The division theorem for smooth functions

by

Pieter Oloff de Wet

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Summary

We discuss Lojasiewicz's beautiful proof of the division theorem for smooth functions. The standard proofs are based on the Weierstrass preparation theorem for analytic functions and use techniques from the theory of partial differential equations. Lojasiewicz's approach is more geometric and synthetic. In the appendices appear new proofs of results which are required for the theorem.

Contents

Notation and prerequisites

We denote by $\mathbf{N}$, $\mathbf{R}$ and $\mathbf{C}$ the natural numbers, the field of real numbers and the field of complex numbers respectively. Familiarity with commutative algebra is assumed with rings, ideals, powers of ideals as well as modules being used frequently. Nakayama's lemma [1] is only used once: If A is a local ring with maximal ideal M and if E is an finitely generated A-module, then $ME = E$ implies $E = 0$.

Some topology is present and, of course, real and complex analysis. A function $f : U \to \mathbf{R}$, where $U \subset \mathbf{R}^n$ is open, is called differentiable if the partial derivatives of *all* orders exist and are continuous. Three forms of Taylor's formula are often used:

- For any natural number n the differentiable function f on $\mathbf{R}$ can be written as

$$f(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + r(x) x^{n+1}$$

  where the c_i are real constants and $r(x)$ is a differentiable function.

- The differentiable function f on $\mathbf{R}^m$ can be written as

$$f(x, y) = c_0(y) + c_1(y)x + c_2(y)x^2 + \cdots + c_n(y)x^n + r(x, y)x^{n+1}$$

  where (x, y) is shorthand for $(x, y_1, \ldots, y_{m-1})$, the $c_i(y)$ are differentiable functions of $y_1, \ldots, y_{m-1}$ and $r(x, y)$ is a differentiable function of $x, y_1, \ldots, y_{m-1}$.

- By repeated use of the above all c_i can be expanded to give for $f(x_1, \ldots, x_m)$ on $\mathbf{R}^m$:

$$f = (\text{a polynomial of order } n) + \sum_j g_j \cdot (\text{primitive monomial of order } n+1)_j$$

  where the summation is over all primitive monomials of order $n+1$ and the g_j are differentiable functions on $\mathbf{R}^m$. (A primitive monomial of order n is of the form $x_1^{v_1} \cdots x_m^{v_m}$ with $v_1, \ldots, v_m$ in $\mathbf{N}$ and $v_1 + \cdots + v_m = n$.) For brevity we might also just write

$$f = (\text{polynomial of order } n) + r(x).$$

Preface

The local theory of the ring of smooth functions is a branch of mathematics that brings together ideas of algebra, topology and advanced calculus. The smooth function is a fundamental concept in pure and applied mathematics and is used for the construction of models of the continuum containing nilpotent infinitesimals. In this thesis, which relies heavily on the exposition [7] of Martinet, we discuss Lojasiewicz's beautiful proof of the division theorem for smooth functions.

This theorem was first proven by Malgrange [6] at the suggestion of René Thom. The standard proofs are based on the Weierstrass preparation theorem for analytic functions and use techniques from the theory of partial differential equations. Such a proof can be found in [3], for instance. Lojasiewicz's approach is more geometric and synthetic and leads to additional insights into the preparation theorem.

In the appendices appear new proofs of results which are required for the theorem.

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Chapter 1

Germs of functions

This chapter introduces the basic concepts. Much follows directly from calculus; proofs being supplied otherwise.

1.1 Germs

A function $f : U \to \mathbf{R}$, where $U \subset \mathbf{R}^n$ is open, is called **smooth** or **differentiable** (or an element of C^∞ on U) if the partial derivatives of all orders exist and are continuous. This definition is independent of the choice of the perpendicular axes in $\mathbf{R}^n$. When referring to functions we will usually assume it to be differentiable functions.

To work locally we define a **germ** f at $x \in \mathbf{R}^n$ of a function f as the equivalence class of functions that agree with f on a neighbourhood (open set) of x. More formally, if U and V are neighbourhoods of x, then the functions $f : U \to \mathbf{R}$ and $g : V \to \mathbf{R}$ are both **representatives** of the germ f at x if an open W exists such that $x \in W$, $W \subset U$, $W \subset V$ and the functions f and g agree on W. We denote the germ as $f : (\mathbf{R}^n, x) \to \mathbf{R}$. We call a germ differentiable if it can be represented by a differentiable function on a neighbourhood of x. When referring to germs we will usually assume it to be differentiable germs.

1.2 The ring of germs

The germs at x can be added and multiplied (just like functions) and thus form a commutative ring with identity ($f = 1$) or an algebra over $\mathbf{R}$ denoted by $E(\mathbf{R}^n, x)$ or simply by E_n when x is 0 (the origin – where we often work for simplicity).

Some examples of ideals in this ring are :

- All germs in E_2 such that $f(x, 0) = 0$.
- All germs in E_n that are zero at 0. (This ideal is maximal for if $f(0) \neq 0$ then f is non-zero on a neighbourhood of 0 and therefore represents a unit in E_n.)
- We have a filtration of ideals $E_n \supset M_1 \supset M_2 \supset \cdots$, where M_{k+1} consists of all those germs whose representatives have all partial derivatives of order k or smaller zero at the origin (a function is seen as its own order zero derivative so that M_1 refers to the previous example). A germ in M_{k+1} is said to be **k-flat**. The fact that M_k is an ideal follows directly from the rules of differentiation. We write M for M_1.

Proposition 1 *The ideal M is generated by the representatives of the coordinate functions $x_1, x_2, \ldots, x_n$.*

Proof: Let $f \in M$ and use Taylor's formula to write:

$$f = 0 + g_1x_1 + g_2x_2 + \cdots + g_nx_n$$

where the g_i are in E_n. QED

Proposition 2 *We have $M_k = M^k$.*

Proof: Using Taylor's formula an element of M_k can be expanded as

$$f = 0 + \sum g_i \cdot (\text{primitive monomials of order k})_i, \ \ g_i \in E_n$$

Using the previous proposition a germ f in the ideal M^k can be written as a product of k elements of the form $(g_1x_1 + g_2x_2 + \cdots + g_nx_n)$.

Clearly these two ideals contain the same elements. QED

We can also consider the ideal $I \subset E_{n+p}$ which consists of those germs at the origin of $\mathbf{R}^{n+p} = \mathbf{R}^n \times \mathbf{R}^p$ that is zero on $\mathbf{R}^p$. Similarly we then get the following.

Proposition 3 *The ideal I is generated by $x_1, \ldots, x_n$. (The representatives of the coordinate functions of $\boldsymbol{R}_n$.)*

1.3 k-Jets

Next we consider the quotient ring (and $\mathbf{R}$-algebra) E_n/M^{k+1}. Using Taylor's formula we see that its members can be represented by polynomials (sometimes referred to as Taylor polynomials) of order k or less. These members are added like normal polynomials and multiplied like normal polynomials except that terms with order higher than k are omitted. The ring is called the $\mathbf{R}$-algebra of **k-jets**, and we denote the natural homomorphism from germs to k-jets by

$$j^k : E_n \to J_n^k$$

where J_n^k denotes E_n/M^{k+1}.

1.4 Flatness

At this stage we note that one can talk of the values of a germ or its derivatives at 0 in an unambiguous manner. (It simply refers to these values of the representative function.) Furthermore local definitions like flatness refer to germs at the origin when not specified otherwise, but could be defined for any other point or subspace in a similar manner. We will use these liberties in the rest of the text for sakes of fluency and brevity.

The subtleties of differentiable germs have much to do with the following concept: $f \in E_n$ is called **flat** if $f \in \cap_{k=1}^{\infty} M^k$. This implies that all partial derivatives at the origin are 0 and such a germ has 0 as its Taylor series. Examples of flat germs are given in the next section. The flat germs form an ideal denoted M^∞.

We can now use M^∞ to extend the idea of k-jets of the previous section to the concept of **jets**. We denote the natural homomorphism from germs to jets by

$$j : E_n \to J_n$$

where J_n denotes E_n/M^∞.

From calculus we have the homomorphism $E_n \to \mathbf{R}[[x_1, \ldots, x_n]]$ of a germ of a function to its formal power series (also called its Taylor series). We now prove the isomorphism:

$$E_n/M^\infty \simeq \mathbf{R}[[x_1, \ldots, x_n]]$$

Thus we have to show that $E_n/M^\infty \to \mathbf{R}[[x_1, \ldots, x_n]]$ is injective and surjective. Since the natural map $E_n \to \mathbf{R}[[x_1, \ldots, x_n]]$ has M^∞ as kernel the injectivity follows. We are left to prove surjectivity:

Theorem 1 (Borel, [2]) *Given any formal power series over the reals (not necessarily convergent) there exists a function which has this power series as its Taylor series at the origin.*

Proof: To simplify notation we only prove the one-dimensional case – the multi-dimensional case being similar.

Let $a_0 + a_1x + a_2x^2 + \cdots$ be any given Taylor series and take the function $\phi : \mathbf{R} \to \mathbf{R}$ to be such that $0 \leq \phi(x) \leq 1$ with $\phi(x) = 1$ for $|x| \leq 1/2$ and $\phi(x) = 0$ for $|x| \geq 1$. Now, for a sequence of reals (t_n) with $1 < t_n$, set

$$f(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} \cdot x^n \cdot \phi(t_n \cdot x) \tag{1.1}$$

This construction gives us the appropriate function if the t_n can be chosen big enough that f is a differentiable function. Thus we need to prove that all the series

$$\sum_{n=0}^{\infty} \frac{\mathrm{d}^m}{\mathrm{d}x^m} \frac{a_n}{n!} \cdot x^n \cdot \phi(t_n \cdot x) \quad , m = 0, 1, 2, \ldots \tag{1.2}$$

converge uniformly. (See term-by-term differentiation [9].)

