Chapter 1

Preliminaries

The notation in this dissertation is generally consistent with that of [DU77] and [KK76]. Suggested references for locally convex spaces are [Sch71], [RR73] and [Rud91]. For references relating to measurable functions and Polish spaces, see [Coh80] and [HJ94].

1.1 Locally Convex Spaces

Here follows a short summary of some of the definitions and results from the theory of locally convex spaces that we shall employ.

Definition 1.1. *Let E be a vector space over a (real or complex) field $\mathbb{F}$. Let A be a subset of E. Then A is*

(i) convex if $tx + (1-t)y \in A$ for all x and $y \in A$ and $0 \leq t \leq 1$;

(ii) balanced (circled) if $tx \in A$ for each $x \in A$ and for all $|t| \leq 1$;

(iii) absorbing (radial) if for each $x \in E$ there exists a $t \in \mathbb{F}$ such that $tx \in A$;

(iv) barrel if it is absorbing, convex, balanced and closed;

(v) relatively compact if its closure is compact (when E is a Hausdorff uniform space);

(vi) *precompact if it is relatively compact in the completion of* E.

Definition 1.2. *Suppose that* τ *is a topology on a vector space* E*, we write* (E, τ)*, such that*

(i) *every point in* E *is a closed set, and*

(ii) *the vector space operations are continuous with respect to* τ*, i.e.* $(x, y) \to x + y$ *for all* $x, y \in E$ *and* $(\alpha, x) \to \alpha x$ *for* $x \in E$ *and scalar* α *are continuous mappings.*

then (E, τ) *is called a topological vector space.*

Here follows a list of some topological vector spaces used in this dissertation.

Definition 1.3. *A topological vector space* (E, τ) *is*

(i) *locally convex if* E *has a base whose members are convex, balanced and absorbent;*

(ii) *a quasi-complete locally convex space if every bounded, closed subset of the space* E *is complete;*

(iii) *metrizable if* τ *is compatible with some metric* d*;*

(iv) *a metric space if its topology is induced by some metric* d*;*

(v) *a Fréchet space if it is a complete metrizable locally convex space;*

(vi) *a barrelled space if it is a locally convex space where every barrel in* E *is a neigbourhood of* 0*;*

(vii) *a nuclear space if it is a locally convex space with a base* B *consisting of convex, balanced,* 0*-neighbourhoods such that for each* $V \in B$*, the canonical map* $E \to \widetilde{E}_V$ *is nuclear.*[1]

Every 0-neighbourhood of a topological vector space, E is an absorbing set.

[1] See Chapter 1.1.1 for the definition of $\widetilde{E}_V$.

A locally convex space can also be represented in an equivalent manner in terms of seminorms. A family of seminorms which topologizes (generates) locally convex space E is denoted by P_E.

If E is a quasi-complete locally convex space then every precompact set in E is relatively compact. A quasi-complete locally convex space is sequentially complete, see [Sch71, p. 7].

A nuclear space E can also be characterized as a space where all of its continuous maps into any Banach space is nuclear, see [Sch71, Theorem III.7.2].

Consider the duality $\langle E, F\rangle$. The weakest locally convex topology under which all seminorms of the form

$$p_y(x) = |\langle x, y\rangle|$$

for all $y \in F$, is continuous, is called the weak topology on E. This topology is denoted by $\sigma(E, F)$.

Note that every $\sigma(E, F)$-bounded subset of a locally convex space is bounded.

The Mackey topology, $\tau(E, F)$ is the finest locally convex topology on E consistent with (E, F). That is, it is the topology of uniform convergence on all $\sigma(F, E)$-compact, convex, balanced subsets of F.

The strong topology, $\beta(E, F)$, is the topology of uniform convergence on all the $\sigma(F, E)$-bounded subsets of F.

Note that if E is a barrelled space, then the Mackey and strong topologies coincide, that is $\tau(E, F) = \beta(E, F)$.

Let E^* indicate the algebraic dual of E, that is all linear functional defined on E.

The topological dual of E is the linear subspace E' of E^* which consists of all continuous (with respect to a certain topology) linear forms in E^*.

The strong dual of E is defined as $(E', \beta(E', E))$, that is, E' endowed with the strong topology $\beta(E', E)$. If we write E' we mean $(E', \beta(E', E))$.

The topological dual of E', given the $\beta(E', E)$, is denoted by E''. It is called the bidual of E.

Consider the duality $\langle E, E' \rangle$. The polar of a set U is defined as

$$U^\circ = \{x' \in E' : \sup |\langle x, x' \rangle| \leq 1, x \in U\}$$

1.1.1 Quotient and Normed Spaces

Let (E, τ) be a locally convex space generated by a family of seminorms denoted by P_E. Then for any $p \in P_E$,

$$\Pi_p : E \to E_p := E \backslash p^{-1}(0)$$

is a canonical quotient map i.e. $\Pi_p(x) = x + p^{-1}(0)$ for all $x \in E$. The space $p^{-1}(0) = \{x \in E : p(x) = 0\}$.

If we let $\|\Pi_p(x)\|_p := p(x)$ for all $x \in E$. Then $\| \cdot \|_p$ is a well-defined norm on E_p. Let $\widetilde{E}_p$ indicate the completion of E_p with respect to $\| \cdot \|_p$. That is, $\widetilde{E}_p$ is a Banach space.

Let V be a convex, balanced 0-neighbourhood of E. Let

$$p_V(x) = \inf\{\lambda > 0 : x \in \lambda V\}$$

be the (Minkowski) gauge for V. Let E_V denote the quotient space $E \backslash p_V$. Let $\widetilde{E}_V$ indicate the completion of E_V.

Let $B \neq \emptyset$ be a convex, balanced, bounded subset of E. Then

$$E_1 := \cup_{\lambda \in \mathbb{R}^+} \lambda B$$

is a subspace of E. The gauge of B,

$$p_B(x) = \inf\{\lambda > 0 : x \in \lambda B\}$$

is a norm on E_1. Let E_B indicate the normed space (E_1, p_B). The embedding map $\Psi_B : E_B \to E$ is continuous. If B is complete in E then E_B is a Banach space.

For any vector measure $m \in fa(\mathcal{F}, E)$ where $\mathcal{F}$ is a field of subsets of a set Ω, let $m_p := \Pi_p \circ m$ define a measure

$$m_p : \Sigma \to E_p \subset \widetilde{E}_p$$

for all $p \in P_E$.

If E and F are vector spaces and $T : E \to F$ a linear map then T defines an isomorphism $T_0 \in L(E \backslash T^{-1}(0), T(E))$ called the bijective map associated with T. If $\phi \in L(E, E \backslash T^{-1}(0))$ is the quotient map and $\psi \in L(T(E), F)$ the embedding map then $T = \psi \circ T_0 \circ \phi$.

1.2 Vector Measures

In this dissertation the notion of "measure" is used for any finitely additive set function.

The range of a measure m over a set $A \in \mathcal{F}$ is denoted by

$$(\mathcal{R}m)(A) := \{m(B) : B \in \mathcal{F}, B \subset A\}$$

Let $\mathcal{R}m := (\mathcal{R}m)(\Omega)$.

Let $\mathcal{P}(\Omega)$ denote the collection of all finite partitions of Ω. If $\mathcal{F}$ is a field of subsets of a set Ω then $\mathcal{P}(\Omega, \mathcal{F})$ denotes the collection of all elements of $\mathcal{P}(\Omega)$ consisting of elements of $\mathcal{F}$. If no confusion can occur we use the notation $\mathcal{P}(\Omega)$ instead of $\mathcal{P}(\Omega, \mathcal{F})$.

A field $\mathcal{F}$ of subsets of a set Ω has the interpolation property (I) if and only if for any two sequences (A_n) and (B_n) in $\mathcal{F}$ satisfying the condition that $A_n \subseteq B_m$ for all n, m there exists a set $C \in \mathcal{F}$ such that $A_n \subseteq C \subseteq B_m$.

1.2.1 Spaces of Measures

Let $ca_+(\mathcal{F})$ (resp. $ba_+(\mathcal{F})$) indicate the space of real valued non-negative bounded σ-additive (resp. finitely additive) measures.

Let E be a locally convex space. The space $ca(\mathcal{F}, E)$ (resp. $fa(\mathcal{F}, E)$), indicates all E-valued σ-additive (resp. finitely additive) measures defined on $\mathcal{F}$. The space of all of the bounded elements in $fa(\mathcal{F}, E)$ is denoted by $ba(\mathcal{F}, E)$. Let $sa(\mathcal{F}, E)$ indicate the space of all strongly additive measures, that is, all E-valued measures m defined on $\mathcal{F}$, with the property that $\sum_{n=1}^{\infty} m(A_n)$ converges and the sum belongs to E, for any collection of pairwise disjoint elements $\{A_n\}_{n=1}^{\infty} \subset \mathcal{F}$. For any $x' \in E'$, let $\langle m, x' \rangle : \mathcal{F} \to \mathbb{C}$ denote the complex measure $A \to \langle m(A), x' \rangle$ for all $A \in \mathcal{F}$. Let $wca(\mathcal{F}, E)$ indicate the space of all weakly σ-additive measures, that is all measures $m : \mathcal{F} \to E$ with the property that $\langle m, x' \rangle \in ca(\mathcal{F}, \mathbb{C})$.

1.2.2 Stone Representation

If $\mathcal{F}$ is a field of subsets of a set Ω then there exists a Boolean isomorphism i from $\mathcal{F}$ onto $\mathcal{F}_1$ the field of all clopen sets of a totally disconnected compact Hausdorff space Ω_1. Under a Boolean isomorphism unions, intersections and complements are continuous. There exists an isomorphism denoted by B from $sa(\mathcal{F}, E)$ onto $ca(\sigma(\mathcal{F}_1), E)$ where for each $m \in sa(\mathcal{F}, E)$, the vector measure $Bm(iA) := m(A)$ for all $A \in \mathcal{F}$. We call the triple $[\Omega_1, \sigma(\mathcal{F}_1), Bm]$ the Stone Representation of $(\Omega, \mathcal{F}, m)$. See for instance [DS58, Section I.12] and [DU77, Theorem I.5.7].

1.2.3 p-semivariation

Let E be a locally convex space topologized by a family of seminorms P_E. For $p \in P_E$, the p-semivariation of a measure $m : \mathcal{F} \to E$ is denoted by the function

$$p(m) : \mathcal{F} \to [0, \infty)$$

defined by

$$p(m)(A) = \sup\{|\langle m, x' \rangle|(A) : x' \in U^\circ\}$$

for all $A \in \mathcal{F}$. The set U° denotes the polar of $U = \{x \in E : p(x) \leq 1\}$ and $|\langle m, x' \rangle|$ denotes the total variation of $\langle m, x' \rangle$.

The p-semivariation of m of any set $A \in \mathcal{F}$ equals the semivariation in the quotient space

$\|\Pi_p \circ m\|_p$, see [Pan08, Proposition 1.2.15], that is

$$p(m)(A) = \|\Pi_p \circ m\|_p \,(A)$$

We have the following connection between boundedness of the range of a vector measure and the boundedness of its variation:

$$\sup\{p(x) : x \in (\mathcal{R}m)(A)\} \leq p(m)(A) \leq 4 \sup\{p(x) : x \in (\mathcal{R}m)(A)\}$$

for all $A \in \mathcal{F}$, see [KK76, Chapter II.1].

Let $m : \mathcal{F} \to E$ be a vector measure and $\mu \in ba_+(\mathcal{F})$. The following notions take the place of "absolute continuity" for measures defined on a field: m is μ-null if $m(A) = 0$ whenever $\mu(A) = 0$; μ is m-null if $\mu(A) = 0$ whenever $m(A) = 0$; μ is m-continuous if $\mu(A) \to 0$ whenever $p(m)(A) \to 0$ for all $p \in P_E$ (equiv. $m(A) \to 0$ in F) and m is μ-continuous if $p(m)(A) \to 0$ for all $p \in P_E$ whenever $\mu(A) \to 0$. A vector measure m is said to be equivalent to $\mu \in ba_+(\mathcal{F})$ if m is μ-continuous and μ is m-continuous.

The space $ba_\mu(\mathcal{F}, E)$ is of all elements in $ba(\mathcal{F}, E)$ equivalent to a scalar measure $\mu \in ba_+(\mathcal{F})$. Likewise, the space $sa_\mu(\mathcal{F}, E)$ is the space of all elements in $sa(\mathcal{F}, E)$ equivalent to a scalar measure $\mu \in ba_+(\mathcal{F})$. The space $ca_\mu(\mathcal{F}, E)$ is the space of all elements in $ca(\mathcal{F}, E)$ equivalent to a scalar measure $\mu \in ca_+(\mathcal{F})$.

1.2.4 Bartle-Dunford-Schwartz-type Theorems

Let E be a locally convex space and Σ a σ-field of subsets of Ω.

For every $p \in P_E$ and $m \in ca(\Sigma, E)$ there exists a $\mu_p \in ca_+(\Sigma)$ such that

$$\mu_p(A) \leq p(m)(A)$$

for every $A \in \Sigma$ and $\mu_p(A) \to 0$ implies that $p(m)(A) \to 0$, see [KK76, Theorem II.1.1].

As a result of the Stone Representation theorem there exists a $\mu_p \in ba_+(\mathcal{F})$ for every $m \in sa(\mathcal{F}, A)$ such that

$$\mu_p(A) \le p(m)(A)$$

for every $A \in \mathcal{F}$ and $\mu_p(A) \to 0$ implies that $p(m)(A) \to 0$. The proof is along the lines of [DU77, Corollary I.5.3].

Let F be a Fréchet space generated by a countable collection of seminorms P_E. For every $m \in ca(\Sigma, F)$ there exists a $\mu \in ca_+(\Sigma)$ such that m and μ are equivalent, see [KK76, Corollary II.1.2]. As a result of the Stone Representation theorem there exists a $\mu \in ba_+(\mathcal{F})$ for every $m \in sa(\mathcal{F}, F)$ such that m and μ are equivalent. The proof is along the lines of [DU77, Corollary I.5.3]. If $m \in sa(\mathcal{F}, F)$ is weakly σ-additive then m has a unique extension to a σ-additive F-valued measure on $\sigma(\mathcal{F})$, see [Klu73a].

1.3 Polish Spaces

For a space E, let $\mathcal{B}(E)$ denote the Borel σ-field of E.

Definition 1.4. *Let E be a Hausdorff topological space. Then,*

(i) E is a Polish space if it is separable and can be metrized by means of a complete metric;

(ii) E is a Lusin space if it is the image of a Polish space under a continuous bijection;

(iii) (E, Σ) is a standard measurable space if there exists a Polish space S such that (E, Σ) and $(S, \mathcal{B}(S))$ are isomorphic i.e. there exists a bijection $f : E \to S$ such that f is $(\Sigma, \mathcal{B}(S))$-measurable and f^{-1} is $(\mathcal{B}(S), \Sigma)$-measurable.

