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The Hurst parameter and option pricing with fractional Brownian motion

by

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Submitted in partial fulfilment of the requirements for the degree

Magister Scientiae

in the Department of Mathematics and Applied Mathematics

in the Faculty of Natural and Agricultural Sciences

University of Pretoria

Pretoria

November 2012

Declaration

I, the undersigned, declare that the dissertation, which I hereby submit for the degree Magister Scientiae at the University of Pretoria is my own work and has not previously been submitted by me for any degree at this or any other tertiary institution.

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Date: November 2012

Acknowledgements

Thank you to my best friend C. Britz and my supervisor Dr A.J. Van Zyl. Thank you to the Department of Mathematics and Applied Mathematics at the University of Pretoria.

Summary

The Hurst parameter $H \in [0, 1]$ is a useful measure for the predictability of stock prices. The Hurst parameter was estimated for different South African stocks over different periods of time to determine if there was persistency in the returns. Fractional Brownian motion (fBm) is a Gaussian process that depends on the Hurst parameter which allows for the modeling of autocorrelation in price returns.

In this dissertation when modeling financial derivatives, the underlying driving process is replaced with fBm. fBm is not a semimartingale, thus arbitrage cannot be excluded by the choice of integration theory. The classical theory of stochastic calculus is not applicable and the solution of the fractional stochastic differential equation is found using fractional Wick Itô Skorohod integrals.

Fractional Black-Scholes and Black formulas are derived in three different frameworks where the underlying is driven by fractional Brownian motion in each case. The mathematics behind the models is discussed, in addition some analysis of the models is done. It was found that there is a range of possible combinations of Hurst and volatility parameters corresponding to a given price in the models.

The performance of the models is investigated by using South African futures option prices and warrants. Assuming a constant Hurst parameter the fractional implied volatilities were backed out and compared to the market volatilities. We found simple relationships between the implied fractional volatilities and the market volatility for each of the models. For fixed Hurst parameters the out-of-sample percentage pricing errors and absolute pricing errors are calculated to investigate the performance of the models.

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Chapter 1

Introduction

In this dissertation we are studying option pricing models when the underlying asset is driven by fractional Brownian motion. In particular we are investigating European options on shares and options on futures. A European call option gives the buyer the right but not the obligation to buy an asset at a certain time in the future for a predetermined price. A European put option gives the buyer the right but not the obligation to sell an asset at a certain time in the future for a predetermined price. The classical Black-Scholes formula is usually used to determine the price when the underlying is a stock and the Black formula when the underlying is a future on a stock.

In financial mathematics the Black-Scholes option pricing model consists of a risky asset, stock $S(t)$, and a risk free asset, a bond. The risky asset is a stochastic process $S(t)$ which follows a geometric Brownian motion and is defined by the stochastic differential equation $dS(t) = \mu S(t)\,dt + \sigma S(t)\,dB(t)$. In the Black-Scholes model the returns are independent of each other, i.e. today's price change has no correlation with previous price changes. Some studies (Mandelbrot, 1967) have shown long-range dependency does exist between the returns in some markets. It is proposed to replace Brownian motion in modelling derivatives with fractional Brownian motion $B^H(t)$.

Mandelbrot (1977) introduced the term fractals to describe objects related to the whole and Mandebrot (2006) describes the ten heresies of finance[1]. When dealing with chaos, complexity, fractals or probabilities a question that brings to mind if "God plays dice?" see Carr (2004).

Long-range dependency has been investigated by many authors[2]. It has been shown that many of the emerging markets do exhibit a Hurst exponent that is larger than $\frac{1}{2}$, thus implying that the returns have long-term memory. Cheung and Lai (1995) investigated long memory in 18 countries and only 5 showed persistent behaviour. Cajueiro and Tabak (2003) investigated the Brazilian equity market and found persistency more importantly their results suggest the the Hurst parameter is time varying even after adjusting for short-range dependency. Cajueiro and Tabak (2004) investigated 11 emerging markets and the U.S. and Japan, their results concluded significant long range-dependency in Asian countries, less in the Latin American countries, except Chile, the U.S. and Japan were the most efficient. Sadique and Silvapulle (2001) found persistency in Korea, New Zealand, Malaysia, Singapore, while no or little evidence of persistency was found in Japan, the U.S. and Australia. The returns of the Standard & Poor's 500 and the Dow Jones Industrial Average returns did not display trend reinforcing behaviour see Grau-Carles (2000). Lo (1989) found little evidence of long term memory in U.S. stock market returns.

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Mandelbrot and Hudson (2006) discuss ten heresies of finance.	
1. Markets are turbulent	6. Markets are deceptive.
2. Markets are very risky.	7. Time is flexible
3. Market timing influences gains and losses.	8. Prices leap.
4. Markets are uncertain and bubbles will occur.	9. Predicting prices is dangerous but future volatility can be estimated.
5. All markets work the same.	10. The idea of financial "value" has limited value.

2 Sewell (2011) gives a list of studies.

Cheung (1993) investigated long memory in foreign exchange rates and found evidence of long-memory. Wei and Leuthold (2000) investigated the agricultural market and found long memory in the sugar market. Jamdee and Los (2005) and (2007) show evidence of long memory on European options through a time-dependent volatility. The South African market showed persistency for some stocks in this dissertation.

Peters (1991) suggests that if a stock time series has a high Hurst exponent, then the stock will be less risky and there will be less noise in the data set. Motivated by these results the application of fractional Brownian motion is proposed. Replacing Brownian motion with the fractional Brownian motion is suggested to reduce model risk. Fractional Brownian motion is self-similar and captures long-range dependency. The fractional option pricing models depend on an extra parameter, the Hurst parameter H.

The Hurst parameter $0 \leq H \leq 1$ classifies a time series into three different groups. If $H = \frac{1}{2}$ then events follow a random walk. The returns are uncorrelated and random. If $0 \leq H < \frac{1}{2}$ then the time series is said to have anti-persistent behaviour, i.e. mean reverting and if $\frac{1}{2} < H \leq 1$ then the time series is said to have persistent behaviour, i.e. trend reinforcing. If the stock prices have a $H > \frac{1}{2}$ this shows that long-range dependence exists in the stock prices. Long-range dependency is the same as a long-memory process where past events have a decaying effect on the future. Mandelbrot (1982) pointed out two characteristics of the stock market price behaviour and called them the Noah and Joseph effects. The Noah-effect refers to the observed instances of large discontinuous jumps in the stock prices, or outliers. The Joseph-effect refers to the tendency of the stock prices to have long term trends with non-periodic cycles see Lo (1989) who investigate long term memory in stock market prices.

Fractional Brownian motion is a continuous Gaussian process that depends on the Hurst parameter H and is defined by its covariance function. When $H = \frac{1}{2}$ fractional Brownian motion becomes the ordinary Brownian motion.

Mandelbrot and Van Ness (1968) defined a stochastic integral representation of fractional Brownian motion. When $H \neq \frac{1}{2}$, $B^H(t)$ is not a semimartingale, and therefore the application of classical Itô calculus is not possible. Incorporating fractional Brownian motion to price options using pathwise integration theory is not possible as it allows for arbitrage possibilities. Under pathwise integration fractional Brownian motion does not have zero expectation, which already implies the possibility of a riskless gain. Duncan and Pasik-Duncan (1991) introduce another integration theory based on the Wick product and a so-called Wick Itô Skorohod integral for fractional Brownian motion. The Wick Itô stochastic integral has a zero expectation.

Delbaen and Schachermayer (1994) proved if the underlying stock price process is not a semimartingale then there exist a weak form of arbitrage called "free lunch with vanishing risk". This statement holds true if the definitions of arbitrage, self-financing and admissibility remain unchanged.

Hu and Øksendal (2000) proposed that, in order to consider non semimartingale models, one needs to modify the underlying definition of the portfolio value. A Wick self-financing condition is imposed on the portfolio. The authors derive a closed form solution to the fractional Black-Scholes formula. The market becomes free of strong arbitrage and completeness can be shown. Elliot and van der Hoek (2003) derived similar results as Hu and Øksendal.

Björk and Hult (2005) criticized the work of both Hu and Øksendal (2003) and Elliot and van der Hoek (2003), stating that the self-financing strategies used by the above authors do not have a reasonable economic interpretation. But Björk and Hult did emphasize that they were not against the usage of fractional Brownian motion in finance, only against the particular application.

Necula (2002) used Wick stochastic calculus to generalize a fractional Black-Scholes formula to price option from any arbitrary time t to the maturity time T using quasi-conditional expectations. Using the results of the quasi-conditional expectations, a fractional risk-neutral valuation theorem is derived and used to price options.

Mathematically, the approaches of Hu and Øksendal and Necula are correct and accurate, but when trading in continuous time the Wick Itô integration theory still admits weak arbitrage. Researchers have proposed that by imposing suitable restrictions, arbitrage can be excluded. Cheridito (2002) proved that when using an arbitrarily small amount of time between two consecutive transactions, arbitrage can be excluded from the models. Therefore, it is assumed that investors cannot react immediately when the information is received and, due to the large number of investors, the prices will be fair. It is suggested to restrict the modeling to a discontinuous trading strategy.

Rostek (2009) derived a formula for pricing fractional European options using conditional expectation in a risk preference based pricing approach by assuming a minimal time between trading strategies. The underlying stock process follows a fractional Brownian motion. This model also assumes that traders are risk neutral but they possess some knowledge of the past. Rostek and Schöbel (2010) derived the same model by assuming that participants have a constant relative risk aversion and trade in discrete time. The investor's wealth and the stock process follow a bivariate log-normal distribution. Under assumed investor objectives a stochastic discount factor is introduced to satisfy an equilibrium condition.

Bender (2003) proves that the law of one price holds in a market where the stock is driven by fractional Brownian motion.

Nualart (2001) investigated stochastic volatility models driven by fractional Brownian motion to price options and showed that the market is incomplete and martingale measures are not unique. Rogers (1997) states that fractional Brownian motion is a absurd candidate for pricing options and suggests replacing the process with similar process that captures long-range dependency of returns while avoiding arbitrage. Bender, Sottinen and Valkeila (2006) states that it is not sensible to use just fractional models but an add on of Brownian motion to fractional Brownian motion should be considered. These models allow less arbitrage possibilities and they include hedges see Bender, Sottinen and Valkeila (2009) . Mishura (2008) investigated the stochastic calculus behind the mixed models. Bratyk and Mishura (2008) investigate the application of Brownian motion and fractional Brownian motion to the modeling of hedge contingent claims and found absence of arbitrage and incompleteness.

The application of various estimation methods of the Hurst parameter, namely the aggregated variance method, absolute moments method, Higuchi method, and the rescaled range analysis, were implemented. The Hurst parameter was estimated over two periods one before and one after the 2008 market crash, for the whole period, as well as at yearly intervals for different South African stocks.

The derivation of the fractional Black-Scholes models was studied and key results and arguments are given for each of the models. We derive a fractional Black model for all the settings because a majority of the options that are traded in South Africa are options on futures. Options on stocks are known as warrants in South Africa.

Using ALSI, SBK and MTN data on calls on futures and warrants, the models are examined using two different perspectives. Fixing a constant empirical Hurst parameter, the fractional implied volatility was backed out. The relationship between the fractional implied volatilities and the market implied volatilities was studied, and the out-of-sample pricing comparison was investigated. Keeping a constant Hurst, the performance of the models is compared with the out-of-sampling pricing performance for different strikes and for different Hurst parameters. The out-of-sample pricing errors reflect the model's static performance.

The goal of this dissertation is to understand the mathematical application of fractional Brownian motion in option pricing. The empirical applicability of these models and to get a deeper insight into how these models perform compared to the performance of the classical Black-Scholes and Black formula.

The dissertation is organized in the following way. Chapter 2 paves in the way by presenting nessary results to option pricing of derivatives where the underlying is driven by Brownian motion. Chapter 3 presents numerical methods for estimating the Hurst parameter and provides evidence of dependency in the South African markets. Chapter 4 provides an introduction to chaos, fractals and fractional Brownian motion. The Wick product as well as the main theorems are introduced in Chapter 5. Results are presented that are needed for the derivation of the models as well as an alternative fractional Brownian motion is presented as done by Bender. Chapter 6 deals with Hu and Øksendal's model. A fractional Black-Scholes option pricing model is derived and a fractional Black formula is proved. Björk and Hult's criticism is also noted. In chapter 7 Necula's model is presented. Rostek and Schöbel 's Black-Scholes model is presented in chapter 8 and a conditional fractional Black formula is proved. The tools needed for the empirical comparison of the models are presented in Chapter 9. Application to the ALSI, SBK, MTN calls on futures and warrants is shown in chapter 10. In chapter 11 a conclusion follows. Appendices A and B contains tables of different Hurst parameters for different sectors of the economy. Appendix C deals with white noise analysis. Appendix D states the Malliavin derivative and appendix E gives a description of an optimization algorithm. Appendix F contains MATLAB code that was used. Appendix G gives the tables of the ALSI pricing errors by option and by day.

Chapter 2

Option Pricing with Brownian Motion

2.1 Introduction

Imagine a market with participants such as speculators, arbitrageurs and hedgers all trying to make a profit at the end of the day, in which Brownian motion is used to drive the process of the underlying stock. Around 1900, Bachelier, did his thesis on the pricing of options assuming the stock price follows a Brownian motion with zero expectation (Merton, 1973) . The Black-Scholes formula allows one to price derivatives such as European or American call or put options. The price returns are independent and the distribution of returns is log-normal. But through historical observation prices returns are known to not be log-normal (Lo and MacKinlay, 1999) and long term memory can be found. Outliers and catastrophes occur as well which no Gaussian character will ever capture.

Black and Scholes (1973) derived a formula to price options that assumes a constant volatility for the underlying. Again through empirical studies the implied volatility smile was found and volatility surfaces through time shows us different behaviour. Thus we see that options cannot be correctly priced with a single volatility thus the Black-Scholes model is incorrect. Regardless though, it is the most popular means of pricing derivatives in practice.

The Black and Scholes world relies heavily on the assumption of no-arbitrage which implies that two assets with identical payoffs cannot sell at different prices.

This is a vital assumption otherwise one can make a risk free profit whilst trading. The participants want to make a risk-free profit thus due to the demand arbitrage opportunities will quickly disappear. There will be an absence of arbitrage in this market if and only if there exists a local martingale measure (Björk, 2004). As an example of arbitrage, buying bottled water or a slice of cake at a shoppingmall is substantially more expensive than buying the water at a reservoir and the ingredients separately and these are forms of arbitrage opportunities. The law of one price states that if we look at two investments that have the same payoff because of no arbitrage through the mathematical modeling the two instruments will have the same price. Efficient markets[1] are priced in such a way that prices move only when new information is received. Therefore, it is assumed that investors react immediately when the information is received and due to the large number of investors the prices will be fair. But it is obvious that markets are not efficient.

In this chapter the modeling of stock price movements is done using Brownian motion $B(t)$. For time t greater than zero we have a stochastic process such that $B(t) - B(s)$ has Gaussian distribution with mean 0 and variance $t - s$. For each sample path, $B(t)$ is a continuous function of t, yet not differentiable. Some main definitions and theorems concerning $B(t)$ will be presented here. Integration with Brownian motion is done using the Itô integral and is important for solving stochastic differential equations driven by $B(t)$.

[1] The assumptions to the Efficient Market Hypothesis are:

1. Investors are rational and risk-adverse.
2. Markets which are made up of large number of investors participate continuously.
3. Today's prices will only be affected by todays news and the prices are uncorrelated with yesterdays prices.
4. Investors react immediately when information is received.

Thereafter we will create a market setup consisting of a risky stock and riskless government security. The markets operates continuously and is efficient implying that all relevant information is already contained in the prices.

In this chapter we will be discussing the Black-Scholes formula and the Black formula, they are used to price vanilla options. An European option gives the right but not the obligation to exercise the claim on a underlying at maturity for the strike price. A forward contract amounts to buying or selling today an underlying with a some delivery date and a future contract is similar to a forward (Bouchaud and Potters, 2000). The Black formula prices a European option on a future on an underlying.

The mathematics behind the European call option pricing model as done by Black and Scholes (1973) by using delta hedging techniques will be discussed here. The objective is to create a replicating a portfolio consisting of positions in the underlying and risk free instruments such that this portfolio through arbitrage will replicate the value of the call option. There are many other ways in which one can derive the option pricing formula for European options some of which include expectations, the binomial lattice, change of numeraire or Monte-Carlo simulations.

2.2 Stochastic Process Driven by Brownian Motion

2.2.1 Brownian motion

Consider the probability space $(\Omega, \mathcal{F}_t, P)$, where Ω is the state space of random events, $\mathcal{F}_t$ is the σ-field generated by all Brownian motion on Ω and P is the underlying measure. We define Brownian motion as

Definition 2.1 (*Durret,* 1996). *A one dimensional Brownian motion starting at zero is the process* $B(t)$, *in* $\mathbb{R}$ *and has the following properties:*

1. *Let* $t_0 < t_1 < ... < t_n$ *then* $B(t_0), B(t_1) - B(t_0), ..., B(t_n) - B(t_{n-1})$ *are independent implying that Brownian motion has independent increments.*

2. *Let* $s, t \geq 0$ *then*

$$P(B(s+t) - B(s)) = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right) dx$$

with probability of 1*. It follows that* $B(s+t) - B(s)$ *has a normal distribution with mean* 0 *and variance* t*.*

3. $B(0) = 0$ *and* $t \longmapsto B(t)$ *is continuous.*

Properties of one dimensional Brownian motion are

1. If $B(0) = 0$ then for any $t > 0$ we have $\{B(st), \ s \geq 0\} \stackrel{d}{=} \left\{t^{\frac{1}{2}} B(s), \ s \geq 0\right\}$ also known as the scaling relation.

2. $B(t)$ is a Gaussian process.

3. $E[B(s)] = 0$, and $E(B(s) B(t)) = s \wedge t = \min\{s, t\}$.

4. We also have $B(t) - B(s) \sim N(0, t - s)$.

The Markov property states that given the present state $B(s)$ what happened before s does not matter for predicting what will came next. What happened before is described by the filtration which is a collection of σ-fields: Define $\mathcal{F}_t = \sigma(B(r) : r \leq t)$; for each $t \geq 0$ then for $s \leq t$, $\mathcal{F}_s \subset \mathcal{F}_t$. We say Brownian motion is measurable with respect to $\mathcal{F}_s$ and set $\mathcal{F}_s^0 = \sigma(B(r) : r \leq s)$ and $\mathcal{F}_s^+ = \cap_{t>s} \mathcal{F}_s^0$ which is right continuous. Let C be a space of continuous coordinate maps $C = \{\omega : t \rightarrow \omega(t)\}$ and $\mathcal{C}$ the σ-field generated by the coordinate maps, then for $t, s \geq 0$ and $\omega \in \Omega$ we let $\theta(s) : C \rightarrow C$ be a shift transformation given by

$$\theta(s)(\omega)(t) = \omega(s+t)$$

see Durret (1996). Let $Y : C \to \mathbb{R}$ is $\mathcal{C}$ measurable. The conditional expectation of $Y \circ \theta(s)$ given $\mathcal{F}_s^+$ is the expected value of Y for a Brownian motion starting at $B(s)$.

Theorem 2.1 *The Markov property. If $s \geq 0$ and Y is bounded and $\mathcal{C}$ measurable then for all $x \in \mathbb{R}^d$ we have*

$$E_x\left[Y \circ \theta(s) \,|\mathcal{F}_s^+\right] = E_{B(s)}Y.$$

For the proof see Durret (1996, *page* 9).

2.2.2 Itô Formula

Let M be a square integrable martingale, M_t be a martingale process with $\sup_{t\geq 0} E\left[M_t^2\right] < \infty$ and $M_0 = 0$. We denote by $\mathcal{M}^2$ be the space of all martingales. Let $\lim_{t\to\infty} E\left[M_t\right] = E\left[M_\infty\right] < \infty$. Then we endow $\mathcal{M}^2$ with the inner product $(M, N) = E\left[M_\infty N_\infty\right]$. It follows that $\mathcal{M}^2$ is a Hilbert space. A random step process is a process of the form $f(t) = \sum_{i=0}^{n-1} \xi_i 1_{[t_i,t_{i+1})}(t)$ where ξ_i is square integrable and ξ_i is $\mathcal{F}_{t_i}$ measurable. The Wiener process $W(t)$ is a martingale with respect to the filtration $\mathcal{F}_t$ and we can define a stochastic process by

$$(f \cdot W)(t) = \sum_{i=1}^{n-1} \xi_i \left(W(t_{i+1}) - W(t_i)\right).$$

And is defined to be the L^2 limit of the stochastic integral

$$\int_0^t f(s)\, dW(s).$$

For $t \geq 0$, a continuous stochastic process $\xi(t)$ is called an Itô process if it has the form

$$\xi(T) = \xi(0) + \int_0^T a(t)\, dt + \int_0^T b(t)\, dW(t)$$

where $b(t) \in M_T^2$, for $T > 0$ and $a(t)$ is $\mathcal{F}_t$ adapted such that $\int_0^T |a(t)|\, dt < \infty$ almost surely for all $T \geq 0$ (Brzeźniak and Zastawniak, 2006).

Lemma 2.1 *A simplified Itô formula in differential notation is given by*

$$df(t, W(t)) = \left(\frac{\partial f(t, W(t))}{\partial t} + \frac{1}{2} \frac{\partial^2 f(t, W(t))}{\partial W^2(t)} \right) dt + \frac{\partial f(t, W(t))}{\partial W(t)} dW(t)$$

Proof. For the proof see Brzeźniak and Zastawniak (2006, page 196). We apply stochastic calculus and by the Taylor expansion we have

$$\begin{aligned} df(t, W(t)) &= f(t + dt, W(t) + dW(t)) - f(t, W(t)) \\ &= \frac{\partial f(t, W(t))}{\partial W(t)} dW(t) + \frac{\partial f(t, W(t))}{\partial t} dt + \frac{1}{2} \frac{\partial^2 f(t, W(t))}{\partial W^2(t)} (dW(t))^2 \\ &\quad + \frac{\partial f(t, W(t))}{\partial W(t)\, \partial t} dW(t)\, dt + \frac{1}{2} \frac{\partial^2 f(t, W(t))}{\partial t^2(t)} (dt)^2 + \ldots \end{aligned}$$

with $dW(t)\, dt = 0$ and $(dt)^2 = 0$. ■

Example 2.1 *(Brzeźniak and Zastawniak,* 2006*). Let $B(\cdot)$ be a Brownian motion then*

$$\int_0^t B(s)\, dB(s) = \frac{1}{2} B^2(t) - \frac{1}{2} t.$$

Example 2.2 *(Durret,* 1996*). Consider a stochastic differential equation of the form $dX(s) = bX(s)\, ds + \sigma X(s)\, dB(s)$ which can be rewritten in integral form as*

$$X(t) = X(0) + \int_0^t bX(s)\, ds + \sigma \int_0^t X(s)\, dB(s) \tag{2.1}$$

using stochastic calculus the solution to this equation is a diffusion process with continuous paths. Let X_0 be a real number and $B(t)$ a standard one dimensional Brownian motion and let $X(t) = X_0 \exp(\mu t + \sigma B(t))$ be the exponential Brownian motion. Using Itô formula the solution is $X(t) = X_0 + \int_0^t \mu X(s)\, ds + \int_0^t \sigma X(s)\, dB(s) + \frac{1}{2} \int_0^t \sigma^2 X(s)\, ds$ thus we get the solution of the stochastic differential equation with $b(x) = \left(\mu + \frac{\sigma^2}{2}\right) x$ and $\sigma(x) = \sigma x$. Exponential Brownian motion is used to represent stock prices.

2.2.3 Girsanov Formula

In finance the Girsanov formula gives us the possibility to change between equivalent measures.

Definition 2.2 (*Schoutens*, 2003). *An equivalent martingale measure Q is equivalent to P if they have the same null sets and the discounted stock-price process is a under the risk neutral measure is a martingale. If the equivalent martingale measure exists then it is related to the absence of arbitrage while the uniqueness of the measure is related to market completeness.*

X is a continuous semimartingale if $X(t)$ can be written as $M(t) + A(t)$ where $M(t)$ is a continuous local martingale and $A(t)$ is a continuous adapted process that is locally of bounded variation. $X(t) = M(t) + A(t)$ is a continuous semimartingale if $M(t)$ and $A(t)$ are continuous process with $A(0) = 0$, and the decomposition is unique see Durret (1996). We denote the quadratic variation as $\langle X \rangle (t)$ and the covariance $\langle X, Y \rangle (t)$ is the same under P and Q.

A collection of semimartingales and the definition of the stochastic integral are not affected by a local change of measure. Two measures Q and P defined on a filtration $\mathcal{F}_t$ are said to be locally equivalent if for each t their restriction to $\mathcal{F}_t$, Q_t, and P_t are equivalent, i.e. mutually absolutely continuous. We set

$$\alpha(t) = \frac{dQ(t)}{dP(t)}.$$

Theorem 2.2 *The Girsanov's formula states that if X is a local martingale under the measure P and let $A(t) = \int_0^t \alpha^{-1}(s)\, d\langle \alpha, X \rangle (s)$, then $X(t) - A(t)$ is a local martingale under the measure Q.*

For the proof see Durret (1996, *page* 91).

A bounded local martingale is a martingale see Durret (1996).

2.3 Derivatives Driven by Brownian Motion

2.3.1 The Market

Consider a Black-Scholes market with an investment in a money account and a stock driven by Brownian motion in a continuos setting $0 \leq t \leq T$. Let $r > 0$ be a constant riskless interest rate and the same for all maturities. Then the money market account $A(t)$ at time t develops according to the equation

$$\begin{aligned} dA(t) &= rA(t)\,dt \\ A(0) &= 1. \end{aligned} \tag{2.2}$$

The solution of equation (2.2) is

$$A(t) = \exp(rt) \tag{2.3}$$

Let $\mu = \mu(t)$ be the drift of the stock and $\sigma \neq 0$ be the corresponding volatility. The stock price process has the following dynamics

$$\begin{aligned} dS(t) &= \mu(t)\,S(t)\,dt + \sigma S(t)\,dB(t) \\ S(0) &= S_0 > 0. \end{aligned} \tag{2.4}$$

If we let

$$d\hat{B}(t) = \gamma dt + dB(t) \tag{2.5}$$

it follows by the Girvsanov theorem $\hat{B}$ is normally distributed with zero mean and variance dt under measure Q. Substituting equation (2.5) into equation (2.4) we get

$$\begin{aligned} dS(t) &= \mu S(t)\,dt + \sigma S(t)\,dB(t) \\ &= \mu S(t)\,dt + \sigma S(t)\left(d\hat{B}(t) - \gamma dt\right) \\ &= S(t)(\mu - \sigma\gamma)\,dt + \sigma S(t)\,d\hat{B}(t) \end{aligned}$$

under the equivalent martingale measure Q. Setting the market price of risk as

$$\gamma = \frac{\mu - r}{\sigma}$$

then it follows

$$\begin{aligned} dS(t) &= S(t)\left(\mu - \sigma\left(\frac{\mu - r}{\sigma}\right)\right)dt + \sigma S(t)\,d\hat{B}(t) \\ &= rS(t)\,dt + \sigma S(t)\,d\hat{B}(t). \end{aligned} \tag{2.6}$$

If we let $H(t, S(t)) = \ln S(t)$ then

$$\frac{\partial H(t, S(t))}{\partial t} = 0, \quad \frac{\partial H(t, S(t))}{\partial S(t)} = \frac{1}{S(t)} \text{ and } \frac{\partial^2 H(t, S(t))}{\partial S^2(t)} = -\frac{1}{S^2(t)}.$$

It follows from the Itô formula, lemma (2.1) that

$$dH(t, S(t)) = \frac{1}{S(t)}dS(t) - \frac{1}{2}\frac{1}{S^2(t)}(dS(t))^2 \tag{2.7}$$

Substituting equation (2.6) into equation (2.7) we obtain

$$\begin{aligned} &dH(t, S(t)) \\ &= \frac{1}{S(t)}\left(rS(t)\,dt + \sigma S(t)\,d\hat{B}(t)\right) - \frac{1}{2}\frac{1}{S^2(t)}\left(rS(t)\,dt + \sigma S(t)\,d\hat{B}(t)\right)^2 \\ &= \left(r - \frac{1}{2}\sigma^2\right)dt + \sigma d\hat{B}(t) \end{aligned}$$

Integrating we get

$$S(t) = S_0 \exp\left(\sigma\hat{B}(t) + rt - \frac{1}{2}\sigma^2 t\right). \tag{2.8}$$

We can create a self-financing portfolio $Z(t)$ consisting of $\Delta(t)$, delta, in the risky asset and $u(t)$ in the riskless asset as follows

$$Z(t) = \Delta(t)\,S(t) + u(t)\,A(t). \tag{2.9}$$

We say the portfolio is admissible if

$$dZ(t) = \Delta(t)\,dS(t) + u(t)\,dA(t). \tag{2.10}$$

2.3.2 Vanilla Model Assumptions

The assumptions to pricing the Black-Scholes formula and the Black formula are as follows:

1. The stock price follows a geometric Brownian motion and changes in the stock price follow a Markov process.
2. Stochastic differentials are interpreted in the Itô Skorohod sense.
3. The stock prices are log-normally distributed implying that the returns have a Gaussian distribution.
4. There exists a unique equivalent martingale measure under Q.
5. Absence of arbitrage, no free lunch with vanishing risk and the law of one price holds.
6. The market is complete and the efficient market hypothesis holds.
7. The drift μ and volatility σ are constant and the r is a constant risk-free rate of interest.
8. The portfolio is self-financing as in equation (2.9).
9. The definition of an admissible portfolio is give in (2.10) and is done using the normal multiplication.
10. Short selling is allowed and there are no penalties for shortselling.
11. There are no transactions costs or taxes.
12. There are no dividends or commissions.
13. Trading is done continuously.

14. All securities are perfectly divisible.

15. The option is European implying it can be only exercised at time T.

2.3.3 Black-Scholes Option Pricing formula

Black and Scholes (1973) derive a call option pricing formula in equilibrium, where the expected return on the hedged position must be equal to the return on the riskless asset.

Theorem 2.3 *Black-Scholes formula. At time t let $S(t)$ be the underlying stock, K the strike price and T the maturity date then the price of an European call option $C(t, S(t))$ is given by*

$$C(t, S(t)) = S(t) N(d_1) - Ke^{-r(T-t)} N(d_2) \tag{2.11}$$

with

$$d_1 = \frac{\ln\left(\frac{S(t)}{K}\right) + r(T-t) + \frac{1}{2}\sigma^2 (T-t)}{\sigma\sqrt{(T-t)}}$$

and

$$d_2 = \frac{\ln\left(\frac{S(t)}{K}\right) + r(T-t) - \frac{1}{2}\sigma^2 (T-t)}{\sigma\sqrt{(T-t)}}$$

where $N(x)$ is the cumulative probability distribution function for a Gaussian distribution

$$N(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} \exp\left(-\frac{x^2}{2}\right) dx.$$

Proof. Black and Scholes (1973).

We create a hedged portfolio consisting of a long position in the stock and a short position in the option. The number of options sold short against one stock long is

$$\frac{1}{\frac{\partial C(t, S(t))}{\partial S(t)}}.$$

This is the inverse of the delta of the option where the delta of the option is defined as the change in the call option with respect to the change of the underlying stock, i.e. the sensitivity of the portfolio with respect to the option stock price. Thus the value of this position is

$$S(t) - \left(\frac{\partial C(t, S(t))}{\partial S(t)}\right)^{-1} C(t, S(t)).$$

Let dt be a increment of infinitesimal time since we can self finance the change in the value of the portfolio value over a infinitesimal period of time is

$$dS(t) - \left(\frac{\partial C(t, S(t))}{\partial S(t)}\right)^{-1} dC(t, S(t)). \tag{2.12}$$

The portfolio value after the change at time $t + dt$ is

$$S(t) + dS(t) - \left(\frac{\partial C(t, S(t))}{\partial S(t)}\right)^{-1} (C(t, S(t)) + dC(t, S(t))).$$

By the Itô formula we have

$$dC(t, S(t)) = \frac{\partial C(t, S(t))}{\partial S(t)} dS(t) + \frac{1}{2}\sigma^2 S^2(t) \frac{\partial^2 C(t, S(t))}{\partial S^2(t)} dt + \frac{\partial C(t, S(t))}{\partial t} dt \tag{2.13}$$

substituting (2.13) into (2.12) we have the change

$$\begin{aligned} & dS(t) - \left(\frac{\partial C(t, S(t))}{\partial S(t)}\right)^{-1} \begin{pmatrix} \frac{\partial C(t,S(t))}{\partial S(t)} dS(t) + \frac{1}{2}\sigma^2 S^2(t) \frac{\partial^2 C(t,S(t))}{\partial S^2(t)} dt \\ + \frac{\partial C(t,S(t))}{\partial t} dt \end{pmatrix} \\ = & -\left(\frac{\partial C(t, S(t))}{\partial S(t)}\right)^{-1} \left(\frac{1}{2}\sigma^2 S^2(t) \frac{\partial^2 C(t, S(t))}{\partial S^2(t)} + \frac{\partial C(t, S(t))}{\partial t}\right) dt. \end{aligned}$$

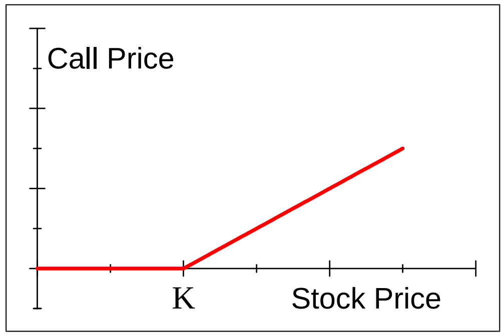
Because no arbitrage holds the value of this portfolio has to equal to the risk free return on the investment otherwise there are obvious arbitrage possibilities. The risk in the hedge position is zero if the short position in the option is adjusted continuously

$$\begin{aligned} & -\left(\frac{\partial C(t, S(t))}{\partial S(t)}\right)^{-1} \left(\frac{1}{2}\sigma^2 S^2(t) \frac{\partial^2 C(t, S(t))}{\partial S^2(t)} + \frac{\partial C(t, S(t))}{\partial t}\right) dt \\ = & \left(S(t) - \left(\frac{\partial C(t, S(t))}{\partial S(t)}\right)^{-1} C(t, S(t))\right) r dt. \end{aligned}$$

Rearranging we have the Black-Scholes partial differential equation

$$rS(t)\frac{\partial C(t,S(t))}{\partial S(t)}+\frac{1}{2}\sigma^2S^2(t)\frac{\partial^2 C(t,S(t))}{\partial S^2(t)}+\frac{\partial C(t,S(t))}{\partial t}=rC(t,S(t)). \quad (2.14)$$

Let K be the exercise price, then the call option looks like



Call Option

thus at time T if $S<K$ the payoff of the call becomes worthless, else if $S\geq K$ the payoff of a call is $S(t)-C(t,S(t))$. We can write the claim as

$$\chi(T)=\max\{S(T)-K,0\}. \quad (2.15)$$

The solution of equation (2.14) given boundary conditions (2.15) is done by making the following substitution

$$\begin{aligned} & C(t,S(t)) \\ = \; & e^{r(t-T)}y\left(\begin{array}{c}\frac{2}{\sigma^2}\left(r-\frac{1}{2}\sigma^2\right)+\ln\frac{S(t)}{K}-\left(r-\frac{1}{2}\sigma^2\right)(t-T)\\ -\frac{2}{\sigma^2}\left(r-\frac{1}{2}\sigma^2\right)^2(t-T)\end{array}\right) \end{aligned} \quad (2.16)$$

where

$$\begin{aligned} y(u,0) &= 0,\; u<0 \\ &= K\left(\exp\left(\frac{u\left(\frac{1}{2}\sigma^2\right)}{\left(r-\frac{1}{2}\sigma^2\right)}\right)-1\right),\; u\geq 0. \end{aligned}$$

The solution follows as

$$y(u,s)=\frac{1}{\sqrt{2\pi}}\int_{-\frac{u}{\sqrt{2s}}}^{\infty}K\left(\exp\left(\frac{\frac{\sigma^2}{2}\left(u+q\sqrt{2s}\right)}{\left(r-\frac{1}{2}\sigma^2\right)}\right)-1\right)\exp\left(-\frac{q^2}{2}\right)dq. \quad (2.17)$$

Substituting (2.17) into equation (2.16) we obtain the price of a European call option as

$$C\left(t,S\left(t\right)\right)=S\left(t\right)N\left(d_1\right)-Ke^{-r(T-t)}N\left(d_2\right).$$

■

Suppose we create a portfolio $X\left(t\right)$ at time t consisting of a long in a European put option on a strike K with maturity T and a long the underlying stock then the value of the portfolio becomes

$$X\left(t\right)=P\left(t,S\left(t\right)\right)+S\left(t\right).$$

At time t suppose we also create another portfolio $Y\left(t\right)$ consisting of a long in a European call option on a strike K with maturity T and we long the amount K of a riskless government security bond $A\left(t\right)$ paying one unit at time T then the portfolio becomes

$$Y\left(t\right)=C\left(t,S\left(t\right)\right)+KA\left(t\right).$$

At time T if $S\left(T\right)<K$ then portfolio $X\left(T\right)$ is worth $K-S\left(T\right)+S\left(T\right)=K$ since the put gets exercised. While portfolio $Y\left(T\right)$ is worth K since the call expires worthless. If $S\left(T\right)\geq K$ we have portfolio $X\left(T\right)$ is worth $S\left(T\right)$ and portfolio $Y\left(T\right)$ is worth $S\left(T\right)-K+K=S\left(T\right)$. Thus at time T portfolio $X\left(T\right)$ is equal to $Y\left(T\right)$ and since the law of one price holds and we must have at time t the put-call parity relationship

$$P\left(t,S\left(t\right)\right)=C\left(t,S\left(t\right)\right)+KA\left(t\right)-S\left(t\right).$$

Through symmetry a European put option can be derived as

$$P\left(t,S\left(t\right)\right)=Ke^{-r(T-t)}N\left(-d_2\right)-S\left(t\right)N\left(-d_1\right).$$

2.3.4 Black Option Pricing formula

The Black formula can be derived using the Black-Scholes option pricing formula by doing a suitable substitution.

Theorem 2.4 *Black formula. The price at every $t \in [0, T]$ of an European call option with strike price K and maturity T on the futures contact F, is given by*

$$c(t, F(t)) = e^{-r(T-t)} \left(F(t) N(d_1) - KN(d_2) \right)$$

with

$$d_1 = \frac{\ln\left(\frac{F(t)}{K}\right) + \frac{1}{2}\sigma^2 (T-t)}{\sigma\sqrt{(T-t)}}$$

and

$$d_2 = \frac{\ln\left(\frac{F(t)}{K}\right) - \frac{1}{2}\sigma^2 (T-t)}{\sigma\sqrt{(T-t)}}$$

where $N(x)$ is the cumulative normal density function.

For the proof see Björk (2004, *page* 104.)

Because the law of one price holds we can create a portfolio consisting of put options on future contacts and some other ingredients and through arbitrage argument and symmetry we derive a Black formula for a European put p at time T which is given by

$$p(t, F(t)) = e^{-r(T-t)} \left(KN(-d_2) - F(t) N(-d_1) \right).$$

2.3.5 The Clark-Ocone formula

Consider the filtration $\{\mathcal{B}_t, t \geq 0\}$ generated by a d-dimensional Brownian motion with $B(0) = 0$. All local martingales adapted to $\{\mathcal{B}_t, t \geq 0\}$ are continuous and every random variable $X \in L^2(\Omega, \mathcal{B}_\infty, P)$ can be written as a stochastic integral. If we let $B(t) = \left(B^1(t), ..., B^d(t)\right)$ with $B(0) = 0$ and we let $\{\mathcal{B}_t^i,\ t \geq 0\}$ be the filtration then by the martingale representation theorem for any $X \in L^2(\Omega, \mathcal{B}_\infty, P)$ there are unique ϕ^i which is a predictable process such that

$$X = E[X] + \sum_{i=1}^{d} \int_0^\infty \phi^i(s)\, dB^i(s).$$

The Clark-Ocone formula provides an explicit martingale representation for certain random variables using the Mallivian derivative see Carmona and Tehranchi (2004).

Theorem 2.5 *For the Clark-Ocone formula we let* $F = \sum_{n=0}^{\infty} I_n(f_n)$ *where* $\{f_n\}_{n=0}^{\infty}$, $f \in \mathbb{R}$, *is a sequence of functions and* $f^{(\alpha_1,...,\alpha_n)}(t_1, ..., t_n)$ *are functions which are symmetric in the variables* $(t_1, ..., t_n)$. *Let* $\otimes$ *be the tensor product then we define*

$$I_n(f) = \sum_{(\alpha_1,.,\alpha_n)\in I^n} \int_{[0,T]^n} f^{(\alpha_1,...,\alpha_n)}(t_1, ..., t_n)\, dM_{t_1}^{\alpha_1} \otimes ... \otimes dM_{t_n}^{\alpha_n}$$

and the norm is $\|F\|_{L^2(\Omega)}^2 = \sum_{n=0}^{\infty} n!\, \|f_n\|_n^2$. *Let the space* $\mathbb{D}_{1,2} \subset L^2(\Omega)$ *with norm* $\sum_{n=1}^{\infty} n \cdot n!\, \|f_n\|_n^2 < \infty$ *and is dense in* $L^2(\Omega)$. *For* $F \in \mathbb{D}_{1,2}$ *let* D *be a operator given by* $D_{t,\alpha}F = \sum_{n=1}^{\infty} nI_{n-1}(f_n^\alpha(\cdot, t))$, *then*

$$F = E[F] + \sum_{\alpha\in I} \int_0^T E[D_{t,\alpha}F|\mathcal{F}_{t-}]\, dM_t^\alpha$$

where $[D_{t,\alpha}F|\mathcal{F}_{t-}]$ *is the projection of* $D_{t,\alpha}F$.

For the proof see Løkka (1999, *page* 12).

Carmona and Tehranchi (2004) derive a hedging formula in the setting of interest rate option using the Clark Ocone formula.

Chapter 3

The Hurst Parameter

3.1 Introduction

In the early 20th century a hydrologist named Harold Edwin Hurst worked on the Nile river dam project. Hurst's (1951) plan involved storing the water inflows from the great lakes into the Nile river basin from the good years for the use in bad years and this necessitated that the dam needed to be large enough to meet the unexpected phenomena. Most hydrologists assumed that water inflows were a random process. Hurst (1956) found that even after shuffling the data the data still approximated the Gaussian distribution and high and low values clustered, grouped together, more than the random[1] process should. Hurst found that there were cycles in the time series even though the series was non-periodic (Long, 2002). He later found that most natural systems, like temperature, river discharges and sunspots do not follow a random walk (Peters, 1991).

The Hurst parameter $0 \leq H \leq 1$ classifies a time series into three different groups. If $H = \frac{1}{2}$ then events follow a random walk, thus present events will not

[1] Mandelbrot (1997, page 123) categorizes randomess into

1. Mild randomness.
2. Slow randomness.
3. Wild randomness.

influence the future and the returns are independent of each other, see figure 3.1. If $0 \leq H < \frac{1}{2}$ then the time series is said to exhibit anti-persistent behaviour, meaning it is mean reverting, see figure 3.2. If $\frac{1}{2} < H \leq 1$ then the time series is said to have persistent behaviour in other words trend reinforcing, see figure 3.3.

Peters (1991) defines a fractal time series as time series which are statistically self-similar with respect to time. If you look at the graph of such a time series it displays jagged lines. An anti-persistance chart would display a time series with many jagged lines as it is subject to reversals. In other words, if the prices experienced an up movement then the next movement is more likely to be down and vice versa. Persistent chart would result in a series that is less jagged and closer to a line. If the stock prices were up, then there is a higher probability that stock prices will be up again in the next movement and vice versa. If it is assumed that the Hurst parameter was 0.6 and that the last move was up, then there is a 60% chance that the next move will again be up. Today's returns have some correlation with the past returns.

Definition 3.1 *(Biagini, Hu, Øksendal and Zhang, 2008). A stochastic function* $\{X(t)\}_{t \geq 0}$ *is self-similar if for every* $a > 0$ *there exists* $b > 0$ *such that*

$$Law\left(X\left(at\right)\right) = Law\left(bX\left(t\right)\right).$$

In other words the two processes $X(at)$ and $bX(t)$ have the same finite-dimensional distribution functions. For every choice $t_0, ..., t_n$ in $\mathbb{R}$,

$$P\left(X\left(at_0\right) \leq x_0, ..., X\left(at_n\right) \leq x_n\right) = P\left(bX\left(t_0\right) \leq x_0, ..., bX\left(t_n\right) \leq x_n\right)$$

where $x_0, ..., x_1 \in \mathbb{R}$. In other words no matter which level of scale is chosen the process qualitatively looks the same. Compressing or uncompressing the process by

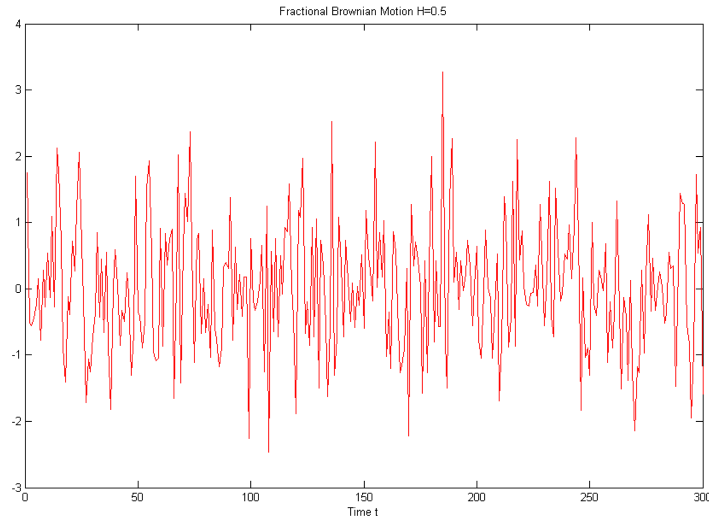


Figure 3.1: Fractional Brownian motion for $H = 0.5$ showing Brownian motion.

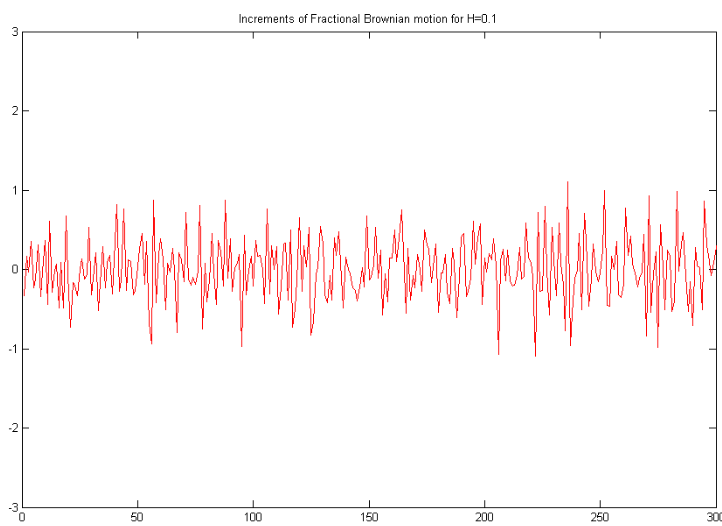


Figure 3.2: Fractional Brownian motion for $H = 0.1$ showing anti-persistency.

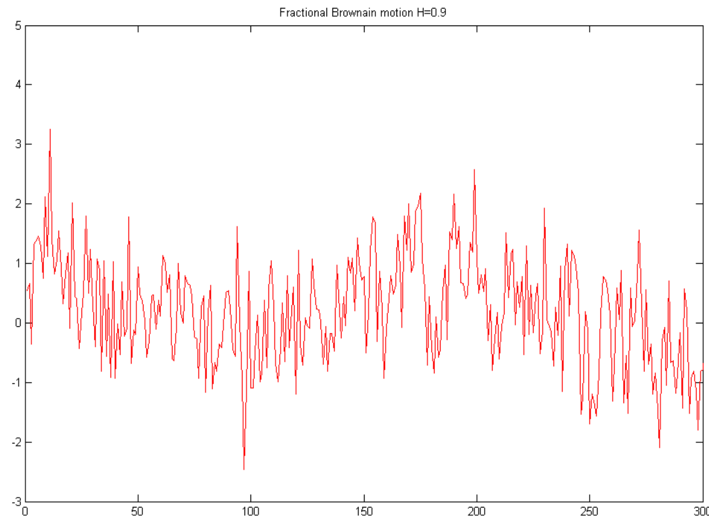


Figure 3.3: Fractional Brownian motion for $H = 0.9$ showing persistency.

a factor only changes the characteristic of the process up to a scaling. The same idea is associated when thinking of fractals. See figures 3.4, 3.5 and 3.6. Let $b = a^{-H}$ in the above definition then stochastic function $\{X(t)\}_{t\geq 0}$ is a self-similar process with the Hurst parameter H. The quantity $D = \frac{1}{H}$ is called the statistical Hurst dimension of X.

Definition 3.2 *(Biagini, Hu, Øksendal and Zhang, 2008). A stationary sequence* $(X(n))_{n\in\mathbb{N}}$ *exhibits long-range dependence if the autocovariance function which is*

$$\rho(n) = cov(X(k), X(k+n))$$

satisfies

$$\lim_{n\to\infty} \frac{\rho_n}{cn^{-\alpha}} = 1$$

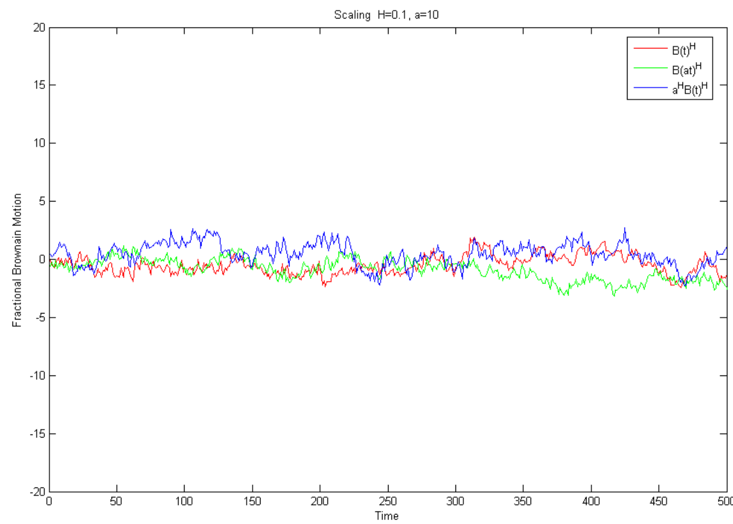


Figure 3.4: Anti-persistency. Fractional Brownian motion scaling for $H = 0.1$ and $a = 10$.

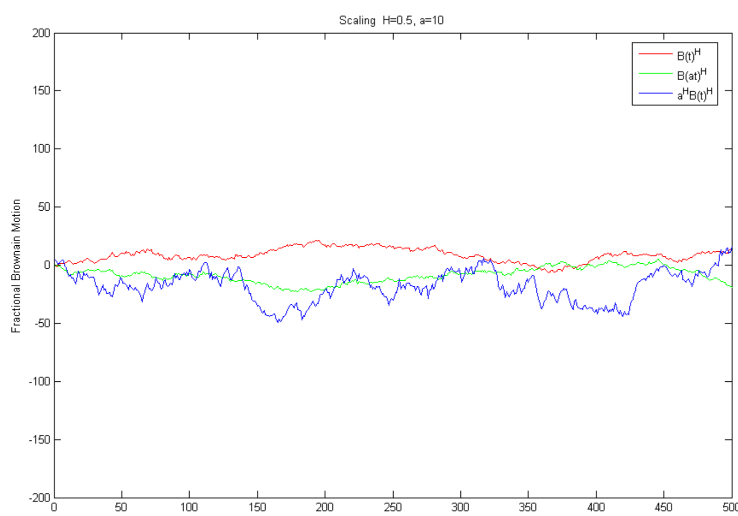


Figure 3.5: Random Walk. Fractional Brownian motion scaling, $H = \frac{1}{2}$, $a = 10$.

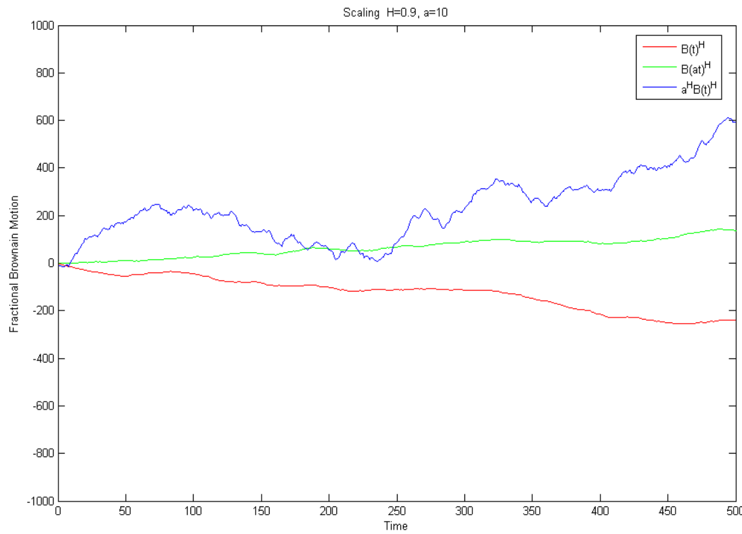


Figure 3.6: Persistent. Fractional Brownian motion scaling, $H = 0.9$, $a = 10$.

for some constant c *and* $\alpha \in (0, 1)$, *then the dependence between the increments decays slowly as* $n \to \infty$ *and*

$$\sum_{n=1}^{\infty} \rho(n) = \infty.$$

If the stock prices have a $H > \frac{1}{2}$ this shows that long-range dependence exists in the stock prices. Long-range dependency is the same as a long-memory process. A long-memory process in a data set implies that the present price increment is autocorrelated with a price increment in the future. This autocorrelation decays over time and the decay follows the power law

$$\rho(n) = H(2H - 1)\, n^{2H-2}$$

where $\rho(n)$ is the autocorrelation function with lag n (Biagini, Hu, Øksendal and Zhang, 2008). Figure 3.7 shows the autocorrelation function for different lags and

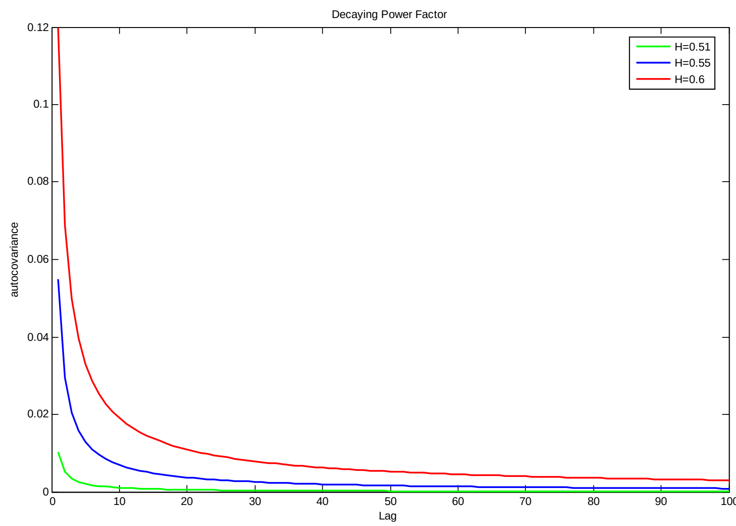


Figure 3.7: Autocorrelation function for different lags and different Hurst parameters.

three Hurst parameters, $H = 0.51$, $H = 0.55$ and $H = 0.6$. It can be seen that the autocorrelation function quickly goes to zero when the Hurst parameter is close to 0.5.[2]

3.2 Numerical Methods

Mandelbrot used the rescaled range analysis (R/S) to model the Hurst parameter for the Nile (Peters, 1991). Rescaled range (R/S) analysis is a common tool for estimating the Hurst parameter in many applications but this analysis is not reliable

[2]Econophysics is the study of financial markets using tools froms physics see Bouchaud (2002) , Farmer and Lux (2008), Lo and Mueller (2010), Sharma, Agrawal, Sharma, Bisen and Sharma (2011) and Stanleya, Amaral, Canning, Gopikrishnan, Lee, Liu (1999) .Researchers in this field study have also studied correlation in finiancial time series see for intance Mantegna and Stanley (2000) .

for small samples (Clark, 2005). Gammel (1997) used the Rescaled Range method to find correlations in generated pseudo random numbers in Monte Carlo simulations. Newer methods have been introduced to improve the approximation.

If a statistic $f(m)$ behaves like m^{aH+b} as $m \to \infty$ for some constant a and b, then

$$\log f(m) \approx (aH + b)\log m + R$$

where R is independent of H. The Hurst parameter H is computed by the estimated slope $(aH + b)$. Using linear regression for the log plot of $f(m)$ against the log plot of m should produce a straight line with this slope see Biagini, Hu, Øksendal and Zhang (2008).

Define a data series $X(t)$ and let N be the length of the vector X. Divide the original series $X(t)$ into k number of blocks of size m for $k = 1, 2, ..., M$, where $M = \left[\frac{N}{m}\right]$ denotes the integer part of $\frac{N}{m}$. The mean of each block is taken and a new aggregated series $X^{(m)}$ is formed

$$X^{(m)}(k) = \frac{1}{m}\sum_{i=(k-1)m}^{km} X_i.$$

Let the sample average of X, $E[X]$ is

$$E[X] = \frac{1}{N}\sum_{i=1}^{N} X_i.$$

Algorithm 3.1 *(Biagini, Hu, Øksendal and Zhang,* 2008)*. The aggregated variance method is based on the self-similarity property of a sample. The variance is given as*

$$Var\left(X^{(m)}(k)\right) = \frac{1}{N/m}\sum_{i=1}^{M}\left(X^{(m)}(i) - E[X]\right)^2.$$

Where $Var\left(X^{(m)}\ (k)\right)$ *behaves like* m^{2H-2} *for large* m*. In particular, log* $\left(Var\left(X^{(m)}(k)\right)\right) = (2H-2)\log m + R$*. The estimator of* H *is obtained by plotting log* $\left(Var\left(X^{(m)}(k)\right)\right)$ *versus* m *on a log-log scale.*

Algorithm 3.2 *(Biagini, Hu, Øksendal and Zhang, 2008). The absolute moments (value) method is a generalization of the aggregated variance method and is given by*

$$AM\left(X^{(m)}(k)\right) = \frac{1}{N/m}\sum_{k=1}^{M}\left|X^{(m)}(k) - E[X]\right|$$

where $AM\left(X^{(m)}(k)\right)$ behaves like m^{H-1} for large m.

Algorithm 3.3 *(Montanari, Taqqu and Teverovsky, 1999). The Higuchi method takes the partial sums $Y(n) = \sum_{i=1}^{n} X(i)$ of the original series. Instead of using non intersecting blocks, a sliding window is used. Let $M(i) = \left[\frac{N-i}{m}\right]$ and*

$$L(m) = \frac{N-1}{m^3}\sum_{i=1}^{m}\frac{1}{M(i)}\sum_{k=1}^{M(i)}\left|Y(i+km) - Y(i+(k-1)m)\right|$$

then $L(m) \sim Cm^{-D}$, where $D = H-2$ and C is a constant. The log-log *plot of $L(m)$ versus m should produce a straight line with a slope of $D = H-2$.*

Algorithm 3.4 *(Biagini, Hu, Øksendal and Zhang, 2008). Rescaled range analysis method uses the time series $X_1, .., X_N$ by dividing the whole series into k non-intersecting blocks that all contain M elements with M being the greatest integer that is smaller than $\frac{N}{k}$. Let $Y(t) = \sum_{i=1}^{t} X(i)$ be partial sums. The range series R is calculated as*

$$\begin{aligned} R(t,k) &= \max_{0\leq i\leq k}\left\{Y(t+i) - Y(t) - \frac{i}{k}(Y(t+k) - Y(t))\right\} \\ &\quad - \min_{0\leq i\leq k}\left\{Y(t+i) - Y(t) - \frac{i}{k}(Y(t+k) - Y(t))\right\}. \end{aligned}$$

The standard deviation of the series is given as

$$S(t,k) = \sqrt{\frac{1}{k}\sum_{i=t+1}^{t+k}\left(X(i) - \frac{1}{k}\sum_{i=t+1}^{t+k}X(i)\right)^2}.$$

The rescaled adjusted range is computed for different numbers of t and k and is given by

$$RS(k) = \frac{R(t,k)}{S(t,k)}.$$

$RS(k)$ behaves like k^H in distribution for large k values.

Morales, Matteo, Gramatica and Aste (2011) applied the weighted generalized Hurst parameter which assigns more weight to the more recent events to find stability or instability in financial companies. This way large outliers in the past influence the present less. Bayraktar, Poor and Sircar (2008) estimated the Hurst parameter of the S&P 500 index using Wavelt analysis and their method exhibits robustness to non-stationarities that are present in the data.

Descriptions of the MATLAB codes used are given in the appendix F.

To investigate the aggregated variance, absolute moments, Higuchi and the Rescaled Range methods we generated a random walk i.e. $H = \frac{1}{2}$. Uniformly distributed random numbers were used for our random walk. The mean and the standard deviation of the Hurst parameter estimates are given for different sample sizes, see table 3.1. It was found the more observation the better the Hurst parameter estimate. When dealing with stocks it is not always possible to obtain historical data over long periods.

Based on the difficultly of distinguishing dependency from a random walk we make the following conservative proposal.

Remark 3.1 *For $0 < H < 0.25$ we say the data series of stock returns displays a strong anti-persistent behaviour. For $0.25 < H < 0.45$ the data series displays a slight anti-persistent behaviour. For $0.45 < H < 0.55$ we say the data series has a Hurst parameter close to a half, i.e. $H \approx \frac{1}{2}$. For $0.55 < H < 0.65$ the data series displays a slight persistent behaviour. For $0.65 < H < 1$ the data series displays a strong persistent behaviour.*

Table 3.1: Hurst's Mean and Standard Deviation.

Aggregated Variance			Absolute Moments		
N	mean	Stdev	N	mean	Stdev
500	0.4705	0.0713	500	0.4730	0.0753
1000	0.4756	0.0619	1000	0.4754	0.0585
10000	0.4826	0.0310	10000	0.4846	0.0354
Higuchi			Rescaled Range		
N	mean	Stdev	N	mean	Stdev
500	0.4951	0.0744	500	0.6043	0.0376
1000	0.4957	0.0570	1000	0.5877	0.0318
10000	0.4962	0.0370	10000	0.5556	0.0141

3.3 South African Stock Market

We refer the reader to interesting articles on the South African market by Wentzel and Maré (2007) who investigate the South African equity market using Extreme Value Theory. Zhou and Sornette (2008) investigated financial bubbles in 45 indices between 2003 and 2006 and five indices showed a fast acceleration in prices.

An investigation of the South African market was done in this investigation to see whether key stocks displayed persistent or anti-persistent behaviour. The South African economy is an emerging market and there are sectors in which we found high persistency. The Hurst parameter was investigated for the log returns of 130 stocks. The data was obtained from Sharenet. We also looked at the Hurst parameter for the period before the recession and after. The Hurst parameter varies across different periods.

- The first interval, the whole interval, was from the first date on which we had data available, for a specific stock, up to 17-Feb-2011.

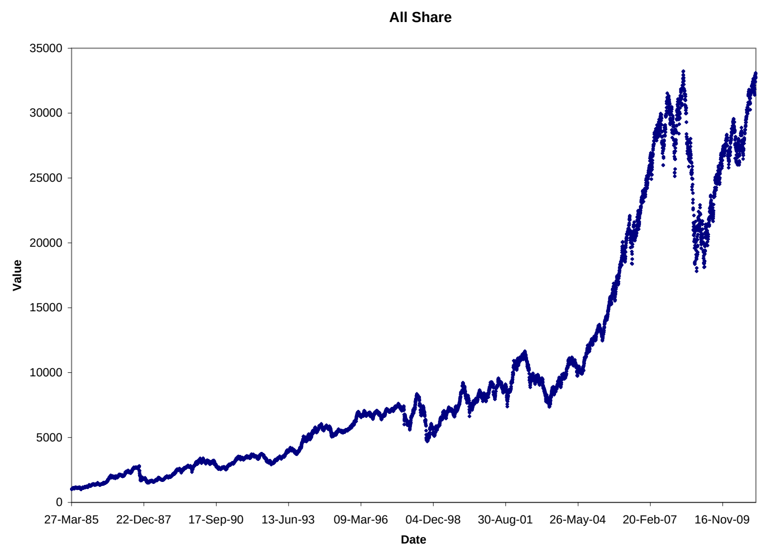


Figure 3.8: JSE-ALSI, 1985/03/27-2011/02/17.

- The second interval was from the beginning up to 23-May-2008, and is the period preceding the recession. The 23rd of May 2008 was the highest value in the domain of the JSE-ALSH index data which was obtained from Sharenet.

- The third interval was after the crash, from the period 23-May-2008 to 17-Feb-2011.

Figure 3.8 and figure 3.9 are the All share index (JSE-ALSH) plots from 27-Mar-1985 till 17-Feb-2011. The investigation of $2, 3$ and 4 yearly intervals for persistency and anti-persistency was done.