We simplify our notation and write the above as

$$\begin{array}{ccccccc} s_{0,0} & + & s_{0,1} & + & s_{0,2} & + & \cdots \\ s_{1,0} & + & s_{1,1} & + & s_{1,2} & + & \cdots \\ s_{2,0} & + & s_{2,1} & + & s_{2,2} & + & \cdots \\ \vdots & & \vdots & & \vdots & & \ddots \end{array}$$

Consider the first n elements of the n-th column, $s_{0,n}, \ldots, s_{n-1,n}$. (All elements in column n above or on the diagonal.) We write such an element $s_{m,n}$ (with $m < n$) as

$$\begin{aligned} s_{m,n} &= \frac{\mathrm{d}^m}{\mathrm{d}x^m} \frac{a_n}{n!} \cdot x^n \cdot \phi(t_n \cdot x) \\ &= \frac{\mathrm{d}^m}{\mathrm{d}x^m} \left(\frac{1}{t_n}\right)^n \frac{a_n}{n!} \cdot (t_n \cdot x)^n \cdot \phi(t_n \cdot x) \\ &= \frac{\mathrm{d}^m}{\mathrm{d}x^m} \left(\frac{1}{t_n}\right)^n \cdot a_n \cdot \psi_n(t_n \cdot x) \end{aligned}$$

where ψ_n vanishes when $|t_n \cdot x| > 1$ and we define

$$H_n = \max\{ | \frac{\mathrm{d}^m}{\mathrm{d}x^m} a_n \cdot \psi_n(x) | : \; m < n, \; x \in \mathbf{R}\}$$

which exists because only finite m are smaller than a specific n.
Now we have

$$\begin{aligned} s_{m,n} &= \frac{\mathrm{d}^m}{\mathrm{d}x^m}\left(\frac{1}{t_n}\right)^n \cdot a_n \cdot \psi_n(t_n \cdot x) \\ |\, s_{m,n} \,| &\leq (t_n)^m \cdot \left(\frac{1}{t_n}\right)^n \cdot H_n \\ &\leq H_n/t_n \end{aligned}$$

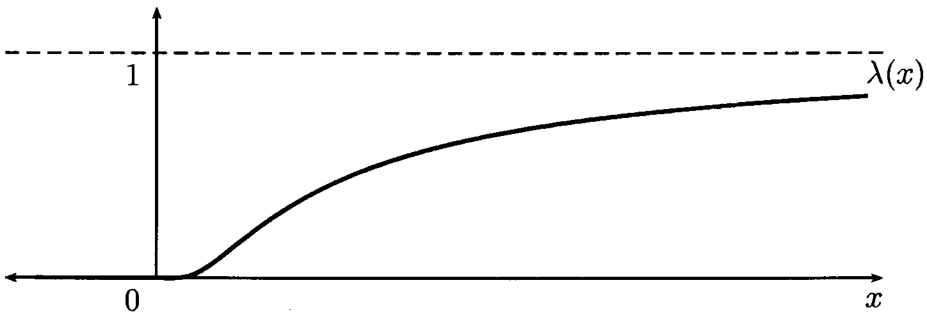
Finally we choose t_n big enough that $H_n/t_n < (1/2)^n$. This insures that $s_{0,n}, \ldots, s_{n-1,n}$ are all smaller than $(1/2)^n$. Thus for any m the sum $\sum_{n=0}^{\infty} s_{m,n}$ converges, for $s_{m,n}$ with $n > m$ (terms right of the diagonal) were chosen appropriately and $s_{m,n}$ with $n \leq m$ (terms left of or on the diagonal) are only finitely many. QED

1.5 Examples and a lemma

As promised, we give some examples, starting with a function that represents a flat germ.

- Example 1

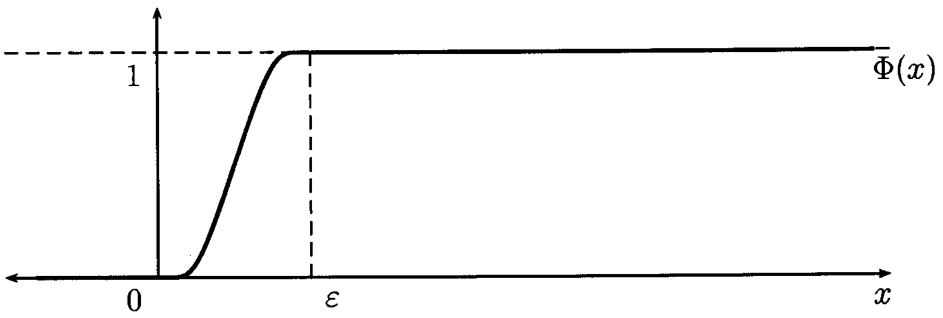
$$\begin{aligned} &\lambda : \mathbf{R} \to \mathbf{R} \\ &\lambda(x) = 0 \qquad \text{for } x \leq 0 \\ &\lambda(x) = e^{-1/x} \qquad \text{for } x > 0 \end{aligned}$$



Due to its composition λ is obviously differentiable everywhere off 0. On closer inspection it is also differentiable at 0 with all derivatives 0, thus rendering the same Taylor series as the zero germ, namely $0 + 0x + 0x^2 + \cdots$.

- Example 2

$$\Phi_\varepsilon : \mathbf{R} \to \mathbf{R} \quad , \varepsilon \text{ a positive constant}$$
$$\Phi_\varepsilon(x) = \frac{\lambda(x)}{\lambda(x) + \lambda(\varepsilon - x)}$$

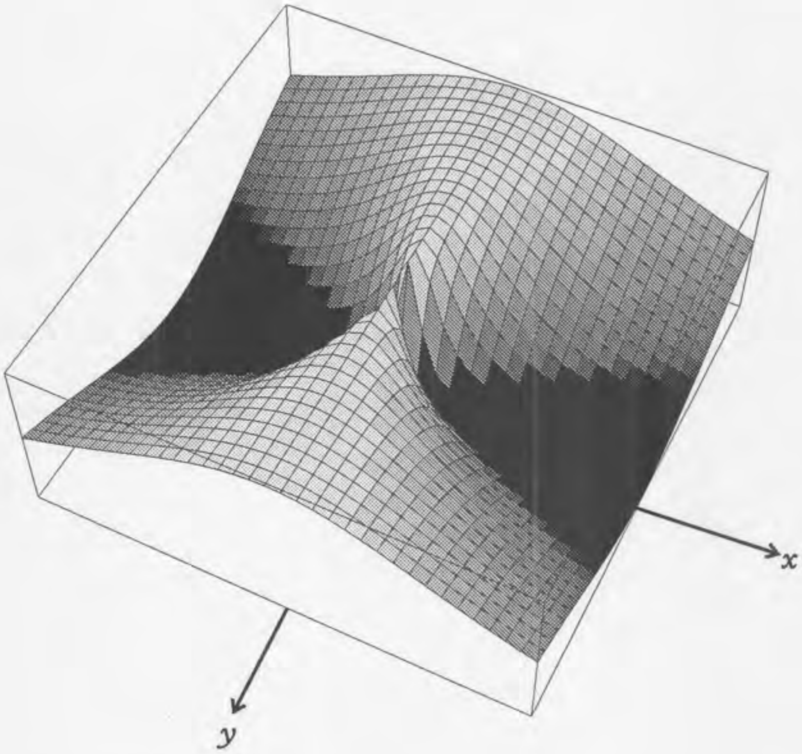


We note that Φ_ε is differentiable with $\Phi_\varepsilon(x) = 0$ when $x \leq 0$ and $\Phi_\varepsilon(x) = 1$ when $x \geq \varepsilon$. The Taylor series at the origin is $0+0x+0x^2+\cdots$ and at ε it is $1 + 0x + 0x^2 + \cdots$.

- Example 3
We use polar coordinates to define:

$$L : \mathbf{R}^2 \to \mathbf{R}$$
$$L(r,\theta) = \Phi_{\pi/2}(\theta) \qquad \text{for } r > 0 \quad \text{and } 0 \leq \theta \leq \pi/2$$

This defines L for $0 \leq \theta \leq \pi/2$. We extend L to $0 \leq \theta \leq \pi$ by taking it to be symmetric in the y-axis and then extend L to the whole plane by letting it be symmetric in the x-axis and by defining $L(0,0) = 0$.



We note that L has the following properties:

1. It is 0 on the x-axis and 1 on the y-axis. (Except for the origin.)
2. It is flat on the x-axis and $L - 1$ is flat on the y-axis.(Except for the origin.)
3. It is not differentiable at the origin.

Although L is not a differentiable germ at the origin, it does however behave like a differentiable germ in the following respect: the product of L and a flat germ is again flat. To see why, assume that g is flat at the origin. Since $0 \leq L \leq 1$ we have $0 \leq |L \cdot g| \leq |g|$. Derivatives can be expressed as limits of sequences but since g dominates $L \cdot g$ all derivatives of $L \cdot g$ will also be zero, making $L \cdot g$ flat.

By replacing the x-axis and y-axis in the previous example with subspaces it can be generalised to the following lemma, which we will use later:

Lemma 1 *Let F_1 and F_2 be two subspaces of $\boldsymbol{R}_n$. There exists a germ L such that:*

1. L is flat on $F_1 - (F_1 \cap F_2)$,

2. *$L - 1$ is flat on $F_2 - (F_1 \cap F_2)$,*

3. *$L \cdot g$ is flat on $F_1 \cap F_2$ if g is flat on $F_1 \cap F_2$.*

1.6 The kernel of j

We have seen in Section 1.4 that we have a natural homomorphism from germs to jets

$$j : E_n \longrightarrow J_n$$

where J_n denotes $E_n/M^\infty \simeq \mathbf{R}[[x_1, \ldots, x_n]]$.

From Borel's Theorem we know that j is surjective. We also know from example 1 in the previous section that j is not injective, for both 0 and λ are in the kernel of j. We can in fact say more about the size of M^∞.