Of particular interest is Lusin spaces. If E is a Lusin space then $(E, \mathcal{B}(E))$ is a standard measurable space, see [Coh80, Propopsition 8.6.12]. Every Hausdorff topology weaker than a Lusin topology is also a Lusin topology. Examples of Lusin spaces are the weak topology on any Banach space X and the weak* topology on X'.

Consider $E_1 \times E_2$ the Cartesian product between two topological spaces E_1 and E_2. The product-σ-field on $E_1 \times E_2$ denoted by

$$\mathcal{B}(E_1) \otimes \mathcal{B}(E_2)$$

is the smallest σ-field making the projections

$$\pi_i : E_1 \times E_2 \to E_i$$

$i = 1, 2$ measurable. It is always true [Coh80, Proposition 8.1.5] that

$$\mathcal{B}(E_1) \otimes \mathcal{B}(E_2) \subset \mathcal{B}(E_1 \times E_2)$$

The question of when

$$\mathcal{B}(E) \otimes \mathcal{B}(E) = \mathcal{B}(E \times E) \tag{1.3.1}$$

is of vital importance. For instance if (E, d) is a metric space then this is a sufficient condition for the mapping $\omega \to d(\omega, \omega)$ to be measurable. Property (1.3.1) is true if E is a Polish space. In fact it is event true if E is a Souslin space.

Talagrand [Tal79] showed that if E is a normed space then (1.3.1) is true if and only if addition, that is $A : E \times E \to E$ where $A : (\omega_1, \omega_2) \to \omega_1 + \omega_2$, is $(\mathcal{B}(E) \otimes \mathcal{B}(E), \mathcal{B}(E))$-measurable.

It must be noted that there are non-separable spaces for which 1.3.1 hold. For instance, Talagrand [Tal79] showed that

$$\mathcal{B}(l^\infty) \otimes \mathcal{B}(l^\infty) = \mathcal{B}(l^\infty \times l^\infty)$$

under the continuum hypothesis and Zermelo-Fraenkel.

1.4 Vector-valued Measurable Functions

The triple (Ω, Σ, μ) indicates a finite measure space. Let E be a topological space and let $\mathcal{B}(E)$ denote the Borel sets of E. A function $f : \Omega \to E$ is called $(\Sigma, \mathcal{B}(E))$-measurable

(Σ-measurable if the meaning is clear) if $f^{-1}(A) \in \Sigma$ for every Borel set $A \in \mathcal{B}(E)$. Hence $\sigma(f) := \{f^{-1}(A) : A \in \mathcal{B}(E)\}$ is a sub-σ-field of Σ. The function f is called a Σ-simple function if there exists $x_1, x_2, \ldots, x_n \in E$ and pairwise disjoint sets $A_1, A_2, \ldots, A_n \in \Sigma$ such that $f = \sum_{i=1}^{n} x_i \chi_{A_i}$ where $\chi_{A_i}(\omega) = 1$ if $\omega \in A_i$ and $\chi_{A_i}(\omega) = 0$ if $\omega \notin A_i$. It is well-known that Σ-simple functions are Σ-measurable.

Definition 1.5. *Let E be a quasi-complete locally convex space. Let (Ω, Σ, μ) be a complete finite measure space. (Following from the discussion in Chapter 1.3 the completeness assumption can be dropped in certain cases). Consider a function $f : \Omega \to E$:*

(i) f is called μ-measurable (strongly measurable) if there exists a sequence (f_n) of E-valued Σ-simple functions such that

$$\lim_n p(f - f_n) = 0 \ \mu\text{-a.e.}$$

for all $p \in P_E$;

(ii) f is measurable by seminorms if for every $p \in P_E$ there exists a family of simple functions $\{f_n^p\}$ such that

$$\lim_n p(f_n^p - f) = 0 \ \mu\text{-a.e.}$$

(iii) f is called weakly μ-measurable (scalarly measurable) if for every $x' \in E'$ the numerical function $x'f$ is μ-measurable i.e. for every $x' \in E'$ there exists a sequence of scalar-valued Σ-simple functions, (r_n) such that

$$\lim_n |r_n - \langle f, x' \rangle| = 0 \ \mu\text{-a.e.}$$

Theorem 1.6. *(The Pettis's Measurability Theorem)(cf. [DU77, Theorem II.1.2]) (Ω, Σ, μ). Let X be a Banach space. A function $f : \Omega \to X$ is μ-measurable if and only if f is weakly-μ-measurable and there exists a set N of measure zero such that $f(\Omega \backslash N)$ is separable.*

The Pettis's Measurability Theorem translates in the following way to locally convex space.

Lemma 1.7. (Ω, Σ, μ). *Let E be a locally convex space. A function $f : \Omega \to E$ is measurable by seminorms if and only if f is weakly μ-measurable and for $p \in P_E$ there exists a set N_p of measure zero such that $f(\Omega \backslash N_p)$ is separable with respect to p.*

1.4.1 Integrability and Integrals

Let (Ω, Σ, μ) be a finite measure space.

Let E be a quasi-complete locally convex space. In general, we shall assume that (Ω, Σ, μ) is complete. (Following from the discussion in Chapter 1.3 the completeness assumption can be dropped in certain cases).

Definition 1.8. *A function $f : \Omega \to E$ is called Dunford integrable if $\langle f, x' \rangle \in L_1(\mu)$ for all $x' \in E'$ and for each $A \in \Sigma$ there exists an element $x''_A \in E''$ such that*

$$\langle x''_A, x' \rangle = \int_A \langle f, x' \rangle d\mu \tag{1.4.1}$$

If for each $A \in \Sigma$, the values of x''_A are essentially contained in E then we say that f is a Pettis integrable.

Let $m_f : \Sigma \to E''$ be defined by $m_f(A) = x''_A$ then m_f is a finitely additive measure. If f is Pettis integrable then it can be verified that m_f is σ-additive.

The following definition are from [Mar07], it generalizes the concept of Bochner integrability to locally convex spaces:

Definition 1.9. *Let E be a locally convex space. A function $f : \Omega \to E$ is Bochner (strongly) integrable if there exists a sequence of simple functions (f_n) such that*

(i) $f_n \to f$ *μ-a.e.;*

(ii) for each $p \in P_E$ and every $n \in \mathbb{N}$, $p(f_n - f) \in L_1(\mu)$ and $\lim_{n\to\infty} \int_\Omega p(f_n - f) d\mu = 0$;

(iii) there exists an $x_A \in E$ *such that* $\lim_{n\to\infty} p(\int_A f_n d\mu - x_A) = 0$ *for all* $A \in \Sigma$.

Condition *(iii)* in Definition 1.9 and Definition 1.10 is superfluous if E is quasi-complete.

Definition 1.10. *Let* E *be a locally convex space. A function* $f : \Omega \to E$ *is integrable by seminorm if for every* $p \in P_E$ *there exists a sequence of simple functions* (f_n^p) *and a set* $N_p \in \Sigma$ *of measure zero such that, for all* $p \in P_E$,

(i) $p(f_n^p - f) \to 0$ *on* $\Omega \backslash N_p$;

(ii) for each $p \in P_E$ *and every* $n \in \mathbb{N}$, $p(f_n^p - f) \in L_1(\mu)$ *and* $\lim_{n\to\infty} \int_\Omega p(f_n - f) d\mu = 0$;

(iii) there exists an $x_A \in E$ *such that* $\lim_{n\to\infty} p(\int_A f_n d\mu - x_A) = 0$ *for all* $A \in \Sigma$.

If E is a Banach space then the concepts of Bochner integrability and integrability by seminorm are the same.

1.5 Nuclear maps and Nuclear spaces

The reader is referred to the definition and discussions on nuclear maps and nuclear spaces in Schaefer [Sch71, p.97 to 99]. Here follows only the most essential definitions for the purposes of this disseration:

The following definition is given in [Jar81, p.376] and is given as a characterization in [Sch71, Theorem 7.1]:

Definition 1.11. *A continuous linear map* $u : E \to F$ *between two arbitrary locally convex spaces is nuclear if and only if it is of the form*

$$u(x) = \Sigma_{n=1}^{\infty} \lambda_n \langle x, f_n \rangle y_n$$

where $\{\lambda_n\} \subset \ell^1$, $\{f_n\}$ *is an equicontinuous sequence in* E' *and* $\{y_n\}$ *is a sequence which converges to* 0 *in the space* F_B *for some balanced, convex, bounded subset* B *of* F *for which* F_B *is complete.*

We continue with remarks in [Sch71]:

Remark 1.12. *([Sch71, Remark and Corollary, p. 99, 100]). The sequence* $\{y_n\}$ *converges to* 0 *in a suitable Banach space* F_B *and we may assume that* $(\lambda_n) \subset l^1$ *is such that* $\sum_{n=1}^{\infty} |\lambda_n| \leq 1$. *Let*

$$U = \{x \in E : |\langle x, f_n \rangle| \leq 1, n \in \mathbb{N}\} \tag{1.5.1}$$

The set U *is convex, balanced and a* 0*-neighbourhood in* E*. The set* $u(U)$ *is contained in the closed, convex, balanced hull* C *of* $\{y_n\}$ *in* F_B*; since* $\{y_n\}$ *is relatively compact in* F_B *and* F_B *is complete,* C *is compact in* F_B *and thus also compact in* F*, since* $F_B \hookrightarrow F$ *is continuous.*

The nuclear map u can be factorized as follows (this is the definition of a nuclear map in [Sch71, Remark and Corollary, p. 98]):

Let E_U indicate the quotient space of E with respect to the gauge p_U on U and $\widetilde{E}_U$ its closure with respect to p_U. If $\phi \in L(E, \widetilde{E}_U)$ is the quotient map and $\psi \in L(F_B, F)$ the embedding map, then

$$u = \psi \circ u_0 \circ \phi \tag{1.5.2}$$

where $u_0 \in L(\widetilde{E}_U, F_B)$ is a nuclear map.

Chapter 2

Liapounoff Convexity-type Theorems

A. Liapounoff [Lia40] showed that if m is a σ-additive measure defined on a σ-field, Σ, taking values in a finite dimensional vector space, E_1, we say $m \in ca(\Sigma, E_1)$, then the range of m denoted by $\mathcal{R}m$ is compact and if m is non-atomic then $\mathcal{R}m$ is convex, see also J. Lindenstrauss [Lin66].

Various well-known related theorems for infinite dimensional vector spaces exists. Some of these theorems are listed below.

Let E be a quasi-complete locally convex space and $m \in ca(\Sigma, E)$ a non-atomic measure:

- (I. Kluvánek [Klu73a, Theorem 1, Corollary 3.1]. The weak closure of $\mathcal{R}m$ coincides with the closure of $co(\mathcal{R}m)$.

Let F be a Fréchet space and $m \in ca(\Sigma, F)$ a non-atomic measure:

- (S. Ohba [Ohb78] see also I. Kluvánek and G. Knowles [KK76, Theorem IV.6.1] and [SS03]). If $\mathcal{R}m$ is relatively compact then the closure of $\mathcal{R}m$ is convex.
- (J.J. Uhl [Uhl69] generalized by S. Ohba [Ohb78]). If F has the Radon-Nikodým prop-

erty and if m is also of bounded variation then the closure of $\mathcal{R}m$ is compact and convex.

In this chapter, these theorems are investigated for the case of finitely additive, bounded finitely additive and strongly additive vector measures defined on fields (of sets) and fields of sets with the interpolation property (I) of G.L Seever [See68].

In Section 2.1, we show that none of the above mentioned theorems may hold if the σ-field is replaced by a field. Here a property stronger than non-atomicity must be considered.

A. Sobczyk and P.C. Hammer [SH44] utilized the concept of "continuous" set function. To avoid confusion, we call this concept strongly continuous as done in [BRBR83]. In Section 2.2, we investigate the relationship between non-atomicity, strong continuity and Darboux properties for the case of non-negative finite measures defined on a fields of sets and fields of sets with property (I). The strong continuity property is introduced for the case of Fréchet space-valued measures in Section 2.3. Finally in Section 2.4 we give the mentioned Liapounoff theorems and discuss conditions under which bounded finitely additive measures are strongly additive.

2.1 Counterexample

The following example is *la raison d'être* for the structures studied in this chapter. This example shows that the classical Liapounoff Convexity theorem and the mentioned theorems by I. Kluvánek , J.J. Uhl and S. Ohba can't be extended to the case of a non-atomic vector measure on a field. In fact these theorems can't even be extended to a non-atomic σ-additive vector measure of bounded variation on a field.

Let $\Omega = [0, 1]$ and let $\mathcal{F}$ be the field generated by all sets of the form $[a, b)$ where $a < b$ and are rational numbers in Ω. Let α be any number in Ω. It is important to note that, since a σ-field isn't under consideration, $\{\alpha\} \notin \mathcal{F}$. Let μ be the "indicator" measure on $\mathcal{F}$

for the point α i.e. for any set $A \in \mathcal{F}$ if $\alpha \in A$ then $\mu(A) = 1$ otherwise $\mu(A) = 0$. Clearly, μ is an atomic measure. Let λ be the restriction of the Lebesgue measure on Ω to $\mathcal{F}$. The non-negative measure λ is non-atomic, since for every set $A \in \mathcal{F}$ such that $\lambda(A) > 0$ there exists a subset B of A in $\mathcal{F}$ such that $0 < \lambda(B) < \lambda(A)$. The vector measure $m : \mathcal{F} \to \mathbb{R}^2$ defined by

$$m = \begin{bmatrix} \lambda \\ \mu \end{bmatrix}$$

is σ-additive since λ and μ are both σ-additive measures on $\mathcal{F}$. For any $\pi \in \mathcal{P}(\Omega, \mathcal{F})$, only one set in π, say set A, can contain the point α. Under the sup-norm of $\mathbb{R}^2$

$$\sum_{D \in \pi} \|m(D)\|_\infty = \|m(A)\|_\infty + \sum_{D \in \pi, D \neq A} \|m(D)\|_\infty < 2$$

Hence, the measure m is of bounded variation and thus also strongly additive, see [DU77, Proposition I.1.9].

Now, since λ is non-atomic, m is also non-atomic. It is obvious that $\mathcal{R}m$ is neither compact, nor convex. Since the rational numbers are dense in the real numbers the closure of $\mathcal{R}m$ denoted by $\widetilde{\mathcal{R}m}$ is compact but non-convex.

Let $\tilde{m}$ denote the extension of m to $\sigma(\mathcal{F})$. Since $\mathcal{R}m$ is dense in $\mathcal{R}\tilde{m}$, it's worth studying the relationship between m and $\tilde{m}$, specifically the non-atomicity relationship. Although m on $\mathcal{F}$ is non-atomic, $\tilde{m}$ on $\sigma(\mathcal{F})$ is atomic, since

$$\tilde{m}(\{\alpha\}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \neq 0$$

but $\{\alpha\}$ does not contain any non-empty subset. We call $\{\alpha\}$ an imbedded atom of $\mathcal{F}$ in terms of m. That is, an imbedded atom of a field $\mathcal{F}$ in terms of a vector measure m is a set in $\sigma(\mathcal{F})$ which is an atom of $\tilde{m}$, the extension of m to $\sigma(\mathcal{F})$.

