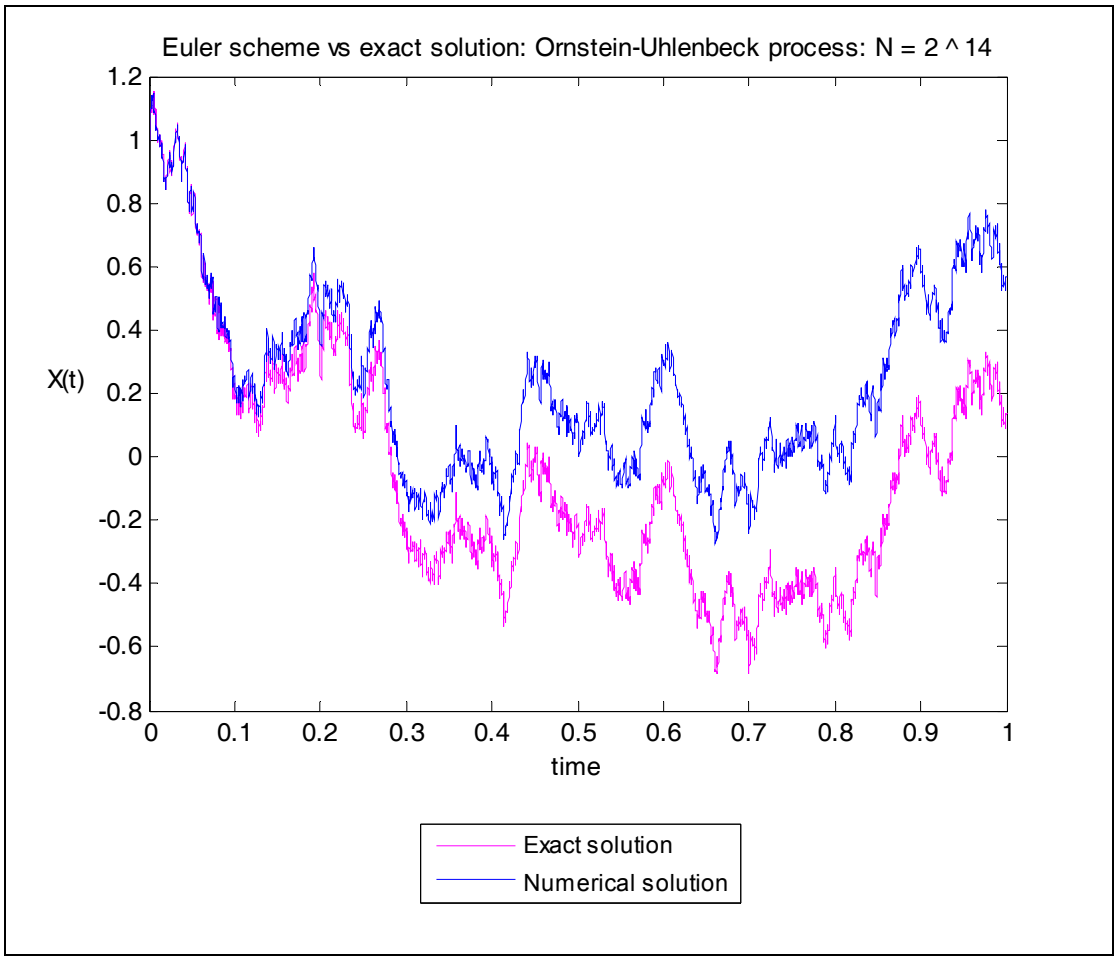

Figure 6.18: Euler scheme and exact solution: Ornstein-Uhlenbeck process when N = 2^14

Figure 6.18 shows that the numerical scheme (for $N = 2^{14}$) is not a very close approximation to the exact solution, especially when T > 0.3. Although the numerical solution traces the exact solution, it does not overlap the exact solution.

As before, N is increased from $N = 2^5$ to $N = 2^{14}$ and the associated errors are recorded in Table 6.7.

Euler Scheme		
N	Error	
	Multiplicative Noise	Ornstein-Uhlenbeck process
2 ^ 5	7.5709	0.7611
2 ^ 6	5.4325	1.1823
2 ^ 7	0.7487	0.9181
2 ^ 8	0.0094	0.2749
2 ^ 9	0.1677	0.0524
2 ^ 10	0.0195	0.2361
2 ^ 11	0.0087	0.8817
2 ^ 12	0.0582	0.6455
2 ^ 13	0.0457	0.6995
2 ^ 14	0.0670	0.4289

Table 6.7: Comparison of errors: SDE with multiplicative noise and Ornstein-Uhlenbeck process

A comparison of the errors in Table 6.7 shows that the Euler scheme produces smaller errors for the SDE with multiplicative noise than the SDE with additive noise (i.e. the Ornstein-Uhlenbeck process).

ii. Milstein Scheme

The Milstein scheme is given by:

$$Y_{n+1} = Y(\tau_n) + a(\tau_n, Y(\tau_n))(\tau_{n+1} - \tau_n) + b(\tau_n, Y(\tau_n))(W_{\tau_{n+1}} - W_{\tau_n}) +$$

$$\frac{1}{2} b(\tau_n, Y(\tau_n)) b'(\tau_n, Y(\tau_n)) \{ (W_{\tau_{n+1}} - W_{\tau_n})^2 - (\tau_{n+1} - \tau_n) \}.$$

For the Ornstein-Uhlenbeck process,

$$dXt = -\alpha X_t \, dt + \beta \, dW_t ,$$

$b\left(\tau_n, Y\left(\tau_n\right)\right)=\beta$, and hence $b'\left(\tau_n, Y\left(\tau_n\right)\right)=0$. Therefore, the Milstein scheme reduces to the Euler scheme for the Ornstein-Uhlenbeck process.

iii. Explicit Order 1 Strong scheme

The explicit order 1 strong scheme is given by equation (6.6):

$$\begin{aligned} Y_{n+1} &= Y_n + a\left(\tau_n, Y_n\right)\Delta n + b\left(\tau_n, Y_n\right)\Delta W_n + \\ &\frac{1}{2\sqrt{\Delta n}}\left\{ b\left(\tau_n, \overline{Y_n}\right) - b\left(\tau_n, Y_n\right)\right\}\left\{\left(\Delta W_n\right)^2 - \Delta n\right\}, \end{aligned}$$

where:

$$\overline{Y_n} = Y_n + a\left(\tau_n, Y_n\right)\Delta n + b\left(\tau_n, Y_n\right)\sqrt{\Delta n}\,.$$