Proposition 4 *The kernel of j, that is M^∞, is not finitely generated as a module over E_n.*

Proof: Assume that M^∞ is a finitely generated module over E_n. Now since M is maximal in E_n and since we have

$$M(M^\infty) = M^\infty$$

we know from Nakayama's Lemma that M^∞ has to be 0. This is not true, as λ is also in M^∞, thus M^∞ can not be finitely generated. QED

Chapter 2

Germs of mappings

This chapter generalises the concept of a germ of a differentiable function to that of a germ of a differentiable mapping. Section one provides the definitions and the other sections state some useful properties of these germs.

2.1 Differentiable mappings and their germs

A mapping $f : U \to \mathbf{R}^k$, where $U \subset \mathbf{R}^n$ is open, is called **differentiable** if the partial derivatives of all orders exist and are continuous. Such a mapping can be written as

$$f(x_1, \ldots, x_n) = (f_1(x_1, \ldots, x_n), \ldots, f_k(x_1, \ldots, x_n))$$

where $f_1, \ldots, f_k$ are differentiable functions. We call $x_1, \ldots, x_n$ the components of x and $f_1(x_1, \ldots, x_n), \ldots, f_k(x_1, \ldots, x_n)$ the components of $f(x)$.

We define the **germ** of a mapping f at $x \in \mathbf{R}^n$ as the equivalence class of mappings that agree with f on a neighbourhood of x. The mapping f is then a **representative** of the germ f and we denote the germ of the mapping as $f : (\mathbf{R}^n, x) \to \mathbf{R}^k$. Once again the differentiability of the germ stems from that of the representative and we will often refer to differentiable mappings simply as mappings.

2.2 Some properties of differentiable mappings

We state without proof some local properties of differentiable mappings and consequently of the germs which they represent.

2.2.1 The inverse function theorem

We can assosiate with a specific point $a \in U$ and a mapping $f : U \to \mathbf{R}^k$, where $U \subset \mathbf{R}^n$, a matrix $\mathrm{D}f(a)$. This matrix, which is called the Jacobian matrix, has real numbers as elements and is defined as:

$$\mathrm{D}f(a) = \left(\frac{\partial f_i}{\partial x_j}(a)\right) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(a) & \frac{\partial f_1}{\partial x_2}(a) & \cdots & \frac{\partial f_1}{\partial x_n}(a) \\ \frac{\partial f_2}{\partial x_1}(a) & \frac{\partial f_2}{\partial x_2}(a) & \cdots & \frac{\partial f_2}{\partial x_n}(a) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_k}{\partial x_1}(a) & \frac{\partial f_k}{\partial x_2}(a) & \cdots & \frac{\partial f_k}{\partial x_n}(a) \end{pmatrix}$$

The Jacobian matrix can be seen as a linear mapping $\mathrm{D}f(a) : \mathbf{R}^n \to \mathbf{R}^k$ which gives a linear approximation of f in the following sense:

$$\begin{pmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{pmatrix} \approx \begin{pmatrix} f_1(a) \\ \vdots \\ f_n(a) \end{pmatrix} + \mathrm{D}f(a) \begin{pmatrix} x_1 - a_1 \\ \vdots \\ x_n - a_n \end{pmatrix}$$

Theorem 2 (The inverse function theorem) *Let $f : U \to \boldsymbol{R}^n$ be a differentiable function and let $x \in U$ where $U \subset \boldsymbol{R}^n$ is open and let $f(x) = y$. Then there exists a neighbourhood of x where f has an inverse f^{-1} which is defined on a neighbourhood of y if and only if the Jacobian matrix $Df(x)$ is invertible (non-singular). We call f a local diffeomorphism.*

Thus the germ $f : (\mathbf{R}^n, x) \to (\mathbf{R}^n, y)$ possesses an inverse germ f^{-1} : $(\mathbf{R}^n, y) \to (\mathbf{R}^n, x)$ if and only if $\mathrm{D}f(x)$ is invertible. Such a germ is called the **germ of a local diffeomorphism**.

2.2.2 The implicit function theorem

Theorem 3 (The implicit function theorem) *Let U be an open set in $\boldsymbol{R}^k \times \boldsymbol{R}^n$ containing the origin. Suppose $f : U \to \boldsymbol{R}^k$ is differentiable with $f(0) = 0$ and suppose that f is a diffeomorphism when restricted to $\boldsymbol{R}^k$. Then there exists a open set V in $\boldsymbol{R}^n$ containing 0 and a differentiable mapping $\sigma : V \to \boldsymbol{R}^k$ such that $f(\sigma(y), y) = 0$ when $y \in V$.*

Speaking informally, for the situation where f restricted to $\mathbf{R}^k$ is a diffeomorphism, we can say that when $y \in \mathbf{R}^n$ is close enough to 0 then an x in $\mathbf{R}^k$ can be found such that $f(x, y) = 0$.

2.3 Complex valued functions

In this section we consider a particular type of mapping; the complex valued function.

Given a differentiable function $f : U \to \mathbf{C}$, with U open in $\mathbf{R}^n$, it can be regarded as $f(x) = f_1(x)+if_2(x)$ where f_1 and f_2 are real valued differentiable functions. Both f_1 and f_2 can be expanded according to Taylor's formula giving us

$$\begin{aligned} f(x) &= f_1(x) + if_2(x) \\ &= (\text{polynomial of order } k)_1 + r_1(x) + i(\text{polynomial of order } k)_2 + ir_2(x) \\ &= (\text{polynomial of order } k) + r(x) \end{aligned}$$

where r_1 and r_2 are in M^{k+1}. In the last line the polynomial now has complex coefficients and r is a complex valued function in the complex equivalent of M^{k+1}. We can thus speak of the Taylor polynomial of a complex valued function and, by letting the order of the polynomial go to infinity, of the Taylor series of a complex valued function. We again denote this natural homomorphism by

$$j : E_n \to J_n$$

and write $jf(x)$ for the jet (or Taylor series) of f.

The following theorem is due to Malgrange [5]. We call a function on $\mathbf{R}^n$ analytic if it is the restriction of an analytic function on $\mathbf{C}^n$.

Theorem 4 *Let $f_1(x)$ be an analytic function on $\mathbf{R}^n$ with values in $\mathbf{C}$. Then any function $f(x)$ in E_n belongs to the ideal of E_n generated by $f_1(x)$ if and only if, at each point x in $\mathbf{R}^n$, $jf(x)$ belongs to the ideal generated by $jf_1(x)$ in the ring J_n.*

This theorem is actually true for ideals generated by any finite number of analytic functions $f_1, \ldots, f_m$. We will however only use it to obtain the following result.

Let ω be defined as:

$$\begin{aligned} &\omega : \mathbf{R}^n \to \mathbf{C} \\ &\omega(x, y, z) = x + iy \end{aligned}$$

where $z \in \mathbf{R}^{n-2}$.

Corollary 1 *A function f in E_n is divisible by ω if and only if its Taylor series is divisible by $x + iy$ at each point where $x = y = 0$.*

Proof: The divisibility of the Taylor series follows from the fact that j is a homomorphism; the divisibility of the function is trivially true when $x \neq 0$ or $y \neq 0$ and follows from the theorem otherwise. QED

Chapter 3

The division theorem: Part 1

In this and the next chapter we state and prove the division theorem. Our exposition is based on the book by Martinet [7], which in turn relies heavily on the ideas of Lojasiewicz [4].

3.1 Statement of the division theorem

Consider the ring $E_{1+n} = E_{x,y}$ of differentiable germs at 0 where we write (x, y) for $(x, y_1, \ldots, y_n) \in \mathbf{R} \times \mathbf{R}^n$.

A germ P in $E_{x,y}$ is **regular of order** k **in** x or k**-regular in** x if we have:

$$P(0) = \frac{\partial P}{\partial x}(0) = \ldots = \frac{\partial^{k-1} P}{\partial x^{k-1}}(0) = 0$$

and

$$\frac{\partial^k P}{\partial x^k}(0) \neq 0$$

The division theorem states that if P is regular of order k in x then any germ $f \in E_{x,y}$ can be divided by P as follows:

$$f(x,y) = P(x,y) \cdot Q(x,y) + \sum_{i=1}^{k} r_i(y) x^{k-i}$$

with some Q in $E_{x,y}$ and the r_i in E_y.

3.2 The canonical division theorem

The function

$$P_k : \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^k \to \mathbf{R}$$
$$P_k(x, y, \sigma) = x^k + \sigma_1 x^{k-1} + \ldots + \sigma_{k-1} x + \sigma_k$$

is called the canonical polynomial of degree k. We will consider P_k to be a germ belonging to $E_{x,y,\sigma}$ and will sometimes write $P_k(x,\sigma)$ since it is independent of y. We note that P_k is k-regular in x.

The canonical division theorem states that any germ $f \in E_{x,y,\sigma}$ can be divided by P_k as follows:

$$f(x,y,\sigma) = P_k(x,\sigma)\cdot Q(x,y,\sigma) + \sum_{i=1}^{k} r_i(y,\sigma)x^{k-i}$$

for some $Q \in E_{x,y,\sigma}$ and the r_i in $E_{y,\sigma}$.

Theorem 5 *The canonical division theorem implies the division theorem.*

Thus we introduce more dimensions (with the variable σ), but this allows us to have to work with only one particular regular function, P_k, instead of all regular functions. In the remainder of the section we prove this theorem.

Proof: We assume the canonical division theorem to hold and prove that it implies the division theorem. We work in two steps.

Step 1: Implication of the canonical division theorem for a regular function.

