Chapter 3

Introduction to Interest Rate Modelling

3.1 Fundamentals

This section presents the fundamentals of interest rate modelling as they pertain to work in this dissertation.

A *zero coupon bond* is an obligation to pay the holder one dollar at a fixed maturity date T. We write the value of the zero coupon bond at time t as $P(t,T)$.

We assume (for this chapter) that the payment will be made with absolute certainty. At any time $t_0 < t$, we let $P(t,T)$ denote the price of a zero coupon bond at time t maturing at time T. The no arbitrage condition gives

$$P(t,T) = P(t,\tau)P(\tau,T) \tag{3.1}$$

for all $\tau \in (t,T)$

A general bond may have *coupons*. These are payments of the same amount c_i which are paid at times t_i, where the t_i's are less than or equal to the final maturity date T. The bond will also pay some *principal amount* p at maturity.[1] All bonds of the above form can be written as a linear combination of zero

[1]The principal amount is generally referred to as the *par value* of the bond.

coupon bonds. If β_t is the value at time t of a bond with principal p, maturity T and coupons c_i to be paid at times t_i, then we have

$$\beta(t) = \sum_{t \le t_i \le T} c_i P(t, t_i) + pP(t, T) \tag{3.2}$$

This implies that it is sufficient to restrict our attention to zero coupon bonds because all coupon bonds are just linear combinations of zero coupon bonds.

The *continuously compounded zero coupon yield*, $y(t, T)$ is given by

$$y(t, T) = -\frac{1}{T - t} \ln(P(t, T)) \tag{3.3}$$

For a fixed t, the function $T \longmapsto y(t, T)$ is called the *(zero coupon) yield curve*.

An *instantaneous forward rate* $f(t, T)$ is defined as

$$f(t, T) = -\frac{\partial \ln(P(t, T))}{\partial T} \tag{3.4}$$

which implies

$$P(t, T) = \exp\left\{-\int_t^T f(t, s) ds\right\} \tag{3.5}$$

The *spot rate*[2] $r(t)$ is defined to be

$$r(t) = \lim_{T \downarrow t} f(t, T) \tag{3.6}$$

The spot rate can also be thought of as the rate of return of a bond with an infinitesimal time to maturity. That is

$$r(t) = -\lim_{T \downarrow t} \frac{1}{T - t} \ln(P(t, T)) \tag{3.7}$$

3.1.1 The Wiener Process

The definition below and related concepts are taken from (and covered in much more detail in) Brzeźniak and Zastawniak (1999). The *Wiener process (or Brownian motion)* is a stochastic process $W(t)$ with values in $\Re$ defined for $t \in [0, \infty)$ such that

[2]The spot rate is sometimes referred to as the *short rate*.

1. $W(0) = 0$ a.s.;

2. the sample paths $t \longmapsto W(t)$ are continuous a.s.;

3. $W(t)$ has stationary independent, normally distributed increments: If

$$0 = t_0 < t_1 < t_2 \ldots < t_n$$

and

$$Y_1 = W(t_1) - W(t_0), Y_2 = W(t_2) - W(t_1), \ldots, Y_n = W(t_n) - W(t_{n-1})$$

then

- $Y_1, Y_2, \ldots, Y_n$ are independent,
- $E[Y_j] = 0 \; \forall j$,
- $Var[Y_j] = t_j - t_{j-1} \; \forall j$.

3.1.2 Itô's Lemma

Let $W(t)$ be a Wiener Process. Let $x(t)$ be an Itô Process with $dx = a(x,t)dt + b(x,t)dW$. Let $V = V(x,t)$, then,

$$\begin{aligned} dV &= \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial x}dx + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}b(x,t)^2 dt \\ &= \left[\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}a(x,t) + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}b(x,t)^2\right]dt + \frac{\partial V}{\partial x}b(x,t)dW \end{aligned}$$

Proof: Using a Taylor expansion

$$dV = \frac{\partial V}{\partial t}dt + \frac{1}{2}\frac{\partial^2 V}{\partial t^2}(dt)^2 + \frac{\partial V}{\partial x}dx + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}(dx)^2 + \frac{\partial^2 V}{\partial t \partial x}dxdt + h.o.t \qquad (3.8)$$

Any term of order $(dt)^{\frac{3}{2}}$ or higher is denoted by h.o.t. and is small relative to terms of order dt . Note that $(dW)^2 = dt$.

