Chapter 6

Structural vs. Reduced-Form Models of Default

6.1 Introduction

In this chapter we review the strengths, drawbacks and inherent properties of structural and reduced-form models. In chapters 4 and 5, we showed that because of the intricate properties of default-risky debt, a model characterizing default-risky debt value requires a fair amount of complexity. The reason for this complexity lies in the number of factors driving default-risky debt value coupled with the interaction of these factors. The two main approaches to credit risk modelling (structural and reduced-form) differ in their treatment of the following factors that drive default-risky debt value, together with the interaction between the default process and the default- risk-free rate process.

- Default- risk-free rate process embodied in the short-rate, r_t
- Default process
- Recovery process

The assumptions made regarding these processes and their interactions affect the tractability, simplicity and practical applicability of the different models for default-risky debt. In general, models for pricing default-risky debt can be expressed simply using the following equation:

$$P(r,T,\cdot) = B(r,T) - \delta(\cdot)Q(\cdot)B(r,T) \tag{6.1}$$

where r is the riskless interest rate, T is maturity, $P(\cdot)$ is the price of a default-risky discount bond, $B(\cdot)$ is the price of riskless debt of the same maturity, $Q(\cdot)$ is the pseudo-probability of default and $\delta(\cdot) = 1 - \beta(\cdot)$, where $\beta(\cdot)$ is the recovery rate on default.

Structural models treat the value of the firm as the underlying stochastic process. $\delta(\cdot)$ and $Q(\cdot)$ are written as functions of firm value and the debt claims issued by the firm. While this approach is well-grounded in theory, it has the practical difficulty of being predicated on a difficult to observe stochastic process, the firm value. Reduced-form models treat $\delta(\cdot)$ and $Q(\cdot)$ as stochastic processes, utilizing the information about these functions that is embedded in observed credit spreads and recovery rates, such as in JLT.

Structural models of default posit some dynamics for the firm value process, and assume that there exists a lower threshold (constant or stochastic) which triggers default should firm value ever reach it. In contrast, reduced-form models of default abstract from the firm value process. They effectively assume that default is a jump process, and directly model the probability of such a jump occurring. Simply said, structural models rely on economic arguments of why firms default whereas reduced-form models eliminate the need for an economic explanation of default. The time at which default might occur in a reduced-form model is a random variable. Even in a structural model such as that of Longstaff and Schwartz (1995), the default time is not known in advance because the value of the firm is a random variable. Yet there are technical conditions that make a crucial distinction between the properties of the default time in most structural models and those in reduced-form models. In general, the default time in reduced-form models is more unpredictable than in structural models, where the time of financial distress can be foretold just before it occurs[1]

[1] More precisely, the time of default is predictable under structural models (based on a diffusion process), meaning that there is an increasing sequence of stopping times that converges

The strength of reduced-form models is also their weakness. Divorcing the firm from the intensity process enables modelling default without much information about why the firm defaults. Herein lies the strength. However, modelling default without theoretical guidance runs the risk of both ignoring market information and drawing erroneous conclusions without the tools to discover the appropriate explanation. Herein lies the weakness. The point to remember is that mathematical tractability - not economics - drives the choice of how to specify a reduced-form model.

Structural models for default-risky bonds are well suited if the relationship between prices of different securities issued by the firm is of importance, e.g. for convertible bonds or callable bonds that can be converted into shares when called by the issuer. Furthermore, the model allows the pricing of default-risky bonds directly from fundamentals, from the firm's value. Thus structural models can give a fair price of a default-risky bond as output. Further, questions from corporate finance like optimal capital structure design or the relative powers of shareholders and creditors can be addressed within a structural framework.