According to our assumption a function $P \in E_{x,y}$ which is regular of order k in x can be considered as in $E_{x,y,\sigma}$ and divided by P_k to give:

$$P(x,y) = (x^k + \sigma_1 x^{k-1} + \ldots + \sigma_k)\cdot Q(x,y,\sigma) + \sum_{i=1}^{k} r_i(y,\sigma)x^{k-i} \qquad (3.1)$$

At $\sigma = 0$ and $y = 0$ this becomes:

$$P(x,0) = x^k \cdot Q(x,0,0) + \sum_{i=1}^{k} r_i(0,0)x^{k-i}$$

Since we chose P to be regular we also get, if we differentiate to different orders, that:

$$r_i(0,0) = 0 \text{ for all } i \text{ and } Q(0,0,0) \neq 0$$

Furthermore we also chose P to be independent of σ. Thus if we differentiate equation (3.1) with respect to σ_j and let $y = 0$ and $\sigma = 0$ we get:

$$0 = x^{k-j}\cdot Q(x,0,0) + x^k \frac{\partial Q}{\partial \sigma_j}(x,0,0) + \sum_{i=1}^{k} \frac{\partial r_i}{\partial \sigma_j}(0,0)x^{k-i}$$

By again differentiating to different orders, we see that:

$$\begin{aligned} \frac{\partial r_i}{\partial \sigma_j}(0,0) &= 0 \quad \text{if } i > j \\ \frac{\partial r_j}{\partial \sigma_j}(0,0) &= -Q(0,0,0) \end{aligned}$$

We can now represent these derivatives in a matrix:

$$\left(\frac{\partial r_i}{\partial \sigma_j}(0,0)\right)$$

The matrix is triangular and the diagonal entries are all $-Q(0,0,0)$ which is non-zero, as we saw, thus the matrix is invertible. This is the Jacobian matrix of the mapping $r : 0 \times \mathbf{R}^k \to \mathbf{R}^k$ where the r_i are the components of r. This mapping must be a diffeomorphism by the inverse function theorem. Consequently, by the implicit function theorem, a differentiable $\sigma(y)$ must exist so that $r(y, \sigma(y)) = 0$. We write $\sigma(y) = (\sigma_1(y), \ldots, \sigma_k(y))$ and substitute into (3.1) to get

$$P(x,y) = \Big(x^k + \sigma_1(y)x^{k-1} + \ldots + \sigma_k(y)\Big) \cdot Q(x,y,\sigma(y))$$

If we simplify our notation we have this result:

If the canonical division theorem is true for P_k then any $P(x,y)$ which is regular of order k in x can be written as

$$P(x,y) = (x^k + \sigma_1(y)x^{k-1} + \cdots + \sigma_k(y)) \cdot Q(x,y)$$

where $Q(0,0) \neq 0$ and the σ_i are in E_y (and in M_y).

Step 2: Implication of the canonical division theorem for any function.

Let $f \in E_{x,y}$ and consider it to be in $E_{x,y,\sigma}$ to write

$$\begin{aligned} f(x,y) &= P_k(x,\sigma) \cdot Q_1(x,y,\sigma) + \sum_{i=1}^{k} s_i(y,\sigma)x^{k-i} \\ &= (x^k + \sigma_1 x^{k-1} + \cdots + \sigma_k) \cdot Q_1(x,y,\sigma) + \sum_{i=1}^{k} s_i(y,\sigma)x^{k-i} \end{aligned}$$

Since f is independent of σ we can choose σ, as we saw in step one, to correlate to any specific regular function $P(x,y)$ we want to divide by and get

$$\begin{aligned}
f(x,y) &= \left(x^k + \sigma_1(y)x^{k-1} + \ldots + \sigma_k(y)\right) Q_1(x,y,\sigma(y)) + \sum_{i=1}^{k} s_i(y,\sigma(y))x^{k-i} \\
&= \left(x^k + \ldots + \sigma_k(y)\right) \frac{Q(x,y)}{Q(x,y)} Q_1(x,y,\sigma(y)) + \sum_{i=1}^{k} s_i(y,\sigma(y))x^{k-i} \\
&= P(x,y) \cdot \frac{Q_1(x,y,\sigma(y))}{Q(x,y)} + \sum_{i=1}^{k} s_i(y,\sigma(y))x^{k-i}
\end{aligned}$$

which completes the proof since we know that Q is a unit in $E_{x,y}$. QED

We have now shown that in order to prove the division theorem it is sufficient to prove the canonical division theorem. The proof of the canonical division theorem of order 1 is quite simple: We have $P_1(x,\sigma) = x + \sigma$ with $x, \sigma \in \mathbf{R}$. Let f be in $E_{x,y,\sigma}$. We set $r(y,\sigma) = f(-\sigma, y, \sigma)$ and consider $g(x,y,\sigma) = f(x,y,\sigma) - r(y,\sigma)$. By its construction the germ g is zero on the hyperplane $x + \sigma = 0$. Thus, by Proposition 3 in Chapter 1, it belongs to the ideal generated by $(x+\sigma)$ so that we have $g(x,y,\sigma) = g_1(x,y,\sigma)\cdot(x+\sigma)$ and thus

$$f(x,y,\sigma) = g_1(x,y,\sigma)\cdot(x+\sigma) + r(y,\sigma)$$

The proof for higher orders is however much more complex and is the aim of the rest of this chapter and the next.

3.3 Real forms and real subsets of $\mathbf{C}^n$

3.3.1 Real forms

We define a **real form** in $\mathbf{C}^n$ as a real vector subspace $\mathbf{F}$ of $\mathbf{C}^n$ with these properties:
a) $\dim \mathbf{F} = n$
b) $\mathbf{F} \cap i\mathbf{F} = \{0\}$

Conditions a) and b) are equivalent to the condition

$$\mathbf{F} + i\mathbf{F} = \mathbf{C}^n$$

and also to the condition that any basis $\{e_1, \ldots, e_n\}$ of $\mathbf{F}$ as a real vector space is a basis of $\mathbf{C}^n$ as a complex vector space. Thus $\mathbf{F} \subset \mathbf{C}^n$ is diffeomorphic to $\mathbf{R}^n \subset \mathbf{C}^n$.

Examples of real forms:

- In $\mathbf{C}$ any line $\{xc| \ x \in \mathbf{R}, \ c \neq 0$ a complex constant$\}$ is a real form.
- In $\mathbf{C}^2$ the subspace $\mathbf{C} \times \{0\} \subset \mathbf{C}^2$ is not a real form but $\mathbf{R} \times \mathbf{R} \subset \mathbf{C}^2$ is a real form.

Now let $f : \mathbf{F} \to \mathbf{C}$ be a differentiable function and let $\{e_1, \ldots, e_n\}$ be a basis of $\mathbf{F}$. This also serves as a basis of $\mathbf{C}^n$ and we let $x_1, \ldots, x_n$ be coordinates in $\mathbf{C}^n$ relative to this basis. Now f is a differentiable function of the real variables $x_1, \ldots x_n$ and can, as we have remarked in the previous chapter, be expanded for every $k \geq 1$ to

$$f(x) = (\text{complex polynomial in } x \text{ of order k}) + r(x)$$

where $r(x)$ is in the complex valued M^{k+1}.

We show that the function described by the polynomial does not depend on the choice of the basis. If we consider another basis with $y_1, \ldots, y_n$ as coordinates there would exist a linear transformation such that $x = T(y)$ which could be substituted into the expansion for $f(x)$ to give:

$$f(y) = (\text{complex polynomial in } y \text{ of order k}) + r(y)$$

We note that since T is a linear transformation it does not change the order of the polynomial and thus from the uniqueness of Taylor's expansion the polynomial found by working in a different basis describes the same function.

We return to the Taylor expansion of $f(x)$ and make the following important observation: The function is only defined for real values of x; and this is also true for $r(x)$ in the expansion. However the polynomial in the expansion also makes sense for complex x. Thus given a differentiable function on a real form $f : \mathbf{F} \to \mathbf{C}$ we can assosiate with it a polynomial of order k on $\mathbf{C}^n$ written as:

$$j_c^k f : \mathbf{C}^n \to \mathbf{C}$$

We call $j_c^k f$ the complex Taylor polynomial of f.

3.3.2 Real subsets

We define a **real subset** $\mathbf{F}$ of $\mathbf{C}^n$ as a finite union of real forms, thus:

$$\mathbf{F} = \cup_{i=1}^r \mathbf{F}_i$$

A mapping $f : U \to \mathbf{C}$, where U is an open subset of $\mathbf{F}$, is called differentiable if:

1. the restriction of f to $\mathbf{F}_i \cap U$, denoted by f_i, is differentiable for any $i = 1, \ldots, r$.

2. for each i and j $(i \neq j)$ and $x \in \mathbf{F}_i \cap \mathbf{F}_j \cap U$ we have $j_c^k f_i(x) = j_c^k f_j(x)$ for all integers $k \geq 0$.

The last condition can also be written as $j_c f_i(x) = j_c f_j(x)$ where $j_c f(x)$ is the complex Taylor series of f. The differentiable germs at the origin in $\mathbf{F}$ form a ring denoted by $E(\mathbf{F}, 0)$ or $E_{\mathbf{F}}$.

Examples:

- If f is an analytic function we have that its restriction to $\mathbf{F}$, a real subset, is differentiable.
- Let $\mathbf{F} = \mathbf{R} \cup i\mathbf{R} \subset \mathbf{C}$ and let $f : \mathbf{F} \to \mathbf{R}$ be defined as

$$\begin{aligned} f(x) &= \lambda(x) \quad \text{for } x \in \mathbf{R} \\ f(x) &= 0 \quad \text{for } x \in i\mathbf{R} \end{aligned}$$

  where $\lambda(x)$ is the flat function which was defined in Chapter 1. We see that f is differentiable but that it is not the restriction of an analytic function to $\mathbf{F}$.

For any differentiable function $f : U \to \mathbf{C}$, where U is an open subset of a real set $\mathbf{F}$ in $\mathbf{C}^n$, we can set

$$j_c^k f(x) = j_c^k f_i(x)$$

for any integer k, where $x \in \mathbf{F}_i \cap U$. This definition is independent of i.