So,

$$\begin{aligned} (dt)^2 &= h.o.t \\ dxdt &= a(x,t)(dt)^2 + b(x,t)dWdt = h.o.t. \\ (dx)^2 &= b(x,t)^2(dW)^2 + h.o.t = b(x,t)^2 dt + h.o.t. \end{aligned}$$

Model	$v(r_t,t)$	$\sigma(r_t,t)$
Merton [4]	θ	σ
GBM[5]	θr_t	σr_t
Ho and Lee	θ_t	σ
Vasicek	$\theta + \alpha r_t$	σ
Brennan & Schwartz	$\theta + \alpha r_t$	σr_t
Cox-Ingersoll-Ross	$\theta + \alpha r_t$	$\sigma\sqrt{r_t}$

Table 3.1: Short rate models

Therefore,

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial x}dx + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}b(x,t)^2 dt \tag{3.9}$$

3.2 Short Rate Models

A common approach to stochastic modelling of interest rates is to take the short rate $r(t)$[3] to be a stochastic process. Models of this form are commonly referred to as *short rate models*. The process r_t is generally taken to be a diffusion process defined by the stochastic differential equation

$$dr_t = v(r_t,t)dt + \sigma(r_t,t)dW_t \tag{3.10}$$

driven by a Wiener process W. We can interpret v as an instantaneous rate of return. Some examples with their specification of v and σ are shown in Table 3.1 above.

It has been shown (in Vasicek (1977), for example) that, in this formulation, the value of a zero coupon bond at time t maturing at time T must be the solution to the partial differential equation

$$\begin{aligned} P_t + \frac{1}{2}\sigma^2(r_t,t)P_{rr} + (v(r_t,t) + \lambda(r_t,t)\sigma(r_t,t))P_r - rP &= 0 \\ P(t,T) &= 1 \end{aligned} \tag{3.11}$$

[3]Sometimes written as r_t.

where λ is the market price of risk.[6] The Feynman-Kač equation gives

$$P(r_t, t, T) = E_{(r_t, t)}\left[\exp\left\{-\int_t^T r_s ds\right\}\right] \tag{3.12}$$

where r_t is now the solution to the SDE

$$dr_t = \mu(r_t, t)dt + \sigma(r_t, t)dW_t \tag{3.13}$$

where $\mu = v + \lambda\sigma$ is a risk-neutral drift.

3.3 HJM Models

Another approach is to model the instantaneous forward rates $f(t, T)$ as the underlying stochastic variables. Models which apply this approach are generally referred to as "Heath, Jarrow and Morton," models (or HJM models) after the authors of Heath, Jarrow and Morton (1992).

Mathematically, an HJM model can be described as follows. Forward rates are modelled as a stochastic process given by

$$df(t, T) = \mu(f, t, T)dt + \sum_{i=1}^{n} \sigma_i(f, t, T)dW_t^i \tag{3.14}$$

where $W_t^1, \ldots, W_t^n$ are independent Brownian motions, the $\sigma_i(f, t, T)$'s are specified by the modeller, and the $\mu(f, t, T)$'s are determined by the no arbitrage condition. This is the requirement that

$$\mu(f, t, T) = \sum_{i=1}^{n} \sigma_i(f, t, T) \int_t^T \sigma_i(f, t, s)ds \tag{3.15}$$

This condition is often referred to as the "HJM drift condition."

[6] There is a theorem which states that the no-arbitrage condition implies that the difference between the instantaneous rate of return of any asset and the spot rate divided by the asset's volatility must be a function of the state variables and calendar time. That function is called the *market price of risk*.

3.4 The Yield Curve

Interest rates vary for different maturities of debt. A graph of the spot yield for different maturities is called a *yield curve*. In general, there is a distinct difference between short and long term interest rates. There are a number of economic theories that are cited to explain the shape of the yield curve. However, the *expectations theory* and the *market segmentation theory* have evolved as the major theories that explain the shape of the yield curve.

The *expectations theory* is based on the premise that current interest rates are somehow related to the market's expectations of future rates. These future interest rates are affected by economic factors such as money supply figures, inflation and trade deficit figures. Market participants have different views on the expected future behaviour of these economic factors and this determines their anticipations of the future interest rates. These expectations are evident from the shape of the yield: a downward sloping yield curve implies that the short term interest rate is expected to fall, whereas the opposite is expected from an upward sloping yield curve. In general, the short term interest rate is more sensitive to the economic factors than the long term interest rate.

The *market segmentation theory* relies on the idea that some investors have restrictions (either legal or practical) on their maturity structure. Examples include money market funds (short-term maturities) or life insurance companies (long-term maturities). The shape of the yield curve is therefore determined by the supply and demand for securities within a given maturity segment.