One hundred thirty different stocks from 45 sectors were investigated using two methods, the Higuchi and the absolute moments method. Using the classification criterion in the remark 3.1 above we found when looking at the whole interval that

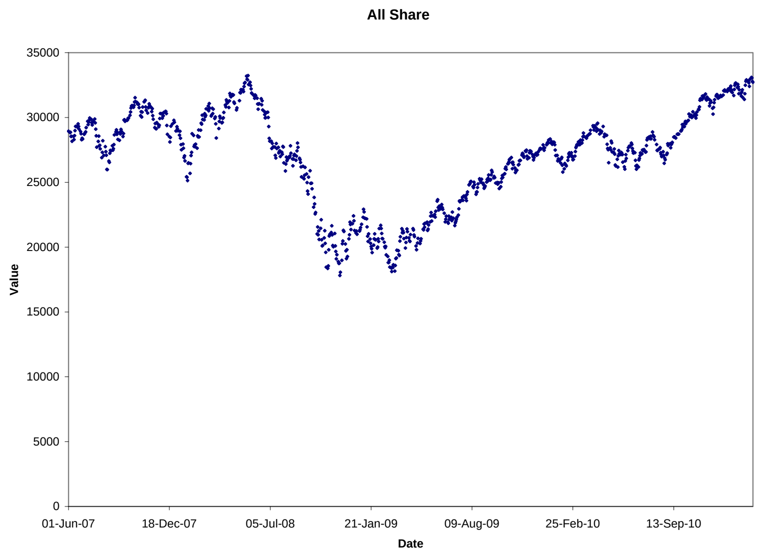


Figure 3.9: JSE-ALSI, 2007/06/01-2011/02/17.

31.54% of the stocks displayed a persistent trend using both methods and that 26.92% of the stocks displayed a persistent trend using one of the methods see figure 3.10. Further, 30% of the stocks displayed a $H \approx \frac{1}{2}$ trend using both methods while 4.62% of the stocks displayed anti-persistent behaviour using both methods, 5.38% of the stocks displayed anti-persistent behaviour using one method and 1.54% of the stocks displayed persistent behaviour for one method and anti-persistent behaviour for one method, see figure 3.10.

When looking at the interval before the crash till 23 May 2008, 25.23% of the stocks displayed a persistent trend using both methods and that 25.23% of the stocks displayed a persistent trend using one of the methods. Further, 30.84% of the stocks displayed a $H \approx \frac{1}{2}$ trend using both methods while 5.62% of the stocks displayed anti-persistent behaviour using both methods, 6.54% of the stocks displayed anti-

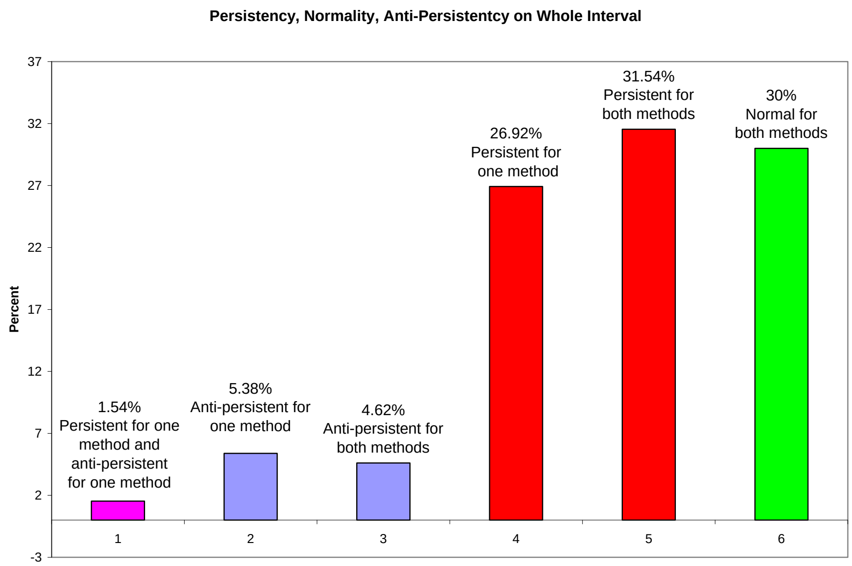


Figure 3.10: Persistency, $H \approx \frac{1}{2}$, Anti-Persistentcy on Whole Interval.

persistent behaviour using one method and 6.54% of the stocks displayed persistent behaviour for one method and anti-persistent behaviour for one method (see Figure 3.11).

When looking at the interval after the crash, from 23 May 2008 till 17 February 2011, 18.88% of the stocks displayed a persistent trend using both methods, 8.49% of the stocks displayed a persistent trend using one of the methods, 18.88% of the stocks displayed a $H \approx \frac{1}{2}$ trend using both methods while 24.53% of the stocks displayed anti-persistent behaviour using both methods and 27.34% of the stocks displayed anti-persistent behaviour using one method. Further, 1.88% of the stocks displayed persistent behaviour for one method and anti-persistent behaviour for one method, (see Figure 3.12).

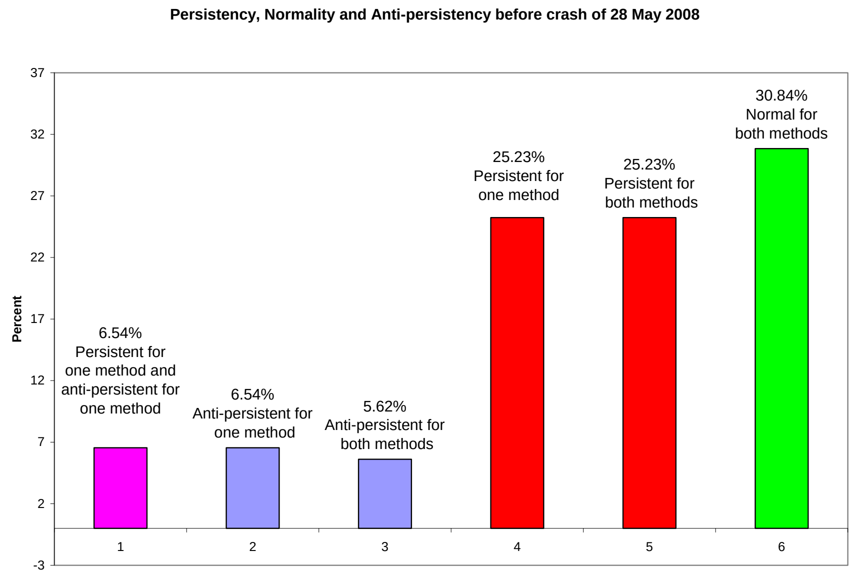


Figure 3.11: Persistency, $H \approx \frac{1}{2}$ and Anti-persistency before crash of 23 May 2008.

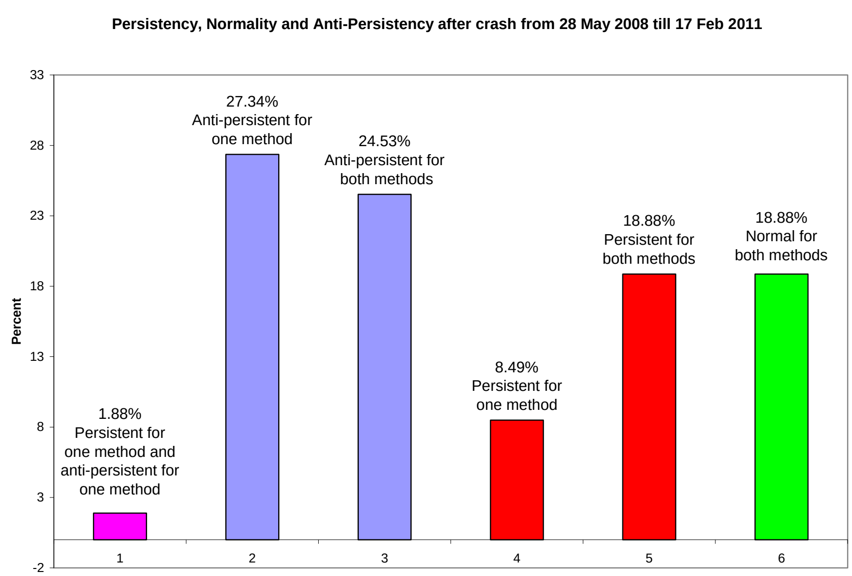


Figure 3.12: Persistency, $H \approx \frac{1}{2}$ and Anti-Persistency after crash from 23 May 2008 till 17 Feb 2011.

Table 3.2: All share index Hurst parameter.

JSE-ALSH		Hurst Parameter			
Period	N	Aggregated Variance	Absolute Moments	Higuchi	R/S
27-Mar-85 to 17-Feb-11	6462	0.4917	0.5143	0.5886	0.6019
27-Mar-85 to 23-May-08	5775	0.4805	0.5043	0.5790	0.6134
23-May-08 to 17-Feb-11	688	0.5300	0.5474	0.5217	0.5816

The **All share index** (ALSI) was investigated using all four methods, see table 3.2. The results of the investigation for the whole period showed $H \approx \frac{1}{2}$ for the aggregated variance and absolute moments methods while the Higuchi and the rescaled range analysis showed slight persistency. Looking just at the period before the recession the same was found. For the period after the recession all the methods except the rescaled range analysis showed $H \approx \frac{1}{2}$. Using the absolute moments method and the Higuchi method the data was divided into two and four year intervals and the estimated Hurst parameter was plotted, see figures 3.13, 3.14, 3.15 and 3.16. For two year intervals the absolute moments method estimate appears to be mean reverting around a Hurst parameter of 0.5 while for two year intervals the Higuchi method estimate appears to be mean reverting above 0.5. We concluded that the ALSI has a $H \approx \frac{1}{2}$ to slightly anti-persistent.

In the metals and minerals sector, **Assore Ltd** (Assore) showed a strong level of persistency throughout the interval. The data was divided into 2 yearly intervals from 27-Mar-1985 to 17-Feb-2011 and the Hurst parameter worked out for each interval. This plotted in figure 3.17, it can be seen that the Hurst parameter is above 0.5 for the entire interval.

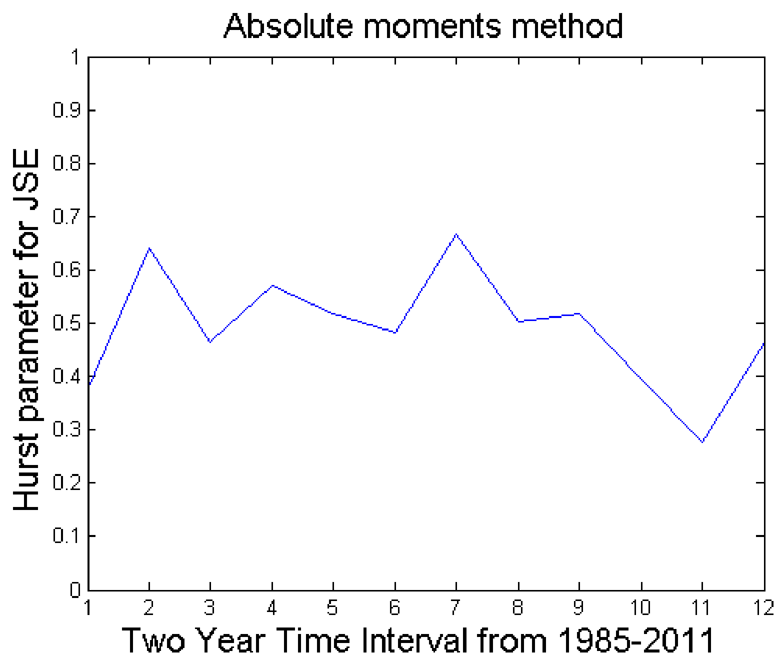


Figure 3.13: The Hurst parameter for ALSI stock for two year intervals using Absolute Moment method.

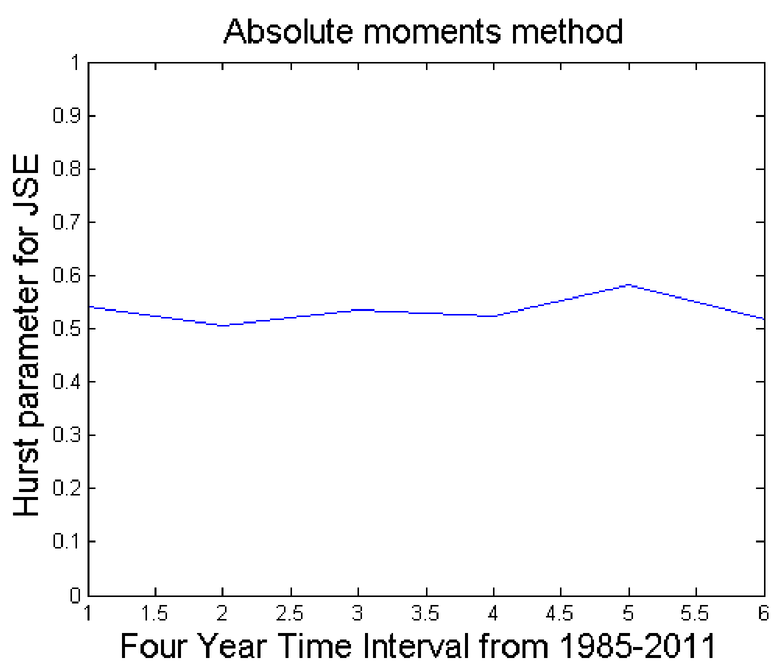


Figure 3.14: The Hurst parameter for ALSI stock for four year intervals using Absolute Moment method.

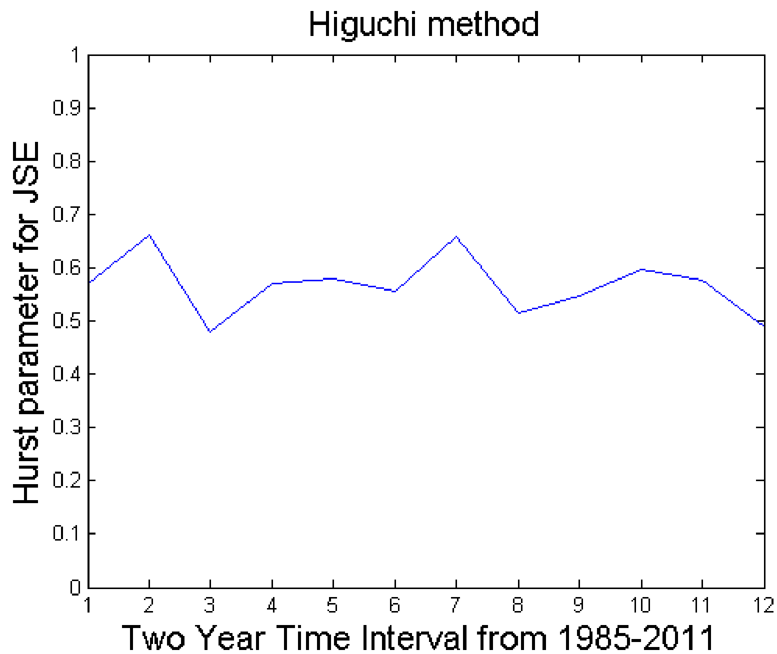


Figure 3.15: The Hurst parameter for ALSI stock for two year intervals using Higuchi method.

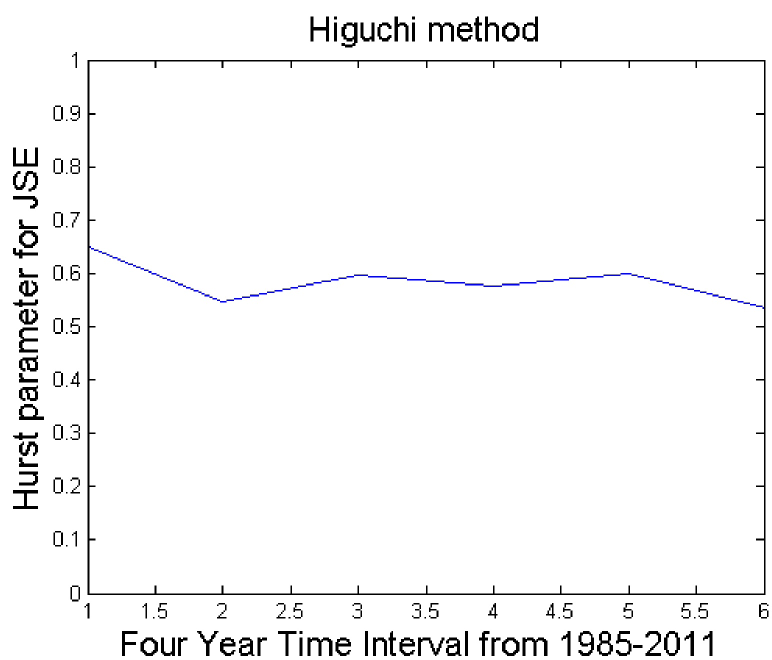


Figure 3.16: The Hurst parameter for ALSI stock for four year intervals using Higuchi method.

Table 3.3: Sector: Metals and Minerals, Stock: Assore Ltd.

Assore		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
27-Mar-85 to 17-Feb-11	5926	0.6244	0.6897
27-Mar-85 to 23-May-08	5238	0.6262	0.6858
23-May-08 to 17-Feb-11	688	0.5788	0.5905

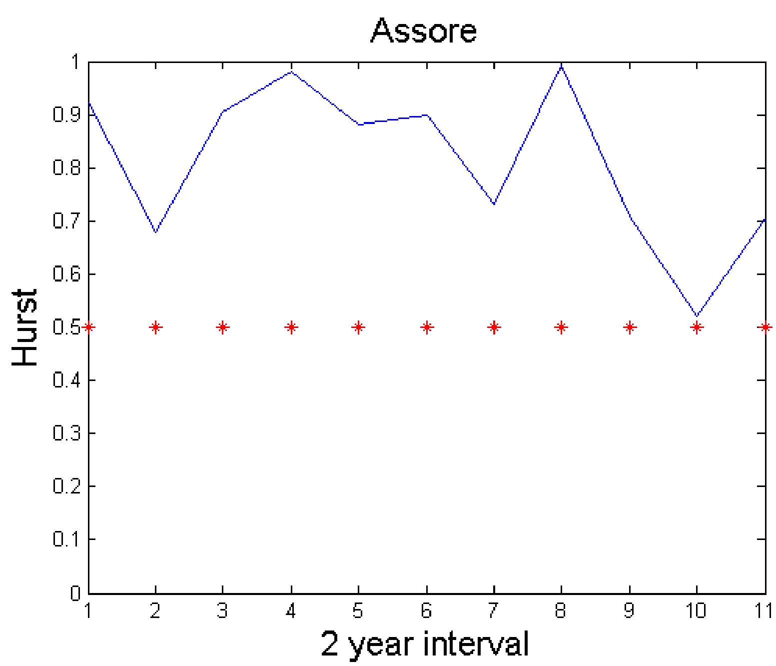


Figure 3.17: Stock Assore, Hurst parameter for two year interval.

Table 3.4: Sector: Metals and Minerals, Stock: Merafe Resources Ltd.

Merafe		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
20-Jul-88 to 17-Feb-11	5635	0.4957	0.4940
20-Jul-88 to 23-May-08	4947	0.4789	0.4766
23-May-08 to 17-Feb-11	688	0.6443	0.6546

Table 3.5: Sector: Metals and Minerals, Stock: Metorex Limited.

Metorex		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
19-Jun-96 to 17-Feb-11	3659	0.5700	0.5627
19-Jun-96 to 23-May-08	2971	0.5227	0.5080
23-May-08 to 17-Feb-11	688	0.6992	0.7033

Merafe Resources Ltd (Merafe) and **Metorex Limited** (Metorex) had a $H \approx \frac{1}{2}$ trend up to the second quarter of 2008 and then rose to a strong persistent behaviour. The Hurst graph for Merafe is given in figure 3.18.

York Timber Holdings Limited (York) from the forestry subsector had an overall slight persistency and after the recession the persistency rose. The Hurst graph of York was plotted for 3 year intervals in figure 3.19.

Adcock Ingram Hlgs Ld (Adcock) from the pharmaceuticals subsector had a strong persistency trend on the overall interval although when the separate intervals

Table 3.6: Sector: Forestry: Stock: York Timber Holdings Limited.

York		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
26-Mar-87 to 17-Feb-11	5963	0.6224	0.6313
26-Mar-87 to 23-May-08	5275	0.6858	0.6077
23-May-08 to 17-Feb-11	688	0.6858	0.7201

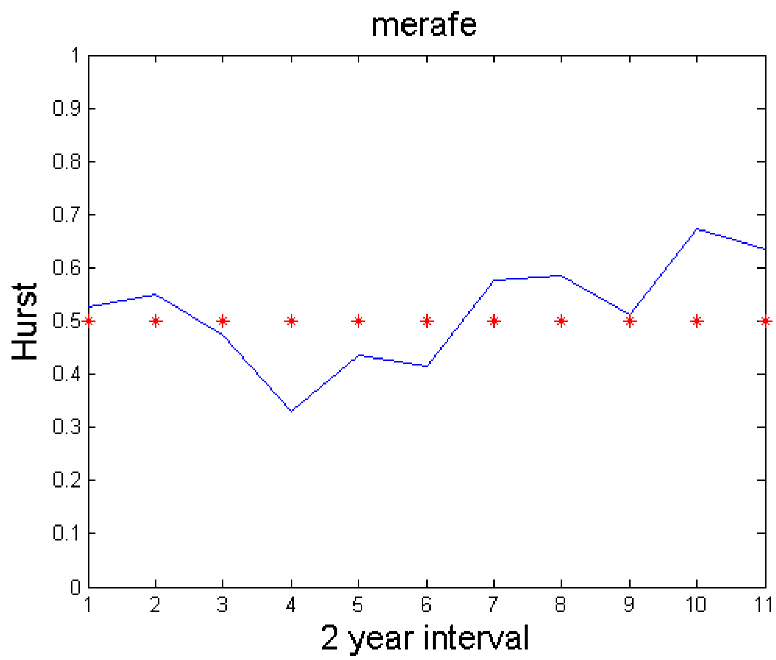


Figure 3.18: Stock Merafe, Hurst parameter for two year interval.

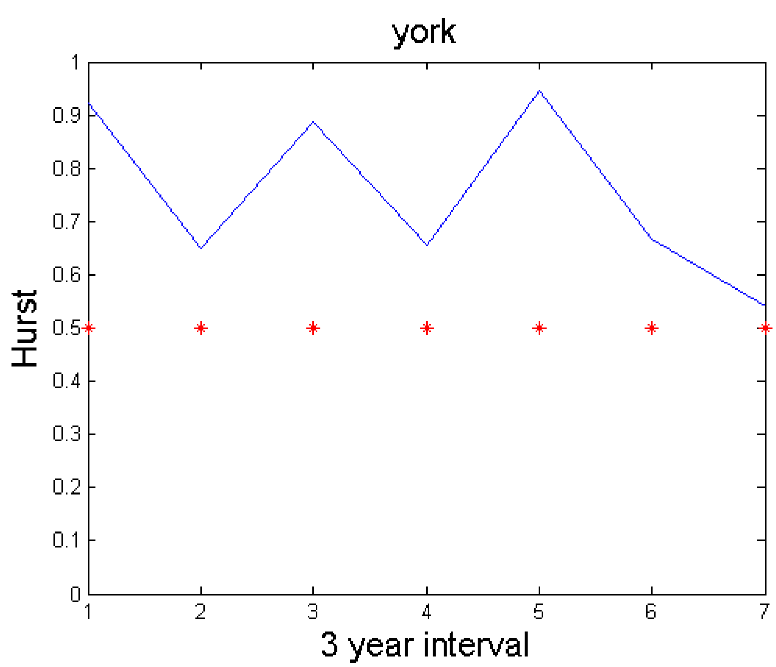


Figure 3.19: Stock: York Timber Holdings Limited Hurst parameter for three year intervals.

Table 3.7: Sector: Pharmaceuticals, Stock: Adcock Ingram Hlgs Ld.

Adcock		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
06-Sep-85 to 17-Feb-11	4233	0.7010	0.7397
06-Sep-85 to 23-May-08	3545	0.7229	0.7367
23-May-08 to 17-Feb-11	688	0.4733	0.6214

were observed it was seen that there is a drop in the persistency level after the recession, see figure 3.20.

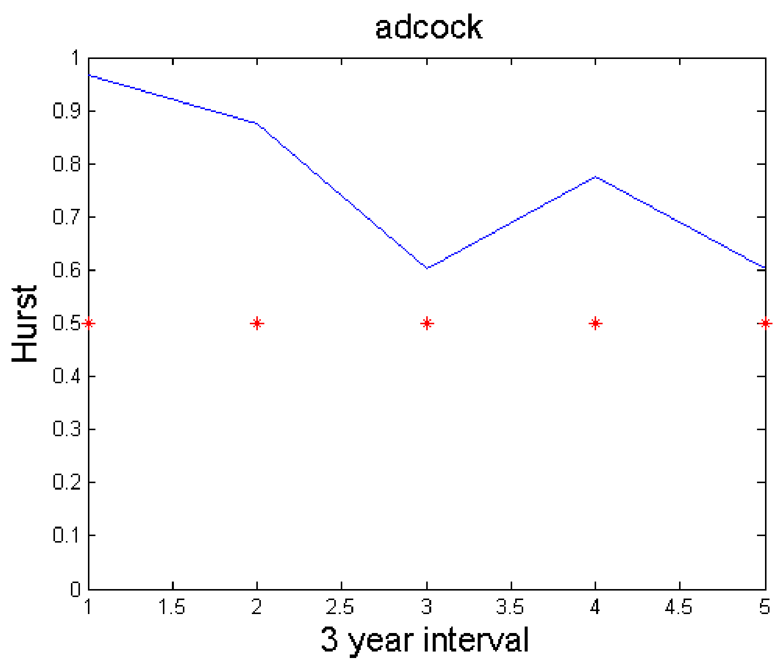


Figure 3.20: Stock Adcock Hurst parameter for three year interval.

Telkom SA Limited (Telkom) from the Fixed-line telecommunications sector had strong persistency over the whole interval, see table 3.8.

Table 3.8: Sector: Fixed-Line Telecom Services, Stock: Telkom SA Limited.

Telkom		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
04-Mar-03 to 17-Feb-11	1991	0.6378	0.6218
04-Mar-03 to 23-May-08	1303	0.5095	0.5890
23-May-08 to 17-Feb-11	688	0.6188	0.6526

Table 3.9: Sector: Brewers, Stock: Awethu Breweries Ltd.

Awethu		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
20-Nov-97 to 17-Feb-11	3306	0.3648	0.4037
20-Nov-97 to 23-May-08	2618	0.3709	0.3799
23-May-08 to 17-Feb-11	688	0.2434	0.2125

In beverages brewers subsector **Awethu Breweries Ltd** (Awethu) showed strong anti-persistency (see table 3.9), while **Sabmiller Plc** (Sabmiller) showed $H \approx \frac{1}{2}$ before the crash and slight anti-persistency afterwards (see table 3.10).

Figure 3.21 displays the stock prices for the different South African banks. Before the global crisis hit South Africa, **ABSA Group Limited** (Absa) had a $H \approx \frac{1}{2}$ trend, however after the crash of mid 2008 the bank's Hurst parameter dropped thus showing an anti-persistent behaviour. For **Nedbank Group Ltd** (Nedcore), **Rand**

Table 3.10: Sector: Brewers, Stock: Sabmiller Plc.

Sabmiller		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
27-Mar-85 to 17-Feb-11	6462	0.4833	0.5599
27-Mar-85 to 23-May-08	5774	0.4808	0.5580
23-May-08 to 17-Feb-11	688	0.4370	0.4532

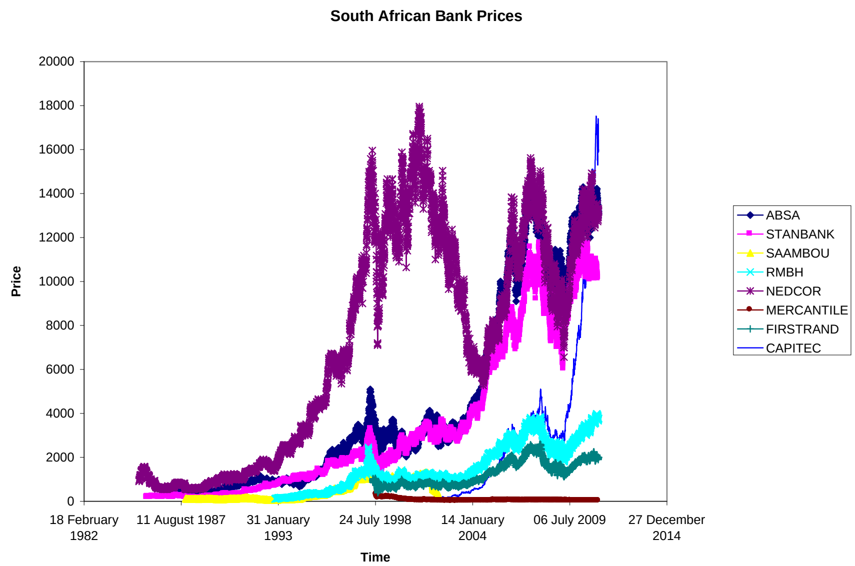


Figure 3.21: South African banks

Merchant Bank Holdings Limited (Rmbh) and **Standard Bank Group Limited** (Stanbank), one method showed $H \approx \frac{1}{2}$ on the whole interval, before the crash and anti-persistency afterwards. The other method showed slight persistency on the whole interval, before the crash and anti-persistency after the crash. **Firstrand Limited** (Firstrand) has a $H \approx \frac{1}{2}$ on the whole interval and anti-persistent on the intervals before the crash and after the crash. **Saambou Holdings limited** (Saambou) Hurst parameter was close to $\frac{1}{2}$ to slightly persistent until the bank collapsed. **Capitec** (Capitec) displayed persistency on all three intervals using Higuchi method and the absolute moments method showed $H \approx \frac{1}{2}$ on the whole interval and the interval after the crash while anti-persistency before the crash.

From figure 3.21 it is clear that **Mercantile Bank Holdings** (Mercantil) stock prices are anti-persistent. The Mecantil data set was divided into 5 groups, first one

Table 3.11: Sector: Banks, Stock: ABSA Group Limited.

ABSA		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
03-Dec-86 to 17-Feb-11	6038	0.4894	0.5473
27-Mar-85 to 23-May-08	5350	0.4834	0.5435
23-May-08 to 17-Feb-11	688	0.3876	0.4180

Table 3.12: Sector: Banks, Stock: Nedbank Group Ltd.

Nedcor		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
27-Mar-85 to 17-Feb-11	6462	0.5321	0.5674
27-Mar-85 to 23-May-08	5775	0.5241	0.5689
23-May-08 to 17-Feb-11	688	0.4433	0.4521

Table 3.13: Sector: Banks, Stock: Rand Merchant Bank Holdings Limited.

Rmbh		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
25-Nov-92 to 17-Feb-11	4551	0.5280	0.5642
25-Nov-92 to 23-May-08	3864	0.5197	0.5689
23-May-08 to 17-Feb-11	688	0.3676	0.4059

Table 3.14: Sector: Banks, Stock: Standard Bank Group Limited.

Stanbank		Hurst	
Period	N	Absolute Moment	Higuchi
06-Sep-85 to 17-Feb-11	6356	0.4530	0.5603
06-Sep-85 to 23-May-08	5669	0.4503	0.5591
23-May-08 to 17-Feb-11	688	0.3809	0.4005

Table 3.15: Sector: Banks, Stock: Firstrand Limited.

Firstrand		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
25-May-98 to 17-Feb-11	3184	0.4631	0.4519
25-May-98 to 23-May-08	2496	0.4292	0.4295
23-May-08 to 17-Feb-11	688	0.3896	0.4136

Table 3.16: Sector: Banks, Stock: Capitec.

Capitec		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
18-Feb-02 to 17-Feb-11	2253	0.5162	0.6570
18-Feb-02 to 23-May-08	1565	0.4417	0.6256
23-May-08 to 17-Feb-11	688	0.5112	0.7109

Table 3.17: Sector: Banks, Stock: Saambou Holdings limited.

Saambou		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
11-Nov-87 to 07-Feb-02	3545	0.5501	0.5774

Table 3.18: Sector: Banks, Stock: Mercantile Bank Holdings.

mercantil		Hurst Parameter		
Period	N	Aggregated Variance	Absolute Moments	Higuchi
12-Aug-98 to 17-Feb-11	3129	0.4671	0.5024	0.4932
12-Aug-98 to 17-Aug-00	500	0.3479	0.3623	0.4200
18-Aug-00 to 16-Aug-02	500	0.3108	0.3623	0.4129
19-Aug-02 to 21-Aug-06	1000	0.3250	0.3197	0.3876
22-Aug-06 to 17-Feb-11	1126	0.2666	0.3464	0.3298

being the whole interval, the second and third interval has 500 observations, the fourth interval has 1000 observations while the fith interval has 1126 observations. Using the aggregated variance, absolute moments and the Higuchi methods a Hurst close to a half was found for the whole interval while all the other intervals showed anti-persistency.

Nawrocki (1995) found empirical results that support persistency in a finite financial data set and no evidence of long term memory assuming an infinite memory series using Rescaled range method. Davidsson (2011) investigated the Standard & Poor's 500 data and found that the returns are serially independent while volatility and expected returns have positive serial correlation.

Results for the other sectors are shown in appendices A and B.

Chapter 4

Fractional Brownian Motion

4.1 Introduction

Chaos is the science of the global nature of systems. Generally, Chaos is defined as a state without order. Mathematically, Chaos is defined as the randomness that is generated by a simple deterministic system (Peitgen, Jürgens and Saupe, 2004). Dynamical systems are systems that move or change in time. Unpredictable systems are non linear systems that display complexity and irregularity. A definition of a fractal is an object in which parts are related to the whole (Mandelbrot, 1977). Leibniz imagined a drop of water containing a universe and within containing another water droplet with its own universe and so forth. As he said, "To see the world in a grain of sand"(Gleick, 1998). Chaos, unpredictability and free will create novelty and novelty is the beginning of new order (Marion, 1999) .

Fractals give structure to complexity and beauty to chaos they are curves whose dimension are greater than one, (Mandelbrot, 1982). Another important aspect of fractals is their self similarity property. Fractals such as cauliflower, trees or sea shells show symmetry across scale. Some time series can be described using fractals, price charts display self similarity as at finer and finer time scales a resembling pattern occurs with a constant measure, (Peters, 1991).

A simple example of a fractal is the Koch Curve which is no where differentiable and has an infinite length in a finite space and the self similarity is built in the construction process. An example of a random fractal is the Sierpinski Triangle (Mandelbrot, 1977). The stock price dynamics are driven by Brownian motion in the Black-Scholes model, which, topologically, is a curve with a dimension of one. Yet, Brownian motion can be seen as the trail of a fine particle that is constantly moving up, down, accelerating, stopping; in other words it is in constant irregular motion. Thus it ends up filling a plane and has a fractal dimension of two (Mandelbrot, 1982). Fractal time series are time series which are statistically self similar with respect to time. If you look at the graph of such a time series it displays jagged lines. They are not one dimensional as they are not straight lines, but neither do they have a dimension of two as they do not fill a plane. Their fractal dimension is between one and two (Peters, 1991). The fractal structure of the market is exhibited in the self-similarity property. Price charts display self-similarity as at finer and finer time scales a resembling pattern occurs. By zooming in on successive monthly price changes this pattern may resemble the structure of weekly price changes.

Mandelbrot (1967) investigated the cotton prices and concluded that the price of cotton did not follow a Gaussian stationary random walk. He applied the Rescaled Range analysis to approximate the Hurst parameter that measures whether the prices of the stock markets has any underlying trend. Mandelbrot showed that prices changes follow a levy stable distribution which is a power law distribution.

Definition 4.1 *Sornette* (2003) . *An observable* $\mathcal{O}$ *depending on* x *is said to be scale invariant under* $x \to \lambda x$ *if there is a number* $\mu(\lambda)$ *such that*

$$\mathcal{O}(x) = \mu \mathcal{O}(\lambda x) . \tag{4.1}$$

The solution of (4.1) *is a power law* $\mathcal{O}(x) = x^{\alpha}$, *where* α *is*

$$\alpha = -\frac{\ln \mu}{\ln \lambda}.$$

Scale invariance in a special case of the self similarity property. In finance, if we let $X(t)$ be the stock return and let $P_{\Delta t}[X(t+\Delta t) - X(t) = 0]$ be the probability of return of the origin as a function of the time change Δt. Mantegna and Stanley, (2000) investigate the S&P500 with $\Delta(t)$ taking on values between one minute and thousand minutes, when plotted on the log-log scale we observe a power law scaling behaviour.

Definition 4.2 *Oświęcimka, Kwapień, Drożdż, Górski and Rak* (2006). *Let* $X(t, \Delta t) = X(t+\Delta t) - X(t)$ *be a stationary process such that*

$$E[X(t, \Delta t)^q] \sim \Delta t^{\tau(q)+1}$$

where $\tau(q)$ *is the scaling exponent. If* $\tau(q)$ *depends on* q *non-linearly we say that* $X(t, \Delta t)$ *possesses a multifractal character. If* $\tau(q)$ *depends on* q *linearly we say that the process is a monofractal.*

Fractional Brownian motion is a example of a monofractal.

Mandelbrot (1996) developed a flexible multifractal model of asset returns (MMAR) that incorporates long-tails and the Lévy-stable distribution. The model contains long-range dependence in the absolute value of the price returns increments, while the price returns are uncorrelated as well as long memory in volatility see Mandelbrot, Fisher and Calvet (1997) . MMAR is a cointinuous-time stochastic process see Calvet and Fisher (2002) .On a time scale multifractality is defined as a set of limitations imposed on the stochastic process moments and is scale consistent. The MMAR is a alternative for ARCH-type models.

Czarnecki and Grech (2009) did multifractal analysis on the Polish stock market and found that multifractal image is obscured for very long time horizon. Mu, Chen, Kertész and Zhou (2009) did multifractal analysis of Chinese stocks. Jamdee and Los (2005) show that the multifractal model of asset returns is consistent with martingale pricing. Barunik, Aste, Matteo and Liu (2012) apply the genealized Hurst parameter to multifractal analysis for different financial data .Ghosh, Jaekel and Petruccione (2012) investigated the multi-fractal structure of the Johannesburg Stock Exchange and found presence of the power law in the fluctuations of the returns.

Researchers started applying fractional Brownian motion to many fields, some of which include hydrology, telecommunications, fluidodynamics, economics and finance (Gradinaru, Nourdin, Russo and Vallois, 2005). Peters (1991) proposed a Fractional Market Hypothesis[1] that combines fractals and chaos theory.

4.2 Fractional Brownian motion

Fractional Brownian motion depends on the Hurst parameter H and is given by the following definition.

Definition 4.3 *(Biagini, Hu, Øksendal and Zhang, 2008). Let the Hurst parameter* H *be a constant with* $0 < H < 1$*. A continuous and centered Gaussian process*

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[1]The assumptions to the Fractional Market Hypothesis are:

1. The markets are stable and have sufficient liquidity when it comprises of investors with different time horizons.
2. Investors stay in their preferred time horizon.
3. Not all information may be reflected in the market prices.
4. A market price trend indicates the changes in expected earnings.
(Peters, 1991)

$\left\{B^H(t)\right\}_{t\geq 0}$ *for all* $t, s \in \mathbb{R}$ *is a fractional Brownian motion (fBm) when it's covariance function is given as*

$$E\left[B^H(t)\, B^H(s)\right] = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right).$$

Properties of fractional Brownian motion:

1. $$B^H(0) = 0.$$

2. The expectation of fractional Brownian motion is

$$E\left[B^H(t)\right] = 0 \text{ for all } t > 0.$$

3. The variance of fractional Brownian motion is

$$\begin{aligned} Var\left[B^H(t)\right] &= E\left[B^H(t)^2\right] - E\left[B^H(t)\right]^2 \\ &= E\left[B^H(t)\, B^H(t)\right] \\ &= \frac{1}{2}\left(|t|^{2H} + |t|^{2H} - |t-t|^{2H}\right) \\ &= t^{2H}. \end{aligned}$$

4. When $H = \frac{1}{2}$ fractional Brownian motion coincides with the classical Brownian motion.

For example, setting $H = \frac{1}{2}$ and for $t > s > 0$ the covariance of fractional Brownian motion is given as

$$\begin{aligned} E\left[B^{\frac{1}{2}}(t), B^{\frac{1}{2}}(s)\right] &= \frac{1}{2}\left(|t|^{2\frac{1}{2}} + |s|^{2\frac{1}{2}} - |t-s|^{2\frac{1}{2}}\right) \\ &= \frac{1}{2}\left(|t| + |s| - |t-s|\right) \\ &= \frac{1}{2}\left((t) + (s) - (t-s)\right) \\ &= \frac{1}{2}(2s) \\ &= s \\ &= \min(s,t). \end{aligned}$$

Fractional Brownian motion $B^H(t)$ has stationary increments implying that the distribution of the increments depends only on the length of the interval and not on the time they occur. When $H = \frac{1}{2}$ then fBm has independent increments. When $H > \frac{1}{2}$ the increments of fBm are positively correlated and the series exhibits persistent behaviour. When $H < \frac{1}{2}$ the increments of fBm are negatively correlated and the series exhibits anti-persistent behaviour. Fractional Brownian motion is self-similar in the sense that $B^H(\alpha t)$ has the same distribution law as $\alpha^H B^H(t)$ for $\alpha > 0$. If $H > \frac{1}{2}$ then fBm has long-range dependence, implying it has long-memory. If $H \neq \frac{1}{2}$ then fBm is non-Markovian and is not a semimartingale. Fractional Brownian motion does not have differentiable sample paths.

Figure 4.1 displays paths of fractional Brownian motion for three different Hurst parameters.

Mandelbrot and Van Ness (1968) defined a stochastic integral representation of fractional Brownian motion as follows

$$B^H(t) = c_H\left(\int_{-\infty}^{0}\left[(t-s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}}\right]dB(s) + \int_0^t (t-s)^{H-\frac{1}{2}}\,dB(s)\right) \quad (4.2)$$

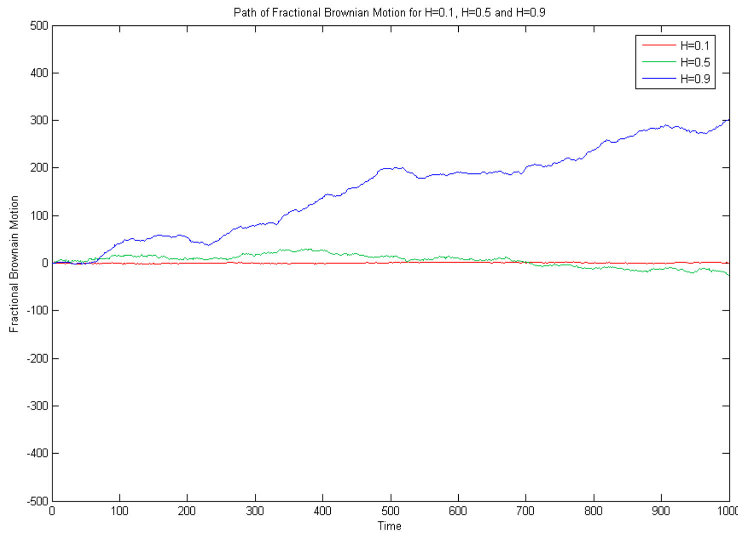

Figure 4.1: Path of fractional Brownian motion for varying Hurst parameter.

where c_H is the normalizing constant and given as

$$c_H = \sqrt{\frac{2H\Gamma\left(\frac{3}{2} - H\right)}{\Gamma\left(\frac{1}{2} + H\right)\Gamma\left(2 - 2H\right)}},$$

and $\Gamma(\cdot)$ is the Gamma function.

For example setting the Hurst parameter $H = \dfrac{1}{2}$ the normalizing constant reduces to

$$\begin{aligned} c_{\frac{1}{2}} &= \sqrt{\frac{2\frac{1}{2}\Gamma\left(\frac{3}{2} - \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2} + \frac{1}{2}\right)\Gamma\left(2 - 2\frac{1}{2}\right)}} \\ &= \sqrt{\frac{\Gamma(1)}{\Gamma(1)\Gamma(1)}} \\ &= 1 \end{aligned}$$

and fractional Brownian motion becomes the standard Brownian motion as

$$\begin{aligned} B^{\frac{1}{2}}(t) &= c_{\frac{1}{2}}\left(\int_{-\infty}^{0}\left[(t-s)^{\frac{1}{2}-\frac{1}{2}}-(-s)^{\frac{1}{2}-\frac{1}{2}}\right]dB(s)+\int_{0}^{t}(t-s)^{\frac{1}{2}-\frac{1}{2}}\,dB(s)\right) \\ &= \int_{0}^{t} dB(s) \\ &= B(t)\,. \end{aligned}$$

Jakubowski (2009) introduced the idea of a local predictor which shows the existence of fractional Brownian motion for $H > \frac{1}{2}$.

Chapter 5

Wick-Itô Stochastic Calculus

5.1 Introduction

In stochastic calculus Brownian motion is used as an input variable to the system. The stock price dynamics depends on an assumed integration theory. In the usual case we assume the Itô stochastic integral. Since fractional Brownian motion is not a semimartingale when $H \neq \frac{1}{2}$ ordinary stochastic calculus is not applicable so different integration theories have to be applied; a fractional pathwise integral was the first integral that was introduced to solve the problem. Under pathwise integration stochastic fractional Brownian motion does not have zero expectation. It was shown that one can create a replicating portfolio which produces arbitrage possibilities (see for instance Biagini, Hu, Øksendal and Zhang, 2008 for the proof) thus pathwise integration cannot be used in the markets. Another integration theory that is based on white noise theory has been established. Duncan, Hu, Pasik-Duncan (1991) developed a Wick calculus universe based on the Wick product, which is denoted by the symbol $\diamond$. A fractional stochastic integral was introduced, that is defined by means of the Wick product. Only some basic Wick product properties will be discussed here. Under this Wick-based integration theory the stochastic integral has a zero expectation and the market is free from strong arbitrage.

Stochastic integration with respect to fractional Brownian motion for $\frac{1}{2} < H < 1$ is briefly discussed in the following chapter and only the necessary theorems and results are presented. Fractional Brownian motion is expressed in terms of fractional white noise and fractional white noise is the derivative of fractional Brownian motion. An in-depth discussion of fractional white noise requires technical concepts so we refer the reader to appendix C for white noise analysis as presented by Hida, Kuo, Potthoff and Striet (1993) and Grothaus, Kondratiev and Us (1998) who show that the wick product can be expressed in terms of the $\mathcal{S}$ transform. Bender (2003) contrast an alternative fractional Brownian motion which has the same covariance structure as the fractional Brownian motion under Biagini, Hu, Øksendal and Zhang (2008). The Clark-Ocone theorem under Biagini, Hu, Øksendal and Zhang (2008) is not well-defined therefore we present the Clark-Ocone theorem as presented by Bender and Elliot (2002).

Dai and Heyde (1996) give a representation of the Itô formula with respect to fractional Brownian motion and show that the solution of the stochastic differential equation is unique.

5.2 Construction of Fractional Brownian Motion

In order to obtain the fractional Wick Itô integral a function ϕ is introduced to handle the existence of the Wick product. For a fixed Hurst parameter $\frac{1}{2} < H < 1$ and for $s, t \in \mathbb{R}$ we introduce a function $\phi : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\phi(t, s) = H(2H-1)|t-s|^{2H-2}.$$

For s, $t > 0$ we have (Biagini, Hu, Øksendal and Zhang, 2008)

$$\int_0^t \int_0^t \phi(u, v)\, du dv = t^{2H}$$

and

$$\int_0^t \int_0^s \phi(u, v)\, du dv = \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right). \tag{5.1}$$

Let $\mathcal{S}(\mathbb{R})$ be a Schwartz space[1] of rapidly decreasing real-valued smooth functions on $\mathbb{R}$. Let $f, g : \mathbb{R} \to \mathbb{R}$ and if $f, g \in \mathcal{S}(\mathbb{R})$ then we define the inner product of the two functions f and g as

$$\langle f, g \rangle_{\phi} = \int_{\mathbb{R}} \int_{\mathbb{R}} f(s)\, g(t)\, \phi(t, s)\, ds dt$$

and the norm as

$$||f||_{\phi}^{2} = \int_{\mathbb{R}} \int_{\mathbb{R}} f(s)\, f(t)\, \phi(t, s)\, ds dt$$

which is finite (Biagini, Hu, Øksendal and Zhang, 2008), i.e.

$$\|f\|_{\phi}^{2} < \infty.$$

We can now construct a Hilbert space $L_{\phi}^{2}(\mathbb{R})$ which is the completion of $\mathcal{S}(\mathbb{R})$. One can similarly construct a Hilbert space $L_{\phi}^{2}(\mathbb{R}_{+})$ for the positive real numbers. Let $\Omega = \mathcal{S}'(\mathbb{R})$ be the dual of $\mathcal{S}(\mathbb{R})$ also called the tempered distributions and let $\mathcal{B}(\Omega)$ be Borel subsets of Ω. Denote $\langle \omega, f \rangle$ the action of $\omega \in \mathcal{S}'(\mathbb{R})$ on $f \in \mathcal{S}(\mathbb{R})$, then we have the map $\mathcal{S}(\mathbb{R}) \times \mathcal{S}'(\mathbb{R}) \to \mathbb{R}$. By the Bochner-Minlos theorem there exists a probability measure μ_{ϕ} on $\mathcal{B}(\Omega)$ such that

$$\int_{\Omega} \exp\left(i \langle \omega, f \rangle\right) d\mu_{\phi}(\omega) = \exp\left(-\frac{1}{2} ||f||_{\phi}^{2}\right).$$

See Biagini, Hu, Øksendal and Zhang (2008).

The expectation under the probability measure μ_{ϕ} (Biagini, Hu, Øksendal and Zhang, 2008) is

$$E_{\mu_{\phi}}\left[\langle \omega, f \rangle\right] = 0$$

[1](Stein and Shakarchi, 2003) A Schwartz space $\mathcal{S}(\mathbb{R})$ on $\mathbb{R}$ consists of the sets of all differentiable functions f so that f and all it's derivatives $f', f'', ..., f^{(l)}$ .. are rapidly decreasing such that for every $k, l \geq 0$

$$\sup_{x \in \mathbb{R}} |x|^{k} \left| f^{(l)}(x) \right| < \infty.$$

And if $f \in \mathcal{S}(\mathbb{R})$, we have

$$f'(x) = \frac{df}{dx} \in \mathcal{S}(\mathbb{R})$$

and $xf(x) \in \mathcal{S}(\mathbb{R})$.

and

$$E_{\mu_\phi}\left[\langle \omega, f\rangle^2\right] = ||f||_\phi^2 .$$

Fractional Brownian motion $B^H(t)$ is defined on a probability space $\left(\Omega, \mathcal{F}_t^H, \mu_\phi\right)$, where $\mathcal{F}_t^H$ is the Borel σ-algebra[2] generated by $B^H(t)$. Let χ be a piecewise function of the form

$$\chi_{[0,t]} = \begin{cases} 1 & 0 \leq s \leq t \\ -1 & t \leq s \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

then for $t \in \mathbb{R}$ we define

$$\tilde{B}^H(t) = \tilde{B}^H(t, \omega) = \left\langle \omega, \chi_{[0,t]}(\cdot)\right\rangle$$

and $\tilde{B}^H(t) \in L^2\left(\mu_\phi\right)$ (Biagini, Hu, Øksendal and Zhang, 2008). Where the space $L^p\left(\mu_\phi\right)$ for each $p \geq 1$ is the space of all random variables $F : \Omega \to \mathbb{R}$ such that

$$\|F\|_{L^p\left(\mu_\phi\right)} = E_{\mu_\phi}\left[|F|^p\right]^{\frac{1}{p}} < \infty.$$

Then there exists a continuous version $B^H(t)$ of $\tilde{B}^H(t)$ which is also a fractional Brownian motion.

5.3 Stochastic Integral for Deterministic Functions

In this section we introduce the stochastic integral of a deterministic function which is driven by fractional Brownian motion using approximating step functions which is a piecewise function. Denote $L_H^2(\mathbb{R})$ the subspace of deterministic functions contained in $L_\phi^2(\mathbb{R})$. In order to represent a stochastic integral with respect to fractional Brownian motion we define the integration of a function f which is an element in

[2](Björk, 2004). Let $X = \mathbb{R}^n$, then we define the Borel algebra $\mathcal{B}(\mathbb{R}^n)$ as the sigma-algebra which is generated by the class of open sets on $\mathbb{R}^n$.

$L_H^2(\mathbb{R})$. A simple function is the sum of a real constant multiplied by the indicator function and we approximate the integrand by using simple functions. For $0 = t_0 < t_i < .. < t_m < T$ we partition the interval and let there be a sequence of functions $\{f_m(t)\} \underset{m\to\infty}{\to} f(t)$ such that

$$f_m(t) = \sum_{i=1}^{m} a_i^m \left(\chi_{[0,i+1]}(t) - \chi_{[0,i]}(t)\right)$$

is piecewise constant and a_i is a sequence of constants. Setting

$$\begin{aligned}
\langle \omega, f_m \rangle &= \omega(f_m) \\
&= \omega\left(\sum_{i=1}^{m} a_i^m \left(\chi_{[0,i+1]}(t) - \chi_{[0,i]}(t)\right)\right) \\
&= \sum_{i=1}^{m} a_i^m \omega\left(\chi_{[0,i+1]}(t) - \chi_{[0,i]}(t)\right) \\
&= \sum_{i=1}^{m} a_i^m \left(\omega\left(\chi_{[0,i+1]}(t)\right) - \omega\left(\chi_{[0,i]}(t)\right)\right) \\
&= \sum_{i=1}^{m} a_i^m \left(\langle \omega, \chi_{[0,i+1]}(t)\rangle - \langle \omega, \chi_{[0,i]}(t)\rangle\right) \\
&= \sum_{i=1}^{m} a_i^m \left(B^H(t_{i+1}) - B^H(t_i)\right) \\
&= \int_{\mathbb{R}} f_m(t)\, dB^H(t).
\end{aligned}$$

Taking limit from both sides

$$\int_{\mathbb{R}} f(t)\, dB^H(t) = \lim_{m\to\infty} \int_{\mathbb{R}} f_m(t)\, dB^H(t)$$

and the integral belongs to $L^2(\mu_\phi)$. By using the approximating step functions the pairing is given as

$$\langle \omega, f \rangle = \int_{\mathbb{R}} f(t)\, dB^H(t, \omega).$$

Then if f and g belong to $L_H^2(\mathbb{R})$, then $\int_{\mathbb{R}} f(s)\, dB^H(s)$ and $\int_{\mathbb{R}} g(s)\, dB^H(s)$ are Gaussian random variables with zero mean

$$E_{\mu_\phi}\left[\int_{\mathbb{R}} f(t)\, dB^H(t)\right] = 0, \tag{5.2}$$

variance (see Dasgupta and Kallianpur, 1999)

$$E_{\mu_\phi}\left[\int_{\mathbb{R}} f(t)\, dB^H(t)\right]^2 = ||f||_\phi^2 \tag{5.3}$$

and covariance

$$\begin{aligned} E_{\mu_\phi}\left[\int_{\mathbb{R}} f(t)\, dB^H(t) \int_{\mathbb{R}} g(s)\, dB^H(s)\right] &= \int_0^t \int_0^s f(u)\, g(v)\, \phi(u,v)\, dudv \quad (5.4)\\ &= \langle f, g\rangle_\phi . \end{aligned}$$

The expectation of $B^H(t)$ with respect to the measure μ_ϕ and using (5.2) is

$$\begin{aligned} E_{\mu_\phi}\left[B^H(t)\right] &= E_{\mu_\phi}\left[\left\langle \omega, \chi_{[0,t]}\right\rangle\right] \\ &= E_{\mu_\phi}\left[\int_{\mathbb{R}} \chi_{[0,t]}(u)\, dB^H(u)\right] \\ &= 0 \end{aligned} \tag{5.5}$$

For all $t, s \in \mathbb{R}$ fractional Brownian motion is a Gaussian process $\left\{B^H(t)\right\}_{t\geq 0}$ with a zero expectation. Setting $t > s$ and using (5.1)and (5.3) the covariance of fractional Brownian motion is given as

$$\begin{aligned} & E_{\mu_\phi}\left[B^H(t)\, B^H(s)\right] \\ = & E_{\mu_\phi}\left[\left\langle \omega, \chi_{[0,t]}\right\rangle \left\langle \omega, \chi_{[0,s]}\right\rangle\right] \\ = & E_{\mu_\phi}\left[\int_{\mathbb{R}} \chi_{[0,t]}(u)\, dB^H(u) \int_{\mathbb{R}} \chi_{[0,s]}(v)\, dB^H(v)\right] \\ = & E_{\mu_\phi}\left[\int_0^t dB^H(u) \int_0^s dB^H(v)\right] \\ = & \int_0^t \int_0^s \phi(u,v)\, dudv \\ = & \frac{1}{2}\left[|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right]. \end{aligned}$$

The variance follows as (Rostek, 2009)

$$\begin{aligned} Var\left[B^H(t)\right] &= E_{\mu_\phi}\left[B^H(t)^2\right] - E_{\mu_\phi}\left[B^H(t)\right]^2 \\ &= t^{2H}. \end{aligned} \tag{5.6}$$

Exponentials functions $\varepsilon(f)$ are defined with deterministic integrands f. Then for any $f \in L_H^2(\mathbb{R})$ we define

$$\begin{aligned}\varepsilon(f) &= \exp\left(\int_{\mathbb{R}} f(t)\, dB^H(t) - \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} f(s) f(t) \phi(s,t)\, ds dt\right) \\ &= \exp\left(\int_{\mathbb{R}} f(t)\, dB^H(t) - \frac{1}{2}\|f\|_\phi^2\right)\end{aligned}$$

then $\varepsilon(f) \in L^p(\mu_\phi)$ for each $p \geq 1$. We have that (see Rostek, 2009)

$$E_{\mu_\phi}[\varepsilon(f)] = 1. \tag{5.7}$$

It also follows by Rostek (2009) that $\int_0^\infty f(t)\, dB^H(t)$ is normally distributed with zero mean and variance $\|f\|_\phi^2$ and $\exp\left(\int_{\mathbb{R}} f(t)\, dB^H(t)\right)$ is log-normally distributed with a mean of $\exp\left(\frac{1}{2}\|f\|_\phi^2\right)$.

5.4 Results from Wick Calculus

The preliminary results from Wick calculus are presented in this section; for further results refer to Holden, Øksendal, Ubøe and Zhang (1996) and Biagini, Hu, Øksendal and Zhang (2008). In the fractional market Hu and Øksendal (2003) change the definition of the portfolio value by using the Wick product.

We do not formally expand on the fractional Hida test function space $(\mathcal{S})_H$ and its dual, the fractional Hida distribution space $(\mathcal{S})_H^*$. Refer theorem (5.6) for the description of a Hida test function space $(\mathcal{S})$ which is the projective limit and its dual, the Hida distribution space $(\mathcal{S})^*$ which is the lnductive limit[3]. The Wick product

[3](See Kelley and Namioka(1963)). An inductive system has the following: an A index set with partial ordering $\geqq$; for each $t \in A$, E_t is a linear space; for $t \geqq s \geqq r$ there is a canonical linear map Q_{ts} of E_s into E_t such that: $Q_{ts} \circ Q_{sr} = Q_{tr}$ and Q_{tt} is the identity map of E_t $\forall t$. Let N be the subspace of the direct sum $\sum\{E_t, t \in A\}$, then the inductive limit is defined as the quotient space $\left(\sum\{E_t, t \in A\}\right)/N$ which is a space of elements which of the form $x + N$ where x in $\sum\{E_t, t \in A\}$. A projective system has the following: an A index set with partial ordering $\geqq$; for each $t \in A$, E_t is a linear space; for $t \geqq s \geqq r$

is not the same as normal multiplication and is defined using Hermite polynomials in $\mathbb{R}$. If we define $F, G \in (\mathcal{S})_H^*$, then the Wick product of $F \diamond G \in (\mathcal{S})_H^*$. Refer to appendix C for the construction of the wick product between two distributions in a generalized Gaussian space as presented by Grothaus, Kondratiev and Us (1998)

Lemma 5.1 *(Holden, Øksendal, Ubøe and Zhang,* 1996*).The Wick product of $F \diamond G$ has the following algebraic properties:*

a) Commutative law $F, G \in (\mathcal{S})_H^*$

$$F \diamond G = G \diamond F.$$

b) Associated law $F, G, H \in (\mathcal{S})_H^*$

$$F \diamond (G \diamond H) = (F \diamond G) \diamond H.$$

c) Distributive law $F, A, B \in (\mathcal{S})_H^*$

$$F \diamond (A + B) = F \diamond A + F \diamond B. \tag{5.8}$$

d) If $F, G, H \in (\mathcal{S})_H^*$ then

$$(F \diamond G) \cdot H \neq F \diamond (G \cdot H).$$

Although suppose that a least one of F and G is deterministic e.g. $F = c \in \mathbb{R}$, then the Wick product coincides with normal multiplication

$$F \diamond G = F \cdot G.$$

Another example is if $F = 0$, then $F \diamond G = 0$. Wick algebra obeys the same rules as the ordinary algebra for example

$$(X + Y)^{\diamond 2} = X^{\diamond 2} + 2X \diamond Y + Y^{\diamond 2}$$

---

there is a caconical transformation P_{st} of E_t into E_s such that: $P_{rt} = P_{rs} \circ P_{st}$ and P_{ss} is the identity transformation $\forall s$ in A. The projective limit of the system is the subspace of the product $\mathcal{X}\{E_t, t \in A\}$ which consists of all x such that $P_{st}(x(t)) = x(s)$.

and

$$\exp^{\diamond}(X+Y) = \exp^{\diamond}(X) \diamond \exp^{\diamond}(Y)$$

(see Øksendal, 1997). The expectation is well defined as

$$E_{\mu_\phi}[F \diamond G] = E_{\mu_\phi}[F]\, E_{\mu_\phi}[G] \tag{5.9}$$

Definition 5.1 *(Biagini, Hu, Øksendal and Zhang, 2008). The Wick Power of* $X \in (\mathcal{S})_H^*$ *is*

$$X^{\diamond n} = \underbrace{X \diamond X \diamond ... \diamond X}_{n}.$$

Examples include $X^{\diamond 0} = 1$ and $X^{\diamond k} = X \diamond X^{\diamond(k-1)}$ for $k = 1, 2, ...$ .If $E_{\mu_\phi}[X] \neq 0$, we can define the Wick inverse $X^{\diamond(-1)}$ with the following property (see Holden, Øksendal, Ubøe and Zhang, 1996)

$$X \diamond X^{\diamond(-1)} = 1.$$

Let $f, g \in L_H^2(\mathbb{R})$, then the Wick product of two integral functions is

$$\left(\int_{\mathbb{R}} f dB^H\right) \diamond \left(\int_{\mathbb{R}} g dB^H\right) = \left(\int_{\mathbb{R}} f dB^H\right) \cdot \left(\int_{\mathbb{R}} g dB^H\right) - \langle f, g\rangle_\phi$$

and is an element of $(\mathcal{S})_H^*$ (Biagini, Hu, Øksendal and Zhang, 2008).

Definition 5.2 *(Biagini, Hu, Øksendal and Zhang, 2008). Wick exponentials of* $X \in (\mathcal{S})_H^*$ *is*

$$e^{\diamond}(X) = \sum_{n=0}^{\infty} \frac{1}{n!} X^{\diamond n}$$

provided that the series converges in $(\mathcal{S})_H^*$.

The following concept will often be used later. For $\omega \in \Omega$ being a random variable and $f \in L_H^2(\mathbb{R})$ then

$$\begin{aligned} \exp^{\diamond}(\langle \omega, f\rangle) &= \exp\left(\langle \omega, f\rangle - \frac{1}{2}\|f\|_\phi^2\right) \\ &= \varepsilon(f). \end{aligned}$$

(See Biagini, Hu, Øksendal and Zhang, 2008). Denote $L_H^2(\mathbb{R}_+)$ the subspace of deterministic functions contained in $L_\phi^2(\mathbb{R}_+)$. Moreover for any $f \in L_H^2(\mathbb{R}_+)$ we have

$$\begin{aligned} \varepsilon(f) &= \exp^{\diamond}\left(\int_0^\infty f(t)\,dB^H(t)\right) \\ &= \exp\left(\int_0^\infty f(t)\,dB^H(t) - \frac{1}{2}\|f\|_\phi^2\right) \\ &= \exp\left(\int_0^\infty f(t)\,dB^H(t) - \frac{1}{2}\int_0^\infty\int_0^\infty f(s)f(t)\phi(s,t)\,dsdt\right). \end{aligned}$$

Wick exponential functions will play an important role when solving stochastic differential equations.