3.4 The extension theorem of Lojasiewicz

Given two real subsets $\mathbf{H}$ and $\mathbf{F}$ of $\mathbf{C}^n$ with $\mathbf{H} \supset \mathbf{F}$, we have that a differentiable function (or germ) on $\mathbf{H}$ is also differentiable on $\mathbf{F}$ when restricted to $\mathbf{F}$, giving us a homomorphism from $E_{\mathbf{H}}$ to $E_{\mathbf{F}}$. In this section we prove the following result:

Theorem 6 (The extention theorem of Lojasiewicz, [4]) *Let $\boldsymbol{H}$ and $\boldsymbol{F}$ be two real subsets of $\boldsymbol{C}^n$ with $\boldsymbol{H} \supset \boldsymbol{F}$. The restriction homomorphism*

$$E_{\boldsymbol{H}} \to E_{\boldsymbol{F}}$$

is surjective.

The theorem says that any differentiable germ at the origin on $\mathbf{F}$ can be extended to a differentiable germ at the origin on $\mathbf{H}$. We begin the proof with two lemmas.

Lemma 2 *Let $\boldsymbol{F} = \boldsymbol{R} \times \boldsymbol{R}^n \subset \boldsymbol{C} \times \boldsymbol{C}^n$ and let $\boldsymbol{F}' = (c \cdot \boldsymbol{R}) \times \boldsymbol{R}^n \subset \boldsymbol{C} \times \boldsymbol{C}^n$, where c is a non-real complex number. Let $\boldsymbol{H} = \boldsymbol{F} \cup \boldsymbol{F}'$. The restriction homomorphism*

$$E_{\boldsymbol{H}} \to E_{\boldsymbol{F}}$$

is surjective.

Proof: From Borel's theorem we know that given any sequence of complex numbers (a_p) there exists a differentiable function $f : \mathbf{R} \to \mathbf{C}$ which has

$$\frac{\mathrm{d}^p f}{\mathrm{d}x^p}(0) = p! a_p$$

for any integer $p \geq 0$. (We proved this theorem in Chapter 1 for real a_p; the complex case is similar.)

We can generalise this to: given a sequence of complex valued differentiable functions $(a_p(y))$, y in $\mathbf{R}^n$, then there exists a differentiable function $f : \mathbf{R}^{n+1} \to \mathbf{C}$ which has

$$\frac{\partial^p f}{\partial x^p}(0, y) = p!\ a_p(y)$$

for any integer $p \geq 0$.

Now since any given differentiable function g on $\mathbf{F}'$ determines the $(a_p(y))$, it can be extended to a differentiable f on $\mathbf{H}$. QED

Lemma 3 *Let $F_1, \ldots, F_p$, with p a positive integer, be vector subspaces of $\boldsymbol{R}^n$. Let f and g be two differentiable functions on $\boldsymbol{R}^n$ such that $f - g$ is flat on $(F_1 \cup \cdots \cup F_{p-1}) \cap F_p$. Then there exists a differentiable function h on $\boldsymbol{R}^n$ such that:*

1. *$f - h$ is flat on $F_1 \cup \cdots \cup F_{p-1}$*

2. *$g - h$ is flat on F_p.*

Thus if f and g have the same Taylor series at each point of $(F_1 \cup \cdots \cup F_{p-1}) \cap F_p$, then there exists an h that has the same Taylor series as f on $(F_1 \cup \cdots \cup F_{p-1})$ and the same Taylor series as g on F_p.

Proof: According to the lemma at the end of Chapter 1 there exist $L_1, \ldots, L_{p-1}$ such that each L_m is flat on F_m with $L_m - 1$ flat on F_p (excluding the intersection). Now let

$$L = \prod_{m=1}^{p-1} L_m$$

Since we have L flat on $(F_1 \cup \cdots \cup F_{p-1})$ and $L-1$ flat on F_p (excluding the intersection), we can set

$$h = f \cdot (1-L) + g \cdot L$$

for the desired function. QED

Proof of the theorem: We want to show that for real subsets $\mathbf{H} \supset \mathbf{F}$ we have that $E_{\mathbf{H}} \to E_{\mathbf{F}}$ is a surjective homomorphism. It is sufficient to prove the theorem for the case $\mathbf{H} = \mathbf{F} \cup \mathbf{F}'$ where F' is a real form of $\mathbf{C}^n$. Let the real subset $\mathbf{F} = \mathbf{F}_1 \cup \ldots \cup \mathbf{F}_p$ with the $\mathbf{F}_i$ real forms and use induction on p.

Step 1: The case $p = 1$.
In this case $\mathbf{F}$ is itself a real form. If $\mathbf{F} \cap \mathbf{F}'$ has dimension $n-1$ the situation is that of Lemma 2.

If the dimension of $\mathbf{F} \cap \mathbf{F}'$ is less than $n-1$, we construct a sequence of real forms

$$\mathbf{F} = \mathbf{F}_0, \mathbf{F}_1, \ldots, \mathbf{F}_k$$

such that:

1. $\mathbf{F}_0 \cap \mathbf{F}' \subset \mathbf{F}_1 \cap \mathbf{F}' \subset \ldots \subset \mathbf{F}_k \cap \mathbf{F}'$
2. $\dim \mathbf{F}_k \cap \mathbf{F}' = n-1$
3. $\dim \mathbf{F}_{i-1} \cap \mathbf{F}_i = n-1 \quad$, for $i = 1, \ldots, k$.

We now apply Lemma 2 k times.

Step 2: The case $p > 1$.
We assume the theorem is true for a real subset consisting of $p-1$ real forms and prove it for a real subset

$$\mathbf{F} = \mathbf{F}_1 \cup \cdots \cup \mathbf{F}_p$$

Let $f \in E_{\mathbf{F}}$ be the germ that we want to extend to $\mathbf{H} = \mathbf{F} \cup \mathbf{F}'$. We restrict f to $\mathbf{F}_1 \cup \cdots \cup \mathbf{F}_{p-1}$ and extend it to $\mathbf{F}_1 \cup \cdots \cup \mathbf{F}_{p-1} \cup \mathbf{F}'$ (according to our assumption), calling it f_1. We also restrict f to $\mathbf{F}_p$ and extend it to $\mathbf{F}_p \cup \mathbf{F}'$, calling it f_2.

We now restrict f_1 and f_2 to $\mathbf{F}'$ and denote $\mathbf{F}_i \cap \mathbf{F}'$ by $\mathbf{F}'_i$. We now have $f_1 - f_2$ flat on $\left(\mathbf{F}'_1 \cup \cdots \cup \mathbf{F}'_{p-1}\right) \cap \mathbf{F}'_p$. According to Lemma 3 we can get a function h which is differentiable on $\mathbf{F}' \simeq \mathbf{R}^n$ such that $f_1 - h$ is flat on $\mathbf{F}'_1 \cup \cdots \cup \mathbf{F}'_{p-1}$ and $f_2 - h$ is flat on $\mathbf{F}'_p$. Thus we can extend f to $\mathbf{H} = \mathbf{F} \cup \mathbf{F}'$ by letting it be h on $\mathbf{F}'$. QED

Chapter 4

The division theorem: Part 2

This chapter will complete the proof of the division theorem. As we have seen in the previous chapter, we are left to prove the canonical division theorem of order k. We define the canonical polynomial of order k slightly differently as

$$P_k(x, \sigma) = x^k - \sigma_1 x^{k-1} + \ldots + (-1)^k \sigma_k$$

for the proof. It should be clear that it is essentially the same as in the previous chapter and that the relevant results remain true.

We want to prove that any germ $f \in E_{x,y,\sigma}, (x, y, \sigma) \in \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^k$, can be written as

$$f(x, y, \sigma) = P_k(x, \sigma) \cdot Q(x, y, \sigma) + \sum_{i=1}^{k} r_i(y, \sigma) x^{k-i}$$

where Q is in $E_{x,y,\sigma}$ and the r_i are in $E_{y,\sigma}$.

We work as follows: Section 1 introduces the Newton mapping and some of its properties, Section 2 proves a proposition and a lemma and Section 3 completes the proof of the division theorem.

4.1 The Newton mapping

The Newton mapping is defined by

$$N : \mathbf{C}^k \to \mathbf{C}^k$$
$$N(z) = \sigma$$

with

$$\begin{aligned}\sigma_1 &= z_1 + z_2 + \cdots + z_k \\ \sigma_2 &= \sum_{1\le i<j\le k} z_i z_j \\ &\vdots \\ \sigma_k &= z_1 z_2 \cdots z_k.\end{aligned}$$

Equivalently we see that the $\sigma_1, \ldots, \sigma_k$ can be determined by

$$\prod_{i=1}^{k}(\xi - z_i) = \xi^k - \sigma_1 \xi^{k-1} + \ldots + (-1)^k \sigma_k. \tag{4.1}$$

Some properties of the Newton mapping are:

- The Newton mapping is surjective: For any σ there exists a z such that $N(z) = \sigma$; this follows from (4.1) since the z_i are simply the roots of the equation.

- The Newton mapping is continuous and open.

- $N^{-1}(\sigma)$ consists of at most $k!$ distinct points. This maximum occurs if σ is such that the roots of (4.1) are all distinct; then the $k!$ permutations of the roots will represent different points in $\mathbf{C}^k$. $N^{-1}(\sigma)$ consists of less than $k!$ points if some of the roots of (4.1) are the same; this happens when the discriminant polynomial of (4.1) is zero:

$$\delta(\sigma) = 0$$

- The determinant of DN (the Jacobian matrix of N at z) is zero when we have $z_i = z_j$ for some $i \neq j$. (See the appendix for proof of this.) The image of such a z is given by $\delta(\sigma) = 0$ according to the previous remark. A set like this, which is defined by a polynomial, is called an algebraic set.

- Let $\tau \in T_k$ where T_k is the group of permutations of $1, \ldots, k$. We abuse our notation to write $\tau(z_1, \ldots, z_k)$ for $(z_{\tau(1)}, \ldots, z_{\tau(k)})$. For the Newton mapping we have invariance under T_k:

$$N \circ \tau = N$$

for any $\tau \in T_k$.