In any nation the lowest interest rates on local currency denominated debt apply to those loans assumed by the sovereign government. These loans take place through the sale of government bonds. Provided that the debt is issued in the sovereign currency, the government has the option of printing money to meet any payments that are due. It is for this reason that sovereign debt is assumed to have no risk of default. This means that the probability that the loan will not be paid is effectively zero and consequently, the interest rate offered on a sovereign loan is regarded as the risk-free rate. A yield curve constructed using government bonds is therefore called a *risk-free yield curve* or a *zero-coupon yield curve*.

The risk-free yield curve is a concept central to economic and financial theory and the pricing of interest rate contingent claims. Together with the no-arbitrage theory, it provides a mechanism for comparing cash flows occurring at different times. Any risk-free financial asset comprised of specified tranches can be assigned a present value that is arbitrage-free. This is because it is possible to lend (borrow) the appropriate amount now that will match each tranche as it occurs. We will illustrate this concept with the arbitrage-free pricing of a South African government bond

3.5 Pricing A South African Government Bond

South African government bonds have fixed rate coupons which are paid semi-annually up to and including the maturity date at which time the principal or face value is also repaid. The coupons are quoted in percentages and indicate the percentage of the principal to be repaid annually. Therefore a 13% semi-annual coupon means that 6.5% of the principal is repaid every six months with the final coupon payment and the repayment of the principal at maturity. South African bonds are priced by *yield-to-maturity* - the price of a bond is quoted as a semi-annual interest rate and the cash price is obtained by discounting the cash flows of the bond to the present using this yield-to-maturity as the interest rate for the discounting.[7]

We now provide the framework for pricing a South African bond, with the assumption that the principal on the bond is 100.

- $P(t, y, n)$ = trading price of a bond
- t = current time
- n = coupons still to be received
- Δn = fractional number of half-years before the first coupon will be received
- c = coupon rate of the bond (13 for 13%)

[7]The yield-to-maturity can thus be regarded as the 'internal rate of return' of the bond's cash flows.

- y = quoted yield of the bond expressed as an interest rate compounded semi-annually

The quoted yield is consistent with the traded price if the following equality holds,

$$P(t,y,n) = \frac{1}{(1+y/2)^{\Delta n}} \left(\sum_{i=0}^{n-1} \frac{c/2}{(1+y/2)^i} + \frac{100}{(1+y/2)^{n-1}} \right) \tag{3.16}$$

A bond has a specified time structure of payments. Assume a function $r(t,T)$ which represents the risk-free, continuously compounded interest rate at time t applicable to loan maturing at time T. Using no-arbitrage theory, the present value of each cash flow of the bond can then be determined. A coupon received at time t_i will have a present value of $(c/2)\exp(-r(t,t_i)(t_i-t))$. The sum of the present values of all of the cash flows comprising the bond will be an arbitrage-free value for the bond. Let $P(t,n)$ be the arbitrage-free price for the bond. Then,

$$P(t,n) = \left(\sum_{i=1}^{n} \frac{c}{2} e^{-r(t,t_i)(t_i-t)} \right) + 100e^{-r(t,t_n)(t_n-t)} \tag{3.17}$$

3.6 Determinants of the Risk-Free Yield Curve

Fundamental to the pricing of interest rate derivative instruments and the management of their risk is the construction of a risk-free yield curve. In liquid fixed-income markets, zero-coupon bonds and money market rates are typically used to construct the risk-free yield curve; in markets where a limited number of zero-coupon bonds are traded, there are usually enough coupon bearing bonds traded to use in constructing this curve. In the South African fixed-income-market, however, only a limited number of liquid financial instruments are available to construct the risk-free yield curve. Under the efficient market hypothesis the most liquid of these instruments will be trading at arbitrage-free prices. A risk-free yield curve must be consistent with these prices. The present value of the cash flows of these instruments should sum to their trade price as in equation (3.16).

The primary financial instruments of South Africa's money market[8] that may be used to reliably fix interest rates at the short end of the risk-free yield curve are the Johannesburg Interbank Acceptance Rate (JIBAR), Negotiable Certificates of Deposits (NCDs) and Treasury Bills (T-bills). The JIBAR J_t is the rate of interest that banks will offer to each other for a t -month loan that begins on that particular day. The most popular period is 3 months but 1, 6, and 12-month JIBAR rates are also available. NCDs, the most liquid of the instruments, are issued by all major banks through private placements. T-bills are issued by the government using an auction and usually have a maturity of 91 days. The secondary market for T-bills is relatively illiquid because local banks use them to meet reserve requirements. This lack of liquidity in the T-bills has led market participants to use Forward Rate Agreements (FRAs) in constructing the short end of the risk-free yield curve instead of T-bills. A Forward Rate Agreement is a forward contract where two parties agree that a certain interest rate will apply to a certain notional loan or deposit during a specified future period of time. A 3×6 FRA is an agreement to fix the rate for the period between three and six months time (i.e., for the 3 month period starting in 3 months time). Other FRAs frequently traded in the South African Market are 6×9's and 9×12's. Settlement is against the relevant JIBAR rate. FRAs are settled at the start of the future period, when the FRA yield rate (i.e., the rate agreed upon in advance under the FRA) and the JIBAR rate are compared. If there is a difference between these rates a discounted cash settlement based on the difference is made.