2.2 Non-negative Scalar Measures

We consider the relationships between the following properties of scalar measures:

Definition 2.1. *Let $\mathcal{F}$ be a field of subset of a set Ω and $\mu \in ba_+(\mathcal{F})$, then*

(i) μ is non-atomic if for any $A \in \mathcal{F}$ such that $\mu(A) \neq 0$, there exists a $B \subset A$ in $\mathcal{F}$ such that $\mu(B) \neq 0$ and $\mu(B) \neq \mu(A)$;

(ii) μ is strongly continuous if for every $\epsilon > 0$ there exists a $\pi \in \mathcal{P}(\Omega, \mathcal{F})$ such that $\mu(D) < \epsilon$ for every $D \in \pi$ i.e.

$$\inf_{\pi} \max_{D \in \pi} \mu(D) = 0$$

where the infimum is taken over all $\pi \in \mathcal{P}(\Omega, \mathcal{F})$;

(iii) μ has the Darboux property if for any $A \in \mathcal{F}$ and $\beta \in (0, \mu(A))$ there exist a set $B \subset A$ in $\mathcal{F}$ such that $\mu(B) = \beta$.

If μ is non-negative then it is trivial to show that $(iii) \Rightarrow (ii) \Rightarrow (i)$.

For a full treatment of these concepts in the setting of non-zero scalar measures, see [BRBR83].

Lemma 2.2. *(Sobczyk-Hammer Decomposition, [SH44]). Let $\mathcal{F}$ be a field of subsets of a set Ω and $\mu \in ba_+(\mathcal{F})$. Then the following decomposition of μ is unique*

$$\mu = \mu_0 + \sum_{n=1}^{\infty} \mu_n$$

where $\mu_0 \in ba_+(\mathcal{F})$ is a strongly continuous and $(\mu_n) \subset ba_+(\mathcal{F})$, is a sequence of distinct, two-valued measures.

N. Dinculeanu [Din67, p.26] showed that if a σ-additive non-negative measure on a δ-ring is non-atomic, then it has the Darboux property. Here we use a construction from this proof to show the same result for the case of a field with the interpolation property (I).

Theorem 2.3. *Let $\mathfrak{F}$ be a field of subsets of a set Ω and $\mu \in ca_+(\sigma(\mathfrak{F}))$. Then the following statements are equivalent:*

(i) *μ has the Darboux property on $\sigma(\mathfrak{F})$;*

(ii) *μ is strongly continuous on $\sigma(\mathfrak{F})$;*

(iii) *μ is non-atomic on $\sigma(\mathfrak{F})$;*

(iv) *$\mu|_{\mathfrak{F}}$ is strongly continuous on $\mathfrak{F}$;*

If $\mathfrak{F}$ has property (I) these results are also equivalent to

(v) *$\mu|_{\mathfrak{F}}$ has the Darboux property on $\mathfrak{F}$;*

(vi) *$\mu|_{\mathfrak{F}}$ is non-atomic on $\mathfrak{F}$.*

Proof. The following are easy to show: $(i) \Rightarrow (ii) \Rightarrow (iii)$; $(v) \Rightarrow (iv)$; $(iv) \Rightarrow (vi)$. $(ii) \Leftrightarrow (iv)$ is from [BRBR83, Proposition 5.3.7].

We now prove that $(vi) \Rightarrow (v)$ and since a σ-field has the interpolation property it also follows that $(iii) \Rightarrow (i)$.

Let $D \in \mathfrak{F}$ be of positive measure and $\alpha \in (0, \mu(D))$. Dinculeanu [Din67, Theorem I.2.7] constructed sequences (A_n) and (B_m) in $\mathfrak{F}$ with the following properties: (Note that this construction only depends on the non-atomicity and finite additivity of μ).

(a) $A_0 \subset A_1 \subset A_2 \subset .. \subset B_2 \subset B_1 \subset D$

(b) If we put

$$a_n = \sup\{\mu(A) : A_{n-1} \subset A \subset B_{n-1}, \mu(A) \leq \alpha\}$$

and

$$b_n = \sup\{\mu(B) : A_n \subset B \subset B_{n-1}, \mu(B) \geq \alpha\}$$

then the sequence (a_n) is monotone decreasing and (b_n) is monotone increasing and we have $a_n \leq \alpha \leq b_n$ for all $n \in \mathbb{N}$.

(c) There exist sequences (k_n) and (l_n) which tend to zero such that

$$a_n - k_n < \mu(A_n) \leq a_n$$

and

$$b_n \leq \mu(B_n) < b_n + l_n$$

Taking the limits of sequences (a_n) and (b_n) to a and b respectively, we have

$$\lim_{n\to\infty} \mu(A_n) = a \leq \alpha \leq b = \lim_{n\to\infty} \mu(B_n)$$

From property (I) of $\mathcal{F}$ there exists a set $C \in \mathcal{F}$ such that

$$A_0 \subset A_1 \subset A_2 \subset .. \subset C \subset .. \subset B_2 \subset B_1$$

and $a \leq \mu(C) \leq b$. If $\mu(C) \leq \alpha$ from Conditions (b) and (c) we deduce that

$$a_n - k_n \leq \mu(C) \leq a_{n+1}$$

for every n. Consequently, $\mu(C) = a$. We can show in a similar manner that $\mu(C) = b$. Hence $\mu(C) = \alpha$. The converse is trivial. □

2.3 Vector measures

Definition 2.4. *Let $\mathcal{F}$ be a field of subsets of a set Ω, E a quasi-complete locally convex space topologized, E' the dual space of E and $m : \mathcal{F} \to E$ an E-valued measure. Then*

(i) m is non-atomic, if for every $A \in \mathcal{F}$ such that $m(A) \neq 0$ there exists a $B \subset A$ in $\mathcal{F}$ such that $m(B) \neq 0$ and $m(A - B) \neq 0$;

(ii) m is strongly continuous if there exists a sequence $\{\pi_n\}_{n\in\mathbb{N}}$ in $\mathcal{P}(\Omega, \mathcal{F})$ such that for every $\epsilon > 0$ and every $p \in P_E$ there exists $N_p \in \mathbb{N}$ such that $p(m)(D) < \epsilon$ for every $D \in \pi_n$ and $n \geq N_p$ i.e.

$$\lim_{n\to\infty} \max_{D\in\pi_n} p(m)(D) = 0$$

(ii′) *m is strongly continuous if there exists a sequence $\{\pi_n\}_{n\in\mathbb{N}}$ in $\mathcal{P}(\Omega)$ such that for every $\epsilon > 0$ and every $p \in P_E$ there exists $N_p \in \mathbb{N}$ such that $\sup\{p(x) : x \in \mathcal{R}(D)\} < \epsilon$ for every $D \in \pi_n$ and $n \geq N_p$;*

(iii) *m is p-strongly continuous if for every $p \in P_E$ there exists a sequence $\{\pi_n\}_{n\in\mathbb{N}}$ in $\mathcal{P}(\Omega, \mathcal{F})$ such that for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $p(m)(D) < \epsilon$ for every $D \in \pi_n$ and $n \geq N_p$;*

(iv) *m is w-strongly continuous if $\langle m, x'\rangle$ is strongly continuous for all $x' \in E'$.*

Condition (ii') is an alternative, but equivalent, version of Condition (ii) since for each $p \in P_E$

$$\sup\{p(x) : x \in (\mathcal{R}m)(A)\} \leq p(m)(A) \leq 4\sup\{p(x) : x \in (\mathcal{R}m)(A)\}$$

for all $A \in \mathcal{F}$, see [KK76, Chapter II.1].

It is obvious that Condition (ii) implies Condition (iii).

Example 2.5. *U.K. Bandyopadhyay [Ban74] studied a Darboux-type property of Banach space valued measures, that is, a Banach space valued measure m on a ring $\mathcal{R}$ has this property if for any set $A \in \mathcal{R}$ and $\alpha \in (0, 1)$ there exist a set $B \subset A$ in $\mathcal{R}$ such $m(B) = \alpha m(A)$. This property implies the non-atomicity of m in fact it implies that the range of m over $\mathcal{R}$ is convex.*

However, unlike the scalar case, a non-atomic σ-additive Banach space valued measure defined on a σ-field need not have this Darboux property. See [Uhl69] for an example of a non-atomic Banach space valued measure on a σ-field with a non-convex range, thus not possessing this Darboux property.

It is obvious that if a quasi-complete locally convex space-valued measure $m : \mathcal{F} \to E$ is strongly continuous then $|\langle m, x'\rangle|$ is strongly continuous for all $x' \in E'$. That is, the strong continuity of m, implies its w-strong continuity.

A Fréchet space F has the Rybokov property if for every F-valued, σ-additive measure m there exists an $x' \in F'$ such that m is $|\langle m, x' \rangle|$-continuous.

A Fréchet space F has the Rybakov property if and only if F does not contain a linear homeomorphic copy of $\mathbb{C}^{\mathbb{N}}$, see [FN97] and [Ric98]. A list of conditions implying the Rybakov property for real Fréchet spaces is stated in [KK76, Theorem VI.3.1]. All Banach spaces have the Rybakov property, see [DU77, Chapter IX.2].

Lemma 2.6. *Let $\mathcal{F}$ be a field of subsets of a set Ω and E a quasi-complete locally convex space. If E has the Rybakov property then the strong continuity property and the w-strong continuity property are equivalent for all strongly additive E-valued measures.*

Proof. We only need to show that w-strong continuity implies strong continuity.

Let $\mathcal{F}$ be a field and $m \in sa(\mathcal{F}, E)$. Let $[\Omega_1, \sigma(\mathcal{F}_1), m_1]$ be the Stone Representation of $(\Omega, \mathcal{F}, m)$. There exists an $x' \in E'$ such that m_1 is $|\langle m_1, x' \rangle|$-continuous. This still holds if we restrict the domain of m_1 to $\mathcal{F}_1$. Since $m_1(iA) = m(A)$ for all $A \in \mathcal{F}$ it follows that m is $|\langle m, x' \rangle|$-continuous.

Let i indicate the Boolean isomorphism from $\mathcal{F}$ onto $\mathcal{F}_1$. Let $x' \in E'$ be such that m is $|\langle m, x' \rangle|$-continuous. For every $\epsilon > 0$ there exists a $\pi \in \mathcal{P}(\Omega, \mathcal{F})$ such that $|\langle m, x' \rangle|(A) < \epsilon$ for every $A \in \pi$. The strong continuity of m follows since m is $|\langle m, x' \rangle|$-continuous. □

Lemma 2.7. *Let $\mathcal{F}$ be a field of subsets of a set Ω and E a quasi-complete locally convex space space. Let $\mu \in ba_+(\mathcal{F})$ and $m \in sa(\mathcal{F}, E)$. Then*

(i) m is non-atomic if m is μ-null and μ is non-atomic;

(ii) μ is non-atomic if μ is m-null and m is non-atomic;

(iii) m is strongly continuous if m is μ-continuous and μ is strongly continuous;

(iv) μ is strongly continuous if μ is m-continuous and m is strongly continuous.

Lemma 2.8 and Theorem 2.9 is a consequence of Lemma 2.7 and Theorem 2.3.

Lemma 2.8. (Ω, Σ). *Let E be a quasi-complete locally convex space and $m \in ca(\sigma(\mathcal{F}), E)$. If m is non-atomic then m is p-strongly continuous.*

Proof. For every seminorm $p \in P_E$, there exists a $\mu_p \in ba_+(\Sigma)$ such that $\mu(A) \leq p(m)(A)$ for every $A \in \Sigma$ and such that $\mu_p(A) \to 0$ implies that $p(m)(A) \to 0$, see Chapter 1.2.4.

If m is non-atomic then μ_p is also non-atomic and hence strongly continuous for every $p \in P_E$. It follows that m is p-strongly continuous.

□

Theorem 2.9. *Let $\mathcal{F}$ be a field of subsets of a set Ω, E a quasi-complete locally convex space and $m \in ca(\sigma(\mathcal{F}), E)$. Consider the following:*

(i) m is strongly continuous on $\sigma(\mathcal{F})$;

(ii) m is non-atomic on $\sigma(\mathcal{F})$;

(iii) $m|_{\mathcal{F}}$ is strongly continuous on $\mathcal{F}$;

(iv) $m|_{\mathcal{F}}$ is non-atomic on $\mathcal{F}$;

(v) $m_0 \in sa(\mathcal{F}_0, F)$ is strongly continuous on a field $\mathcal{F}_0$ of subsets of a set Ω_0, this is if $[\Omega, \sigma(\mathcal{F}), m]$ is the Stone Representation of $(\Omega_0, \mathcal{F}_0, m_0)$.

Then (i) ⇒ (ii), (iii) ⇒ (iv), (iii) ⇔ (v) and (i) ⇔ (iii). If $m \in ca_\mu(\sigma(\mathcal{F}), E)$ then (ii) ⇒ (i) and (iv) ⇒ (iii). If $\mathcal{F}$ has property (I) then (iii) ⇒ (iv).

Proof. For every seminorm $p \in P_E$, there exists a $\mu_p \in ba_+(\sigma(\mathcal{F}))$ such that $\mu(A) \leq p(m)(A)$ for every $A \in \sigma(\mathcal{F})$ and such that $\mu_p(A) \to 0$ implies that $p(m)(A) \to 0$, see Chapter 1.2.4.

The fact that $(i) \Leftrightarrow (iii)$ follows immediately from Theorem 2.3.

$(i) \Rightarrow (ii)$. If m is strongly continuous then μ_p is also strongly continuous for every $p \in P_E$. Since μ_p is defined on a σ-field, it follows from Theorem 2.3 that μ_p is non-atomic. Hence, if

$A \in \sigma(\mathcal{F})$ such that $m(A) \neq 0$ then there exists a seminorm $p \in P_E$ such that $p(m(A)) \neq 0$ and therefor $p(m)(A) \neq 0$. Since $p(m)(A) \neq 0$, it follows that $\mu_p(A) \neq 0$. There exists a subset $B \in \sigma(\mathcal{F})$ of A such that $\mu_p(B) \neq 0$ and $\mu_p(\Omega \backslash B) \neq 0$ hence $p(m)(B) \neq 0$ and $p(m)(\Omega \backslash B) \neq 0$. We can find two sets $B_1 \subset B$ and $B_2 \subset \Omega \backslash B$ of Σ such that $p(m(B_1)) \neq 0$ and $p(m(B_2)) \neq 0$. Hence $m(B_1) \neq 0$ and $m(B_2) \neq 0$ it follows that m is non-atomic. The fact that $(iii) \Rightarrow (iv)$ follows directly.

$(iii) \Leftrightarrow (v)$. For every $\epsilon > 0$ there exists a partition $\pi \in \mathcal{P}(\Omega_0, \mathcal{F}_0)$ such that $p(m|_{\mathcal{F}})(iA) = p(m_0)(A) < \epsilon$ for every $A \in \mathcal{F}_0$. It is trivial to verify that $i\pi := \{iA : A \in \pi\}$ is a finite partition of Ω consisting of elements of $\mathcal{F}$. The converse is proved in the same way.