- The analytic theorem of Newton: Let $f(z_1, \ldots, z_k)$ be an analytic function which is symmetric in $z_1, \ldots, z_k$. (We have $f \circ \tau = f$ for all τ.) Then there exists a unique analytic function $g(\sigma_1, \ldots, \sigma_k)$ such that $f = g \circ N$. (This is also true with parameters: If $f(y, z_1, \ldots, z_k)$ is symmetric in $z_1, \ldots, z_k$ then there is a $g(y, \sigma)$ with $f(y, z) = g(y, N(z))$.)

The proof of the analytic theorem of Newton is fairly simple and we omit it since we do not need it for our final goal. In the next section, however, we will prove a similar looking result (for real subsets) which we do need. That result furthermore leads directly to the differentiable theorem of Newton, but we will not pursue it. We end this section by proving the analytic division theorem – mainly to demonstrate a technique which we will later use.

Consider the mapping

$$\mathbf{C} \times \mathbf{C}^n \times \mathbf{C}^k \to \mathbf{C} \times \mathbf{C}^n \times \mathbf{C}^k$$
$$(x, y, z) \mapsto (x, y, N(z))$$

which we will still call N. For any function $f(x, y, \sigma)$, let $\bar{f}(x, y, z) = f(x, y, N(z))$, i.e., $\bar{f} = f \circ N$. Thus we have

$$\bar{P}(x, z) = (x - z_1)(x - z_2)\ldots(x - z_k),$$

where P is the canonical polynomial.

Let $f(x, y, \sigma)$ be any analytic function and consider $\bar{f}(x, y, z)$. We divide $\bar{f}$ by $\bar{P}$ using the following algorithm:
Consider the difference $\bar{f}(x, y, z) - \bar{f}(z_1, y, z)$ and note that it is zero on the set $x - z_1 = 0$. Therefore, by the analytic version of Proposition 3 in Chapter 1, it is divisible by $(x - z_1)$ in the ring of analytic functions. We thus obtain

$$\bar{f}(x, y, z) = (x - z_1)\bar{Q}_1(x, y, z) + \bar{f}(z_1, y, z).$$

Repeat this process using $\bar{Q}_1$ and $(x - z_2)$ in place of $\bar{f}$ and $(x - z_1)$, etc. By substitution, we eventually obtain

$$\bar{f}(x, y, z) = (x - z_1)\ldots(x - z_k)\bar{Q}(x, y, z) \; + \; \sum_{i=1}^{k} \bar{r}_i(y, z)x^{k-i},$$

where $\bar{Q}$ and the $\bar{r}_i$ are analytic.

The functions $\bar{Q}$ and $\bar{r}_i$ are in fact symmetric in $z_1, \ldots, z_k$. To see this, let y and z be given such that the coordinates $z_1, \ldots, z_k$ are all distinct. Then the equation yields

$$\begin{aligned}\bar{f}(z_1, y, z) &= \sum_{i=1}^{k} \bar{r}_i(y, z) z_1^{k-i} \\ \vdots \quad & \quad \vdots \\ \bar{f}(z_k, y, z) &= \sum_{i=1}^{k} \bar{r}_i(y, z) z_k^{k-i}.\end{aligned}$$

These equations determine the numbers $\bar{r}_1(y, z), \ldots, \bar{r}_k(y, z)$ since the determinant of this system is a non-zero Van der Monde determinant. Since the set of z with coordinates all distinct is open and dense, and since this is also true for $\bar{P}(x, z) \neq 0$, we have uniqueness for division by $\bar{P}$. The symmetry of $\bar{Q}$ and of the $\bar{r}_i$ in z follows immediately from the symmetry of $\bar{f}$ and $\bar{P}$.

We now apply the analytic theorem of Newton (with x and y as parameters) to obtain the desired result.

4.2 A proposition and a lemma

Set $\mathbf{F}$ equal to $N^{-1}(\mathbf{R}^k)$, the inverse image of $\mathbf{R}^k$ by N. Now $\mathbf{F}$ is the set of points $(z_1, \ldots, z_k)$ such that the complex numbers $z_1, \ldots, z_k$ are roots of an equation

$$x^k - \sigma_1 x^{k-1} + \ldots + (-1)^k \sigma_k = 0,$$

where the coefficients $\sigma_1, \ldots, \sigma_k$ are real. We know that complex roots will occur in pairs as complex conjugates, thus there exists a permutation τ in T_k, which is an involution, such that

$$z_i = \bar{z}_{\tau(i)} \quad \text{for } i = 1, \ldots, k.$$

For each involution τ of $1, \ldots, k$, let

$$\mathbf{F}_\tau = \{z \mid z_i - \bar{z}_{\tau(i)} = 0, \quad i = 1, \ldots, k\}.$$

Each $\mathbf{F}_\tau$ is now a real form of $\mathbf{C}^k$.

Examples:

- $\mathbf{F}_1 = \mathbf{R}^k$, where 1 is the identity permutation of $1, \ldots, k$.
- For $k = 2$ we have two involutions (1 and τ). We now have:
  $\mathbf{F}_1 = \mathbf{R}^2$ and
  $\mathbf{F}_\tau = \{z \mid z_1 = \bar{z}_2\}$.
  We can see that $\mathbf{F}_\tau$ is a real form from the fact that $(1, 1)$ and $(-i, i)$ form a basis for the real form over $\mathbf{R}$ and a basis for $\mathbf{C}^2$ over $\mathbf{C}$.

We now have that

$$\mathbf{F} = \cup \mathbf{F}_\tau$$

and thus that $\mathbf{F}$ is a real subset of $\mathbf{C}^k$. It is invariant under the group of permutations T_k:

$$\rho(\mathbf{F}_\tau) = \mathbf{F}_{\rho \circ \tau \circ \rho^{-1}}, \text{ for } \rho \text{ in } T_k.$$

We note that in $\mathbf{F}$ the set of points belonging to two distinct real forms is defined by $\prod(z_i - z_j) = 0$.

During the rest of this chapter, we will denote by

$$N : \mathbf{F} \to \mathbf{R}^k$$

the restriction of the Newton mapping to $\mathbf{F}$. This is a differentiable mapping from $\mathbf{F}$ to $\mathbf{R}^k$ in the sense of Section 3.2 of the previous chapter since it is the restriction to $\mathbf{F}$ of an analytic mapping. It remains continuous, open and surjective. We can now prove the proposition to which we referred in the previous section.

Proposition 5 *If f is a symmetric differentiable function on $\boldsymbol{F}$, then there exists a unique differentiable function g on $\boldsymbol{R}^k$ such that*

$$f = g \circ N.$$

Proof: The equation $f = g \circ N$ defines (setwise) a mapping g on $\mathbf{R}^k$ since f is constant on the fibres of N and since N is surjective. Furthermore g is continuous on $\mathbf{R}^k$ since N is open.

We have that g is differentiable on $\mathbf{R}^k - \Sigma$, since N is a local diffeomorphism over each point of $\mathbf{R}^k - \Sigma$. Since $\Sigma \subset \mathbf{R}^k$ is an algebraic subset, to show that g is differentiable everywhere, we need only prove that its derivatives of all orders have *continuous extensions* to all of $\mathbf{R}^k$. (See the appendix.) We will demonstrate this by induction on the order of the derivative, but we will only present the first step in the induction to keep the notation simple. From the chain rule it follows that

$$Df(z) = Dg(N(z)) \cdot DN(z) \tag{4.2}$$

for all non-singular points of N. We will consider the linear equation on $\mathbf{F}$

$$Df(z) = G_1(z) \cdot DN(z) \tag{4.3}$$

with

$$G_1(z) \in L(\mathbf{C}^n, \mathbf{C}).$$

Let us assume that (4.3) admits a unique solution G_1 which is symmetric and differentiable on $\mathbf{F}$. We first show that this result implies the proposition. Comparing (4.2) and (4.3), we have

$$G_1 = g_1 \circ N$$

where g_1 is defined and continuous on $\mathbf{R}^k$ by the symmetry of G_1 and the properness of N. Thus g_1 is a continuous extension of Dg to $\mathbf{R}^k$. Next, replace f by G_1 and repeat the same argument to show that the second derivative of g extends to a continuous function on $\mathbf{R}^k$ and so on for all derivatives.

We are left to prove our assumption: equation (4.3) admits a unique solution G_1 which is symmetric and differentiable on $\mathbf{F}$.
If we multiply (4.3) term by term by the adjoint matrix A of DN we get

$$Df(z) \cdot A = |DN| \cdot G_1 = \sum_{1 \leq i < j \leq k} (z_i - z_j) \cdot G_1. \tag{4.4}$$

The first term is clearly differentiable on $\mathbf{F}$ and is antisymmetric in $(z_1, \ldots, z_k)$. This means that

$$(Df \cdot A) \circ \tau = -Df \cdot A$$

for any transposition τ of $\{1, \ldots, k\}$.

If $z_i - z_j = 0$ (with $i < j$) we have by the antisymmetry property that the Taylor series of $Df \cdot A$ is divisible by $z_i - z_j$. It follows from Theorem 4 in Chapter 2 that $Df \cdot A$ is divisible by $z_i - z_j$. This is true for all $z_i - z_j = 0$ (with $i < j$) so that the first term of (4.4) is divisible by the product of these forms. The existence and symmetry of G_1 follows. The uniqueness is clear since G_1 has been determined beforehand on an open and dense subset of $\mathbf{F}$. QED

We end this section with a lemma.

Lemma 4 *Consider a real subset consisting of real forms in $\boldsymbol{C}^n$:*

$$\boldsymbol{F} = \boldsymbol{F}_1 \cup \ldots \cup \boldsymbol{F}_p.$$

Suppose $f \in E_{\boldsymbol{F}}$ and suppose $\omega : \boldsymbol{F} \to \boldsymbol{C}$ is the restriction of a linear form to $\boldsymbol{F}$ such that at each $x \in \boldsymbol{F}$ where ω is zero, we have $f(x) = 0$ and an $\boldsymbol{F}_i$ such that $x \in \boldsymbol{F}_i$ and ω is real on $\boldsymbol{F}_i$. Then f is divisible by ω.