The main strength of the structural approach, the orientation towards fundamentals is also the model's weakness. Often it is hard to define a meaningful process for the firm's value, let alone observe it continuously. It can be very hard to calibrate such a firm's value process to market prices. Furthermore the model may very quickly become too complex to analyse in a real-world application. If one were to model the full set of claims on the value of the assets of a medium sized firm one may very well have to price twenty or more classes of claims: from banks, shareholders and private creditors down to workers' wages, taxes and supplier demands. This obviously quickly becomes unfeasible. Another drawback of the structural models is that they cannot incorporate credit-rating changes that occur quite frequently for default-risky bonds. Many default-risky bonds undergo credit downgrades by credit rating agencies before they actually default, and bond prices react to these rating changes either in anticipation or when they occur.

Both structural and reduced-form models cannot readily incorporate financial restructuring that often occurs upon default, such as renegotiating of the

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to the default time, and therefore "foretells" the event of default.

terms of the debt contract by extending the maturity or lowering/postponing the promised payments, exchanging the debt for other forms of securities, or some combination of the above. Also, debt restructurings anticipated by the market will be priced into the value of a defaultable bond in ways that none of these models captures.

6.2 Credit Spreads and default Probability

Reduced-form models predict significantly different term structures of credit spreads than structural models. At the short end of the yield curve, structural models predict that the credit spread drops to zero as maturity goes to zero (i.e., upward sloping term structure of credit spreads), while reduced-form models predict that the spreads remain positive. The theoretical prediction that the term structure of credit spreads should be upward sloping at the short end is not an inherent property of the structural model framework, but rather is due to the assumption that the evolution of firm value follows a diffusion process.

Before considering the implications of the two frameworks at the long end of the yield curve, we describe below a simple reduced-form and a simple structural model that we will use in our analysis.

A simple reduced-form model assumes that default is triggered by a Poisson jump process with stochastic intensity h_t . Suppose we have a standard Poisson process N and define the counting process

$$N_t^* \equiv N(h_t) \tag{6.2}$$

where

$$H_t = \int_0^t h_s ds \tag{6.3}$$

The function h is the *intensity or hazard rate* function of the counting process N_t^* . Let $T_1, T_2, \ldots$ denote the arrival times of the jumps by N. We model the time τ of default as the first time that N^* jumps, so we have

$$H_\tau = T_1 \tag{6.4}$$

We now assume that the hazard rate process is defined in some way and then take an *independent* Poisson process N and define τ by way of equation (6.3) and equation (6.4). From this we have immediately, using *iterated expectations*, that the default probability is given by

$$\begin{aligned} P[\tau \leq T] &= P[T_1 \leq H(T)] \\ &= 1 - P[T_1 > H(T)] \\ &= 1 - E[P[T_1 > H(T)] \mid F_T] \\ &= 1 - E[\exp(-H(T))] \\ &= 1 - \underbrace{E\left[\exp\left(-\int_0^T h_s ds\right)\right]}_{\theta} \end{aligned} \tag{6.5}$$

where F is a sigma field with respect to which is h measurable but which is independent of N and θ is the survival probability.

From equation (6.5), we have that

$$P[\tau \leq \infty \mid F_t] = 1 \tag{6.6}$$

That is, reduced-form models predict that default will occur with certainty at some finite date. In other words, the probability of a firm never defaulting is zero in reduced-form models.

In contrast to reduced-form models, standard structural models assume that firm value V_t evolves dynamically as

$$\begin{aligned} \frac{dV_t}{V_t} &= \mu dt + \sigma dW_t \\ V_0 &> 0 \end{aligned} \tag{6.7}$$

where $\mu \in \Re$ is a drift parameter, $\sigma > 0$ is a volatility parameter, and W is a standard Brownian motion. For a given default threshold process $D = (D_t)_{t \leq 0}$ with $0 < D_0 < V_0$, the default time τ is given by

$$\tau = \inf\{t > 0 : V_t \leq D_t\} \tag{6.8}$$

so that τ is a random variable valued in $(0, \infty]$.