$(ii) \Rightarrow (i)$. If $m \in ca_\mu(\sigma(\mathcal{F}), E)$ then there exists a $\mu \in ca(\sigma(\mathcal{F}))$ equivalent to m. Hence if m is non-atomic, so is μ. Theorem 2.3 implies that μ is strongly continuous which in turns implies that m is strongly continuous. The fact that $(iv) \Rightarrow (iii)$ follows directly. □

Corollary 2.10 is in the Fréchet space setting. Its proof is much simpler because there exists a $\mu \in ca_+(\sigma(\mathcal{F}))$ equivalent to m.

Corollary 2.10. *Let $\mathcal{F}$ be a field of subsets of a set Ω, F a Fréchet space and $m \in ca(\sigma(\mathcal{F}), F)$. The following statements are equivalent:*

(i) m is strongly continuous on $\sigma(\mathcal{F})$;

(ii) m is non-atomic on $\sigma(\mathcal{F})$;

(iii) $m|_{\mathcal{F}}$ is strongly continuous on $\mathcal{F}$;

(iv) $m|_{\mathcal{F}}$ is non-atomic on $\mathcal{F}$ if $\mathcal{F}$ has property (I);

(v) any of the statements in Theorem 2.3 for any $\mu \in ca_+(\sigma(\mathcal{F}))$ equivalent to m;

(vi) $m_0 \in sa(\mathcal{F}_0, F)$ is strongly continuous on a field $\mathcal{F}_0$ of subsets of a set Ω_0, this is if $[\Omega, \sigma(\mathcal{F}), m]$ is the Stone Representation of $(\Omega_0, \mathcal{F}_0, m_0)$.

Lemma 2.11. *Let $\mathcal{F}$ be a field, E a quasi-complete locally convex space and $m_\mu \in sa(\mathcal{F}, E)$. Let $[\Omega_1, \mathcal{F}_1, m_1]$ be the Stone Representation of $(\Omega, \mathcal{F}, m)$. Then the closures in E of the sets $\mathcal{R}m$ and $\mathcal{R}m_1$ are equal i.e. $\widetilde{\mathcal{R}m} = \widetilde{\mathcal{R}m_1}$.*

Proof. From its definition, $\mathcal{R}m = \mathcal{R}m_1|_{\mathcal{F}_1}$ and $\mathcal{R}m_1|_{\mathcal{F}_1} \subseteq \mathcal{R}m_1$ are obvious. If i is the Boolean isomorphism then $\mu_1(\cdot) := \mu(i(\cdot)) \in ba_+(\mathcal{F}_1)$ is equivalent to m_1.

For every set $A \in \sigma(\mathcal{F}_1)$ there exists a sequence $\{D_k\}_{k=1}^{\infty}$ in $\mathcal{F}_1$ such that $\mu_1(D_k \Delta A) \to 0$, with Δ the symmetric difference. This is from [DK67] or classical scalar valued measure theory [Din67, Proposition I.5.13]. Thus $m_1(D_k \Delta A) \to 0$. Since $\{D_k\}_{k=1}^{\infty}$ is in $\mathcal{F}_1$, it follows that $m_1|_{\mathcal{F}_1}(D_k) \to m_1(A)$, thus $\mathcal{R}m_1 \subset \widetilde{\mathcal{R}m_1|_{\mathcal{F}_1}}$. □

2.4 Liapounoff Convexity-type Theorems

Lemma 2.12. *Let $\mathcal{F}$ be a field of subsets of a set Ω. Let E be a quasi-complete locally convex space. If $m : \mathcal{F} \to E$ is of bounded variation or $\widetilde{\mathcal{R}m}$ is compact then m is strongly continuous.*

This result must be well-known. For each $p \in P_E$ let $m_p := \Pi_p \circ m$. If $\mathcal{R}m$ is precompact, then $\mathcal{R}m_p$ is also precompact since Π_p is a continuous map or if m is of bounded variation then it is easy to verify that m_p is also of bounded variation. In both cases it follows from the Banach space-valued case that m_p is strongly continuous for all $p \in P_E$ and hence m is strongly continuous.

Theorem 2.13. *Let $\mathcal{F}$ be a field of subsets of a set Ω, E a quasi-complete locally convex space and $m \in sa_\mu(\mathcal{F}, E)$. Then the weak closure of $\mathcal{R}m$ coincides with its closed convex hull and is weakly compact.*

If $\mathcal{F}$ has the interpolation property (I) then the strong continuity condition can be replaced by non-atomicity.

Proof. Let $[\Omega_1, \sigma(\mathcal{F}_1), m_1]$ be the Stone Representation of $(\Omega, \mathcal{F}, m)$. From the definition of

Stone Representation, m_1 is σ-additive on $\sigma(\mathcal{F}_1)$. Also, m_1 is non-atomic since m is strongly continuous or alternatively if $\mathcal{F}$ has property (I) and m is non-atomic, see Corollary 2.10. From Lemma 2.11 we know that $\widetilde{\mathcal{R}m} = \widetilde{\mathcal{R}m_1}$. Since the weak closure and closed convex hulls of $\mathcal{R}m$ and $\mathcal{R}m_1$ are equal. An appeal to [Klu73b] completes the proof. □

Theorem 2.14. *Let $\mathcal{F}$ be a field of subsets of a set Ω, F a Fréchet space and $m \in fa(\mathcal{F}, F)$ strongly continuous. Then*

(i) if $m \in sa(\mathcal{F}, F)$ then the weak closure of $\mathcal{R}m$ coincides with its closed convex hull and is weakly compact;

(ii) if $\widetilde{\mathcal{R}m}$ is compact then $\widetilde{\mathcal{R}m}$ is convex;

(iii) if F has the Radon-Nikodým property and m is of bounded variation then the closure of $\mathcal{R}m$ is compact and convex.

If $\mathcal{F}$ has the interpolation property (I) then the strong continuity condition can be replaced by non-atomicity.

Proof. The proof of (i) follows from above since the existence of a measure $\mu \in ba_+(\mathcal{F})$ is guaranteed, since F is a Fréchet space.

If m has a precompact range or is of bounded variation then m is strongly continuous, see Lemma 2.12. Let $[\Omega_1, \sigma(\mathcal{F}_1), m_1]$ be the Stone Representation of $[\Omega, \mathcal{F}, m]$. From the definition of Stone Representation, m_1 is σ-additive on $\sigma(\mathcal{F}_1)$. Also, m_1 is non-atomic since m is strongly continuous or alternatively if $\mathcal{F}$ has property (I) and m is non-atomic, see Corollary 2.10. From Lemma 2.11 we know that $\widetilde{\mathcal{R}m} = \widetilde{\mathcal{R}m_1}$. To complete each proof:

(ii) the proof follows immediately from [Klu73b] since $\widetilde{\mathcal{R}m} = \widetilde{\mathcal{R}m_1}$;

(iii) since m is of bounded variation, m_1 is also of bounded variation. An appeal to [Ohb78] completes the proof.

□

We now consider a case where the strong additivity assumption in Theorem 2.14(i) can be relaxed.

Definition 2.15. *A field $\mathcal{F}$ of subsets of a set Ω has the Vitali-Hahn-Saks property, if every sequence $\{\mu_n\} \subset ba_+(\mathcal{F})$, where $\{\mu_n(A)\}$ converges for every $A \subset \mathcal{F}$, is uniformly strongly additive.*

Lemma 2.16. *Let X be a Banach space and let $\mathcal{F}$ be a field of subsets of a set Ω and let $m \in ba(\mathcal{F}, F)$. If m takes values in a finite dimensional subspace of X then m is of bounded variation and hence strongly additive.*

The proof of this lemma follows easily from the case of signed measures.

Lemma 2.17. *The space $C(-\infty, \infty)$, of all continuous functions on the reals equipped with the topology of uniform convergence on compact sets is a Fréchet space with a Schauder basis.*

Proof. $C(-\infty, \infty)$ equipped with the topology of uniform convergence on compact sets is a Fréchet space, see [Rud91, Example 1.44]. $C(-\infty, \infty)$ is isomorphic to $C([0, 1])^N$, a countable product of copies of $C([0, 1])$, this follows easily from [Val82, Theorem 3.3.6.2, p 496]. It is well-known that $C([0, 1])$ has a Schauder basis [Woj91, II.B.12], hence $C([0, 1])^N$ and thus $C(-\infty, \infty)$ also have Schauder bases.

□

The following theorem contains some ideas in [Die73] applied to a Fréchet space setting.

Theorem 2.18. *Let F be a separable Fréchet space generated by an increasing family of seminorms denoted by P_F and let $\mathcal{F}$ be a field of subsets of a set Ω with the Vitali-Hahn-Saks property. Let $m \in ba(\mathcal{F}, F)$ then m is strongly additive.*

Proof. Every separable Fréchet space is linearly homeomorphic to a subspace of $C(-\infty, \infty)$ equipped with the topology of uniform convergence on compact sets, see [MO53, p. 144].

Denote the Schauder basis of $C(-\infty, \infty)$ by (e_n) and let (f_n) denote the associated sequence of coefficient functionals. Each $f_n \in F'$ and each $x \in F$ can be uniquely represented in the form $x = \Sigma_n f_n(x) e_n$, hence $m(A) = \Sigma_n f_n(m(A)) e_n$. The vector measure

$$m_k(A) := \Sigma_{n=1}^k f_n(m(A)) e_n$$

takes its values in a finite dimensional subspace of $C(-\infty, \infty)$.

Let $p \in P_F$ and $(\widetilde{F}_p, \|\cdot\|_p)$ be the Banach space defined in terms of the quotient map $\Pi_p : F \to F/p^{-1}(0)$. Since m is bounded, m_k is also bounded and it follows that $\Pi_p \circ m_k$ is a bounded finitely additive measure; hence $\Pi_p \circ m_k$ is strongly additive. There exists a measure $\mu_k \in ba_+(\mathcal{F})$ such that m_k is μ_k-continuous, see [DU77, Corollary I.5.3]. Hence $\|\Pi_p \circ m_k\|_p(D) \to 0$ as $\mu_k(D) \to 0$ where $\|\Pi_p \circ m_k\|_p(D)$ indicates the semivariation of $\Pi_p \circ m_k$ over a set $D \in \mathcal{F}$, see [DU77, p.2].

Since $\mathcal{F}$ has the Vitali-Hahn-Saks property, $\widehat{\mu}_p := \sup\{\mu_k : k \in \mathbb{N}\}$ is strongly additive, hence bounded. If $\widehat{\mu}_p(A) \to 0$ then $\|\Pi_p \circ m_k\|_p(A) \to 0$ for each $k \in \mathbb{N}$ which implies that $p(m(A)) = \|\Pi_p \circ m(A)\| \to 0$, see [DU77, Corollary I.5.4].

We can construct a single measure $\widehat{\mu} \in ba_+(\mathcal{F})$ from the set $\{\widehat{\mu}_p\}_{p \in P_F}$ as done in [KK76, Corollary II.2.2], with the property that $\widehat{\mu}(A) \to 0$ implies that $\widehat{\mu}_p(A) \to 0$ for all $p \in P_F$. If $\{A_m\} \subset \mathcal{F}$ is a mutually disjoint sequence of sets then $\widehat{\mu}(A_m) \to 0$ which implies that $p(m(A_m)) = \|\Pi_p \circ m(A_m)\|_p \to 0$ for each $p \in P_F$. □

Chapter 3

Barrelled spaces

In this chapter we investigate the existence of weak (Dunford, Gel'fand, Pettis) integrals in locally convex space. In general, the existence of the Dunford (and Gel'fand) integral depends on whether the closed graph theorem for the dual of the space under consideration holds.

Since barrelled spaces can be characterized in terms of the validity of the closed graph theorem, we consider locally convex spaces whose duals are barrelled spaces.

3.1 Existence of the Dunford Integral

All of the proof of the existence of the Dunford integral relies on the closed graph theorem. We consider barrelled spaces because of the following theorem:

Theorem 3.1. *(Closed Graph Theorem). Let (E, τ) be a barrelled space, X a Banach space and $f : E \to X$ a linear mapping with closed graph in $E \times X$. Then f is continuous.*

This is not the most general version of the closed graph theorem. But will suffice for our purposes. For a more complete discussion on the closed graph theorem, see [PCB87], specifically Chapters 4 and 7.

Theorem 3.2. *Let X be a Banach space and E a locally convex space. If the fact that a* *linear mapping $u : E \to X$ with closed graph in $E \times X$ implies that u is continuous then E is a barrelled space.*

Lemma 3.3. *(cf. [Sch71]) Fréchet spaces (and hence Banach spaces) are barrelled spaces.*

We now prove the existence of the Dunford integral.

Theorem 3.4. *Let (E, τ) be a locally convex space and suppose that E', equipped with the $\beta(E', E)$-topology, is a barrelled space and $f : \Omega \to E$ is a weakly μ-measurable function and $\langle f(\cdot), x' \rangle \in L_1(\mu)$ for all $x' \in E'$. Then for each $A \in \Sigma$ there exists an $x''_A \in E''$ satisfying*

$$\langle x''_A, x' \rangle = \int_A \langle f(\cdot), x' \rangle d\mu$$

Proof. We first prove that the integration operation is a continuous linear map.

Let $A \in \Sigma$ and define $T_A : E' \to L_1(\mu)$ by $T_A(x') = \langle f\chi_A, x' \rangle$.

Let $\{x'_\alpha\}$ be a net in E' which converges to x' in the $\beta(E', E)$-topology of E', hence also in the $\sigma(E, E')$-topology, and suppose that $T_A(x'_\alpha) = \langle f\chi_A, x'_\alpha \rangle$ converges to a function g in the norm topology of $L_1(\mu)$.

Then $\{\langle f\chi_A, x'_\alpha \rangle\}$ has (a subsequence which also converges to g, which has) a subsequence $\{\langle f\chi_A, x'_{\alpha_j} \rangle\}$ which converges μ-a.e. to g, but

$$\lim_{j \to \infty} \langle f\chi_A, x'_{\alpha_j} \rangle = \langle f\chi_A, x' \rangle$$

everywhere. Hence $\langle f\chi_A, x' \rangle = g$ μ-a.e. This means that the mapping T_A has a closed graph, hence T_A is continuous by the closed graph theorem. Now,

$$\begin{aligned} |\int_A \langle f, x' \rangle d\mu| &\leq \|\langle f\chi_A, x' \rangle\|_1 \\ &= \|T_A(x')\|_1 \end{aligned}$$

Thus, the mapping $x' \to \int_A \langle f, x'\rangle d\mu$ defines a continuous linear functional x''_A on E' for each $A \in \Sigma$. □

The existence of the Gel'fand integral is an immediate consequence of the same duality argument.

Let $m_f : \Sigma \to E''$ be defined by $m_f(A) := x''_A$. It is easy to see that $m \in fa(\Sigma, E'')$.