For the Ornstein-Uhlenbeck process,

$$\begin{aligned} a\left(\tau_n, Y_n\right) &= -\alpha Y_n, \\ b\left(\tau_n, Y_n\right) &= \beta\ , \\ b\left(\tau_n, \overline{Y_n}\right) &= \beta\overline{Y_n}\ , \\ \overline{Y_n} &= Y_n - \alpha Y_n\,\Delta n + \beta\sqrt{\Delta n}\ , \end{aligned}$$

and

$$\begin{aligned} b\left(\tau_n, \overline{Y_n}\right) - b\left(\tau_n, Y_n\right) &= \beta\overline{Y_n} - \beta \\ &= \beta Y_n - \alpha\beta\, Y_n\,\Delta n + \beta^2\sqrt{\Delta n} - \beta\ . \end{aligned}$$

Therefore the explicit order 1 strong scheme is:

$$Y_{n+1} = Y_n - \alpha Y_n \Delta n + \beta \Delta W_n + \frac{1}{2\sqrt{\Delta n}} \left\{ \beta Y_n - \alpha \beta Y_n \Delta n + \beta^2 \sqrt{\Delta n} - \beta \right\} \left\{ \left(\Delta W_n \right)^2 - \Delta n \right\}.$$

Figure 6.19 shows the explicit order 1.0 strong scheme approximation to the exact solution, when $N = 2^{10}$. The associated error is 0.0400. Again, the numerical scheme traces the exact solution, but it does not overlap the exact solution to provide a good approximation.

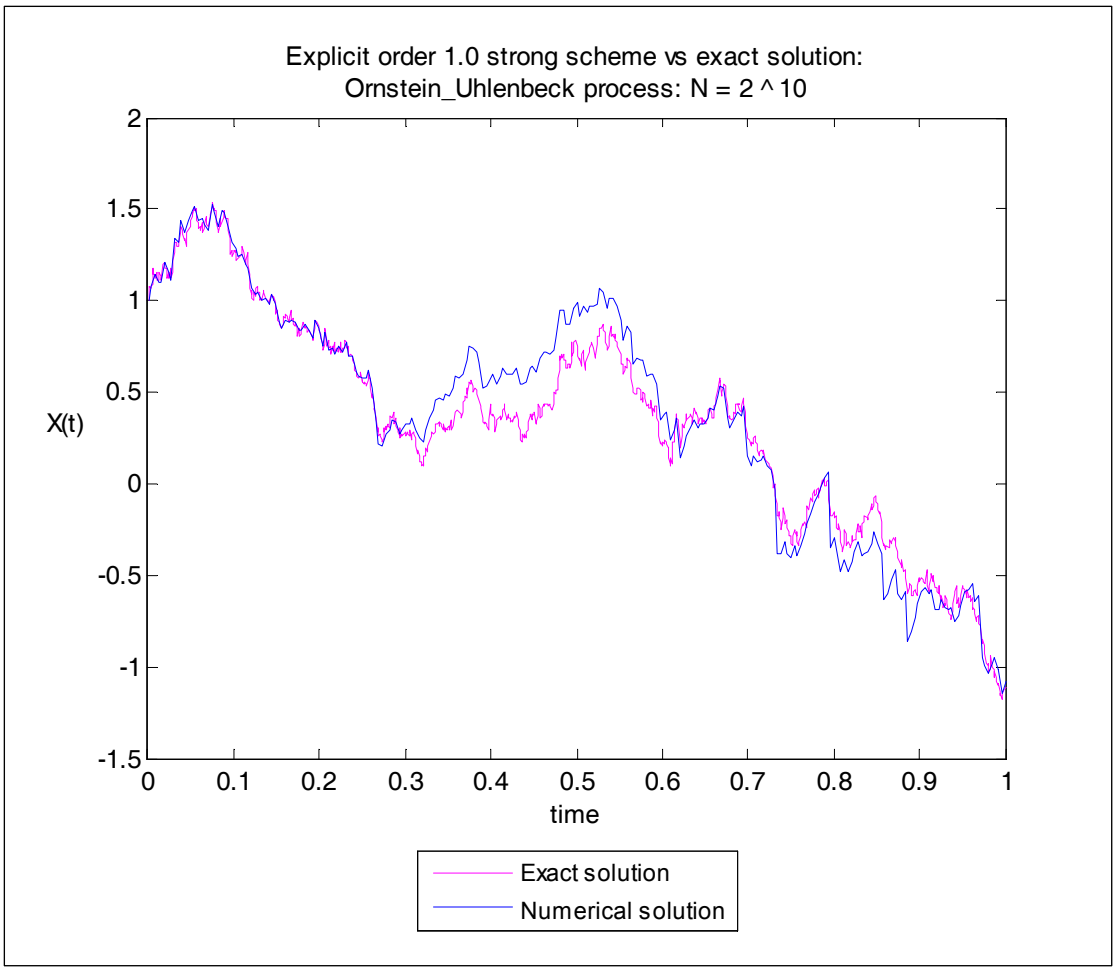

Figure 6.19: Explicit order 1.0 strong scheme and exact solution:
Ornstein-Uhlenbeck process when N = 2^10

The simulation is carried out for $N = 2^5$ and the number of time steps is increased to $N = 2^{14}$. The associated errors are recorded in Table 6.8. The errors do not, in general, decrease as the number of time steps is increased.