In order to calculate default probabilities in this model, we define the *running minimum log-asset process* $M = (M_t)_{t\geq 0}$ by

$$M_t = \min_{s\leq T}(ms + \sigma W_s), \tag{6.9}$$

that is, M keeps track of the historic low of the log-asset value. With equation (6.8) we then find for the default probability (using the unique solution for the SDE in (6.7) given by proposition 1 in Section 4.2.)

$$\begin{aligned} P[\tau \leq T] &= P\left[\min_{s\leq T} V_s \leq D_T\right] \\ &= P\left[\min_{s\leq T}(V_0 e^{ms+\sigma W_s}) \leq D_T\right] \\ &= P\left[M_T \leq \ln(D_T/V_0)\right] \end{aligned} \tag{6.10}$$

That is, the event of default by time T is equivalent to the running minimum log-asset value at time T being below the adjusted default threshold $\ln(D_T/V_0)$ at time T. Assuming that D is a deterministic function of time and using the fact that the distribution of M_t is inverse Gaussian[2], we have

$$\begin{aligned} P[\tau \leq T] &= 1 - \Phi\left(\frac{mT - \ln(D_T/V_0)}{\sigma\sqrt{T}}\right) \\ &+ e^{\frac{2m\ln(D_T/V_0)}{\sigma^2}} \Phi\left(\frac{\ln(D_T/V_0) + mT}{\sigma\sqrt{T}}\right) \end{aligned} \tag{6.11}$$

where $\Phi(\cdot)$ is the cumulative normal distribution function. If m is positive, which is a common occurrence in practice, then the structural model predicts that the probability of the firm never defaulting is

$$P[\tau > \infty] = 1 - e^{\frac{2m\ln(D_T/V_0)}{\sigma^2}} \tag{6.12}$$

The probability of the firm never defaulting is positive in structural models of default. The implication of this is that, if the firm does not default relatively 'soon', then structural models predict that the firm value will most likely continue to drift away from the default threshold forever.

[2]To find that distribution, one first calculates the joint distribution of the pair (W_t, M_t^W), where M^W is the running minimum of W, by the reflection principle. Girsanov's theorem is then used to extend to the case of Brownian motion with drift.

6.3 Previous Empirical Research

Despite the rich arrays of theories for pricing default-risky bonds, the empirical literature is rather thin. There is especially little to tell us how well different models perform and what is the nature of the errors they make in predicting credit spreads. Indeed, only a few empirical papers attempt to implement a structural or reduced-form model to test its ability to predict prices or credit spreads. Lack of good bond data, noisiness in even the best bond data, and the apparent inefficiency of the default-risky bond markets contribute to the dearth of good empirical evidence in this area.

6.3.1 Testing Structural Models

The empirical literature on structural bond pricing models is rather small, especially in comparison to the theoretical literature. Partly, this reflects the fact that reliable bond data have only become recently available to academics. The empirical studies fall into two categories: (1) tests of predictions that are generated by the structural models and (2) analyses of the empirical implementation of the models. The first group includes tests of the shape of the term structure of credit spreads and tests related to changes in bond prices. The latter group consists of papers by Jones, Mason and Rosenfeld (1984) and Wei and Guo (1997) and Eom, Hewege and Huang (2000).

Sarig and Warga (1989), estimated the term structure of credit spreads using a small number of zero coupon corporate bonds and zero coupon U.S. Treasuries. They demonstrated curve shapes (slightly upward sloping for investment-grade bonds, humped shaped for lower-grade bonds, and downward sloping for speculative-grade bonds) as predicted by Merton. Helwege and Turner (1999), show that Sarig and Warga's results largely reflect sample selection bias by maturity, and that the term structure of credit spreads facing low-grade issuers is actually mostly upward sloping, if one controls for firm specific credit risk. Helwege and Turner conclude that structural models place too much emphasis on the upside potential of speculative-grade bonds, perhaps through excessively high volatility parameter or from overstating the typical leverage of a speculative-grade bond.

Another aspect of the structural models that has been tested is the implied risk-neutral default rates. Delianedes and Geske (1998), find that rating migrations (using S&P credit ratings) and defaults are detected months before in the equity markets.

The second line of empirical research on structural models of default-risky bond prices involves implementation of the structural model with available data. These studies compare the actual prices in the market with those predicted by the model.