Corollary 3.5. *Let E be a locally convex space and suppose that E' is a barrelled space. Let $f : \Omega \to E$ be a Dunford integrable function. Then $\mathcal{R}m_f$ is bounded in the $\sigma(E'', E')$-topology of E''.*

Proof. Let $x' \in E'$. For every $A \in \Sigma$,

$$\begin{aligned}|\langle m_f(A), x'\rangle| &= |\int_A \langle f, x'\rangle d\mu| \\ &\leq \int_A |\langle f, x'\rangle| d\mu \\ &\leq \|T_\Omega(x')\|_1\end{aligned}$$

□

Corollary 3.6. *Let E be a locally convex space and suppose that E' is a barrelled space. Let $f : \Omega \to E$ be a Dunford integrable function. Then the Dunford integral of f is σ-additive in the $\sigma(E'', E')$-topology of E''.*

Proof. If we consider a set $A = \cup_{i=1}^\infty A_i$ in Σ where $\{A_i\}$ is a collection of pairwise disjoint sets of Σ then the series $\Sigma_j m_f(A_j)$ converges in the $\sigma(E'', E')$ because for any $k \in \mathbb{N}$ and for all $x' \in E'$,

$$\begin{aligned}\Sigma_{j=1}^k |\langle m_f(A_j), x'\rangle| &\leq \Sigma_{j=1}^k \int_{A_j} |\langle f, x'\rangle| d\mu \\ &= \int_{\cup_{j=1}^k A_j} |\langle f, x'\rangle| d\mu \\ &\leq \|T_\Omega(x')\|_1\end{aligned}$$

From which it follows that

$$\begin{aligned}\langle \Sigma_j m_f(A_j), x' \rangle &= \Sigma_j \int_{A_j} \langle f(\cdot), x' \rangle d\mu \\ &= \int_A \langle f(\cdot), x' \rangle d\mu \\ &= \langle m_f(A), x' \rangle\end{aligned}$$

for all $x' \in E'$. Hence the Dunford integral is σ-additive in the $\sigma(E'', E')$-topology. □

We now turn to the existence of Pettis integrable functions. First consider the following result due to A. Grothendieck in the locally convex space setting.

Theorem 3.7. *([Gro53]). Let (E, τ) be a locally convex space then a finitely additive measure $m : \Sigma \to E$ is weakly σ-additive, that is, σ-additive in the $\sigma(E, E')$-topology of E if and only if m is σ-additive.*

Corollary 3.8. *Let E be a locally convex space and suppose that E' is a barrelled space. Let $f : \Omega \to E$ be a Dunford integrable function. Then f is Pettis integrable if and only if the Dunford integral of f is σ-additive (strongly additive).*

Chapter 4

Nuclear spaces and Nuclear maps

In this chapter we find that measures and measurable functions of nuclear spaces have "improved" properties, above those of general locally convex spaces and Banach spaces. For example the range of a bounded nuclear space-valued measure is precompact, from which a Liapounoff convexity result follows. Also weakly-μ-measurable functions are measurable by seminorm.

J. Diestel [Die72] discovered that the composition of an absolutely summing map (between two Banach spaces) with a Pettis integrable function has "improved" integrability properties, compared to that of the integrable function considered on its own. This naturally leads to the investigation of the composition of nuclear maps (between two locally convex spaces) with weakly measurable functions in locally convex spaces.

A reminder of some of the notation which is used in this chapter: For a locally convex (Hausdorff topological vector) space E, a family of seminorms which topologizes E will be denoted by P_E. Throughout this chapter we will use the notation Π_p for the extension of the quotient map between E and its quotient space, the normed space $E_p := E\backslash p^{-1}(0)$, to the completion $\widetilde{E}_p$ of E_p.

4.1 Measures

Theorem 4.1. *Let $\mathcal{F}$ be a field of subsets of a set Ω. Let E and F be locally convex spaces. Let $u : E \to F$ be a nuclear map. If $m \in ba(\mathcal{F}, E)$, then $u \circ m$ is of bounded q-variation for every $q \in P_F$.*

Proof. We first show that $u \circ m$ is of bounded q-semivariation.

Let $q \in P_F$. Then $\Pi_q \circ u$ is a nuclear map and $m_q := \Pi_q \circ u \circ m \in ba(\mathcal{F}, \widetilde{F}_p)$. Since $\mathcal{R}m$ is bounded it follows that $\mathcal{R}m_q = (\Pi_q \circ u)(\mathcal{R}m)$ is relatively compact in $\widetilde{F}_q$, see [Sch71, Corollary III.7.1]. From this it immediately follows that $m_q \in sa(\mathcal{F}, \widetilde{F}_q)$, see [DU77, Corollary I.5.3].

The q-semivariation of m on any set $A \in \mathcal{F}$ equals the semivariation in the quotient space $\|m_q\|_q$, see [Pan08, Proposition 1.2.15], that is

$$q(m)(A) = \|m_q\|_q \ (A)$$

Since a strongly additivity, Banach spaces-valued measure is of bounded semivariation, the fact that $m_q \in sa(\mathcal{F}, F_q)$ implies that

$$q(m)(A) \leq q(m)(\Omega) < \infty$$

for all $A \in \mathcal{F}$.

Now to show that $u \circ m$ is of bounded q-variation.

For each $q \in P_F$, we consider the factorization of the nuclear map $(\Pi_q \circ u)$, therefore

$$(\Pi_q \circ u)(x) = \Sigma_{n=1}^{\infty} \lambda_n \langle x, f_n \rangle y_n$$

where $\{\lambda_n\} \subset \ell^1$, $\{f_n\}$ is an equicontinuous sequence in E' and $\{y_n\}$ is a sequence which converges to 0 in the space F_B for some balanced, convex, bounded subset B of F for which F_B is complete.

Since $\{f_n\}$ is equicontinuous and hence uniformly bounded, there exist a finite number $M_1 > 0$ such that

$$|\langle x, f \rangle| < M_1$$

for every $x \in U_q$ and $f \in \{f_n\}$, hence $\{f_n/M_1\} \subset U_q^0$.

For any finite set $\pi \in \mathcal{P}(\Omega, \mathcal{F})$, where $\mathcal{P}(\Omega, \mathcal{F})$ consists of the elements of $\mathcal{F}$ which forms a pairwise disjoint partitions of Ω, we have

$$\begin{aligned}\sum_{A\in\pi} \|\Pi_p \circ u \circ m(A)\|_q &\leq \sum_{A\in\pi}\sum_{n=1}^{\infty} \|\lambda_n \langle m(A), f_n\rangle y_n\|_q \\ &= \sum_{n=1}^{\infty}\sum_{A\in\pi} |\langle m(A), f_n\rangle| \|\lambda_n y_n\|_q \\ &= M_1 \sum_{n=1}^{\infty} \|\lambda_n y_n\|_q \sum_{A\in\pi} |\langle m(A), f_n/M_1\rangle|.\end{aligned}$$

Now

$$\sum_{n=1}^{\infty} \|\lambda_n y_n\|_q \leq M_2 \sum_{n=1}^{\infty} |\lambda_n| \leq M_2$$

where $\|y_n\|_q \leq M_2$ for all n since $\{y_n\}$ is contained in a bounded set and the summation can be taken as $\sum_{n=1}^{\infty} |\lambda_n| \leq 1$. Also,

$$\sum_{A\in\pi} |\langle m(A), f_n/M_1\rangle| \leq q(m)(\Omega) < \infty$$

because m is of bounded q-semivariation.

In conclusion, $u \circ m$ is of bounded q-variation because

$$\sum_{A\in\pi} \|\Pi_q \circ u \circ m(A)\|_q \leq M_1 M_2 q(m)(\Omega)$$

is bounded for all elements of $\mathcal{P}(\Omega, \mathcal{F})$. □

Corollary 4.2. *Let $\mathcal{F}$ be a field of subsets of a set Ω. Let E and F be locally convex spaces. Let $u : E \to F$ be a nuclear map. If $m \in ba(\mathcal{F}, E)$, then*

(i) $\mathcal{R}(u \circ m)$ is a precompact set;

(ii) there exists a measure $\mu \in ba_+(\mathcal{F})$ such that $u \circ m$ is μ-continuous;

(iii) the closure of $\mathcal{R}(u \circ m)$ is a convex set if the closure is contained in F and m is strongly continuous (or non-atomic and $\mathcal{F}$ has property (I)).

Proof. We use the factorization of u as described in (1.5.2).

(i). Since $m \in ba(\mathcal{F}, E)$, the set $\mathcal{R}m$ is bounded. Thus the set $\phi(\mathcal{R}m) = \mathcal{R}(\phi \circ m)$ is bounded and $(u_0 \circ \phi)(\mathcal{R}m) = \mathcal{R}(u_0 \circ \phi \circ m)$ is relatively compact in F_B. Hence $\psi(\mathcal{R}(u_0 \circ \phi \circ m)) = \mathcal{R}(u \circ m)$ is precompact in F, by the continuity of ψ.

(ii). Since $\mathcal{R}(u_0 \circ \phi \circ m)$ is relatively compact in F_B, it follows from [DU77, Corollary I.5.3] that $u_0 \circ \phi \circ m \in sa(\mathcal{F}, \widetilde{E}_U)$ and there exists a control measure $\mu \in ba(\mathcal{F})$ equivalent to $u_0 \circ \phi \circ m$. Hence $u \circ m$ is μ-continuous.

(iii). If m is strongly continuous then it follows directly from, for instance the proof of Theorem 2.9, that $u_0 \circ \phi \circ m$ is also strongly continuous in F_B because of the continuity of $u_0 \circ \phi$. In addition it follows from Theorem 2.14 that the closure of $\mathcal{R}(u_0 \circ \phi \circ m)$ is convex because the closure of $\mathcal{R}(u_0 \circ \phi \circ m)$ is compact. From the continuity of ψ it follow that the closure of $\psi(\mathcal{R}(u_0 \circ \phi \circ m))$ is also convex if it is contained in F.

The same proof will hold if "strong continuity" is replaced by "non-atomic" and if in adddition $\mathcal{F}$ has property (I). □

Corollary 4.3. *Let $\mathcal{F}$ be a field of subsets of a set Ω. Let E be a nuclear space. If $m \in ba(\mathcal{F}, E)$ then m is of bounded p-variation for every $p \in P_E$, that is, the variation with respect to the quotient space induced by every p is bounded.*

Proof. Let $p \in P_E$. Since $\Pi_p : E \to \widetilde{E}_p$ is a continuous linear map from a nuclear space to a Banach space, it is also nuclear. The result follows from Corollary 4.2. □

Corollary 4.4. *Let $\mathcal{F}$ be a field of subsets of a set Ω. Let E be a nuclear space and X a Banach space. Let $u : E \to X$ be a continuous linear map. If $m \in ba(\mathcal{F}, E)$ then $u \circ m$ is of bounded variation.*

Theorem 4.5. *(cf. [Sch71, Corollary III.7.2.1]). Every bounded subset of a nuclear space is precompact.*

The following Liapounoff-type results for nuclear spaces follow:

Theorem 4.6. *Let $\mathcal{F}$ be a field of subsets of a set Ω. Let E be a nuclear space. If $m \in ba(\mathcal{F}, E)$ then $\mathcal{R}m$ is precompact.*

If in addition, if E is quasi-complete and m is strongly continuous (or non-atomic and $\mathcal{F}$ has property (I)), then the closure of $\mathcal{R}m$ is a compact, convex set.

Proof. If $m \in ba(\mathcal{F}, E)$ then $\mathcal{R}m$ is a bounded subset which is precompact.

Let m be strongly continuous. If τ is a neighbourhood base of 0 in E consisting of convex, balanced sets, then there exists an isomorphism, v, on E to a subspace of $\Pi_{B\in\tau}\widetilde{E}_B$ defined by

$$v : x \to \{\phi_B(x) : B \in \tau\}$$

see [Sch71, Theorem II.5.4 and Corollary II.5.4.2].

Let $B \in \tau$. Since the quotient map $\phi_B : E \to \widetilde{E}_B$ is continuous, the measure $\phi_B \circ m$ is strongly continuous (and relatively compact). Then according to Theorem 2.14, the closure of the range of $\phi_B \circ m$ is convex.

For any A and A_0 in $\mathcal{F}$ and scalar $r \in (0, 1)$,

$$\begin{aligned} &v(rm(A) + (1-r)m(A_0)) \\ &= \{r\phi_B(m(A)) + (1-r)\phi_B(m(A_0)) : B \in \tau\} \\ &\in \{\widetilde{\phi_B(\mathcal{R}m)} : B \in \tau\} = \widetilde{v(\mathcal{R}m)} \end{aligned}$$

where $\widetilde{\phi_B(\mathcal{R}m)}$ indicates the closure of $\phi_B(\mathcal{R}m)$, etc. For the last equality, see [Eng68, Theorem 2, p. 74].

This means that the closure of measure $v \circ m$ is convex. Since v is an isomorphism it follows

$$\widetilde{\mathcal{R}m} \subseteq v^{-1}(\widetilde{v(\mathcal{R}m)}) \subseteq v^{-1}(\widetilde{v(\mathcal{R}m)}) = \widetilde{\mathcal{R}m}$$

hence

$$rm(A) + (1-r)m(A_0) \in \widetilde{\mathcal{R}m}$$

This means that the closure of measure m is also convex. □

4.2 Measurability and Integrability

In this section, unless stated differently, we consider the triple (Ω, Σ, μ) which indicates a complete finite measure space.

The following lemma holds true for Banach spaces, see [Pie72, Proposition p. 52]. In general, this results follows from the factorization of the nuclear map.

Lemma 4.7. *Each nuclear mapping between two locally convex space has a separable range.*

Theorem 4.8. *Let E be a nuclear space and $f : \Omega \to E$ a weakly μ-measurable function. Then f is measurable by seminorms.*

Proof. Let $p \in P_E$, notice that Π_p is a nuclear map. Now, the range of Π_p is separable hence $\Pi_p \circ f(\Omega)$ is a separable set. The mapping $\Pi_p \circ f$ is also weakly μ-measurable since $e \circ \Pi_p \in E'$ for every $e \in \widetilde{E}'_p$. Hence, by Pettis's measurability theorem, $\Pi_p \circ f$, is μ-measurable.

It means that there exists a sequence $\{g_k\}$ of $\widetilde{E}_p$-valued simple functions such that

$$\|\Pi_p \circ f - g_k\|_p \to 0$$

on $\Omega \backslash N$ for some $N \in \Sigma$ where $\mu(N) = 0$. Since Π_p is linear and surjective, we can construct a sequence of E-valued simple functions, $\{g'_k\}$ such that $\Pi_p \circ g'_k = g_k$. Hence

$$p(f - g'_k) = \|\Pi_p \circ f - g_k\|_p \to 0$$

on $\Omega \backslash N$. This means that f is measurable by seminorms.

□

For a discussion on the integrability properties of functions measurable by seminorm, see [Mar07] authored by V. Marraffa.

Corollary 4.9. *Let E be a nuclear space and X a Banach Space. Let $u : E \to X$ be a continuous linear map. If $f : \Omega \to E$ is a weakly μ-measurable function then $u \circ f$ is μ-measurable.*

Theorem 4.10. *Let E be a locally convex space, F a quasi-complete locally convex space and $u : E \to F$ a nuclear map. If $f : \Omega \to E$ is a weakly μ-measurable function then $u \circ f$ is a μ-measurable function.*

Proof. Since $u_0 \circ \phi$ is a nuclear map, $(u_0 \circ \phi)(E)$ is a separable subset of F_B. Hence $u_0 \circ \phi \circ f$ takes its values μ-essentially in a separable set.