Explicit Order 1 Strong scheme	
N	**Error**
2 ^ 5	1.2113
2 ^ 6	1.1855
2 ^ 7	0.8197
2 ^ 8	0.0443
2 ^ 9	0.4011
2 ^ 10	0.0400
2 ^ 11	0.8699
2 ^ 12	1.9973
2 ^ 13	0.4754
2 ^ 14	0.2456

Table 6.8: Errors generated by the explicit order 1.0 strong scheme: Ornstein-Uhlenbeck process

iii. Implicit Euler Scheme

In this scheme, only the drift term depends on Y_{n+1} and not β, the co-efficient of the diffusion term. The Implicit Euler scheme is:

$$Y_{n+1} = Y_n + a\left(\tau_{n+1}, Y_{n+1}\right)\Delta t + b\left(\tau_n, Y_n\right)\Delta W .$$

For the Ornstein-Uhlenbeck process, the implicit Euler scheme is:

$$Y_{n+1} = Y_n - \alpha Y_{n+1}\Delta t + \beta\ \Delta W ,$$

since

$$a\left(\tau_{n+1}, Y_{n+1}\right) = -\alpha Y_{n+1},$$

$$b\left(\tau_n, Y_n\right) = \beta .$$

Thus, for the Ornstein-Uhlenbeck process, the implicit Euler reduces to:

$$\begin{aligned} Y_{n+1} + \alpha Y_{n+1} \Delta t &= Y_n + \beta \, \Delta W \\ \Rightarrow \quad Y_{n+1} &= \frac{Y_n + \beta \Delta W}{1 + \alpha \Delta t}. \end{aligned}$$

Figure 6.20 shows the implicit Euler scheme approximation to the exact solution for the Ornstein-Uhlenbeck process when $N = 2^{10}$ and the associated error is 0.2433.

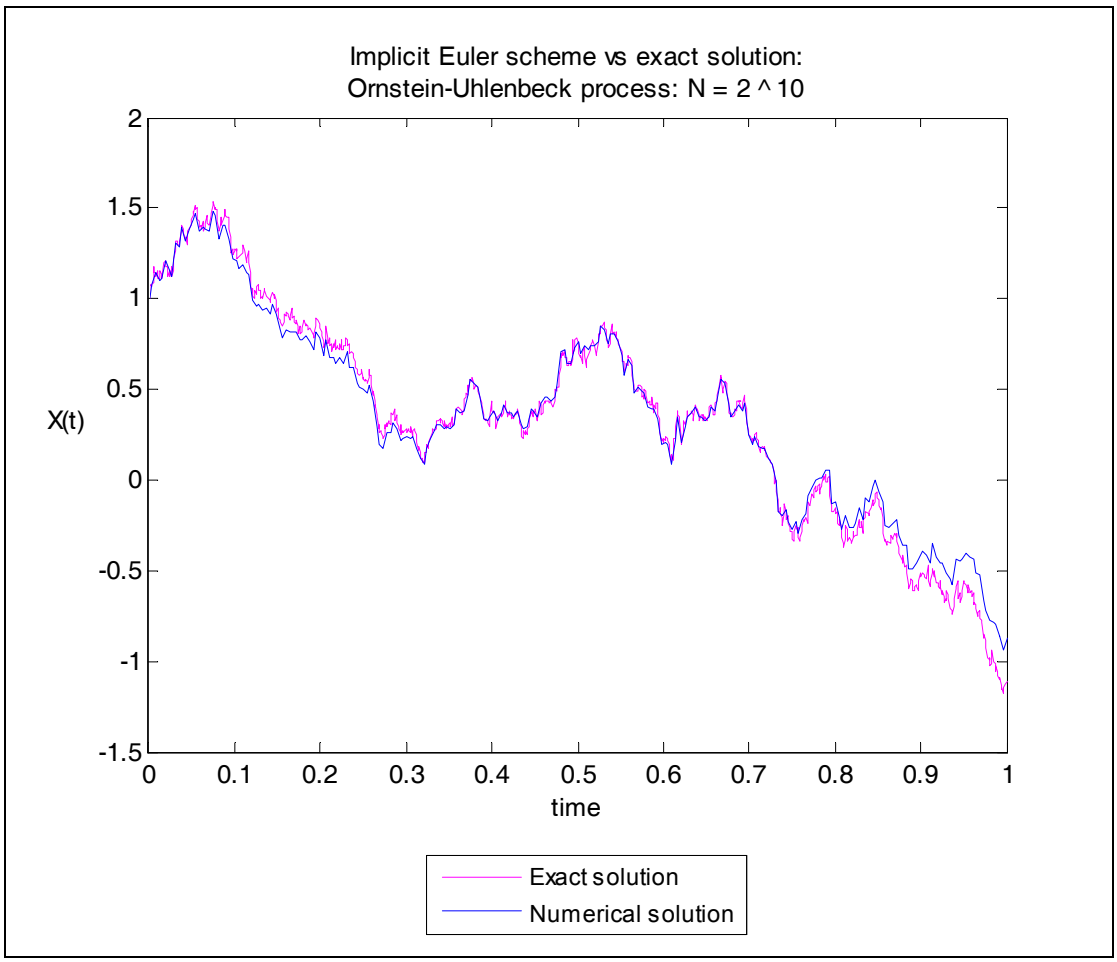

Figure 6.20: Implicit Euler scheme and exact solution:
Ornstein-Uhlenbeck process when N = 2^10

Just as before, the simulation is carried out and the number of time steps is increased from $N = 2^5$ to $N = 2^{14}$. The errors are recorded in Table 6.9.

Implicit Euler Scheme	
N	**Error**
2 ^ 5	0.8437
2 ^ 6	1.1697
2 ^ 7	0.8670
2 ^ 8	0.2545
2 ^ 9	0.0501
2 ^ 10	0.2433
2 ^ 11	0.8842
2 ^ 12	0.6463
2 ^ 13	0.6997
2 ^ 14	0.4289

Table 6.9: Errors generated by the implicit Euler scheme: Ornstein-Uhlenbeck process

iv. Implicit Milstein Scheme

The implicit Milstein scheme is defined as:

$$Y_{n+1} = Y(\tau_n) + a(\tau_{n+1}, Y(\tau_{n+1}))(\tau_{n+1} - \tau_n) + b(\tau_n, Y(\tau_n))(W_{\tau_{n+1}} - W_{\tau_n}) +$$

$$\frac{1}{2} b(\tau_n, Y(\tau_n)) b'(\tau_n, Y(\tau_n)) \left\{ (W_{\tau_{n+1}} - W_{\tau_n})^2 - (\tau_{n+1} - \tau_n) \right\},$$

where again, only the drift term is depends on Y_{n+1}.