The most extensive attempt at implementation of a structural model is found in Jones, Mason and Rosenfeld (1984) (henceforth JMR). JMR's implementation of the model shows that model prices are too high, or alternatively, that credit spreads from the model are too low relative to those observed in the secondary market. The errors are largest for speculative-grade firms, but they conclude that the Merton model works better for low-grade bonds than high-grade bonds because the Merton model has greater incremental explanatory power for speculative-grade bonds than a naïve model (discounting cash flows at the risk-free rate). They also find that pricing errors are significantly related to maturity, estimated equity volatility, leverage and time period. Later Franks and Torous (1989), confirmed the finding that actual credit spreads were much greater than predicted credit spreads.

Wei and Guo (1997)], implement two structural models to determine their predictive abilities: Merton (1974), and Longstaff and Schwartz (1995). The authors find that neither model is able to predict credit spreads that are statistically equal to those found in the Eurodollar market[3], and the prediction errors are higher the longer the maturity. Wei and Guo draw two conclusions concerning the performance of the two models: (1) the two models have similar powers in predicting spreads and (2) the Longstaff and Schwartz model suffers from the assumption of a constant recovery rate, while benefiting from its more general treatment of the default event that the Merton model. The problem with this study concerns what the spread in the Eurodollar market actually represents. While some portion of that spread undoubtedly compensates for credit risk, other non-credit characteristics likely explain the bulk of this spread.

More recently, Eom, Hewege and Huang (2000), (henceforth EHH), test

[3]The Eurodollar market is largely a market of short-term debt issued by banks.

four structural models to determine their predictive abilities: a naïve one-factor model based on Merton (1974), Geske (1977), Longstaff and Schwartz (1995), and Leland and Toft (1996). EHH find that both the naïve one-factor model and the Geske model underpredict spreads (overprices bonds). In contrast, EHH find that under reasonable assumptions, the Longstaff and Schwartz model generates credit spreads that are too high on average and the Leland and Toft model, under all circumstances predicts excessively high credit spreads.

The conventional wisdom, while praising the theoretical insights into the default process gained from structural models, dismisses them as impractical for actual bond valuation. However, small sample sizes used in some of the empirical research, and doubts about the quality of bond pricing data leave us without conclusive evidence regarding the power of structural models. The resolution of these empirical issues awaits further research.

6.3.2 Testing Reduced-Form Models

Empirical implementation of reduced-form models is still in its infancy. Partly, this is due the fact that these models require that credit spread data accurately reflect market expectations about credit risk, recovery in the event of default, and liquidity. Accurate bond data is difficult to find.[4] The question remains whether reduced-form models can describe the behaviour of default-risky bonds successfully.

So far the papers that attempt to answer this question are by Duffee (1999), Frühwirth and Sögner (2001)(FS henceforth). Duffee estimates the parameters for the stochastic process of the credit spread for the Duffie and Singleton (1999), framework. FS estimate default intensities within the Jarrow and Turnbull (1995) framework. Duffee finds that reduced-form models based on the Duffie and Singleton (1997) framework have difficulty explaining the observed term structure of credit spreads across firms of different credit qualities. For example, the model produces both flat term structures of credit spreads for investment-grade bonds with less default risk and steeper term structures of credit spreads

[4]Duffee (1996) suggests that the firms' observed bond yields that make up his dataset are "flawed" because these yields are traders' indicative bid prices that appear to react very slowly to information in the firms' stock prices.

for investment-grade bonds with relatively more default risk. He also finds that the model's parameter estimates imply that regardless of how much the firm's financial health improves, the firm's credit spread remains positive. This suggests that the model successfully captures a non-default component in credit spreads. On average, the model appears to fit investment-grade, corporate bonds well; term structures of credit spreads for lower quality firms are more steeply sloped than are term structures of credit spreads for higher quality firms. Duffee makes significant strides in implementing this modelling framework; and concludes that "The results here can be used both as benchmarks for models of corporate bond pricing and as directions for future research." FS show that estimated default intensities strongly depend on the date of estimation and the bond. They also show that liquidity has no significant influence. FS also show that there is a statistically significant correlation between default intensity and default-free interest rates and they conclude that "... further research should engage in models where the default rate is a function of some relevant parameters and in the estimation of these models."