Proof: Consider any $x \in \mathbf{F}$ where ω is zero. The restriction f_i of f to $F_i \simeq \mathbf{R}^n$ is divisible by ω_i (the restriction to F_i), since ω_i is real on F_i and can be considered as a coordinate function, and f_i is zero when ω_i is zero. (By Proposition 3 in Chapter 1.) It follows that $jf_i(x)$ is divisible by $j\omega_i(x)$ and thus that $j_c f(x)$ is divisible by $j_c\omega(x)$. By Theorem 4 in Chapter 2 we now have that f is divisible by ω. QED

4.3 The proof completed

Consider again the mapping

$$\mathbf{C} \times \mathbf{C}^n \times \mathbf{C}^k \to \mathbf{C} \times \mathbf{C}^n \times \mathbf{C}^k$$
$$(x, y, z) \mapsto (x, y, N(z))$$

which we will still call N. Let $\mathbf{F} = N^{-1}(\mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^k)$. It is again a real subset

$$\mathbf{F} = \cup \mathbf{F}_\tau$$

with each $\mathbf{F}_\tau$ a real form associated with an involution τ of $1, \ldots, k$.

We now consider

$$N : \mathbf{F} \to \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^k$$

and for a function $f(x, y, \sigma)$, we set

$$\bar{f} = f \circ N$$

to obtain a differentiable function on $\mathbf{F}$ which is symmetric in $(z_1, \ldots, z_k)$. We again have

$$\bar{P}(x, z) = (x - z_1) \ldots (x - z_k).$$

We now define in the space $\mathbf{C} \times \mathbf{C}^n \times \mathbf{C}^k$ a real subset $\tilde{\mathbf{F}} \supset \mathbf{F}$ as follows:

$$\tilde{\mathbf{F}} = \mathbf{F} \cup (\cup_{\tau,i} \mathbf{F}_{\tau,i})$$

where for each involution τ of $\{1, \ldots, k\}$ and $i = 1, \ldots, k$ we have that $\mathbf{F}_{\tau,i}$ is the set of (x, y, z) in $\mathbf{C} \times \mathbf{C}^n \times \mathbf{C}^k$ such that

$$\begin{cases} z_j - \bar{z}_{\tau(j)} = 0 & j = 1, \ldots, k \\ (x - z_i) - (\bar{x} - \bar{z}_i) = 0, & (i.e., x - z_i \in \mathbf{R}) \\ y \in \mathbf{R}. \end{cases}$$

Again each $\mathbf{F}_{\tau,i}$ is a real form, and the set $\tilde{\mathbf{F}}$ is invariant under T_k.

Given any f in $E(\mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^k)$, set $\bar{f} = f \circ N \in E_{\mathbf{F}}$ and then extend $\bar{f}$ to a function $\tilde{f}$ which is differentiable on $\tilde{\mathbf{F}}$, by using the extension theorem of Lojasiewicz. We can assume that $\tilde{f}$ is symmetric in $(z_1, \ldots, z_k)$. (Use the sum $1/k!(\sum_\tau \tilde{f} \circ \tau)$.)

We are now in the situation where we can divide $\tilde{f}$ on $\tilde{\mathbf{F}}$ by $(x - z_1)(x - z_2) \ldots (x - z_k)$ in the same way that we did in the analytic case in the first section. Form

$$\tilde{f}(x, y, z) - \tilde{f}(z_1, y, z), \qquad (x, y, z) \in \tilde{\mathbf{F}}.$$

The function is well-defined ($(z_1, y, z) \in \mathbf{F}_{\tau,1}$ for some τ) and is differentiable on $\tilde{\mathbf{F}}$. It is divisible in $E_{\tilde{\mathbf{F}}}$ by $(x - z_1)$, since it is zero by construction if $x - z_1 = 0$, since every point of $\tilde{\mathbf{F}}$ where $x = z_1$ belongs to $\mathbf{F}_{\tau,1}$ for some τ and since $x - z_1$ is real on $\mathbf{F}_{\tau,1}$. (Lemma 4.)

We continue this process as in the analytic case until we obtain

$$\tilde{f}(x, y, z) = (x - z_1) \ldots (x - z_k) \cdot \tilde{Q}(x, y, z) + \sum_{i=1}^{k} \tilde{r}_i(y, z) \cdot x^{k-i}.$$

We can show uniqueness of the above division process just as in the analytic case. The symmetry of $\tilde{Q}$ and the $\tilde{r}_i$ in z follows from that of $\tilde{f}$ and $\bar{P}$. We now restrict the above equation to $\mathbf{F}$ and use Proposition 1 (with parameters) to complete the existence of the division.

Appendix A

The determinant of the Newton mapping

A.1 Introduction

Let N denote the Newton mapping, DN the Jacobian matrix of the Newton mapping and $|DN|$ the determinant of this matrix. We discuss a combinatorial proof to show that the absolute value of $|DN|$ is equal to that of the Van der Monde determinant, and subsequently provide a formula for $|DN|$. As far as the author is aware, the proof is new.

A.2 Notation

The Newton mapping is defined by

$$N : \mathbf{C}^k \to \mathbf{C}^k$$
$$N(z) = \sigma$$

with

$$\begin{array}{rcccl} \sigma_1 & = & z_1 + z_2 + \cdots + z_k & = & \sum z_* \\ \sigma_2 & = & \sum\limits_{1 \le i < j \le k} z_i z_j & = & \sum z_*^2 \\ \vdots & & \vdots & & \vdots \\ \sigma_k & = & z_1 z_2 \cdots z_k & = & \sum z_*^k \end{array}$$

where we use the notation $\sum z_*^p$ to indicate the sum of all the products consisting of p different variables z_i. We also use $\sum_{z_m=0} z_*^p$ to indicate that all terms containing z_m are omitted; for example if $k = 4$ we have

$$\sum z_*^3 = z_1z_2z_3 + z_1z_2z_4 + z_1z_3z_4 + z_2z_3z_4$$

and

$$\sum_{z_2=0} z_*^3 = z_1z_3z_4.$$

With this notation, the Jacobian matrix of the Newton mapping can be written as

$$DN = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \sum\limits_{z_1=0} z_* & \sum\limits_{z_2=0} z_* & \cdots & \sum\limits_{z_k=0} z_* \\ \vdots & \vdots & & \vdots \\ \sum\limits_{z_1=0} z_*^{k-1} & \sum\limits_{z_2=0} z_*^{k-1} & \cdots & \sum\limits_{z_k=0} z_*^{k-1} \end{pmatrix}.$$

A.3 The absolute value of $|DN|$ is equal to that of the Van der Monde determinant

Given a square matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{pmatrix}$$

we begin by noting a way in which A can be altered without changing the absolute value of its determinant.

Lemma 5 *Replacing a row $a_{n1}, \ldots, a_{nk}$ of A by a new row $b_1, \ldots, b_k$ to form matrix B changes only the sign of the determinant if there exists another row $a_{m1}, \ldots, a_{mk}$ of A such that for all $i \neq j$ we have*

$$-\begin{vmatrix} a_{mi} & a_{mj} \\ a_{ni} & a_{nj} \end{vmatrix} = \begin{vmatrix} a_{mi} & a_{mj} \\ b_i & b_j \end{vmatrix}.$$

We say in this case that we use the row $a_{m1}, \ldots, a_{mk}$ as a hinge for the replacement.

Proof: We work in three steps.

Step 1 Interchange the rows of A so that $a_{n1}, \ldots, a_{nk}$ is the last and $a_{m1}, \ldots, a_{mk}$ the second to last row to form A'. Do the same with B so that $b_1, \ldots, b_k$ is the last and $a_{m1}, \ldots, a_{mk}$ the second to last row of B'.

Step 2 By beginning at the top and working down, using the usual algorithm for calculating the determinant, we eventually obtain

$$|A'| = \sum_{1 \leq i < j \leq k} c_{ij} \begin{vmatrix} a_{mi} & a_{mj} \\ a_{ni} & a_{nj} \end{vmatrix},$$

where the c_{ij} are complex constants.

Step 3 We do the same for B' and obtain

$$|B'| = \sum_{1 \leq i < j \leq k} c_{ij} \begin{vmatrix} a_{mi} & a_{mj} \\ b_i & b_j \end{vmatrix} = -|A'|.$$

Since interchanging rows can only change the sign of a determinant and since A and B were treated similarly in Step 1, this completes the proof. QED

We are now ready to show that the absolute value of $|DN|$ is equal to that of the Van der Monde determinant; that is

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \sum_{z_1=0} z_* & \sum_{z_2=0} z_* & \cdots & \sum_{z_k=0} z_* \\ \vdots & \vdots & & \vdots \\ \sum_{z_1=0} z_*^{k-1} & \sum_{z_2=0} z_*^{k-1} & \cdots & \sum_{z_k=0} z_*^{k-1} \end{vmatrix} = \pm \begin{vmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_k \\ \vdots & \vdots & & \vdots \\ z_1^{k-1} & z_2^{k-1} & \cdots & z_k^{k-1} \end{vmatrix}.$$

Our aim is to alter the left-hand side until it corresponds to the right-hand side, by using induction on the indexes of the rows. We note that the first rows are already similar. Thus let us assume that the first p rows at the left can be replaced by rows that are similar to those at the right without changing the absolute value of the determinant. This means that we have

$$|DN| = \pm \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ z_1^{p-1} & z_2^{p-1} & \cdots & z_k^{p-1} \\ \sum_{z_1=0} z_*^p & \sum_{z_2=0} z_*^p & \cdots & \sum_{z_k=0} z_*^p \\ \vdots & \vdots & & \vdots \\ \sum_{z_1=0} z_*^{k-1} & \sum_{z_2=0} z_*^{k-1} & \cdots & \sum_{z_k=0} z_*^{k-1} \end{vmatrix}.$$