For the Ornstein-Uhlenbeck process,

$$b(\tau_n, Y_n) = \beta,$$

and hence

$$b'(\tau_n, Y_n) = 0 .$$

Therefore, the implicit Milstein scheme reduces to the implicit Euler scheme.

6.6 Summary of numerical results

i. SDE with multiplicative Noise

Table 6.10 provides a summary of the numerical schemes and the associated errors for the SDE with multiplicative noise.

Scheme	N = 2^5	N = 2^6	N = 2^7	N = 2^8	N = 2^9
Euler Scheme	7.5079	5.4325	0.7487	0.0094	0.1677
Milstein Scheme	5.8242	3.4856	0.0786	0.0064	0.0241
Explicit Order 1 Strong Scheme	5.1262	2.8150	0.0996	0.0041	0.0500
Implicit Euler Scheme	0.6587	1.4605	0.6119	0.0227	0.2174
Implicit Milstein Scheme	1.7499	1.0197	0.1247	0.0198	0.0210
Order 2 Weak Taylor Scheme	2.1735	0.9191	0.0778	0.0173	0.0127

Scheme	N = 2^10	N = 2^11	N = 2^12	N = 2^13	N = 2^14
Euler Scheme	0.0195	0.0087	0.0582	0.0457	0.0670
Milstein Scheme	0.0068	0.0025	0.0023	0.0011	7.3684E-04
Explicit Order 1 Strong Scheme	0.0092	0.0032	0.0052	0.0027	0.0023
Implicit Euler Scheme	0.0232	0.0103	0.0592	0.0461	0.0666
Implicit Milstein Scheme	0.0032	0.0009	0.0014	0.0007	0.0011
Order 2 Weak Taylor Scheme	6.6027E-05	4.0001E-05	3.4032E-04	1.2792E-04	3.8124E-04

Table 6.10: Comparison of errors: SDE with multiplicative noise

The error of the Euler scheme decreases when the number of discretisation points increases from $N = 2^6$ to $N = 2^8$. The Milstein scheme performs better than the Euler scheme owing to the additional information contained in the stochastic double integral. The explicit order 1.0 strong scheme improves on the

Milstein scheme by using additional information contained in the multiple stochastic integrals. Thus, if improved accuracy is required, one could consider higher order explicit strong schemes.

The implicit schemes do not perform as well as the other strong schemes (when the number of time steps is beyond $N = 2^8$) although the implicit Milstein scheme performs better than the implicit Euler scheme, as expected. The implicit Euler and Milstein schemes do not perform as well as the Euler and Milstein schemes, respectively. One reason is the fact that example (6.3) could be written in closed form for the implicit schemes and thus some information is lost that otherwise would have been included in the approximation.

When $N = 2^5$, the implicit Euler scheme outperforms the other schemes, in that this scheme produces the smallest error while the Euler scheme has the largest error (Table 6.10).

When $N = 2^6$, the implicit Milstein scheme has the smallest error amongst the strong schemes, as expected, while the Euler scheme still has the largest error. The order 2.0 weak Taylor scheme has the smallest error for both the strong and weak schemes (Table 6.10).

The explicit order 1.0 strong scheme outperforms both the Euler and Milstein schemes when $N = 2^5$ and $N = 2^6$. But as the number of time steps increase, the Milstein scheme (in its simplicity) produces lower errors that the explicit order 1.0 strong scheme. This is clear for N between $N = 2^9$ and $N = 2^{14}$ (Table 6.10).