Since f is weakly-μ-measurable and $y' \circ u_0 \circ \phi \in E'$ for all $y' \in F'_B$, there exists a sequence of scalar-valued simple functions (r_n) such that

$$r_n \to \langle f, y' \circ u_0 \circ \phi \rangle = \langle u_0 \circ \phi \circ f, y' \rangle$$

Hence $u_0 \circ \phi \circ f$ is weakly-μ-measurable. It follows from Pettis' measurability theorem that $u_0 \circ \phi \circ f$ is μ-measurable. It follows that there exists a sequence of F_B-valued simple functions (g_n) that tends to $u_0 \circ \phi \circ f$, μ-a.e. on a set $\Omega \backslash N$ where $\mu(N) = 0$. Hence $\Psi \circ g_n$ tends to $u \circ f$, μ-a.e. on $\Omega \backslash N$. The theorem is proved. □

Lemma 4.11. *(cf. [Sch71, Lemma 1, p. 169]). Let E and F be locally convex spaces, $u : E \to F$ a continuous linear map which maps a suitable 0-neighbourhood of E into a weakly compact subset of F. Then u'', the second adjoint of u, maps the bidual E'' into $F \subset F''$.*

Theorem 4.12. *Let E be a locally convex space, F a quasi-complete locally convex space, $u : E \to F$ a continuous linear map which maps a suitable 0-neighbourhood of E into a weakly compact subset of F. If $f : \Omega \to E$ is a Dunford integrable function then $u \circ f$ is a Pettis integrable function.*

Proof. Since f is a Dunford integrable, there exists a finitely additive measure $m_f : \Sigma \to E''$ where

$$\langle m_f(A), x' \rangle = \int_A \langle f, x' \rangle d\mu$$

for some $A \in \Sigma$. Then $u \circ f$ is Pettis integrable because

$$\begin{aligned}
\int_A \langle u \circ f, y' \rangle d\mu &= \int_A \langle f, u' \circ y' \rangle d\mu \\
&= \langle m_f(A), u' \circ y' \rangle \\
&= \langle u'' \circ m_f(A), y' \rangle
\end{aligned}$$

where u' and u'' are the adjoint and second adjoint, respectively, of u. It follows from Theorem 4.11 that, $u'' \circ m_f$ takes its values in F, hence $u \circ f$ is Pettis integrable. □

It is of course important to note that this means that

$$m_{u \circ f} := u'' \circ m_f : \Sigma \to F$$

is a bounded σ-additive measure.

Theorem 4.13. *([Rod06]). Let $u : X \to Y$ be an absolutely summing mapping between two Banach spaces and $f : \Omega \to X$ a Dunford integrable function. Let $g : \Omega \to Y$ be scalarly equivalent to $u \circ f$. Then g is Bochner integrable if and only if g is μ-measurable.*

Theorem 4.14. *Let E be a locally convex space, F a quasi-complete locally convex space, $u : E \to F$ a nuclear map and $f : \Omega \to E$ a Dunford integrable function. Then $u \circ f$ is Bochner integrable.*

Proof. We verify that $u \circ f$ satisfies Definition 1.9.

It follows from Theorem 4.10 that the function $u \circ f$ is μ-measurable. According to the same theorem, it also follows that $u_0 \circ \phi \circ f$ is μ-measurable since u is a nuclear map if and only if u_0 is nuclear.

It is trivial to show that since f is weakly μ-measurable, so is $\phi \circ f$. Hence $u_0 \circ \phi \circ f$ is Bochner integrable, since it is μ-measurable and scalarly equivalent to itself.

Hence, there exists a sequence of (defining) Σ-simple functions (h_n) taking values in F_B such that

$$\int_\Omega \|h_n - u_0 \circ \phi \circ f\|_{F_B} d\mu \to 0$$

Let $p \in P_F$ and let $\Phi_p : F \to F_p$ denote a quotient map.

The map

$$\Phi_p \circ \Psi : F_B \to F_p$$

is a continuous linear functional between two Banach spaces which means that $\Phi_p \circ \Psi$ is a bounded linear functional. Hence there exists a finite scalar $M_p > 0$ such that for every $h \in F_B$,

$$\|\Phi_p \circ \Psi(h)\|_p \leq M_p \|h\|_{F_B}$$

from which it follows that

$$p(u \circ f) = \|\Phi_p \circ u \circ f\|_p \leq M_p \|u_0 \circ \phi \circ f\|_{F_B}$$

hence

$$\begin{aligned}\int_\Omega p(u \circ f) d\mu &= \int_\Omega \|\Phi_p \circ u \circ f\|_p d\mu \\ &\leq M_p \int_\Omega \|u_0 \circ \phi \circ f\|_{F_B} d\mu < \infty\end{aligned}$$

and further

$$\begin{aligned}\int_\Omega p(\Psi \circ h_n - u \circ f) d\mu &= \int_\Omega \|\Phi_p \circ \Psi \circ (h_n - u_0 \circ \phi \circ f)\|_{F_B} d\mu \\ &\leq M_p \int_\Omega \|h_n - u_0 \circ \phi \circ f\|_{F_B} d\mu\end{aligned}$$

Notice that $(\Psi \circ h_n)$ is a sequence of Y-valued Σ-simple functions.

This inequality implies that $p(\Psi \circ h_n - u \circ f) \in L_1(\mu)$ since $h_n - u_0 \circ \phi \circ f \in L_1(\mu, X)$ and since $\int_\Omega \|h_n - u_0 \circ \phi \circ f\|_{F_B} d\mu \to 0$ it follows that $\int_\Omega p(\Psi \circ h_n - u \circ f) d\mu \to 0$. Hence $u \circ f$ is Bochner integrable. $\square$

Chapter 5

Factorization of Measurable Functions

In this chapter we derive results concerning the factorization of vector-valued, set-valued and operator-valued measurable functions. Of special interest is the existence of an operator between two L_1-spaces.

In this chapter, to correspond with the main related reference material, we shall refer to mappings as operators.

We utilize a generalization of a celebrated result in classical stochastic processes, sometimes called Doob-Dynkin's Lemma, which we refer to as the Factorization Theorem.

Theorem 5.1. *(cf. [Rao84, Proposition 3, p. 7]). If (Ω_i, Σ_i), $i = \alpha, \beta$ are two measure spaces, $f : \Omega_\alpha \to \Omega_\beta$ is a $(\Sigma_\alpha, \Sigma_\beta)$-measurable function and $g : \Omega_\alpha \to \mathbb{R}$ is a $(\Sigma_\alpha, \mathcal{B}(\mathbb{R}))$-measurable function, where $\mathcal{B}(\mathbb{R})$ indicates the Borel sets on $\mathbb{R}$, then there exists a $(\Sigma_\beta, \mathcal{B}(\mathbb{R}))$-measurable function $h : \Omega_\beta \to \mathbb{R}$ such that $g = h \circ f$ if and only if g is measurable with respect to the smallest σ-field generated by f.*

In Chapter 5.1 we generalize the Factorization Theorem to the case where g takes its values in a Polish space and more generally in a standard measurable space. We then give a similar result for the case where g is a μ-measurable Fréchet space-valued function.

We find an application of the Factorization Theorem in the theory of multifunctions presented in Chapter 5.2.1 and in the theory of operator-valued measures which is presented in Chapter 5.2.2. A well-known application of the Factorization Theorem is found in the theory of conditional expectations. In Chapter 5.2.3 we show a similar result for conditional expectation with respect to μ-measurable functions. Finally we consider the factorization of operators on $L_1(\mu)$ in Chapter 5.2.4.

5.1 Core Results

Here we give a complete proof of the Factorization Theorem for the case, referring back to the introduction, where the function g and hence h take their values in a Polish space. From there we can extend this result further to where g and h take values in a standard measure space and to the case where g is μ-measurable.

Throughout this section, unless mentioned otherwise, P is a Polish space, Ω_α a set and $(\Omega_\beta, \Sigma_\beta)$ a measurable spaces. Let

$$f : \Omega_\alpha \to \Omega_\beta$$

be a function with $\sigma(f)$ the smallest σ-field on Ω_α making f measurable.

Lemma 5.2. *Let P be an uncountable Polish space, Ω_α a set and let $(\Omega_\beta, \Sigma_\beta)$ be a measurable spaces. A function $g : \Omega_\alpha \to P$ is $\sigma(f)$-measurable i.e. $\sigma(g) \subseteq \sigma(f)$ if and only if there exists a Σ_β-measurable function $h : \Omega_\beta \to P$ with the property that $g = h \circ f$.*

Proof. There exists a Borel isomorphism $i : P \to \mathbb{R}$, i.e. a bijection i which is $(\mathcal{B}(P), \mathcal{B}(\mathbb{R}))$-measurable and i^{-1} is $(\mathcal{B}(\mathbb{R}), \mathcal{B}(P))$-measurable, see [Coh80, Theorem 8.3.6]. It follows that $i \circ g$ is a $(\Sigma_\alpha, \mathcal{B}(\mathbb{R}))$-measurable function. According to the original Factorization Theorem, cf.

[Rao84, Proposition 3, p. 7], there exists a $(\Sigma_\beta, \mathcal{B}(\mathbb{R}))$-measurable function $\widetilde{h} : \Omega_\beta \to \mathbb{R}$ such that $i \circ g = \widetilde{h} \circ f$, hence $h := i^{-1} \circ \widetilde{h}$ is a $(\Sigma_\beta, \mathcal{B}(P))$-measurable function and $g = h \circ f$. □

It is known that for every Polish space P, the measurable space $(P, \mathcal{B}(P))$ is isomorphic to one of the following Polish spaces: $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$ and $(\mathbb{N}_f, \mathcal{P}(\mathbb{N}_f))$ where $\mathcal{P}$ indicates power sets and $\mathbb{N}_f$ is any finite subset of $\mathbb{N}$.

We now proceed to give a proof that is valid for all three cases. The proof is along the lines of the original case, again see [Rao84, Proposition 3, p. 7].

We know that if $A_1, A_2 \in \sigma(f)$ are disjoint then it is not guaranteed that $B_1, B_2 \in \Sigma_\beta$, where $f^{-1}(B_1) = A_1$ and $f^{-1}(B_2) = A_2$, are disjoint.

Proposition 5.3. *If $(A_i) \subset \mathcal{P}(\Omega_\alpha, \sigma(f))$ is a sequence of pairwise disjoint sets and for each A_i there exists a set $B_i \in \Sigma_\beta$ such that $A_i = f^{-1}(B_i)$, then it can be assumed that the elements of the collection (B_i) are pairwise disjoint.*

Proof. We prove the theorem for the case of two sets $A_1, A_2 \in \sigma(f)$. Let $B_1, B_2 \in \Sigma_\beta$ be any two sets such that $A_1 = f^{-1}(B_1)$ and $A_2 = f^{-1}(B_2)$. Let $C = B_2 - B_1$ then $C \cap B_1 = \emptyset$ and $f^{-1}(C) = f^{-1}(B_2 - B_1) = f^{-1}(B_2) \cap [f^{-1}(B_1)]^c = A_2 \cap A_1^c = A_2$. The full result follows by induction. □

The Factorization Theorem for the case where P is a finite Polish space is contained in the following lemma. This is because measurable functions on finite Polish spaces are essentially simple functions.

Lemma 5.4. *Let $g : \Omega_\alpha \to P$ be a Σ-simple function where $\Sigma \subseteq \sigma(f)$ is a σ-field then there exists a Σ_β-simple function $h : \Omega_\beta \to P$ such that $g = h \circ f$.*

Proof. There exists a finite set $(s_i) \subset P$ and a set $(A_i) \subset \mathcal{P}(\Omega_1, \Sigma)$ such that $g = \sum_{i=1}^n s_i \chi_{A_i}$. From Proposition 5.3 we can construct a sequence of pairwise disjoint sets $(B_i) \subset \Sigma_\beta$. Let

$$h = \sum_{i=1}^n s_i \chi_{B_i}$$

then h is a simple function on $(\Omega_\beta, \Sigma_\beta)$. Next,

$$\begin{aligned} h(f(\omega)) &= \sum_{i=1}^{n} s_i \chi_{B_i}(f(\omega)) \\ &= \sum_{i=1}^{n} s_i \chi_{f^{-1}(B_i)}(\omega) \\ &= g(\omega) \end{aligned}$$

□

Theorem 5.5. *Let P be a Polish space, Ω_α a set and let $(\Omega_\beta, \Sigma_\beta)$ be a measurable space. A function $g : \Omega_\alpha \to P$ is $\sigma(f)$-measurable i.e. $\sigma(g) \subseteq \sigma(f)$ if and only if there exists a Σ_β-measurable function $h : \Omega_\beta \to P$ with the property that $g = h \circ f$.*

Proof. Since g is a measurable function, there exists a sequence (g_k) of $\sigma(g)$-simple functions such that $g_k(\omega) \to g(\omega)$ for all $\omega \in \Omega_\alpha$.

Let (h_k) be a sequence of Σ_β-simple functions where $h_k : \Omega_\beta \to P$ is defined in terms of g_k as in Lemma 5.4. Now, we know from the proof of Theorem 5.7 that

$$L = \{\omega \in \Omega_\beta : (h_k(\omega)) \text{ is Cauchy}\} \in \Sigma_\beta$$

and

$$h(\omega) := \lim_{k\to\infty} h_k(\omega)\chi_L(\omega)$$

is Σ_β-measurable. Since $(h_k \circ f)(\omega) = g_k(\omega) \to g(\omega)$ for all $\omega \in \Omega_\alpha$, it follows that $f(\omega) \in L$ for all $\omega \in \Omega_\alpha$, hence $(h \circ f)(\omega) = g(\omega)$ for all $\omega \in \Omega_\alpha$.

For the converse, let A be any set in $\sigma(g)$ then there exists a set $B \in \mathcal{B}(P)$ such that

$$\begin{aligned} A &= \{\omega \in \Omega_\alpha : g(\omega) \in B\} \\ &= \{\omega \in \Omega_\alpha : h(f(\omega)) \in B\} \\ &= \{\omega \in \Omega_\alpha : f(\omega) \in h^{-1}(B)\} \\ &= f^{-1}(h^{-1}(B)) \in \sigma(f) \end{aligned}$$

since $h^{-1}(B) \in \Sigma_\beta$. □

Corollary 5.6. *Let (S, Σ) be a standard measurable space and let $(\Omega_\beta, \Sigma_\beta)$ be a measurable space. A function $g : \Omega_\alpha \to S$ is $\sigma(f)$-measurable i.e. $\sigma(g) \subseteq \sigma(f)$ if and only if there exists a Σ_β-measurable function $h : \Omega_\beta \to S$ with the property that $g = h \circ f$.*

Proof. Let i be the isomorphism that associates S with a Polish space P. Since $i \circ g : \Omega \to P$ it follows that $\{(i \circ g)^{-1}(A) : A \in \mathcal{B}(P)\} = \{g^{-1}(A) : A \in \Sigma\}$ thus $\sigma(i \circ g) = \sigma(g)$.