5.5 Wick-Itô Skorohod Integral

In the previous section we dealt with integrals of deterministic functions. In this section we show an extension to the general case. Biagini, Hu, Øksendal and Zhang (2008) show the fractional stochastic integral of Itô type can be represented using fractional white noise. We denote fractional white noise at time t as $W^H(t) \in (\mathcal{S})_H^*$ for all t and it is defined in Biagini, Hu, Øksendal and Zhang (2008, page 56). For $0 \leq s \leq t$ and $t \to W^H(t)$ is a continuous function from $\mathbb{R}$ into $(\mathcal{S})_H^*$. Fractional white noise $W^H(t)$ is integrable in $(\mathcal{S})_H^*$ and the integral is

$$\int_0^t W^H(s)\,ds = B^H(t).$$

When regarded as a map $B^H(\cdot) : \mathbb{R} \to (\mathcal{S})_H^*$ fractional Brownian motion is differentiable with respect to t,

$$\frac{d}{dt}B^H(t) = W^H(t). \tag{5.10}$$

Definition 5.3 *(Biagini, Hu, Øksendal and Zhang, 2008). Given a function $Y : \mathbb{R} \to (\mathcal{S})_H^*$ such that $Y(t) \diamond W^H(t)$ is integrable in $(\mathcal{S})_H^*$ then the fractional Wick-Itô Skorohod integral is defined as*

$$\int_{\mathbb{R}} Y(t)\,dB^H(t) = \int_{\mathbb{R}} Y(t) \diamond W^H(t)\,dt. \tag{5.11}$$

Let $F \in L^p\left(\mu_\phi\right)$ be a random variable then Rostek (2009) shows

$$E_{\mu_\phi}\left[\int_0^T F(s)\, dB^H(s)\right] = 0.$$

Example 5.1 *(Biagini, Hu, Øksendal and Zhang, 2008). Suppose* $F_i \in (\mathcal{S})_H^*$ *and we set*

$$Y(t) = \sum_{i=1}^{n} F_i(\omega)\, I_{[t_i,t_{i+1})}(t).$$

Then the integral can be expressed as

$$\begin{aligned}\int_{\mathbb{R}} Y(t)\, dB^H(t) &= \int_{\mathbb{R}} \sum_{i=1}^{n} F_i(\omega)\, I_{[t_i,t_{i+1})}(t)\, dB^H(t) \\ &= \sum_{i=1}^{n} \int_{\mathbb{R}} F_i(\omega)\, I_{[t_i,t_{i+1})}(t) \diamond W^H(t)\, dt \\ &= \sum_{i=1}^{n} \int_{t_i}^{t_{i+1}} F_i(\omega) \diamond W^H(t)\, dt\end{aligned}$$

but we see that

$$\int_{t_i}^{t_{i+1}} F_i(\omega) \diamond W^H(t)\, dt = \lim_{\substack{m\to\infty \\ \Delta s_j \to 0}} \sum_{j=0}^{m-1} F_i(\omega) \diamond W^H\left(\xi_j\right)\left(s_j - s_{j-1}\right)$$

where $\xi_j \in [s_{j-1}, s_j]$ *and* $\Delta s_j = (s_j - s_{j-1})$. *Since* $F_i(\omega)$ *does not depend on* t *and by the distributive law for Wick products* (5.8) *we have*

$$\lim_{\substack{m\to\infty \\ \Delta s_j \to 0}} \sum_{j=0}^{m-1} F_i(\omega) \diamond W^H\left(\xi_j\right)\left(s_j - s_{j-1}\right) = F_i(\omega) \diamond \lim_{\substack{m\to\infty \\ \Delta s_j \to 0}} \sum_{j=0}^{m-1} W^H\left(\xi_j\right)\left(s_j - s_{j-1}\right).$$

It follows

$$\begin{aligned}\int_{\mathbb{R}} Y(t)\, dB^H(t) &= \sum_{i=1}^{n} F_i(\omega) \diamond \int_{t_i}^{t_{i+1}} W^H(t)\, dt \\ &= \sum_{i=1}^{n} F_i(\omega) \diamond \left(B^H(t_{i+1}) - B^H(t_i)\right).\end{aligned}$$

Example 5.2 *(Biagini, Hu, Øksendal and Zhang, 2008). Using stochastic Wick calculus we obtain the following integral*

$$\begin{aligned}
&\int_0^t B^H(s)\, dB^H(s) \\
=\ & \int_0^t B^H(s) \diamond W^H(s)\, ds \\
=\ & \int_0^t B^H(s) \diamond \frac{d}{ds} B^H(s)\, ds \\
=\ & \frac{1}{2}\left(B^H(t)\right)^{\diamond 2} \\
=\ & \frac{1}{2}\left(B^H(t) \diamond B^H(t)\right) \\
=\ & \frac{1}{2}\left(\int_0^t \chi_{[0,t]}(u)\, dB^H(u) \diamond \int_0^t \chi_{[0,t]}(u)\, dB^H(u)\right) \\
=\ & \frac{1}{2}\left(\int_0^t \chi_{[0,t]}(u)\, dB^H(u) \int_0^t \chi_{[0,t]}(u)\, dB^H(u) - \int_0^t \int_0^t \phi(t,s)\, dsdt\right) \\
=\ & \frac{1}{2}\left(B^H(t)\right)^2 - \frac{1}{2} t^{2H}.
\end{aligned}$$

Example 5.3 *(Biagini, Hu, Øksendal and Zhang, 2008). Consider a fractional stochastic differential equation, with x, μ and σ constants, $\sigma \neq 0$ which is given as*

$$\begin{aligned}
dX(t) &= \mu X(t)\, dt + \sigma X(t)\, dB^H(t) \\
X(0) &= x > 0.
\end{aligned}$$

The stochastic equation can be rewritten as

$$X(t) = X(0) + \int_0^t \mu X(s)\, ds + \int_0^t \sigma X(s)\, dB^H(t).$$

Using the Wick Itô Skorohod integral we get

$$X(t) = X(0) + \int_0^t \mu X(s)\, ds + \int_0^t \sigma X(s) \diamond W^H(s)\, ds.$$

It follows

$$\begin{aligned}
\frac{d}{dt} X(t) &= \mu X(t) + \sigma X(t) \diamond W^H(t) \\
&= X(t) \diamond \left(\mu + \sigma W^H(t)\right).
\end{aligned}$$

The solution of this differential equation is similar to that of the ordinary differential for exponentials function but the ordinary multiplication sign is replaced with the Wick product.

$$\begin{aligned} X(t) &= X(0) \diamond \exp^{\diamond}\left(\int_0^t \mu ds + \sigma \int_0^t W^H(s)\, ds\right) \\ &= x \exp^{\diamond}\left(\mu t + \sigma B^H(t)\right) \end{aligned}$$

To show that the above is a solution we differentiate

$$\begin{aligned} \frac{dX(t)}{dt} &= \frac{d}{dt}\left(x \exp^{\diamond}\left(\mu t + \sigma B^H(t)\right)\right) \\ &= x \exp^{\diamond}\left(\mu t + \sigma B^H(t)\right) \diamond \frac{d}{dt}\left(\mu t + \sigma B^H(t)\right) \\ &= x \exp^{\diamond}\left(\mu t + \sigma B^H(t)\right) \diamond \left(\mu + \sigma W^H(t)\right) \\ &= X(t) \diamond \left(\mu + \sigma W^H(t)\right). \end{aligned}$$

Using Wick calculus the solution can be rewritten as

$$\begin{aligned} X(t) &= x \exp^{\diamond}\left(\mu t + \sigma B^H(t)\right) \\ &= x \exp^{\diamond}(\mu t) \diamond \exp^{\diamond}\left(\sigma B^H(t)\right) \\ &= x \exp(\mu t) \exp^{\diamond}\left(\sigma B^H(t)\right). \end{aligned}$$

Setting $f = \sigma \chi_{[0,t]}$ *hence* $\langle \omega, f \rangle = \sigma B^H(t)$ *and using Wick exponentials we have*

$$\begin{aligned} \exp^{\diamond}\left(\sigma B^H(t)\right) &= \exp\left(\sigma B^H(t) - \frac{1}{2}\sigma^2 \int_0^t \int_0^t \phi(s,t)\, ds dt\right) \\ &= \exp\left(\sigma B^H(t) - \frac{1}{2}\sigma^2 t^{2H}\right). \end{aligned}$$

It follows that the solution is

$$X(t) = x \exp\left(\sigma B^H(t) + \mu t - \frac{1}{2}\sigma^2 t^{2H}\right).$$

5.6 Fractional Girsanov Theorem

The fractional Girsanov theorem in finance gives us the possibility of converting between equivalent measures which describes the probability of fractional Brownian motion refer to theorem (2.2) for the Girsanov's formula under Brownian motion.

Theorem 5.1 *Let $T > 0$ and let γ be a continuous function with the support, supp $\gamma \subset [0, T]$. Let K a function with $K \subset [0, T]$ such that for all $f \in \mathcal{S}(\mathbb{R})$ and the supp $f \subset [0, T]$ we have*

$$\langle K, f \rangle_{\phi} = \langle \gamma, f \rangle_{L^2(\mathbb{R})}$$

then for $0 \leq t \leq T$ it follows

$$\int_{\mathbb{R}} K(s)\, \phi(s, t)\, ds = \gamma(t).$$

On σ-algebra $\mathcal{F}_T^H$ generated by $\left\{B^H(s), 0 \leq s \leq T\right\}$, define a probability measure $\mu_{\phi,\gamma}$

$$\frac{d\mu_{\phi,\gamma}}{\mu_{\phi}} = \exp^{\diamond}\{-\langle \omega, K \rangle\}.$$

Then for $0 \leq t \leq T$ it follows that

$$\hat{B}^H(t) = B^H(t) + \int_0^t \gamma(s)\, ds$$

is a fractional Brownian motion under the new measure $\mu_{\phi,\gamma}$.

For the proof see Biagini, Hu, Øksendal and Zhang (2008, page 60).

Lemma 5.2 *(Biagini, Hu, Øksendal and Zhang, 2008). Wick products on different white noise spaces. Let μ_{ϕ} be the measure on $B^H(t)$ and let $\mu_{\phi,\gamma}$ be the new measure defined by the fractional Girsanov theorem on $\hat{B}^H(t) = B^H(t) + \int_0^t \gamma_s ds$. Let the Wick products corresponding to μ_{ϕ} and $\mu_{\phi,\gamma}$ be denoted by $\diamond_{\mu_{\phi}}$ and $\diamond_{\mu_{\phi,\gamma}}$. Then*

$$F \diamond_{\mu_{\phi}} G = F \diamond_{\mu_{\phi,\gamma}} G \tag{5.12}$$

for all $F, G \in (\mathcal{S})_H^*$.

5.6.1 Quasi-Conditional Expectation and the Fractional Clark-Ocone Theorem

When modelling financial instruments, especially when the derivation of the Black-Scholes option pricing formula is done using expectations, the conditional expectation of the fractional Brownian motion process $E_{\mu_\phi}\left[B^H(t)\left|\mathcal{F}_s^H\right.\right]$ is difficult to compute due to the correlation with the past. For this reason a system of quasi-conditional expectations is developed for fractional Brownian motion. The fractional conditional expectation $\tilde{E}$ is different from the ordinary expectation E. The definition of quasi-conditional expectation is quite involved and we will therefore not give it here, refer to Biagini, Hu, Øksendal and Zhang (2008).

Let $\mathcal{F}_t^H$ be the σ-algebra generated by fractional Brownian motion for $0 \leq s \leq t$, then we denote the fractional quasi-conditional expectation of G with respect to $\mathcal{F}_t^H$ by $\tilde{E}_{\mu_\phi}\left[G\left|\mathcal{F}_t^H\right.\right]$. We have for $0 \leq s \leq t$

$$\tilde{E}_{\mu_\phi}\left[B^H(t)\left|\mathcal{F}_s^H\right.\right] = B^H(s)$$

and fractional Brownian motion $B^H(t)$ is thus called a quasi-martingale. If $G \in L^2\left(\mu_\phi\right)$ then

$$\tilde{E}_{\mu_\phi}\left[G\left|\mathcal{F}_t^H\right.\right] = G \ a.s. \Leftrightarrow E_{\mu_\phi}\left[G\left|\mathcal{F}_t^H\right.\right] = G \ \ a.s.$$

Unfortunately the normal properties of conditional expectation do not hold. For instance for the normal conditional expectation if G is $\mathcal{F}_t^H$ measurable then

$$E\left[FG|\mathcal{F}_t^H\right] = GE\left[F|\mathcal{F}_t^H\right]$$

but this may not hold for quasi-conditional expectation (Hu and Peng, 2009). Another property of conditional expectation that does not hold is monotonicity. Namely if $\xi_1 \geq \xi_2$, then

$$E\left[\xi_1|\mathcal{F}_t^H\right] \geq E\left[\xi_2|\mathcal{F}_t^H\right].$$

In fact Hu and Peng (2009) show there may be $\xi \geq 0$ a.s. such that $\tilde{E}\left[\xi|\mathcal{F}_t^H\right] < 0$ with a positive probability. However quasi-conditional expectation will be monotone on some subsets. The quasi-conditional expectation of the wick product of F, G is

$$\tilde{E}_{\mu_\phi}\left[F \diamond G\left|\mathcal{F}_t^H\right.\right] = \tilde{E}_{\mu_\phi}\left[F\left|\mathcal{F}_t^H\right.\right] \diamond \tilde{E}_{\mu_\phi}\left[G\left|\mathcal{F}_t^H\right.\right].$$

Denote $D_s^\phi F = \int_{\mathbb{R}} \phi(s,t) D_t F dt$, where $D_t F$ refers to the Malliavin derivative. Refer to the appendix D for the Malliavin derivative. Let $\mathcal{L}_\phi^{1,2}(\mathbb{R})$ denote the completion of the set of all $\mathcal{F}_t^H$-adapted processes $f(t) = f(t,\omega)$ such that

$$\|f\|^2_{\mathcal{L}_\phi^{1,2}(\mathbb{R})} = E_{\mu_\phi}\left[\int_{\mathbb{R}}\int_{\mathbb{R}} f(s) f(t) \phi(s,t) \, ds dt\right] + E_{\mu_\phi}\left[\left(\int_{\mathbb{R}} D_s^\phi f(s) \, ds\right)^2\right] < \infty. \tag{5.13}$$

Lemma 5.3 *(Necula, 2002).*

a) If we let $f \in L_H^2(\mathbb{R})$ and

$$\varepsilon(t) = \exp\left(\int_0^t f(s) \, dB^H(s) - \frac{1}{2}\left|\left|f\chi_{[0,t]}\right|\right|_\phi^2\right)$$

then $\varepsilon(t)$ is a quasi-martingale.

b) Let $f \in \mathcal{L}_\phi^{1,2}(\mathbb{R})$ and

$$M(t) = \int_0^t f(s,\omega) \, dB^H(s) \tag{5.14}$$

then $M(t)$ is a quasi-martingale.

We refer the reader to equation (2.5) for the Clark-Ocone theorem. Before we introduce the fractional Clark-Ocone theorem we have to define the Hida Malliavin derivative.

Definition 5.4 *(Biagini, Hu, Øksendal and Zhang, 2008). Let $F : \mathcal{S}'(\mathbb{R}) \to \mathbb{R}$ be a given function and let $\gamma \in \mathcal{S}'(\mathbb{R})$. Then F has a directional derivative in $(\mathcal{S})_H^*$ in the direction of γ if the following*

$$D_\gamma^H F(\omega) = \lim_{\varepsilon \to 0} \frac{F(\omega + \varepsilon\gamma) - F(\omega)}{\varepsilon}$$

exists in $(\mathcal{S})_H^*$. *The function* $F : \mathcal{S}'(\mathbb{R}) \to \mathbb{R}$ *is differentiable if there exists a map* $\Psi : \mathbb{R} \to (\mathcal{S})_H^*$ *such that*

$$\Psi(t)\gamma(t) = \Psi(t,\omega)\gamma(t)$$

is integrable in $(\mathcal{S})_H^*$. *Then for all* $\gamma \in L^2(\mathbb{R})$ *we have*

$$D_\gamma^H F(\omega) = \int_{\mathbb{R}} \Psi(t,\omega)\gamma(t)\,dt.$$

The stochastic gradient (Hida Malliavin derivative) of F *at* t *is defined as*

$$\begin{aligned} D_t^H F(\omega) &= \frac{dF}{d\omega}(t,\omega) \\ &= \Psi(t). \end{aligned}$$

Example 5.4 *(Biagini, Hu, Øksendal and Zhang, 2008). If* $F(\omega) = \langle \omega, f \rangle = \int_{\mathbb{R}} f(t)\,dB^H(t,\omega)$ *then for some* $f \in \mathcal{S}(\mathbb{R})$ *and* $\gamma \in L^2(\mathbb{R})$ *then the directional derivative in the direction of* γ *is*

$$\begin{aligned} D_\gamma^H F(\omega) &= \lim_{\varepsilon \to 0} \frac{F(\omega + \varepsilon\gamma) - F(\omega)}{\varepsilon} \\ &= \lim_{\varepsilon \to 0} \frac{[\langle \omega + \varepsilon\gamma, f \rangle - \langle \omega, f \rangle]}{\varepsilon} \\ &= \lim_{\varepsilon \to 0} \frac{\langle \varepsilon\gamma, f \rangle}{\varepsilon} \\ &= \langle \gamma, f \rangle \\ &= \int_{\mathbb{R}} f(t)\gamma(t)\,dt. \end{aligned}$$

It follows by the definition that F *is differentiable and the stochastic gradient of* $F(\omega)$ *is given as*

$$D_t^H F(\omega) = f(t)$$

for almost all (t,ω).

We say G is $\mathcal{F}_T^H$-measurable$_{H,\emptyset}$ if

$$\tilde{E}_{\mu_\phi}\left[G | \mathcal{F}_t^H\right] = G.$$

Note that this differs from the usual definition of $\mathcal{F}_T^H$-measurability.

Hu and Øksendal (2000) use the fractional Clark-Ocone theorem to show completeness in the market.

Theorem 5.2 *A fractional Clark-Ocone theorem.*

a) Let $G(\omega)$ *be* $\mathcal{F}_T^H$*-measurable*$_{H,\emptyset}$*, then the quasi-conditional expectation* $\tilde{E}_{\mu_\phi}\left[D_t^H G \left| \mathcal{F}_t^H\right.\right] \diamond W^H(t)$ *is integrable in* $(S)_H^*$ *and*

$$G(\omega) = E_{\mu_\phi}[G] + \int_0^T \tilde{E}_{\mu_\phi}\left[D_t^H G \left| \mathcal{F}_t^H\right.\right] \diamond W^H(t)\, dt.$$

b) Suppose $G(\omega)$ *is* $\mathcal{F}_T^H$*-measurable*$_{H,\emptyset}$*. Define*

$$\Psi(t,\omega) = \tilde{E}_{\mu_\phi}\left[D_t^H G \left| \mathcal{F}_t^H\right.\right]$$

for $t \in [0,T]$*. Then*

$$G(\omega) = E_{\mu_\phi}[G] + \int_0^T \tilde{E}_{\mu_\phi}\left[D_t^H G \left| \mathcal{F}_t^H\right.\right] dB^H(t).$$

*For the proof see Biagini, Hu, Øksendal and Zhang (*2008*, page 85).*

Bender (2003) proves that theorem (5.2) is not well-defined and we give an alternative fractional Clark-Ocone theorem in theorem (5.5).

5.7 An Alternative Fractional Brownian Motion

In this section we look at Bender (2003)(a), Bender (2003) (b), Bender (2003) (c) and Bender and Elliot (2002) perspective on fractional Brownian motion, the fractional Clark-Ocone theorem and fractional White noise.

5.7.1 Setup

For $0 \leq t < \infty$, let $B^{(1)}(t)$ and $B^{(2)}(t)$ be two continuous independent Brownian motions on a probability space $(\Omega, \mathcal{A}, P)$. Let $\mathcal{F}'$ denote the σ-field generated by these two Brownian motions and let $\mathcal{N}$ be the Null sets of $\mathcal{F}'$ given as

$$\mathcal{N} = \{G; \exists_{F \in \mathcal{F}'} G \subset F \wedge P(F) = 0\}$$

Let $\mathcal{F} = \sigma(\mathcal{F}' \cup \mathcal{N})$ and let $(L^2) = L^2(\Omega, \mathcal{F}, P)$ be the space of square integrable random variables on a probability space $(\Omega, \mathcal{F}, P)$ and $\|\cdot\|_0$ be the norm. For $i = 1, 2$ and $0 \leq s \leq t$, denote $\mathcal{F}'(i)$ the augmentation of the σ-fields $\sigma(B^i(s))$. Let $s, t \in \mathbb{R}$ be real numbers and let $0 < H < 1$ be the Hurst parameter, then a continuous stochastic process $B^H(t)$ is a called a two sided fractional Brownian motion if it follows a covariance structure as defined in definition 3.1. When $H = \frac{1}{2}$ we get a two sided Brownian motion. For $i = 1, 2$ let $\mathcal{F}^{(i)} = \sigma(\mathcal{F}'(i) \cup \mathcal{N})$ and $\mathcal{F}_t^{(i)}$ be measurable process. Let $X(t)$ be $\mathcal{F}_t^{(1)}$-adapted and $X(-t)$ be $\mathcal{F}_t^{(2)}$-adapted measurable process such that $X : \Omega \times \mathbb{R} \to \mathbb{R}$ and

$$\int_{\mathbb{R}} E_P\left[X^2(t)\right] dt < \infty.$$

Lemma 5.4 *A process $X(t)$ is a two sided Brownian motion if and only if the processes $X^{(1)}(t) = X(t)$ and $X^{(2)}(t) = X(-t)$ are independent Brownian motions.*

From lemma 5.4 we set

$$B(t) = B^{\frac{1}{2}}(t) = \begin{cases} B^{(1)}(t), & \text{if } t \geq 0 \\ B^{(2)}(-t), & \text{if } t < 0 \end{cases}.$$

Define stochastic integrals driven by Brownian motion as a two sided Itô integral

$$\int_{\mathbb{R}} X(t)\, dB(t) = \int_0^\infty X(s)\, dB^{(1)}(s) - \int_0^\infty X(-s)\, dB^{(2)}(s) \qquad (5.15)$$

and we have

$$E_P\left[\int_{\mathbb{R}} X(s)\, dB(s) \int_{\mathbb{R}} Y(s)\, dB(s)\right] = E_P\left[\int_{\mathbb{R}} X(t)\, Y(t)\, dt\right].$$

For $a, b \in \mathbb{R}$ let the indicator function be given as

$$1(a,b)(t) = \begin{cases} 1, & \text{if } a \leq t < b \\ -1, & \text{if } b \leq t < a \\ 0, & \text{otherwise} \end{cases},$$

then we can represent the classical Brownian motion as follows

$$B(t) = \int_{\mathbb{R}} 1(0,t)(s)\, dB(s).$$

Let $f \in L^2(\mathbb{R})$ be a deterministic function and denote the Wiener integral by

$$I(f) = \int_{\mathbb{R}} f(t)\, dB(t).$$

For f and $g \in L^2(\mathbb{R})$ we define the inner product as

$$(f,g)_0 = \int_{\mathbb{R}} f(s)\, g(s)\, ds$$

with corresponding norm $|\cdot|_0$. For $x > 0$ the Gamma function is given as

$$\Gamma(x) = \int_0^{\infty} s^{x-1} e^{-s} ds.$$

Let

$$s_+^{H-\frac{1}{2}} = \begin{cases} s^{H-\frac{1}{2}} & \text{if } s > 0 \\ 0 & \text{otherwise} \end{cases},$$

then for $H \neq \frac{1}{2}$ we define a continuous modified fractional Brownian motion

$$\tilde{B}^H(t) = \tilde{K}_H \int_{\mathbb{R}} \left((t-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}} \right) dB(s)$$

with normalizing constant

$$\tilde{K}_H = \Gamma \left(\frac{2H\Gamma\left(\frac{3}{2} - H\right)}{\Gamma(H+1)\,\Gamma(2-2H)} \right)^{\frac{1}{2}}.$$

(See Bender (2003) (b), page 3). Denote fractional Brownian motion by $B^H(t)$. It follows the same properties as in chapter 3. Recall that $B^H(t)$ not a semi-martingale and consider a partition $a = \tau_0 < \tau_1 < ... < \tau_N = b$, then for $i = 0, .., N$, $B^H(\tau)$ is nowhere differentiable, in fact

$$\lim_{\tau \to \tau_0} \sup \left| \frac{B^H(\tau) - B^H(\tau_0)}{\tau - \tau_0} \right| = \infty.$$

We are now going to define the stochastic integral with respect to fractional Brownian motion.

Consider stopping times $0 \leq t_0 \leq t_1 \leq ... \leq t_n$, for $k \in \mathbb{R}$ let H_k be $\mathcal{F}_{t_{k-1}}$ measurable random variables and let $Y(0)$ be a constant. Consider a simple process of the form

$$Y(t) = Y(0) + \sum_{k=1}^{n} H_k 1(t_{k-1}, t_k)(t).$$

Consider a stochastic process $\mathbf{X}$ which is a semimartingale and has the form $(X_t, \mathcal{F}_t)$. Then $\mathbf{YX}$ is a stochastic process defined to be

$$\mathbf{YX} = Y(0) X(0) + \sum_{k=1}^{n} H_k (X(t_k) - X(t_{k-1})).$$

Let $\diamond$ denote the Wick product (see Bender 2003 (b), page 7), then we have

$$\sum_{k=1}^{n} H_k (B(t_k) - B(t_{k-1})) = \sum_{k=1}^{n} H_k \diamond (B(t_k) - B(t_{k-1}))$$

and

$$\mathbf{Y}_\diamond \mathbf{B}^H = \sum_{k=1}^{n} H_k \diamond \left(B^H(t_k) - B^H(t_{k-1})\right)$$

with $E\left[\mathbf{Y}_\diamond \mathbf{B}^H\right] = 0$.

Wick exponentials of $I(f)$ are defined as

$$: e^{I(f)} := e^{I(f) - \frac{1}{2}|f|_0^2}$$

which is an element of (L^2) with

$$E_P\left[: e^{I(f)} :\right] = \frac{1}{\sqrt{2\pi}\,|f|_0} \int_{\mathbb{R}} \exp\left\{u - \frac{1}{2}\left(|f|_0^2 + \frac{u^2}{|f|_0^2}\right)\right\} du = 1$$

and $E\left[: e^{I(f)} :: e^{I(f)} :\right] = E\left[e^{(f,g)_0} : e^{I(f+g)} :\right] = e^{(f,g)_0}$. Bender $(2003(b)$, page 9) proves that if we let f_n converge to f in $L^2(\mathbb{R})$, then $: e^{I(f_n)} :$ converges to $: e^{I(f)} :$ in (L^2). We can also write

$$\begin{aligned} : e^{I(f)} : &= \exp\left\{\int_0^\infty f(s)\,dB^{(1)}(s) - \frac{1}{2}\int_0^\infty f(s)^2\,ds\right\} \\ &\quad \times\left\{-\int_0^\infty f(-s)\,dB^{(2)}(s) - \frac{1}{2}\int_0^\infty f(-s)^2\,ds\right\} \end{aligned}$$

then from the Girsanov theorem we see that we can use the wick exponential to define a new measure Q_f equivalent to P as

$$dQ_f =: e^{I(f)} : dP$$

under which $\left(B^{(1)}(t) - \int_0^t f(s)\,ds,\ B^{(2)}(t) + \int_0^t f(-s)\,ds\right)$ for $0 \leq t < \infty$ is a two-dimensional Brownian motion and $B(t) + \int_0^t f(s)\,ds$ for $t \in \mathbb{R}$ is a two-sided Brownian motion (see Bender $(2003(b)$, page 10)).

We refer the reader to theorem (5.1) for the fractional Girsanov as presented by Biagini, Hu, Øksendal and Zhang (2008).

We will now define the $\mathcal{S}$-transform, refer to definition $(C.3)$ in appendix C for the $\mathcal{S}$-transform in terms of the Wick exponential. The $\mathcal{S}$-transform transforms stochastic problems into deterministic ones. Let $F \in (L^2)$ be a random variable and let $\mathcal{S}(\mathbb{R})$ be the Schwartz space on $\mathbb{R}$, then for $\eta \in \mathcal{S}(\mathbb{R})$ the $\mathcal{S}$-transform is defined as

$$\mathcal{S}F(\eta) = E_{Qn}[F]$$

which is a mapping from (L^2) into $\mathcal{S}(\mathbb{R})$ and $E_{Q_n}$is the expectation under the measure Q_n. The $\mathcal{S}$-transform is linear and injective. The $\mathcal{S}$-transform of (5.15) is given as

$$\mathcal{S}\left(\int_{\mathbb{R}} X(s)\,dB(s)\right) = \int_{\mathbb{R}} \mathcal{S}(X(s))(\eta)\,\eta(s)\,ds,$$

see Bender $2003(b)$, page 11.

5.7.2 Fractional Brownian motion

Fractional Brownian motion can be expressed in terms of operators and the indicator function. We introduce the Riemann-Liouville fractional integral (see Miana (2005) for further details on the integral) as the integral provides us with the link to get the covariance structure of fractional Brownian motion.

For $0 < \alpha < 1$ define fractional integrals of Riemann-Liouville type as

$$\left(I_{\pm}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_{0}^{\infty} f(x \mp t)\, t^{\alpha-1} dt.$$

Let $I^{\alpha}(L^{p})$ be a space see Bender 2003 (b). Let $\alpha \in (0,1)$, $1 \leq p < \alpha^{-1}$ and $\varepsilon > 0$ and $f \in I^{\alpha}(L^{p})$ then fractional derivatives of Marchaud's type are

$$\left(I_{\pm,\varepsilon}^{-\alpha} f\right)(x) = \lim_{\varepsilon \to 0+} \frac{\alpha}{\Gamma(1-\alpha)} \int_{\varepsilon}^{\infty} \frac{f(x) - f(x \mp t)}{t^{\alpha+1}} dt$$

if the limit exist is $L^{p}(\mathbb{R})$. For $\eta \in \mathcal{S}(\mathbb{R})$ we have the Riemann-Liouville fractional derivative

$$\left(I_{\pm,\varepsilon}^{-\alpha} \eta\right)(s) = \pm \frac{d}{ds} \left(I_{\pm,\varepsilon}^{-\alpha} \eta\right)(s)$$

and Bender $(2003\,(b)$, page 29) provides us with a useful integration by parts rule

$$\int_{\mathbb{R}} f(x) \left(I_{+}^{\alpha} g\right)(x)\, dx = \int_{\mathbb{R}} g(x) \left(I_{-}^{\alpha} f\right)(x)\, dx$$

that holds for some conditions and for all $t \in \mathbb{R}$ and $\alpha \in \left(-\frac{1}{2}, 0\right) \cup \left(0, \frac{1}{2}\right)$ we have

$$\left(I_{-}^{\alpha} 1(0,t)\right)(s) = \frac{1}{\Gamma(\alpha+1)} \left[(t-s)_{+}^{\alpha} - (-s)_{+}^{\alpha}\right].$$

Put M to be a martingale with zero expectation and covariance. For $0 < H < 1$ define the operators $M_{\pm}^{H}$ by

$$M_{\pm}^{H} f = \begin{cases} K_{H} I_{\pm}^{H-\frac{1}{2}} f, & H \neq \frac{1}{2} \\ f, & H = \frac{1}{2} \end{cases}$$

with

$$K_H = \Gamma\left(H+\frac{1}{2}\right)\Gamma\left(\frac{2H\Gamma\left(\frac{3}{2}-H\right)}{\Gamma\left(H+1\right)\Gamma\left(2-2H\right)}\right)^{\frac{1}{2}}$$

see Bender 2003 (b), page 29.

We summarize some important results.

Lemma 5.5 *(Bender 2003 (a), 2003 (b) and 2003 (c)). In some spaces under some conditions we have*

1. *M_-^H and M_+^H are dual operators such that $\left(f, M_-^H g\right)_0 = \left(M_+^H f, g\right)_0$.*
2. *$\left(f, M_-^H 1(0,t)\right)_0 = \int_0^t \left(M_+^H f\right)(s)\,ds$.*
3. *$M_+^H f$ is continuous.*
4. *$\left(f, M_-^H 1(0,t)\right)_0$ is differentiable.*
5. *$\frac{d}{dt}\left(f, M_-^H 1(0,t)\right)_0 = M_+^H f(t)$.*
6. *$\frac{d^n}{ds^n}\left(M_\pm^H \eta\right)(s) = M_\pm^H\left(\frac{d^n}{ds^n}\eta\right)(s)$.*
7. *$B_t^H = I\left(M_-^H 1(0,t)\right)$ is a fractional Brownian motion.*
8. *$\int_{\mathbb{R}} f(s)\,dB^H(s) = \int_{\mathbb{R}}\left(M_-^H f\right)(s)\,dB(s)$.*
9. *$\mathcal{S}B_t^H(\eta) = \int_0^t M_+^H \eta(s)\,ds$.*
10. *$\mathcal{S}\left(\int_{\mathbb{R}} f(s)\,dB^H(s)\right)(\eta) = \int_{\mathbb{R}} f(s)\left(M_+^H \eta\right)(s)\,ds$.*
11. *$\mathcal{S}(F \diamond G)(\eta) = \mathcal{S}F(\eta)\,\mathcal{S}G(\eta)$.*

5.7.3 Fractional Clark-Ocone

Let $n \in \mathbb{N}$. We consider an n-dimensional space of symmetric functions and let $L_H^2(\mathbb{R}^n)$ be the muliti-dimensional analogue of $L_H^2(\mathbb{R})$. For $f \in L^p(\mathbb{R}^n)$ and $f : \mathbb{R}^n \to \mathbb{R}$ and for $0 < \alpha < 1$ we let $I_{\pm}^{\alpha,n} f = \left(I_{\pm}^{\alpha}\right)^{\otimes n} f$ be the n-dimensional Riemann-Liouville fractional integrals which are symmetric; a generalized version in terms of tensor powers. Define n-dimensional operators as $M_{\pm}^{H,n} f = K_H^n I_{\pm}^{H-\frac{1}{2},n} f$. For $H = \frac{1}{2}$ the operators $M_{\pm}^{\frac{1}{2},n}$ are identity mappings on the space of measurable functions from $\mathbb{R}^n$ to $\mathbb{R}^n$. For $0 < H < \frac{1}{2}$ we have a space $\widehat{L_H^2(\mathbb{R}^n)} = \left\{ I_{-}^{\frac{1}{2}-H,n} f,\ f \in L^2(\mathbb{R}^n) \text{ symmetric} \right\}$ endowed with a inner product $(f,g)_{L_H^2(\mathbb{R})} = \left(M_{-}^{H,n} f, M_{-}^{H,n} g \right)_0$. For $\frac{1}{2} < H < 1$ we consider the space $\left| \widehat{L_H^2(\mathbb{R}^n)} \right|$ which is a multi-dimensional analogue of the space $\left| L_H^2(\mathbb{R}^n) \right|$, this is the space of symmetric functions $f \in L^p(\mathbb{R}^n)$ for some $1 < p < \frac{1}{\left(H - \frac{1}{2}\right)}$.

Theorem 5.3 *For $n \in \mathbb{N}$ and if $H \in \left(0, \frac{1}{2}\right)$ then $f_n \in \widehat{L_H^2(\mathbb{R}^n)}$ or if $H \in \left(\frac{1}{2}, 1\right)$ then $f_n \in \left| \widehat{L_H^2(\mathbb{R}^n)} \right|$. Then we can denote the iterated fractional Itô integral of order n as*

$$\begin{aligned} I_n^H(f_n) &= I_n\left(M_{-}^{H,n} f_n\right) \\ &= n! \int_{\mathbb{R}} \int_{-\infty}^{t_n} \cdots \int_{-\infty}^{t_2} f_n(t_1, .., t_n)\, dB_{t_1}^H \ldots dB_{t_{n-1}}^H dB_{t_n}^H \end{aligned}$$

if it exists.

For the proof see Bender (2003(b), page 85).

We now present the fractional chaos expansion theorem in terms of iterated fractional Itô integrals.

Theorem 5.4 *For $H \in \left(0, \frac{1}{2}\right)$, $F \in (L^2)$, $f_0 \in \mathbb{R}$ and $f_n \in \widehat{L_H^2(\mathbb{R}^n)}$ there is a unique sequence $(f_n)_{n \in \mathbb{N}_0}$ such that F has a fractional chaos decomposition given as*

$$\begin{aligned} F &= \sum_{n=0}^{\infty} I_n^H(f_n) \qquad (5.16) \\ &= \sum_{n=0}^{\infty} n! \int_{\mathbb{R}} \int_{-\infty}^{t_n} \cdots \int_{-\infty}^{t_2} f_n(t_1, .., t_n)\, dB_{t_1}^H \ldots dB_{t_{n-1}}^H dB_{t_n}^H \end{aligned}$$

which converges in (L^2). *The* (L^2)*-norm of* F *is given in terms of the fractional chaos decomposition by*

$$E_P\left[F^2\right] = \sum_{n=0}^{\infty} n! \left|f_n\right|^2_{L^2_H(\mathbb{R}^n)}.$$

For the proof see Bender (2003(b), page 86).

For $H > \frac{1}{2}$, $F \in (L^2)$ and $f_n \in \left|\widehat{L^2_H(\mathbb{R}^n)}\right|$ the fractional chaos decomposition is the same but now an element of (L^2_H) which is a subspace of (L^2). See Bender 2003(b), page 87 for further details.

For $-\infty \leq a \leq b \leq \infty$, let $\mathcal{F}^H_{[a,b]}$ denote the augmentation of the filtration generated by $\left(B^H(s)\right)_{s\in[a,b]}$. Let $0 < H < \frac{1}{2}$ and $f \in L^2_H(\mathbb{R})$ with support in $[a,b]$, then $\int_{\mathbb{R}} f(s)\, dB^H(s)$ is $\mathcal{F}^H_{[a,b]}$-measurable. We define the H-quasi-conditional expectation of $F \in (L^2_H)$ with respect to $\mathcal{F}^H_{[a,b]}$ as

$$\begin{aligned}
\tilde{E}_P\left[F|\mathcal{F}^H_{[a,b]}\right] &= \sum_{n=0}^{\infty} n! \int_0^T \int_0^{t_n} ... \int_0^{t_2} f_n(t_1,..,t_n)\, dB^H_{t_1}...dB^H_{t_{n-1}} dB^H_{t_n} \\
&= \sum_{n=0}^{\infty} I^H_n\left(1(a,b)^{\otimes n} f_n\right) \\
&= \sum_{n=0}^{\infty} I_n\left(M^{H,n}_{-}\left(1(a,b)^{\otimes n} f_n\right)\right)
\end{aligned}$$

if the series converges in (L^2). This is the same as Hu and Øksendal but Bender proves that this is not well-defined because the fractional chaos is not unique. For $H > \frac{1}{2}$ Bender (2003)(b) constructs a (L^2)-random variable for which not all quasi-conditional expectation exists in (L^2).

We note that $I^H_n(f_n) = I(f)^{\diamond n}$ where $I(f)^{\diamond n} = I(f)^{\diamond n-1} \diamond I(f)$. For $\frac{1}{2} < H < 1$ let $F \in (L^2_H)$ be given as in (5.16), then F is called fractional Malliavin differentiable if

$$D^H_t F = \sum_{n=1}^{\infty} n I^H_{n-1}\left(f_n(\cdot,t)\right)$$

converges in (L^2) for almost all $t \in \mathbb{R}$. Let $\mathcal{D}_H^{1,2}$ be the space of H-fractionally Malliavin differentiable random variables such that

$$\int_{\mathbb{R}} \int_{\mathbb{R}} E_P \left[\left| D_t^H F \right| \cdot \left| D_s^H F \right| \right] |t-s|^{2H-2} \, ds dt < \infty.$$

If $F \in (L_H^2)$ and is H-fractionally Malliavin differentiable then the fractional Clark-Ocone derivative is the integral representation for a class of (L^2)-random variables in terms of fractional Itô integrals at time t and is given by

$$\nabla_t^H F = \tilde{E}_P \left[D_t^H F | \mathcal{F}_{[0,t]}^H \right]$$

if expectation exist in (L^2), but this again does not hold for some variables. So we put $\left| \mathcal{D}_H^{1,2} \right|$ to be a space of f_n such that

$$\sum_{n=1}^{\infty} n n! \int_{\mathbb{R}^n} \left(M^{H,n} \left(|f_n| \right) (t) \right)^2 dt < \infty$$

and by proposition 3.3.3 in Bender 2003(b) , page 93 we have$\left| \mathcal{D}_H^{1,2} \right| \subset \mathcal{D}_H^{1,2}$.

Theorem 5.5 *Let* $F \in \left| \mathcal{D}_H^{1,2} \right|$ *be* $\mathcal{F}_T^H$*-measurable*$_{H,\emptyset}$ *then at almost every time* $t \in [0,T]$ *the fractional Clark-Ocone derivative of* F *exists which satisfies*

$$\int_0^T \int_0^T E_P \left[\left| \nabla_t^H F \right| \cdot \left| \nabla_s^H F \right| \right] |t-s|^{2H-2} \, ds dt < \infty$$

and is given as

$$F = E_P [F] + \int_0^T \nabla_t^H F dB^H (t) \, .$$

For the proof see Bender (2003(b), page 94).

Bender (2003) also explicitly shows the fractional Clark-Ocone derivative for functionals of fractional Wiener integrals.

5.7.4 Fractional White Noise

Let E be a Hilbert space (function space) and let $\mathcal{E}$ (space of test functions) be a subspace that will be endowed with a metric, and $\mathcal{E}'$ (space of generalized functions) it's dual and $\mathcal{E} \subset E \subset \mathcal{E}'$. Let $(E, (\cdot,\cdot)_0)$ be a separable, infinite dimensional (real or complex) Hilbert space and let $B : E \to E$ be a self-adjoint, injective, compact operator with $\|B\| < 1$. For $n \in \mathbb{N}$ we define the linear space $E_n = B^n(E)$, where B^n is related to B. We define an inner product on the space E_n by

$$(f, g)_n = \left(B^{-n} f, B^{-n} g\right)_0 ,$$

with this choice $(E_n, (\cdot,\cdot)_n)$ becomes a Hibert space which can be shown to be isometrically isomorphic to $(E, (\cdot,\cdot)_0)$.

Theorem 5.6 *Let $\mathcal{E} = \cap_{n\in\mathbb{N}} E_n$. Then*

1. *$(\cdot,\cdot)_n$ defines an increasing ordered inner product on $\mathcal{E}$.*

2. *The corresponding sequence of norms $|\cdot|_n$ is compatible.*

3. *$\mathcal{E}$ is a complete space with respect to the metric*

$$d(x, y) = \sum_{n=0}^{\infty} 2^{-n} \frac{|x - y|_n}{1 + |x - y|_n}.$$

4. *Let E_n' denote the completion of $\mathcal{E}$ with respect to the norm $|\cdot|_n$. For $m \in \mathbb{N}$ there is an $n > m$ such that the embedding of E_n' into E_m' is a Hilbert-Schmidt operator.*

For the proof see Bender (2003(b), page 113).

Definition 5.5 *(Bender 2003(b), page 113). A linear space $\mathcal{E}$ together with a sequence of inner products $(\cdot,\cdot)_n$ satisfying conditions (1)-(4) is a nuclear space.*

On $\mathbb{R}^d$, recall that $\mathcal{S}\left(\mathbb{R}^d\right)$ denotes the space of tempered distributions and $\mathcal{S}'\left(\mathbb{R}^d\right)$ is the space of continuous linear functionals on $\mathcal{S}\left(\mathbb{R}^d\right)$, i.e. the dual space. We can represent a triplet $\mathcal{S}'\left(\mathbb{R}^d\right) \subset L^2\left(\mathbb{R}^d\right) \subset \mathcal{S}\left(\mathbb{R}^d\right)$. The nuclear space $\mathcal{S}\left(\mathbb{R}^d\right)$ can be constructed from $L^2\left(\mathbb{R}^d\right)$ see Bender (2003, page) Let $A \subset \mathbb{R}^m$ be a Borel set, $\eta_1, .., \eta_m \in \mathcal{S}\left(\mathbb{R}^d\right)$, then a set of the form $\left\{F \in \mathcal{S}'\left(\mathbb{R}^d\right); (F(\eta_1), .., F(\eta_m)) \in A\right\}$ is called a cylinder set in $\mathcal{S}'\left(\mathbb{R}^d\right)$. Since $\mathcal{S}(\mathbb{R})$ is a nuclear space then by the Bochner-Minlos theorem there exists a unique probability measure P on the space $(\mathcal{S}'(\mathbb{R}), \mathcal{F})$, where $\mathcal{F}$ is the σ-field generated by the cylinder sets, such that for all $\eta \in \mathcal{S}(\mathbb{R})$ we have

$$\int_{\mathcal{S}'(\mathbb{R})} \exp\{i\langle\omega,\eta\rangle\}\, dP(\omega) = \exp\left\{-\frac{1}{2}|\eta|_0^2\right\}$$

where $\langle\omega,\eta\rangle$ denotes the dual pairing $\omega(\eta)$ and P is the standard Gaussian measure on $\mathcal{S}'(\mathbb{R})$. The probability space $(\mathcal{S}'(\mathbb{R}), \mathcal{F}, P)$ is called the white noise space. For $\eta \in \mathcal{S}(\mathbb{R})$ the random variable $\langle\cdot,\eta\rangle$ is a centered Gaussian with covariance

$$E_P\left[\langle\cdot,\eta\rangle^2\right] = |\eta|_0^2.$$

Bender 2003 (b), page 116 constructs a nuclear space starting from (L^2) and the operator $\Gamma(B)$ defined in terms of the chaos decomposition We define the inductive limit of the sequence $(\mathcal{S})_{-n}$ to be the completion of (L^2) with respect to $\|\cdot\|_{-n}$. The nuclear space is obtained as projective limit of the sequence of Hilbert spaces $(\mathcal{S})_n$.

Definition 5.6 *(Bender 2003 (b), page 117). The space of Hida test functions is*

$$(\mathcal{S}) = \cap_{n\in\mathbb{N}} (\mathcal{S})_n$$

and the space of Hida distributions is

$$(\mathcal{S})^* = \cup_{n\in\mathbb{N}} (\mathcal{S})_{-n}.$$

It follows that $(\mathcal{S})^*$ is the dual space of $(\mathcal{S})$. Let $F \in (\mathcal{S})^*$ and $\Phi \in (\mathcal{S})$ then we denote the dual pairing by $\langle\langle F, \Phi\rangle\rangle$.

Example 5.5 *(Bender 2003 (b), page 117). For all $\eta \in \mathcal{S}(\mathbb{R})$ the Wick exponential $: e^{I(\eta)} :$ is a Hida test function.*

The Hida distribution is uniquely determined by it's $\mathcal{S}$-transform.

Definition 5.7 *(Bender 2003 (b), page 117). The generalized $\mathcal{S}$-transform is a mapping from $(\mathcal{S})^*$ into $\mathcal{S}(\mathbb{R})$ and is defined by*

$$\mathcal{S}F(\eta) = \left\langle\left\langle F; : e^{I(\eta)} : \right\rangle\right\rangle.$$

The generalized $\mathcal{S}$-transform is injective.

Let $I \subset \mathbb{R}$ be an interval then a mapping $X : I \to (\mathcal{S})^*$ is called a stochastic distribution process. Then X is differentiable if $\lim_{h\to 0} h^{-1}(X(t+h) - X(t))$ exists in $(\mathcal{S})^*$. Let $F : I \to \mathcal{S}'(\mathbb{R})$ be differentiable then $\langle \cdot, F(t)\rangle$ is a differentiable stochastic distribution process and $\frac{d}{dt}\langle \cdot, F(t)\rangle = \left\langle \cdot, \frac{d}{dt}F(t)\right\rangle$. For $0 < H < 1$ we have the operator $M_-^H \mathbf{1}(0, \cdot)$ which maps $\mathbb{R} \to \mathcal{S}'(\mathbb{R})$ and is differentiable and continuous. We can expand this operator in terms of Hermite polynomials so for $n = 0, 1, ...$the n-th Hermite polynomial is defined as $H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$ and the n-th Hermite function is defined as

$$\xi_n(x) = \pi^{-\frac{1}{4}} (2^n n!)^{-\frac{1}{2}x^2} H_n(x).$$

The derivative of the operator is

$$\frac{d}{dt} M_-^H 1(0,t) = \sum_{k=0}^{\infty} \left(M_+^H \xi_k\right)(t)\, \xi_k$$

with limit in $\mathcal{S}'(\mathbb{R})$. Using fractional integration by parts we have

$$\begin{aligned}
\left\langle \frac{d}{dt} M_-^H \mathbf{1}(0,t), \eta \right\rangle &= \frac{d}{dt}\left\langle M_-^H 1(0,t), \eta\right\rangle \\
&= \frac{d}{dt}\left\langle \sum_{k=0}^{\infty} \left(M_-^H 1(0,t), \xi_k\right)_0 \xi_k, \eta \right\rangle \\
&= \frac{d}{dt}\left\langle \sum_{k=0}^{\infty} \int_0^t \left(M_+^H \xi_k\right)(s)\, ds \xi_k, \eta \right\rangle \\
&= \left(M_+^H \eta\right)(t).
\end{aligned}$$

A modified fractional Brownian motion is given as $\mathbf{B}^H = \left\langle \cdot, M_-^H \mathbf{1}(0,t) \right\rangle$ and we have

$$\frac{d}{dt} B^H(t) = \left\langle \cdot, \sum_{k=0}^{\infty} \left(M_+^H \xi_k \right)(t)\, \xi_k \right\rangle.$$

Let δ_t be the Dirac δ-function and define a distribution as

$$\left\langle \delta_t \circ M_+^H, f \right\rangle = \left(M_+^H f \right)(t)$$

where $\delta_t \circ M_+^H$ is a generalized Wiener integral. The derivative of $\mathbf{B}^H$ is referred to as fractional White noise and is defined as

$$W^H(t) = \left\langle \cdot, \delta_t \circ M_+^H \right\rangle \tag{5.17}$$

which is a mapping from $\mathbb{R}$ to $(\mathcal{S})^*$ (see Bender 2003 (b), page 120).

Equation (5.17) is fractional White noise and a representation of White noise is different and can be found in Hida, Kuo, Potthoff and Striet (1993).

Bender (2005) investigates solutions of fractional backward stochastic differential equations. Bender and Parczewski (2010) deal with discrete Wick calculus for fractional Brownian motion.

Chapter 6

Hu and Øksendal's fBm Pricing Model

6.1 Introduction

In this section the application of fractional Brownian motion using Wick Itô type integration to the markets will be discussed. Hu and Øksendal (2000) developed a fractional Black-Scholes market based on the Wick Itô type integration. The market is shown to be complete and free from strong arbitrage but the model is economically meaningless. Hu and Øksendal (2000) prove an explicit formula for a fractional Black-Scholes price of a European call option and also show explicitly the replicating portfolio of such an option. A Black formula is proven under their framework.

Consider a fractional market with an investment in a money market account or zero-coupon bond and a stock driven by fractional Brownian motion in a continuous setting $0 \leq t \leq T$. Let $r > 0$ be a constant riskless interest rate then the money market account $A(t)$ at time t develops according to the equation

$$\begin{aligned} dA(t) &= rA(t)\,dt \\ A(0) &= 1. \end{aligned}$$

Let μ be the drift of the stock and $\sigma \neq 0$ be the corresponding volatility, fixed for the tenure of the option. The stock price process has the following dynamics

$$\begin{aligned} dS(t) &= \mu S(t)\,dt + \sigma S(t)\,dB^H(t) \qquad (6.1) \\ S(0) &= s > 0. \end{aligned}$$

Applying (5.11) and using Wick calculus the solution of the system is given as

$$S(t) = s\exp\left(\sigma B^H(t) + \mu t - \frac{1}{2}\sigma^2 t^{2H}\right).$$

An economic interpretation is given in Biagini, Hu, Øksendal and Zhang (2008). We consider $S(t,\omega)$ as a stochastic distribution in ω which is an element of $(\mathcal{S})^*_H$. Assume that $S(t) = S(t,\cdot)$ is a generalized stock price process representing the total value of a company. This value is not observed directly. We let the actual observed stock price $\bar{S}(t)$ be a result of applying the stochastic distribution of $S(t)$ to a stochastic test function ψ known as market observers. We will assume that they have the following form

$$\begin{aligned}\psi(f) &= \exp^{\diamond}\left(\int_{\mathbb{R}} h(t)\,dB^H(t)\right)\\ &= \exp\left(\int_{\mathbb{R}} h(t)\,dB^H(t) - \frac{1}{2}\|f\|^2_\phi\right)\end{aligned}$$

for some $h \in L^2_H(\mathbb{R})$. We let $\mathcal{D}$ denote the set of all market observers. The observed price $\bar{S}(t)$ has the following form

$$\begin{aligned}\bar{S}(t) &= \langle S(t,\cdot), \psi(\cdot)\rangle\\ &= \langle S(t), \psi\rangle.\end{aligned}$$

The economic interpretation is that the price is obtained when a market observer or stock market confronts the general state of a company.

A portfolio or trading strategy $Z^\theta(t,\omega) = \theta(t,\omega) = (u(t), v(t))$ is an $\mathcal{F}^H_t$ adapted 2-dimensional process where $u(t)$ is the amount that is invested in the money account and $v(t)$ is the amount that is invested in the stock at time t. Let $v(t)$ be a buy and hold strategy. The generalized total wealth process is $Z^\theta(t) = Z^\theta(t,\cdot)$ and is given by

$$Z^\theta(t) = u(t)\,A(t) + v(t) \diamond S(t).$$

Lemma 6.1 *Let* $F, G \in (\mathcal{S})_H^*$. *Then*

$$\langle F \diamond G, \psi \rangle = \langle F, \psi \rangle \cdot \langle G, \psi \rangle$$

for all $\psi \in \mathcal{D}$.

For the proof see Biagini, Hu, Øksendal and Zhang, (2008, page 174).

Let the actual observed wealth value be

$$\begin{aligned} \bar{Z}^\theta(t) &= u(t) A(t) + \langle (v(t) \diamond S(t)), \psi \rangle \\ &= u(t) A(t) + \langle v(t), \psi \rangle \times \langle S(t), \psi \rangle \\ &= u(t) A(t) + \langle v(t, \cdot), \psi(\cdot) \rangle \times \langle S((t, \cdot), \psi(\cdot)) \rangle \\ &= u(t) A(t) + \bar{v}(t) \times \bar{S}(t) \end{aligned}$$

where the actual number of stocks is given as

$$\bar{v}(t) = \langle v(t, \cdot), \psi(\cdot) \rangle.$$

6.2 Self-Financing, Arbitrage and Completeness

Assume that the value process $Z^\theta(t) = Z^\theta(t, \omega)$ is given by

$$Z^\theta(t, \omega) = u(t) A(t) + v(t) \diamond S(t).$$

The portfolio is called self-financing if $t \in [0, T]$ we have

$$\begin{aligned} dZ^\theta(t, \omega) &= u(t) dA(t) + v(t) \diamond dS(t) \\ &= u(t) dA(t) + v(t) \diamond \left(\mu S(t) dt + \sigma S(t) dB^H(t)\right) \\ &= u(t) dA(t) + \mu v(t) \diamond S(t) dt + \sigma v(t) \diamond S(t) dB^H(t). \end{aligned} \quad (6.5)$$

Lemma 6.2 *Assets driven by the same fractional Brownian motion have the same market price of risk.*

Proof. Assume that we have two tradable assets S_1 and S_2 which are driven by fractional Brownian motion and a riskless asset $A(t)$. The associated dynamics follow

$$dS_1(t) = \mu_1 S_1(t)\,dt + \sigma_1 S_1(t)\,dB^H(t),$$

$$dS_2(t) = \mu_2 S_2(t)\,dt + \sigma_2 S_2(t)\,dB^H(t)$$

and

$$dA(t) = rA(t)\,dt.$$

Consider a self-financing portfolio consisting of an investment of $\alpha(t)$ units in $S^1(t)$, $\beta(t)$ units in $S^2(t)$ and $u(t)$ units in $A(t)$. Hence the portfolio becomes

$$\Pi(t) = \alpha(t) \diamond S_1(t) + \beta(t) \diamond S_2(t) + u(t)\,A(t).$$

Since the portfolio is self-financing we have

$$\begin{aligned}
& d\Pi(t) \\
= \; & \alpha(t) \diamond dS_1(t) + \beta(t) \diamond dS_2(t) + u(t)\,dA(t) \\
= \; & \alpha(t) \diamond \left(\mu_1 S_1(t)\,dt + \sigma_1 S_1(t)\,dB^H(t)\right) + \beta(t) \diamond \left(\mu_2 S_2(t)\,dt + \sigma_2 S_2(t)\,dB^H(t)\right) \\
& + u(t)\,(rA(t)\,dt) \\
= \; & \mu_1 \alpha(t) \diamond S_1(t)\,dt + \sigma_1 \alpha(t) \diamond S_1(t)\,dB^H(t) + \mu_2 \beta(t) \diamond S_2(t)\,dt \\
& + \sigma_2 \beta(t) \diamond S_2(t)\,dB^H(t) + u(t)\,rA(t)\,dt \\
= \; & (\mu_1 \alpha(t) \diamond S_1(t) + \mu_2 \beta(t) \diamond S_2(t) + u(t)\,rA(t))\,dt \\
& + (\sigma_1 \alpha(t) \diamond S_1(t) + \sigma_2 \beta(t) \diamond S_2(t))\,dB^H(t).
\end{aligned}$$

Setting

$$\sigma_1 \alpha(t) \diamond S_1(t) + \sigma_2 \beta(t) \diamond S_2(t) = 0$$

we let

$$\alpha(t) = \sigma_2 S_2(t) \text{ and } \beta(t) = -\sigma_1 S_1(t). \tag{6.6}$$

With this choice the stochastic term $dB^H(t)$ gets eliminated. Thus the portfolio dynamics reduces to

$$d\Pi(t) = (\mu_1 \alpha(t) \diamond S_1(t) + \mu_2 \beta(t) \diamond S_2(t) + u(t) rA(t))\, dt.$$

Because the portfolio is now riskless we have the following dynamics

$$d\Pi(t) = r\Pi(t)\, dt.$$

Substituting the dynamics of $\Pi(t)$ and the values for $\alpha(t)$ and $\beta(t)$ the following is obtained

$$\mu_1 \alpha(t) \diamond S_1(t) + \mu_2 \beta(t) \diamond S_2(t) + u(t) rA(t) = r(\alpha(t) \diamond S_1(t) + \beta(t) \diamond S_2(t) + u(t) A(t))$$

rearranging the terms and using Wick calculus properties we have

$$(\mu_1 - r)\alpha(t) \diamond S_1(t) + (\mu_2 - r)\beta(t) \diamond S_2(t) = 0$$

substituting in (6.6)

$$\sigma_2(\mu_1 - r) S_2(t) \diamond S_1(t) - \sigma_1(\mu_2 - r) S_1(t) \diamond S_2(t) = 0$$

and

$$\begin{aligned} & \sigma_2(\mu_1 - r)(S_1(t) \diamond S_2(t)) \diamond (S_1(t) \diamond S_2(t))^{\diamond(-1)} \\ = \; & \sigma_1(\mu_2 - r)(S_1(t) \diamond S_2(t)) \diamond (S_1(t) \diamond S_2(t))^{\diamond(-1)}. \end{aligned}$$

It follows

$$\sigma_2(\mu_1 - r) = \sigma_1(\mu_2 - r)$$

and

$$\frac{(\mu_1 - r)}{\sigma_1} = \frac{(\mu_2 - r)}{\sigma_2}.$$

■

By the Girsanov theorem, theorem (5.1) for fractional Brownian motion $\hat{B}^H(t)$ is a fractional Brownian motion under the new measure defined as $\hat{\mu}_\phi$ and is expressed as

$$\hat{B}^H(t) = \frac{\mu - r}{\sigma} dt + dB^H(t).$$

The measure $\hat{\mu}_\phi$ is defined on $\mathcal{F}_T^H$ by

$$d\hat{\mu}_\phi(\omega) = \exp\left(-\int_0^T K(s)\, dB^H(s) - \frac{1}{2}|K|_\phi^2\right) d\mu_\phi(\omega),$$

where $K(s) = K(T,s)$ is defined by the following properties: supp $K \subset [0,T]$ and

$$\int_0^T K(T,s)\,\phi(t,s)\, ds = \frac{\mu - r}{\sigma}$$

for $0 \le t \le T$. Hu and Øksendal (2000) prove an explicit expression for $K(T,s)$ which is given as

$$K(T,s) = \frac{(\mu - r)}{2\sigma H(2H-1)\Gamma(2H-1)\Gamma(2-2H)\cos\left(\pi\left(H - \frac{1}{2}\right)\right)} \left(Ts - s^2\right)^{\frac{1}{2}-H}.$$

Lemma 6.3 *A self-financing portfolio has the following value process under* $\hat{\mu}_\phi$

$$dZ^\theta(t,\omega) = rZ^\theta(t,\omega)\, dt + \sigma v(t) \diamond S(t)\, d\hat{B}^H(t).$$

Proof. (Hu and Øksendal, 2000).

Assume that $\theta = (u,v)$ is a self-financing portfolio and given as

$$Z^\theta(t,\omega) = u(t)\, A(t) + v(t) \diamond S(t).$$

One can solve for the units invested in the riskless asset

$$u(t) = \frac{Z^\theta(t,\omega) - v(t) \diamond S(t)}{A(t)}. \tag{6.7}$$

Substituting (6.7) into (6.5), using Wick calculus and applying Girsanov theorem for fractional Brownian motion the dynamics of the value process reduces to

$$\begin{aligned}
& dZ^{\theta}(t,\omega) \\
= \; & u(t)\, dA(t) + \mu v(t) \diamond S(t)\, dt + \sigma v(t) \diamond S(t)\, dB^{H}(t) \\
= \; & \left(\frac{Z^{\theta}(t,\omega) - v(t) \diamond S(t)}{A(t)} \right) rA(t)\, dt + \mu v(t) \diamond S(t)\, dt + \sigma v(t) \diamond S(t)\, dB^{H}(t) \\
= \; & rZ^{\theta}(t,\omega)\, dt - rv(t) \diamond S(t)\, dt + \mu v(t) \diamond S(t)\, dt + \sigma v(t) \diamond S(t)\, dB^{H}(t) \\
= \; & rZ^{\theta}(t,\omega)\, dt + (\mu v(t) - rv(t)) \diamond S(t)\, dt + \sigma v(t) \diamond S(t)\, dB^{H}(t) \\
= \; & rZ^{\theta}(t,\omega)\, dt + v(t)(\mu - r) \diamond S(t)\, dt + \sigma v(t) \diamond S(t)\, dB^{H}(t) \\
= \; & rZ^{\theta}(t,\omega)\, dt + \sigma v(t) \diamond S(t) \left[\frac{(\mu - r)}{\sigma} dt + dB^{H}(t) \right] \\
= \; & rZ^{\theta}(t,\omega)\, dt + \sigma v(t) \diamond S(t)\, d\hat{B}^{H}(t).
\end{aligned}$$

■

We now want to obtain the discounted value of the portfolio

$$\begin{aligned}
dZ^{\theta}(t,\omega) - rZ^{\theta}(t,\omega)\, dt &= \sigma v(t) \diamond S(t)\, d\hat{B}^{H}(t) \\
\frac{dZ^{\theta}(t,\omega)}{dt} - rZ^{\theta}(t,\omega) &= \sigma v(t) \diamond S(t) \frac{d\hat{B}^{H}(t)}{dt}.
\end{aligned}$$

For $0 \leq t \leq T$, we multiply throughout by e^{-rt} and integrate

$$\begin{aligned}
\frac{e^{-rt} dZ^{\theta}(t,\omega)}{dt} - re^{-rt} Z^{\theta}(t,\omega) &= e^{-rt} \sigma v(t) \diamond S(t) \frac{d\hat{B}^{H}(t)}{dt} \\
\frac{d}{dt}\left[e^{-rt} Z^{\theta}(t,\omega) \right] &= e^{-rt} \sigma v(t) \diamond S(t) \frac{d\hat{B}^{H}(t)}{dt} \\
e^{-rt} Z^{\theta}(t,\omega) - Z^{\theta}(0,\omega) &= \int_0^t e^{-r\tau} \sigma v(\tau) \diamond S(\tau) \frac{d\hat{B}^{H}(\tau)}{d\tau} d\tau \\
e^{-rt} Z^{\theta}(t) &= z + \int_0^t e^{-r\tau} \sigma v(\tau) \diamond S(\tau)\, d\hat{B}^{H}(\tau) \qquad (6.8)
\end{aligned}$$

where $z = Z^{\theta}(0)$ is the initial capital that is invested into the portfolio.

Definition 6.1 *(Hu and Øksendal, 2000). A portfolio is Wick Itô Skorohod admissible if it is self-financing and*

$$v \diamond S \in \hat{\mathcal{L}}_{\phi}^{1,2}(\mathbb{R})$$

where $\hat{\mathcal{L}}_{\phi}^{1,2}(\mathbb{R})$ *is the space defined similarly to* $\mathcal{L}_{\phi}^{1,2}(\mathbb{R})$ *(see* (5.13)*) but with* μ_{ϕ} *and* $B^{H}(t)$*, replaced with* $\hat{\mu}_{\phi}$ *and* $\hat{B}^{H}(t)$*.*

Definition 6.2 *(Hu and Øksendal, 2000). An admissible portfolio* θ *is called a strong arbitrage for the market* $(A(t), S(t))$ *for* $t \in [0, T]$ *if the corresponding total wealth process* $Z^{\theta}(t)$ *satisfies the following conditions:*

$$\begin{aligned} Z^{\theta}(0) &= 0 \\ Z^{\theta}(T) &\geq 0 a.s. \\ \mu_{\phi}\left(\left\{\omega; Z^{\theta}(T,\omega) > 0\right\}\right) &> 0. \end{aligned}$$

Theorem 6.1 *There is no strong arbitrage admitted in the Wick Itô Skorohod fractional Black-Scholes market* $(A(t), S(t))$*.*

Proof. (Biagini, Hu, Øksendal and Zhang, 2008).