The JIBAR rate is quoted as a yield rate. This means that a discount bond maturing in three months time would be traded as

$$P(t, t+3) = \frac{100}{(1 + J_3 \frac{91}{365})} \tag{3.18}$$

where $P(t,T)$ is a zero coupon bond of maturity based on a notional principal of 100.

FRAs are also quoted as yield rates implying

[8]The money market is the universe of instruments for the relatively short term (<2years) borrowing and lending of cash.

$$P(t, t+3i) = \frac{100}{\left(1 + J_3 \frac{91}{365}\right)\left(1 + FRA_{3\times 6}\frac{91}{365}\right)\dots\left(1 + FRA_{3(i-1)\times 3i}\frac{91}{365}\right)} \quad (3.19)$$

NCDs are quoted as yield rates i.e.,

$$P(t, t+s) = \frac{100}{\left(1 + NCD_s \frac{s}{365}\right)} \quad (3.20)$$

where s is the term of the NCD in days.

T-Bills are quoted as discount rates i.e.,

$$P(t, t+s) = \left(1 - TB_s \frac{s}{365}\right) 100 \quad (3.21)$$

where s is the term of the T-Bill in days.

The arbitrage-free price of $P(t,T)$ is $100e^{-r(t,T)(T-t)}$, where the (continuously compounded) risk-free rate $r(t,T)$ of the bond is related to its price by

$$r(t,T) = \frac{1}{T-t} \ln\left(\frac{100}{P(t,T)}\right) \quad (3.22)$$

Assume that there are k of these money market instruments. Using equation (3.22), the zero coupon bond values for these instruments can be converted into continuously compounded zero coupon interest rates. This implies that the risk-free rate $r(t)$ is known at distinct times $\{t_j^* : j = 1, \dots, k\}$ where at each t_j^* there is the following restriction on $r(t)$

$$r(t_j^*) = r_j^* \; \forall j = 1, \dots, k \quad (3.23)$$

In South Africa we can use the government bond market and the interest rate swaps markets to obtain information about interest rates for longer periods. The South African bond market is a relatively developed fixed income market with bond maturities of up to 30 years but it suffers from a lack of a complete set of well-traded bonds with well-spaced maturities. For sectors of the yield curve that don't offer good tradable liquidity to reliably fix interest rates, market participants use interest rate swaps - which can be seen as par yield bonds. An interest rate swap is an exchange of cash flows based upon different interest rate indices denominated in the same currency on a pre-set notional amount with a pre-determined schedule of payments and calculations. Usually, one party will receive fixed flows (the swap rate) in exchange for making floating payments (according to JIBAR). In South Africa, interest rate swaps are quoted form 1 to 25 years, with the most liquid swaps being in the 1 to 10 years area. The most popular interest rate reset period is for 3 months, but reset periods can also be 1, 6 or 12 months. Settlement is against the relevant JIBAR rate. On every reset date the agreed swap rate and the JIBAR rate are compared, and if there is a difference between these rates, the settlement is made at the end of the reset period.

3.7 Estimating the Risk-Free Yield Curve

The Fundamental Theorem of Asset Pricing [see Dybvig and Ross (1989), for example] implies that in a world of certain cash flows, $c(t)$, and frictionless markets, absence of arbitrage is equivalent to the existence of a linear pricing rule, $\delta(t) > 0 \ \forall t$, such that

$$P = \sum_{t=1}^{T} c(t)\delta(t) \tag{3.24}$$

If markets are incomplete, there exists multiple sets of $\delta(t)$ which satisfy this equation. In the term structure of interest rates literature, $\delta(t)$, called the "discount function" is usually transformed into a zero coupon curve by $r(t) = -\ln\delta(t)/t$ The discount function $\delta(t)$ is the current price of a risk-free zero coupon bond paying one unit of currency at time t. Clearly if we exclude the possibility of negative interest rates we must have the following for a discount function $\delta : [0,\infty) \longmapsto [1,0)$:

$$\begin{aligned} \delta(0) &= 1, \\ \delta(t_1) < \delta(t_2) &\iff t_1 > t_2 \end{aligned}$$

Estimating the risk-free yield curve requires three decisions:

1. A pricing function relating instrument market prices, P_i, to the discount rate function, $r(t_j)$, via promised cash flows, c_i at time t_j, for $1 \leq j \leq K$.