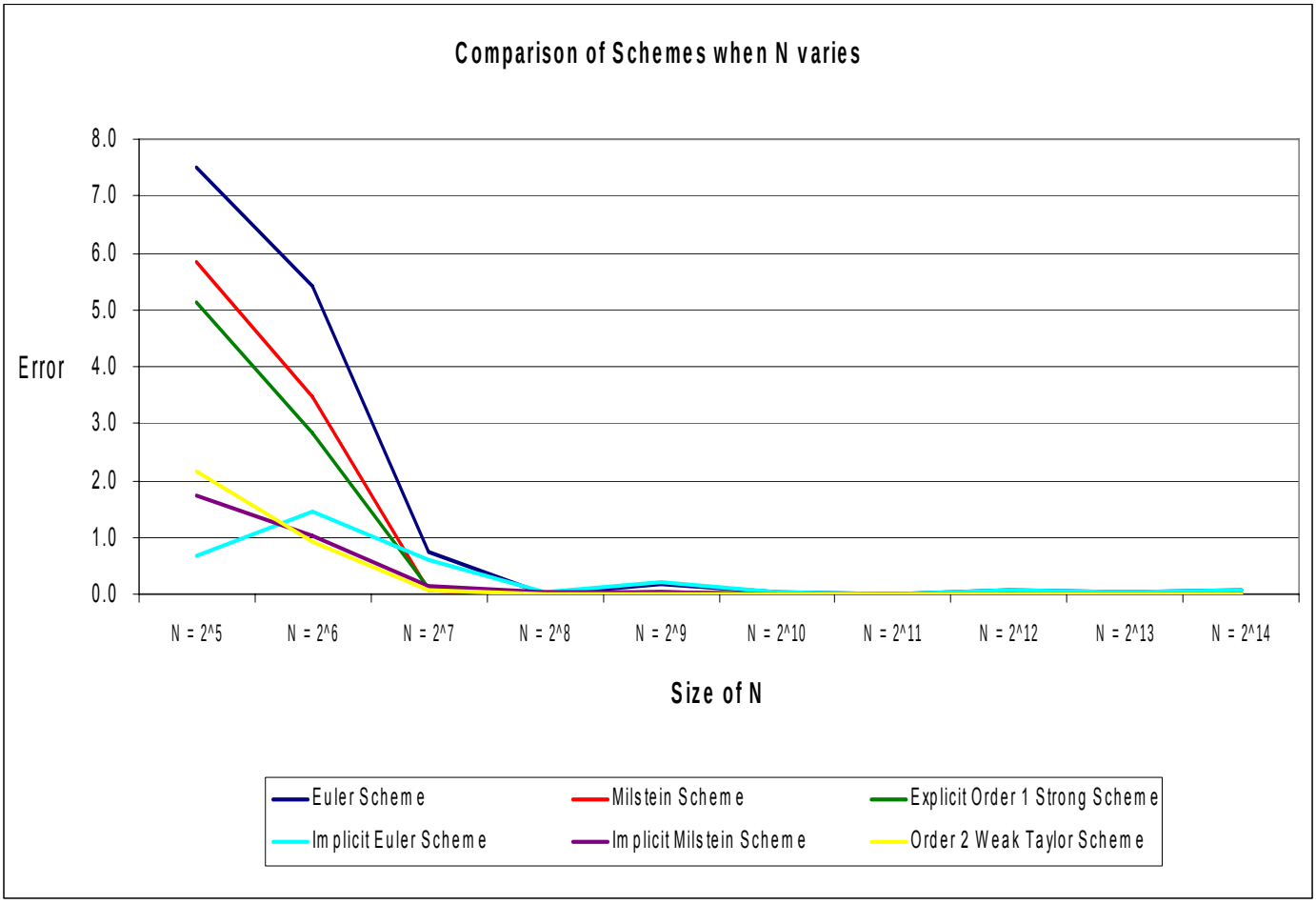

Figure 6.21: Comparison of schemes when the number of time steps increase from N = 2^5

These results suggest that numerical schemes that use more terms (in their definition) perform much better than numerical schemes with fewer terms.

As a general trend, the errors decrease as N increases, for a particular scheme. Out of all the six schemes considered, it is only the implicit Euler scheme where the error increases as the number of discretisation points is increased from $N = 2^5$ to $N = 2^6$ (Figure. 6.21).

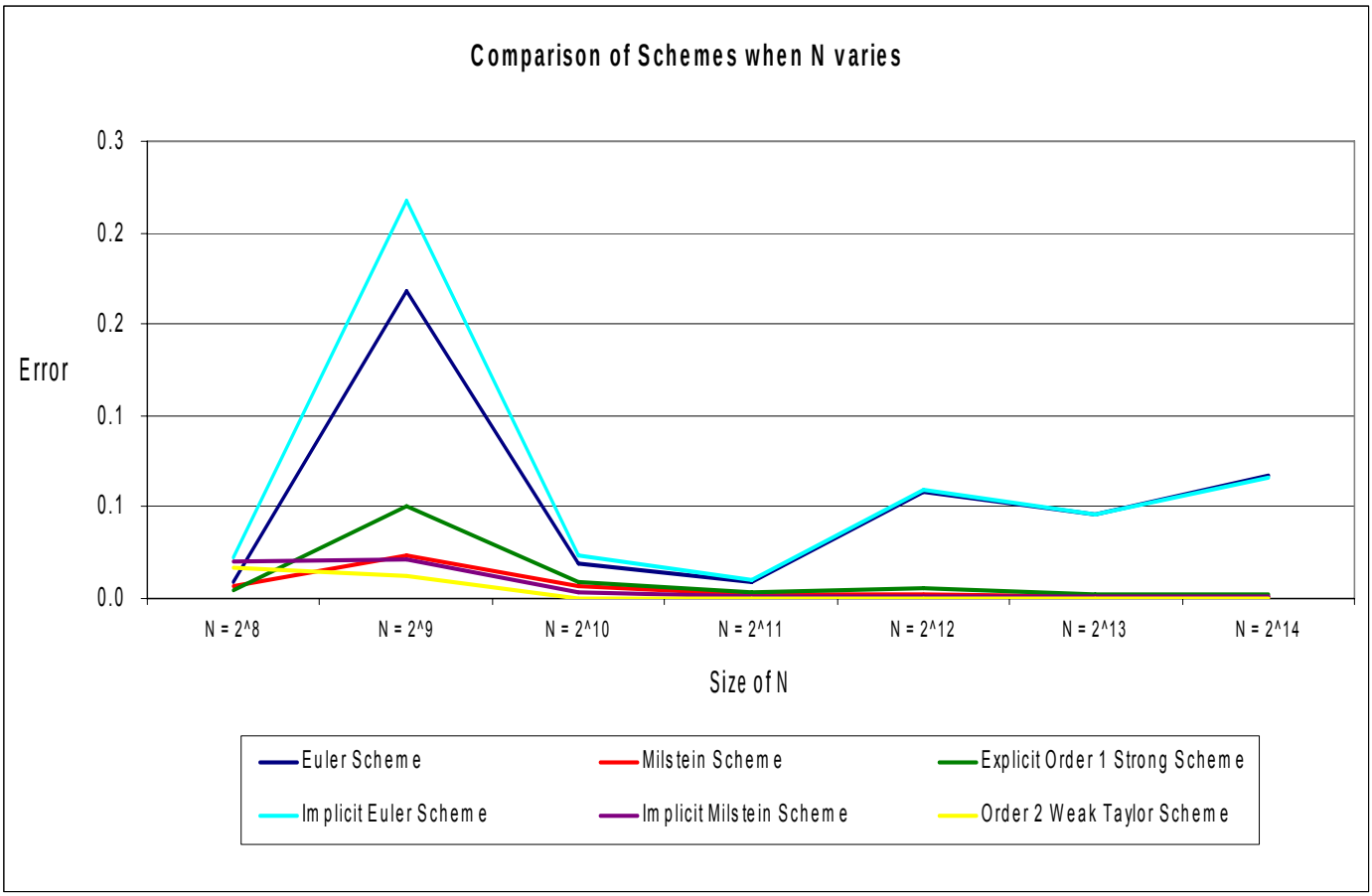

Figure 6.22: Comparison of schemes when the number of time steps increases from N = 2^8

However, on closer analysis of the results, there is a spike in the graph at $N = 2^9$ and the error increases as the number of discretisation points is increased from $N = 2^8$ to $N = 2^9$ (Figure 6.22). It is only the implicit Milstein scheme where the error continues to decrease as N increases from $N = 2^8$ to $N = 2^9$. The error generated by the implicit Euler scheme decreases from $N = 2^{10}$ to $N = 2^{11}$, but continues to increase as the number of time steps is increased from $N = 2^{11}$ to $N = 2^{14}$. The errors generated from the explicit order 1.0 strong scheme, the Milstein scheme and the implicit Milstein scheme continue to decrease as N increases.