We want to show that in row $p+1$

$$\sum_{z_1=0} z_*^p, \ldots, \sum_{z_k=0} z_*^p$$

can be replaced by

$$z_1 \sum_{z_1=0} z_*^{p-1}, \ldots, z_k \sum_{z_k=0} z_*^{p-1},$$

then by

$$z_1^2 \sum_{z_1=0} z_*^{p-2}, \ldots, z_k^2 \sum_{z_k=0} z_*^{p-2},$$

and so on, until we have reached row $p+1$ as

$$z_1^p, \ldots, z_k^p.$$

Thus it would suffice to show that we can replace

$$z_1^n \sum_{z_1=0} z_*^{p-n}, \ldots, z_k^n \sum_{z_k=0} z_*^{p-n},$$

for any $n < p$, by

$$z_1^{n+1} \sum_{z_1=0} z_*^{p-n-1}, \ldots, z_k^{n+1} \sum_{z_k=0} z_*^{p-n-1}.$$

We use Lemma 5 with row $n+1$ as hinge. This row is $z_1^n, \ldots, z_k^n$ by our inductive hypothesis. For $i \neq j$ we obtain

$$\begin{vmatrix} z_i^n & z_j^n \\ z_i^n \sum_{z_i=0} z_*^p & z_j^n \sum_{z_j=0} z_*^p \end{vmatrix} = z_i^n z_j^n \left(\sum_{z_j=0} z_*^p - \sum_{z_i=0} z_*^p \right).$$

Omitting terms which negate each other gives

$$z_i^n z_j^n \left(z_i \sum_{\substack{z_j = 0 \\ z_i = 0}} z_*^{p-1} - z_j \sum_{\substack{z_i = 0 \\ z_j = 0}} z_*^{p-1} \right)$$

and then, allowing some terms which negate each other, gives

$$z_i^n z_j^n \left(z_i \sum_{z_i=0} z_*^{p-1} - z_j \sum_{z_j=0} z_*^{p-1} \right) = - \begin{vmatrix} z_i^n & z_j^n \\ z_i^{n+1} \sum_{z_i=0} z_*^{p-1} & z_j^{n+1} \sum_{z_j=0} z_*^{p-1} \end{vmatrix} .$$

This completes the proof.

A.4 A formula for $|DN|$

The Van der Monde determinant is given by the well known formula

$$\prod_{1 \leq i < j \leq k} (z_j - z_i).$$

Since we used Lemma 5 exactly $(k-1) + (k-2) + \ldots + 1$ times in the proof of the previous section, we have

$$|DN| = (-1)^{(k-1)+(k-2)+\ldots+1} \prod_{1 \leq i < j \leq k} (z_j - z_i) = \prod_{1 \leq i < j \leq k} (z_i - z_j).$$

Appendix B

Extending the derivative of g to an algebraic subset

B.1 Introduction

Given that

$\Sigma \subset \mathbf{R}^k$ is an algebraic subset,

$g : \mathbf{R}^k \backslash \Sigma \to \mathbf{R}$ is in C^∞,

$\overline{g} : \mathbf{R}^k \to \mathbf{R}$ (the extension of g to $\mathbf{R}^k$) is continuous,

$\overline{\frac{\partial g}{\partial x_1}} : \mathbf{R}^k \to \mathbf{R}$ (the extension of $\frac{\partial g}{\partial x_1}$ to $\mathbf{R}^k$) is continuous,

we want to show that

$$\frac{\partial \overline{g}}{\partial x_1} = \overline{\frac{\partial g}{\partial x_1}}.$$

The proof uses two lemmas which are proved at the end. The author was unable to find a proof of this result in the literature. The proof is due to him.

B.2 Proof

We need to show that for any $x \in \Sigma$ we have

$$\frac{\partial \overline{g}}{\partial x_1}(x) = \overline{\frac{\partial g}{\partial x_1}}(x).$$

We can assume that $x = 0$ and that $\frac{\partial g}{\partial x_1}(0) = 0$. Thus we want to show that $\frac{\partial \overline{g}}{\partial x_1}(0) = 0$. We regard the positive x_1-axis and show that the right-hand derivative is zero; the left-hand derivative is similar and the result follows.

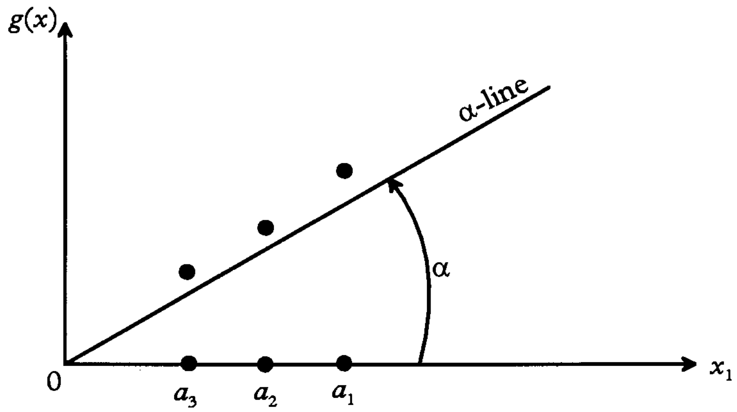
According to Lemma 6 we either have that the x_1-axis lies within Σ or that we can choose ϵ small enough for the whole line

$$(0, \epsilon) = \{x \text{ is on the positive } x_1\text{-axis}, \quad 0 < |x| < \epsilon\}$$

to be outside Σ.

B.2.1 Line $(0, \epsilon)$ is outside Σ

Assume that $\frac{\partial \overline{g}}{\partial x_1}(0) \neq 0$. Assuming $g(x)$ positive (without loss) implies that there exist an angle α and a sequence of points $a_1, a_2, \ldots$ on the positive x_1-axis which approach 0 such that all $g(a_i)$ are above the α-line. (Thus $g(a_i) > \alpha(a_i)$.)



It now follows (from the mean-value theorem for derivatives) that between any a_i and 0 there is a point where the derivative will be at least that of the α-line. Thus the limit of the derivatives can not be zero, that is $\frac{\partial \overline{g}}{\partial x_1}(0) \neq 0$, which is a contradiction.

B.2.2 The x_1-axis lies within Σ

The argument is similar to that of the above, but we need Lemma 7. We again assume $\frac{\partial \overline{g}}{\partial x_1}(0) \neq 0$ and find an angle α and a sequence of points $a_1, a_2, \ldots$ on the positive x_1-axis which approach 0 such that all $g(a_i)$ are above the α-line. As a_i approach 0 we can now find line segments (a, b), according to Lemma 7, which lies outside Σ with a arbitrarily close to 0 and b arbitrary close to a_i. Thus, by the continuity of $\overline{g}$, we can choose (a, b) such that g has gradient equal to or higher than that of the α-line. Since (a, b) lies outside Σ g is differentiable on it and we again use the mean-value theorem for derivatives to obtain the desired contradiction.

B.3 Lemmas

Lemma 6 *If $\Sigma \subset \boldsymbol{R}^k$ is an algebraic subset then we either have that the x_1-axis lies within Σ or that we can choose ϵ small enough for the whole line*

$$(0, \epsilon) = \{x \text{ is on the positive } x_1\text{-axis}, \quad 0 < |x| < \epsilon\}$$

to be outside Σ.

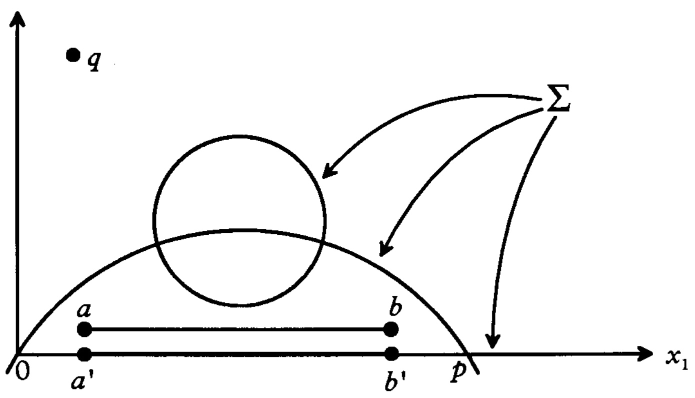
Proof: Regard $\Sigma \cap \{x_1\text{–axis}\}$. The intersection of two algebraic sets is again an algebraic set. In one dimension this is either the whole line or a finite number of points, so that we choose ϵ small enough for $(0, \epsilon)$ to be outside Σ. QED

Lemma 7 *Let Σ be an algebraic subset such that*

$$\{x_1\text{–axis}\} \subset \Sigma \subset \boldsymbol{R}^k.$$

Then there exists a line (a, b) parallel and next to the positive x_1-axis such that it does not intersect with Σ and with a arbitrarily close to 0.

Proof: Find any point q off Σ and consider the plane through this point and the x_1-axis. The intersection of this plane and Σ must again be an algebraic set (in 2 dimensions) and, since it is not the whole plane, it consists of a finite number of points and smooth curves with a finite number of intersections.



Suppose p is the closest point of intersection to 0 of $\Sigma - \{x_1\text{–axis}\}$ on the positive x_1-axis. We can now regard any closed line segment $[a', b']$ which lies strictly between 0 and p to find, by means of a continuity argument, that the smooth curves of the algebraic set (excluding the x_1-axis) attains a minimum distance from $[a', b']$. We can thus construct $[a, b]$ parallel and opposite to $[a', b']$ and close enough to the x_1-axis to miss Σ. QED

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DECLARATION

I, the undersigned, hereby declare that the dissertation submitted herewith for the degree Magister Scientiae to the University of Pretoria contains my own, independent work and has not been submitted for any degree at any other university.

Name: PO de Wet

Date: 2003-02-25