6.4 Summary

Currently, we can choose from many theoretical models to price default-risky bonds. The assembling and analyzing of quality pricing data is critical at this stage because without empirical results, choosing the best model remains a difficult task. A valuable extension to the structural approach would be to incorporate jumps in the value of the firm in a reasonable way. With jumps incorporated in the evolution of firm value, a firm can default instantaneously because of a sudden drop in its value. Within the reduced-form framework, we need to explore further parametric forms of the intensity and recovery processes.

Chapter 7

Conclusion

In this dissertation, we have tried to highlight the growing importance of valuing credit risk and give an overview of *structural models* and *reduced-form models*. A summary of the salient features of some of the models within these two categories is found in Appendix B. A summary of the models' strengths and drawbacks is found in Appendix C.

Despite the natural and elegant way of modelling default by the first time the firm value hits some barrier, structural models have been criticized for several reasons. The major points of criticism are that they require estimating the parameters of processes that cannot be directly observed, such as firm value and default boundary. Therefore, they are of limited use in arriving at precise valuations. They are, however, useful in gaining intuition about the effects of various variables on credit risk. The Merton model's strengths are its relative simplicity and intuition. However, its lack of interest rate dynamics, correlation modelling, and tractability limit its practical applications. The Longstaff and Schwartz model, though not closed form addresses some of the weaknesses of the Merton model.

Unlike structural models, reduced-form models are based upon more direct assumptions about the default process. These models can be parameterized to fit the current term structure of credit spreads. The Jarrow, Lando and Turnbull (JLT) model provided a simple framework that is easy to calculate. However, its simplicity results in inflexibility: bonds within a given class are considered homogenous and there is no correlation with interest rates modelled. The Duffie and Singleton (DS) model outlines an interesting framework for modelling credit risk, with an increasingly complex model of credit risk suggested. However very little specific guidance is given of model implementation, which leaves a great deal of unanswered questions. Of particular interest, in future research would be the possible combination of the JLT approach to modelling the default process (as a Markov chain in the credit ratings) in combination with the analytic tractability of the *recovery of market value* (RMV) assumption of DS on the recovery process.

While the structural approach is economically sound and often generates more conceptual insights on default behaviour, it implies less plausible credit spreads properties. The reduced-form approach is ad hoc though, but tractable and implies plausible credit spread properties. Can we have a model which not only has the flexibility of the reduced-form approach to fit data but also provides the theoretical insights on the economic mechanism behind default events of the traditional structural approach? Can we have a model which allows for both expected and unexpected defaults in a single framework? How can we reconcile the different implications of the traditional reduced-form and structural approaches? A valuable extension to the literature on credit risk modelling would be a model that answers these questions.

Madan and Unal (2000), (hereafter, MU) recently proposed a *structural reduced-form model* in closed form that attempts to answer these questions. The distinguishing feature of this model is that it incorporates the attractive features of structural models with the reduced-form approach. A key assumption of this model is that default is a consequence of a single jump loss event that drives the equity value to zero and requires cash outlays that cannot be externally financed. The authors provide examples such as the near collapse of Long Term Capital Management (LTCM) in 1998 and the collapse of Barings bank in 1995 to justify this questionable assumption. MU also state that such jump loss events can be the result of the outcome of lawsuits, sudden default of a creditor, supplier, or a customer and unexpected devaluations.

MU obtained parameter values for their model by calibrating the model to a small set of data on the term structure of credit spreads. They report that the resulting model for credit spreads is tractable and can be readily implemented.

When calibrated to data on credit spreads, the model yielded a variety of realistic credit shapes. An important addition to the literature on the empirical testing of credit risk models would be a full-scale empirical study of corporate bond yields using the MU model.