If $\sigma(g) \subseteq \sigma(f)$ then there exists a $(\Sigma_\beta, \mathcal{B}(P))$-measurable function $\widetilde{h} : \Omega_\beta \to P$ such that $i \circ g = \widetilde{h} \circ f$. Let $h := i^{-1} \circ \widetilde{h}$ then $g = h \circ f$. The converse is trivial. □

We now consider the link between Borel-type measurability and μ-measurability.

Theorem 5.7. (Ω, Σ, μ). *Let F be a Fréchet space and $k : \Omega \to F$ a μ-measurable function, then*

(i) there exists a complete, separable, (Polish) subspace S of F and a $(\Sigma, B(S))$-measurable function $\widetilde{k} : \Omega \to S$ such that $k = \widetilde{k}$ μ-a.e.;

(ii) $\widetilde{k}$ is μ-a.e. unique, that is, if S_1 is another Polish spaces and $\widetilde{k}_1 : \Omega \to S_1$ a Σ-measurable function where $k = \widetilde{k}_1$ μ-a.e. then $\widetilde{k} = \widetilde{k}_1$ μ-a.e.;

(iii) if (Ω, Σ, μ) is complete then k is a $(\Sigma, \mathcal{B}(S))$-measurable function.

Proof. (i). Since k is μ-measurable, there exists a sequence (k_n) of Σ-simple functions which converges strongly to k on a set $E \in \Sigma$ where $\mu(E^c) = 0$. The set

$$T := \cup_{n=1}^{\infty} \{k_n(\omega) : \omega \in \Omega\}$$

is denumerable hence $S := \widetilde{sp}\ T$ is a separable closed subspace of F, that is, a Polish space. Thus (k_n) takes its values in S. Since (k_n) is a sequence of Σ-simple functions, its elements are $(\Sigma, \mathcal{B}(S))$-measurable. Therefore

$$\begin{aligned} L &= \{\omega \in \Omega : (k_n(\omega)) \text{ is Cauchy}\} \\ &= \cap_{i=1}^{\infty} \cup_{n=1}^{\infty} \cap_{l=1}^{\infty} \cap_{j=1}^{\infty} \{\omega \in \Omega : d(k_{l+n}(\omega), k_{j+n}(\omega)) < \frac{1}{2^i}\} \in \Sigma, \end{aligned}$$

since $\omega \to d(k_{l+n}(\omega), k_{j+n}(\omega))$ is continuous hence $(\Sigma, \mathcal{B}(S))$-measurable and S is Polish, see [Coh80, Proposition 8.1.9]. Then $\mu(E^c \cap L^c) = 0$ and $\widetilde{k} := k\chi_L = k$ μ-a.e.. Since $(k_n\chi_L)$ and $\widetilde{k}$ take their values in S, it follows that $\widetilde{k}$ is Σ-measurable.

(ii). There exist sets E and E_1 in Σ such that $\mu(E^c) = 0 = \mu(E_1^c)$ and $k(\omega) = \widetilde{k}(\omega)$ for all $\omega \in E$ and $k(\omega) = \widetilde{k}_1(\omega)$ for all $\omega \in E_1$. Thus $\widetilde{k}(\omega) = \widetilde{k}_1(\omega)$ for all $\omega \in E \cap E_1$ and

$$\mu((E \cap E_1)^c) = \mu(E^c \cup E_1^c) \leq \mu(E^c) + \mu(E_1^c) = 0$$

(iii). If $A \in \mathcal{B}(F)$ then $k^{-1}(A) = k^{-1}(A \cap S) \cup k^{-1}(A \backslash S)$. Now, $k^{-1}(A \backslash S) \subset E^c$ and $\mu(E^c) = 0$ thus $k^{-1}(A \backslash S) \in \Sigma$. Since $k^{-1}(A \cap S) = \widetilde{k}^{-1}(A \cap S) \in \Sigma$, it follows that k is Σ-measurable. □

Corollary 5.8. *Let $\mu_\alpha : \sigma(f) \to \mathbb{R}$ be a finite σ-additive measure, F a Fréchet space and $g : \Omega_\alpha \to F$ a μ_α-measurable function. There exist a complete separable (Polish) subspace S of F and a $(\Sigma_\beta, \mathcal{B}(S))$-measurable function $h : \Omega_\beta \to S \subseteq X$, with the property that $g = h \circ f$ μ_α-a.e. if and only if $\sigma(\widetilde{g}) \subseteq \sigma(f)$ where $\widetilde{g} = g$ μ_α-a.e. and $\widetilde{g} : \Omega_\alpha \to S$ is a $(\Sigma_\alpha, \mathcal{B}(S))$-measurable function.*

Proof. According to Theorem 5.7 there exist a complete separable subspace S of F and a measurable function $\widetilde{g} : \Omega_1 \to S$ such that $\widetilde{g} = g$ μ_α-a.e. Theorem 5.5 gives the existence of a Σ_β-measurable function $h : \Omega_\beta \to S$ such that $\widetilde{g} = h \circ f$ then $g = h \circ f$ μ_α-a.e. □

For $i = \alpha, \beta$, let $(\Omega, \Sigma_i, \mu_i)$ be a finite measure space and $r_i : \Omega \to F$ μ_i-measurable. According to Lemma 5.7 there exists a Σ_i-measurable function $\widetilde{r}_i : \Omega \to S_i$ such that $\widetilde{r}_i = r_i$ μ_i-a.e. where S_i is some complete separable (Polish) subspace of F.

A natural question is, if Σ_β is a sub-σ-field of Σ_α, is S_β a subset of S_α and is $\sigma(r_\beta)$ a sub-σ-field of $\sigma(r_\alpha)$? The following example shows that this is not the case, not even if $\sigma(r_\alpha)$ and $\sigma(r_\beta)$ are closely related.

Example 5.9. *Let $X = \mathbb{R}^3$, $\Omega_\alpha = [0,1] = \Omega_\beta$, Σ_α the Borel sets on Ω_α and let μ_α be the Lebesgue measure on Ω_α. Let Σ_β a sub-σ-field of Σ_α on Ω_β and μ_β the restriction of μ_α to Σ_β. Let $r_\beta := x\chi_A + y\chi_B$ with A, B non-empty subset of Σ_β and hence of Σ_α and x and y linearly independent vectors in X and let $r_\alpha := x\chi_A$. Clearly, r_i is a Σ_i-simple function and hence μ_i-measurable, for $i = \alpha, \beta$. $\mathbb{R}^3$ is of course a Polish space, but we can construct smaller Polish spaces as done in Lemma 5.7 as follows: $S_\alpha = sp\{x\}$ and $S_\beta = sp\{x, y\}$, so S_α is a line contained in the plain S_β. Hence, $\widetilde{r}_i : \Omega_i \to S_i$ is measurable functions and $\widetilde{r}_i = r_i$ for $i = \alpha, \beta$. It is then obvious that $\{\widetilde{r}_\alpha(\omega) : \omega \in \Omega_\alpha\} = \{0, x\} \subset \{0, x, y\} = \{\widetilde{r}_\beta(\omega) : \omega \in \Omega_\beta\}$ and thus that $\sigma(r_\alpha) \subset \sigma(r_\beta)$.*

5.2 Applications

5.2.1 Set-valued Operators

In this section we investigate the factorization of set-valued operators (multifunctions). Let (Ω, Σ) be a measurable space and let C_X (O_X) indicate all the closed (open) subsets of a topological space X. A multifunction $M : \Omega \to C_X$ is a mapping of subsets from Ω to closed subsets of a topological space X. The multifunction M is Σ-measurable if

$$M^{-}(A) := \{\omega \in \Omega : M(\omega) \cap A \neq \emptyset\} \in \Sigma$$

whenever $A \in O_X$.

Theorem 5.10. *([Iof79, Theorem 1]). Let X be a Polish space (compact metrizable space). There exists a Polish space (compact metrizable space) Z such that for any Σ-measurable multifunction $M : \Omega \to C_X$ with $\Sigma := \sigma(M)$, that is the σ-field on Ω generated by all sets of the form $M^{-}(A)$ where $A \in O_X$, there exists a mapping $f : \Omega \times Z \to X$ such that*

(i) $f(\omega, \cdot)$ is continuous for all $\omega \in \Omega$ and $f(\cdot, z)$ is Σ-measurable for all $z \in Z$;

(ii) for all $\omega \in dom(M) := \{\omega \in \Omega : M(\omega) \neq \emptyset\}$, one has $M(\omega) = f(\omega, Z) := \{f(\omega, z) : z \in Z\}$.

We say that (Z, f) represents M.

It is important to note that the construction of the Polish space Z only depends on the Polish space X and is independent of M.

Theorem 5.11. *Let X be a Polish space and let $M_i : \Omega \to C_X$ be a Σ-multifunction with $i = \alpha, \beta$. Let (Z, f_i) indicate a representation of M_i which is a consequence of Theorem 5.10. Then (i) $\leftrightarrow$ (ii) and (ii) $\to$ (iii) $\to$ (iv) where*

(i) $\sigma(f_\beta) \subseteq \sigma(f_\alpha)$;

(ii) there exists a $(\mathcal{B}(X), \mathcal{B}(X))$-measurable function $h : X \to X$ such that $f_\beta = h \circ f_\alpha$;

(iii) there exists a $(\mathcal{B}(X), \mathcal{B}(X))$-measurable function $h : X \to X$ such that $M_\beta(\omega) = h(M_\alpha(\omega))$ for all $\omega \in \Omega$;

(iv) $\sigma(M_\beta) \subseteq \sigma(M_\alpha)$.

Proof. $(i) \Leftrightarrow (ii)$ is as a consequence of Theorem 5.5 and $(ii) \Rightarrow (iii)$ follows from Theorem 5.10(ii).

$(iii) \Rightarrow (iv)$: We first show that

$$\sigma(\{f(\cdot, z) : z \in Z\}) = \sigma(M)$$

Let $M : \Omega \to C_X$ be a Σ-multifunction represented by (Z, f). Since $f(\cdot, z)$ is $\sigma(M)$-measurable for every $z \in Z$ it immediately follows that $\sigma(\{f(\cdot, z) : z \in Z\}) \subseteq \sigma(M)$. Now,

$$\begin{aligned} M^-(A) &= \{\omega \in \Omega : M(\omega) \cap A \neq \emptyset\} \\ &= \{\omega \in \Omega : f(\omega, z) \in A \text{ for some } z \in Z\} \\ &\in \sigma(\{f(\cdot, z) : z \in Z\}) \end{aligned}$$

for every $A \in O_X$. Since $\sigma(M)$ is generated by sets of the for $M^-(A)$ for all $A \in O_X$, it follows that,

$$\sigma(\{f(\cdot, z) : z \in Z\}) = \sigma(M) \tag{5.2.1}$$

Since $\sigma(M_\beta)$ is generated by sets of the form $M_\beta^-(A)$ for all $A \in O_X$, it follows that,

$$\begin{aligned} M_\beta^-(A) &= \{\omega \in \Omega : h \circ M_\alpha(\omega) \cap A \neq \emptyset\} \\ &= \{\omega \in \Omega : h \circ f_\alpha(\omega, z) \in A \text{ for some } z \in Z\} \\ &\in \sigma(\{f_1(\cdot, z) : z \in Z\}) = \sigma(M_\alpha) \end{aligned}$$

hence $\sigma(M_\beta) \subseteq \sigma(M_\alpha)$ □

It is obvious that (iii) need not imply (ii) because each multifunction can be represented by different (point-valued) functions. The following example illustrates that (iv) need not imply (i).

Example 5.12. (Ω, Σ). *Let* $Z = [0,1]$, $X = [0,2]$ *where* $f_\alpha(\omega, z) := \chi_A(\omega)$ *for a set* $A \in \Sigma$ *and* $f_\beta(\omega, z) := z$ *then it follows from (5.2.1) that*

$$\sigma(M_\beta) = \sigma(\{f_\beta(\cdot, z) : z \in Z\}) \subset \sigma(\{f_\alpha(\cdot, z) : z \in Z\}) = \sigma(M_\alpha)$$

but

$$\sigma(\{f_\alpha(\omega, \cdot) : \omega \in \Omega\}) \subset \sigma(\{f_\beta(\omega, \cdot) : \omega \in \Omega\})$$

Hence $\sigma(f_\alpha)$ *and* $\sigma(f_\beta)$ *are not comparable.*

5.2.2 Operator-valued Measurable Functions

Let $\mathcal{L}(E, F)$ indicate the space of all continuous linear operators between Fréchet spaces E and F. Let $\mathcal{L}_C(E, F)$ indicate the space $\mathcal{L}(E, F)$ equipped with the topology of uniform convergence on the compact subsets of E, the space $\mathcal{L}_C(E, F)$ is a Lusin space, see [Sch73, Theorem 7, p.112]. It follows that all Hausdorff topologies on $\mathcal{L}(E, F)$ weaker than $\mathcal{L}_C(E, F)$ are also Lusin. These topologies are discussed in [BJY95, p.1818]. They include the topology of simple convergence on $\mathcal{L}(E, F)$ which we shall denote by $\mathcal{L}_S(E, F)$. Let (Ω, Σ, μ) indicate a finite measure space and $\mathcal{L}(E)$ indicate the space of all continuous linear operators on E. An operator-valued function $A : \Omega \to \mathcal{L}(E)$ is strong operator measurable if and only if $A(\cdot)x : \Omega \to E$ is μ-measurable for all $x \in E$. If (Ω, Σ, μ) is a complete measure space and E a separable Banach space then strong operator measurability is equivalent to $(\Sigma, \mathcal{B}(\mathcal{L}_S(E)))$-measurability, see [Joh93].

Theorem 5.13. *Let* E, F *and* G *be three separable Fréchet spaces and let* (Ω, Σ) *be a measurable space. Further let*

$$A : \Omega \to \mathcal{L}(E, F)$$

be $(\Sigma, \mathcal{B}(\mathcal{L}_C(E, F)))$*-measurable and let*

$$B : \Omega \to \mathcal{L}(E, G)$$

be $(\Sigma, \mathcal{B}(\mathcal{L}_C(E,G)))$*-measurable. Then there exists a function*

$$T : \mathcal{L}(E,F) \to \mathcal{L}(E,G)$$

which is

$$(\mathcal{B}(\mathcal{L}_C(E,F)), \mathcal{B}(\mathcal{L}_C(E,G)))\text{-}measurable$$

such that $B = T \circ A$ *if and only if* $\sigma(B) \subseteq \sigma(A)$.

The existence of T follows immediately from Corollary 5.6. The same statement holds if the topology of uniform convergence is replaced by a weaker Hausdorff topology. T does have some linearity properties. Let $u, v \in E$ and $\alpha \in \mathcal{F}$ then $T \circ (\alpha A(u) + A(v)) = T \circ A((\alpha u + v)) = B(\alpha u + v) = \alpha B(u) + B(v) = \alpha T \circ A(u) + T \circ A(v)$.