Taking the expectation under $\hat{\mu}_{\phi}$ and setting $t = T$ of (6.8), applying (5.12) and using the expectation of an Itô integral we have

$$\begin{aligned} E_{\hat{\mu}_{\phi}}\left[e^{-rt} Z^{\theta}(t)\right] &= Z^{\theta}(0) + E_{\hat{\mu}_{\phi}}\left[\int_0^t e^{-rt} \sigma v(\tau) \diamond S(\tau)\, d\hat{B}^{H}(\tau)\right] \\ &= Z^{\theta}(0) + E_{\mu_{\phi}}\left[\int_0^t e^{-rt} \sigma v(\tau) \diamond S(\tau)\, dB^{H}(\tau)\right] \\ &= Z^{\theta}(0). \end{aligned}$$

It follows

$$e^{-rT} E_{\hat{\mu}_{\phi}}\left[Z^{\theta}(T)\right] = Z^{\theta}(0). \tag{6.9}$$

Thus the condition will not be satisfied. ■

By (6.9) $\hat{\mu}_\phi$ is a risk neutral measure. From the fractional Girsanov theorem we have

$$\begin{aligned}\hat{B}^H(t) &= \frac{\mu - r}{\sigma}t + B^H(t) \\ B^H(t) &= \hat{B}^H(t) - \frac{\mu - r}{\sigma}t.\end{aligned}$$

The stock price is given by

$$\begin{aligned}S(t) &= S(0)\exp\left(\sigma B^H(t) + \mu t - \frac{1}{2}\sigma^2 t^{2H}\right) \\ &= S(0)\exp\left(\sigma\left(\hat{B}^H(t) - \frac{\mu - r}{\sigma}t\right) + \mu t - \frac{1}{2}\sigma^2 t^{2H}\right) \\ &= S(0)\exp\left(\sigma\hat{B}^H(t) + rt - \frac{1}{2}\sigma^2 t^{2H}\right).\end{aligned}$$

Thus the stock price dynamics under the risk neutral measure are given as

$$dS(t) = rS(t)\,dt + \sigma S(t)\,d\hat{B}^H(t). \tag{6.10}$$

Definition 6.3 *(Biagini, Hu, Øksendal and Zhang, 2008). A Wick Itô Skorohod admissible portfolio* $(u(t), v(t))$ *admits weak arbitrage if the total wealth process* $Z^\theta(t)$ *satisfies*

$$\begin{aligned}Z^\theta(0) &= 0 \\ \left\langle Z^\theta(T), \psi\right\rangle &\geq 0 \\ \left\langle Z^\theta(T), \psi\right\rangle &> 0\end{aligned}$$

for every stochastic test function ψ*.*

There is weak arbitrage in the fractional Black-Scholes market $(A(t), S(t))$. An example can be found in Biagini, Hu, Øksendal and Zhang (2008).

Definition 6.4 *(Hu and Øksendal, 2000). The fractional Black-Scholes market* $(A(t), S(t))$ *for* $t \in [0, T]$ *is called complete if for every* $\mathcal{F}_t^H$*-measurable*$_{H,\varnothing}$

bounded random variable F *there exists* $z = Z^{\theta}(0) \in \mathbb{R}$ *and an admissible portfolio* $\theta(t) = (u(t), v(t))$ *such that*

$$F(\omega) = Z^{\theta}(T, \omega).$$

Hu and Øksendal (2000) show that, similar to (6.8) this is the same as requiring

$$e^{-rT} F(\omega) = z + \int_0^T e^{-rt} \sigma v(t) \diamond S(t)\, d\hat{B}^H(t). \tag{6.11}$$

Consider $G(\omega) = e^{-rT} F(\omega)$, applying the fractional Clark-Ocone theorem 5.2 to $G(\omega) = e^{-rT} F(\omega)$ with $\tilde{E}_{\hat{\mu}_\phi}$ denoting the quasi-conditional expectation defined under the new measure $\hat{\mu}_\phi$ and $\hat{D}_t^H$ is the fractional Hida-Malliavin derivative with respect to $\hat{B}^H(t)$ we have

$$e^{-rT} F(\omega) = E_{\hat{\mu}_\phi}\left[e^{-rT} F\right] + \int_0^T \tilde{E}_{\hat{\mu}_\phi}\left[e^{-rT} \hat{D}_t^H F \left| \mathcal{F}_t^H \right.\right] d\hat{B}^H(\tau). \tag{6.12}$$

Comparing (6.11) and (6.12) the market is complete and there is a unique initial value

$$\begin{aligned} z &= E_{\hat{\mu}_\phi}\left[e^{-rT} F\right] \\ &= e^{-rT} E_{\hat{\mu}_\phi}[F]. \end{aligned} \tag{6.13}$$

Lemma 6.4 *The corresponding replicating / hedging portfolio* $\theta(t) = (u(t), v(t))$ *for the claim* F *is*

$$v(t) = e^{-r(T-t)} \sigma^{-1} S^{\diamond(-1)}(t) \diamond \tilde{E}_{\hat{\mu}_\phi}\left[\hat{D}_t^H F \left| \mathcal{F}_t^H \right.\right]$$

and

$$u(t) = \frac{Z^{\theta}(t) - v(t) \diamond S(t)}{A(t)}.$$

Proof. (Hu and Øksendal, 2000).

Equating the random terms in (6.11) and (6.12) we get

$$\begin{aligned}
\int_0^T E_{\hat{\mu}_\phi}\left[e^{-rT}\hat{D}_t^H F\left|\mathcal{F}_t^H\right.\right]d\hat{B}^H(\tau) &= \int_0^T e^{-rt}\sigma v(\tau)\diamond S(\tau)\,d\hat{B}^H(\tau)\\
E_{\hat{\mu}_\phi}\left[e^{-rT}\hat{D}_t^H F\left|\mathcal{F}_t^H\right.\right] &= e^{-rt}\sigma v(t)\diamond S(t)\\
e^{-(T-t)}\sigma^{-1}E_{\hat{\mu}_\phi}\left[\hat{D}_t^H F\left|\mathcal{F}_t^H\right.\right] &= v(t)\diamond S(t)\\
e^{-(T-t)}\sigma^{-1}E_{\hat{\mu}_\phi}\left[\hat{D}_t^H F\left|\mathcal{F}_t^H\right.\right]\diamond S^{\diamond(-1)}(t) &= v(t)\diamond S(t)\diamond S^{\diamond(-1)}(t)\\
e^{-(T-t)}\sigma^{-1}S^{\diamond(-1)}(t)\diamond E_{\hat{\mu}_\phi}\left[\hat{D}_t^H F\left|\mathcal{F}_t^H\right.\right] &= v(t).
\end{aligned}$$

The rest follows. ■

6.3 Criticism of the Market Model

Björk and Hult (2005) prove that in the fractional Wick Itô Skorohod integration model a portfolio can consist of a positive number of shares in a stock with a positive price, with positive probability and yet the portfolio can have a negative value.

Example 6.1 *Consider a portfolio consisting of a risky asset with* $Z^\theta(t,\omega) = \theta(t,\omega) = (0, v(t))$*. The portfolio consists of the amount* $v(t) = S(t) - s_0$ *shares in a stock and is given as* $S(t) = s_0\exp\left(B^H(t) - \frac{1}{2}\right)$ *for a discrete point in time. We will express the stock as*

$$\begin{aligned}
S(t) &= \varepsilon\left(1_{[0,t]}\right)\\
&= s_0\exp\left(B^H(t) - \frac{1}{2}t^{2H}\right).
\end{aligned}$$

Let $\Omega' = \left\{\omega\in\Omega\left|B^H(1,\omega)\in\left(\frac{1}{2},\frac{3}{2}\right)\right.\right\}$*. Hence* $S(t) > s_0$*. The portfolio at time* $t = 1$ *is given* $Z^\theta(1,\omega) = \theta(1,\omega) = (0, v(1))$ *such that* $v(1) > 0$ *and the amount*

$u(1) = 0$ *in the bank account. It follows for* $\omega \in \Omega'$

$$
\begin{aligned}
& Z^{\theta}(1,\omega) \\
= \; & v(1) \diamond S(1) \\
= \; & (S(1) - s) \diamond S(1) \\
= \; & S(1) \diamond S(1) - s_0 S(1) \\
= \; & \varepsilon\left(1_{[0,1]}\right) \diamond \varepsilon\left(1_{[0,1]}\right) - s_0 S(1) \\
= \; & s_0 \exp^{\diamond}\left(\int_0^{\infty} 1_{[0,1]} dB^H(t)\right) \diamond s_0 \exp^{\diamond}\left(\int_0^{\infty} 1_{[0,1]} dB^H(t)\right) - s_0 S(1) \\
= \; & s_0^2 \exp^{\diamond}\left(\int_0^{\infty} 1_{[0,1]} dB^H(t) + \int_0^{\infty} 1_{[0,1]} dB^H(t)\right) - s_0 S(1) \\
= \; & s_0^2 \exp\left(\int_0^{\infty} 1_{[0,1]} dB^H(t) + \int_0^{\infty} 1_{[0,1]} dB^H(t) - \frac{1}{2}\left\|1_{[0,1]} + 1_{[0,1]}\right\|_{\phi}^2\right) - s_0 S(1) \\
= \; & s_0^2 \exp\left(B^H(1) + B^H(1) - \frac{1}{2}\left|\left|1_{[0,1]} + 1_{[0,1]}\right|\right|_{\phi}^2\right) - s_0 S(1) \\
= \; & s_0^2 \exp\left(\begin{array}{c} 2B^H(1) \\ -\frac{1}{2}\int_0^{\infty}\int_0^{\infty}\left(1_{[0,1]}(s) + 1_{[0,1]}(s)\right)\left(1_{[0,1]}(t) + 1_{[0,1]}(t)\right)\phi(t,s)\,dsdt \end{array}\right) \\
& -s_0 S(1) \\
= \; & s_0^2 \exp\left(2B^H(1) - \frac{1}{2}\int_0^1\int_0^1 (2)(2) H(2H-1)|t-s|^{2H-2} dsdt\right) - s_0 S(1) \\
= \; & s_0^2 \exp\left(2B^H(1) - \frac{1}{2}(4)\frac{1}{2}\left(1^{2H} + 1^{2H} - |1-1|^{2H}\right)\right) - s_0 S(1) \\
= \; & s_0^2 \exp\left(2B^H(1) - 2\right) - s_0^2 \exp\left(\int_0^{\infty} 1_{[0,1]} dB^H(t) - \frac{1}{2}\left\|1_{[0,1]}\right\|_{\phi}^2\right) \\
= \; & s_0^2 \exp\left(2B^H(1) - 2\right) - s_0^2 \exp\left(\begin{array}{c} B^H(1) \\ -\frac{1}{2}\int_0^{\infty}\int_0^{\infty}\left(1_{[0,1]}(s)\right)\left(1_{[0,1]}(t)\right)\phi(t,s)\,dsdt \end{array}\right) \\
= \; & s_0^2 \exp\left(2B^H(1) - 2\right) - s_0^2 \exp\left(B^H(1) - \frac{1}{2}\int_0^1\int_0^1 1H(2H-1)|t-s|^{2H-2} dsdt\right) \\
= \; & s_0^2 \exp\left(2B^H(1) - 2\right) - s_0^2 \exp\left(B^H(1) - \frac{1}{2}\frac{1}{2}\left(1^{2H} + 1^{2H} - |1-1|^{2H}\right)\right) \\
= \; & s_0^2 \left(\exp\left(2B^H(1) - 2\right) - \exp\left(B^H(1) - \frac{1}{2}\right)\right)
\end{aligned}
$$

Then there exists a portfolio with $P(\Omega') > 0$ *such that* $v(1) \diamond S(1) < 0$ *on* Ω'.

This shows that the market setup has significant problems.

6.4 Option Pricing Formula

The assumptions to pricing the fractional Black-Scholes formula and the fractional Black formula are as follows:

1. The stock price follows equation (6.1) .
2. Stochastic differentials are interpreted in the Wick Itô Skorohod sense.
3. The Hurst parameter is $\frac{1}{2} < H < 1$ and is constant over time.
4. The drift μ and volatility σ are constant and the r is a constant risk-free rate of interest and the same for all maturities.
5. The definition of the portfolio is done using the Wick product.
6. The definition of an admissible portfolio is given by definition 6.1.
7. The portfolio is Wick self-financing as in equation (6.5) .
8. Short selling is allowed.
9. There are no transactions costs or taxes.
10. There are no dividends.
11. Trading is done continuously.
12. All securities are perfectly divisible.

6.4.1 Fractional Black-Scholes Formula

In this section the fractional Black-Scholes option pricing model for European call options is presented as derived by Hu and Øksendal (2000).

Theorem 6.2 *Fractional Black-Scholes formula. The price of a fractional European call option given the claim*

$$F(\omega) = \max\{S(T,\omega) - K, 0\}$$

is given as

$$C^H(0, S(0)) = S(0)\, N\left(d_1^H\right) - Ke^{-rT} N\left(d_2^H\right) \tag{6.14}$$

where

$$d_1^H = \frac{\ln\left(\frac{S(0)}{K}\right) + rT + \frac{1}{2}\sigma^2 T^{2H}}{\sigma\sqrt{T^{2H}}},$$

and

$$d_2^H = \frac{\ln\left(\frac{S(0)}{K}\right) + rT - \frac{1}{2}\sigma^2 T^{2H}}{\sigma\sqrt{T^{2H}}}$$

and

$$N(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} \exp\left(-\frac{x^2}{2}\right) dx$$

is the standard normal cumulative distribution function.

Proof. (Hu and Øksendal, 2000).

Consider the claim

$$F(\omega) = \max\{S(T,\omega) - K, 0\}.$$

Using (6.13)

$$\begin{aligned} C^H(0,S(0)) &= e^{-rT}E_{\hat{\mu}_\phi}[F] \\ &= e^{-rT}E_{\hat{\mu}_\phi}\left[\max\left\{(S(T,\omega)-K),0\right\}\right] \\ &= e^{-rT}E_{\hat{\mu}_\phi}\left[\max\left\{S(0)\exp\left(\sigma B^H(T)+\mu T-\frac{1}{2}\sigma^2T^{2H}\right)-K,0\right\}\right] \\ &= e^{-rT}E_{\hat{\mu}_\phi}\left[\max\left\{S(0)\exp\left(\sigma \hat{B}^H(T)+rT-\frac{1}{2}\sigma^2T^{2H}\right)-K,0\right\}\right] \end{aligned}$$

doing a measure change we have

$$\begin{aligned} C^H(0,S(0)) &= e^{-rT}E_{\mu_\phi}\left[\max\left\{S(0)\exp\left(\sigma B^H(T)+rT-\frac{1}{2}\sigma^2T^{2H}\right)-K,0\right\}\right] \\ &= e^{-rT}E_{\mu_\phi}\left[S(T)\,1_{\{S(T)>K\}}\right]-e^{-rT}KE_{\mu_\phi}\left[1_{\{S(T)>K\}}\right]. \end{aligned}$$

Solving for the boundary

$$\begin{aligned} S(0)\exp\left(\sigma z+rT-\frac{1}{2}\sigma^2T^{2H}\right) &> K \\ z &> \frac{\ln\left(\frac{K}{S(0)}\right)-rT+\frac{1}{2}\sigma^2T^{2H}}{\sigma} \\ &= -\frac{\ln\left(\frac{S(0)}{K}\right)+rT-\frac{1}{2}\sigma^2T^{2H}}{\sigma}. \end{aligned}$$

Setting

$$\hat{d}_1=\frac{\ln\left(\frac{S(0)}{K}\right)+rT-\frac{1}{2}\sigma^2T^{2H}}{\sigma}.$$

Calculating the first expectation

$$E_{\mu_\phi}\left[S(T)\,1_{\{S(T)>K\}}\right]=\int_{-\hat{d}_1}^{\infty}\frac{1}{T^H\sqrt{2\pi}}\exp\left(-\frac{y^2}{2T^{2H}}\right)x\exp\left(\sigma y+rT-\frac{1}{2}\sigma^2T^{2H}\right)dy.$$

Since the variance of $B^H(T)$ is T^{2H}, see (5.6) and the mean is 0, see (5.5) and using the Gaussian character of fractional Brownian motion, it follows that

$$\begin{aligned}
E_{\mu_\phi}\left[S(T)1_{\{S(T)>K\}}\right] &= e^{rT}\int_{-\hat{d}_1}^{\infty}\frac{1}{T^H\sqrt{2\pi}}\exp\left(-\frac{y^2}{2T^{2H}}+\sigma y-\frac{1}{2}\sigma^2T^{2H}\right)xdy \\
&= xe^{rT}\int_{-\hat{d}_1}^{\infty}\frac{1}{T^H\sqrt{2\pi}}\exp\left(-\frac{1}{2T^{2H}}\left(y^2-2\sigma yT^{2H}+\sigma^2T^{4H}\right)\right)dy \\
&= xe^{rT}\int_{-\hat{d}_1}^{\infty}\frac{1}{T^H\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left(\frac{y-\sigma T^{2H}}{T^H}\right)^2\right)dy.
\end{aligned}$$

Let

$$z=\frac{y-\sigma T^{2H}}{T^H}\Rightarrow y=zT^H+\sigma T^{2H}$$

then

$$dy=T^Hdz.$$

It follows that

$$\begin{aligned}
E_{\mu_\phi}\left[S(T)1_{\{S(T)>K\}}\right] &= xe^{rT}\int_{\frac{-\hat{d}_1-\sigma T^{2H}}{T^H}}^{\infty}\frac{1}{T^H\sqrt{2\pi}}\exp\left(-\frac{1}{2}(z)^2\right)T^Hdz \\
&= xe^{rT}\int_{-\infty}^{\frac{\hat{d}_1+\sigma T^{2H}}{T^H}}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}(z)^2\right)dz \\
&= xe^{rT}N\left(\frac{\hat{d}_1+\sigma T^{2H}}{T^H}\right) \\
&= xe^{rT}N\left(d_1^H\right)
\end{aligned}$$

where

$$
\begin{aligned}
d_1^H &= \frac{\hat{d}_1 + \sigma T^{2H}}{T^H} \\
&= \frac{\frac{\ln\left(\frac{S(0)}{K}\right) + rT - \frac{1}{2}\sigma^2 T^{2H}}{\sigma} + \sigma T^{2H}}{T^H} \\
&= \frac{\ln\left(\frac{S(0)}{K}\right) + rT - \frac{1}{2}\sigma^2 T^{2H} + \sigma^2 T^{2H}}{\sigma T^H} \\
&= \frac{\ln\left(\frac{S(0)}{K}\right) + rT + \frac{1}{2}\sigma^2 T^{2H}}{\sigma T^H}.
\end{aligned}
$$

Calculating the second integral

$$
E_{\mu_\phi}\left[1_{\{S(T)>K\}}\right] = \int_{-\hat{d}_1}^{\infty} \frac{1}{T^H\sqrt{2\pi}} \exp\left(-\frac{y^2}{2T^{2H}}\right) dy
$$

and setting

$$
w = \frac{y}{T^H} \Rightarrow y = wT^H
$$

differentiating we have

$$
dy = T^H dz.
$$

Then it follows that

$$
\begin{aligned}
E_{\mu_\phi}\left[1_{\{S(T)>K\}}\right] &= \int_{\frac{-\hat{d}_1}{T^H}}^{\infty} \frac{1}{T^H\sqrt{2\pi}} \exp\left(-\frac{1}{2}w^2\right) T^H dw \\
&= \int_{-\infty}^{\frac{\hat{d}_1}{T^H}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}w^2\right) dw \\
&= N\left(\frac{\hat{d}_1}{T^H}\right) \\
&= N\left(d_2^H\right)
\end{aligned}
$$

where

$$
d_2^H = \frac{\ln\left(\frac{S(0)}{K}\right) + rT - \frac{1}{2}\sigma^2 T^{2H}}{\sigma T^H}.
$$

It follows that the price is

$$C^H(0, S(0)) = S(0) N\left(d_1^H\right) - Ke^{-rT} N\left(d_2^H\right).$$

■

Setting $H = \frac{1}{2}$ the classical Black-Scholes option pricing formula is achieved.

Theorem 6.3 *The corresponding replicating portfolio $\theta(t) = (u(t), v(t))$ which is given by* (6.7) *and* (6.8) *is given as*

$$v(t) = e^{-r(T-t)} \sigma^{-1} S^{\diamond(-1)}(t) \diamond \kappa(S(t)).$$

Where

$$\kappa(y) = \frac{1}{\sqrt{2\pi\left(T^{2H} - t^{2H}\right)}} \int_{\mathbb{R}} \exp\left(-\frac{(\hat{y} - z)^2}{2\left(T^{2H} - t^{2H}\right)}\right) h(z)\, dz$$

with

$$\hat{y} = \frac{\ln y - \ln s - rt + \frac{1}{2}\sigma^2 t^{2H}}{\sigma}$$

and

$$h(z) = \sigma \chi_{[K,\infty)}\left(s \exp\left(\sigma z + rT - \frac{1}{2}\sigma^2 T^{2H}\right)\right).$$

For the proof see Hu and Øksendal (2000 page 26).

6.4.2 Fractional Black Formula

We will now prove a fractional Black formula for pricing European options on futures. Consider a fractional market with an investment in a money account and a stock in a continuos setting $0 \leq t \leq T$. Let $r > 0$ be a constant riskless interest rate then the money account $A(t)$ at time t develops according to the equation

$$\begin{aligned} dA(t) &= rA(t)\, dt \\ A(0) &= 1. \end{aligned}$$

Let μ be the drift of the stock and $\sigma \neq 0$ be the corresponding volatility. We assume the future price process is a martingale under $\hat{\mu}_\phi$ and has the following dynamics

$$\begin{aligned} dF(t) &= \sigma F(t)\, dB^H(t) \\ F(0) &> 0. \end{aligned} \tag{6.15}$$

Applying (5.11) and using Wick calculus the solution is

$$F(t) = F(0) \exp\left(\sigma B^H(t) - \frac{1}{2}\sigma^2 t^{2H}\right).$$

We consider $F(t,\omega)$ as a stochastic distribution in ω which is an element of $(\mathcal{S})^*_H$.

A portfolio or trading strategy $Z^\theta(t,\omega) = \theta(t,\omega) = (u(t), v(t))$ is an $\mathcal{F}^H_t$ adapted 2-dimensional process where $u(t)$ is the amount that is invested in the money account and $v(t)$ is the amount that is invested in the futures contract at time t. Since the cost of taking a long or short position in a futures contract is ignored since the initial margin requirements is small, then the value process $Z^\theta(t) = Z^\theta(t,\omega)$ is given by

$$Z^\theta(t,\omega) = u(t)\, A(t).$$

Using similar arguments as in Musiela and Rutkowski (2011), for the standard case, the portfolio is called self-financing if $t \in [0,T]$ and we have the dynamics

$$\begin{aligned} dZ^\theta(t,\omega) &= u(t)\, dA(t) + v(t) \diamond dF(t) \\ &= u(t)\, dA(t) + v(t) \diamond \left(\sigma F(t)\, dB^H(t)\right) \\ &= u(t)\, dA(t) + \sigma v(t) \diamond F(t)\, dB^H(t). \end{aligned} \tag{6.17}$$

Lemma 6.5 *A self-financing portfolio has the following value process under* $\hat{\mu}_\phi$

$$dZ^\theta(t,\omega) = rZ^\theta(t,\omega)\, dt + \sigma v(t) \diamond S(t)\, d\hat{B}^H(t).$$

Proof. Assume that $\theta = (u, v)$ is a self-financing portfolio given as

$$Z^{\theta}(t, \omega) = u(t) A(t).$$

One can solve for the units invested riskless asset

$$u(t) = \frac{Z^{\theta}(t, \omega)}{A(t)}. \tag{6.18}$$

Substituting (6.18) into (6.17), using Wick calculus the dynamics of the value process reduces to

$$\begin{aligned} & dZ^{\theta}(t, \omega) \\ = \; & u(t) \, dA(t) + \sigma v(t) \diamond F(t) \, d\hat{B}^{H}(t) \\ = \; & \left(\frac{Z^{\theta}(t, \omega)}{A(t)} \right) rA(t) \, dt + \sigma v(t) \diamond F(t) \, d\hat{B}^{H}(t) \\ = \; & rZ^{\theta}(t, \omega) \, dt + \sigma v(t) \diamond F(t) \, d\hat{B}^{H}(t). \end{aligned}$$

■

Using similar arguments to the above we obtain the discounted wealth process as

$$e^{-rt} Z^{\theta}(t) = z + \int_0^t e^{-r\tau} \sigma v(\tau) \diamond F(\tau) \, d\hat{B}^{H}(\tau) \tag{6.19}$$

where $z = Z^{\theta}(0)$ is the initial capital that is invested in the portfolio. There is no strong arbitrage in the Wick Itô Skorohod fractional Black market $(A(t), S(t))$. The proof is similar to that of theorem 6.1.

Definition 6.5 *The fractional Black market* $(A(t), S(t))$ *for* $t \in [0, T]$ *is called complete if for every* $\mathcal{F}_t^H$*-measurable*$_{H,\emptyset}$ *bounded random variable* χ *there exists* $z = Z^{\theta}(0) \in \mathbb{R}$ *and an admissible portfolio* $\theta(t) = (u(t), v(t))$ *such that*

$$\chi(\omega) = Z^{\theta}(T, \omega).$$

Similarly the market is complete and there is a unique initial value

$$\begin{aligned} z &= E_{\hat{\mu}_\phi}\left[e^{-rT}\chi\right] \\ &= e^{-rT}E_{\hat{\mu}_\phi}\left[\chi\right]. \end{aligned} \tag{6.20}$$

Theorem 6.4 *Fractional Black formula. The price of an European call option with strike price K and maturity T on the futures contact F, is given as*

$$c^H\left(0, F(0)\right) = e^{-rT}\left[F\left(0\right)N\left(d_1^{H*}\right) - KN\left(d_2^{H*}\right)\right] \tag{6.21}$$

where

$$d_1^{H*} = \frac{\ln\left(\frac{F\left(0\right)}{K}\right) + \frac{1}{2}\sigma^2 T^{2H}}{\sigma\sqrt{T^{2H}}}$$

and

$$d_2^{H*} = \frac{\ln\left(\frac{F\left(0\right)}{K}\right) - \frac{1}{2}\sigma^2 T^{2H}}{\sigma\sqrt{T^{2H}}}.$$

Proof. Using equation(6.20)

$$\begin{aligned} c^H\left(0, F\left(0\right)\right) &= e^{-rT}E_{\hat{\mu}_\phi}\left[c^H\left(T, F\left(T\right)\right)\right] \\ &= e^{-rT}E_{\hat{\mu}_\phi}\left[\max\left\{\left(F\left(T\right) - K\right), 0\right\}\right] \\ &= e^{-rT}E_{\hat{\mu}_\phi}\left[\max\left\{F\left(0\right)\exp\left(\sigma\hat{B}^H\left(T\right) - \frac{1}{2}\sigma^2T^{2H}\right) - K, 0\right\}\right] \end{aligned}$$

doing a measure change we have

$$\begin{aligned} c^H\left(0, F\left(0\right)\right) &= e^{-rT}E_{\mu_\phi}\left[\max\left\{F\left(0\right)\exp\left(\sigma B^H\left(T\right) - \frac{1}{2}\sigma^2T^{2H}\right) - K, 0\right\}\right] \\ &= e^{-rT}E_{\mu_\phi}\left[F\left(T\right)1_{\{F(T)>K\}}\right] - e^{-rT}KE_{\mu_\phi}\left[1_{\{F(T)>K\}}\right]. \end{aligned}$$

Calculating the first expectation

$$E_{\mu_\phi}\left[F\left(T\right)1_{\{F(T)>K\}}\right] = \int_{\mathbb{R}} \frac{1}{T^H\sqrt{2\pi}}\exp\left(-\frac{y^2}{2T^{2H}}\right)x\exp\left(\sigma y - \frac{1}{2}\sigma^2T^{2H}\right)dy.$$

Since the variance of $B^H(T)$ is T^{2H}, see (5.6) and the mean is 0, see (5.5) and using the Gaussian character of fractional Brownian motion. Solving for the boundary

$$\begin{aligned} F(0)\exp\left(\sigma z - \frac{1}{2}\sigma^2 T^{2H}\right) &> K \\ z &> \frac{\ln\left(\frac{K}{F(0)}\right) + \frac{1}{2}\sigma^2 T^{2H}}{\sigma} \\ &= -\frac{\ln\left(\frac{F(0)}{K}\right) - \frac{1}{2}\sigma^2 T^{2H}}{\sigma} \end{aligned}$$

and setting

$$\hat{d}_1 = \frac{\ln\left(\frac{F(0)}{K}\right) - \frac{1}{2}\sigma^2 T^{2H}}{\sigma}.$$

Then it follows that

$$\begin{aligned} E_{\mu_\phi}\left[F(T)1_{\{F(T)>K\}}\right] &= \int_{-\hat{d}_1}^{\infty} \frac{1}{T^H\sqrt{2\pi}} \exp\left(-\frac{y^2}{2T^{2H}} + \sigma y - \frac{1}{2}\sigma^2 T^{2H}\right) x dy \\ &= x\int_{-\hat{d}_1}^{\infty} \frac{1}{T^H\sqrt{2\pi}} \exp\left(-\frac{1}{2T^{2H}}\left(y^2 - 2\sigma y T^{2H} + \sigma^2 T^{4H}\right)\right) dy \\ &= x\int_{-\hat{d}_1}^{\infty} \frac{1}{T^H\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{y-\sigma T^{2H}}{T^H}\right)^2\right) dy. \end{aligned}$$

Let

$$z = \frac{y-\sigma T^{2H}}{T^H} \Rightarrow y = zT^H + \sigma T^{2H}$$

then

$$dy = T^H dz.$$

It follows

$$\begin{aligned} E_{\mu_\phi}\left[F(T)\,1_{\{F(T)>K\}}\right] &= x\int_{\frac{-\hat{d}_1-\sigma T^{2H}}{T^H}}^{\infty}\frac{1}{T^H\sqrt{2\pi}}\exp\left(-\frac{1}{2}(z)^2\right)T^H dz \\ &= x\int_{-\infty}^{\frac{\hat{d}_1+\sigma T^{2H}}{T^H}}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}(z)^2\right)dz \\ &= xN\left(\frac{\hat{d}_1+\sigma T^{2H}}{T^H}\right) \\ &= xN\left(d_1^{H*}\right) \end{aligned}$$

where

$$\begin{aligned} d_1^{H*} &= \frac{\hat{d}_1+\sigma T^{2H}}{T^H} \\ &= \frac{\dfrac{\ln\left(\dfrac{F(0)}{K}\right)-\dfrac{1}{2}\sigma^2T^{2H}}{\sigma}+\sigma T^{2H}}{T^H} \\ &= \frac{\ln\left(\dfrac{F(0)}{K}\right)-\dfrac{1}{2}\sigma^2T^{2H}+\sigma^2T^{2H}}{\sigma T^H} \\ &= \frac{\ln\left(\dfrac{F(0)}{K}\right)+\dfrac{1}{2}\sigma^2T^{2H}}{\sigma T^H}. \end{aligned}$$

Calculating the second integral

$$E_{\mu_\phi}\left[1_{\{F(T)>K\}}\right] = \int_{-\hat{d}_1}^{\infty}\frac{1}{T^H\sqrt{2\pi}}\exp\left(-\frac{y^2}{2T^{2H}}\right)dy$$

and setting

$$w = \frac{y}{T^H} \Rightarrow y = wT^H$$

differentiating we have

$$dy = T^H dz.$$

It follows

$$\begin{aligned} E_{\mu_\phi}\left[1_{\{F(T)>K\}}\right] &= \int_{\frac{-\hat{d}_1}{T^H}}^{\infty} \frac{1}{T^H\sqrt{2\pi}} \exp\left(-\frac{1}{2}w^2\right) T^H dw \\ &= \int_{-\infty}^{\frac{\hat{d}_1}{T^H}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}w^2\right) dw \\ &= N\left(\frac{\hat{d}_1}{T^H}\right) \\ &= N\left(d_2^{H*}\right) \end{aligned}$$

where

$$d_2^{H*} = \frac{\ln\left(\frac{F(0)}{K}\right) - \frac{1}{2}\sigma^2 T^{2H}}{\sigma T^H}.$$

It follows that the claim is

$$c^H(0, F(0)) = e^{-rT}\left[F(0)\,N\left(d_1^{H*}\right) - KN\left(d_2^{H*}\right)\right].$$

■

Setting $H = \frac{1}{2}$ the classical Black option pricing formula is achieved.

6.4.3 Extending Fractional Black-Scholes and Fractional Black Formulae to an Arbitrary Time t

The above models assume an absolute time T. In the model, time 0 is not specified. We could also do the modelling from any time t. We will assume that time 0 is when we want to price the option. This means T becomes the time to maturity in (6.14) or (6.21). In other words we have the following two conjectures.

Conjecture 6.1 *Fractional Black-Scholes formula. The price of a fractional European call option at time t is given as*

$$C^H(t, S(t)) = S(t) N\left(d_1^H\right) - Ke^{-r(T-t)} N\left(d_2^H\right)$$

where

$$d_1^H = \frac{\ln\left(\frac{S(t)}{K}\right) + r(T-t) + \frac{1}{2}\sigma^2 (T-t)^{2H}}{\sigma\sqrt{(T-t)^{2H}}}$$

and

$$d_2^H = \frac{\ln\left(\frac{S(t)}{K}\right) + r(T-t) - \frac{1}{2}\sigma^2 (T-t)^{2H}}{\sigma\sqrt{(T-t)^{2H}}}.$$

Since the Law of one price holds in this market, put-call parity will hold regardless of the pricing model. We will prove the fractional European put option using this put-call parity

$$C^H(t, S(t)) - P^H(t, S(t)) = S(t) - Ke^{-r(T-t)}.$$

Thus

$$\begin{aligned} P^H(t, S(t)) &= C^H(t, S(t)) - S(t) + Ke^{-r(T-t)} \\ &= S(t) N\left(d_1^H\right) - e^{-r(T-t)} K N\left(d_2^H\right) - S(t) + Ke^{-r(T-t)} \\ &= S(t)\left(N\left(d_1^H\right) - 1\right) + Ke^{-r(T-t)}\left(1 - N\left(d_2^H\right)\right). \end{aligned}$$

It follows

$$
\begin{aligned}
N\left(d_1^H\right)-1 &= \int_{-\infty}^{d_1^H}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{y^2}{2}\right)dy-\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{y^2}{2}\right)dy\\
&= -\left(\int_{-\infty}^{d_1^H}-\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{y^2}{2}\right)dy+\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{y^2}{2}\right)dy\right)\\
&= -\left(\int_{d_1^H}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{y^2}{2}\right)dy\right)\\
&= -\left(\int_{-\infty}^{-d_1^H}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{y^2}{2}\right)dy\right)\\
&= -N\left(-d_1^H\right).
\end{aligned}
$$

Following similar arguments we have

$$1-N\left(d_2^H\right)=N\left(-d_2^H\right).$$

Thus the price at every $t\in[0,T]$ of a fractional European put option with strike price K and maturity T is given by

$$P^H\left(t,S\left(t\right)\right)=Ke^{-r(T-t)}N\left(-d_2^H\right)-S\left(t\right)N\left(-d_1^H\right).$$

Conjecture 6.2 *The price at every $t\in[0,T]$ of an European call option with strike price K and maturity T on the futures contact F, is given as*

$$c^H\left(t,F\left(t\right)\right)=e^{-r(T-t)}\left[F\left(t\right)N\left(d_1^{H*}\right)-KN\left(d_2^{H*}\right)\right]$$

where

$$d_1^{H*}=\frac{\ln\left(\frac{F(t)}{K}\right)+\frac{1}{2}\sigma^2\left(T-t\right)^{2H}}{\sigma\sqrt{\left(T-t\right)^{2H}}}$$

and

$$d_2^{H*} = \frac{\ln\left(\frac{F(t)}{K}\right) - \frac{1}{2}\sigma^2 (T-t)^{2H}}{\sigma\sqrt{(T-t)^{2H}}}.$$

A put-call parity relation holds for futures (Hull, 2006) regardless of the pricing model

$$c^H + Ke^{-r(T-t)} = p^H + F(t)\, e^{-r(T-t)}.$$

Hence we have

$$\begin{aligned} p^H &= c^H + Ke^{-r(T-t)} - F(t)\, e^{-r(T-t)} \\ &= e^{-r(T-t)}\left[F(t)\left(N\left(d_1^{H*}\right) - 1\right) + K\left(1 - N\left(d_2^{H*}\right)\right)\right]. \end{aligned}$$

A Black formula for a European put p^H at time T is

$$p^H(t, F(t)) = e^{-r(T-t)}\left(KN\left(-d_2^{H*}\right) - F(t)\, N\left(-d_1^{H*}\right)\right).$$

6.5 Model Analysis

In the figure 6.1 Hurst parameters of 0.2, 0.5 and 0.8 were chosen and the power factor $(T)^{2H}$, which replaces the expiration time in the pricing formula, is plotted against expiration time. We used a Hurst parameter less than $\frac{1}{2}$ for comparison with the other models even though in this model it is assumed that $H > \frac{1}{2}$. When the expiration time is longer than a year the persistent power factor is bigger than the anti-persistent case and the normal case. For shorter expiration times the persistent power factor is less than the normal case.

The price of a fractional European call option with respect to different spot prices S is graphed for the anti-persistent, persistent and normal case as well as the

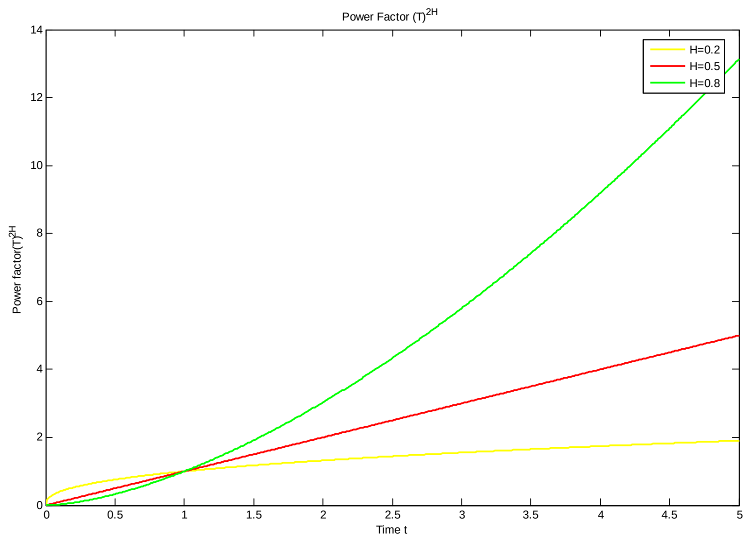

Figure 6.1: Hu and Øksendal power factor $(T)^{2H}$ for a varying time T.

intrinsic value line which is $\max\{S(t) - K, 0\}$. In figure 6.2 the time to expiration is $T = 0.25$, and the graph displays that the price of the persistent case will be less than the normal and the anti-persistent case. In figure 6.3 the time to expiration is $T = 1$ and all the prices will be the same for varying spot prices. In figure 6.4 the time to expiration is $T = 2$ and the graph shows that the price of persistent case will be larger than that of the normal case. We see that when the time to maturity increase the prices move away from the intrinsic value line.

Using the conjectured model we plot the pricing formula for varying Hurst parameters for a fixed time T and for three different t. For shorter time to expiration $T = 0.5$ and $t = 0.1$, $t = 0.25$ and $t = 0.4$ figure 6.5 shows the price of a call option decreases as the Hurst parameter increases. For larger time to expiration $T = 5$ and

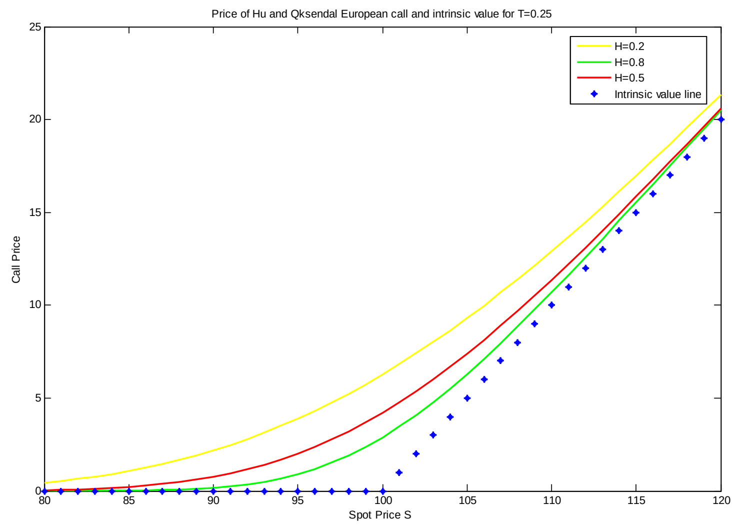

Figure 6.2: Hu and Øksendal price of European call for varying spot for fixed parameters $K = 100$, $r = 0.02$, $\sigma = 0.2$ and $T = 0.25$ and the intrinsic value line.

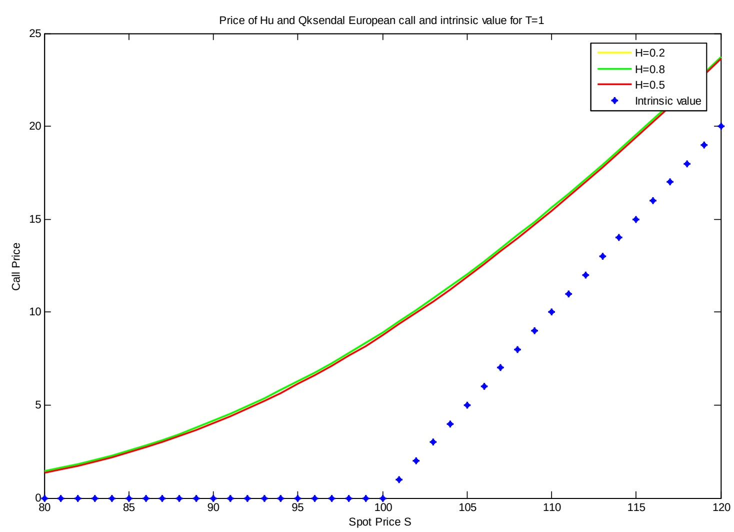

Figure 6.3: Hu and Øksendal price of European call for varying spot for fixed parameters $K = 100$, $r = 0.02$, $\sigma = 0.2$ and $T = 1$ and the intrinsic value line.

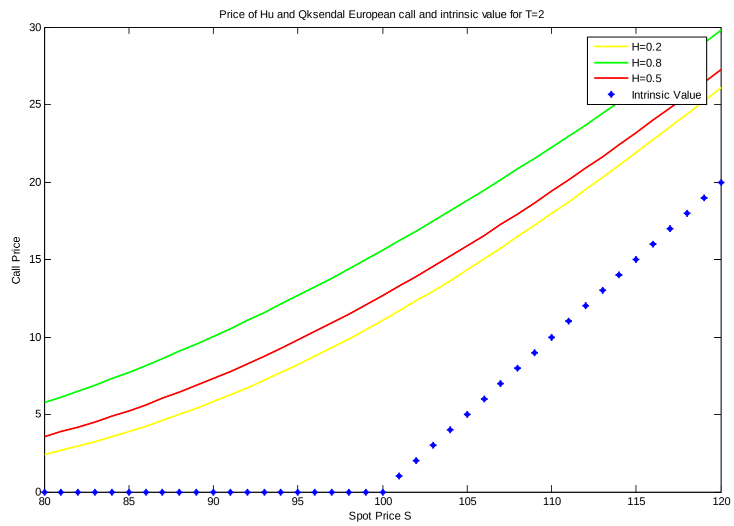

Figure 6.4: Hu and Øksendal price of European call for varying spot and fixed parameters $K = 100$, $r = 0.02$, $\sigma = 0.2$ and $T = 2$ and intrinsic value.

Table 6.1: Time to maturity effects

	Persistent	**Anti-Persistent**
Short-Run	Less volatile, lower price	More volatile, higher price
Long-Run	More volatile, higher price	Less volatile, lower price

$t = 1$, $t = 2.5$ and $t = 4$ figure 6.6 shows the price of a call option increases as the Hurst parameter increases.

Rostek and Schöbel (2009) explained this effect known as the power effect for their model. For a persistent time series there will be less short run deviations from the mean and more long run deviations from the mean. For an anti-persistent series the effect will be the opposite, see table 6.1.

For $\sigma \in (0, 1)$ and $H \in (0, 1)$ we generated numbers and the price of the option is plotted against volatility and the Hurst parameter for three different time to

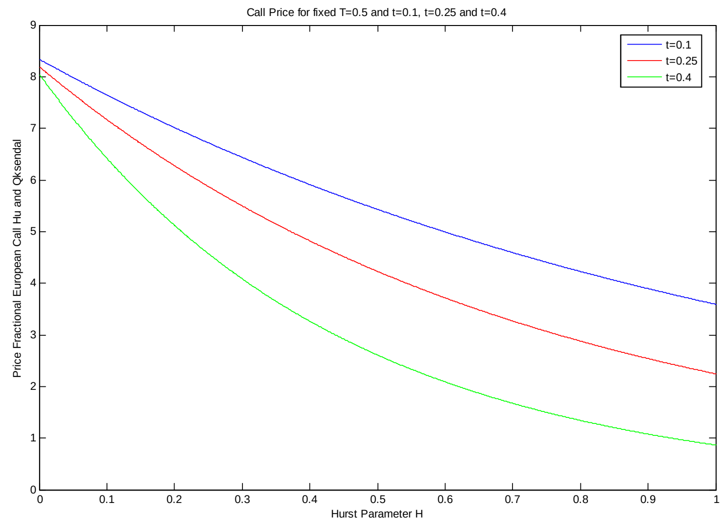

Figure 6.5: Hypothetical Hu and Øksendal price of European call with varying Hurst and $t = 0.1$, $t = 0.25$ and $t = 0.4$. Fixed parameters $K = 100$, $S = 100$, $\sigma = 0.2$, $r = 0.02$ and $T = 0.5$.

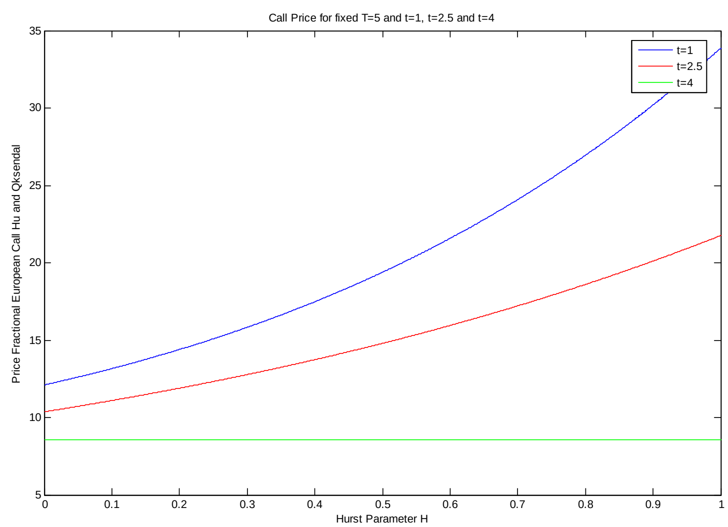

Figure 6.6: Hypothetical Hu and Øksendal price of European call with varying Hurst and $t = 1$, $t = 2.5$ and $t = 4$. Fixed parameters $K = 100$, $S = 100$, $\sigma = 0.2$, $r = 0.02$ and $T = 5$.

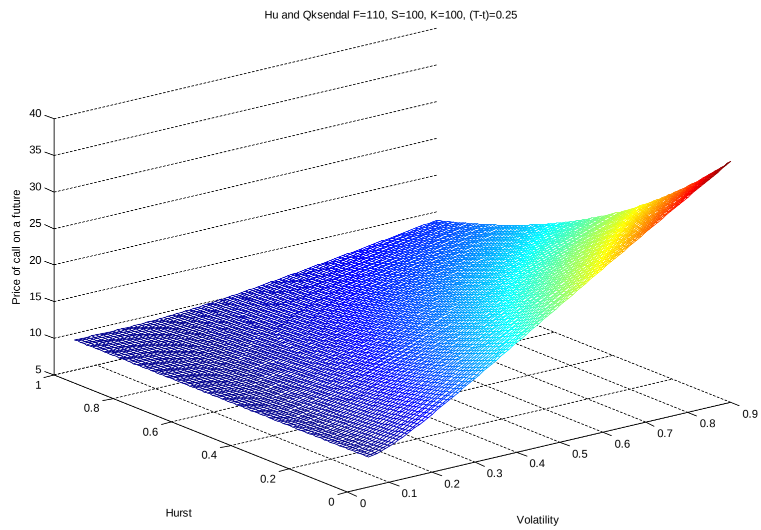

Figure 6.7: Black Hu and Øksendal Price vs Hurst vs Volatility $F_t = 110, S_t = 100, K = 100, (T - t) = 0.25$.

expirations. On the z-axis we have the price of the call on a future contract, on the x-axis the volatility is plotted and on the y-axis we plot the Hurst parameter. Setting $F(t) = 110,\ S(t) = 100,\ K = 100$ and $(T - t) = 0.25$ we obtain concave upward figure 6.7. We see that when $\sigma \to 1$ and $H \to 0$ the price increases significantly.

Choosing $(T - t) = 0.75$ we obtain figure 6.8 which is more linear and as the volatility increase for all Hurst parameters the price increases.

Choosing $(T - t) = 5$ we obtain concave down figure 6.9 and we see that when $\sigma \to 1$ and $H \to 1$ the prices are the largest.

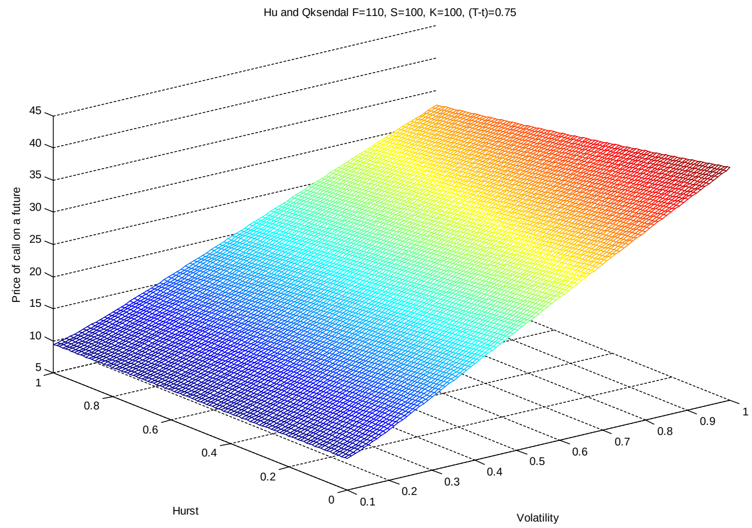

Figure 6.8: Black Hu and Øksendal Price vs Hurst vs Volatility $F_t = 110, S_t = 100, K = 100$, $(T - t) = 0.75$

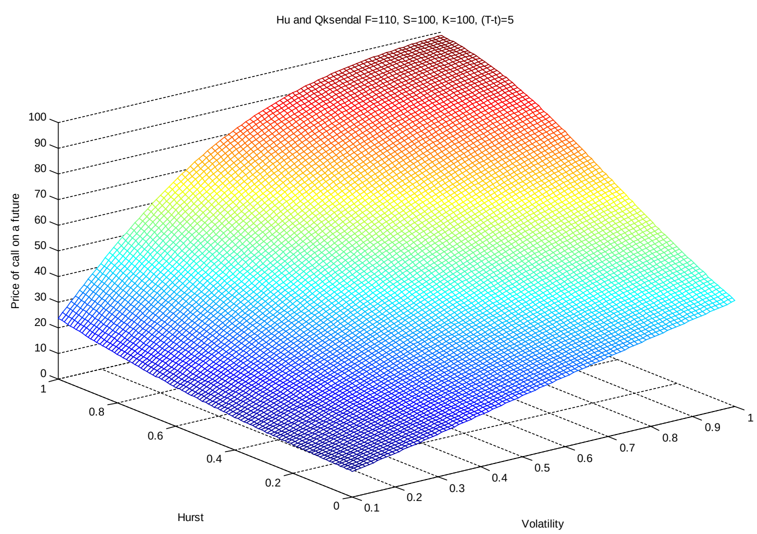

Figure 6.9: Black Hu and Øksendal Price vs Hurst vs Volatility $F_t = 110, S_t = 100, K = 100, (T - t) = 5$.

Chapter 7

Necula's fBm Pricing Model

7.1 Introduction

Necula (2002) applied Wick stochastic calculus to generalize a fractional Black-Scholes formula to price European puts and calls from any arbitrary time t to the maturity time T if the underlying is driven by fractional Brownian motion for the case $H \in \left(\frac{1}{2}, 1\right)$. Necula presented results regarding quasi-conditional expectation for fractional Brownian motion. Under the results of quasi-conditional expectation we compute the expected value of an exponential when it is driven by fractional Brownian motion. The fractional Clark-Ocone theorem is applied to prove the fractional risk-neutral evaluation theorem. A change in the risk neutral measure is assured by the fractional Girsanov transform and using properties of normality a quasi-conditional Black-Scholes formula for pricing European options and a quasi-conditional Black is proved.

Theorem 7.1 *Let $B^H(t)$ be a fractional Brownian motion with respect to the measure μ_ϕ. For every $0 < t < T$ and $\lambda \in \mathbb{C}$, where $\mathbb{C}$ is a set of complex number then*

$$\tilde{E}_{\mu_\phi}\left[e^{\lambda B^H(T)} \left| \mathcal{F}_t^H \right.\right] = \exp\left(\lambda B^H(t) + \frac{\lambda^2}{2}\left(T^{2H} - t^{2H}\right)\right).$$

Proof. (Necula, 2002).

Consider the following stochastic fractional differential equation

$$\begin{aligned} dX(t) &= \lambda X(t)\, dB^H(t) \\ X(0) &= 1 \end{aligned}$$

using Wick calculus the solution of the system is

$$X(t) = \exp\left(\lambda B^H(t) - \frac{1}{2}\lambda^2 t^{2H}\right). \tag{7.1}$$

Since

$$X(t) = \int_0^t \lambda X(s)\, dB^H(s)$$

and using the property of quasi-conditionality from (5.14), the expectation under the measure μ_ϕ follows as

$$\tilde{E}_{\mu_\phi}\left[X(T)\left|\mathcal{F}_t^H\right.\right] = X(t). \tag{7.2}$$

Then substituting (7.1) into (7.2)

$$\tilde{E}_{\mu_\phi}\left[\exp\left(\lambda B^H(t) - \frac{1}{2}\lambda^2 T^{2H}\right)\left|\mathcal{F}_t^H\right.\right] = \exp\left(\lambda B^H(t) - \frac{1}{2}\lambda^2 t^{2H}\right).$$

Then it follows that

$$\tilde{E}_{\mu_\phi}\left[e^{\lambda B^H(T)}\left|\mathcal{F}_t^H\right.\right] = \exp\left(\lambda B^H(t) + \frac{\lambda^2}{2}\left(T^{2H} - t^{2H}\right)\right).$$

■

Theorem 7.2 *Let f be a function such that $\left[f\left(B^H(T)\right)\right] < \infty$. Then for every $t \leq T$, the quasi–conditional expectation is*

$$\tilde{E}_{\mu_\phi}\left[f\left(B^H(T)\right)\left|\mathcal{F}_t^H\right.\right] = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\left(T^{2H} - t^{2H}\right)}} \exp\left(-\frac{\left(x - B^H(t)\right)^2}{2\left(T^{2H} - t^{2H}\right)}\right) f(x)\, dx.$$

For the proof Necula (2002, page 8).

Corollary 7.1 *(Necula, 2002). Let $A \in \mathcal{B}(\mathbb{R})$, where $\mathcal{B}(\mathbb{R})$ is a Borel σ-algebra. Then*

$$\tilde{E}_{\mu_\phi}\left[1_A\left(B^H(T)\right)\left|\mathcal{F}_t^H\right.\right] = \int_A \frac{1}{\sqrt{2\pi\left(T^{2H}-t^{2H}\right)}}\exp\left(-\frac{\left(x-B^H(t)\right)^2}{2\left(T^{2H}-t^{2H}\right)}\right)dx.$$

Consider the process

$$\begin{aligned} Z(t) &= \varepsilon\left(-\theta\chi_{[0,t]}\right) \\ &= \exp\left(\begin{array}{c} \int_0^\infty -\theta\chi_{[0,t]}(u)\,dB^H(u) \\ -\frac{1}{2}\int_0^\infty\int_0^\infty -\theta\chi_{[0,t]}(u)-\theta\chi_{[0,t]}(v)\,\phi(u,v)\,dudv \end{array}\right) \\ &= \exp\left(\int_0^t -\theta dB^H(u) - \frac{1}{2}\int_0^t\int_0^t \theta^2\phi(u,v)\,dudv\right) \\ &= \exp\left(-\theta B^H(t) - \frac{\theta^2}{2}t^{2H}\right). \end{aligned}$$

Let $\theta \in \mathbb{R}$ and for $0 \le t \le T$ consider the process

$$\begin{aligned} \hat{B}^H(t) &= B^H(t) + \theta t^{2H} \\ &= B^H(t) + \int_0^t 2H\theta\tau^{2H-1}d\tau. \end{aligned} \tag{7.3}$$

By fractional Girsanov formula, refer to theorem (5.1) it follows that $\hat{B}^H(t)$ is a fractional Brownian motion under the measure $\mu_{\phi,\gamma}$. We will denote $\tilde{E}_{\mu_{\phi,\gamma}}$ the quasi-conditional expectation under the measure $\mu_{\phi,\gamma}$.

Theorem 7.3 *Let f be a function such that $E_{\mu_\phi}\left[f\left(B^H(T)\right)\right] < \infty$. Then for every $t \le T$*

$$\tilde{E}_{\mu_{\phi,\gamma}}\left[f\left(B^H(T)\right)\left|\mathcal{F}_t^H\right.\right] = \frac{1}{Z(t)}\tilde{E}_{\mu_\phi}\left[f\left(B^H(T)\right)Z(T)\left|\mathcal{F}_t^H\right.\right].$$

Proof. (Necula, 2002).

Let $\hat{f}$ be the Fourier transform of f.

$$\hat{f}(\xi) = \int_{\mathbb{R}} \exp(-ix\xi)\,f(x)\,dx.$$

Then the inverse Fourier transform f of $\hat{f}$ is

$$f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \exp(ix\xi)\,\hat{f}(\xi)\,d\xi.$$

It follows

$$f\left(B^H(T)\right) = \frac{1}{2\pi}\int_{\mathbb{R}} \exp\left(iB^H(T)\,\xi\right)\hat{f}(\xi)\,d\xi.$$

The left hand side

$$\begin{aligned}
&\tilde{E}_{\mu_\phi}\left[f\left(B^H(T)\right)Z(T)\,\middle|\mathcal{F}_t^H\right] \\
=\;&\tilde{E}_{\mu_\phi}\left[\frac{1}{2\pi}\int_{\mathbb{R}}\exp\left(iB^H(T)\,\xi\right)\exp\left(-\theta B^H(T)-\frac{\theta^2}{2}T^{2H}\right)\hat{f}(\xi)\,d\xi\,\middle|\mathcal{F}_t^H\right] \\
=\;&\tilde{E}_{\mu_\phi}\left[\frac{1}{2\pi}\int_{\mathbb{R}}\exp\left(B^H(T)(i\xi-\theta)\right)\exp\left(-\frac{\theta^2}{2}T^{2H}\right)\hat{f}(\xi)\,d\xi\,\middle|\mathcal{F}_t^H\right] \\
=\;&\frac{1}{2\pi}\exp\left(-\frac{\theta^2}{2}T^{2H}\right)\int_{\mathbb{R}}\tilde{E}_{\mu_\phi}\left[\exp\left(B^H(T)(i\xi-\theta)\right)\middle|\mathcal{F}_t^H\right]\hat{f}(\xi)\,d\xi
\end{aligned}$$

applying theorem 7.1 we have

$$\begin{aligned}
&\tilde{E}_{\mu_\phi}\left[f\left(B^H(T)\right)Z(T)\,\middle|\mathcal{F}_t^H\right] \\
=\;&\frac{1}{2\pi}\exp\left(-\frac{\theta^2}{2}T^{2H}\right)\int_{\mathbb{R}}\exp\left(B^H(t)(i\xi-\theta)+\frac{1}{2}(i\xi-\theta)^2\left(T^{2H}-t^{2H}\right)\right)\hat{f}(\xi)\,d\xi \\
=\;&\frac{1}{2\pi}\exp\left(-\frac{\theta^2}{2}T^{2H}\right)\int_{\mathbb{R}}\exp\left(\begin{array}{c}B^H(t)(i\xi-\theta)\\+\left(-\frac{\xi^2}{2}-i\xi\theta+\frac{\theta^2}{2}\right)\left(T^{2H}-t^{2H}\right)\end{array}\right)\hat{f}(\xi)\,d\xi \\
=\;&\frac{1}{2\pi}\exp\left(-\theta B^H(t)-\frac{\theta^2}{2}t^{2H}\right)\int_{\mathbb{R}}\exp\left(\begin{array}{c}B^H(t)(i\xi)\\+\left(-\frac{\xi^2}{2}-i\xi\theta\right)\left(T^{2H}-t^{2H}\right)\end{array}\right)\hat{f}(\xi)\,d\xi \\
=\;&Z(t)\,\frac{1}{2\pi}\int_{\mathbb{R}}\exp\left(B^H(T)(i\xi)+\left(-\frac{\xi^2}{2}-i\xi\theta\right)\left(T^{2H}-t^{2H}\right)\right)\hat{f}(\xi)\,d\xi.
\end{aligned}$$

The right hand side

$$\begin{aligned}
&\tilde{E}_{\mu_{\phi,\gamma}}\left[f\left(B^H(T)\right)\middle|\mathcal{F}_t^H\right] \\
=\;&\tilde{E}_{\mu_{\phi,\gamma}}\left[\frac{1}{2\pi}\int_{\mathbb{R}}\exp\left(i\xi\left(\hat{B}^H(T)-\theta T^{2H}\right)\right)\hat{f}(\xi)\,d\xi\,\middle|\mathcal{F}_t^H\right] \\
=\;&\frac{1}{2\pi}\int_{\mathbb{R}}\tilde{E}_{\mu_{\phi,\gamma}}\left[\exp\left(i\xi\hat{B}^H(T)\right)\middle|\mathcal{F}_t^H\right]\exp\left(-i\xi\theta T^{2H}\right)\hat{f}(\xi)\,d\xi
\end{aligned}$$

applying theorem 7.1 we have

$$\begin{aligned} &= \frac{1}{2\pi}\int_{\mathbb{R}} \exp\left(i\xi\hat{B}^H(t)+\frac{1}{2}(i\xi)^2\left(T^{2H}-t^{2H}\right)\right)\exp\left(-i\xi\theta T^{2H}\right)\hat{f}(\xi)\,d\xi \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} \exp\left(i\xi\hat{B}^H(t)-\frac{\xi^2}{2}\left(T^{2H}-t^{2H}\right)\right)\exp\left(-i\xi\theta T^{2H}\right)\hat{f}(\xi)\,d\xi \end{aligned}$$

and using (7.3)

$$\begin{aligned} &= \frac{1}{2\pi}\int_{\mathbb{R}} \exp\left(i\xi\left(B^H(t)+\theta t^{2H}\right)-\frac{\xi^2}{2}\left(T^{2H}-t^{2H}\right)\right)\exp\left(-i\xi\theta T^{2H}\right)\hat{f}(\xi)\,d\xi \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} \exp\left(i\xi B^H(t)-\frac{\xi^2}{2}\left(T^{2H}-t^{2H}\right)+i\xi\theta t^{2H}-i\xi\theta T^{2H}\right)\hat{f}(\xi)\,d\xi \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} \exp\left(i\xi B^H(t)+\left(-\frac{\xi^2}{2}-i\xi\theta\right)\left(T^{2H}-t^{2H}\right)\right)\hat{f}(\xi)\,d\xi. \end{aligned}$$

It follows that

$$\frac{1}{Z(t)}\tilde{E}_{\mu_\phi}\left[f\left(B^H(T)\right)Z(T)\left|\mathcal{F}_t^H\right.\right]=\tilde{E}_{\mu_{\phi,\gamma}}\left[f\left(B^H(T)\right)\left|\mathcal{F}_t^H\right.\right].$$

■

7.2 The Market

Consider a market with an investment in a money account and a stock that is driven by fractional Brownian motion. For $0\le t\le T$ the money account $A(t)$ at time t develops according to the equation

$$\begin{aligned} dA(t) &= rA(t)\,dt \\ A(0) &= 1 \end{aligned}$$

where $r>0$ is a constant riskless interest rate. The stock price $S(t)$ has the following the dynamics

$$\begin{aligned} dS(t) &= \mu S(t)\,dt+\sigma S(t)\,dB^H(t) \\ S(0) &= s>0 \end{aligned}$$

where μ and $\sigma \neq 0$ are constants, for $0 \leq t \leq T$. From (6.10) the stock price under the risk neutral measure follows the following process

$$dS(t) = rS(t)\,dt + \sigma S(t)\,dB^H(t).$$

Necula's states that he is working under Hu and Øksendal's (2000) framework where they show that the market is complete and that there is a self-financing replicating portfolio of the claim, $(u(t), v(t))$. It is not clear that this is the case as Necula does not use the Wick definition of the portfolio. Let $\tilde{E}_{\mu_\phi}$ denote the quasi-conditional expectation with respect to the risk-neutral measure.