The error from the Order 2 weak Taylor scheme is much lower than the implicit schemes. One could repeat the approximations a substantial number of times

and then calculate the mean error from the approximations. This could then be used to construct confidence intervals for the error of the weak Taylor scheme.

These results suggest that the higher order schemes perform much better that the lower order ones. Thus, analysts should try to use the scheme with the highest order that can be easily implemented, taking note of the computational time, especially for non-linear stochastic differential equations.

ii. SDE with additive noise – Ornstein Uhlenbeck process

Table 6.11 provides a summary of the errors produced by the different numerical schemes when applied tot eh Ornstein-Uhlenbeck process.

N	Errors		
	Euler	Explicit order 1	Implicit Euler
2 ^ 5	0.7611	1.2113	0.8437
2 ^ 6	1.1823	1.1855	1.1697
2 ^ 7	0.9181	0.8197	0.8670
2 ^ 8	0.2749	0.0443	0.2545
2 ^ 9	0.0524	0.4011	0.0501
2 ^ 10	0.2361	0.0400	0.2433
2 ^ 11	0.8817	0.8699	0.8842
2 ^ 12	0.6455	1.9973	0.6463
2 ^ 13	0.6995	0.4754	0.6997
2 ^ 14	0.4289	0.2456	0.4289

Table 6.11: Comparison of errors: Ornstein-Uhlenbeck process

For the Euler scheme, the error decreases as the number of time steps is increased from $N = 2^6$ to $N = 2^9$. However, as the number of time steps is increased beyond $N = 2^9$, the errors also increase.

The implicit Euler scheme produces similar results to the Euler scheme. This is evident from Figure 6.23.

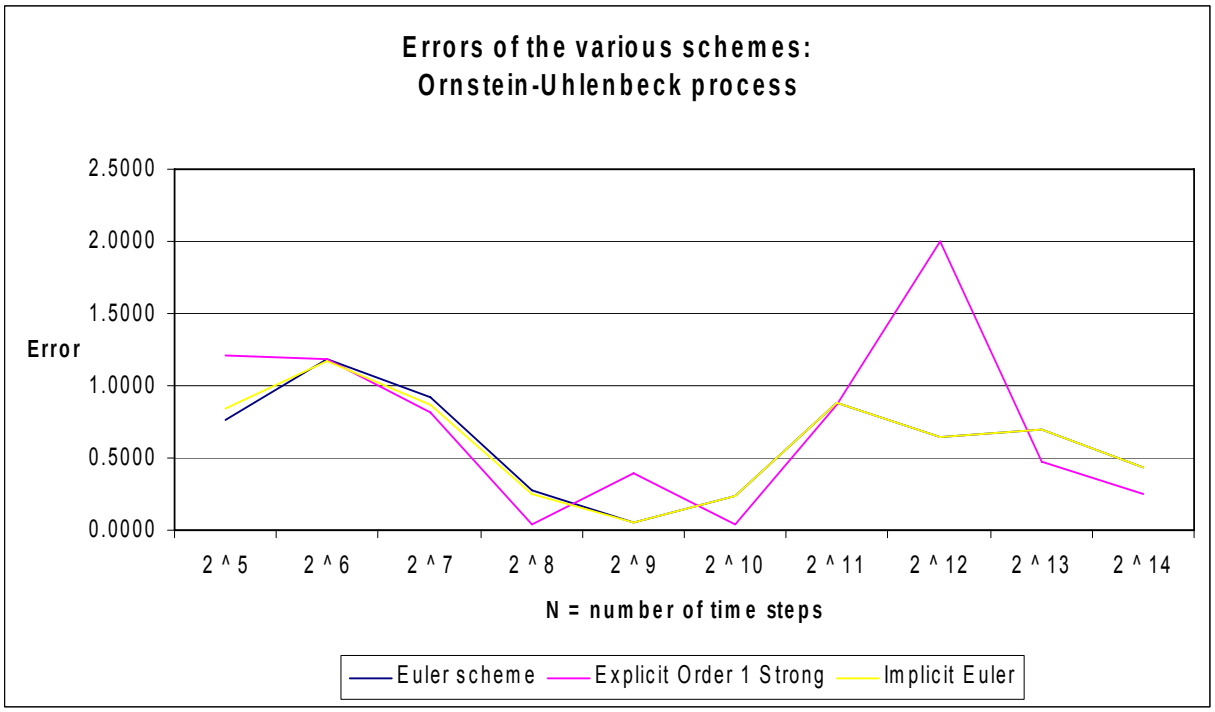

Figure 6.23: Comparison of errors for the Ornstein-Uhlenbeck process

When the number of time steps N lies between $N = 2^6$ and $N = 2^9$, the explicit order 1.0 strong scheme produces smaller errors than both the Euler and implicit Euler schemes; and the implicit Euler also produces smaller errors than the Euler. This is expected since the higher order strong schemes involves more terms in the approximation.

However, beyond $N = 2^9$, the errors are not consistent, in that errors produced by the implicit Euler and Euler scheme are much smaller than the errors produced explicit order 1.0 strong scheme. This is evident when $N = 2^{12}$.

These numerical results suggest that the numerical schemes performed much better for the SDE with the multiplicative noise that the SDE with the additive noise (Ornstein-Uhlenbeck process).

One of the reasons could be the fact that the term $\beta \Delta W$ in the Ornstein-Uhlenbeck process,

$$Y_{n+1} = Y_n - \alpha Y_n \Delta n + \beta \Delta W ,$$

does not contain the previous estimated value Y_n in the simulation of Y_{n+1}. Thus there is much less information for which to estimate Y_{n+1} during the simulation of Y_{n+1}.

CHAPTER VII - CONCLUSION

The numerical schemes for stochastic differential equations considered in this thesis have been derived from the stochastic Taylor formula. The Euler scheme is the simplest numerical approximation in that it uses the first three terms of the Taylor formula (3.20). The Euler scheme for the SDE is similar to the Euler scheme for deterministic ordinary differential equations, however for the SDE, one needs to generate random increments of the discretised Brownian motion. The Euler scheme attains the order of strong convergence γ = 0.5.