Corollary 5.14. *Let* (Ω, Σ, μ) *be a complete measure space and* E *a separable Banach space. If operators* $A, B : \Omega \to \mathcal{L}(E)$ *are both strong operator measurable, then there exists a function*

$$T : \mathcal{L}(E) \to \mathcal{L}(E)$$

which is

$$(\mathcal{B}(\mathcal{L}_S(E)), \mathcal{B}(\mathcal{L}_S(E))\text{-}measurable$$

such that $B = T \circ A$ *if and only if* $\sigma(B) \subseteq \sigma(A)$.

Theorem 5.15. *Let* (Ω, Σ, μ) *be a finite measure space and* E *be a Banach space. Let* $A, B : \Omega \to \mathcal{L}(E)$ *be two strong operator measurable functions then* AB *is a strong operator measurable function.*

Proof. A is strong operator measurable i.e. $A(\cdot)(x) : \Omega \to E$ is μ-measurable for every $x \in E$, then there exists a $(\Sigma, \mathcal{B}(P))$-measurable (resp. $(\Sigma, \mathcal{B}(P_1))$-measurable) function $\widetilde{A}(\cdot)(x) : \Omega \to P$ ($\widetilde{B}(\cdot)(x) : \Omega \to P_1$) for some separable complete (Polish) subspace P (resp. P_1) of E such that $\widetilde{A}(\cdot)(x) = A(\cdot)(x)$ μ-a.e. (resp. $\widetilde{B}(\cdot)(x) = B(\cdot)(x)$ μ-a.e.) for all $x \in E$. Fix $x \in E$. Since $A(\cdot)(x)$ only takes its values in P, we can restrict the domain of B to P thus $\widetilde{B}|_P(y) : \Omega \to P_1$ for all $y \in P$. □

5.2.3 Conditional Expectation

Definition 5.16. *Let (Ω, Σ, μ) be a finite measure space, $\mathcal{F}$ a sub-σ-field of Σ and let $d \in L_1(\mu, X)$ where X is a Banach space. An element $g \in L_1(\mu, X)$ is called the conditional expectation of d relative to $\mathcal{F}$ if g is $\mu_{\mathcal{F}}$-measurable and*

$$\int_E g d\mu_{\mathcal{F}} = \int_E d d\mu_{\mathcal{F}} \text{ for all } E \in \mathcal{F} \tag{5.2.2}$$

In this case g is denoted by $E(d|\mathcal{F})$.

The integral above is a Bochner integral, see [DU77, Chapter II and V].

Some ideas of Conditional Expectation is contained in the following lemma.

Lemma 5.17. *Let $(\Omega_\alpha, \Sigma_\alpha, \mu_\alpha)$ be a finite measure space, $(\Omega_\beta, \Sigma_\beta)$ a measurable space and $f : \Omega_\alpha \to \Omega_\beta$ a measurable function, that is, $\sigma(f)$ is a sub-σ-field of Σ_α. Let $d : \Omega_\alpha \to X$ be a μ-measurable function and $g : \Omega_\alpha \to X$ a $\mu_{\sigma(f)}$-measurable function such that $g = d$ $\mu_{\sigma(f)}$-a.e. Then there exists a Polish subspace S of X and Σ_β-measurable function $h : \Omega_\beta \to S$ such that $g = h \circ f$ $\mu_{\sigma(f)}$-a.e.*

It should be noted that h depends on more than just the σ-field $\sigma(f)$. Even if two measurable functions $f_1, f_2 : \Omega_\alpha \to \Omega_\beta$, have the same range and generate the same σ-field it still doesn't mean that there exists a single Σ_β-measurable function $h : \Omega_\beta \to X$ such that $g = h \circ f_1$ $\mu_{\sigma(f_1)}$-a.e. and $g = h \circ f_2$ $\mu_{\sigma(f_2)}$-a.e..

Example 5.18. *In the notation above, let Ω_β and X equal $\mathbb{R}$. Let $f_1 : \Omega_\alpha \to \mathbb{R}$ where $f_1 := \chi_{A_+} - \chi_{A_-}$ for any pairwise disjoint sets $A_+, A_- \in \Sigma_\alpha$ of non-zero μ_α-measure. Let $f_2 := -f_1$. Then it is clear that f_1 and f_2 are both μ_α-measurable and measurable i.e. $\sigma(f_1)$ and $\sigma(f_1)$ are both σ-fields on Ω_α. The range of f_1 equals $\{0, -1, 1\}$ which equals the range of f_2 and $\sigma(f_1) = \sigma(f_2)$. Choose $C_+, C_- \in \mathcal{B}(\mathbb{R})$ such that they are pairwise disjoint and C_+ contains 1 but neither -1 nor 0 and C_- contains -1 but neither 0 nor 1. Then $\chi_{A_+}^{-1}(C_+) = A_+$ and*

$(-\chi_{A_-})^{-1}(C_-) = A_-$ *and also* $\chi_{A_-}^{-1}(C_+) = A_-$ *and* $(-\chi_{A_+})^{-1}(C_-) = A_+$. *Let* $h = \chi_{C_+} - \chi_{C_-}$ *then*

$$\begin{aligned} h(f_1(\omega)) &= \chi_{C_+}(f_1(\omega)) - \chi_{C_-}(f_1(\omega)) \\ &= \chi_{C_+}(\chi_{A_+}(\omega)) - \chi_{C_-}(-\chi_{A_-}(\omega)) \\ &= \chi_{A_+}(\omega) - \chi_{A_-}(\omega) \\ &:= g(\omega) \end{aligned}$$

but

$$\begin{aligned} h(f_2(\omega)) &= \chi_{C_+}(f_2(\omega)) - \chi_{C_-}(f_2(\omega)) \\ &= \chi_{C_+}(\chi_{A_-}(\omega)) - \chi_{C_-}(-\chi_{A_+}(\omega)) \\ &= \chi_{A_-}(\omega) - \chi_{A_+}(\omega) \\ &:= -g(\omega) \end{aligned}$$

Theorem 5.19. *(cf. [DU77, Theorem V.I.4]). Let* (Ω, Σ, μ) *be a finite measure space and let* $\mathcal{F}$ *be a sub-*σ*-field of* Σ*. Then* $E(d|\mathcal{F})$ *exists for every* $d \in L_1(\mu, X)$*.*

We continue with the notation of Lemma 5.17

Corollary 5.20. *Let* $d \in L_1(\mu, X)$ *then* $E(d|\sigma(f)) := h \circ f$ $\mu_{\sigma(f)}$*-a.e., that is, conditional expectation is* $\mu_{\sigma(f)}$*-a.e. equal to a* Σ_β*-measurable function on* Ω_β *which takes values in a Polish subspace of* X*.*

5.2.4 Operators on $L_1(\mu)$ and $L_1(\mu, X)$

Definition 5.21. *A Banach space valued bounded linear operator* $T : L_1(\mu) \to X$ *is Riesz representable (or simply representable) if there exists a* $g \in L_\infty(\mu, X)$ *such that*

$$Tf = \int_\Omega fg d\mu$$

for all $f \in L_1(\mu)$*.*

For a complete discussion on representable operators, see [DU77, Chapter III].

Theorem 5.22. *(Factorization Operator). Let $(\Omega_i, \Sigma_i, \mu_i)$ be finite measure spaces, for $i = \alpha, \beta$ and let $f : \Omega_\alpha \to \Omega_\beta$. Let $(X, \|\cdot\|)$ be a Banach space and let $\|\cdot\|_{1,i}$ indicate the norm on $L_1(\mu_i, X)$ where $i = \alpha, \beta$. There exists a bounded linear operator $F : L_1(\mu_\alpha, X) \to L_1(\mu_\beta, X)$ with the property that*

$$F(g) \circ f = g \ \mu_\alpha\text{-a.e.}$$

for all $g \in L_1(\mu_\alpha, X)$ if and only if $\Sigma_\alpha := \{f^{-1}(A) : A \in \Sigma_\beta\}$.

This proof is along the lines of [DU77, Lemma III.2.1].

Proof. Let $F_{\pi_\beta} : L_1(\mu_\alpha, X) \to L_1(\mu_\beta, X)$ be defined by

$$F_{\pi_\beta}(s) := \Sigma_{B \in \pi_\beta} \frac{\int_{f^{-1}(B)} s d\mu_\alpha}{\mu_\beta(B)} \chi_B \tag{5.2.3}$$

(observing the convention that 0/0=0) for all $s \in L_1(\mu_\alpha, X)$ where $\pi_\beta \in \mathcal{P}(\Omega_\beta, \Sigma_\beta)$.

Let $s \in L_1(\mu_\alpha, X)$, then

$$\begin{aligned} \|F_{\pi_\beta}(s)\|_{1,\beta} &= \|\Sigma_{B \in \pi_\beta} \frac{\int_{f^{-1}(B)} s d\mu_\alpha}{\mu_\beta(B)} \chi_B\|_{1,\beta} \\ &= \Sigma_{B \in \pi_\beta} \| \int_{f^{-1}(B)} s d\mu_\alpha \| \\ &\le \int_{\Omega_\alpha} \|s\| d\mu_\alpha = \|s\|_{1,\alpha} \end{aligned}$$

hence $\|F_{\pi_\beta}\| \le 1$. We first consider the dense linear subspace, M, of $L_1(\mu_\alpha, X)$ consisting of all simple functions. A quick calculation shows that if $s \in M$, say, $s = \Sigma_{A \in \pi} s_A \chi_A$ then the net $(F_{\pi_\beta}(s))_{\pi_\beta}$ is eventually constant. Let $\pi' \in \mathcal{P}(\Omega_\beta, \Sigma_\beta)$ be such a partition, then,

$$F_{\pi'}(s) = \Sigma_{B \in \pi'} s_A \chi_B$$

where $A = f^{-1}(B)$, hence

$$\begin{aligned}(F_{\pi'}(s))(f(\alpha)) &= \Sigma_{B\in\pi'} s_A \chi_B(f(\alpha)) \\ &= \Sigma_{B\in\pi'} s_A \chi_A(\alpha) = s(\alpha)\end{aligned}$$

for all $\alpha \in \Omega_\alpha$.

Since the pointwise limit of bounded linear operators is again a bounded linear operator, the necessity is proved.

The converse is proven in the same way as Corollary 5.8.

□

Let $K_\infty(\mu_i, X)$ indicate the subspace of $L_\infty(\mu_i, X)$ consisting of all members of $L_\infty(\mu_i, X)$ whose ranges are essentially relatively compact.

Corollary 5.23. *(Factorization Operator). There exists a bounded linear operator*

$$F : K_\infty(\mu_\alpha, X) \to K_\infty(\mu_\beta, X)$$

with

$$F(g) \circ f = g \ \mu_\alpha\text{-}a.e.$$

for all $g \in K_\infty(\mu_\alpha, X)$.

Operators constructed in this way will be called factorization operators.

Lemma 5.24. $F_{\pi_\beta}(X_\alpha)$ *is isometrically isomorphic to* ℓ_p^∞, *where* $n = dim(F_{\pi_\beta}(X_\alpha))$

Theorem 5.25. *Let* X *be a Banach space and* $G : L_1(\mu_\alpha) \to X$ *a representable operator. Then there exists a bounded linear operator* $F : L_1(\mu_\alpha) \to L_1(\mu_\beta)$ *and a representable operator* $H : L_1(\mu_\beta) \to X$ *such that* $G = HF$.

Proof. Since G is representable, there exists a $g \in L_\infty(\mu_\alpha, X)$ such that if $s \in L_1(\mu_\alpha)$ then

$$\begin{aligned} G(s) &= \int_{\Omega_\alpha} sgd\mu_\alpha \\ &= \int_{\Omega_\alpha} (F(s) \circ f)(h \circ f) d\mu_\alpha \\ &= \int_{\Omega_\beta} F(s) h d\mu_\beta \\ &= (HF)(s) \end{aligned}$$

where $F : L_1(\mu_\alpha) \to L_1(\mu_\beta)$ is a factorization operator and $h : \Omega_\beta \to X$ is a μ_β-measurable function with $g = h \circ f$. From the above lemma, it follows that $g \in L_\infty(\mu_\beta)$. The operator $H(s) := \int_{\Omega_\beta} shd\mu_\beta$ for all $s \in L_1(\Omega_\beta, \mu_\beta)$.

□

Example 5.26. *If $\alpha = \beta$, that is $\mu_\alpha = \mu_\beta$ and $L_1(\mu_\alpha) = L_1(\mu_\beta)$ then F is the identity operator.*

Corollary 5.27. *Let X be a Banach space and $G : L_1(\mu_\alpha) \to X$ be a compact linear operator. Then there exists a bounded linear operator $F : L_1(\mu_\alpha) \to L_1(\mu_\beta)$ and a compact linear operator $H : L_1(\mu_\beta) \to X$ such that $G = HF$.*

Proof. There exists a sequence of conditional expectation operators, $G_\pi := L_1(\mu_\alpha) \to L_1(\mu_\alpha)$ defined by

$$G_{\pi_\alpha}(s) := \Sigma_{A \in \pi_\alpha} \frac{\int_{A \in \pi_\alpha} sd\mu_\alpha}{\mu_\alpha(A)} \chi_A \tag{5.2.4}$$

such that GG_{π_α} is a finite rank operator and $\lim_{\pi_\alpha} \|GG_{\pi_\alpha} - G\| = 0$, see [DU77, Corollary III.2.3]. From [DU77, Theorem III.2.2] we know that there exist a $g_{\pi_\alpha} \in K_\infty(\mu_\alpha, X)$, defined by

$$g_{\pi_\alpha} = \Sigma_{A\in\pi_\alpha}\frac{G(\chi_A)}{\mu_\alpha(A)}\chi_A$$

such that if $s \in L_1(\mu_\alpha)$ then

$$\begin{aligned}
GG_{\pi_\alpha}(s) &= \int_{\Omega_\alpha} s g_{\pi_\alpha} d\mu_\alpha \\
&= \int_{\Omega_\alpha} (F(s)\circ f)(\hat{F}(g_{\pi_\alpha})\circ f) d\mu_\alpha \\
&= \int_{\Omega_\beta} F(s)\hat{F}(g_{\pi_\alpha}) d\mu_\beta \\
&= (H_{\pi_\beta}F)(s)
\end{aligned}$$

where $F : L_1(\mu_\alpha) \to L_1(\mu_\beta)$ and $\hat{F} : K_\infty(\mu_\alpha, X) \to K_\infty(\mu_\beta, X)$ are factorization operators. Since $\hat{F}(g_{\pi_\alpha})$ is a simple function it follows that $H_{\pi_\beta} : L_1(\mu_\beta) \to X$ defined by $H_{\pi_\beta}(t) := \int_{\Omega_\beta} t\hat{F}(g_{\pi_\alpha}) d\mu_\beta$ for all $t \in L_1(\mu_\beta)$, is a finite rank operator. Since we assume that (π_β) is directed by refinement it follows that the operator $H : L_1(\mu_\beta) \to X$ is a compact linear operator. A quick computation reveals that $G = HF$.

□