Theorem 7.4 *Fractional risk neutral evaluation. For every $t \in [0, T]$, the price of a bounded $\mathcal{F}_T^H$-measurable$_{H,\text{Ø}}$ claim where $F \in L^2(\mu_\phi)$ is given by*

$$F(t) = e^{-r(T-t)}\tilde{E}_{\mu_\phi}\left[F(T)\,\middle|\mathcal{F}_t^H\right].$$

Proof. (Necula, 2002).

Let $u(t)$ be the amount invested in the money account and $v(t)$ be the amount invested in the stock. The portfolio value has the following form

$$F(t) = u(t)A(t) + v(t)S(t).$$

The dynamics of the self-financing portfolio follows as

$$\begin{aligned} dF(t) &= u(t)\,dA(t) + v(t)\,dS(t) \\ &= u(t)\,rA(t)\,dt + v(t)\left(rS(t)\,dt + \sigma S(t)\,dB^H(t)\right) \\ &= r\left(u(t)A(t) + v(t)S(t)\right)dt + v(t)S(t)\,\sigma dB^H(t) \\ &= rF(t)\,dt + \sigma v(t)S(t)\,dB^H(t). \end{aligned}$$

Denoting $F(T) = F$ we have

$$dF = rFdt + \sigma vSdB^H.$$

Multiplying both sides by the integrating factor e^{-rt} we have

$$\begin{aligned} e^{-rt}dF &= re^{-rt}Fdt + e^{-rt}\sigma v S dB^H \\ e^{-rt}\frac{dF}{dt} + re^{-rt}F &= e^{-rt}\sigma v S \frac{dB^H}{dt} \\ \frac{d}{dt}\left[e^{-rt}F\right] &= e^{-rt}\sigma v S \frac{dB^H}{dt}. \end{aligned}$$

For $0 \leq t \leq T$ we integrate

$$e^{-rt}F(t) = F(0) + \int_0^t e^{-r\tau}\sigma v(\tau) S(\tau) dB^H(\tau) \tag{7.4}$$

and by the fractional Clack-Ocone theorem it follows

$$F(\omega) = E_{\mu_\phi}[F] + \int_0^t \tilde{E}_{\mu_\phi}\left[D_t^H F | \mathcal{F}_t^H\right] dB^H(t).$$

Multiplying both sides by e^{-rT} we obtain

$$e^{-rT}F = E_{\mu_\phi}\left[e^{-rT}F\right] + e^{-rT}\int_0^T \tilde{E}_{\mu_\phi}\left[D_t^H F | \mathcal{F}_t^H\right] dB^H(\tau). \tag{7.5}$$

From the completeness of the market it follows

$$e^{-rt}\sigma v(\tau) S(\tau) = e^{-rT}\tilde{E}_{\mu_\phi}\left[D_t^H F | \mathcal{F}_t^H\right].$$

For $0 \leq \tau \leq T$ we have

$$\tilde{E}_{\mu_\phi}\left[D_t^H F | \mathcal{F}_t^H\right] = e^{r(T-t)}\sigma v(\tau) S(\tau). \tag{7.6}$$

Substituting (7.6) into (7.5) we have

$$e^{-rT}F = E_{\mu_\phi}\left[e^{-rT}F\right] + e^{-rT}\int_0^T e^{r(T-\tau)}\sigma v(\tau) S(\tau) dB^H(\tau),$$

and taking the quasi-conditional expectation

$$\tilde{E}_{\mu_\phi}\left[e^{-rT}F \left| \mathcal{F}_t^H\right.\right] = E_{\mu_\phi}\left[e^{-rT}F\right] + \tilde{E}_{\mu_\phi}\left[\int_0^T e^{-r\tau}\sigma v(\tau) S(\tau) dB^H(\tau) \left| \mathcal{F}_t^H\right.\right].$$

It follows from (5.14) that

$$\tilde{E}_{\mu_\phi}\left[e^{-rT}F \left| \mathcal{F}_t^H\right.\right] = E_{\mu_\phi}\left[e^{-rT}F\right] + \int_0^T e^{-rt}\sigma v(\tau) S(\tau) dB^H(\tau) \tag{7.7}$$

since the discounted price process are martingales under the risk neutral measure we have

$$F(0) = E_{\mu_\phi}\left[e^{-rT}F\right]$$

comparing (7.4) and (7.7), it follows that

$$F(t) = e^{-r(T-t)}\tilde{E}_{\mu_\phi}\left[F\left|\mathcal{F}_t^H\right.\right].$$

■

7.3 Option Pricing Formula

The assumptions for pricing the quasi-conditional fractional Black-Scholes formula and the quasi-conditional fractional Black formula are as follows:

1. The stock price follows equation (6.1).

2. Stochastic differentials are interpreted in the Wick Itô Skorohod sense.

3. The Hurst parameter is $\frac{1}{2} < H < 1$ and the Hurst parameter is constant over time.

4. The drift μ and volatility σ are constant and the r is a constant risk-free rate of interest and the same for all maturities.

5. Working within Hu and Øksendal's (2000) framework.

6. Short selling is allowed.

7. There are no transactions costs or taxes.

8. There are no dividends.

9. Trading is done continuously.

10. All securities are perfectly divisible.

7.3.1 Quasi-Conditional Fractional Black-Scholes Formula

Theorem 7.5 *Quasi-conditional fractional Black-Scholes formula. The price at every $t \in [0,T]$ of an European call option with strike price K and maturity T is given as*

$$C^H(t, S(t)) = S(t) N\left(d_1^H\right) - Ke^{-r(T-t)} N\left(d_2^H\right)$$

where

$$d_1^H = \frac{\ln\left(\frac{S(t)}{K}\right) + r(T-t) + \frac{1}{2}\sigma^2\left(T^{2H} - t^{2H}\right)}{\sigma\sqrt{T^{2H} - t^{2H}}}$$

and

$$d_2^H = \frac{\ln\left(\frac{S(t)}{K}\right) + r(T-t) - \frac{1}{2}\sigma^2\left(T^{2H} - t^{2H}\right)}{\sigma\sqrt{T^{2H} - t^{2H}}}$$

where $N(\cdot)$ is the cumulative probability of the standard normal distribution.

Proof. (Necula, 2002).

The price of the claim under the measure μ_ϕ using theorem 7.4 is given as

$$\begin{aligned} C^H(t, S(t)) &= e^{-r(T-t)} \tilde{E}_{\mu_\phi}\left[\max\left\{(S(T) - K), 0\right\} \middle| \mathcal{F}_t^H\right] \\ &= e^{-r(T-t)} \tilde{E}_{\mu_\phi}\left[\max\left\{S(0)\exp\left(\begin{array}{c}\sigma B^H(T) + rT \\ -\frac{1}{2}\sigma^2 T^{2H}\end{array}\right) - K, 0\right\} \middle| \mathcal{F}_t^H\right] \\ &= e^{-r(T-t)} \tilde{E}_{\mu_\phi}\left[S(T) 1_{\{S(T)>K\}} \middle| \mathcal{F}_t^H\right] - e^{-r(T-t)} K \tilde{E}_{\mu_\phi}\left[1_{\{S(T)>K\}} \middle| \mathcal{F}_t^H\right]. \end{aligned}$$

The boundary for the option to be in the money is

$$S(T) > K$$

in other words

$$S(0)\exp\left(\sigma B^H(T) + rT - \frac{1}{2}\sigma^2 T^{2H}\right) > K.$$

Denote

$$\hat{d}_2 = \frac{\ln\left(\frac{K}{S(0)}\right) - rT + \frac{1}{2}\sigma^2 T^{2H}}{\sigma}.$$

For the option to be in the money is the same as requiring

$$B^H(T) > \hat{d}_2.$$

For the second expectation we apply corollary 7.1

$$\begin{aligned} & \tilde{E}_{\mu_\phi}\left[1_{\{S(T)>K\}}\left|\mathcal{F}_t^H\right.\right] \\ = \; & \tilde{E}_{\mu_\phi}\left[1_{\left\{x>\hat{d}_2\right\}}\left(B^H(T)\right)\left|\mathcal{F}_t^H\right.\right] \\ = \; & \int_{\hat{d}_2}^{\infty} \frac{1}{\sqrt{2\pi\left(T^{2H}-t^{2H}\right)}} \exp\left(-\frac{1}{2}\left(\frac{x-B^H(t)}{\sqrt{T^{2H}-t^{2H}}}\right)^2\right) dx. \end{aligned}$$

For the transformation set

$$\begin{aligned} w &= \frac{x - B^H(t)}{\sqrt{T^{2H}-t^{2H}}} \\ &\Rightarrow \; dw = \frac{1}{\sqrt{T^{2H}-t^{2H}}} dx. \end{aligned}$$

It follows

$$\begin{aligned} \tilde{E}_{\mu_\phi}\left[1_{\{S(T)>K\}}\left|\mathcal{F}_t^H\right.\right] &= \int_{-\infty}^{\frac{B^H(t)-\hat{d}_2}{\sqrt{T^{2H}-t^{2H}}}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{w^2}{2}\right) dw \\ &= N\left(d_2^H\right) \end{aligned}$$

with

$$\begin{aligned} d_2^H &= \frac{B^H(t) - \hat{d}_2}{\sqrt{T^{2H}-t^{2H}}} \\ &= \frac{\sigma B^H(t) - \left(\ln K - \ln S(0) - rT + \frac{1}{2}\sigma^2 T^{2H}\right)}{\sigma\sqrt{T^{2H}-t^{2H}}}. \end{aligned}$$

We see that

$$\ln S(t) = \ln S(0) + \sigma B^H(t) + rt - \frac{1}{2}\sigma^2 t^{2H}$$

rearranging the terms

$$\ln S(0) = \ln S(t) - \sigma B^H(t) - rt + \frac{1}{2}\sigma^2 t^{2H}$$

it follows that

$$\begin{aligned} d_2^H &= \frac{\sigma B^H(t) - \left(\ln K - \left(\ln S(t) - \sigma B^H(t) - rt + \frac{1}{2}\sigma^2 t^{2H}\right) - rT + \frac{1}{2}\sigma^2 T^{2H}\right)}{\sqrt{T^{2H} - t^{2H}}} \\ &= \frac{\sigma B^H(t) - \left(\ln K - \ln S(t) + \sigma B^H(t) + rt - \frac{1}{2}\sigma^2 t^{2H} - rT + \frac{1}{2}\sigma^2 T^{2H}\right)}{\sqrt{T^{2H} - t^{2H}}} \\ &= \frac{\sigma B^H(t) - \ln \dfrac{K}{S(t)} - \sigma B^H(t) - rt + \frac{1}{2}\sigma^2 t^{2H} + rT - \dfrac{1}{2}\sigma^2 T^{2H}}{\sqrt{T^{2H} - t^{2H}}} \\ &= \frac{\ln\left(\dfrac{S(t)}{K}\right) + r(T-t) - \dfrac{1}{2}\sigma^2\left(T^{2H} - t^{2H}\right)}{\sigma\sqrt{T^{2H} - t^{2H}}}. \end{aligned}$$

Now consider the process for $0 \leq t \leq T$

$$\begin{aligned} \hat{B}^H(t) &= B^H(t) - \sigma t^{2H} \\ B^H(t) &= \hat{B}^H(t) + \sigma t^{2H}. \end{aligned}$$

The fractional Girsanov theorem insures that there is a measure $\mu_{\phi,\gamma}$ such that $\hat{B}^H(t)$ is as fractional Brownian motion under $\mu_{\phi,\gamma}$. Let

$$Z(t) = \exp\left(\sigma B^H(t) - \frac{\sigma^2}{2} t^{2H}\right).$$

Solving for the first expectation and using theorem 7.3 we have the quasi-conditional expectation as

$$\begin{aligned} & \tilde{E}_{\mu_\phi}\left[S(T) 1_{\{S(T)>K\}} \left| \mathcal{F}_t^H\right.\right] \\ = \;& \tilde{E}_{\mu_\phi}\left[\left(S(0)\exp\left(rT + \sigma B^H(T) - \frac{1}{2}\sigma^2 T^{2H}\right)\right) 1_{\left\{x>\hat{d}_2\right\}}\left(B^H(T)\right) \left| \mathcal{F}_t^H\right.\right] \\ = \;& e^{rT} S(0)\, \tilde{E}_{\mu_\phi}\left[Z(T) 1_{\{S(T)>K\}} \left| \mathcal{F}_t^H\right.\right] \\ = \;& e^{rT} S(0)\, Z(t)\, \tilde{E}_{\mu_{\phi,\gamma}}\left[1_{\{S(T)>K\}} \left| \mathcal{F}_t^H\right.\right]. \end{aligned}$$

The stock price has the following form under the measure $\mu_{\phi,\gamma}$

$$\begin{aligned} S(T) &= S(0)\exp\left(rT - \frac{1}{2}\sigma^2 T^{2H} + \sigma B^H(T)\right) \\ &= S(0)\exp\left(rT - \frac{\sigma^2}{2}T^{2H} + \sigma\left(\hat{B}^H(T) + \sigma T^{2H}\right)\right) \\ &= S(0)\exp\left(rT + \frac{\sigma^2}{2}T^{2H} + \sigma\hat{B}^H(T)\right). \end{aligned}$$

The stock price under the new measure brings about a new boundary change

$$S(T) > K$$

implying

$$S(0)\exp\left(rT + \frac{1}{2}\sigma^2 T^{2H} + \sigma\hat{B}^H(T)\right) > K$$

therefore

$$\sigma\hat{B}^H(T) > \ln K - \ln S(0) - rT - \frac{1}{2}\sigma^2 T^{2H}.$$

Denote

$$\hat{d}_1 = \frac{\ln\left(\frac{K}{S(0)}\right) - rT - \frac{1}{2}\sigma^2 T^{2H}}{\sigma}.$$

Applying theorem 7.3 we have

$$\begin{aligned} \tilde{E}_{\mu_{\phi,\gamma}}\left[1_{\{S(T)>K\}} \middle| \mathcal{F}_t^H\right] &= \tilde{E}_{\mu_{\phi,\gamma}}\left[1_{\{x>\hat{d}_1\}}\hat{B}^H(T) \middle| \mathcal{F}_t^H\right] \\ &= \int_{\hat{d}_1}^{\infty} \frac{1}{\sqrt{2\pi\left(T^{2H} - t^{2H}\right)}} \exp\left(-\frac{1}{2}\left(\frac{x - \hat{B}^H(t)}{\sqrt{T^{2H} - t^{2H}}}\right)^2\right) dx. \end{aligned}$$

For the transformation set

$$y = \frac{x - \hat{B}^H(T)}{\sqrt{T^{2H} - t^{2H}}} \Rightarrow x = y\sqrt{T^{2H} - t^{2H}} + \hat{B}^H(T)$$

differentiating

$$dy = \frac{1}{\sqrt{T^{2H} - t^{2H}}}dx \Rightarrow dx = \sqrt{T^{2H} - t^{2H}}dy.$$

It follows

$$\begin{aligned}
\tilde{E}_{\mu_{\phi,\gamma}}\left[1_{\{S(T)>K\}}\left|\mathcal{F}_t^H\right.\right] &= \int_{\frac{\hat{d}_1-\hat{B}^H(T)}{\sqrt{T^{2H}-t^{2H}}}}^{\infty} \frac{1}{\sqrt{2\pi\left(T^{2H}-t^{2H}\right)}}\exp\left(-\frac{1}{2}y^2\right)\sqrt{T^{2H}-t^{2H}}dy \\
&= \int_{-\infty}^{\frac{\hat{B}^H(T)-\hat{d}_1}{\sqrt{T^{2H}-t^{2H}}}} \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{y^2}{2}\right)dy \\
&= N\left(\frac{\hat{B}^H\left(T\right)-\hat{d}_1}{\sqrt{T^{2H}-t^{2H}}}\right) \\
&= N\left(d_1^H\right).
\end{aligned}$$

Following similar steps as above we get

$$d_1^H = \frac{\ln\left(\frac{S\left(t\right)}{K}\right)+r\left(T-t\right)+\frac{1}{2}\sigma^2\left(T^{2H}-t^{2H}\right)}{\sigma\sqrt{T^{2H}-t^{2H}}}.$$

And since

$$\begin{aligned}
S\left(t\right) &= S\left(0\right)\exp\left(rt-\frac{\sigma^2}{2}t^{2H}+\sigma B^H\left(t\right)\right) \\
&= S\left(0\right)e^{rt}Z\left(t\right)
\end{aligned}$$

we can rewrite

$$Z\left(t\right) = \frac{S\left(t\right)}{S\left(0\right)}e^{-rt}.$$

Thus the first expectation follows as

$$\begin{aligned}
\tilde{E}_{\mu_\phi}\left[S\left(T\right)1_{\{S(T)>K\}}\left|\mathcal{F}_t^H\right.\right] &= e^{rT}S\left(0\right)Z\left(t\right)N\left(d_1^H\right) \\
&= e^{rT}e^{-rt}S\left(t\right)N\left(d_1^H\right) \\
&= e^{r(T-t)}S\left(t\right)N\left(d_1^H\right).
\end{aligned}$$

Therefore the price of the call option follows as

$$\begin{aligned}
& C^H\left(t,S\left(t\right)\right) \\
&= e^{-r(T-t)}\tilde{E}_{\mu_\phi}\left[S\left(T\right)1_{\{S(T)>K\}}\left|\mathcal{F}_t^H\right.\right]-e^{-r(T-t)}K\tilde{E}_{\mu_\phi}\left[1_{\{S(T)>K\}}\left|\mathcal{F}_t^H\right.\right] \\
&= S\left(t\right)N\left(d_1^H\right)-e^{-r(T-t)}KN\left(d_2^H\right).
\end{aligned}$$

■

Since Law of one price holds in the fractional market, the put-call parity will hold. Using similar arguments as above the price at every $t \in [0,T]$ of a quasi-conditional fractional European put option with strike price K and maturity T is given by

$$P^H(t, S(t)) = Ke^{-r(T-t)}N\left(-d_2^H\right) - S(t)N\left(-d_1^H\right).$$

7.3.2 Quasi-Conditional Fractional Black Formula

The quasi-conditional future model was used for the empirical comparison that was done in the later chapters. We now assume that the futures prices, instead of the stock prices, follow a fractional Brownian motion with dynamics under the measure μ_ϕ given by

$$dF(t) = F(t)\,\sigma dB^H(t)$$

and using Wick calculus the solution follows as

$$F(T) = F(t)\exp\left(\sigma B^H(t) - \frac{1}{2}\sigma^2 t^{2H}\right).$$

Theorem 7.6 *Quasi-conditional fractional Black formula. The price at every $t \in [0,T]$ of an European call option on a futures contract with strike price K and maturity T is given as*

$$c^H(t, F(t)) = e^{-r(T-t)}\left[F(t)N\left(d_1^{H*}\right) - KN\left(d_2^{H*}\right)\right]$$

where

$$d_1^{H*} = \frac{\ln\left(\frac{F(t)}{K}\right) + \frac{1}{2}\sigma^2\left(T^{2H} - t^{2H}\right)}{\sigma\sqrt{T^{2H} - t^{2H}}}$$

and

$$d_2^{H*} = \frac{\ln\left(\frac{F(t)}{K}\right) - \frac{1}{2}\sigma^2\left(T^{2H} - t^{2H}\right)}{\sigma\sqrt{T^{2H} - t^{2H}}}.$$

Proof. The formula can be obtained using similar arguments as in the proof above in theorem 7.5 (see Liu, 2007). ■

Also at time t the price of a quasi-conditional European put option, with exercise date T and exercise price K on a futures contract is given by

$$p^H(t, F(t)) = e^{-r(T-t)} \left[KN\left(-d_2^{H*}\right) - F(t) N\left(-d_1^{H*}\right) \right].$$

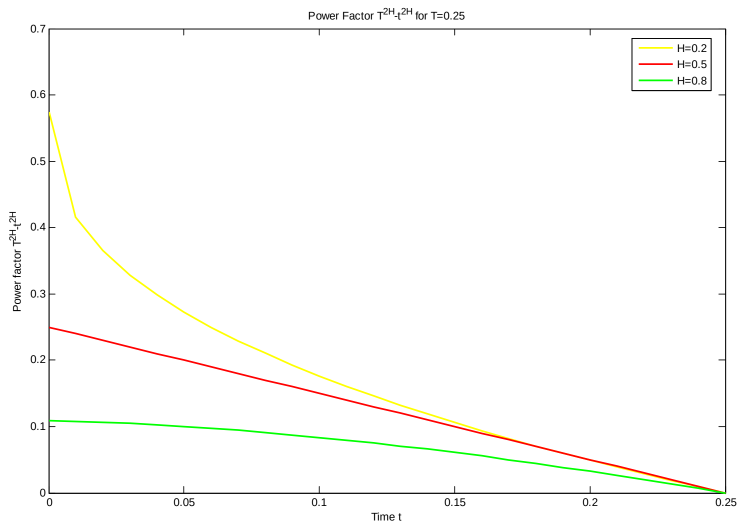

Figure 7.1: Necula factor $\left(T^{2H} - t^{2H}\right)$ for a fixed $T = 0.25$.

7.4 Model Analysis

Note for $t = 0$ fixed, the graph of the factor $T^{2H} - t^{2H}$ plotted against time to maturity will be the same as figure 6.1. We used a Hurst parameter less than $\frac{1}{2}$ for comparison with the other models even though in this model it is assumed that $H > \frac{1}{2}$. Fixing T and varying t we plot the factor for varying Hurst parameters. Figure 7.1 is for $T = 0.25$, we see that the anti-persistent case decreases faster and is above the normal, while the persistent case decreases at a much slower rate. Figure 7.2 is for a fixed $T = 0.5$, where we have a cross-over between the anti-persistent and the persistent case. Figure 7.3 is for a fixed $T = 1$ in this case the persistent case is above the normal.

We plot the pricing formula for varying Hurst parameters, for a fixed time T and for three different t. For shorter time to expiration $T = 0.5$ and $t = 0.1$, $t = 0.25$

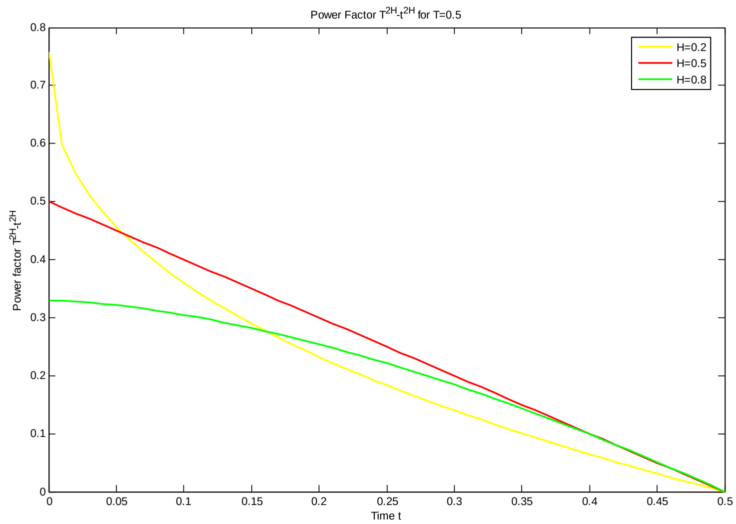

Figure 7.2: Necula factor $\left(T^{2H} - t^{2H}\right)$ for a fixed $T = 0.5$.

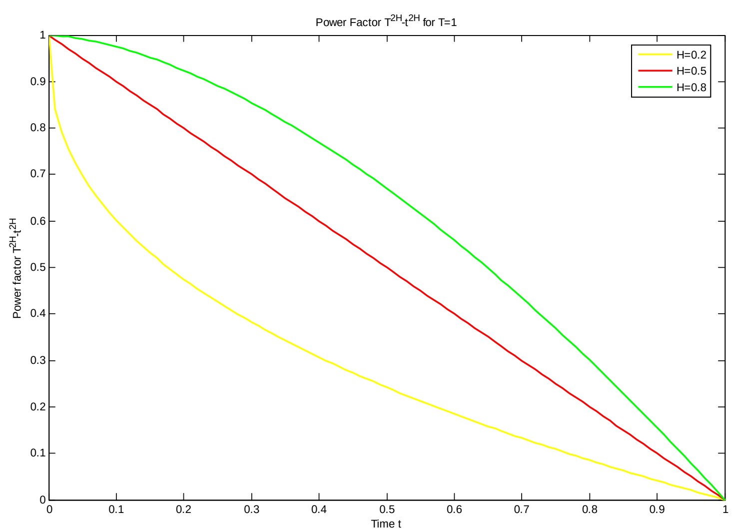

Figure 7.3: Necula factor $\left(T^{2H} - t^{2H}\right)$ for a fixed $T = 1$.

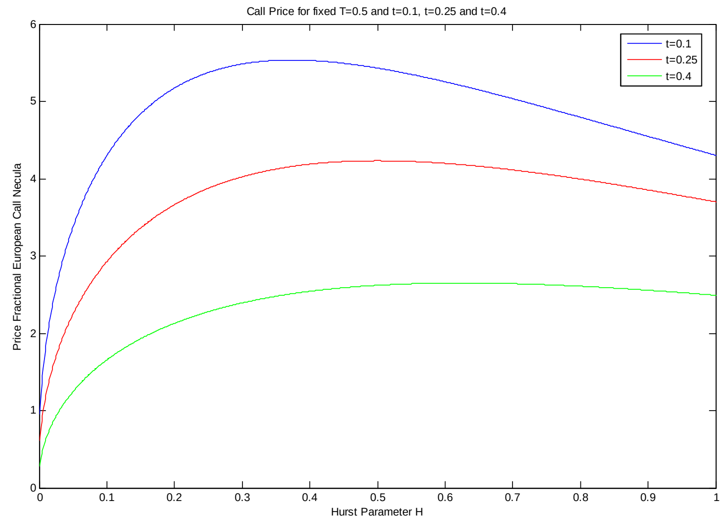

Figure 7.4: Necula price of European call with varying Hurst and $t = 0.1$, $t = 0.25$ and $t = 0.4$. Fixed parameters $K = 100$, $S = 100$, $\sigma = 0.2$, $r = 0.02$ and $T = 0.5$.

and $t = 0.4$ figure 7.4 shows the price of a call option increases then decreases as the Hurst parameter increases. For larger time to expiration $T = 5$ and $t = 1$, $t = 2.5$ and $t = 4$ figure 7.5 shows the price of a call option increases as the Hurst parameter increases.

The quasi-conditional Black function was plotted in three dimensions for $H \in \left(\frac{1}{2}, 1\right)$ and $\sigma \in (0,1)$. Setting $F(t) = 110$, $S(t) = 100$, $K = 100$ and choosing a time to maturity to be $(T-t) = 0.25$ we obtain figure 7.6. As $\sigma \to 0$ and for all $H \in \left(\frac{1}{2}, 1\right)$ we see that the prices are the largest. Choosing $(T-t) = 0.75$ we obtain figure 7.7 and as $\sigma \to 0$ and $H \to \frac{1}{2}$ the prices are the largest.

Choosing $(T-t) = 5$ we obtain figure 7.8 and as $\sigma \to 1$ and $H \to 1$ the prices are the largest.

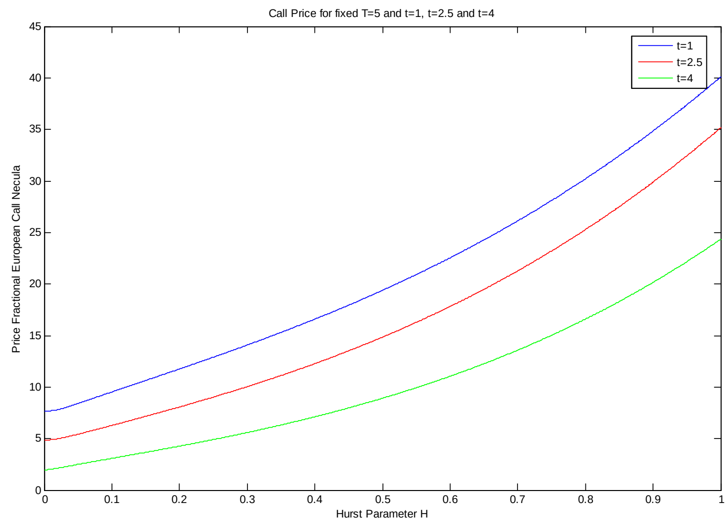

Figure 7.5: Necula price of European call with varying Hurst and $t = 1$, $t = 2.5$ and $t = 4$. Fixed parameters $K = 100$, $S = 100$, $\sigma = 0.2$, $r = 0.02$ and $T = 5$.

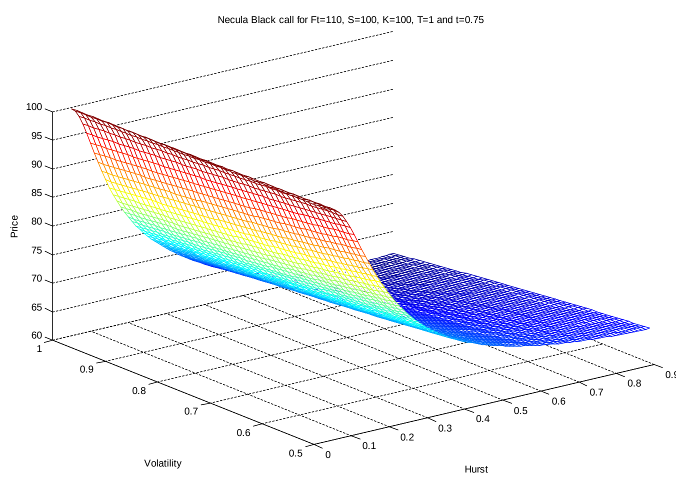

Figure 7.6: Quasi-Conditional Black function Price vs Hurst vs Volatility $F_t = 110$, $S_t = 100, K = 100, T = 1$ and $t = 0.75$.

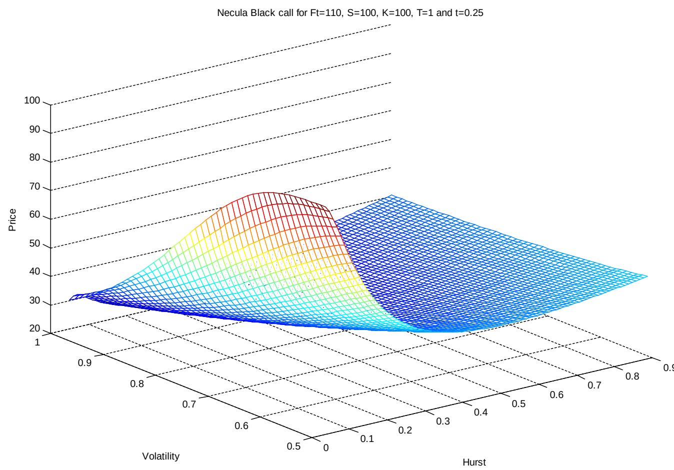

Figure 7.7: Quasi-Conditional Black function Price vs Hurst vs Volatility $F_t = 110,\ S_t = 100, K = 100, T = 1$ and $t = 0.25$.

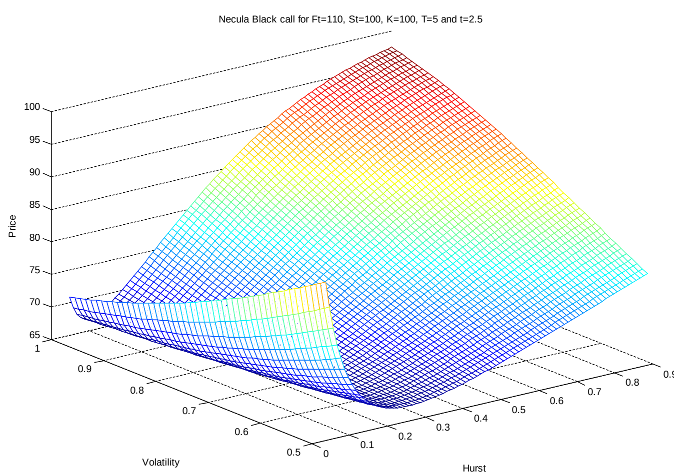

Figure 7.8: Quasi-Conditional Black function price vs Hurst vs volatility $F_t = 110,\ S_t = 100, K = 100, T = 5$ and $t = 2.5$.

For smaller times to maturity the graph is concave up while for larger time horizons the graph is concave down.

Necula (2003) and (2007) priced Barrier options in the same quasi-conditional market setting for $H > \frac{1}{2}$.

Chapter 8

Rostek and Schöbel's fBm Pricing Model

8.1 Introduction

There is a weak form of arbitrage know as "free lunch with vanishing risk" present in the fractional Brownian motion models considered in the previous chapters 6 and 7. Efficient markets are priced in such a way that prices move only when new information is received. Therefore, it is assumed that investors react immediately when the information is received and due to the large number of investors the prices will be fair. If one assumes that a single investor cannot be as fast as the market, i.e. under a very small time intervals all arbitrage possibilities are eliminated (see Cheridito, 2003). More precisely a restriction is put on the minimal time between two consecutive transactions. We will assume this in the following model.

Rostek and Schöbel (2010) derived a formula for fractional European options using conditional expectation and an equilibrium pricing approach, assuming risk averse traders that trade in discrete time. The underlying stock is assumed to follow a conditional fractional Brownian motion. Recall that a market is complete if for any contingent claim there exists a self-financing trading strategy such that at any time t, the strategy replicates the claim. In other words all possible gambles on future states of the world can be constructed with existing assets. A fundamental theorem of derivative pricing states that if the model is arbitrage free then the market is complete if and only if the martingale measure is unique (Björk, 2004). Since there is a minimal time between transactions, dynamical hedging and the possibility of continuous adjustment of the replicating portfolio is not possible; hence we cannot dynamically replicate all contingent claims. Due to the fact that fractional Brownian markets are dynamically incomplete, in this model a stochastic discount factor is derived and used for pricing instead of a change to a risk neutral measure.

Consider a filtered probability space $\left(\Omega, \mathcal{F}_t^H, P\right)$, where Ω is the state space of random events, $\mathcal{F}_t^H$ is the σ-field generated by all $B^H(s)$, for $s \leq t$ on Ω and P is the distribution of $B^H(s)$. Rostek (2009) defines an equivalence class for the conditional distribution of fractional Brownian motion based on an infinite knowledge of the past. For $s \in \mathbb{R}$ and $T > t$ let $B^H(s)$ be a fractional Brownian motion with $\frac{1}{2} < H < 1$ for all $\omega \in \Omega$. The equivalence class is defined as

$$[\omega_1]_t = \left\{\omega \in \Omega \left| B^H(s)(\omega) = B^H(s)(\omega_1)\right., \text{for all } s \in (-\infty, t]\right\}.$$

The distribution of $B^H(T)$ is conditional on the σ-field generated by all $s \leq t$

$$\mathcal{F}_t^H = \sigma\left(B^H(s),\ s \leq t\right).$$

The conditional expectation of fractional Brownian motion $B^H(T)$, $T > t > 0$, based on $\mathcal{F}_t^H$ and concerning all information about the past is a random variable. $E\left[B^H(T)\,|\mathcal{F}_t^H\right](\omega_1)$ is the expectation of $B^H(T)$ over all $\omega \in [\omega_1]_t$.

Lemma 8.1 *Let $B^H(s)$, $s \in \mathbb{R}$ be a fractional Brownian motion with $0 < H < 1$, then for each $T > t > 0$, the conditional expectation of $B^H(T)$ based on $\mathcal{F}_t^H$ can be represented as*

$$\begin{aligned}
\hat{B}^H(T,t) &= E_P\left[B^H(T)\left|\mathcal{F}_t^H\right.\right] \\
&= B^H(t) + (T-t)^{H+\frac{1}{2}} \int_{-\infty}^{t} g(T,t,s)\,ds \\
&= B^H(t) + \hat{\mu}_{T,t}
\end{aligned}$$

where $\hat{\mu}_{T,t}$ is the mean historical evolution of the past and

$$g(T,t,s) = \frac{\sin\left(\pi\left(H-\frac{1}{2}\right)\right)\left(B^H(s)-B^H(t)\right)}{\pi(t-s)^{H+\frac{1}{2}}(T-s)}.$$

See Rostek, 2009.

Lemma 8.2 *The conditional variance of fractional Brownian motion $B^H(T)$ for $T>t>0$, based on $\mathcal{F}_t^H$ is represented by*

$$\begin{aligned} Var\left[B^H(T)\left|\mathcal{F}_t^H\right.\right] &= \nu_H(T-t)^{2H} \\ &= \hat{\sigma}^2_{T,t} \end{aligned}$$

where

$$v_H = \begin{cases} \dfrac{\sin\left(\pi\left(H-\frac{1}{2}\right)\right)}{\pi\left(H-\frac{1}{2}\right)}\dfrac{\Gamma\left(\frac{3}{2}-H\right)^2}{\Gamma(2-2H)} & \text{if } H \neq \dfrac{1}{2} \\ 1 & \text{if } H = \dfrac{1}{2} \end{cases} \tag{8.1}$$

and $\Gamma(\cdot)$ is the Gamma function.

Proof. (Rostek, 2009).

We have that

$$\begin{aligned}
\hat{\sigma}^2_{T,t} &= Var\left[B^H(T)\left|\mathcal{F}^H_t\right.\right] \\
&= E_P\left[\left(B^H(T)-\hat{B}^H(T)\right)^2\left|\mathcal{F}^H_t\right.\right] \\
&= E_P\left[\left(\left(B^H(T)-B^H(t)\right)-\hat{\mu}_{T,t}\right)^2\left|\mathcal{F}^H_t\right.\right] \\
&= E_P\left[\left(B^H(T)-B^H(t)\right)^2-2\left(B^H(T)-B^H(t)\right)\hat{\mu}_{T,t}+\hat{\mu}^2_{T,t}\left|\mathcal{F}^H_t\right.\right] \\
&= E_P\left[E\left[\left(B^H(T)-B^H(t)\right)^2\left|\mathcal{F}^H_t\right.\right]\right]-2E_P\left[\left(B^H(T)-B^H(t)\right)\hat{\mu}_{T,t}\left|\mathcal{F}^H_t\right.\right] \\
&\quad +E_P\left[\hat{\mu}_{T,t}\left|\mathcal{F}^H_t\right.\right]^2 \\
&= E_P\left[B^H(T)-B^H(t)\right]^2-2\left[E_P\left[B^H(T)\left|\mathcal{F}^H_t\right.\right]-B^H(t)\right]E_P\left[\hat{\mu}_{T,t}\left|\mathcal{F}^H_t\right.\right] \\
&\quad +E_P\left[\hat{\mu}_{T,t}\left|\mathcal{F}^H_t\right.\right]^2 \\
&= E_P\left[B^H(T)-B^H(t)\right]^2-2\left[B^H(t)+\hat{\mu}_{T,t}-B^H(t)\right]E_P\left[\hat{\mu}_{T,t}\left|\mathcal{F}^H_t\right.\right] \\
&\quad +E_P\left[\hat{\mu}_{T,t}\left|\mathcal{F}^H_t\right.\right]^2 \\
&= E_P\left[B^H(T)-B^H(t)\right]^2-2E_P\left[\hat{\mu}_{T,t}\right]^2+E_P\left[\hat{\mu}_{T,t}\right]^2 \\
&= E_P\left[B^H(T)-B^H(t)\right]^2-E_P\left[\hat{\mu}_{T,t}\right]^2
\end{aligned}$$

and

$$\begin{aligned}
E_P\left[B^H(T)-B^H(t)\right]^2 &= E_P\left[B^H(T)\right]^2-2E_P\left[B^H(T)B^H(t)\right]-E_P\left[B^H(t)\right]^2 \\
&= T^{2H}-2\frac{1}{2}\left[T^{2H}+t^{2H}-(T-t)^{2H}\right]+t^{2H} \\
&= (T-t)^{2H}
\end{aligned}$$

and

$$\begin{aligned}
E_P\left[\hat{\mu}_{T,t}\right]^2 &= E_P\left[\int_{-\infty}^{t} g(T-t,s-t)\,dB^H(s)\right]^2 \\
&= E_P\left[\int_{-\infty}^{t} g(T-t,s-t)\,dB^H(s)\int_{-\infty}^{t} g(T-t,s-t)\,dB^H(s)\right] \\
&= \int_{-\infty}^{t}\int_{-\infty}^{t} g((T-t),(\upsilon-t))\,g((T-t),(\omega-t))\,\phi(\upsilon,\omega)\,d\upsilon d\omega.
\end{aligned}$$

Setting $x = t_0 - \upsilon$ and $y = t_0 - \omega$ it follows that

$$\begin{aligned} E_P \left[\hat{\mu}_{T,t}\right]^2 &= \int_0^\infty \int_0^\infty g\left((T-t),(-x)\right) g\left((T-t),(-y)\right) \phi\left(x,y\right) dxdy \\ &= (T-t)^{2H}\left(1-\nu_H\right) \end{aligned}$$

and we obtain

$$\begin{aligned} \hat{\sigma}^2_{T,t} &= (T-t)^{2H} - (T-t)^{2H}\left(1-\nu_H\right) \\ &= (T-t)^{2H}\left[1-\left(1-\nu_H\right)\right] \\ &= \nu_H (T-t)^{2H}. \end{aligned}$$

■

Theorem 8.1 *The conditional distribution of $B^H(T)$ based on the observation $[\omega_1]_t$ is normal with expectation*

$$\begin{aligned} E_P\left[B^H(T)\left|\mathcal{F}^H_t\right.\right](\omega_1) &= B^H(t) + (T-t)^{H+\frac{1}{2}} \int_{-\infty}^t g(T,t,s)\,ds \\ &= B^H(t) + \hat{\mu}_{T,t} \end{aligned} \tag{8.2}$$

and variance

$$\begin{aligned} Var\left[B^H(T)\left|\mathcal{F}^H_t\right.\right](\omega_1) &= \nu_H (T-t)^{2H} \\ &= \hat{\sigma}^2_{T,t}. \end{aligned} \tag{8.3}$$

Proof. (Rostek, 2009).

This result follows from theorem 8.1 and lemma 8.2 and the Gaussian property of the process $B^H(t)$. ■

8.2 Conditional Stock Process

The stock price process which is driven by fractional Brownian motion has the following dynamics

$$dS(t) = \mu S(t)\,dt + \sigma S(t)\,dB^H(t).$$

Consider the conditional distribution of $S(T)$ given all the history up to time $t_1 < T$, in other words we restrict $S(T)$ to the equivalence class $[\omega_1]_{t_1}$. Let the conditional stock process be denoted as $\hat{S}(t) = S(t) \left| [\omega_1]_{t_1} \right.$ and the process is restricted to only part of the probability space $\left(\Omega, \mathcal{F}_t^H, P\right)$ namely to a space that is generated by the equivalence class $[\omega_1]_{t_1}$ which is $\left([\omega_1]_{t_1}, \sigma\left([\omega_1]_{t_1}\right), \hat{P}\right)$, where $\hat{P}$ is the conditional probability.

Theorem 8.2 *For $t > t_1$ let $\hat{S}(t)$ be the conditional process of geometric fractional Brownian motion. Then the following conditional fractional Itô theorem holds*

$$\begin{aligned} & F\left(T, \hat{S}(T)\right) \\ = \; & F\left(t_1, \hat{S}(t_1)\right) + \int_{t_1}^{T} \frac{\partial F}{\partial t}\left(t, \hat{S}(t)\right) dt + \int_{t_1}^{T} \frac{\partial F}{\partial x}\left(t, \hat{S}(t)\right) \mu(t) \hat{S}(t)\, dt \\ & + \int_{t_1}^{T} \frac{\partial F}{\partial x}\left(t, \hat{S}(t)\right) \sigma(t) \hat{S}(t)\, d\hat{B}^H(t) \\ & + \nu_H H \sigma(t)^2 \int_{t_1}^{T} (t - t_1)^{2H-1} \frac{\partial^2 F}{\partial x^2}\left(t, \hat{S}(t)\right) \hat{S}(t)^2\, dt. \end{aligned}$$

For the proof see Rostek (2009, page 89).

Let

$$F\left(t, \hat{S}(t)\right) = \ln\left(\hat{S}(t)\right).$$

The partial derivatives are

$$\frac{\partial F}{\partial t}\left(t, \hat{S}(t)\right) = 0, \quad \frac{\partial F}{\partial x}\left(t, \hat{S}(t)\right) = \frac{1}{\hat{S}(t)}, \quad \frac{\partial^2 F}{\partial x^2}\left(t, \hat{S}(t)\right) = -\frac{1}{\hat{S}(t)^2}.$$

Applying the conditional Itô formula we get

$$F\left(T, \hat{S}(T)\right) - F\left(t_1, \hat{S}(t_1)\right) = \int_{t_1}^{T} \mu(t)\, dt - \nu_H H \sigma^2 \int_{t_1}^{T} t^{2H-1} dt + \sigma \int_{t_1}^{T} d\hat{B}^H(t)$$

and differentiating we have

$$\ln \hat{S}(T) - \ln \hat{S}(t_1) = \mu(T - t_1) - \frac{1}{2} \nu_H \sigma^2 (T - t_1)^{2H} + \sigma\left(\hat{B}^H(T) - \hat{B}^H(t_1)\right). \quad (8.4)$$

Therefore the solution of the conditional stock process is

$$\hat{S}(T) = \hat{S}(t_1)\exp\left(\mu(T-t_1) - \frac{1}{2}\nu_H\sigma^2(T-t_1)^{2H} + \sigma\left(\hat{B}^H(T) - \hat{B}^H(t_1)\right)\right).$$

Remark 8.1 *For the rest of the section we will write the conditional stock price process $\hat{S}$ as S, the conditional fractional Brownian motion $\hat{B}^H$ as B^H and the probability measure $\hat{P}$ as P for notational convenience.*

We define M_S differently than in (Rostek and Schöbel, 2010). Rostek and Schöbel subtract $\ln S(t)$ from the conditional expected value of $\ln S(T)$. We get the same final results.

Lemma 8.3 *The conditional log-normal process $\ln S(T)$ is normally distributed. The expectation M_S and variance Σ_S^2 is given as*

$$M_S = \ln S(t) + \mu(T-t) - \frac{1}{2}\nu_H\sigma^2(T-t)^{2H} + \sigma\hat{\mu}_{T,t} \tag{8.5}$$

and

$$\Sigma_S^2 = \nu_H\sigma^2(T-t)^{2H}. \tag{8.6}$$

Proof. (Rostek and Schöbel, 2010).

From equation (8.4) we have

$$\ln S(T) = \ln S(t) + \mu(T-t) - \frac{1}{2}\nu_H\sigma^2(T-t)^{2H} + \sigma\left(B^H(T) - B^H(t)\right).$$

Applying (8.2) and theorem 8.1 the expectation is

$$\begin{aligned} & M_S \\ = \; & E_P\left[\ln S(T)\,\middle|\,\mathcal{F}_t^H\right] \\ = \; & E_P\left[\ln S(t) + \mu(T-t) - \frac{1}{2}\nu_H\sigma^2(T-t)^{2H} + \sigma\left(B^H(T) - B^H(t)\right)\,\middle|\,\mathcal{F}_t^H\right] \\ = \; & \ln S(t) + \mu(T-t) - \frac{1}{2}\nu_H\sigma^2(T-t)^{2H} + \sigma\left(E_P\left[B^H(T)\,\middle|\,\mathcal{F}_t^H\right] - B^H(t)\right) \\ = \; & \ln S(t) + \mu(T-t) - \frac{1}{2}\nu_H\sigma^2(T-t)^{2H} + \sigma\hat{\mu}_{T,t} \end{aligned}$$

and the variance follows as

$$\begin{aligned}
&\Sigma_S^2 \\
= \;& Var\left[\ln S(T)\left|\mathcal{F}_t^H\right.\right] \\
= \;& E_P\left[\ln S(T) - M_S\left|\mathcal{F}_t^H\right.\right]^2 \\
= \;& E_P\left[\begin{array}{c}\left(\ln S(t) + \mu(T-t) - \frac{1}{2}\nu_H\sigma^2(T-t)^{2H} + \sigma\left(B^H(T) - B^H(t)\right)\right) \\ -\left(\ln S(t) + \mu(T-t) - \frac{1}{2}\nu_H\sigma^2(T-t)^{2H} + \sigma\hat{\mu}_{T,t}\right)\end{array}\left|\mathcal{F}_t^H\right.\right]^2 \\
= \;& E_P\left[\sigma\left(B^H(T) - B^H(t)\right) - \sigma\hat{\mu}_{T,t}\left|\mathcal{F}_t^H\right.\right]^2 \\
= \;& E_P\left[\sigma^2\left(B^H(T) - B^H(t)\right)^2 - 2\sigma^2\hat{\mu}_{T,t}\left(B^H(T) - B^H(t)\right) + \sigma^2\hat{\mu}_{T,t}^2\left|\mathcal{F}_t^H\right.\right] \\
= \;& \sigma^2 E_P\left[\left(B^H(T) - B^H(t)\right)^2\left|\mathcal{F}_t^H\right.\right] - 2\sigma^2 E_P\left[\hat{\mu}_{T,t}\left|\mathcal{F}_t^H\right.\right]E_P\left[B^H(T) - B^H(t)\left|\mathcal{F}_t^H\right.\right] \\
& +\sigma^2 E_P\left[\hat{\mu}_{T,t}\left|\mathcal{F}_t^H\right.\right]^2 \\
= \;& \sigma^2 E_P\left[\left(B^H(T) - B^H(t)\right)^2\left|\mathcal{F}_t^H\right.\right] - 2\sigma^2\left(\hat{\mu}_{T,t}\right)\left(E_P\left[B^H(T)\left|\mathcal{F}_t^H\right.\right] - B^H(t)\right) \\
& +\sigma^2\left(\hat{\mu}_{T,t}\right)^2 \\
= \;& \sigma^2 E_P\left[\left(B^H(T) - B^H(t)\right)^2\left|\mathcal{F}_t^H\right.\right] - 2\sigma^2\left(\hat{\mu}_{T,t}\right)^2 + \sigma^2\left(\hat{\mu}_{T,t}\right)^2 \\
= \;& \sigma^2 E_P\left[\left(B^H(T) - B^H(t)\right)^2\left|\mathcal{F}_t^H\right.\right] - \sigma^2\left(\hat{\mu}_{T,t}\right)^2.
\end{aligned}$$

And

$$\begin{aligned}
& E_P\left[\left(B^H(T) - B^H(t)\right)^2\left|\mathcal{F}_t^H\right.\right] \\
= \;& E_P\left[\left(B^H(T)\right)^2 - 2B^H(T)B^H(t) + \left(B^H(t)\right)^2\left|\mathcal{F}_t^H\right.\right] \\
= \;& E_P\left[\left(B^H(T)\right)^2\left|\mathcal{F}_t^H\right.\right] - 2E_P\left[B^H(T)\left|\mathcal{F}_t^H\right.\right]B^H(t) + \left(B^H(t)\right)^2 \\
= \;& Var\left[B^H(T)\left|\mathcal{F}_t^H\right.\right] + E_P\left[B^H(T)\left|\mathcal{F}_t^H\right.\right]^2 - 2E_P\left[B^H(T)\left|\mathcal{F}_t^H\right.\right]B^H(t) + \left(B^H(t)\right)^2 \\
= \;& \nu_H(T-t)^{2H} + \left(B^H(t) + \hat{\mu}_{T,t}\right)^2 - 2\left(B^H(t) + \hat{\mu}_{T,t}\right)B^H(t) + \left(B^H(t)\right)^2 \\
= \;& \nu_H(T-t)^{2H} + \left(\hat{\mu}_{T,t}\right)^2.
\end{aligned}$$

It follows

$$\begin{aligned} \Sigma_S^2 &= \sigma^2 \left(\nu_H (T-t)^{2H} + (\hat{\mu}_{T,t})^2 \right) - \sigma^2 (\hat{\mu}_{T,t})^2 \\ &= \nu_H \sigma^2 (T-t)^{2H}. \end{aligned}$$

■

Using properties of the log-normal distribution, the stock price process $S(T)$ is log-normally distributed with the following conditional expectation

$$\begin{aligned} & E_P\left[S(T) \middle| \mathcal{F}_t^H \right] \\ = \ & \exp(M_S + \frac{1}{2}\Sigma_S^2) \\ = \ & \exp\left(\ln S(t) + \mu (T-t) - \frac{1}{2}\nu_H \sigma^2 (T-t)^{2H} + \sigma \hat{\mu}_{T,t} + \frac{1}{2}\nu_H \sigma^2 (T-t)^{2H} \right) \\ = \ & \exp\left(\ln S(t) + \mu (T-t) + \sigma \hat{\mu}_{T,t} \right) \\ = \ & S(t) \exp\left(\mu (T-t) + \sigma \hat{\mu}_{T,t} \right). \end{aligned}$$

8.3 The Market

Rostek and Schöbel (2010) constructs a wealth process that consists of a number of investments in stocks and a bond. Now consider a market setting which consists of a riskless bond or money account which has the following dynamics

$$dS_0(t) = rS_0(t)\, dt$$

where $r > 0$ is a constant riskless interest rate. For $H > \frac{1}{2}$ let $H_j = (H_1, ..., H_M)$ be the Hurst vector. Consider a finite number of assets $N \in \mathbb{N}$, $S_i(t) = (S_1(t), ..., S_N(t))$ that are driven by a finite number $M \in \mathbb{N}$, of fractional Brownian motions $\left(B_1^{H_1}, ..B_M^{H_M}\right)$. For $0 \leq t \leq T$, let $\mu_i = (\mu_1, .., \mu_N)$ and $\sigma_{ij} = \{\sigma_{NM}\}$ be vectors representing the drift and volatility respectively. Consider a finite number of assets S_i with the following dynamics

$$dS_i(t) = \mu_i S_i(t)\,dt + \sum_{j=1}^{M} \sigma_{ij} S_i(t)\,dB_j^{H_j}(t)\,. \tag{8.7}$$

The stochastic processes are assumed independent. The wealth process $W(t)$ is defined as n_0 units invested in the riskless bond S_0 and n_i units invested the asset S_i and is given as

$$W(t) = \sum_{i=0}^{N} n_i S_i(t)$$

its associated self-financing stochastic differential equation as

$$dW(t) = \sum_{i=0}^{N} n_i dS_i(t)\,.$$

Let $\omega_i = \dfrac{n_i S_i}{W}$ be the weights of the assets in the portfolio. The associated stochastic differential equation of the wealth process $W(t)$ is

$$\begin{aligned}
dW(t) &= \sum_{i=0}^{N} n_i dS_i(t) \\
&= n_0 r S_0(t)\,dt + \sum_{i=1}^{N} n_i \left(\mu_i S_i(t)\,dt + \sum_{j=1}^{M} \sigma_{ij} S_i(t)\,dB_j^{H_j}(t) \right) \\
&= \omega_0 r W(t)\,dt + \sum_{i=1}^{N} \omega_i \left(\mu_i W(t)\,dt + \sum_{j=1}^{M} \sigma_{ij} W(t)\,dB_j^{H_j}(t) \right) \\
&= \omega_0 r W(t)\,dt + \sum_{i=1}^{N} \omega_i \mu_i W(t)\,dt + \sum_{i=1}^{N} \omega_i \sum_{j=1}^{M} \sigma_{ij} W(t)\,dB_j^{H_j}(t) \\
&= \left(\omega_0 r + \sum_{i=1}^{N} \omega_i \mu_i \right) W(t)\,dt + \sum_{j=1}^{M} \left(\sum_{i=1}^{N} \omega_i \sigma_{ij} \right) W(t)\,dB_j^{H_j}(t)\,.
\end{aligned}$$

Denoting

$$\mu_W = \omega_0 r + \sum_{i=1}^{N} \omega_i \mu_i$$

and

$$\sigma_{W_j} = \sum_{i=1}^{N} \omega_i \sigma_{ij}.$$

It follows that

$$dW(t) = \mu_W W(t)\,dt + \sum_{j=1}^{M} \sigma_{W_j} W(t)\,dB_j^{H_j}(t).$$

The solution of the stochastic differential equation follows from the conditional multi-dimensional Itô theorem, (see Biagini and Øksendal, 2003 for the multi-dimensional fractional Itô formula)

$$W(T) = W(t)\exp\begin{pmatrix} \mu_W(T-t) - \frac{1}{2}\sum_{j=1}^{M}\nu_{Hj}\sigma_{W_j}^2 (T-t)^{2H_j} \\ +\sum_{j=1}^{M}\sigma_{W_j}\left(B_j^{H_j}(T) - B_j^{H_j}(t)\right) \end{pmatrix}.$$

We define M_W differently than in (Rostek and Schöbel, 2010). Rostek and Schöbel subtract $\ln W(t)$ from the conditional expected value of $\ln W(T)$.

Lemma 8.4 *(Rostek and Schöbel, 2010). The conditional log-normal wealth process* $\ln W(T)$ *is normally distributed with expectation* M_W *and variance* Σ_W^2 *which is given as*

$$M_W = \ln W(t) + \mu(T-t) - \frac{1}{2}\sum_{j=1}^{M}\nu_{H_j}\sigma_{W_j}^2(T-t)^{2H_j} + \sum_{j=1}^{M}\sigma_{W_j}\hat{\mu}_{T,t}^{H_j}$$

and

$$\Sigma_W^2 = \sum_{j=1}^{M}\sigma_{W_j}^2\nu_{H_j}(T-t)^{2H_j}. \tag{8.8}$$

Consider one of the N assets S_h and assume that this asset is only driven by a single fractional Brownian motion. We will denote this asset by $S(t)$, this Hurst parameter as H, the fractional Brownian motion as $B^H(t)$ and the drift and volatility as μ and σ respectively. Due to the independence of the stochastic components the stock process and the wealth process follow a bivariate log-normal distribution. The joint distributions of the stock and wealth process is presented by the following lemmas.

Lemma 8.5 *(Rostek and Schöbel, 2010). The conditional covariance of random variables* $\ln S(t)$ *and* $\ln W(t)$ *is given as*

$$\begin{aligned}\Sigma_{SW} &= cov\left(\ln S(T), \ln W(T) \left|\mathcal{F}_t^H\right.\right) \\ &= \sigma\sigma_{W_h}\nu_H (T-t)^{2H}.\end{aligned}$$

Lemma 8.6 *(Rostek and Schöbel, 2010). The conditional correlation of the random variables* $\ln S(t)$ *and* $\ln W(t)$ *is*

$$\rho_{SW} = \frac{\sigma_{W_h}\sqrt{\nu_H}(T-t)^H}{\sqrt{\sum_{j=1}^{M}\sigma_{W_j}^2\nu_{H_j}(T-t)^{2H_j}}}. \tag{8.9}$$

The conditional wealth process given the conditional stock process will be discussed next.

Lemma 8.7 *The conditional expectation* $M_{W|S}$ *and the conditional variance* $\Sigma_{W|S}^2$ *of* $\ln W(t)$ *given* $S(t)$ *is normally distributed with*

$$M_{W|S} = M_W + \rho_{SW}\frac{\Sigma_W}{\Sigma_S}(\ln S(T) - M_S)$$

and

$$\Sigma_{W|S}^2 = \Sigma_W^2\left(1-\rho_{SW}^2\right).$$

Proof. Using properties of the bivariate normal distribution, the conditional expectation is

$$\begin{aligned}M_{W|S} &= E_P[\ln W(T)|S(T)] \\ &= E_P\left[\ln W(T)\left|\mathcal{F}_t^H\right.\right] + \rho_{SW}\frac{\Sigma_W}{\Sigma_S}\left(\ln S(T) - E_P\left[\ln S(T)\left|\mathcal{F}_t^H\right.\right]\right) \\ &= M_W + \rho_{SW}\frac{\Sigma_W}{\Sigma_S}(\ln S(T) - M_S)\end{aligned}$$

and the conditional variance is

$$\begin{aligned}\Sigma_{W|S}^2 &= Var[\ln W(T)|S(T)] \\ &= \Sigma_W^2\left(1-\rho_{SW}^2\right).\end{aligned}$$

■

Using the property of log-normal distribution, $W(T)$ is conditionally log-normally distributed with the following expectation

$$\begin{aligned}
& E_P\left[W(T)\,|S(T)\right] \\
&= \exp\left(M_{W|S}+\frac{1}{2}\Sigma_{W|S}^2\right) \\
&= \exp\left(M_W+\rho_{SW}\frac{\Sigma_W}{\Sigma_S}\left(\ln S(T)-M_S\right)+\frac{1}{2}\Sigma_W^2\left(1-\rho_{SW}^2\right)\right) \\
&= \exp\left(M_W+\ln S(T)^{\rho_{SW}\frac{\Sigma_W}{\Sigma_S}}-\rho_{SW}\frac{\Sigma_W}{\Sigma_S}M_S+\frac{1}{2}\Sigma_W^2\left(1-\rho_{SW}^2\right)\right) \\
&= S(T)^{\rho_{SW}\frac{\Sigma_W}{\Sigma_S}}\exp\left(M_W-\rho_{SW}\frac{\Sigma_W}{\Sigma_S}M_S+\frac{1}{2}\Sigma_W^2\left(1-\rho_{SW}^2\right)\right). \qquad (8.10)
\end{aligned}$$

8.4 Stochastic Discount Factor

Assume an investor wants to maximizes his utility by choosing today's consumption level and the number of units invested in each of the N risky assets. Denote $c(t)$ as the initial consumption and let $U(\cdot)$ be the utility function defined over the initial consumption period. Denote $V(\cdot)$ as the utility function defined over the end of the period wealth $W(T)$. The objective function is defined by Rostek and Schöbel (2010) as

$$\max_{\{c(t);\, n_i\}}\left\{U(c(t))+E_P\left[V\left(\begin{array}{c}(W(t)-c(t))\,e^{r(T-t)}\\ +\sum_{i=1}^N n_i\left(S_i(T)-S_i(t)\,e^{r(T-t)}\right)\end{array}\right)\big|\mathcal{F}_t^H\right]\right\}.$$

This can be written as

$$\max_{\{c(t);\, n_i\}}\left\{U(c(t))+E_P\left[V(W(T))\,\big|\mathcal{F}_t^H\right]\right\}.$$

Investors are assumed to have a constant relative risk aversion also known as iso-elastic utility. Denote γ as the parameter of risk aversion and assume the utility function $V(W)$ is a power function of the following form

$$V(W(T)) = \frac{1}{1-\gamma} W(T)^{1-\gamma} \tag{8.11}$$

with its first derivative

$$V'(W(T)) = W(T)^{-\gamma}.$$

In a discrete time framework it is in general not possible to construct a risk-free replicating portfolio (Brennan, 1979). In order to price derivatives using an equilibrium approach we need to solve the aggregation problem, that is when can utility functions of all investors be combined to give a average investor. Rubinstein (1974) showed that this could be done assuming all investors have identical cautiousness and beliefs. For further details see for instance Rubinstein (1974) and Brennan (1979). Rubinstein (1976) proved the Black-Scholes formula in a discrete time model assuming constant proportional risk aversion and bivariate log-normality, see Brennan (1979).

Optimization of the utility function as well as the pricing equilibrium condition will be done in order to get the adjusted drift, so it can be applied to pricing options.

Lemma 8.8 *The first order conditions are*

$$U'(c(t)) - e^{r(T-t)} E_P\left[V'(W(T)) \left| \mathcal{F}_t^H \right.\right] = 0$$

and

$$E_P\left[V'(W(T)) S_i(T) \left| \mathcal{F}_t^H \right.\right] - S_i(t) e^{r(T-t)} E_P\left[V'(W(T)) \left| \mathcal{F}_t^H \right.\right] = 0 \textit{ for all } i. \tag{8.12}$$

Proof. (Rostek and Schöbel, 2010).