For the Milstein scheme, an additional term involving the stochastic double integral is included in the numerical approximation. By adding just one more term to the Euler scheme, the order of strong convergence increases to γ = 1. In order to increase the order of strong convergence, one needs to consider additional terms for the stochastic Taylor expansion in the numerical approximation. These higher order Taylor schemes generally involve multiple stochastic integrals, which contain additional information and generally improve the numerical approximation. However, these higher order schemes can become computationally complex, even for relatively simple SDE's. One should use these schemes if the structure of the SDE will improve the efficiency of the computations.

In the strong Taylor schemes, the derivatives of the drift and diffusion coefficients must also be determined at each step in addition to the coefficients themselves. To overcome this disadvantage, the derivatives are replaced by finite differences in a similar way that Runge-Kutta schemes are use for deterministic differential equations. These are the explicit strong schemes.

There are also implicit strong schemes in which the drift term is also a function of $(n+1)^{st}$ value of the approximation. These schemes are practically important in

that a wide range of step sizes can be implemented. Therefore, these schemes are well suited for simulating the solution of *stiff*[3] stochastic differential equations.

Whereas in the strong schemes a good pathwise approximation is required, for the weak schemes, one is interested in approximating functionals of the Itô process. This is relevant in many practical problems when the functionals cannot be determined analytically.

The explicit order 2.0 weak scheme is a derivative free scheme in that the derivatives of the order 2.0 weak Taylor scheme are replaced by finite differences. For the implicit order 2.0 weak Taylor scheme and the implicit order 2.0 weak scheme, only the drift terms are implicit while for the implicit Euler scheme, one can have both the drift and diffusion terms being implicit. However, because we have unknown on both sides of the equation, difficulties arise in the computations when the co-efficient of the drift and diffusion terms are non-constants and when there is multiplicative noise in the diffusion term.

For example (6.3), the linear SDE had an explicit solution. Further, the functional form $b(t, X_t) = \beta X_t$ of the coefficient of the diffusion term is fairly straightforward so that there is an explicit expression for its derivative, which in this case is β and is independent of X_t. Thus, for example (6.3), there is no need to evaluate the derivative in each step of the approximation.

In cases where the derivative is a function of X_t, the explicit strong scheme provides a computational advantage over other schemes where the derivatives must be evaluated at each step in the approximation.

[3] Stiff – refers to two or more widely differing time scales in the solutions of stochastic processes

The numerical results indicate that as the order of the scheme increases, the error decreases. Further, the results indicate the ease with which numerical approximations can be performed on a digital computer. Thus, one can use the simple Euler and Milstein schemes to obtain reasonable solutions of the stochastic differential equations.

This thesis has not considered the issue of numerical stability. However, according to Kloeden and Platen (1992), all the one step stochastic schemes proposed in this thesis are numerically stable under sufficient smoothness and regularity conditions on the drift and diffusion coefficients.

The analysis in the thesis used the fact the time step that is constant during simulations. However, one could use varying time steps during a particular simulation. Lehn, *et al*, (2002) reviews adaptive schemes which use variable step sizes in the numerical scheme. Thus, further research could focus on convergence issues when variable step sizes are used in the numerical scheme. The variable step size could also be researched in higher order numerical methods, which are complicated when a constant step size is used.

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APPENDIX A

A.1: Discretised Brownian path

```
% Discretised Brownian Motion

randn ('state', 200);                    % Sets the state so that the same
                                         % random numbers are generated
T = 1; N = 500; dt = T/N;
dW = sqrt(dt)*randn(1,N);                % Since dw is N(0,dt)
W = cumsum(dW);                          % Since W(j) = sum dw(i) for i = 1...j
plot([0:dt:T],[0,W],'m-');
xlabel('time');                          % Adds x labels
ylabel ('W(t)', 'rotation',0);           % Adds y labels
Title ('Discretised Brownian Path');     % Adds title to the plot
```

A.2: Exact Solution and Euler approximation

```
% Generates Brownian Motion - Discretised paths

randn('state',200)
alpha = 1.5; beta = 1; Xzero = 1;        % problem parameters
T = 1; N = 2^6; dt = T/N;
dW = sqrt(dt)*randn(1,N);                % Brownian increments
W = cumsum(dW);                          % discretized Brownian path

% Exact Solution

Xtrue = Xzero*exp((alpha-0.5*beta^2)*([dt:dt:T])+beta*W);
plot([0:dt:T],[Xzero,Xtrue],'m-'), hold on

% Euler Scheme

R = 4; Dt = R*dt; L = N/R;               % L EM steps of size Dt = R*dt
Xem = zeros(1,L);                        % preallocate for efficiency
Xtemp = Xzero;
for j = 1:L
```

```
Winc = sum(dW(R*(j-1)+1:R*j));
Xtemp = Xtemp + Dt*alpha*Xtemp + beta*Xtemp*Winc;
Xem(j) = Xtemp;
end

plot([0:Dt:T],[Xzero,Xem],'b-*'), hold off
xlabel('t','FontSize',12)
ylabel('X','FontSize',16,'Rotation',0,'HorizontalAlignment','right')

emerr = abs(Xem(end)-Xtrue(end))
```

A.3: Milstein scheme and exact solution

```
% Generates Brownian Motion - Discretised paths

randn('state',200)
alpha = 1.5; beta = 1; Xzero = 1;                % problem parameters
T = 1; N = 2^8; dt = T/N;
dW = sqrt(dt)*randn(1,N);                        % Brownian increments
W = cumsum(dW);                                  % discretized Brownian path

% Exact Solution

Xtrue = Xzero*exp((alpha-0.5*beta^2)*([dt:dt:T])+beta*W);
plot([0:dt:T],[Xzero,Xtrue],'m-'), hold on

% Milstein Scheme

R = 4; Dt = R*dt; L = N/R;                       % L EM steps of size Dt = R*dt
Xem = zeros(1,L);                                % preallocate for efficiency
Xtemp = Xzero;
for j = 1:L
Winc = sum(dW(R*(j-1)+1:R*j));
Xtemp = Xtemp + Dt*alpha*Xtemp + beta*Xtemp*Winc +0.5*beta*beta*Xtemp*(Winc^2-Dt);
Xem(j) = Xtemp;
end
```

```
plot([0:Dt:T],[Xzero,Xem],'b-*'), hold off
xlabel('t','FontSize',12)
ylabel('X','FontSize',16,'Rotation',0,'HorizontalAlignment','right')

emerr = abs(Xem(end)-Xtrue(end))
```