2. A functional form to be used to approximate the yield curve function, $r(t)$, or the discount function, $\delta(t)$.

3. An econometric method for estimating the parameters of the yield curve function.

3.7.1 Pricing function

The simplest pricing function, appropriate to a world with complete markets and no taxes or transaction costs, is just the present value of the promised cash flows:

$$\begin{aligned} P_j &= \sum_{j=1}^{K} c_j \delta(t_j) \\ &= \sum_{j=1}^{K} c_j \exp(-t_j r(t_j)) \end{aligned} \tag{3.25}$$

Let $\{B_i\}_{1\leq i\leq N}$ be a set of observed market instruments, let $\tau_1 < \tau_2 < \ldots < \tau_K$ be the set of dates at which cash flows occur, let $c_{i,j}$ be the cash flow of the i^{th} instrument on date τ_j, and let P_i be the market price of the i^{th} instrument. The pricing function becomes

$$P_i = \hat{P}_i + \varepsilon_i \tag{3.26}$$

where $\hat{P}_i$ is defined by

$$\begin{aligned} \hat{P}_i &= \sum_{j=1}^{K} c_{i,j} \delta(\tau_j) \\ &= \sum_{j=1}^{K} \exp(-\tau_j r(\tau_j)) \end{aligned} \tag{3.27}$$

Since equation (3.25) omits such obvious factors as taxes and liquidity, the error term, ε_i, will contain both systematic and random factors.

3.7.2 Approximating function

After deciding on the appropriate pricing function, the next step is to decide on the functional form to be used to approximate the yield curve function $r(t)$ or the discount function, $\delta(t)$. It is not possible to estimate the value of the yield curve at each possible horizon as the number of cash flows points will usually exceed the number of available instruments. The usual practice is to select an approximating function and then estimate the parameters of this function. Examples of approximating functions include polynomials, cubic splines, step functions, piecewise linear and exponential forms.

Given a proposed yield curve function $\hat{r}_\psi(t)$ such that

$$\hat{r}_\psi(\tau_j^*) = \hat{r}_j^* \ \forall j = 1, \ldots, K$$

the resultant theoretical price for the i^{th} instrument is, from equation (3.25)

$$\hat{P}_i(\psi) = \sum_{j=1}^{K} c_{i,j} \exp(-\tau_j r_\psi(\tau_j)) \tag{3.28}$$

The yield curve function, ψ, could be chosen such that it minimizes the objective function

$$E_\psi = \sum_{i=1}^{n} (P_i - \hat{P}_i(\psi))^2 \tag{3.29}$$

The problem of solving for the optimal representation of the "true" yield curve becomes an exercise in finding the most efficient technique for choosing $\hat{r}_{\psi+1}(\tau)$ such that

$$E_{\psi+1} < E_\psi$$

and that convergence occurs "rapidly enough".

3.7.3 Estimation method

Lastly, the method of approximating the parameters of the approximating function must be selected. Methods used in the past include weighted least squares, maximum likelihood and linear programming. Related discussions include error weighting functions and how to handle the bid-ask spread (usually by collapsing the bid and ask quotes into a single price by taking their mean).

3.8 Summary

According to Vasicek and Fong (1982), the objective when estimating the term structure empirically is to fit a zero-coupon yield curve that both fits the data sufficiently well and is a smooth function. In this chapter, we introduced some techniques for determination of the zero-coupon yield curve that have these requirements as their objective. We have shown that the modelling is difficult, and in general not computationally straightforward or unique. In South Africa, *bootstrapping* is a popular technique for determining the zero coupon yield curve. The fundamental idea behind bootstrapping is to discount the coupons prior to maturity from a bond using the zero coupon rates already determined from money market instruments. The zero-coupon rate for a specific term obtained this way is then used in the bootstrap process for the next bond. In this way rates for longer and longer periods are obtained and these rates are then approximated by a curve.

The problem with the bootstrap procedure described above is the assumption of the existence of a complete series of regularly spaced coupon bearing bonds - this is not the case in South Africa. Also, according to Smit & van Niekerk (1997), the commonly used approximating functions such as polynomials and cubic splines are not always suitable for the South Africa yield curve due to structural inefficiencies in the fixed-income market and the resultant dispersion of data points. The problem of yield curve determination, especially in a sparse and illiquid market such as South Africa, is not trivial and represents significant opportunities for research for students of financial economics.