We want to maximizes the objective function therefore we differentiate with respect to the initial wealth and number of units invested for all assets.

$$\begin{aligned}
&\frac{d}{dc(t)}\left[U(c(t)) + E_P\left[V\left(\begin{array}{c}(W(t)-c(t))\,e^{r(T-t)} \\ +\sum_{i=1}^{N} n_i\left(S_i(T) - S_i(t)\,e^{r(T-t)}\right)\end{array}\right)\left|\mathcal{F}_t^H\right.\right]\right] \\
= &\frac{d}{dc(t)}U(c(t)) + E_P\left[\frac{d}{dc(t)}V\left(\begin{array}{c}(W(t)-c(t))\,e^{r(T-t)} \\ +\sum_{i=1}^{N} n_i\left(S_i(T) - S_i(t)\,e^{r(T-t)}\right)\end{array}\right)\left|\mathcal{F}_t^H\right.\right] \\
= &U'(c(t)) + E_P\left[-e^{r(T-t)}V'\left(\begin{array}{c}(W(t)-c(t))\,e^{r(T-t)} \\ +\sum_{i=1}^{N} n_i\left(S_i(T) - S_i(t)\,e^{r(T-t)}\right)\end{array}\right)\left|\mathcal{F}_t^H\right.\right] \\
= &U'(c(t)) - e^{r(T-t)}E_P\left[V'(W(T))\left|\mathcal{F}_t^H\right.\right]
\end{aligned}$$

Setting the above equal to zero we obtain the first equation.

$$\begin{aligned}
&\frac{d}{dn_i}\left[U(c(t)) + E_P\left[V\left(\begin{array}{c}(W(t)-c(t))\,e^{r(T-t)} \\ +\sum_{i=1}^{N} n_i\left(S_i(T) - S_i(t)\,e^{r(T-t)}\right)\end{array}\right)\left|\mathcal{F}_t^H\right.\right]\right] \\
= &E_P\left[\frac{d}{dn_i}V\left((W(t)-c(t))\,e^{r(T-t)} + \sum_{i=1}^{N} n_i\left(S_i(T) - S_i(t)\,e^{r(T-t)}\right)\right)\left|\mathcal{F}_t^H\right.\right] \\
= &E_P\left[\left(S_i(T) - S_i(t)\,e^{r(T-t)}\right)V'\left(\begin{array}{c}(W(t)-c(t))\,e^{r(T-t)} \\ +\sum_{i=1}^{N} n_i\left(S_i(T) - S_i(t)\,e^{r(T-t)}\right)\end{array}\right)\left|\mathcal{F}_t^H\right.\right] \\
= &E_P\left[S_i(T)\,V'(W(T)) - S_i(t)\,e^{r(T-t)}V'(W(T))\left|\mathcal{F}_t^H\right.\right] \\
= &E_P\left[S_i(T)\,V'(W(T))\left|\mathcal{F}_t^H\right.\right] - S_i(t)\,e^{r(T-t)}E_P\left[V'(W(T))\left|\mathcal{F}_t^H\right.\right]
\end{aligned}$$

$S_i(t)$ is measurable with respect to $\mathcal{F}_t^H$. Setting the above equal to zero yields the second equation. ■

Theorem 8.3 *The pricing equilibrium condition is*

$$S(t) = E_P\left[z(t,T)\,S(T)\left|\mathcal{F}_t^H\right.\right]$$

where $z(t,T)$ *is the stochastic discount factor given by*

$$z(t,T) = e^{-r(T-t)}\frac{E_P\left[V'(W(T))\,|S(T)\right]}{E_P\left[V'(W(T))\,|\mathcal{F}_t^H\right]}.$$

Proof. (Rostek and Schöbel, 2010).

From the second constraint, equation (8.12) we have that

$$E_P\left[V'(W(T))\,S(T)\left|\mathcal{F}_t^H\right.\right] = e^{r(T-t)}S(t)\,E_P\left[V'(W(T))\left|\mathcal{F}_t^H\right.\right].$$

Solving for the initial price $S(t)$ and using properties of conditional expectation we have

$$\begin{aligned} S(t) &= e^{-r(T-t)}\frac{E_P\left[V'(W(T))\,S(T)\left|\mathcal{F}_t^H\right.\right]}{E_P\left[V'(W(T))\,|\mathcal{F}_t^H\right]} \\ &= e^{-r(T-t)}E_P\left[\frac{\left[V'(W(T))\,S(T)\right]}{E_P\left[V'(W(T))\,|\mathcal{F}_t^H\right]}\,|\mathcal{F}_t^H\right]. \end{aligned}$$

Since $E_P\left[V'(W(T))\left|\mathcal{F}_t^H\right.\right]$ is $\mathcal{F}_t^H$ measurable and using the tower property of conditional expectation we have

$$\begin{aligned} S(t) &= e^{-r(T-t)}E_P\left[E_P\left[\frac{\left[V'(W(T))\,S(T)\right]}{E_P\left[V'(W(T))\,|\mathcal{F}_t^H\right]}|S(T)\right]|\mathcal{F}_t^H\right] \\ &= e^{-r(T-t)}E_P\left[\left[\frac{E_P\left[V'(W(T))\,S(T)\,|S(T)\right]}{E_P\left[V'(W(T))\,|\mathcal{F}_t^H\right]}\right]|\mathcal{F}_t^H\right] \\ &= e^{-r(T-t)}E_P\left[S(T)\left[\frac{E_P\left[V'(W(T))\,|S(T)\right]}{E_P\left[V'(W(T))\,|\mathcal{F}_t^H\right]}\right]|\mathcal{F}_t^H\right]. \end{aligned}$$

Since $S(T)$ is $S(T)$ measurable and setting

$$z(t,T) = \frac{E_P\left[V'(W(T))\,|S(T)\right]}{E_P\left[V'(W(T))\,|\mathcal{F}_t^H\right]} \tag{8.13}$$

it follows

$$S(t) = E_P\left[z(t,T)\,S(T)\left|\mathcal{F}_t^H\right.\right].$$

■

A contingent claim $C(t,S(t))$ whose payoff at time T depends solely on $S(T)$ can be priced as follows

$$C(t,S(t)) = E_P\left[z(t,T)\,C(T,S(T))\left|\mathcal{F}_t^H\right.\right]. \tag{8.14}$$

Lemma 8.9 *The stochastic discount factor with respect to the utility function* $V(W(T))$ *is*

$$z(t,T) = e^{-r(T-t)} S(T)^{-\gamma\rho_{SW}\frac{\Sigma_W}{\Sigma_S}} \exp\left(\gamma\rho_{SW}\frac{\Sigma_W}{\Sigma_S}M_S - \frac{1}{2}\gamma^2\Sigma_W^2\rho_{SW}^2\right).$$

Proof. (Rostek and Schöbel, 2010).

The utility function $V(W(T))$ as defined in (8.11) and substituting into (8.13) we have

$$z(t,T) = e^{-r(T-t)}\frac{E_P\left[W(T)^{-\gamma}|S(T)\right]}{E_P\left[W(T)^{-\gamma}|\mathcal{F}_t^H\right]}.$$

For a constant γ, the distribution of the process $\ln W(T)^{-\gamma} = -\gamma\ln W(T)$ is normally distributed with $-\gamma M_W$ and $\gamma^2\Sigma_W^2$ being the first and second moments respectively of the process, refer to lemma 8.4. Using properties of the log-normal distribution it follows that $W(t)^{-\gamma}$ is log-normally distributed with the following expectation

$$E_P\left[W(t)^{-\gamma}\left|\mathcal{F}_t^H\right.\right] = \exp\left(-\gamma M_W + \frac{1}{2}\gamma^2\Sigma_W^2\right).$$

The distribution of the conditional process $W(t)^{-\gamma}$ given $S(t)$ has a conditional expectation $-\gamma M_{W|S}$ and conditional variance $\gamma^2\Sigma_{W|S}^2$, refer to lemma 8.7. From (8.10) it follows that $W(t)^{-\gamma}$ given $S(T)$ is log-normally distributed with the expectation

$$E_P\left[W(T)^{-\gamma}|S(T)\right] = S(T)^{-\gamma\rho_{SW}\frac{\Sigma_W}{\Sigma_S}}\exp\begin{pmatrix} -\gamma M_W + \gamma\rho_{SW}\frac{\Sigma_W}{\Sigma_S}M_S \\ +\frac{1}{2}\gamma^2\Sigma_W^2\left(1-\rho_{SW}^2\right)\end{pmatrix}.$$

The discount factor follows as

$$\begin{aligned} z(t,T) &= e^{-r(T-t)}\frac{S(T)^{-\gamma\rho_{SW}\frac{\Sigma_W}{\Sigma_S}}\exp\begin{pmatrix} -\gamma M_W + \gamma\rho_{SW}\frac{\Sigma_W}{\Sigma_S}M_S \\ +\frac{1}{2}\gamma^2\Sigma_W^2\left(1-\rho_{SW}^2\right)\end{pmatrix}}{\exp\left(-\gamma M_W + \frac{1}{2}\gamma^2\Sigma_W^2\right)} \\ &= e^{-r(T-t)}\frac{S(T)^{-\gamma\rho_{SW}\frac{\Sigma_W}{\Sigma_S}}\exp\begin{pmatrix} -\gamma M_W + \gamma\rho_{SW}\frac{\Sigma_W}{\Sigma_S}M_S \\ +\frac{1}{2}\gamma^2\Sigma_W^2 - \frac{1}{2}\gamma^2\Sigma_W^2\rho_{SW}^2\end{pmatrix}}{\exp\left(-\gamma M_W + \frac{1}{2}\gamma^2\Sigma_W^2\right)} \\ &= e^{-r(T-t)}S(T)^{-\gamma\rho_{SW}\frac{\Sigma_W}{\Sigma_S}}\exp\left(\gamma\rho_{SW}\frac{\Sigma_W}{\Sigma_S}M_S - \frac{1}{2}\gamma^2\Sigma_W^2\rho_{SW}^2\right). \end{aligned}$$

■

Lemma 8.10 *In equilibrium we have*

$$M_S = \ln S(t) + r(T-t) - \frac{1}{2}\Sigma_S^2 + \gamma\rho_{SW}\Sigma_W\Sigma_S. \tag{8.15}$$

Proof. (Rostek and Schöbel, 2010).

From the pricing equation we have

$$\begin{aligned} S(t) &= E_P\left[z(t,T)S(T)\left|\mathcal{F}_t^H\right.\right] \\ &= E_P\left[e^{-r(T-t)}S(T)^{-\gamma\rho_{SW}\frac{\Sigma_W}{\Sigma_S}}\exp\left(\begin{array}{c}\gamma\rho_{SW}\frac{\Sigma_W}{\Sigma_S}M_S \\ -\frac{1}{2}\gamma^2\Sigma_W^2\rho_{SW}^2\end{array}\right)S(T)\left|\mathcal{F}_t^H\right.\right] \\ &= e^{-r(T-t)}E_P\left[S(T)^{1-\gamma\rho_{SW}\frac{\Sigma_W}{\Sigma_S}}\exp\left(\begin{array}{c}\gamma\rho_{SW}\frac{\Sigma_W}{\Sigma_S}M_S \\ -\frac{1}{2}\gamma^2\Sigma_W^2\rho_{SW}^2\end{array}\right)\left|\mathcal{F}_t^H\right.\right] \\ &= e^{-r(T-t)}E_P\left[S(T)^{1-\gamma\rho_{SW}\frac{\Sigma_W}{\Sigma_S}}\left|\mathcal{F}_t^H\right.\right]\exp\left(\gamma\rho_{SW}\frac{\Sigma_W}{\Sigma_S}M_S - \frac{1}{2}\gamma^2\Sigma_W^2\rho_{SW}^2\right). \end{aligned}$$

The distribution of $\ln S(T)^{1-\gamma\rho_{SW}\frac{\Sigma_W}{\Sigma_S}}$ is normally distributed with the following moments, refer to lemma 8.3

$$\left(1-\gamma\rho_{SW}\frac{\Sigma_W}{\Sigma_S}\right)\ln S(T) \sim N\left(\left(1-\gamma\rho_{SW}\frac{\Sigma_W}{\Sigma_S}\right)M_S, \left(1-\gamma\rho_{SW}\frac{\Sigma_W}{\Sigma_S}\right)^2\Sigma_S^2\right).$$

It follows that $S(T)^{1-\gamma\rho_{SW}\frac{\Sigma_W}{\Sigma_S}}$ is log-normally distributed with the following conditional expectation

$$E_P\left[S(T)^{1-\gamma\rho_{SW}\frac{\Sigma_W}{\Sigma_S}}\left|\mathcal{F}_t^H\right.\right] = \exp\left(\left(1-\gamma\rho_{SW}\frac{\Sigma_W}{\Sigma_S}\right)M_S + \frac{1}{2}\Sigma_S^2\left(1-\gamma\rho_{SW}\frac{\Sigma_W}{\Sigma_S}\right)^2\right).$$

Substituting the expectation into the pricing equation we have

$$\begin{aligned}
& S(t)\, e^{r(T-t)} \\
= \; & \exp\left(\begin{array}{c} \left(1-\gamma\rho_{SW}\frac{\Sigma_W}{\Sigma_S}\right) M_S + \frac{1}{2}\Sigma_S^2\left(1-\gamma\rho_{SW}\frac{\Sigma_W}{\Sigma_S}\right)^2 + \gamma\rho_{SW}\frac{\Sigma_W}{\Sigma_S} M_S \\ -\frac{1}{2}\gamma^2\Sigma_W^2\rho_{SW}^2 \end{array}\right) \\
= \; & \exp\left(\begin{array}{c} M_S - \gamma\rho_{SW}\frac{\Sigma_W}{\Sigma_S} M_S + \frac{1}{2}\Sigma_S^2\left(1-2\gamma\rho_{SW}\frac{\Sigma_W}{\Sigma_S}+\gamma^2\rho_{SW}^2\frac{\Sigma_W^2}{\Sigma_S^2}\right) \\ +\gamma\rho_{SW}\frac{\Sigma_W}{\Sigma_S} M_S - \frac{1}{2}\gamma^2\Sigma_W^2\rho_{SW}^2 \end{array}\right) \\
= \; & \exp\left(\begin{array}{c} M_S - \gamma\rho_{SW}\frac{\Sigma_W}{\Sigma_S} M_S + \frac{1}{2}\Sigma_S^2 - \Sigma_S^2\gamma\rho_{SW}\frac{\Sigma_W}{\Sigma_S} + \frac{1}{2}\Sigma_S^2\gamma^2\rho_{SW}^2\frac{\Sigma_W^2}{\Sigma_S^2} \\ +\gamma\rho_{SW}\frac{\Sigma_W}{\Sigma_S} M_S - \frac{1}{2}\gamma^2\Sigma_W^2\rho_{SW}^2 \end{array}\right) \\
= \; & \exp\left(M_S + \frac{1}{2}\Sigma_S^2 - \gamma\rho_{SW}\Sigma_W\Sigma_S\right)
\end{aligned}$$

therefore

$$\exp\left(\ln S(t) + r(T-t)\right) = \exp\left(M_S + \frac{1}{2}\Sigma_S^2 - \gamma\rho_{SW}\Sigma_W\Sigma_S\right).$$

Taking logs on both sides we get

$$\ln S(t) + r(T-t) = M_S + \frac{1}{2}\Sigma_S^2 - \gamma\rho_{SW}\Sigma_W\Sigma_S.$$

■

Lemma 8.11 *Equation* (8.15) *can be rewritten as*

$$\mu(T-t) + \sigma\hat{\mu}_{T,t} = r(T-t) + \gamma\sigma\sigma_{W_h}\nu_H (T-t)^{2H}. \tag{8.16}$$

Proof. (Rostek and Schöbel, 2010).

Equating (8.5) and (8.15) we have

$$r(T-t) = \mu(T-t) - \frac{1}{2}\nu_H\sigma^2(T-t)^{2H} + \sigma\hat{\mu}_{T,t} + \frac{1}{2}\Sigma_S^2 - \gamma\rho_{SW}\Sigma_W\Sigma_S.$$

Substituting in (8.9), (8.8) and (8.6) we get

$$\begin{aligned} r\left(T-t\right) &= \mu\left(T-t\right)-\frac{1}{2}\nu_{H}\sigma^{2}\left(T-t\right)^{2H}+\sigma\hat{\mu}_{T,t}+\frac{1}{2}\sigma^{2}\nu_{H}\left(T-t\right)^{2H} \\ &\quad -\gamma\frac{\sigma\sigma_{W_h}\nu_{H}\left(T-t\right)^{2H}\sqrt{\sigma^{2}\nu_{H}\left(T-t\right)^{2H}}\sqrt{\sum_{j=1}^{M}\sigma_{W_j}^{2}\nu_{H_j}\left(T-t\right)^{2H_j}}}{\sqrt{\sigma^{2}\nu_{H}\left(T-t\right)^{2H}}\sqrt{\sum_{j=1}^{M}\sigma_{W_j}^{2}\nu_{H_j}\left(T-t\right)^{2H_j}}} \\ &= \mu\left(T-t\right)+\sigma\hat{\mu}_{T,t}-\gamma\sigma\sigma_{W_h}\nu_{H}\left(T-t\right)^{2H}. \end{aligned}$$

■

The left hand side of (8.16) is the conditional drift of the stock process which is composed of the unconditional drift and an adjustment resulting from the price history.

8.5 Option Pricing Formula

The assumptions to pricing the fractional Black-Scholes formula and the fractional Black formula are as follows:

1. The market consists of a bond and a finite number of assets with dynamics given in equation (8.7) .
2. Stochastic differentials are interpreted in the Wick Itô Skorohod sense.
3. The stochastic processes driving the asset prices are independent.
4. The Hurst parameters $0 < H_j < 1$, in equation (8.7) are constant over time.
5. The drifts μ_i and volatilities σ_{ij} are constant and the r is a constant risk-free rate of interest and the same for all maturities.
6. We consider a single stock price following the dynamics of equation (6.1) .

7. Market participants have a constant relative risk aversion.

8. The investors maximizes his utility over a single period.

9. Market participants have the same cautiousness and beliefs.

10. Short selling is allowed.

11. There are no transactions costs or taxes.

12. There are no dividends.

13. Trading is done in a discrete time.

14. All securities are perfectly divisible.

8.5.1 Conditional Fractional Black-Scholes Formula

Theorem 8.4 *The price of a conditional fractional European call option with a strike price K and maturity T is*

$$\tilde{C}^H(t, S(t)) = S(t)\, N\left(\tilde{d}_1^H\right) - Ke^{-r(T-t)} N\left(\tilde{d}_2^H\right)$$

where

$$\tilde{d}_1^H = \frac{\ln\left(\dfrac{S(t)}{K}\right) + r(T-t) + \dfrac{1}{2}\nu_H \sigma^2 (T-t)^{2H}}{\sqrt{\nu_H}\sigma (T-t)^H}$$

and

$$\tilde{d}_2^H = \frac{\ln\left(\dfrac{S(t)}{K}\right) + r(T-t) - \dfrac{1}{2}\nu_H \sigma^2 (T-t)^{2H}}{\sqrt{\nu_H}\sigma (T-t)^H}.$$

Proof. (Rostek and Schöbel, 2010).

Let $S(T) = X$, then $\ln X \sim N(M_S, \Sigma_S^2)$, i.e. normally distributed. For $0 < x < \infty$ it follows that X is log-normally distributed with the following density function

$$f(x) = \frac{1}{x\sqrt{2\pi\Sigma_S^2}} \exp\left(-\frac{1}{2}\frac{(\ln x - M_S)^2}{\Sigma_S^2}\right).$$

Consider a contingent claim that has a payoff $\max\{x-K,0\}$ when $x>K$. The function $\ln(x-K)$ is defined for all $K<x<\infty$. From (8.14) we have

$$\begin{aligned}
C^H(t,S(t)) &= E_P\left[C^H(T,x)\,z(x)\,\middle|\mathcal{F}_t^H\right] \\
&= E_P\left[\max\{(x-K),0\}\,z(x)\,\middle|\mathcal{F}_t^H\right] \\
&= \int_K^\infty f(x)(x-K)\,z(x)\,dx \\
&= \int_K^\infty \frac{1}{x\sqrt{2\pi\Sigma_S^2}}(x-K)\exp\left(-\frac{1}{2}\frac{(\ln x-M_S)^2}{\Sigma_S^2}\right)z(x)\,dx.
\end{aligned}$$

It follows

$$\begin{aligned}
&\exp\left(-\frac{1}{2}\frac{(\ln x-M_S)^2}{\Sigma_S^2}\right)z(x) \\
=\;& e^{-r(T-t)}x^{-\gamma\rho_{SW}\frac{\Sigma_W}{\Sigma_S}}\exp\left(-\frac{1}{2\Sigma_S^2}(\ln x-M_S)^2+\gamma\rho_{SW}\frac{\Sigma_W}{\Sigma_S}M_S-\frac{1}{2}\gamma^2\Sigma_W^2\rho_{SW}^2\right) \\
=\;& e^{-r(T-t)}x^{-\gamma\rho_{SW}\frac{\Sigma_W}{\Sigma_S}}\exp\begin{pmatrix} -\frac{1}{2\Sigma_S^2}\left(2\ln x-2\ln xM_S+M_S^2\right) \\ -\frac{1}{2\Sigma_S^2}\left(-2\gamma\rho_{SW}\Sigma_S\Sigma_WM_S+\gamma^2\Sigma_W^2\Sigma_S^2\rho_{SW}^2\right)\end{pmatrix} \\
=\;& e^{-r(T-t)}x^{-\gamma\rho_{SW}\frac{\Sigma_W}{\Sigma_S}}\exp\left(-\frac{1}{2\Sigma_S^2}\begin{pmatrix} 2\ln x-2\ln xM_S+M_S^2 \\ -2\gamma\rho_{SW}\Sigma_S\Sigma_WM_S+\gamma^2\Sigma_W^2\Sigma_S^2\rho_{SW}^2\end{pmatrix}\right) \\
=\;& e^{-r(T-t)}\exp\left(-\gamma\rho_{SW}\frac{\Sigma_W}{\Sigma_S}\ln x\left(-\frac{1}{2\Sigma_S^2}\begin{pmatrix} 2\ln x-2\ln xM_S+M_S^2 \\ -2\gamma\rho_{SW}\Sigma_S\Sigma_WM_S+\gamma^2\Sigma_W^2\Sigma_S^2\rho_{SW}^2\end{pmatrix}\right)\right) \\
=\;& e^{-r(T-t)}\exp\left(-\frac{1}{2\Sigma_S^2}\begin{pmatrix} 2\gamma\rho_{SW}\Sigma_S\Sigma_W\ln x+2\ln x-2\ln xM_S+M_S^2 \\ -2\gamma\rho_{SW}\Sigma_S\Sigma_WM_S+\gamma^2\Sigma_W^2\Sigma_S^2\rho_{SW}^2\end{pmatrix}\right) \\
=\;& e^{-r(T-t)}\exp\left(-\frac{1}{2\Sigma_S^2}\left(2\ln x-2\ln x\,(M_S-\gamma\rho_{SW}\Sigma_S\Sigma_W)+(M_S-\gamma\rho_{SW}\Sigma_S\Sigma_W)^2\right)\right) \\
=\;& e^{-r(T-t)}\exp\left(-\frac{1}{2\Sigma_S^2}\left([\ln x-(M_S-\gamma\rho_{SW}\Sigma_S\Sigma_W)]^2\right)\right) \\
=\;& e^{-r(T-t)}\exp\left(-\frac{1}{2\Sigma_S^2}\left(\left[\ln x-\left(\ln S(t)+r(T-t)-\frac{1}{2}\Sigma_S^2\right)\right]^2\right)\right).
\end{aligned}$$

Setting

$$\begin{aligned} m &= \ln S(t) + r(T-t) - \frac{1}{2}\Sigma_S^2 \\ &= \ln S(t) + r(T-t) - \frac{1}{2}\sigma^2 \nu_H (T-t)^{2H} \end{aligned}$$

and

$$\begin{aligned} v &= \Sigma_S^2 \\ &= \sigma^2 \nu_H (T-t)^{2H}. \end{aligned}$$

Then we have

$$\begin{aligned} C^H(t, S(t)) &= E_P\left[C^H(T,x)\, z(t,T) \,\middle|\, \mathcal{F}_t^H\right] \\ &= e^{-r(T-t)} \int_K^\infty \frac{1}{x\sqrt{2\pi v}} (x-K) \exp\left(-\frac{1}{2v}\left([\ln x - m]^2\right)\right) dx \\ &= e^{-r(T-t)} \int_K^\infty \frac{1}{x\sqrt{2\pi v}} \exp\left(-\frac{1}{2}\frac{(\ln x - m)^2}{v}\right) x dx \\ &\quad -K e^{-r(T-t)} \int_K^\infty \frac{1}{x\sqrt{2\pi v}} \exp\left(-\frac{1}{2}\frac{(\ln x - m)^2}{v}\right) dx. \end{aligned}$$

Calculation of the first integral

$$\begin{aligned} & e^{-r(T-t)} \int_K^\infty \frac{1}{x\sqrt{2\pi v}} \exp\left(-\frac{1}{2}\frac{(\ln x - m)^2}{v}\right) x dx \\ = \; & e^{-r(T-t)} \int_K^\infty \frac{1}{\sqrt{2\pi v}} \exp\left(-\frac{1}{2}\left(\frac{\ln x - m}{\sqrt{v}}\right)^2\right) dx. \end{aligned}$$

For the transformations set

$$y = \frac{\ln x - m}{\sqrt{v}} \Rightarrow dy = \frac{1}{x\sqrt{v}} dx$$

solving for x we get

$$x = \exp\left(m + \sqrt{v} y\right) \Rightarrow dx = x\sqrt{v} dy.$$

Solving boundary condition, since $x > K$ it follows

$$\begin{aligned} \exp\left(m + \sqrt{v} y\right) &> K \Rightarrow \\ y &> \frac{\ln K - m}{\sqrt{v}}. \end{aligned}$$

Denote

$$-\tilde{d}_2^H = \frac{\ln K - m}{\sqrt{v}}$$

and

$$\tilde{d}_2^H = \frac{m - \ln K}{\sqrt{v}}.$$

Thus

$$\begin{aligned}
& e^{-r(T-t)} \int_{-\tilde{d}_2^H}^{\infty} \frac{1}{\sqrt{2\pi}\sqrt{v}} \exp\left(-\frac{1}{2}y^2\right) \exp\left(m + \sqrt{v}y\right) \sqrt{v} dy \\
= & e^{-r(T-t)} e^m \int_{-\tilde{d}_2^H}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}y^2 + \sqrt{v}y\right) dy \\
= & e^{-r(T-t)} e^m \int_{-\tilde{d}_2^H}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(y^2 - 2\sqrt{v}y\right)\right) dy \\
= & e^{-r(T-t)} e^m \int_{-\tilde{d}_2^H}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(y - \sqrt{v}\right)^2 + \frac{1}{2}v\right) dy \\
= & \exp\left(m + \frac{1}{2}v - r\left(T-t\right)\right) \int_{-\tilde{d}_2^H}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(y - \sqrt{v}\right)^2\right) dy.
\end{aligned}$$

Setting

$$z = \left(y - \sqrt{v}\right) \Rightarrow dz = dy.$$

Therefore it follows

$$\begin{aligned}
& \exp\left(m + \frac{1}{2}v - r\left(T-t\right)\right) \int_{-\tilde{d}_2^H - \sqrt{v}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}z^2\right) dz \\
= & \exp\left(m + \frac{1}{2}v - r\left(T-t\right)\right) \int_{-\infty}^{\tilde{d}_2^H + \sqrt{v}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}z^2\right) dz \\
= & \exp\left(m + \frac{1}{2}v - r\left(T-t\right)\right) N\left(\tilde{d}_2^H + \sqrt{v}\right) \\
= & \exp\left(m + \frac{1}{2}v - r\left(T-t\right)\right) N\left(\tilde{d}_1^H\right)
\end{aligned}$$

with

$$\begin{aligned}\tilde{d}_1^H &= \tilde{d}_2^H + \sqrt{v} \\ &= \frac{m - \ln K}{\sqrt{v}} + \sqrt{v} \\ &= \frac{m + v - \ln K}{\sqrt{v}}.\end{aligned}$$

Calculation of the second integral

$$\begin{aligned}& Ke^{-r(T-t)} \int_K^\infty \frac{1}{x\sqrt{2\pi v}} \exp\left(-\frac{1}{2}\frac{(\ln x - m)^2}{v}\right) dx \\ = \;& Ke^{-r(T-t)} \int_{-\tilde{d}_2^H}^\infty \frac{1}{\exp\left(m + \sqrt{v}y\right)\sqrt{2\pi v}} \exp\left(-\frac{1}{2}y^2\right) \exp\left(m + \sqrt{v}y\right)\sqrt{v}dy \\ = \;& Ke^{-r(T-t)} \int_{-\infty}^{\tilde{d}_2^H} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}y^2\right) dy \\ = \;& Ke^{-r(T-t)} N\left(\tilde{d}_2^H\right).\end{aligned}$$

The price of the conditional fractional European call option is

$$\begin{aligned}& E_P\left[C^H\left(t, S\left(t\right)\right) \middle| \mathcal{F}_t^H\right] \\ = \;& \exp\left(m + \frac{1}{2}v - r\left(T - t\right)\right) N\left(\tilde{d}_1^H\right) - Ke^{-r(T-t)} N\left(\tilde{d}_2^H\right) \\ = \;& \exp\left(\begin{array}{c} \ln S\left(t\right) + r\left(T-t\right) - \frac{1}{2}\sigma^2\nu_H\left(T-t\right)^{2H} \\ + \frac{1}{2}\sigma^2\nu_H\left(T-t\right)^{2H} - r\left(T-t\right)\end{array}\right) N\left(\tilde{d}_1^H\right) \\ & - Ke^{-r(T-t)} N\left(\tilde{d}_2^H\right) \\ = \;& S\left(t\right) N\left(\tilde{d}_1^H\right) - Ke^{-r(T-t)} N\left(\tilde{d}_2^H\right)\end{aligned}$$

where

$$\begin{aligned}\tilde{d}_1^H &= \frac{m + v - \ln K}{\sqrt{v}} \\ &= \frac{\ln S\left(t\right) + r\left(T-t\right) - \frac{1}{2}\nu_H\sigma^2\left(T-t\right)^{2H} + \nu_H\sigma^2\left(T-t\right)^{2H} - \ln K}{\sqrt{\sigma^2\nu_H\left(T-t\right)^{2H}}}\end{aligned}$$

and

$$\begin{aligned}\tilde{d}_2^H &= \frac{m - \ln K}{\sqrt{v}} \\ &= \frac{\ln S(t) + r(T-t) - \frac{1}{2}\nu_H \sigma^2 (T-t)^{2H} - \ln K}{\sqrt{\sigma^2 \nu_H (T-t)^{2H}}}.\end{aligned}$$

■

Following similar arguments to those of a conditional fractional European call, the price of a conditional fractional European put option with a strike price K and maturity T is

$$P_H(t, S(t)) = Ke^{-r(T-t)} N\left(-\tilde{d}_2^H\right) - S(t) N\left(-\tilde{d}_1^H\right).$$

Thus one can obtain the put-call parity relationship

$$\begin{aligned}& C^H(t, S(t)) - P^H(t, S(t)) \\ = \;& S(t) N\left(\tilde{d}_1^H\right) - Ke^{-r(T-t)} N\left(\tilde{d}_2^H\right) - \left(Ke^{-r(T-t)} N\left(-\tilde{d}_2^H\right) - S(t) N\left(-\tilde{d}_1^H\right)\right) \\ = \;& S(t) - Ke^{-r(T-t)}.\end{aligned}$$

8.5.2 Conditional Fractional Black Formula

A futures contract is an agreement between two parties in which an individual agrees at time t to buy an asset $S(t)$ from the other party at time T for a price $F(t)$. Futures contracts are traded on exchanges. A forward contract is an agreement between two parties in which an individual agrees at time t to buy an asset $S(t)$ from the other party at time T for a price $f(t)$. The difference between a futures contact and a forward contact is that there are daily settlements with the futures. In other words if the futures price falls on a day then the party who is short has to pay the exchange money which is then transferred to the party who is long and vice versa.

Lemma 8.12 *Assume a dividend yield of zero and if we denote a forward contract by f then the price is*

$$f(t) = e^{r(T-t)} S(t).$$

Proof. If we assume that

$$f(t) > S(t) e^{r(T-t)}$$

then at time t one could borrow the amount $S(t)$ rands and buy the stock $S(t)$. Then the individual could agree to sell the stock at time T for the price $F(t)$. At time T this individual owes $S(t) e^{r(T-t)}$ to the bank and gains $f(t)$. Thus the individual will make a riskless profit of $f(t) - S(t) e^{r(T-t)}$. If we assume that

$$f(t) < S(t) e^{r(T-t)}$$

then at time t the individual could short sell the stock and the individual invests the proceeds into the bank. The individual also enters a forward contract where he agrees to buy $f(t)$. At time T he receives $S(t) e^{r(T-t)}$ from the bank and pays $f(t)$, thus making a riskless profit of $S(t) e^{r(T-t)} - f(t)$. Therefore

$$f(t) = S(t) e^{r(T-t)}.$$

■

Lemma 8.13 *If interest rates are constant then a futures price is the same as a forward price*

$$F(t) = e^{r(T-t)} S(t). \tag{8.17}$$

Proof. (Hull, 2006).

Let δ be a constant daily interest rate. Let F be a futures contact expiring in n days. Let $F(i)$ be the price at the end of day i for $0 < i < n$. Consider the following strategy, take a total long futures position of $e^{i\delta}$ at the end of day $i-1$. The profit and loss from the position on day i is

$$(F(i) - F(i-1))\, e^{i\delta}.$$

Compounded at a risk free rate till the end of day n this becomes

$$(F(i) - F(i-1))\, e^{\delta i} e^{(n-1)\delta} = (F(i) - F(i-1))\, e^{n\delta}.$$

The value of the entire investment strategy at the end of day n is

$$\begin{aligned} & \sum_{i=1}^{n} (F(i) - F(i-1))\, e^{n\delta} \\ = \;& (F(n) - F(0))\, e^{n\delta} \\ = \;& (S(T) - F(0))\, e^{n\delta}. \end{aligned}$$

Since the sum is telescoping and the futures price at the end of day n is the same as the terminal asset spot price $S(T)$. An investment of $F(0)$ in a risk-free bond combined with this strategy has initial cost of $F(0)$ as it costs nothing to enter into a futures contract. At time T this new portfolio has value

$$F(0)\, e^{\delta n} + (S(T) - F(0))\, e^{n\delta} = S(T)\, e^{n\delta}.$$

Suppose that a forward price at the end of day 0 is $f(0)$. Investing $f(0)$ in a riskless bond and taking a long forward position of $e^{n\delta}$ forward contract also gives final wealth of

$$S(T)\, e^{n\delta}.$$

To avoid arbitrage we must have

$$f(0) = F(0).$$

■

Theorem 8.5 *The price at every $t \in [0,T]$ of an European call option with strike price K and maturity T on the futures contact F, is given as*

$$\tilde{c}^H(t, F(t)) = e^{-r(T-t)}\left(F(t)\, N\left(\tilde{d}_1^{H*}\right) - KN\left(\tilde{d}_2^{H*}\right)\right)$$

with

$$\tilde{d}_1^{H*} = \frac{\ln \frac{F(t)}{K} + \frac{1}{2}\sigma^2 \nu_H (T-t)^{2H}}{\sqrt{\nu_H \sigma^2 (T-t)^{2H}}}$$

and

$$\tilde{d}_2^{H*} = \frac{\ln \frac{F(t)}{K} - \frac{1}{2}\sigma^2 \nu_H (T-t)^{2H}}{\sqrt{\nu_H \sigma^2 (T-t)^{2H}}}.$$

Proof. Adapted from (Björk, 2004).

Consider a futures contract $F(t, T_1, S(t))$ on delivery of $S(T_1)$ at time T_1 and a fractional European call option with exercise price K on the underlying future. It follows that the call option price is

$$\begin{aligned}
\tilde{c}^H(T) &= \max\{(F(T, T_1, S(T)) - K), 0\} \\
&= \max\left\{\left(S(T)\, e^{r(T_1-T)} - K\right), 0\right\} \\
&= e^{r(T_1-T)} \max\left\{\left(S(T) - e^{-r(T_1-T)}K\right), 0\right\}.
\end{aligned}$$

Thus the futures option consists of $e^{r(T_1-T)}$ call options on the underlying asset S with exercise date T and exercise price is $Ke^{-r(T_1-T)}$. Applying theorem 8.4 it follows

$$\tilde{c}^H(t) = e^{r(T_1-T)}\left(S(t)\, N\left(\tilde{d}_1^{H*}\right) - Ke^{-r(T-t)}e^{-r(T_1-T)}N\left(\tilde{d}_2^{H*}\right)\right)$$

where

$$\tilde{d}_1^{H*} = \frac{\ln\left(\frac{S(t)}{Ke^{-r(T_1-T)}}\right) + r(T-t) + \frac{1}{2}\nu_H \sigma^2 (T-t)^{2H}}{\sqrt{\nu_H}\sigma (T-t)^H}$$

and

$$\tilde{d}_2^{H*} = \frac{\ln\left(\frac{S(t)}{Ke^{-r(T_1-T)}}\right) + r(T-t) - \frac{1}{2}\nu_H \sigma^2 (T-t)^{2H}}{\sqrt{\nu_H}\sigma (T-t)^H}.$$

From (8.17) we have

$$S(t) = F(t)\, e^{-r(T_1-t)}.$$

The price follows as

$$\begin{aligned}\tilde{c}^H(t, F(t)) &= e^{r(T_1-T)}\left(F(t)\, e^{-r(T_1-t)} N\left(\tilde{d}_1^{H*}\right) - K e^{-r(T-t)} e^{-r(T_1-T)} N\left(\tilde{d}_2^{H*}\right)\right) \\ &= e^{-r(T-t)}\left(F(t)\, N\left(\tilde{d}_1^{H*}\right) - K N\left(\tilde{d}_2^{H*}\right)\right)\end{aligned}$$

where

$$\begin{aligned}\tilde{d}_1^{H*} &= \frac{\ln\left(\dfrac{F(t)\, e^{-r(T_1-t)}}{K e^{-r(T_1-T)}}\right) + r(T-t) + \frac{1}{2}\nu_H \sigma^2 (T-t)^{2H}}{\sqrt{\nu_H}\sigma (T-t)^H} \\ &= \frac{\ln\left(\dfrac{F(t)}{K}\right) - r(T_1-t) + r(T_1-T) + r(T-t) + \frac{1}{2}\nu_H \sigma^2 (T-t)^{2H}}{\sqrt{\nu_H}\sigma (T-t)^H} \\ &= \frac{\ln\left(\dfrac{F(t)}{K}\right) + \frac{1}{2}\nu_H \sigma^2 (T-t)^{2H}}{\sqrt{\nu_H}\sigma (T-t)^H}\end{aligned}$$

and following similar arguments we have

$$\tilde{d}_2^{H*} = \frac{\ln\left(\dfrac{F(t)}{K}\right) - \frac{1}{2}\nu_H \sigma^2 (T-t)^{2H}}{\sqrt{\nu_H}\sigma (T-t)^H}.$$

■

Conditional Black formula for a put option $\tilde{p}^H$ at time T is

$$\tilde{p}^H(t, F(t)) = e^{-r(T-t)}\left(K N\left(-\tilde{d}_2^{H*}\right) - F(t)\, N\left(-\tilde{d}_1^{H*}\right)\right).$$

8.6 Model Analysis

Rostek (2009) looks at the values of the fractional European call option for varying Hurst parameters. Rostek defines two effects, the narrowing effect ν_H, see equation

Table 8.1: Power effect.

	Persistent	**Anti-Persistent**
Short-Run	Less volatile, lower price	More volatile, higher price
Long-Run	More volatile, higher price	Less volatile, lower price

(8.1) and the power effect $(T-t)^{2H}$. The narrowing effect $0 \leq \nu_H \leq 1$ only depends on the Hurst parameter $0 \leq H \leq 1$. When

$$\lim_{H \to 0} \nu_H = \frac{1}{2},$$

$$\lim_{H \to \frac{1}{2}} \nu_H = 1$$

and

$$\lim_{H \to 1} \nu_H = 0.$$

The critical point is when $H = \frac{1}{2}$ which is the maximum and $\nu_H = 1$, see figure 8.1. For $H > \frac{1}{2}$ the function is decreasing, thus the fractional variance is multiplied by a smaller number, which reduces the prices. Rostek's explanation for this is that the further away from $\frac{1}{2}$ the less the uncertainty. The power effect depends on the time to maturity. For a persistent time series there will be less short run deviations from the mean and more long run deviations from the mean. For an anti-persistent series the effect will be the opposite, see table 8.1.

The total effect of the Hurst parameter on the call price is a result of these two effects.

The price of a conditional fractional European call option with respect to different spot prices S is graphed for the anti-persistent, persistent and normal case. In figure 8.2 the time to expiration is $T = 0.25$, and the graph displays that the price of the persistent case will be less than the normal and the anti-persistent case. While in figure 8.3 the time to expiration is $T = 1$ and all the prices will be close together with

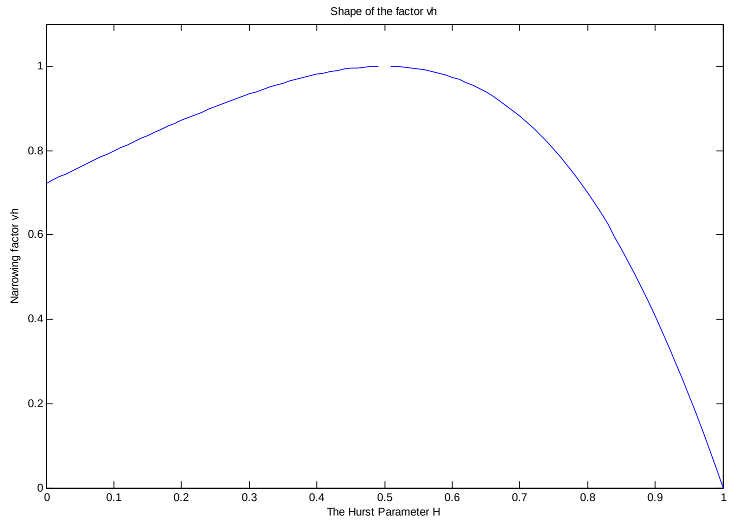

Figure 8.1: Rostek and Schöbel. Narrowing factor ν_H.

the normal one being higher and the persistent case will be the lowest for varying spot prices. In figure 8.4 the time to expiration is $T = 2$ and the graph shows that the price of the persistent case will be larger than that of the normal case.

We plot the pricing formula for varying Hurst parameters, for a fixed time T and for three different t. For shorter time to expiration $T = 0.5$ and $t = 0.1$, $t = 0.25$ and $t = 0.4$ figure 8.5 shows the price of a call option decreases as the Hurst parameter increases. For larger time to expiration $T = 5$ and $t = 1$, $t = 2.5$ and $t = 4$ figure 8.6 shows the price of a call option increases and then rapidly decreases as the Hurst parameter increases. This is due to the two above-mentioned effects.

The conditional Black functions were plotted for varying volatility $\sigma \in (0, 1)$ and varying Hurst parameters $H \in \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right)$ and for different times to maturity. Setting $F(t) = 110$, $S(t) = 100$, $K = 100$ and choosing a time to maturity to be

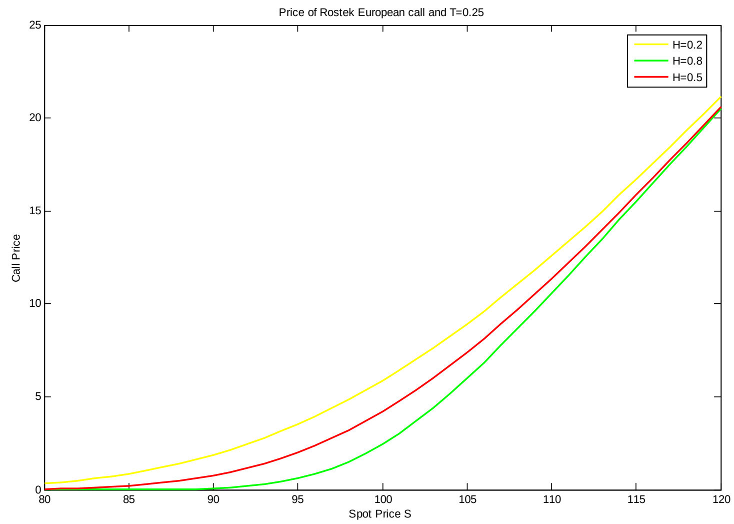

Figure 8.2: Rostek and Schöbel price of European call for varying spot for H=0.2, H=0.5 and H=0.8. Fixed parameters $K = 100,\ r = 0.02,\ \sigma = 0.2$ and $T = 0.25$ and $t = 0$.

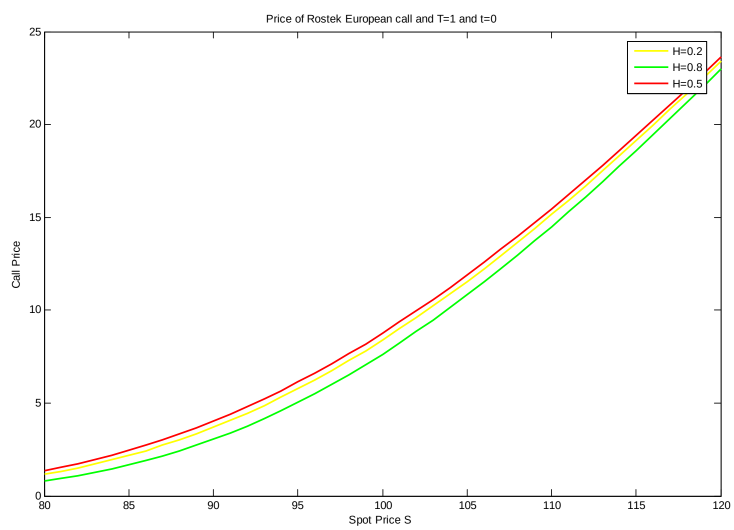

Figure 8.3: Rostek and Schöbel price of European call for varying spot for H=0.2, H=0.5 and H=0.8. Fixed parameters $K = 100,\ r = 0.02,\ \sigma = 0.2$ and $T = 1$ and $t = 0$.

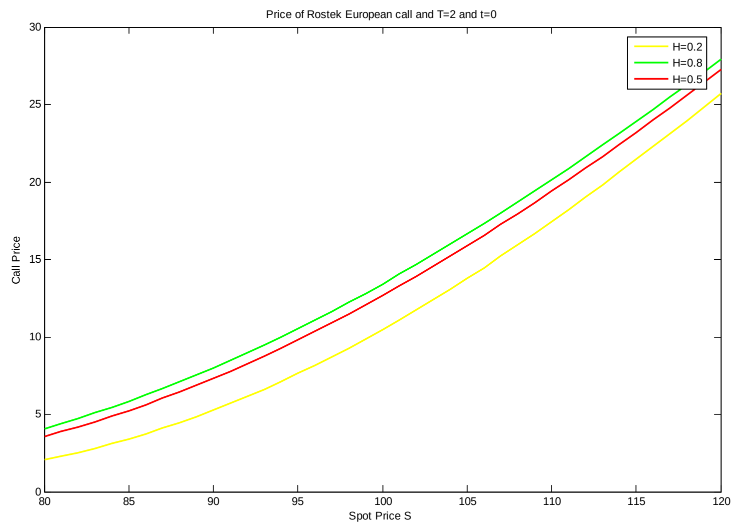

Figure 8.4: Rostek and Schöbel's price of European call for varying spot for H=0.2, H=0.5 and H=0.8. Fixed parameters $K = 100$, $r = 0.02$, $\sigma = 0.2$ and $T = 2$ and $t = 0$.

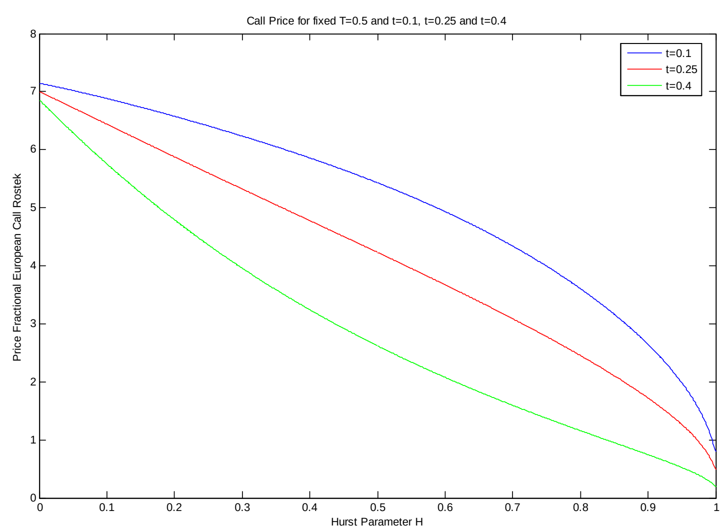

Figure 8.5: Rostek and Schöbel price of European call with varying Hurst and $t = 0.1$, $t = 0.25$ and $t = 0.4$. Fixed parameters $K = 100$, $S = 100$, $\sigma = 0.2$, $r = 0.02$ and $T = 0.5$.

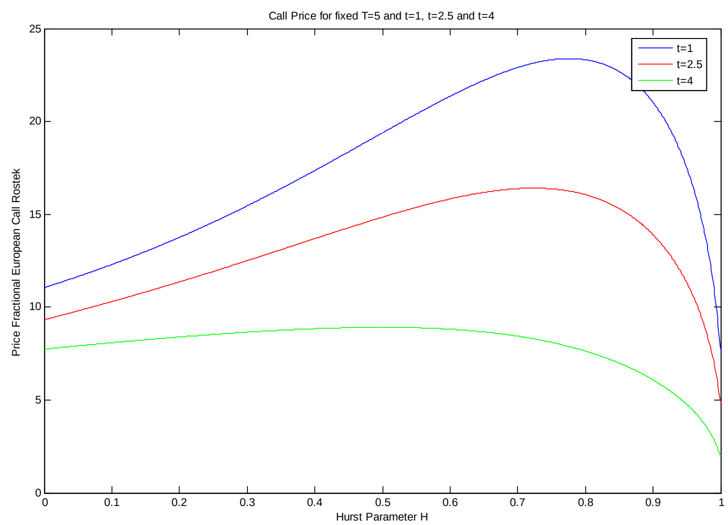

Figure 8.6: Rostek and Schöbel price of European call with varying Hurst and $t = 1$, $t = 2.5$ and $t = 4$. Fixed parameters $K = 100$, $S = 100$, $\sigma = 0.2$, $r = 0.02$ and $T = 5$.

$(T - t) = 0.25$ we obtain figure 8.7. As $\sigma \to 1$ and for all $H \to 0$ we see that the prices are the largest.

Choosing $(T-t) = 0.75$ we obtain figure 8.8 and as $\sigma \to 1$ and $H \to \frac{1}{2}$ the prices are the largest.

Setting the time to maturity $(T - t) = 5$ we obtain figure 8.9 and as $\sigma \to 1$ and for the persistent case the prices are the largest.

In general for all time horizons the curves are parabolic increasing concave down. As the time to maturity increase the prices for the persistency case increase. A small volatility and a vary Hurst has little affect on the price.

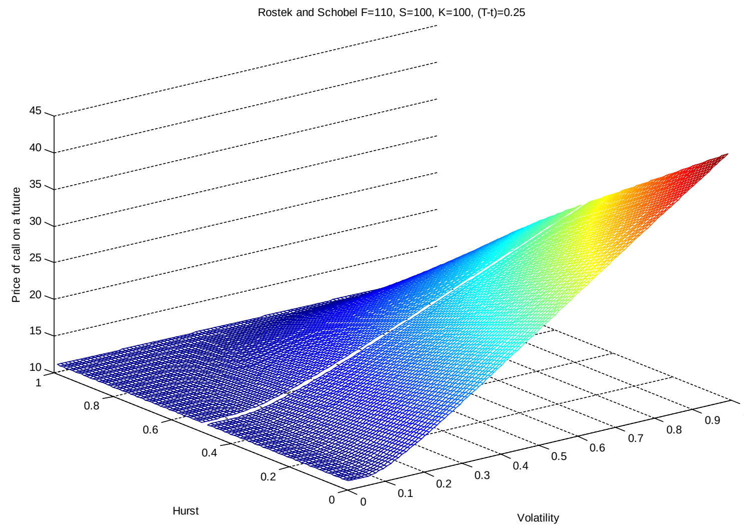

Figure 8.7: Black under Rostek and Schöbel Price vs Hurst vs Volatility $F_t = 110, S_t = 100, K = 100, (T - t) = 0.25$.

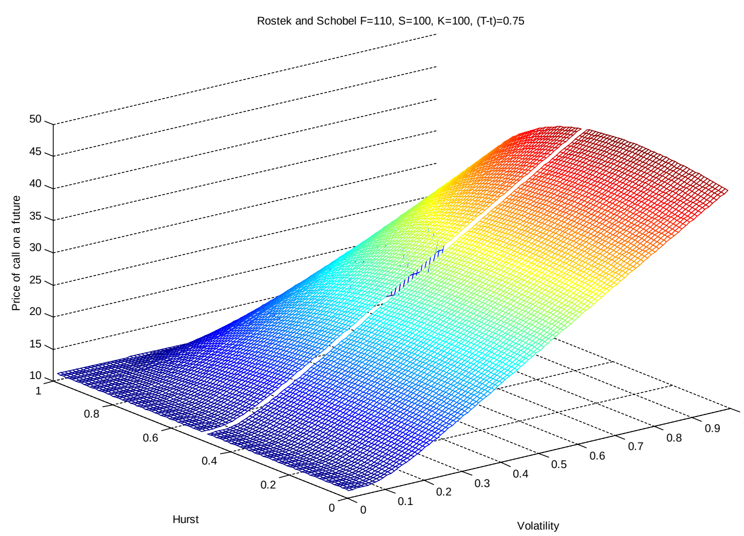

Figure 8.8: Rostek and Schöbel Price vs Hurst vs Volatility $F_t = 110, S_t = 100, K = 100$, $(T - t) = 0.75$

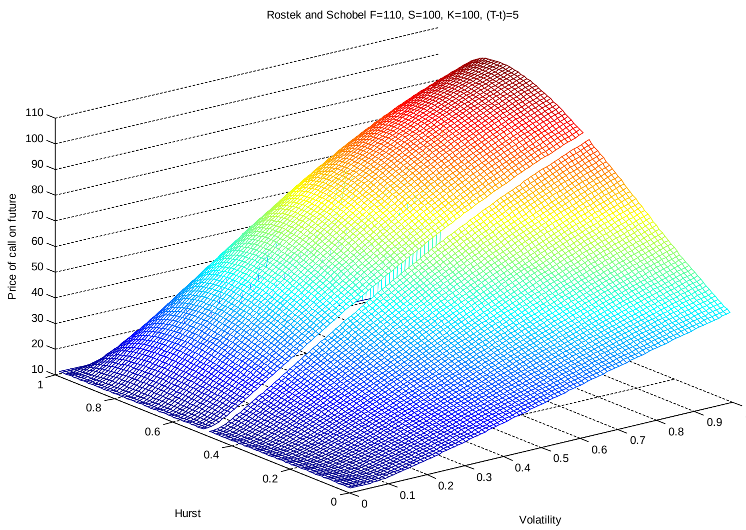

Figure 8.9: Rostek and Schöbel Price vs Hurst vs Volatility $F_t = 110, S_t = 100, K = 100, (T-t) = 5$

Chapter 9

Empirical Performance Part 1: Techniques

9.1 Introduction

An investigation of the South African stock market was done in chapter 3 and it was found that the price changes of stocks returns are not always independent thus the application of Brownian motion is not always sensible and newer methods should be looked into. Wilcox and Gebbie (2008) did interesting research in the South African market, in particular they used a modified Rescaled Range analysis and found long term memory in the first five eigenvector components of the estimated covariance matrix in the Johannesburg Stock Exchange (JSE) data.

Three fractional Black-Scholes and Black formulas were presented in chapters 6, 7 and 8 where the pricing formula depends on two unknown parameters namely the volatility parameter σ and the Hurst parameter H. The option pricing formula provided by Hu and Øksendal (2003) prices a European call option at a certain point in time namely at $t = 0$. Due to this we conjectured a hypothetical Black-Scholes and Black pricing formula under Hu and Øksendal's framework where the volatility enters the pricing equation through the term $\sigma^2 (T - t)^{2H}$, see conjectures 6.1 and 6.2 in chapter 6.

Necula (2002) generalized the European call option pricing formula to an arbitrary current time t and a quasi-conditional Black formula is proven under

Necula's framework where the volatility enters the pricing equation through the term $\sigma^2\left(T^{2H}-t^{2H}\right)$, see theorem 7.6 in chapter 7.

Rostek and Schöbel (2010) proves a fractional European option pricing formula using equilibrium pricing of options where the traders are assumed to have a constant relative risk aversion and trade in discrete time. The pricing formula now also depends on a narrowing factor ν_H which depends on the Hurst parameter. A conditional Black formula is proved as an extension of Rostek and Schöbel's option pricing formula for European options where the volatility enters the pricing equation through the term $\nu_H\sigma^2\left(T-t\right)^{2H}$, see theorem 8.5 in chapter 8.

Comparing this to the classical Black-Scholes formula, see theorem 2.3 in chapter 2, where the volatility enters the pricing equation through the term $\sigma\left(T-t\right)$ and it is clear that when setting $H=\frac{1}{2}$ in the fractional Black-Scholes formulas the classical Black-Scholes formula is obtained in all cases. The same holds for the Black formulas.

The following questions arise: How do these fractional Black models compare to each other and to the classical Black model? Which of these model results have the lowest pricing errors? How does the volatility pattern compare between these models? Is there an implied Hurst parameter? What is the relationship between the volatility parameter and the Hurst parameter?

The option value of the classical Black formula depends critically on the expected future volatility. Realized volatilities are obtained using historical stock data while implied volatilities are obtained by inverting the Black-Scholes (Black) option pricing formula. Implied volatility has to be inferred from option prices. The fractional models cannot be used if they are not calibrated to the market. Without market calibration there would be obvious arbitrage opportunities. Implied volatility is important in many applications including option pricing, risk management, hedging and financial modeling. The implied volatility that is derived from option prices depends on the assumption that is made concerning the underlying asset price distribution. The classical Black-Scholes model assumes that the underlying distribution of the stock returns follows a log-normal distribution. Under this assumption the implied volatilities of this model should be the same for options that have the same underlying asset irrespective of the strike prices and time to maturity. However in practice the implied volatilities differ with different time to maturity and different strike prices. The implied volatilities are a function of time and the underlying stock and plotting the implied volatilities against the strike yields a volatility pattern that resembles a skew. Before the 1987 market crash the smile curve was flat and afterwards it formed a skew, see Derman (2007). On a day the implied volatility is a quoting convention that incorporates the underlying price, strike price, time to maturity and the trading of option is like trading volatility, see for example SAVI squared futures on the Johannesburg Stock Exchange (JSE). The implied volatility is a number that equates the price of the Black-Scholes (Black) option to that of the observed market price of the option (Rebonato, 2004). The Black-Scholes (Black) option pricing formulas cannot be analytically inverted such that we get the implied volatility as a explicit function of the stock, strike, time to maturity, interest rates and the option price, and numerically methods such as Newton-Raphson should be used see Hurst, Platen and Rachev (1999). Imagine a three dimensional graph with the implied volatility on the z-axis, the strike on the x-axis and time on the y-axis, this is known as the volatility surface see Kotzé, Joseph, Naido, Boardman and de Wet (2009). Gatheral and Lynch (2002) stochastic volatility models incorporate different strikes and different times to maturity. It has been noted that implied volatilities decrease as the stock price rises and vice versa. Black (1976) wrote " I have believed for a long time that stocks are related to volatility changes. When stocks go up, volatilities seems to go down; and when stocks go down volatilities seems to go up." Option prices are sensitive to volatility thus small changes in the volatility have a large effect on the price of an option.

Cajueiro and Barbachan (2003) compared Necula's fractional Black-Scholes option pricing formula with that of the classical Black-Scholes model using Brazilian stock returns and found that the model does not deal with fat tails. Krzywda (2011) compared Rostek and Schobël's Black option pricing formula with that of the classical Black model using Warsaw stock returns. In both of the investigations, a constant historical Hurst parameter was obtained using the Rescaled Range analysis and the volatility was estimated by a method that was introduced by Kukush, Mishura and Valkeila (2002). İnkaya (2011) used the Dow Jones Industrial Average index to price options using fractional Brownian motion under Hu and Øksendal and Necula's framework and followed the methodology of Zhang, Xiao and He (2009).

The different fractional Black formulas were investigated as a function of a varying Hurst and volatility parameters. A large collection of volatility and Hurst parameter combinations was generated and input to obtain prices with all other parameters fixed. Figures 9.1, 9.2 and 9.3 plot the obtained prices on the horizontal axis against all the Hurst and volatilities that correspond to each price. It can clearly be seen that there is a range of possible Hurst and volatility parameters that give the same price, and there is hence no unique volatility and Hurst parameter combination that yields a specific price.

Fixing a constant Hurst parameter the implied volatility can be computed by inverting the option valuation formula and solving for the implied volatility. The

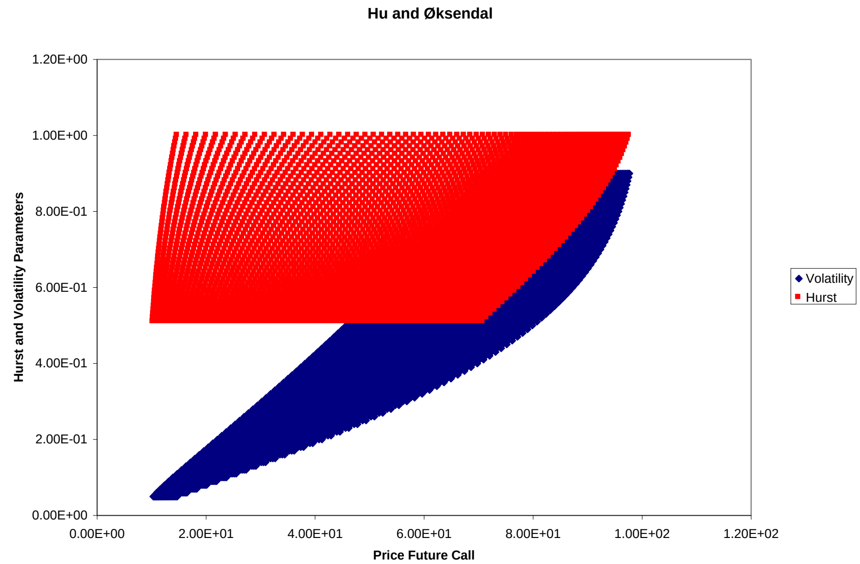

Figure 9.1: Conjectured Hu and Øksendal Black formula for parameters $F = 110, S = 100, K = 100, (T - t) = 5$.

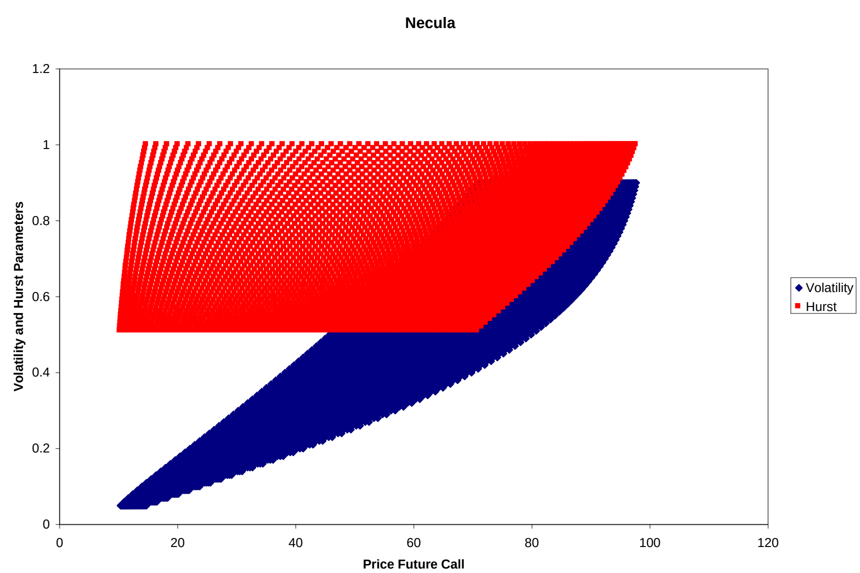

Figure 9.2: Necula quasi-conditional Black formula for parameters $F = 110$, $S = 100$, $K = 100$, $T = 5$ and $t = 0$.

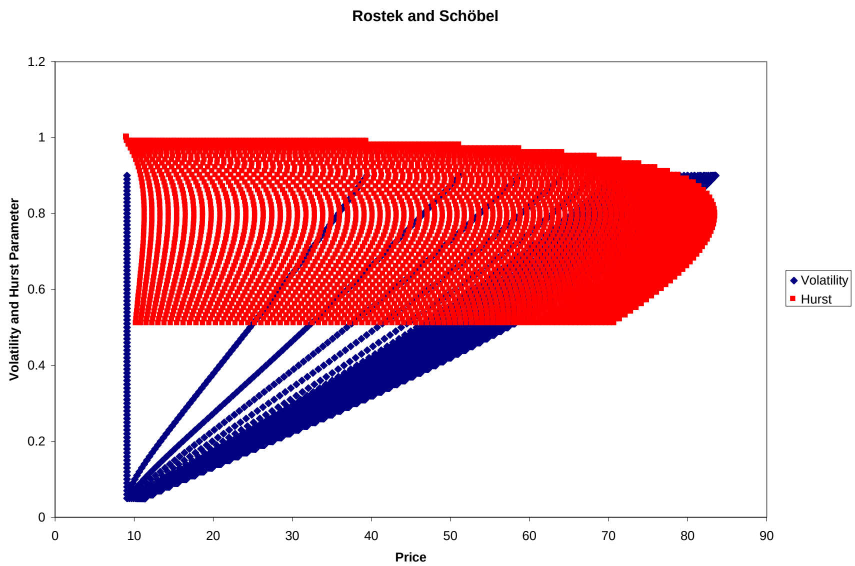

Figure 9.3: Black formula under Rostek and Schöbel framework for parameters $F = 110, S = 100, K = 100, (T - t) = 5$.

option prices are observed in the markets and the remaining parameters can be observed directly from historical data. A heuristic optimization algorithm, simulated annealing, was used to back out the volatility for collections of options. In pricing the fractional Black-Scholes (Black) models both the volatility parameter and the Hurst parameter influence the price of the option, therefore the out of sampling pricing performance is done to investigate their combined presence in the pricing formula. The two parameters both affect the price significantly.