A.4: Explicit order 1 strong scheme

```
% Generates Brownian Motion - Discretised paths

randn('state',200)
alpha = 1.5; beta = 1; Xzero = 1;                % problem parameters
T = 1; N = 2^8; dt = T/N;
dW = sqrt(dt)*randn(1,N);                        % Brownian increments
W = cumsum(dW);                                  % discretized Brownian path

% Exact Solution

Xtrue = Xzero*exp((alpha-0.5*beta^2)*([dt:dt:T])+beta*W);
plot([0:dt:T],[Xzero,Xtrue],'m-'), hold on

% Explicit order 1 strong Scheme

R = 4; Dt = R*dt; L = N/R;                       % L EM steps of size Dt = R*dt
Xem = zeros(1,L);                                % preallocate for efficiency
Xtemp = Xzero;
for j = 1:L
Winc = sum(dW(R*(j-1)+1:R*j));
Xtemp = Xtemp + Dt*alpha*Xtemp + beta*Xtemp*Winc + 1/(2*sqrt(Dt))*(beta*Xtemp*(alpha*Dt +
beta*sqrt(Dt)))*(Winc^2-Dt);
Xem(j) = Xtemp;
end

plot([0:Dt:T],[Xzero,Xem],'b-*'), hold off
xlabel('t','FontSize',12)
ylabel('X','FontSize',16,'Rotation',0,'HorizontalAlignment','right')
```

```
emerr = abs(Xem(end)-Xtrue(end))
```

A.5: Implicit Euler scheme

```
% Generates Brownian Motion - Discretised paths

randn('state',200)
alpha = 1.5; beta = 1; Xzero = 1;                % problem parameters
T = 1; N = 2^8; dt = T/N;
dW = sqrt(dt)*randn(1,N);                        % Brownian increments
W = cumsum(dW);                                  % discretized Brownian path

% Exact Solution

Xtrue = Xzero*exp((alpha-0.5*beta^2)*([dt:dt:T])+beta*W);
plot([0:dt:T],[Xzero,Xtrue],'m-'), hold on

% Implicit Euler Scheme

R = 4; Dt = R*dt; L = N/R;                       % L EM steps of size Dt = R*dt
Xem = zeros(1,L);                                % preallocate for efficiency
Xtemp = Xzero;
for j = 1:L
Winc = sum(dW(R*(j-1)+1:R*j));
Xtemp = Xtemp * (1 + beta*Winc) / (1 - alpha*Dt);
Xem(j) = Xtemp;
end

plot([0:Dt:T],[Xzero,Xem],'b-*'), hold off
xlabel('t','FontSize',12)
ylabel('X','FontSize',16,'Rotation',0,'HorizontalAlignment','right')

emerr = abs(Xem(end)-Xtrue(end))
```

A.6: Implicit Milstein scheme

```
% Generates Brownian Motion - Discretised paths

randn('state',200)
alpha = 1.5; beta = 1; Xzero = 1;                % problem parameters
T = 1; N = 2^8; dt = T/N;
dW = sqrt(dt)*randn(1,N);                        % Brownian increments
W = cumsum(dW);                                  % discretized Brownian path

% Exact Solution

Xtrue = Xzero*exp((alpha-0.5*beta^2)*([dt:dt:T])+beta*W);
plot([0:dt:T],[Xzero,Xtrue],'m-'), hold on

% Implicit Milstein Scheme

R = 4; Dt = R*dt; L = N/R;                       % L EM steps of size Dt = R*dt
Xem = zeros(1,L);                                % preallocate for efficiency
Xtemp = Xzero;
for j = 1:L
Winc = sum(dW(R*(j-1)+1:R*j));
Xtemp = (Xtemp + beta*Xtemp*Winc +0.5*beta*beta*Xtemp*(Winc^2-Dt))/(1 - alpha*Dt);
Xem(j) = Xtemp;
end

plot([0:Dt:T],[Xzero,Xem],'b-*'), hold off
xlabel('t','FontSize',12)
ylabel('X','FontSize',16,'Rotation',0,'HorizontalAlignment','right')

emerr = abs(Xem(end)-Xtrue(end))
```

A.7: The order 2 weak Taylor scheme

```
% Generates Brownian Motion - Discretised paths

randn('state',200)
```

```
alpha = 1.5; beta = 1; Xzero = 1;                % problem parameters
T = 1; N = 2^8; dt = T/N;
G1 = sqrt(dt)*randn(1,N);                        % Brownian increments
W = cumsum(G1);                                  % discretized Brownian path

% Generate random variable Z

G2=randn(1,N);
dZ=0.5*(dt^1.5)*(G1+1/sqrt(3)*G2);

% Exact Solution

Xtrue = Xzero*exp((alpha-0.5*beta^2)*([dt:dt:T])+beta*W);
plot([0:dt:T],[Xzero,Xtrue],'m-'), hold on

% Weak Taylor Scheme

R = 4; Dt = R*dt; L = N/R;                       % L EM steps of size Dt = R*dt
Xem = zeros(1,L);                                % preallocate for efficiency
Xtemp = Xzero;
for j = 1:L
Winc = sum(G1(R*(j-1)+1:R*j));
Zinc = sum(G2(R*(j-1)+1:R*j));
Xtemp = Xtemp + Dt*alpha*Xtemp + beta*Xtemp*Winc+0.5*beta*beta*Xtemp*(Winc^2-Dt)+
alpha*beta*Xtemp*Zinc+0.5*alpha*alpha*Xtemp*(Dt^2)+alpha*beta*Xtemp*(Winc*Dt-Zinc);
Xem(j) = Xtemp;
end

plot([0:Dt:T],[Xzero,Xem],'b-*'), hold off
xlabel('t','FontSize',12)
ylabel('X','FontSize',16,'Rotation',0,'HorizontalAlignment','right')

emerr = abs(Xem(end)-Xtrue(end))
```