Techniques are presented in order to compare empirically the performance of these models with that of the classical Black-Scholes (Black) formula. Bakshi, Cao and Chen (1997) present measures of empirical sampling performance, we investigate the implied volatility and out of sampling methods.

9.2 Backing Out Fractional Implied Volatility

The objective is to minimize the objective function which is the square absolute difference between the actual observed market price and the estimated prices. The implementing procedure is as follows: collect $n = 1, ..N$ futures contacts F on the same stock S which is taken from the same point in time t. Let T_n and K_n be the time to expiration of the futures contract and strike price on the n-th option respectively. Let σ_n be the implied volatility associated with the classical Black formula for the n-th option, this is how prices are quoted. Collect historical parameters F, S, K_n, t and T_n from the market which can be easily done as the efficient market hypothesis holds. Assume the Hurst parameter H is a constant. Let $\sigma_{H,n}$ be the implied volatility associated with the fractional Black formula for the n-th option. The implied volatilities $\sigma_{H,n}$ are unknown and have to be estimated empirically from the market. Let the price of the classical Black formula be denoted as $c_n\left(F, S, K_n, t, T_n, \sigma_n\right)$ and let $c_n^H\left(F, S, K_n, t, T_n, \sigma_{H,n}, H\right)$ be the price of the fractional Black formula for the n-th option. For each n, the error function $\varepsilon_n\left[\sigma_{H,n}\right]$ is defined as the model price less the market price

$$\varepsilon_n\left[\sigma_{H,n}\right] = c_n^H\left(F, S, K_n, t, T_n, \sigma_{H,n}, H\right) - c_n\left(F, S, K_n, t, T_n, \sigma_n\right). \tag{9.1}$$

The objective is to minimizes the objective function SSE which is given as the sum of absolute squared differences of the estimated model and the actual price. The sum of squared pricing errors is given by

$$\min_{\sigma_{H,n}} SSE\left(\sigma_{H,n}, t\right) = \min_{\sigma_{H,n}} \sum_{n=1}^{N} \left(\left|\varepsilon_n\left[\sigma_{H,n}\right]\right|^2\right). \tag{9.2}$$

Using appropriate optimization techniques the implied volatility $\sigma_{H,n}$ can be estimated for each n by minimizing the objective function. Calculations are done similarly for put options on futures and the method is similar for the Black-Scholes formula.

9.3 Out-of-Sample Pricing

By fixing a constant Hurst parameter the implied volatility is backed out for either a single option or a collection of options with the same underlying. Then use the previous day's implied volatility, the same constant Hurst parameter, the current day's observed future and stock price, to price the options. The average pricing error is calculated as the sum of the differences between the market price and the model price. The average absolute pricing error is calculated as the sample average of the absolute difference between the market price and the model price and the average percentage pricing error is calculated as the sample average of the market price minus the model's price divided by the market price.

The model is implemented in the following way: at time $t-1$ assume the future price $F(t-1)$, stock price $S(t-1)$, strike K_n and the option $c(t-1)$ can be observed directly from the market. Assume a constant Hurst parameter H which can be empirically estimated from historical data. Using this information the implied volatility $\sigma_H(t-1)$ can be backed out at time $t-1$ by optimizing (9.2). At time t compute the price of the fractional future option using the parameters $F(t)$, $S(t)$, K_n and using the backed out implied volatility from the previous day $\sigma_H(t-1)$ and the same Hurst parameter that was used to back out the implied volatility. Next subtract the model determined price from the option price in the market at time t. Let $t = 2, ..., M$ be the number of days and let $n = 1, .., N$ be the number of options. Repeat the procedure for numerous days.

At time t the average pricing error is calculated as the sample average of the difference between the market price and the model price and is given by

$$E[|\varepsilon[\sigma_{H,n}]|] = \frac{1}{M}\frac{1}{N}\sum_{t=1}^{M}\sum_{n=1}^{N}\left(c(F_t, S_t, K_n, t, T_n, \sigma_t) - c^H(F_t, S_t, K_n, t, T_n, \sigma_{H,t-1}, H)\right). \tag{9.3}$$

At time t the average absolute pricing error is calculated as the sample average of the absolute difference between the market price and the model price and is given by

$$E\left[\left|\varepsilon\left[\sigma_{H,n}\right]\right|\right] = \frac{1}{M}\frac{1}{N}\sum_{t=1}^{M}\sum_{n=1}^{N}\left|c\left(F_t, S_t, K_n, t, T_n, \sigma_t\right) - c^H\left(F_t, S_t, K_n, t, T_n, \sigma_{H,t-1}, H\right)\right|. \tag{9.4}$$

The average percentage pricing error is calculated as the sample average of the market price minus the model's price divided by the market price and is given by

$$E\left[\varepsilon\left[PE\left(t\right)\right]\right] = \frac{1}{M}\frac{1}{N}\sum_{t=1}^{M}\sum_{n=1}^{N}\left[\frac{c\left(F_t, S_t, K_n, t, T_n, \sigma_t\right) - c^H\left(F_t, S_t, K_n, t, T_n, \sigma_{H,t-1}, H\right)}{c\left(F_t, S_t, K_n, t, T_n, \sigma_t\right)}\right]. \tag{9.5}$$

9.4 Optimization Techniques

According to Hamida and Cont (2005) equation (9.2) is very difficult to solve due to the function not being convex or having any particular structure to assist gradient based methods. Also computing its gradient is difficult as the function has to be computed numerically. For this reason we use a heuristic known as simulated annealing. We used Vandekerckhove's (2006) MATLAB code, for details about the simulated annealing algorithm see appendix E.

Chapter 10

Empirical Performance Part 2: Results

10.1 Introduction

In this chapter three listed futures are considered, namely the ALSI (JSE Top 40 index), SBK (Standard Bank) and MTN (South African cell phone company). Historical data were taken from Safex, the South African futures exchange.

Volatility changes over time and the implied volatilities differ for different strike prices. An investigation of in-the-money (ITM), at-the-money (ATM), out-the-money (OTM) futures is done. A relationship known as put-call parity holds between a European call and the European put option, due to this the implied volatility for a European call option is the same as the implied volatility for the European put options, with the same strike and maturity and all other inputs kept the same (Hull, 2006). Put-call parity relation also holds for fractional call and puts. The implied volatilities were backed out for different Hurst parameters and a volatility pattern is created by plotting the implied volatility of an option as a function of its strike price. This pattern is known as volatility skew, of which one type of the volatility skew is the volatility smile. The volatility smile is a U shaped curve while another common skew pattern is the volatility smirk reverse skew which is what appears for the ALSI, SBK and MTN options. This means that the implied volatilities for options with lower strikes are higher than those with higher strikes. One explanation for the reverse skew is that investors buy more puts for protection as insurance against large drops in the market (Chance, 2008).

If S is the stock price and K is the strike price then a call option is in-the-money when $K < S$. A call option is at-the-money when $S = K$ and a call option is out-the-money when $S < K$. The backed out implied volatility patterns are plotted for each of the fractional models and the actual observed volatilities are plotted to see whether they are comparable and whether these models exhibit changes across in-the-money, at-the-money, out-the-money options and across maturity. Plotting the implied volatilities against time also gives an indication how these fractional models capture different market trends across time. The implied volatilities are compared to that of the actual market volatilities and thus their movements and deviations can be observed. Using these daily implied volatilities the out of sampling pricing performance of the models is done in order to compare the pricing to that of the classical Black formula. The pricing errors are obtained using all the parameters from the previous day's. The objective of this chapter is to empirically compare the performance of these models with that of the classical formulas.

10.2 Data

We will now introduce the data which has been used in this empirical comparison.

10.2.1 South African Options

In particular we are dealing with options written on futures. The JSE only supports American style options. An American option is the same as a European option but it can be exercised prematurely; this is not a problem as it is never optimal to exercise these options early (JSE, 2008). Therefore we can still use the Black formula to price them.

Data were obtained from the SAFEX (South African Futures Exchange) website. ALSI, MTN and SBK options on futures were collected from 1 April 2011 till 1 June 2011. The futures price, stock price, strike price, maturity date and implied volatility were obtained. We used equation (8.17) to estimate the interest rates. The ALSI call options expiring on 15 June 2011 were categorized into four different groups on the 1st of April 2011. The first group was the in-the-money group with moneyness $\left(\frac{S}{K}\right) > 1.03$ there were four strikes. The second group was the at-the-money group with moneyness $\left(\frac{S}{K}\right) \in [0.97, 1.03]$, there were six strikes. The third group was the out-of-the money group with the moneyness $\left(\frac{S}{K}\right) < 0.97$, there were eight strikes and finally the fourth category was all the calls with 21 strikes. The MTN call options expiring on 15 June 2011 was categorized into one group consisting of five strikes on the 1st of April 2011. The SBK call options expiring on 15 June 2011 were categorized into one group consisting of four strikes on the 1st of April 2011.

10.2.2 South African Warrants

Warrants were introduced by the Deutsche Bank in October 1997 on the JSE Securities Exchange (JSE, 2002). Warrants are securities that are issued by a party independent of the underlying asset, giving the right but not the obligation to buy or sell an underlying asset. The warrant style can be either American or European.

TOPSBE and TOPSBF were the European call warrants which were taken into consideration from 1 September 2011 till 31 October 2011; these warrants are issued by Standard Bank. From the Sharenet website we gathered the underlying stock price and the warrant price. From Dr A. Kotzé at the Johannesburg Stock Exchange we obtained a reliable yield curve. We assumed the yield curve was linear between the maturities we had available and used linear interpolation to obtain interests rates for the in-between maturities. Using an algorithm from Benninga (2000), (see appendix F for the code) we backed out the implied volatilities for the individual options.

10.3 Implied Volatilities Across Time

Using (9.2) we backed out the implied volatilities for collections of ALSI, MTN and SBK calls on futures, using the classical Black formula and the Black formula's based on Hu and Øksendal's, Necula's and Rostek and Schöbel's frameworks using simulated annealing. For the ALSI calls on futures, seven Hurst parameters were chosen $H = \{0.5043;\ 0.51426;\ 0.54736;\ 0.6;\ 0.7;\ 0.8;\ 0.9\}$ for the analysis. Necula's model using the absolute times, $T^{2H} - t^{2H}$, this equation necessitates that we pick a starting time. The starting time of 1st of April 2011 was chosen as the beginning of our investigation.

For the entire collection of ALSI calls figures 10.1, 10.2 and 10.3 display the implied volatilities for the above mentioned fractional Black formulas with fixed Hurst parameters. The volatilities backed out using the standard Black formula are also given in each figure. In all the figures it can be seen that the higher the Hurst parameter the higher the backed out implied volatilities for the fractional models. In particular, it can be seen that as the calls approach expiration Hu and Øksendal's and Rostek and Schöbel's volatilities diverge while Necula's backed out implied volatilities converge. In other words when the Hurst parameter is higher as the option approaches expiration the implied volatilities get bigger much faster for the fractional Black models under the Hu and Øksendal and Rostek and Schöbel framework. The effect is opposite for implied volatilities for the Black under Necula's framework. This behaviour will be made clearer later on.

Figures 10.4, 10.5, 10.6, 10.7, 10.8, 10.9 and 10.10 each show for a single Hurst parameter the models implied volatilities. The fractional implied volatilities followed the movements of the Black implied volatilities. It can be seen that Hu and Øksendal's and Rostek and Schöbel's implied volatilities are quite close below a certain Hurst threshold, but for higher Hurst parameters they diverge and Rostek's and Schöbel's implied volatilities are larger.

Three Hurst parameters $H = \{0.5211;\ 0.5454;\ 0.597\}$ were chosen to back out the implied volatilities for the MTN calls on futures. Figure 10.11 displays the implied volatilities for the models for only a Hurst $H = 0.5454$. For the SBK calls on futures four Hurst parameters were chosen $H = \{0.4005;\ 0.45298;\ 0.51;\ 0.5603;\ 0.6\}$. Figure 10.12 shows the implied volatility for the models with $H = 0.4005$, it can be seen that for the anti-persistency case the fractional implied volatilities are lower than the normal Black implied volatilities. For the TOPSBE and TOPSBF seven Hurst parameters were chosen $H = \{0.5043;\ 0.51426;\ 0.54736;\ 0.6;\ 0.7;\ 0.8;\ 0.9\}$ for the backing out the implied volatilities. Figure 10.13 shows the implied volatilities for a $H = 0.6$.

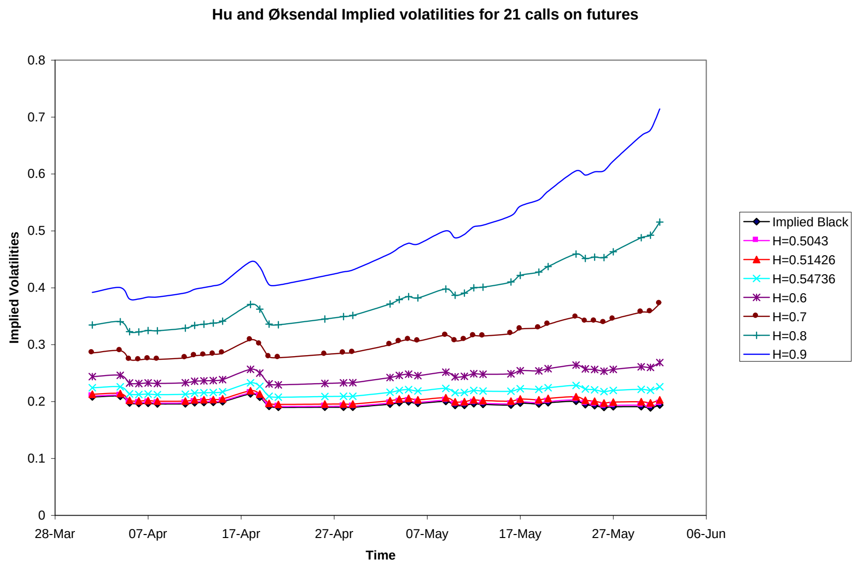

Figure 10.1: Hu and Øksendal implied volatilities for 21 calls on futures.

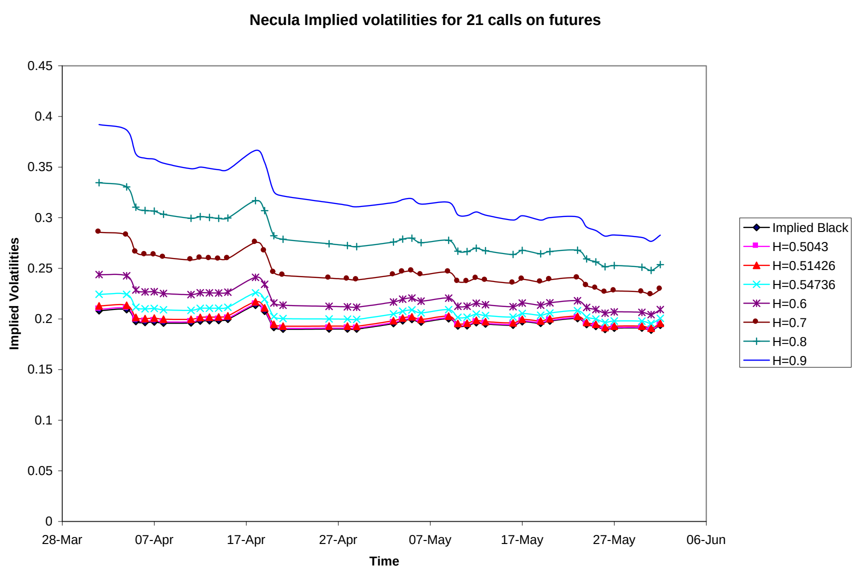

Figure 10.2: Necula implied volatilities for 21 calls on futures.

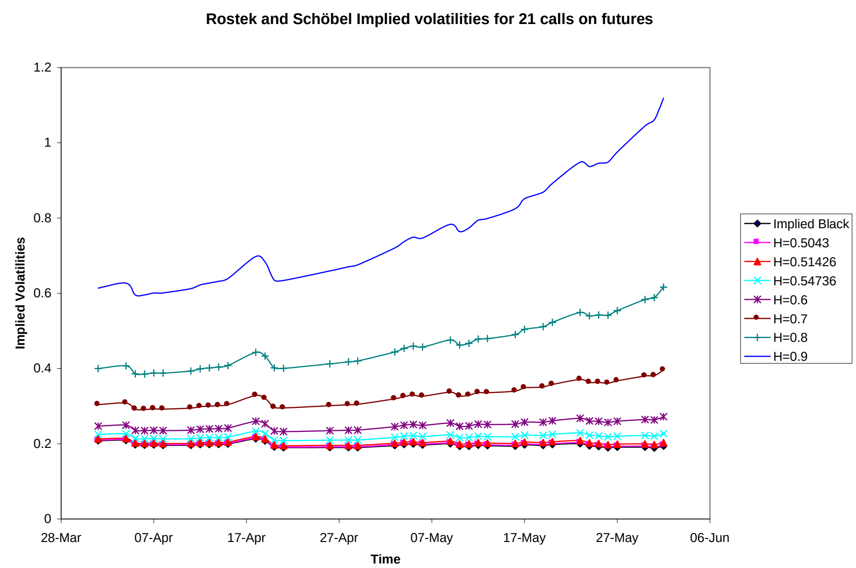

Figure 10.3: Rostek and Schöbel implied volatilities for 21 calls on futures.

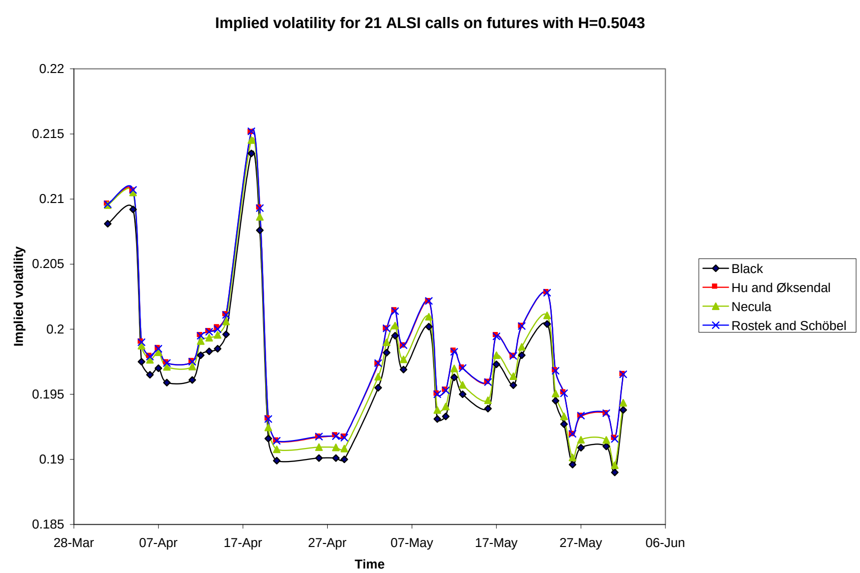

Figure 10.4: Implied volatility for 21 ALSI calls on futures with H=0.5043.

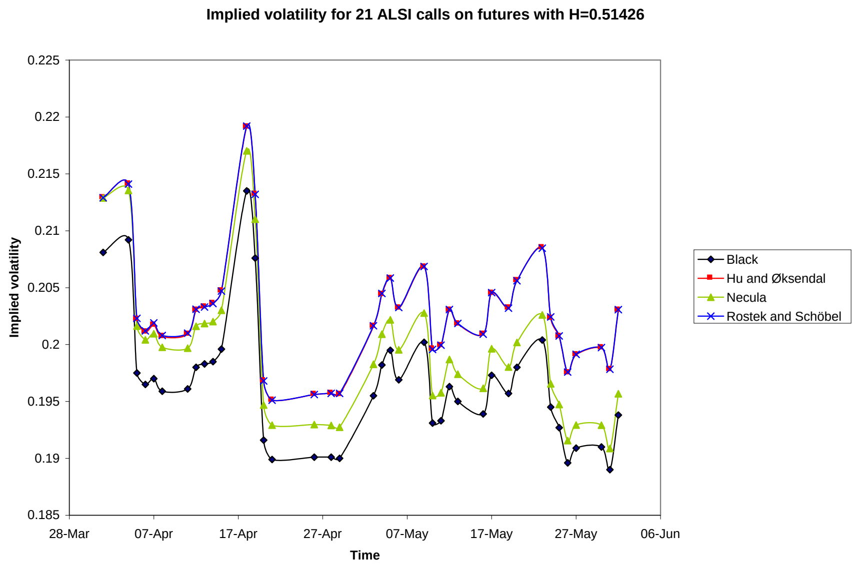

Figure 10.5: Implied volatility for 21 ALSI calls on futures with H=0.51426.

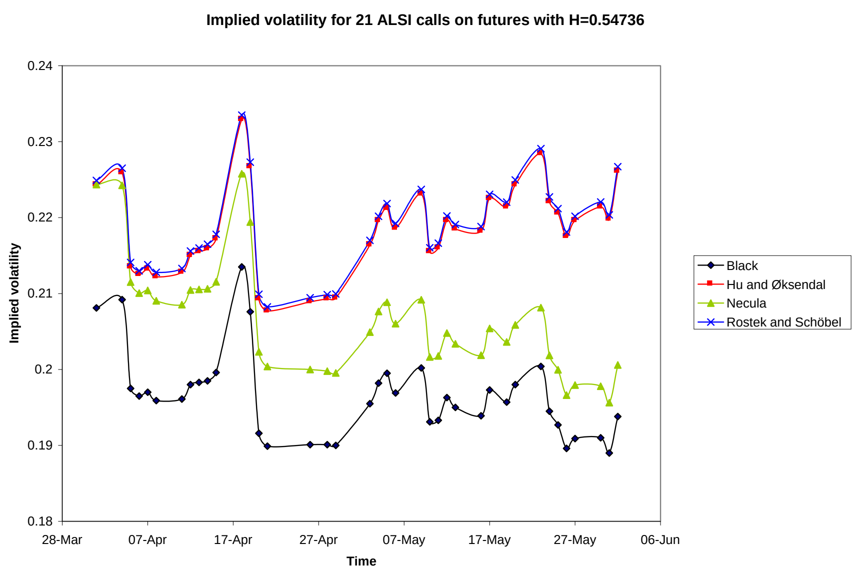

Figure 10.6: Implied volatility for 21 ALSI calls on futures with H=0.54736.

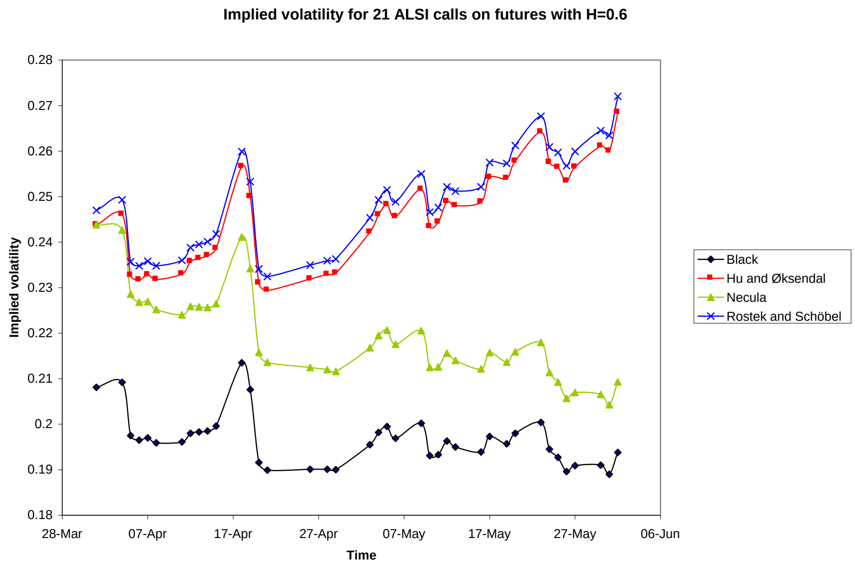

Figure 10.7: Implied volatility for 21 ALSI calls on futures with H=0.6.

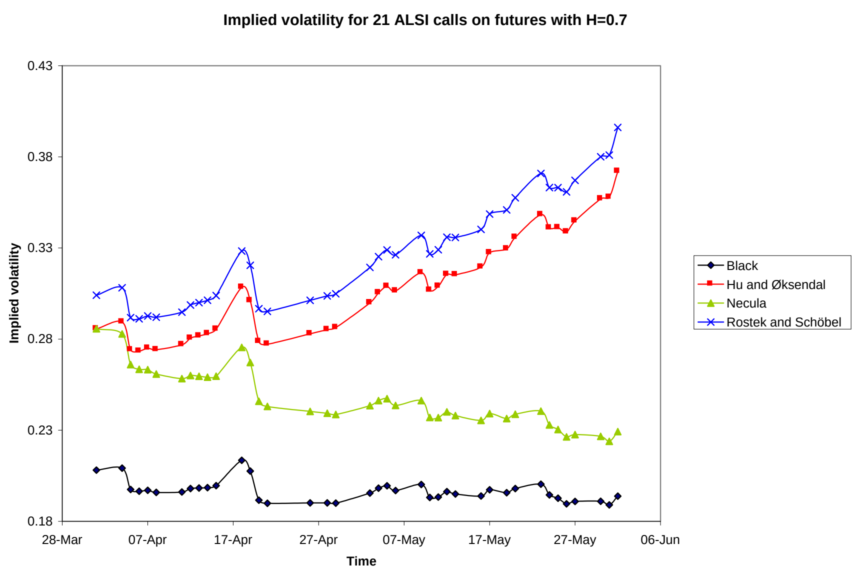

Figure 10.8: Implied volatility for 21 ALSI calls on futures with H=0.7.

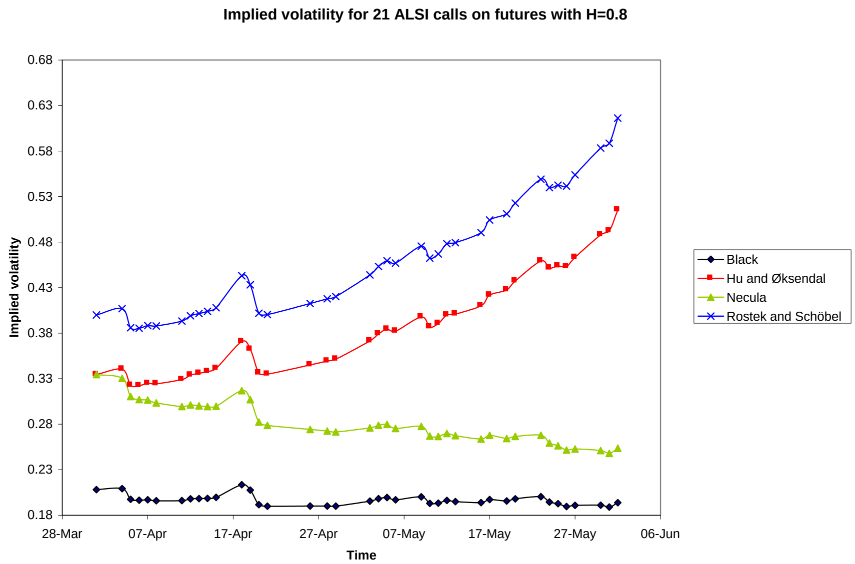

Figure 10.9: Implied volatility for 21 ALSI calls on futures with H=0.8.

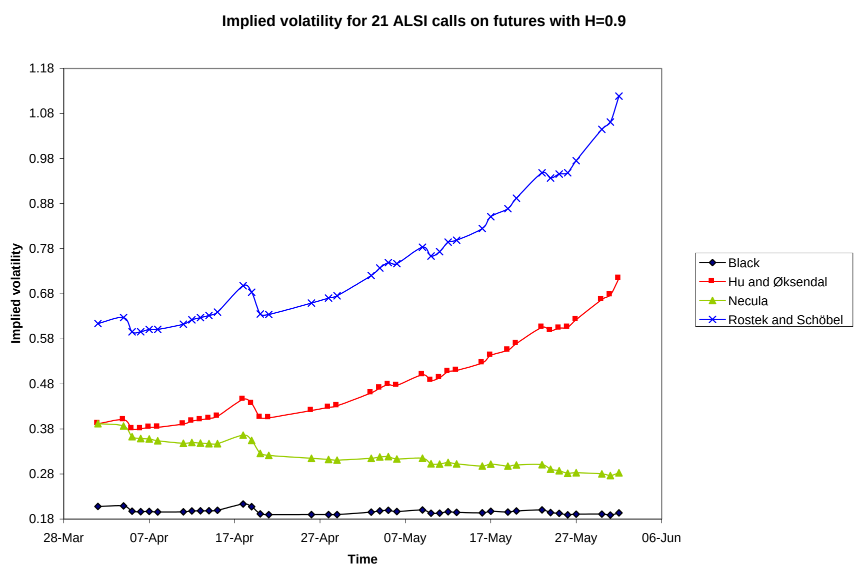

Figure 10.10: Implied volatility for 21 ALSI calls on futures with H=0.9.

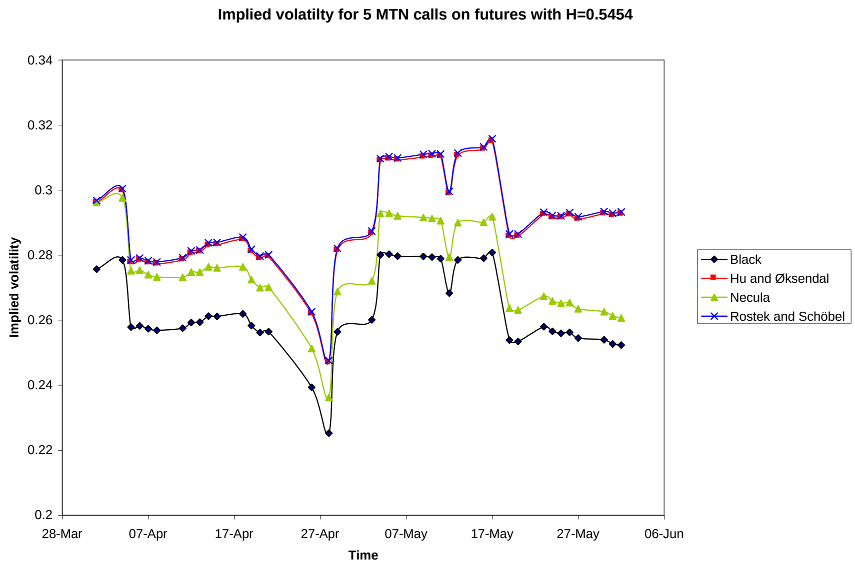

Figure 10.11: Implied volatilty for 5 MTN calls on futures with H=0.5454.

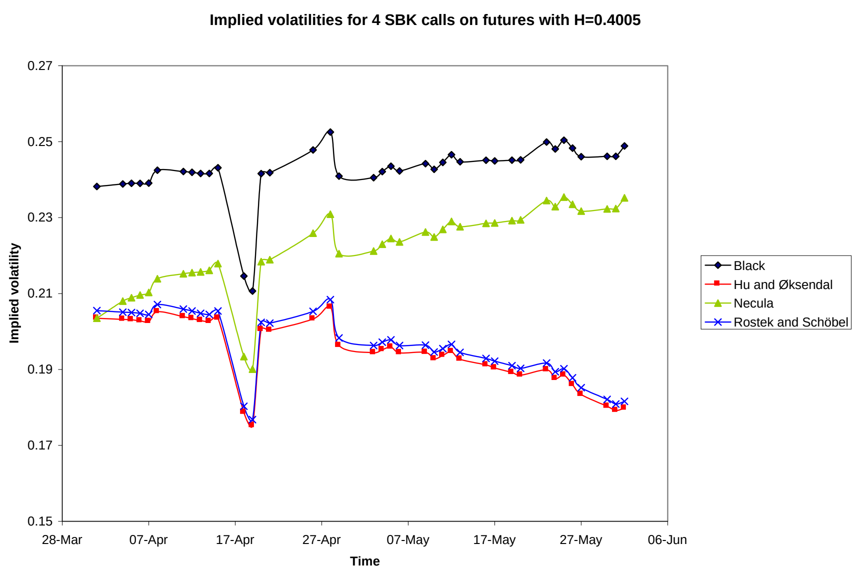

Figure 10.12: Implied volatilities for 4 SBK calls on futures with H=0.4005.

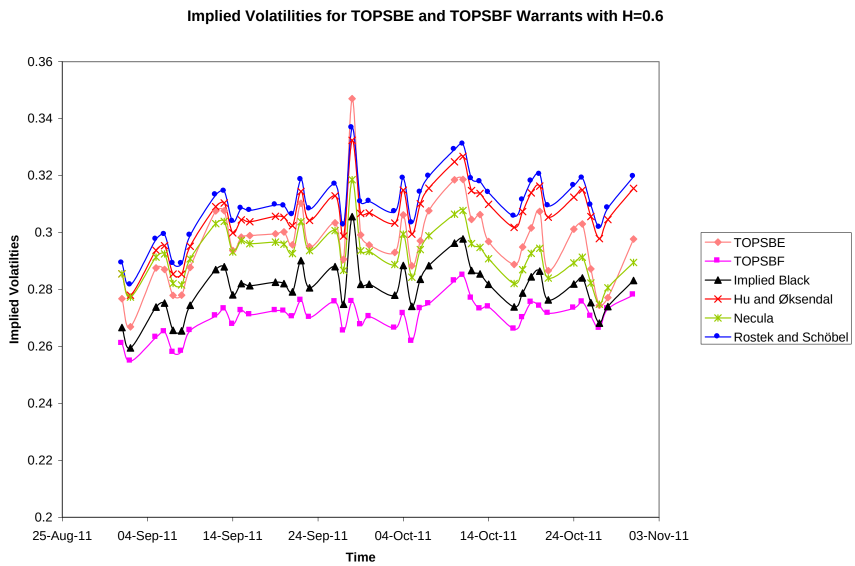

Figure 10.13: Implied volatilities for TOPSBE and TOPSBF warrants with H=0.6.

10.3.1 Explanation

The difference between the classical Black and Black-Scholes models and the conjectured models under Hu and Øksendal's fractional pricing framework is that $\sigma^2(T-t)$ is replaced by $\sigma_{H,1}^2(T-t)^{2H}$ in the pricing formula, where we denote $\sigma_{H,1}$ the fractional volatility for Hu and Øksendal's model. We can write

$$\sigma^2(T-t) = \sigma_{H,1}^2(T-t)^{2H}$$

or

$$\sigma_{H,1}^2 = \sigma^2 \frac{(T-t)}{(T-t)^{2H}}. \tag{10.1}$$

Figure 10.14 displays the fractional volatility $\sigma_{H,1}$ for $\sigma = 0.2$ and time to expiration $T = 2$. When $(T-t)$ is smaller than one then the fractional volatilities are

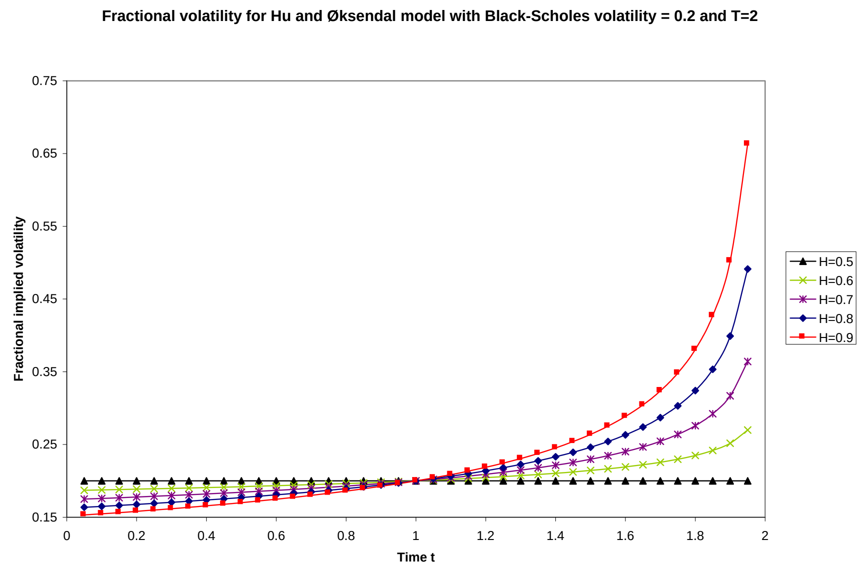

Figure 10.14: Fractional volatility for Hu and Øksendal model with Black-Scholes volatility = 0.2 and T=2.

greater than the Black-Scholes (Black) volatilities. This can also be seen from the implied volatilities graphs above. As $t \to T$ the fractional volatilities increase and this is the same behaviour that can be seen with the empirical backed out volatilities for Hu and Øksendal's model.The difference between the classical Black and Black-Scholes models and the models under Necula's quasi-conditional fractional pricing framework is that $\sigma^2 (T-t)$ is replaced by $\sigma_H^2 \left(T^{2H} - t^{2H}\right)$ in the pricing formula, where we denote $\sigma_{H,2}$ the fractional volatility for Necula's model. We can write

$$\sigma^2 (T-t) = \sigma_{H,2}^2 \left(T^{2H} - t^{2H}\right)$$

or

$$\sigma_{H,2}^2 = \sigma^2 \frac{(T-t)}{\left(T^{2H} - t^{2H}\right)}. \tag{10.2}$$

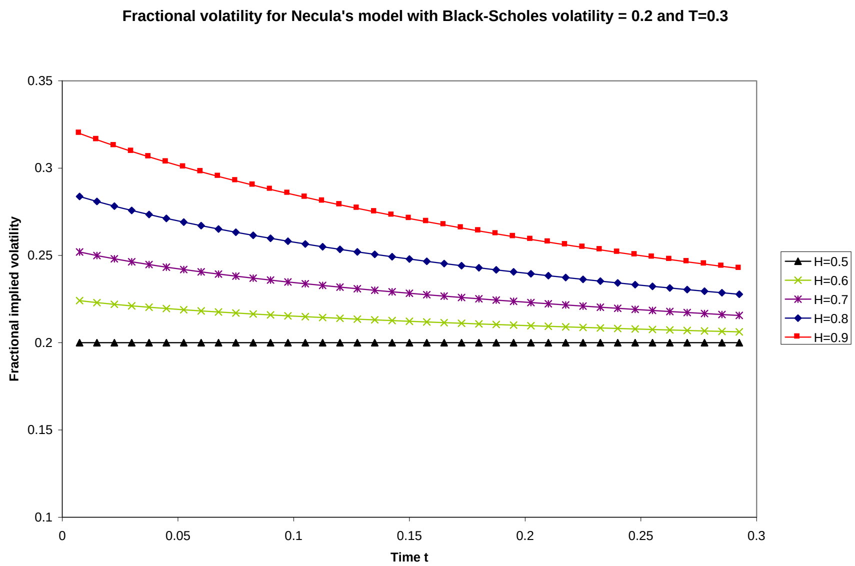

Figure 10.15: Fractional volatility for Necula's model with Black-Scholes volatility = 0.2 and T=0.3.

Figure 10.15 displays the fractional volatility $\sigma_{H,2}$ for $\sigma = 0.2$ and time to expiration $T = 0.3$. We picked t_0 as zero. As $t \to T$ the fractional volatilities decrease and this is the same behaviour that can be seen with the empirical backed out volatilities for Necula's model and are greater than Black-Scholes (Black) volatility. While figure 10.16 displays the fractional volatility $\sigma_{H,2}$ for $\sigma = 0.2$ and time to expiration $T = 2$. In this case the fractional volatilities are below the the Black-Scholes (Black) volatility.

The difference between the classical Black and Black-Scholes models and the models under Rostek and Schöbel's conditional fractional pricing framework is that $\sigma^2 (T - t)$ is replaced by $\nu_H \sigma_H^2 (T - t)^{2H}$ in the pricing formula, where we denote

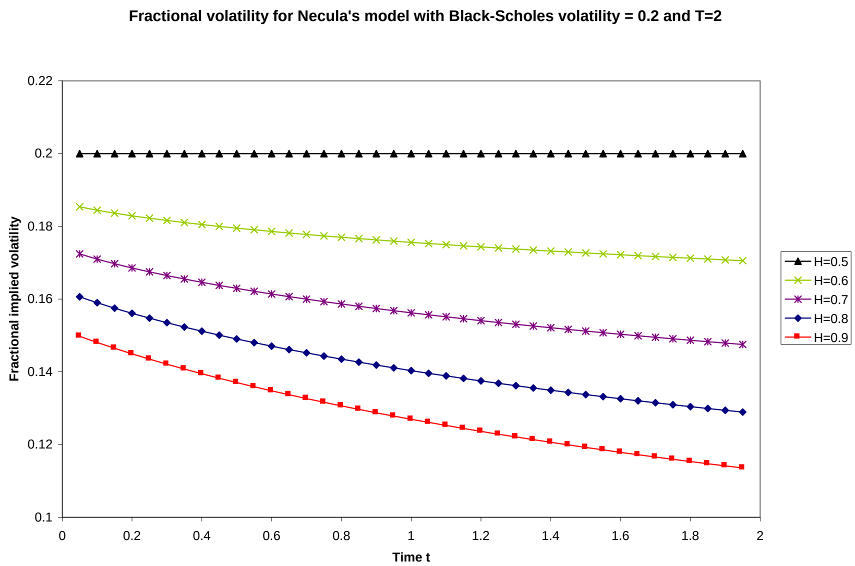

Figure 10.16: Fractional volatility for Necula's model with Black-Scholes volatility = 0.2 and T=2.

$\sigma_{H,3}$ the fractional volatility for Rostek and Schöbel's model. We can write

$$\sigma^2 (T-t) = \sigma_{H,3}^2 \nu_H (T-t)^{2H}$$

or

$$\sigma_{H,3}^2 = \sigma^2 \frac{(T-t)}{\nu_H (T-t)^{2H}}. \tag{10.3}$$

Figure 10.17 displays the fractional volatility $\sigma_{H,3}$ for $\sigma = 0.2$ and time to expiration $T = 2$. As $t \to T$ the fractional volatilities increase and this is the same behaviour that can be seen with the empirical backed out volatilities for Rostek and Schöbel's model. The fractional volatilities are also larger than the Black-Scholes volatility close to maturity.

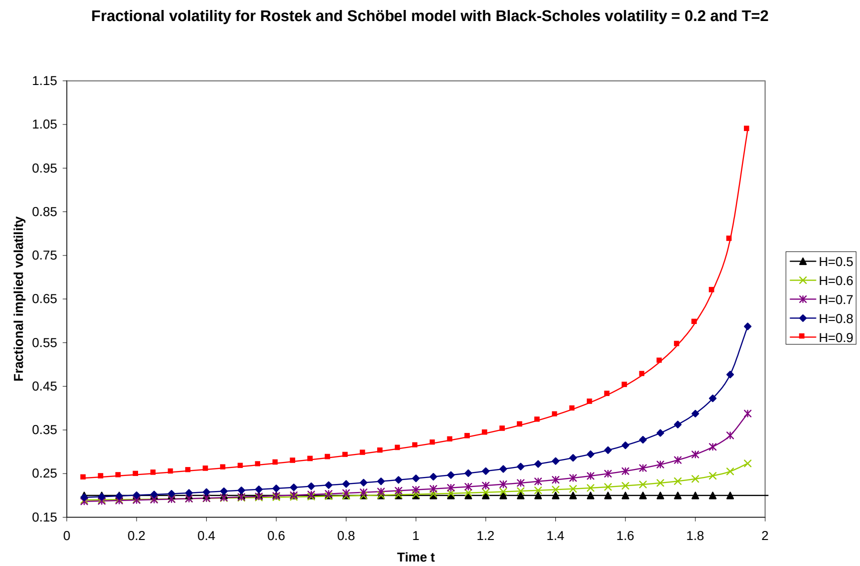

Figure 10.17: Fractional volatility for Rostek and Schöbel model with Black-Scholes volatility = 0.2 and T=2.

10.4 Volatility Smile

We used equations (10.1) , (10.2) and (10.3) to get the fractional implied volatilities for the models. The implied volatilities were backed out on the 1st of June 2011 for the ALSI options expiring on the 15th of June 2011. Figures 10.18, 10.19 and 10.20 display the volatility smile curve for Hurst parameters $H = \{0.5043;\ 0.6;\ 0.9\}$. For $H > \frac{1}{2}$ we see that as the Hurst parameter increases the implied fractional volatility smile curves move upward for the three fractional Black models. In particular we see that when the strike is deep in-the-money the deviations from the observed implied volatilities are greater than the other strikes. Necula's fractional volatilities are closer to the Black volatilities while Hu and Øksendal's fractional volatilities are in between

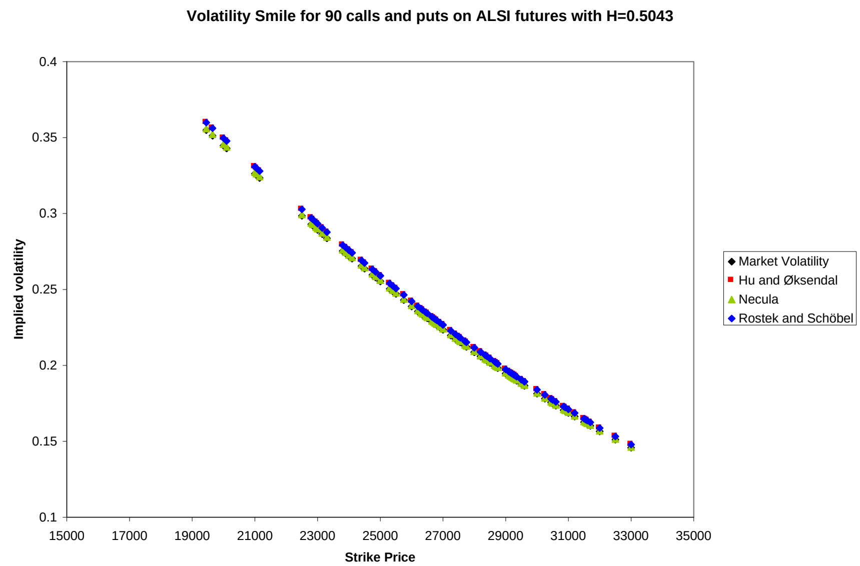

Figure 10.18: Volatility Smile for 90 calls and puts on ALSI futures with H=0.5043.

Necula's and Rostek and Schöbel's fractional volatilities and Rostek and Schöbel's fractional volatilities are the largest.

The fractional volatilities are inconsistent with the assumptions made in chapter 6, section 6.4, chapter 7, section 7.3 and chapter 8, section 8.5 where we assumed a constant volatility on a underlying for all maturities. The fractional Black-Scholes (Black) option pricing models imply the existence are multiple fractional implied volatilities. The fractional vanilla option pricing models are incorrect. On a given day with a certain strike in mind the implied volatility smile curve for different Hurst parameters gives a good number to put into three fractional Black-Scholes (Black) formulas to just get a price.

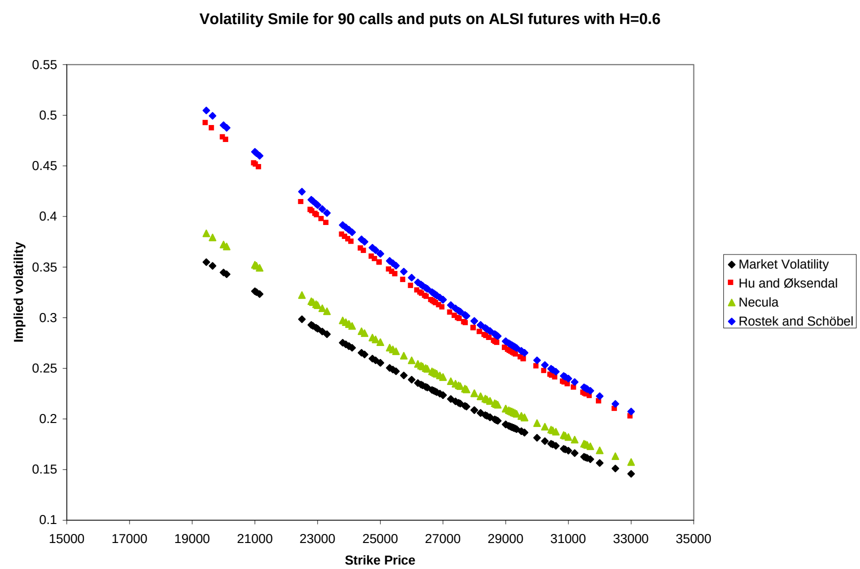

Figure 10.19: Volatility Smile for 90 calls and puts on ALSI futures with H=0.6.

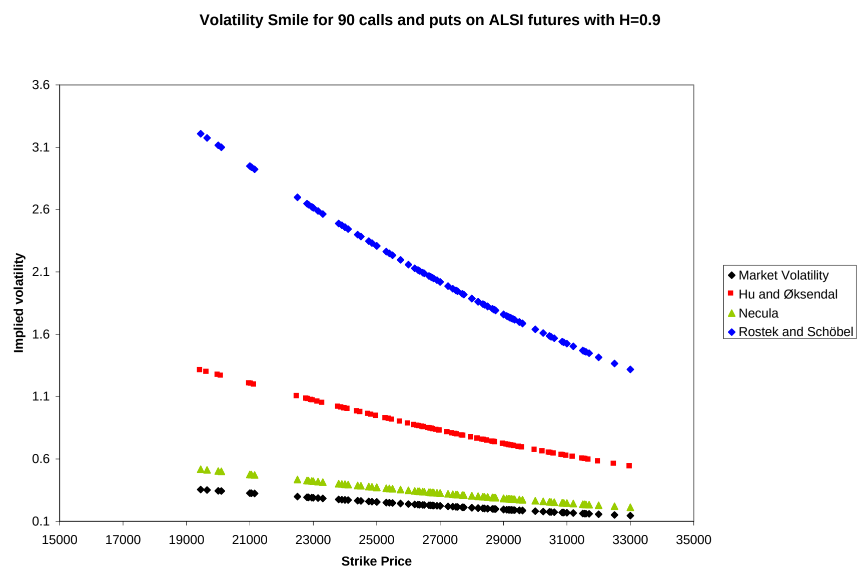

Figure 10.20: Volatility Smile for 90 calls and puts on ALSI futures with H=0.9.

10.5 Pricing Performance

The pricing performance of the above mentioned models is done using equations (9.4) and (9.5) . Tables 10.1, 10.2, 10.3 and 10.4 display the average absolute pricing error and the percentage pricing error for the four groups of ALSI calls on futures with the above chosen Hurst parameters. In appendix G tables G.1, G.2 and G.3 give the pricing errors and percentage pricing errors by option and by day for all of the ALSI calls on futures with a Hurst parameter $H = 0.54736$.

Tables 10.5, 10.6 and 10.7 display the average absolute pricing error and the percentage pricing error for the TOPSBE and TOPSBF warrants, MTN calls on futures and SBK calls on futures for the above mentioned Hurst parameters.

As the Hurst parameter increases Hu and Øksendal's model and Rostek and Schöbel's model average percentage pricing error becomes more positive implying that the prices are getting lower. As the Hurst parameter increases Necula's average percentage pricing error becomes more negative implying that the prices are getting larger. In the data studied the average percentage errors tend to be negative which indicates the models are overpricing the options from day to day. This is not true for all the options in the sample, but only on average see for instance G.1, the in-the-money options are underpriced. It seems that by finding the right Hurst parameter Hu and Øksendal and Rostek and Schöbel's models can give a smaller average percentage pricing error to that of the other models. For the average absolute pricing error the Black model seemed to do the best, otherwise no other patterns were found.

Table 10.1: ALSI calls on futures. In the money

ALSI Futures Calls Expiring on 2011/06/15							
In the money							
	Number of Options		4	Start Date	2011/04/04	End Date	2011/06/01
	Hurst Parameter						
	0.5043	0.51426	0.54736	0.6	0.7	0.8	0.9
	Average Absolute Pricing Error						
Black	18.012498	18.012498	18.012498	18.012498	18.012498	18.012498	18.012498
Hu & Øksendal	18.026863	18.062631	18.196415	18.394307	18.820003	19.4093	20.260574
Necula	17.996028	17.97009	17.879994	17.786036	17.755098	17.769587	17.781566
Rostek & Schöbel	18.026475	18.063042	18.19366	18.393112	18.82475	19.408665	20.251002
	Average Percentage Pricing Error						
Black	-0.006689	-0.006689	-0.006689	-0.006689	-0.006689	-0.006689	-0.006689
Hu & Øksendal	-0.0025574	0.0088644	0.0432929	0.1015847	0.2068055	0.3122478	0.417679
Necula	-0.0097838	-0.0168441	-0.036981	-0.0664033	-0.1157451	-0.1567427	-0.1922872
Rostek & Schöbel	-0.0024995	0.0074958	0.0439561	0.0999051	0.2071511	0.3123004	0.4166065

Table 10.2: ALSI calls on futures. At the money

ALSI Futures Calls Expiring on 2011/06/15							
At the money							
	Number of Options		6	Start Date	2011/04/04	End Date	2011/06/01
	Hurst Parameter						
	0.5043	0.51426	0.54736	0.6	0.7	0.8	0.9
	Average Absolute Pricing Error						
Black	14.430497	14.430497	14.430497	14.430497	14.430497	14.430497	14.430497
Hu & Øksendal	14.452676	14.440618	14.52466	14.819902	15.858503	17.237387	18.924485
Necula	14.39895	14.426945	14.406008	14.478622	14.71953	14.990342	15.236046
Rostek & Schöbel	14.422105	14.448497	14.525178	14.83624	15.870418	17.242916	18.922303
	Average Percentage Pricing Error						
Black	-0.6932005	-0.6932005	-0.6932005	-0.6932005	-0.6932005	-0.6932005	-0.6932005
Hu & Øksendal	-0.6784631	-0.6339232	-0.462463	-0.2110048	0.2734154	0.7558047	1.2317745
Necula	-0.7122691	-0.7287771	-0.8041787	-0.9054734	-1.0872416	-1.247004	-1.384377
Rostek & Schöbel	-0.6784547	-0.6317236	-0.4665112	-0.2116756	0.2743207	0.7560512	1.2323684

Table 10.3: ALSI calls on futures. Out the money

ALSI Futures Calls Expiring on 2011/06/15							
Out the money							
	Number of Options		8	Start Date	2011/04/04	End Date	2011/06/01
	Hurst Parameter						
	0.5043	0.51426	0.54736	0.6	0.7	0.8	0.9
	Average Absolute Pricing Error						
Black	12.201549	12.201549	12.201549	12.201549	12.201549	12.201549	12.201549
Hu & Øksendal	12.2096	12.211497	12.222444	12.262141	12.404469	12.587885	12.814268
Necula	12.205685	12.211891	12.214277	12.234941	12.322668	12.435347	12.545072
Rostek & Schöbel	12.207748	12.21362	12.22434	12.261178	12.399524	12.590628	12.813643
	Average Percentage Pricing Error						
Black	-22.265037	-22.265037	-22.265037	-22.265037	-22.265037	-22.265037	-22.265037
Hu & Øksendal	-22.112735	-21.809025	-20.903119	-19.447209	-16.765899	-14.184275	-11.699985
Necula	-22.294281	-22.412552	-22.702132	-23.135782	-23.955058	-24.685726	-25.344389
Rostek & Schöbel	-22.144483	-21.839857	-20.909077	-19.454685	-16.745921	-14.174272	-11.709711

Table 10.4: ALSI calls on futures. All the calls

ALSI Futures Calls Expiring on 2011/06/15							
All the calls							
	Number of Options		21	Start Date	2011/04/04	End Date	2011/06/01
	Hurst Parameter						
	0.5043	0.51426	0.54736	0.6	0.7	0.8	0.9
	Average Absolute Pricing Error						
Black	40.215745	40.215745	40.215745	40.215745	40.215745	40.215745	40.215745
Hu & Øksendal	40.215881	40.229523	40.263591	40.350366	40.584295	40.915756	41.327648
Necula	40.211863	40.199113	40.175294	40.137286	40.11996	40.123938	40.138608
Rostek & Schöbel	40.222672	40.227293	40.263947	40.345247	40.583643	40.911719	41.329093
	Average Percentage Pricing Error						
Black	-40.654181	-40.654181	-40.654181	-40.654181	-40.654181	-40.654181	-40.654181
Hu & Øksendal	-40.520855	-40.332567	-39.613166	-38.527657	-36.500182	-34.560021	-32.666029
Necula	-40.65847	-40.715018	-40.933274	-41.240659	-41.803962	-42.300578	-42.791527
Rostek & Schöbel	-40.545148	-40.31065	-39.652264	-38.537832	-36.520877	-34.565609	-32.668201

Table 10.5: ALSI warrants, TOPSBE and TOPSBF

ALSI Warrants							
TOPSBE Expiring on 2012/03/06 & TOPSBF Expiring on 2012/03/02							
	Number of Options		2	Start Date	2011/09/01	End Date	2011/10/31
	Hurst Parameter						
	0.5043	0.51426	0.54736	0.6	0.7	0.8	0.9
	Average Absolute Pricing Error						
Black-Scholes	46.07612	46.07612	46.07612	46.07612	46.07612	46.07612	46.07612
Hu & Øksendal	45.998364	45.791276	45.135141	44.037378	43.632498	43.052791	42.552301
Necula	46.039266	45.887605	46.847187	48.206766	50.008029	50.697162	51.356258
Rostek & Schöbel	45.984981	45.780222	45.126892	44.260013	43.65303	43.007768	42.498123
	Average Percentage Pricing Error						
Black	-2.6470886	-2.6470886	-2.6470886	-2.6470886	-2.6470886	-2.6470886	-2.6470886
Hu & Øksendal	-2.6486919	-2.6413768	-2.6302007	-2.5900719	-2.5411396	-2.4838111	-2.4300221
Necula	-2.6644783	-2.6777991	-2.7330269	-2.7774054	-2.8265722	-2.8259458	-2.8238863
Rostek & Schöbel	-2.6432217	-2.6368541	-2.6268133	-2.6006108	-2.5420556	-2.4818034	-2.4276074

Table 10.6: MTN calls on futures

MTN Futures Calls Expiring on 2011/06/15			
Start Date	2011/04/04	End Date	2011/06/01
	Number of Options		5
	Hurst Parameter		
	0.5211	0.5454	0.597
	Average Absolute Pricing Error		
Black	0.1133448	0.1133448	0.1133448
Hu & Øksendal	0.1128939	0.1124757	0.1124616
Necula	0.1136021	0.113997	0.1151322
Rostek & Schöbel	0.1128327	0.1124925	0.112427
	Average Percentage Pricing Error		
Black	-13.734046	-13.734046	-13.734046
Hu & Øksendal	-13.406401	-13.055125	-12.299014
Necula	-13.875013	-14.014264	-14.335713
Rostek & Schöbel	-13.411999	-13.059188	-12.292762

Table 10.7: SBK calls on futures

SBK Futures Calls Expiring on 2011/06/15					
		Start Date	2011/04/04	End Date	2011/06/01
		Number of Options		4	
	Hurst Parameter				
	0.4005	0.45298	0.51	0.5603	0.6
	Average Absolute Pricing Error				
Black	0.0663259	0.0663259	0.0663259	0.0663259	0.0663259
Hu & Øksendal	0.06784	0.0669916	0.0662428	0.0659333	0.0661693
Necula	0.0668705	0.0661655	0.066421	0.06673	0.0671855
Rostek & Schöbel	0.0678554	0.067012	0.066258	0.0659335	0.066162
	Average Percentage Pricing Error				
Black	-11.058446	-11.058446	-11.058446	-11.058446	-11.058446
Hu & Øksendal	-12.681148	-11.778788	-10.91622	-10.111813	-9.5119925
Necula	-10.360816	-10.767519	-11.149505	-11.434228	-11.683333
Rostek & Schöbel	-12.651815	-11.811202	-10.930367	-10.127603	-9.5259285

10.5.1 Explanation

Setting $\sigma_1 = \sigma$ and rewriting (10.1) we have

$$\sigma_1 = \sigma_{H,1}\sqrt{\frac{(T-t)^{2H}}{(T-t)}} \tag{10.4}$$

we call (10.4) the Black-Scholes (Black) equivalent volatility for Hu and Øksendal's framework. As substituting this volatility into the classical Black-Scholes (Black) model will yield the fractional Hu and Øksendal's Black-Scholes (Black) model. Figure 10.21 shows the Black-Scholes equivalent volatility for $\sigma_{H,1} = 0.2$ and for time to expiration $T = 2$. It can be seen that as $t \to T$, σ_1 decreases. Thus it can be seen that the options prices for Hu and Øksendal's framework will be underpriced relative to the Black option prices. As we move from one day to the next the Black-Scholes (Black) volatility will decrease from what was backed out the previous day; thus the

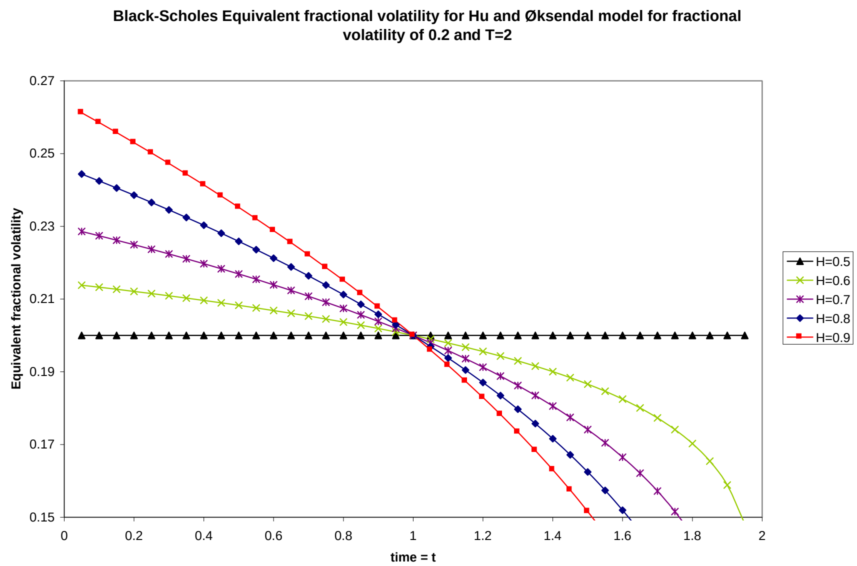

Figure 10.21: Black-Scholes Equivalent fractional volatility for Hu and Øksendal model for fractional volatility of 0.2 and T=2.

option prices will decrease from those predicted by the Black (Black-Scholes) model. For $H = 0.9$ the decrease is more rapid. This can be seen in the average percentage pricing errors.

Setting $\sigma_2 = \sigma$ and rewriting (10.2) we have

$$\sigma_2 = \sigma_{H,2}\sqrt{\frac{T^{2H} - t^{2H}}{(T-t)}} \tag{10.5}$$

we call (10.5) the Black-Scholes (Black) equivalent volatility for Necula's framework. Figure 10.22 shows the Black-Scholes equivalent volatility for $\sigma_{H,2} = 0.2$ and for time to expiration $T = 2$. It can be seen that as $t \to T$, σ_2 increases. Thus it can be seen that the options prices for Necula's framework will be higher relative to the Black option prices. As we move from one day to the next the Black-Scholes (Black) volatility will increase from what was backed out the previous day thus the option

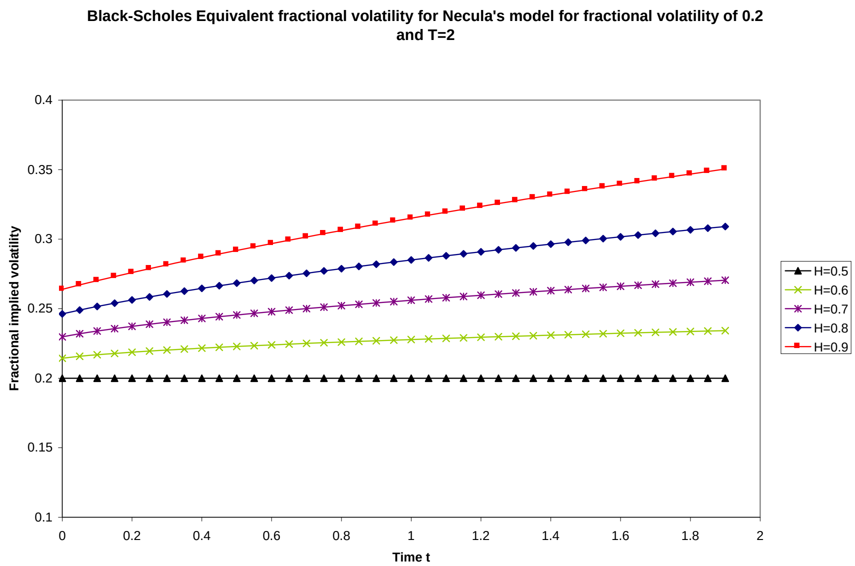

Figure 10.22: Black-Scholes Equivalent fractional volatility for Necula's model for fractional volatility of 0.2 and T=2.

prices will increase from those predicted by the Black (Black-Scholes) model. This can be seen in the above average percentage pricing errors.

Setting $\sigma_3 = \sigma$ and rewriting (10.3) we have

$$\sigma_3 = \sigma_{H,3} \sqrt{\frac{\nu_H (T-t)^{2H}}{(T-t)}} \tag{10.6}$$

we call (10.6) the Black-Scholes (Black) equivalent volatility for Rostek and Schöbel's framework. Figure 10.23 shows the Black-Scholes equivalent volatility for $\sigma_{H,3} = 0.2$ and for time to expiration $T = 2$. It can be seen that as $t \to T$, σ_1 decreases. Thus it can be seen that the options prices for Rostek's framework will be lower relative to the Black option prices, using a similar argument as above for Hu and Øksendal. In these practical applications Hu and Øksendal's model and Rostek and Schöbel model's are very similar as v_H is a constant.

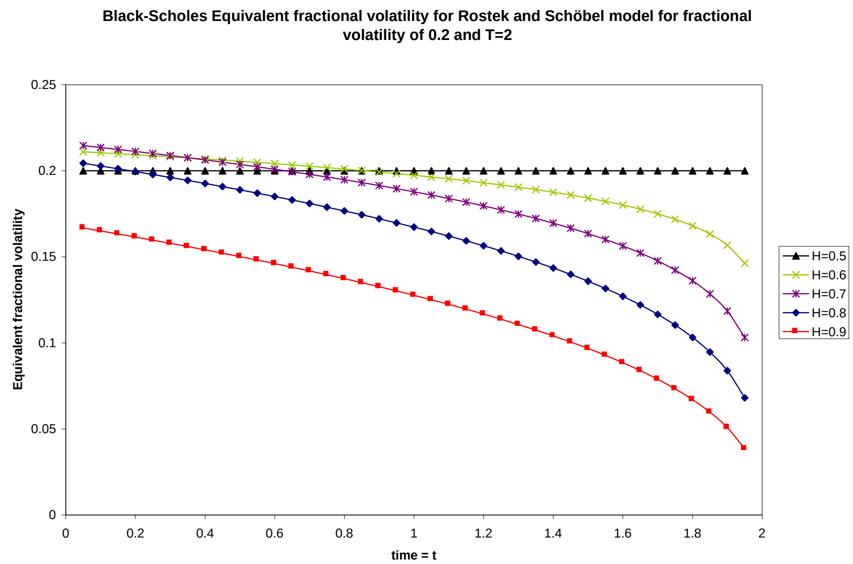

Figure 10.23: Black-Scholes Equivalent fractional volatility for Rostek and Schöbel model for fractional volatility of 0.2 and T=2.

10.5.2 Pricing Using a Single Call and Put Option on Futures Contact

To further investigate the models we priced ALSI calls on futures expiring on the 15th of June 2011 with strikes deep in-the-money $K = 25000$, at-the-money, $K = 29000$ and deep out-the-money, $K = 31700$. The volatilities were backed out using equations (10.1), (10.2) and (10.3). Initially the volatilities were backed out for the following Hurst parameters $H = \{0.5043;\ 0.51426;\ 0.54736;\ 0.6;\ 0.7;\ 0.9\}$. The Rostek and Schöbel average percentage pricing errors increased from negative to positive in all the cases; we conjectured that there should be a Hurst parameter giving optimum pricing with respect to the average percentage pricing error. Through trial and error we found Hurst parameters for Rostek and Schöbel's model see table 10.8.

Table 10.8: Hurst parameter giving the smallest percentage pricing error for Black formula under Rostek and Schöbel's framework.

ALSI Futures Call Expiring on 2011/06/15		
Black under Rostek & Schöbel framework		
Strike	**Hurst**	**Average Percentage Pricing Error**
25000	0.5415	0.000009343
29000	0.552	0.00045002
31700	0.902	0.006568214

Table 10.9: Hurst parameter giving the smallest average pricing error for Black formula under Rostek and Schöbel's framework.

ALSI Futures Call Expiring on 2011/06/15		
Black under Rostek & Schöbel framework		
Strike	**Hurst**	**Average Pricing Error**
25000	0.5638	0.000111928
29000	0.5834	0.000630687
31700	0.7193	0.000627825

We also investigated the average pricing errors using equation (9.3). Through trial and error we found Hurst parameters for Rostek and Schöbel's framework see table 10.9.

Tables 10.10, 10.11 and 10.12 display the errors for the above mentioned Hurst parameters. It can be seen that while we are able to get low average percentage pricing errors and average pricing errors for Rostek and Schöbel's framework the average absolute pricing errors were the highest in these models for all the Hurst parameters we found. These Hurst parameters seem to have no correspondence with the Hurst parameters estimated from the data and depended on the strike.

Table 10.10: ALSI call on futures for K=25000.

ALSI Futures Call Expiring on 2011/06/15								
In the money								
	Strike	25000	Start Date		11/04/04	End Date	11/06/01	
	Hurst Parameter							
	0.5043	0.51426	0.5415	0.54736	0.5638	0.6	0.7	0.9
	Average Pricing Error							
Black	-0.45171	-0.45171	-0.45171	-0.45171	-0.45171	-0.45171	-0.45171	-0.45171
Hu & Øksendal	-0.42107	-0.3502	-0.15713	-0.11574	0.000112	0.253838	0.944989	2.285553
Necula	-0.48119	-0.54811	-0.72176	-0.75743	-0.8545	-1.05378	-1.51829	-2.18428
Rostek & Schöbel	-0.42107	-0.3502	-0.15713	-0.11574	0.000112	0.253838	0.944989	2.285553
	Average Absolute Pricing Error							
Black	2.247492	2.247492	2.247492	2.247492	2.247492	2.247492	2.247492	2.247492
Hu & Øksendal	2.254344	2.272486	2.341487	2.359458	2.410349	2.540477	2.992456	4.00815
Necula	2.238997	2.229163	2.210559	2.207161	2.211702	2.29557	2.584855	3.055914
Rostek & Schöbel	2.254344	2.272486	2.341487	2.359458	2.410349	2.540477	2.992456	4.00815
	Average Percentage Pricing Error							
Black	-0.00726	-0.00726	-0.00726	-0.00726	-0.00726	-0.00726	-0.00726	-0.00726
Hu & Øksendal	-0.0065	-0.00476	9.34E-06	0.001031	0.00389	0.01015	0.027196	0.060228
Necula	-0.00794	-0.0095	-0.01353	-0.01436	-0.01663	-0.02129	-0.03227	-0.04827
Rostek & Schöbel	-0.0065	-0.00476	9.34E-06	0.001031	0.00389	0.01015	0.027196	0.060228

Table 10.11: ALSI call on futures for K=29000.

ALSI Futures Call Expiring on 2011/06/15								
At the money								
	Strike	29000	Start Date		11/04/04	End Date	11/06/01	
	Hurst Parameter							
	0.5043	0.51426	0.54736	0.552	0.5834	0.6	0.7	0.9
	Average Pricing Error							
Black	-2.45658	-2.45658	-2.45658	-2.45658	-2.45658	-2.45658	-2.45658	-2.45658
Hu & Øksendal	-2.32954	-2.03544	-1.0595	-0.92286	0.000631	0.488048	3.412655	9.202508
Necula	-2.54239	-2.73763	-3.35313	-3.43556	-3.97068	-4.23857	-5.6671	-7.83139
Rostek & Schöbel	-2.32954	-2.03544	-1.0595	-0.92286	0.000631	0.488048	3.412656	9.202508
	Average Absolute Pricing Error							
Black	7.105268	7.105268	7.105268	7.105268	7.105268	7.105268	7.105268	7.105268
Hu & Øksendal	7.127347	7.182072	7.511283	7.566766	8.051174	8.336223	10.32012	14.74298
Necula	7.106727	7.111038	7.133287	7.137223	7.201994	7.315385	8.189455	9.903722
Rostek & Schöbel	7.127346	7.182072	7.511283	7.566766	8.051173	8.336223	10.32012	14.74297
	Average Percentage Pricing Error							
Black	-0.23907	-0.23907	-0.23907	-0.23907	-0.23907	-0.23907	-0.23907	-0.23907
Hu & Øksendal	-0.21922	-0.17328	-0.02088	0.00045	0.144598	0.220657	0.676713	1.577956
Necula	-0.24839	-0.26971	-0.33786	-0.3471	-0.40774	-0.43854	-0.60835	-0.88608
Rostek & Schöbel	-0.21922	-0.17328	-0.02088	0.00045	0.144598	0.220657	0.676713	1.577956

Table 10.12: ALSI call on futures for K=31700.

ALSI Futures Call Expiring on 2011/06/15								
Out the money								
	Strike	31700	Start Date		11/04/04	End Date	11/06/01	
	Hurst Parameter							
	0.5043	0.51426	0.54736	0.6	0.7	0.7193	0.9	0.902
	Average Pricing Error							
Black	-1.67561	-1.67561	-1.67561	-1.67561	-1.67561	-1.67561	-1.67561	-1.67561
Hu & Øksendal	-1.64222	-1.56496	-1.30902	-0.90455	-0.14474	0.000628	1.342187	1.356842
Necula	-1.71379	-1.80023	-2.06891	-2.44438	-3.01954	-3.11376	-3.80784	-3.81399
Rostek & Schöbel	-1.64222	-1.56496	-1.30902	-0.90455	-0.14474	0.000628	1.342187	1.356842
	Average Absolute Pricing Error							
Black	3.286138	3.286138	3.286138	3.286138	3.286138	3.286138	3.286138	3.286138
Hu & Øksendal	3.302112	3.339072	3.466208	3.677424	4.11973	4.212421	5.106352	5.116342
Necula	3.288403	3.294153	3.317489	3.365744	3.706009	3.773356	4.329673	4.334973
Rostek & Schöbel	3.302112	3.339073	3.466208	3.677424	4.11973	4.212421	5.106352	5.116342
	Average Percentage Pricing Error							
Black	-11.8437	-11.8437	-11.8437	-11.8437	-11.8437	-11.8437	-11.8437	-11.8437
Hu & Øksendal	-11.7042	-11.3821	-10.3231	-8.6744	-5.65784	-5.09245	-0.04687	0.006568
Necula	-11.8911	-11.9999	-12.3515	-12.8828	-13.8123	-13.9813	-15.4317	-15.4466
Rostek & Schöbel	-11.7042	-11.3821	-10.3231	-8.6744	-5.65784	-5.09245	-0.04687	0.006568

10.5.3 Error Minimizing Hurst Parameter

We used Rostek and Schöbel's Black option pricing model to find a Hurst parameter that satisfies $E\left[\left|\varepsilon\left[\sigma_{H,n}\right]\right|\right]=0$, where $\varepsilon\left[\sigma_{H,n}\right]$ is defined in equation (9.1) using call and put options for different strikes on the same underlying with the same maturity. We will refer to this Hurst parameter as the error minimizing Hurst parameter. There were 37 put options on ALSI future contacts and 20 call options on ALSI futures contacts. We had 6 put options on SBK futures and 4 call options on SBK futures, and 5 put options on MTN futures and 5 call options on MTN futures. All the options on futures were dated from the 1 April 2011 till the 1 June 2011 expiring on the 15 of June 2011. Through trial and error we found the Hurst parameters that minimizes the average percentage error, equation (9.5). Plotting the error minimizing Hurst parameter against the strike yields a interesting pattern. Figure 10.24 displays the error minimizing Hurst parameter for call and put options on SBK futures. The error minimizing Hurst parameter for the put options on futures is lower than the error minimizing Hurst parameter for call options on futures. For example, for a strike $K=115$ the error minimizing Hurst parameter for a call option on a SBK futures contact is 0.572104 while the error minimizing Hurst parameter for a put option on a SBK future contracts is 0.41219. Figure 10.26 and 10.25 exhibit the error minimizing Hurst parameter for call and put option on ALSI future contacts respectively. In addition we plotted the average fractional volatilities given the error minimizing Hurst parameter and the average market volatilities for each of the strikes. For a given strike the average fractional volatilities are larger than the average market volatilities, and the average fractional volatilities for put options on futures is lower than the average volatilities for the call option on futures, and obviously the market average volatilities for put options on futures is the same as for call options on futures. Figure 10.27 shows the error minimizing Hurst parameter, the average frac-

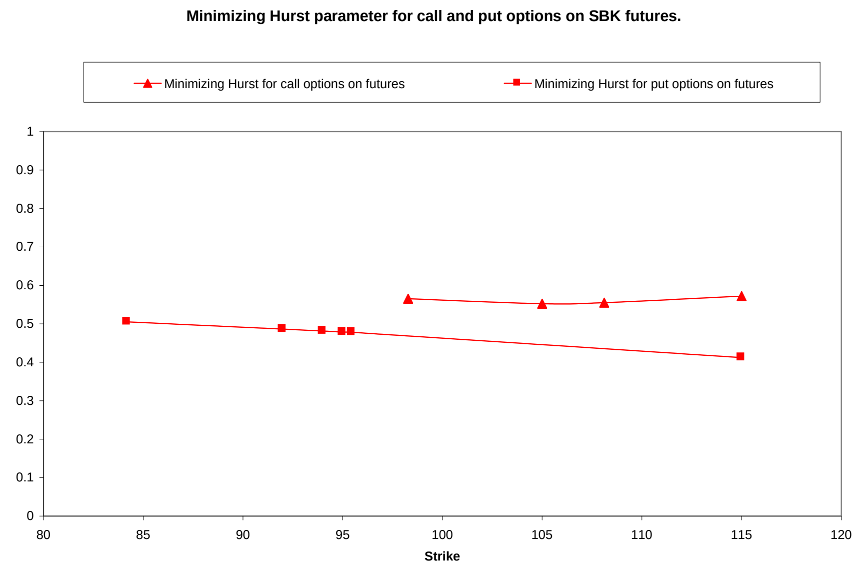

Figure 10.24: Error minimizing Hurst parameter for call and put options on SBK futures using Rostek and Schöbel options on futures.

tional volatilities and the average market volatilities for both call and put options on MTN futures. When the strike is deep-in-and-out money the error minimizing Hurst parameter and the average fractional volatilities is larger than at-the money strikes. In conclusion, we see a broken smile. Further investigation is needed to see whether there is any practical application for pricing with this method as well as the statistical properties of these error minimizing Hurst parameters.

A reason for the break in the fractional volatility smile is that since put options offer protection there is a higher demand for them in the market thus they are not priced the same way. This also shows that the implied fractional volatility smile does not hold.

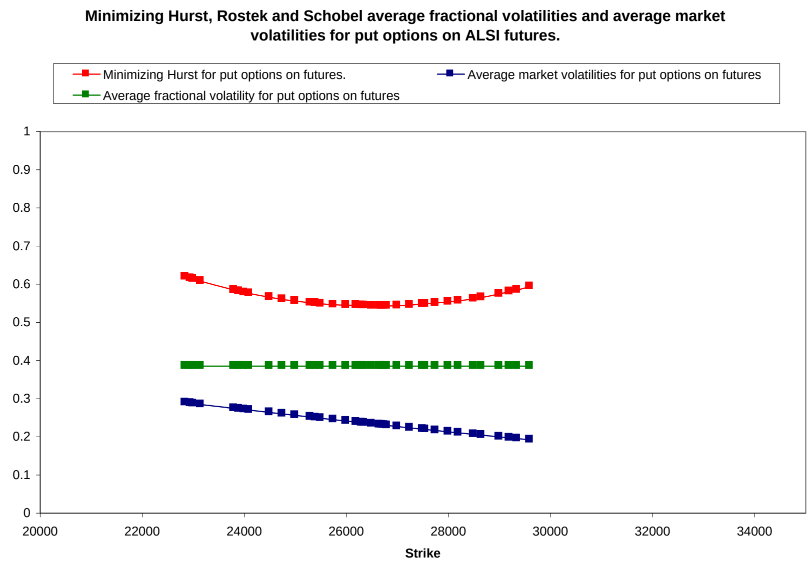

Figure 10.25: Error minimizing Hurst, Rostek and Schöbel's average fractional volatilities and average market volatilities for put options on ALSI futures.

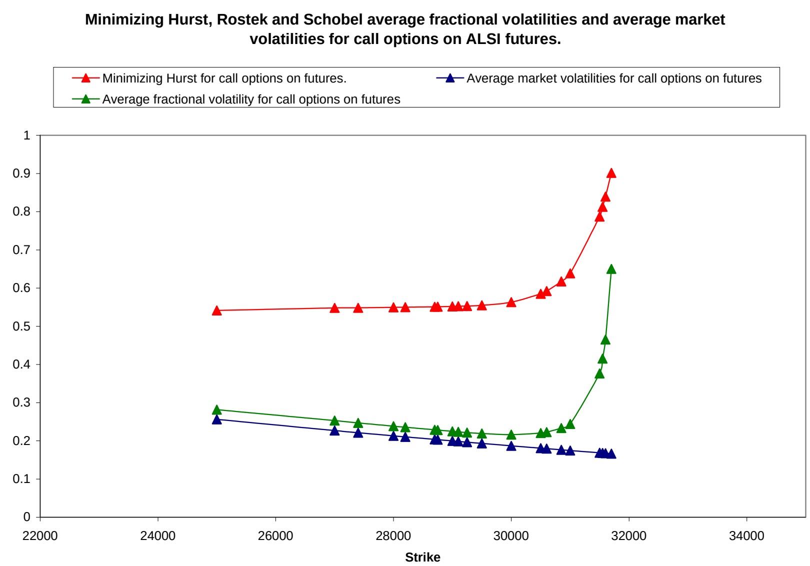

Figure 10.26: Minimizing Hurst, Rostek and Schobel average fractional volatilities and average market volatilities for call options on ALSI futures.

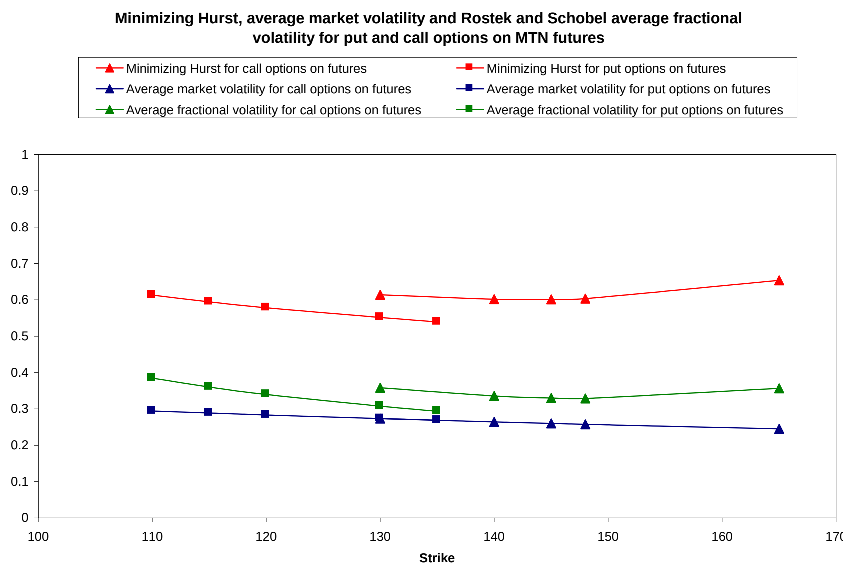

Figure 10.27: Error minimizing Hurst, average market volatility and Rostek and Schöbel's average fractional volatility for put and call options on MTN futures.

Chapter 11

Conclusion

Originally, there was found to be a problem with the assumption of independence of subsequent price changes since long term dependencies have been found in some underlying stocks. The Hurst parameter provides an estimate of the persistency of a time series. The Hurst parameter for a data set provides a measure of whether a time series is a random walk, whether increments are uncorrelated, or if the data set has some degree of autocorrelation. Four methods of estimating the Hurst parameter where investigated, namely the aggregated variance method, absolute moments method, Higuchi method and the rescaled range analysis. The rescaled range analysis gave sizable errors especially when the data set was small. The absolute moments method and Higuchi method was used to investigate the autocorrelation for different stocks from different sectors in the South African stock market.

Many of the South African stocks showed persistent behaviour. High persistent behaviour was found in many individual stocks. Many of the stocks showed a decrease in persistency after the crash of May 2008.

One reason for the application of fractional Brownian motion is that it has long-memory when the Hurst parameter is greater than a $\frac{1}{2}$. Fractional Brownian motion is not a semimartingale and the application of pathwise integrals is not possible as it allows for arbitrage possibilities. The introduction of a new integration theory under the Wick product is discussed in chapter 5. The introduction of Wick-Itô calculus inspired some authors who proved closed form fractional European option pricing formulas. In chapter 6 the application of Wick calculus was discussed under Hu and Øksendal's (2000) framework. Even though these results appeared promising Björk and Hult (2005) showed that their assumptions about changing the meaning of the portfolio value has no economical meaning. In chapter 7 we discussed Necula's (2002) extension of the fractional Black-Scholes option pricing equation to an arbitrary time using quasi-conditional expectation. Due to the fact that Necula's formula contains the term $\left(T^{2H} - t^{2H}\right)$ using it in application is problematical as absolute times are included instead of the normal time to maturity $(T - t)$ which just depends on the difference between the times. In chapter 8 a conditional fractional Brownian motion was presented as in Rostek and Schöbel (2010). Where the underlying stock and wealth of an investor are assumed to follow a bivariate log-normal distribution and investors are assumed to be risk-averse.

Under certain assumptions, three fractional Black option pricing formulas are presented. A Black formula is proved under Hu and Øksendal framework and its extension is conjectured, a quasi-conditional Black formula under Necula's framework is proved and lastly a conditional Black formula is proved under Rostek and Schöbel's work. For shorter times to maturity Hu and Øksendal's option pricing models give lower prices for higher Hurst parameters, while for longer time to maturities these models give higher prices for higher Hurst parameters. With all the other inputs including volatility fixed. For shorter times to maturity Necula's option pricing models seemed to give lower prices for higher Hurst parameters, while for longer times to maturity these models seemed to give higher prices for higher Hurst parameters. For shorter times to maturity Rostek and Schöbel's option pricing models give lower prices for higher Hurst parameters and higher prices for lower Hurst parameters. For longer times to maturity these models give higher prices then lower prices as the Hurst parameter increased. These effects can be motivated as follows: for the persistency case for shorter times to maturity the underlying is less variable while for longer times to maturity the underlying is more variable. For the anti-persistency case (Rostek and Schöbel only) the opposite is true. The Rostek and Schöbel model also has an additional effect where prices are lower as the Hurst parameter moves away from $\frac{1}{2}$.

It was found that there is a range of possible combinations of the Hurst H and volatility σ parameter corresponding to a given price in the models for all three models, thus an implied Hurst cannot be found using a single option on a specific day.

In chapter 9 methods of empirical comparison of the models are explained. Finally in chapter 10, using selected historical South African option and warrant data, and using different constant Hurst parameters, the implied volatilities for the three models were backed out and compared to that of the actual market implied volatilities. The out of sampling performance was done to see how the Hurst parameter affects the price of the models.

To use these models in pricing they must be calibrated to the market. In chapter 10 we backed out the implied volatilities for the ALSI, MTN and SBK calls on futures as well as TOPSBE and TOPSBF warrants by using a heuristic optimization method to minimize an objective function. We observed that the higher the Hurst parameter the higher the fractional volatility and that the implied fractional volatilities capture the market movements. As the option approached time to expiration Hu and Øksendal's and Rostek and Schöbel's implied volatilities increased while Necula's implied fractional volatilities decreased. We explained this behaviour by a simple formula, and from the formula we also found that depending on the time interval the fractional volatilities might be lower. The volatility smile curve was graphed for the fractional Black models for the ALSI future calls with chosen Hurst parameters. The pricing performance of the models was done and it was found that Hu and Øksendal and Rostek and Schöbel's model are very similar. Using the equivalent Black-Scholes volatility factor we could explain that as the Hurst parameter increased Hu and Øksendal and Rostek and Schöbel's model gave lower prices to that of the Black option model. As the Hurst parameter increased Necula's option prices were higher than those of the Black option prices. Our results were confirmed as Hu and Øksendal and Rostek and Schöbel's average percentage pricing error kept rising as the Hurst parameter increased while Necula's average percentage pricing error become more negative as the Hurst parameter increased. The implied fractional volatilities were also backed out for single options that were deep in-the-money, at-the-money and deep out-the-money and it was noted that the pricing errors for these options changed from positive to negative thus we could find an optimum Hurst parameters that could give the lowest average percentage pricing error or lowest average pricing error. We found that these Hurst parameters depended on the strike. These Hurst parameters differ from the estimated Hurst parameter from a time series of the underlying. It remains to be seen if this has any application in the real world option pricing.

We have discussed and achieved an understanding of the mathematical application of fractional Brownian motion in option pricing. We have seen that there is very little empirical applicability of these models when using a historical Hurst parameter. Through formulas and empirical work we gained deeper insight into how these models perform compared to the performance of the classical Black-Scholes and Black formula. Some of the fractional models will perform better if an appropriate Hurst parameter can be chosen.

Further investigation would included option pricing models where the underlying is driven by fractional Brownian motion and Brownian motion and stochastic volatility driven by fractional Brownian motion and Brownian motion. As well as the sensitivities of these mixed models and hedging strategies.

Appendix A

Hurst Tables: Part 1

The tables display the Hurst parameter for different stocks from different sectors. The Hurst parameter was modelled on the whole interval, the interval before the recession and after the recession. The Hurst parameter for the stocks from the sectors will be discussed using the criterion given in remark 3.1.

Table A.1 shows that **Comair Limited** (Comair) from sector airlines and airports had a $H \approx \frac{1}{2}$ on the whole interval and before the crash for both absolute moment and Higuchi methods and anti-persistent after the crash. In the sector builders merchants, **Austro Group Limited** (Austro) had a persistent behaviour on the whole interval for both methods, see table A.2. **Iliad Africa Limited** (Iliad) was persistent on the whole interval and before the crash for both methods. After the crash the stock fell to an anti-persistent behaviour with Higuchi method and a $H \approx \frac{1}{2}$ trend with the absolute moments method, see table A.3. **Marshall Monteagle HD SA** (Mt-egle) had a $H \approx \frac{1}{2}$ on the whole interval and before the crash,

Table A.1: Sector: Airlines and Airports, Stock: Comair Limited.

Comair		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
22-Jul-98 to 17-Feb-11	3143	0.5054	0.5036
22-Jul-98 to 23-May-08	2457	0.5019	0.5017
23-May-08 to 17-Feb-11	688	0.4230	0.3806

Table A.2: Sector: Builders Merchants, Stock: Austro Group Limited.

Austro		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
01-Feb-07 to 17-Feb-11	1013	0.6305	0.6344

Table A.3: Sector: Builders Merchants, Stock: Iliad Africa Limited.

Iliad		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
10-Jun-98 to 17-Feb-11	3172	0.5943	0.5737
27-Mar-85 to 23-May-08	2484	0.5834	0.5622
23-May-08 to 17-Feb-11	688	0.4605	0.4367

after the crash the stock fell to an anti-persistent behaviour for both methods, see table A.4. **Winhold Limited** (Winhold) stock was persistent on the whole interval and before the crash, while it had an anti-persistent stock after the crash with the absolute method. With the Higuchi method the stock had a $H \approx \frac{1}{2}$ on all three intervals, see table A.5.

From the broadcasting contractors sector stock, **Naspers Limited** (Naspersn), was persistent on the whole interval and before the crash, after the crash the stock had a $H \approx \frac{1}{2}$ for both the methods, see table A.6. From the building and construction materials sector stock, **Afrimat Ltd** (Afrimat) was persistent on the whole interval

Table A.4: Sector: Builders Merchants, Stock: Marshall Monteagle HD SA.

Mt-egle		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
04-May-95 to 17-Feb-11	3939	0.5033	0.5175
04-May-95 to 23-May-08	3251	0.5232	0.5479
23-May-08 to 17-Feb-11	688	0.3147	0.3319

Table A.5: Sector: Builders Merchants, Stock: Winhold Limited.

Winhold		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
11-Nov-86 to 17-Feb-11	6055	0.5706	0.5078
11-Nov-86 to 23-May-08	5367	0.5825	0.5078
23-May-08 to 17-Feb-11	688	0.4230	0.4932

Table A.6: Sector: Broadcasting Contractors, Stock: Naspers Limited.

Naspersn		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
08-Jan-96 to 17-Feb-11	3799	0.5573	0.5903
08-Jan-96 to 23-May-08	3110	0.5833	0.6002
23-May-08 to 17-Feb-11	688	0.4525	0.5289

and after the crash while $H \approx \frac{1}{2}$ before the crash for both the methods, see table A.7. **Distribution and Warehousing Network Limited** (Dawn) was persistent on the whole interval and before the crash, after the crash the stock had a $H \approx \frac{1}{2}$ for both the methods, see table A.8. **Mazor Group Ltd** (Mazor) was persistent on the whole interval for both methods, see table A.9. **Pretoria Port Cemnt** (Ppc) was persistent on the whole interval and before the crash and anti-persistent after the crash with the Higuchi method. With the absolute moments method the stock had a $H \approx \frac{1}{2}$ and after the crash it was anti-persistent, see table A.10.

Table A.7: Sector: Building and Construction Materials, Stock: Afrimat Ltd.

Afrimat		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
07-Nov-06 to 17-Feb-11	1072	0.6695	0.6469
07-Nov-06 to 23-May-08	384	0.5424	0.5107
23-May-08 to 17-Feb-11	688	0.6848	0.6384

Table A.8: Sector: Building and Construction Materials, Stock: Distribution and Warehousing Network Limited.

Dawn		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
09-Dec-87 to 17-Feb-11	5783	0.5931	0.6020
09-Dec-87 to 23-May-08	5095	0.5878	0.5779
23-May-08 to 17-Feb-11	688	0.4929	0.5370

Table A.9: Sector: Building and Construction Materials, Stock: Mazor Group Ltd.

Mazor		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
21-Nov-07 to 17-Feb-11	811	0.6058	0.6491

From the sector chemicals -speciality, stock **A E C I Limited** (Aeci), showed persistency on all intervals for both methods except for one interval after the crash using absolute moments that showed a $H \approx \frac{1}{2}$, see table A.11. **African Oxygen Ltd** (Afrox) had a $H \approx \frac{1}{2}$ throughout all the intervals with absolute moments method and persistent on the whole interval and before the crash and the stock return had a $H \approx \frac{1}{2}$ with the Higuchi method, see table A.12. **Freeworld Coatings Ltd** (Freeworld) had a $H \approx \frac{1}{2}$ on the whole interval for both methods, see table A.13. **Omnia Holdings Ltd** (Omnia) was persistent on the whole interval and before the

Table A.10: Sector: Building and Construction Materials, Stock: Pretoria Port Cemnt.

Ppc		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
06-Sep-85 to 17-Feb-11	6352	0.5476	0.6071
06-Sep-85 to 23-May-08	5664	0.5388	0.6080
23-May-08 to 17-Feb-11	688	0.3279	0.2919

Table A.11: Sector: Chemicals - Speciality, Stock: A E C I Limited.

Aeci		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
06-Sep-85 to 17-Feb-11	6352	0.5554	0.5670
06-Sep-85 to 23-May-08	5664	0.5626	0.5668
23-May-08 to 17-Feb-11	688	0.5322	0.5732

Table A.12: Sector: Chemicals - Speciality, Stock: African Oxygen Ltd.

Afrox		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
27-Mar-85 to 17-Feb-11	6462	0.5287	0.5817
27-Mar-85 to 23-May-08	5775	0.5257	0.5827
23-May-08 to 17-Feb-11	688	0.5472	0.5367

crash and had a $H \approx \frac{1}{2}$ afterwards for both the methods, see table A.14. From the sector clothing and footware, stock **Compagnie Fin Richemont** (Richemont) had a $H \approx \frac{1}{2}$ with the absolute moment method on the whole interval and before the crash and there was strong persistent behaviour after the crash. With the Higuchi method there was persistency on all the intervals, see table A.15. **Seardel Investment Corporation Limited** (Seardel) was persistent on the whole interval, had a $H \approx \frac{1}{2}$ before the crash and had a strong persistent behaviour after the crash for both methods, see table A.16. From the coal sector stock, **Coal of Africa Ltd** (Coal) was persistent on all three intervals using Higuchi method. With the absolute moments

Table A.13: Sector: Chemicals - Speciality, Stock: Freeworld Coatings Ltd.

Freeworld		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
03-Dec-07 to 17-Feb-11	803	0.5171	0.5121

Table A.14: Sector: Chemicals - Speciality, Stock: Omnia Holdings Ltd.

Omnia		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
22-May-87 to 17-Feb-11	5926	0.5514	0.6077
22-May-87 to 23-May-08	5238	0.5639	0.6111
23-May-08 to 17-Feb-11	688	0.4597	0.4825

Table A.15: Sector: Clothing and Footware, Stock: Compagnie Fin Richemont.

Richemont		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
03-Nov-88 to 17-Feb-11	5564	0.5230	0.5878
03-Nov-88 to 23-May-08	4876	0.4840	0.5745
23-May-08 to 17-Feb-11	688	0.6623	0.6745

Table A.16: Sector: Clothing and Footware, Stock: Seardel Investment Corporation Limited.

Seardel		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
06-Sep-85 to 17-Feb-11	6352	0.5593	0.5589
06-Sep-85 to 23-May-08	5664	0.5127	0.5324
23-May-08 to 17-Feb-11	688	0.7254	0.6942

Table A.17: Sector: Coal, Stock: Coal of Africa Ltd.

Coal		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
01-Dec-06 to 17-Feb-11	1054	0.6270	0.6245
01-Dec-06 to 23-May-08	367	0.5225	0.5777
23-May-08 to 17-Feb-11	688	0.5696	0.6128

Table A.18: Sector: Coal, Stock: Keaton Energy Holdings Ltd.

Keaton		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
22-Apr-08 to 17-Feb-11	708	0.5099	0.6102

method it was persistent on the whole interval and after the crash, while the $H \approx \frac{1}{2}$ on the interval before the crash, see table A.17. **Keaton Energy Holdings Ltd** (Keaton) had a $H \approx \frac{1}{2}$ using absolute moments method and persistent using the Higuchi method, see table A.18. **Optimum Coal Holdings Ltd** (Optimum) had a $H \approx \frac{1}{2}$ using Higuchi method and persistent using the absolute moments method, see table A.19. **Wescoal Holdings Ltd** (Wescoal) had a $H \approx \frac{1}{2}$ on the whole interval, anti-persistent before the crash and persistent after the crash for both methods, see table A.20.

Exxaro Resources Limited (Exxaro) was anti-persistent on the whole interval and before the crash and a $H \approx \frac{1}{2}$ afterwards using absolute moments method. While

Table A.19: Sector: Coal, Stock: Optimum Coal Holdings Ltd.

Optimum		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
29-Mar-10 to 17-Feb-11	225	0.5705	0.5275

Table A.20: Sector: Coal, Stock: Wescoal Holdings Ltd.

Wescoal		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
20-Jul-05 to 17-Feb-11	1398	0.5107	0.5207
20-Jul-05 to 23-May-08	710	0.3481	0.2979
23-May-08 to 17-Feb-11	688	0.5901	0.5776

Table A.21: Sector: Coal, Stock: Exxaro Resources Limited.

Exxaro		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
27-Nov-06 to 17-Feb-11	1087	0.4491	0.5271
27-Nov-06 to 23-May-08	400	0.3259	0.4242
23-May-08 to 17-Feb-11	688	0.5129	0.4625

using the Higuchi method the $H \approx \frac{1}{2}$ on the whole interval and after the crash and anti-persistent on the interval before the crash, see table A.21. From the diamond sector, stock, **BRC DiamondCore Ltd** (Brc) had a $H \approx \frac{1}{2}$ using absolute moments method and persistent using the Higuchi method, see table A.22. **Trans Hex Group Limited** (Trnshex) was persistent on the whole interval and before the crash and displayed a strong persistent behaviour after the crash for both methods, see table A.23. Sector distillers and vintners, stock, **Capevin Inv Ltd** (Capevin) displayed persistency on the whole interval and before the crash while after the crash was anti-persistent for both methods, see table A.24.

Table A.22: Sector: Diamond, Stock: BRC DiamondCore Ltd.

Brc		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
04-Feb-08 to 17-Feb-11	762	0.5218	0.5728

Table A.23: Sector: Diamond, Stock: Trans Hex Group Limited.

Trnshex		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
04-Feb-08 to 17-Feb-11	6462	0.5785	0.5716
03-Nov-88 to 23-May-08	5774	0.5565	0.5756
23-May-08 to 17-Feb-11	688	0.6935	0.7150

Table A.24: Sector: Distillers and Vintners, Stock: Capevin Inv Ltd.

Capevin		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
22-May-87 to 17-Feb-11	5926	0.5654	0.6442
22-May-87 to 23-May-08	5239	0.5705	0.6475
23-May-08 to 17-Feb-11	687	0.4468	0.4235

Using absolute moments method **Distell** (Distell) had a $H \approx \frac{1}{2}$ on the whole interval and anti-persistent before and after the crash, while with the Higuchi method the stock was persistent on the whole interval and before the crash and had a $H \approx \frac{1}{2}$ after the crash, see table A.25. From the sector education, bus training and employment, stock **Adcorp Holdings Limited** (Adcorp) had a strong persistent behaviour on the whole interval and before the crash and after the crash the stock had a $H \approx \frac{1}{2}$ for both methods, see table A.26. **Kelly Group Ltd** (Kelly) was persistent on the whole interval for both methods, see table A.27. **Primeserv Group Lim-**

Table A.25: Sector: Distillers and Vintners, Stock: Distell.

Distell		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
23-Mar-01 to 17-Feb-11	2479	0.4758	0.6002
23-Mar-01 to 23-May-08	1791	0.4353	0.6040
23-May-08 to 17-Feb-11	688	0.4337	0.4559

Table A.26: Sector: Education, Bus Training and Employment, Stock: Adcorp Holdings Limited.

Adcorp		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
09-Oct-87 to 17-Feb-11	5827	0.6754	0.6899
09-Oct-87 to 23-May-08	5139	0.6822	0.7050
23-May-08 to 17-Feb-11	688	0.4861	0.4520

Table A.27: Sector: Education,Bus Training and Employment, Stock: Kelly Group Ltd.

Kelly		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
04-Apr-07 to 17-Feb-11	970	0.6099	0.6360

ited (Primeserv) had a $H \approx \frac{1}{2}$ on the whole interval and before the crash while the stock was anti-persistent after the crash for both the methods, see table A.28. From the sector electrical equipment, stock, **Allied Electronics Corporation Limited** (Altron-p) had a $H \approx \frac{1}{2}$ on all three intervals using absolute moments method and persistent on the whole interval and before the crash and had a $H \approx \frac{1}{2}$ after the crash using the Higuchi method, see table A.29.

Table A.28: Sector: Education,Bus Training and Employment, Stock: Primeserv Group Limited.

Primeserv		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
29-Apr-98 to 17-Feb-11	3200	0.4945	0.5101
09-Oct-87 to 23-May-08	2512	0.4915	0.5089
23-May-08 to 17-Feb-11	688	0.4226	0.4450

Table A.29: Sector: Electrical Equipment, Stock: Allied Electronics Corporation Limited.

Altron-p		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
18-Nov-94 to 17-Feb-11	4050	0.5300	0.5766
18-Nov-94 to 23-May-08	3362	0.5286	0.5625
23-May-08 to 17-Feb-11	688	0.5458	0.5330

Table A.30: Sector: Electrical Equipment, Stock: Allied Electronics Corporation Limited.

Altron		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
22-May-87 to 17-Feb-11	5926	0.5109	0.5227
22-May-87 to 23-May-08	5238	0.4861	0.5005
23-May-08 to 17-Feb-11	688	0.5355	0.5345

Allied Electronics Corporation Limited (Altron) had a $H \approx \frac{1}{2}$ on all three intervals for both methods, see table A.30. Using absolute moments method **Arb Holdings Ltd** (Arb) was persistent on the whole interval and anti-persistent before the crash and after the crash, while using the Higuchi method the stock return had a $H \approx \frac{1}{2}$ on all three intervals, see table A.31. **Delta Emd Ltd** (Delta) had a $H \approx \frac{1}{2}$ on all three intervals for both methods, see table A.32. **South Ocean Holdings Ltd** (S-Ocean) was persistent on the whole interval and the interval before the

Table A.31: Sector: Electrical Equipment, Stock: Arb Holdings Ltd.

Arb		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
20-Nov-07 to 17-Feb-11	812	0.5521	0.5425
20-Nov-07 to10 March-09	325	0.3926	0.5284
10 March-09 to 17-Feb-11	487	0.3644	0.4645

Table A.32: Sector: Electrical Equipment, Stock: Delta Emd Ltd.

Delta		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
22-May-87 to 17-Feb-11	5926	0.5267	0.5491
22-May-87 to 23-May-08	5238	0.5117	0.5461
23-May-08 to 17-Feb-11	688	0.4533	0.4555

Table A.33: Sector: Electrical Equipment, Stock: South Ocean Holdings Ltd.

S-Ocean		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
28-Feb-07 to 17-Feb-11	994	0.5645	0.5948
28-Feb-07 to 10 March-09	509	0.5579	0.5633
10 March-09 to 17-Feb-11	487	0.4558	0.4623

crash, after the crash the stock had a $H \approx \frac{1}{2}$ for both methods, see table A.33. Sector, exchange traded funds, stock **New Gold Issuer Ltd** (Newgold) showed an anti-persistent behaviour for the whole interval and the interval before the crash and after the crash a $H \approx \frac{1}{2}$ was found using absolute moment method. The Higuchi method showed a persistent behaviour on the whole interval and before the crash and an anti-persistent bahaviour after the crash, see table A.34. From the sector farming and fishing, stock **Afgri Limited** (Afgri) was anti-persistent on the whole

Table A.34: Sector: Exchange Traded Funds, Stock: New Gold Issuer Ltd.

Newgold		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
02-Nov-04 :17-Feb-11	1605	0.4398	0.5810
02-Nov-04 to 23-May-08	917	0.4460	0.5583
23-May-08 to 17-Feb-11	688	0.4703	0.5227

Table A.35: Sector: Farming and Fishing, Stock: Afgri Limited.

Afgri		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
11-Nov-96 to 17-Feb-11	3562	0.4194	0.4358
11-Nov-96 to 23-May-08	2874	0.4510	0.4626
23-May-08 to 17-Feb-11	688	0.4411	0.4065

Table A.36: Sector: Farming and Fishing, Stock: Astral Foods.

Astral		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
09-Apr-01 to 17-Feb-11	2468	0.5298	0.6210
09-Apr-01 to 23-May-08	1780	0.4734	0.6408
23-May-08 to 17-Feb-11	688	0.2998	0.3591

interval and the interval after the crash, before the crash the stock return had a $H \approx \frac{1}{2}$, see table A.35.

Astral Foods (Astral) had a $H \approx \frac{1}{2}$ on the whole interval and before the crash and anti-persistent using absolute method. While using the Higuchi method the stock was persistent on the whole interval and before the crash and anti-persistent after the crash, see table A.36. **Oceana Group Limited** (Oceana) had a $H \approx \frac{1}{2}$ on the whole interval and before the crash and anti-persistent after the crash using absolute moments method, while it was persistent on the whole interval and before the crash and anti-persistent using the Higuchi method, see table A.37. From the sector food and drug retailers, stock **Clicks Group Limited** (Clicks) had a $H \approx \frac{1}{2}$ on the whole interval and before the crash, after the crash absolute moments method showed anti-persistency while the Higuchi method showed persistency, see table A.38. **Pick n Pay Stores Limited** (Picknpay) was anti-persistent on all three intervals using absolute moments method and using the Higuchi method the stock returns had a $H \approx \frac{1}{2}$ on the whole interval and interval before the crash and was anti-persistent

Table A.37: Sector: Farming and Fishing, Stock: Oceana Group Limited.

Oceana		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
06-Sep-85 to 17-Feb-11	6352	0.5359	0.5972
06-Sep-85 to 23-May-08	5664	0.5470	0.6141
23-May-08 to 17-Feb-11	688	0.3050	0.4360

Table A.38: Sector: Food and Drug Retailers, Stock: Clicks Group Limited.

Clicks		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
20-Mar-96 to 17-Feb-11	3717	0.4543	0.5072
06-Sep-85 to 23-May-08	3029	0.4606	0.4995
23-May-08 to 17-Feb-11	688	0.4309	0.5901

after the crash, see table A.39. **Pick n Pay Holdings Limited** (Pikwik) had a $H \approx \frac{1}{2}$ on the whole interval and the interval before the crash and anti-persistent on the interval after the crash for both the methods, see table A.40.

Shoprite Holdings Limited (Shoprit) was persistent on the whole interval and before the crash for both the methods while absolute moments method showed anti-persistency after the crash and the Higuchi had a $H \approx \frac{1}{2}$ behaviour, see table A.41. **The Spar Group Ltd** (Spar) showed anti-persistent behaviour on all three intervals

Table A.39: Sector: Food and Drug Retailers, Stock: Pick n Pay Stores Limited.

Picknpay		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
27-Mar-85 to 17-Feb-11	6461	0.4214	0.5073
27-Mar-85 to 23-May-08	5773	0.4139	0.5031
23-May-08 to 17-Feb-11	688	0.3462	0.3462

Table A.40: Sector: Food and Drug Retailers, Stock: Pick n Pay Holdings Limited.

Pikwik		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
22-May-87 to 17-Feb-11	5925	0.4745	0.5317
22-May-87 to 23-May-08	5237	0.4878	0.5348
23-May-08 to 17-Feb-11	688	0.2861	0.4124

Table A.41: Sector: Food and Drug Retailers, Stock: Shoprite Holdings Limited.

Shoprit		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
05-Dec-86 to 17-Feb-11	5810	0.6178	0.6724
05-Dec-86 to 23-May-08	5122	0.6299	0.6730
23-May-08 to 17-Feb-11	688	0.2805	0.5195

using absolute moments method while the Higuchi method showed persistency on the whole interval and before the crash and had a $H \approx \frac{1}{2}$ after the crash, see table A.42.Sector gaming, stock **Gold Reef Resorts Limited** (Goldreef) showed persistency on the whole interval and before the crash and anti-persistency after the crash for both methods, see table A.43. **Phumelela Gaming And Leisure Limited** (Phumelela) showed persistency on all intervals for both methods except for one after the crash, where $H \approx \frac{1}{2}$, see table A.44. **Sun International Ltd**

Table A.42: Sector: Food and Drug Retailers, Stock: The Spar Group Ltd.

Spar		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
18-Oct-04 to 17-Feb-11	1587	0.4092	0.5557
18-Oct-04 to 23-May-08	899	0.3654	0.5723
23-May-08 to 17-Feb-11	688	0.3640	0.4818

Table A.43: Sector: Gaming, Stock: Gold Reef Resorts Limited.

Goldreef		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
14-Mar-91 to 17-Feb-11	4971	0.5601	0.5803
14-Mar-91 to 23-May-08	4283	0.5545	0.5628
23-May-08 to 17-Feb-11	688	0.4430	0.3524

Table A.44: Sector: Gaming, Stock: Phumelela Gaming And Leisure Limited.

Phumelela		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
14-Jun-02 to 17-Feb-11	2173	0.6173	0.6192
14-Jun-02 to 23-May-08	1486	0.5838	0.6532
23-May-08 to 17-Feb-11	688	0.5558	0.5219

(Sunint) had a $H \approx \frac{1}{2}$ on the whole interval and before the crash, after the crash there was anti-persistency for both methods, see table A.45.

In the sector gold mining, stock, **Anglo Gold Ashanti Limited** (Anggold) was anti-persistent on all three intervals using absolute moments method and anti-persistent on the whole interval while anti-persistent on the interval before and after the crash using Higuchi, see table A.46. **Gold Fields Limited** (Gfields) had a $H \approx \frac{1}{2}$ for both methods except for the Higuchi method after the crash where anti-

Table A.45: Sector: Gaming, Stock: Sun International Ltd

Sunint		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
27-Mar-85 to 17-Feb-11	6462	0.5081	0.5338
27-Mar-85 to 23-May-08	5774	0.5003	0.5159
23-May-08 to 17-Feb-11	688	0.4095	0.4363

Table A.46: Sector: Gold Mining, Stock: Anglo Gold Ashanti Limited.

Anggold		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
19-Jun-96 to 17-Feb-11	3659	0.4195	0.4277
19-Jun-96 to 23-May-08	2971	0.4500	0.4394
23-May-08 to 17-Feb-11	688	0.4850	0.4410

Table A.47: Sector: Gold Mining, Stock: Gold Fields Limited.

Gfields		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
02-Feb-98 to 17-Feb-11	3259	0.4685	0.4723
02-Feb-98 to 23-May-08	2571	0.4686	0.4807
23-May-08 to 17-Feb-11	688	0.4538	0.4247

persistency was found, see table A.47. **Gold One International Ld** (Goldone) had a $H \approx \frac{1}{2}$ on all three intervals for both methods, see table A.48. **Central Rand Gold Ltd** (Cenrand) had a $H \approx \frac{1}{2}$ using absolute moments method and persistent using Higuchi method, see table A.49. **Randgold and Exploration Company Limited** (Rangold) was persistent for both methods except for one interval after the crash where anti-persistent behaviour was found, see table A.50. Using absolute moments method **Witwatersrand Cons Gold** (Witsgold) had a $H \approx \frac{1}{2}$ on the whole interval and before the crash and persistent after the crash, while using the

Table A.48: Sector: Gold Mining, Stock: Gold One International Ld.

Goldone		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
24-Mar-87 to 17-Feb-11	5965	0.5079	0.4870
24-Mar-87 to 23-May-08	5277	0.4862	0.4824
23-May-08 to 17-Feb-11	688	0.5066	0.5077

Table A.49: Sector: Gold Mining, Stock: Central Rand Gold Ltd.

Cenrand		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
08-Nov-07 to 17-Feb-11	820	0.4686	0.6123

Table A.50: Sector: Gold Mining, Stock: Randgold and Exploration Company Limited.

Rangold		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
01-Feb-93 to 17-Feb-11	4500	0.6918	0.6747
01-Feb-93 to 23-May-08	3811	0.6941	0.6952
23-May-08 to 17-Feb-11	688	0.6558	0.4491

Higuchi method there was persistency on the whole interval and after the crash a $H \approx \frac{1}{2}$ was found before the crash, see table A.51.

Drdgold Ltd (Drdgold) had a $H \approx \frac{1}{2}$ on all three intervals for both methods, see table A.52. **Simmer and Jack Mines Limited** (Simmers) had a $H \approx \frac{1}{2}$ for both methods except for one interval after the crash where anti-persistency was found, see table A.53. **Village Main Reef Ltd** (Village) had a $H \approx \frac{1}{2}$ on all three intervals using absolute moments method and anti-persistent on all three intervals using the Higuchi method, see table A.54.From the sector hospital management and long term

Table A.51: Sector: Gold Mining, Stock: Witwatersrand Cons Gold.

Witsgold		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
24-Apr-06 to 17-Feb-11	1208	0.54158542858070	0.5710
24-Apr-06 to 23-May-08	520	0.45083555368597	0.5448
23-May-08 to 17-Feb-11	688	0.56749183577265	0.5790

Table A.52: Sector: Gold Mining, Stock: Drdgold Ltd.

Drdgold		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
27-Mar-85 to 17-Feb-11	6462	0.5003	0.5157
27-Mar-85 to 23-May-08	5775	0.5054	0.5259
23-May-08 to 17-Feb-11	688	0.4863	0.5107

Table A.53: Sector: Gold Mining, Stock: Simmer and Jack Mines Limited.

Simmers		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
27-Mar-85 to 17-Feb-11	6462	0.5094	0.5108
27-Mar-85 to 23-May-08	5775	0.4938	0.4875
23-May-08 to 17-Feb-11	688	0.3939	0.5325

care, stock **Life Healthcare Grp Holdings Ltd** (Lifehc) was anti-persistent using absolute moments method and had a $H \approx \frac{1}{2}$ using the Higuchi method, see table A.55.**Litha Healthcare Group Ltd** (Litha) was persistent using both the methods, see table A.56.

Mediclinic International (Medclin) had a $H \approx \frac{1}{2}$ using absolute moments method on all three intervals while using the Higuchi method the data series was persistent on the whole interval and before the crash and had a $H \approx \frac{1}{2}$ after the crash,

Table A.54: Sector: Gold Mining, Stock: Village Main Reef Ltd.

Village		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
27-Mar-85 to 17-Feb-11	6462	0.4527	0.4310
27-Mar-85 to 23-May-08	5775	0.4509	0.4383
23-May-08 to 17-Feb-11	688	0.4513	0.4363

Table A.55: Sector: Hospital Management and Long Term Care, Stock: Life Healthcare Grp Holdings Ltd.

Lifehc		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
10-Jun-10 to 17-Feb-11	175	0.4346	0.5068

Table A.56: Sector: Hospital Management and Long Term Care, Stock: Litha Healthcare Group Ltd.

Litha		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
17-May-10 to 17-Feb-11	193	0.6426	0.6987

see table A.57.Using absolute moments method **Netcare Limited** (Netcare) had a $H \approx \frac{1}{2}$ on all three intervals while persistent on the whole interval and before the crash and had a $H \approx \frac{1}{2}$ after the crash using the Higuchi method, see table A.58.From sector insurance - non-life, stock **Santam Ltd** (Santam) had a $H \approx \frac{1}{2}$ on the whole interval and before the crash and anti-persistent using absolute moments method while it was persistent on the whole interval and before the crash and anti-persistent using the Higuchi method, see table A.59.Sector insurance brokers, stock **Glenrand M-I-B Ltd** (Glenmib) had a $H \approx \frac{1}{2}$ on all three intervals using absolute moments method and anti-persistent on all three intervals using the Higuchi method, see table

Table A.57: Sector: Hospital Management and Long Term Care, Stock: Mediclinic International.

Medclin		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
22-May-87 to 17-Feb-11	5925	0.5423	0.6021
22-May-87 to 23-May-08	5237	0.5422	0.6028
23-May-08 to 17-Feb-11	688	0.5083	0.5380

Table A.58: Sector: Hospital Management and Long Term Care, Stock: Netcare Limited.

Netcare		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
04-Dec-96 to 17-Feb-11	3546	0.5367	0.5945
04-Dec-96to 23-May-08	2857	0.5370	0.5829
23-May-08 to 17-Feb-11	688	0.4025	0.4708

Table A.59: Sector: Insurance - Non-Life, Stock: Santam Ltd.

Santam		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
03-Mar-86 to 17-Feb-11	6232	0.4900	0.5873
03-Mar-86 to 23-May-08	5544	0.4647	0.5833
23-May-08 to 17-Feb-11	688	0.3927	0.4367

A.60.In the sector investment banks, stock **Investec Plc** (Invplc) had a $H \approx \frac{1}{2}$ on the whole interval, persistent on the interval before the crash and anti-persistent after the crash using absolute moments methods, while the Higuchi method showed persistency on the whole interval and before the crash and anti-persistency after the crash, see table A.61.Sector Kruger Rands, stock **Kruger Rand** (Kr) had a $H \approx \frac{1}{2}$ on the whole interval and on the interval after the crash and before the crash it was persistent using absolute moments method while the Higuchi method

Table A.60: Sector: Insurance Brokers, Stock: Glenrand M-I-B Ltd.

Glenmib		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
26-Jun-98 to 17-Feb-11	3161	0.4601	0.4365
26-Jun-98 to 23-May-08	2473	0.4903	0.4389
23-May-08 to 17-Feb-11	688	0.5086	0.4471

Table A.61: Sector: Investment Banks, Stock: Investec Plc.

Invplc		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
22-Jul-02 to 17-Feb-11	2176	0.5495	0.5885
22-Jul-02 to 23-May-08	1488	0.5667	0.5993
23-May-08 to 17-Feb-11	688	0.3729	0.4456

Table A.62: Sector: Kruger Rands, Stock: Kruger Rand.

Kr		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
27-Mar-85 to 17-Feb-11	6465	0.5461	0.5636
27-Mar-85 to 23-May-08	5777	0.5522	0.5466
23-May-08 to 17-Feb-11	688	0.4651	0.5440

showed persistency on the whole interval and anti-persistency on the interval before and after the crash, see table A.62.From sector life assurance, stock **Clientele Ltd** (Clientele) showed persistency using both methods, see table A.63.

Discovery Holdings Limited (Discovery) had a $H \approx \frac{1}{2}$ on the whole interval and before the crash while after the crash the stock was anti-persistent for both methods, see table A.64.**Old Mutual Plc** (Oldmutual) had a $H \approx \frac{1}{2}$ on the whole interval and anti-persistent using absolute moments method and had a $H \approx \frac{1}{2}$ using the Higuchi method before the crash and persistent using both methods after the crash, see table A.65. Using absolute moments method **Sanlam Limited** (Sanlam)

Table A.63: Sector: Life Assurance, Stock: Clientele Ltd.

Clientele		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
19-May-08 to 17-Feb-11	692	0.5847	0.6165

Table A.64: Sector: Life Assurance, Stock: Discovery Holdings Limited.

Discovery		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
21-Oct-99 to 17-Feb-11	2830	0.5110	0.5278
06-Sep-85 to 23-May-08	2142	0.5165	0.5299
23-May-08 to 17-Feb-11	688	0.2512	0.4058

Table A.65: Sector: Life Assurance, Stock: Old Mutual Plc.

Oldmutual		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
12-Jul-99 to 17-Feb-11	2901	0.5333	0.5272
06-Sep-85 to 23-May-08	2213	0.4458	0.4575
23-May-08 to 17-Feb-11	688	0.5864	0.5812

was anti-persistent on the whole interval and before the crash and had a $H \approx \frac{1}{2}$ after the crash, while the Higuchi method showed a $H \approx \frac{1}{2}$on the whole interval and before the crash and anti-persistency on the interval after the crash, see table A.66. **Grindrod Ltd** (Grindrod) was persistent on the whole interval and the interval before the crash, after the crash there was a $H \approx \frac{1}{2}$, see table A.67. From the sector metals and minerals, stock **Anglo American Plc** (Anglo) had a $H \approx \frac{1}{2}$ on all three interval for both methods, see table A.68. Using absolute moments method **African Rainbow Minerals** (Arm) was persistent on the whole interval and had a $H \approx \frac{1}{2}$

Table A.66: Sector: Life Assurance, Stock: Sanlam Limited.

Sanlam		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
30-Nov-98 to 17-Feb-11	3052	0.4407	0.5077
06-Sep-85 to 23-May-08	2364	0.4454	0.4958
23-May-08 to 17-Feb-11	688	0.4511	0.4319

Table A.67: Sector: Marine Transportation, Stock: Grindrod Ltd.

Grindrod		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
26-Mar-87 to 17-Feb-11	5963	0.5943	0.6293
26-Mar-87 to 23-May-08	5275	0.5887	0.6254
23-May-08 to 17-Feb-11	688	0.5167	0.4933

Table A.68: Sector: Metals and Minerals, Stock: Anglo American Plc.

Anglo		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
27-Mar-85 to 17-Feb-11	6462	0.5010	0.5472
27-Mar-85 to 23-May-08	5774	0.4973	0.5414
23-May-08 to 17-Feb-11	688	0.5436	0.5457

before and after the crash while the other method displayed persistency on the whole interval and the interval before the crash and had a $H \approx \frac{1}{2}$ after the crash, see table A.69.

BHP Billiton Plc (Billiton) had a $H \approx \frac{1}{2}$ on the whole interval and before the crash and anti-persistent after the crash for both methods, see table A.70. From the sector mining, stock **Firestone** (Firestone) was anti-persistent on the whole interval for both methods, see table A.71. **Sephaku Holdings Ltd** (Sephaku) was

Table A.69: Sector: Metals and Minerals, Stock: African Rainbow Minerals.

Arm		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
27-Mar-85 to 17-Feb-11	6462	0.5519	0.5667
27-Mar-85 to 23-May-08	5774	0.5483	0.5581
23-May-08 to 17-Feb-11	688	0.5170	0.5215

Table A.70: Sector: Metals and Minerals, Stock: BHP Billiton Plc.

Billiton		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
28-Jul-97 to 17-Feb-11	3418	0.4663	0.5473
28-Jul-97 to 23-May-08	1789	0.4598	0.5450
23-May-08 to 17-Feb-11	688	0.4341	0.3876

Table A.71: Sector: Mining, Stock: Firestone.

Firestone		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
16-Apr-09 to 17-Feb-11	462	0.2170	0.3058

persistent on the whole interval for both methods see table A.72. **First Uranium Corporation** (Fiuranium) was persistent on the whole interval for both methods, see table A.73. From the sector nonferrous metals, stock **Metmar Ltd** (Metmar) was persistent on all three intervals using Higuchi method and had a $H \approx \frac{1}{2}$ on the whole interval and the interval after the crash and anti-persistent on the interval before the crash using the absolute moments method, see table A.74.

Palabora Mining Company Limited (Palamin) had a $H \approx \frac{1}{2}$ on the whole interval and the interval before the crash and after the crash the stock showed a persistent behaviour for both the methods, see table A.75. From the sector, oil integrated, stock **Oando Plc** (Oando) had an anti-persistency behaviour on all three

Table A.72: Sector: Mining, Stock: Sephaku Holdings Ltd.

Sephaku		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
21-Aug-09 to 17-Feb-11	376	0.6002	0.5564

Table A.73: Sector: Nonferrous Metals, Stock: First Uranium Corporation.

Fiuranium		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
30-Mar-07 to 17-Feb-11	973	0.5686	0.6521

Table A.74: Sector: Nonferrous Metals, Stock: Metmar Ltd.

Metmar		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
22-May-06 to 17-Feb-11	1190	0.5363	0.5652
22-May-06 to 23-May-08	502	0.4222	0.5715
23-May-08 to 17-Feb-11	688	0.5352	0.5670

intervals using absolute moments method. Using the Higuchi method a $H \approx \frac{1}{2}$ was found on the whole interval and the interval after the crash while anti-persistency was found before the crash, see table A.76. **Sacoil Holding Ld** (Sacoil) stock returns had a $H \approx \frac{1}{2}$ on the whole interval, an anti-persistent behaviour on the interval before the crash and a persistent behaviour after the crash for both methods, see table A.77. **Sasol Limited** (Sasol) stock returns had a $H \approx \frac{1}{2}$ on the whole interval and the interval before the crash and after the crash the stock fell to an anti-persistent behaviour, see table A.78. Sector, paper, stock **Mondi Limited** (Mondiltdp) had a $H \approx \frac{1}{2}$ on the whole interval for both methods, see table A.79.

Table A.75: Sector: Nonferrous Metals, Stock: Palabora Mining Company Limited.

Palamin		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
27-Mar-85 to 17-Feb-11	6462	0.5000	0.5193
27-Mar-85 to 23-May-08	5774	0.4930	0.5096
23-May-08 to 17-Feb-11	688	0.6202	0.6174

Table A.76: Sector: Oil Integrated, Stock: Oando Plc.

Oando		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
25-Nov-05 to 17-Feb-11	1307	0.4451	0.4860
25-Nov-05 to 23-May-08	619	0.4142	0.3995
23-May-08 to 17-Feb-11	688	0.3565	0.4547

Table A.77: Sector: Oil Integrated, Stock: Sacoil Holding Ld.

Sacoil		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
19-Oct-94 to 17-Feb-11	4072	0.4764	0.4676
19-Oct-94 to 23-May-08	3384	0.3776	0.3580
23-May-08 to 17-Feb-11	688	0.6530	0.6254

Table A.78: Sector: Oil Integrated, Stock: Sasol Limited.

Sasol		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
27-Mar-85 to 17-Feb-11	6462	0.4626	0.5507
27-Mar-85 to 23-May-08	5774	0.4771	0.5454
23-May-08 to 17-Feb-11	688	0.4241	0.3946

Table A.79: Sector: Paper, Stock: Mondi Limited.

Mondiltdp		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
02-Jul-07 to 17-Feb-11	911	0.5073	0.5213

Table A.80: Sector: Paper, Stock: Sappi Limited.

Sappi		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
27-Mar-85 to 17-Feb-11	6461	0.5370	0.5460
27-Mar-85 to 23-May-08	5774	0.5307	0.5579
23-May-08 to 17-Feb-11	687	0.6007	0.5966

Table A.81: Sector: Pharmaceuticals, Stock: Aspen Pharmacare Holdings.

Aspen		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
25-May-87 to 17-Feb-11	5171	0.5482	0.5881
25-May-87 to 23-May-08	4484	0.5724	0.5927
23-May-08 to 17-Feb-11	687	0.3865	0.5214

Using absolute moments method **Sappi Limited** (Sappi) showed a $H \approx \frac{1}{2}$ on the whole interval and the interval before the crash, after the crash persistency was found. Using the Higuchi method the stock had a $H \approx \frac{1}{2}$ on the whole interval and persistency was found of the interval before and after the crash, see table A.80. From the sector, pharmaceuticals, stock, **Aspen Pharmacare Holdings** (Aspen) showed a $H \approx \frac{1}{2}$ on the whole interval, persistency on the interval before the crash and anti-persistency after the crash using absolute moments method. While using the Higuchi method persistency was found on the whole interval and the interval before the crash, after the crash a $H \approx \frac{1}{2}$ was found, see table A.81. **Cipla Medpro SA Ltd** (Ciplamed) showed persistency using absolute moments method and had a $H \approx \frac{1}{2}$ using the Higuchi method, see table A.82. From the sector, platinum, stock **Anglo American Platinum Corporation Limited** (Angoplat) showed a $H \approx \frac{1}{2}$ except for absolute moments method after the crash where persistency was found, see table A.83. **Anooraq Resources Corporation** (Anooraq) was persistent on the whole interval for both methods, see table A.84.

Table A.82: Sector: Pharmaceuticals, Stock: Cipla Medpro SA Ltd.

Ciplamed		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
10-Jun-05 to 17-Feb-11	1425	0.5584	0.5097

Table A.83: Sector: Platinum, Stock: Anglo American Platinum Corporation Limited.

Angoplat		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
27-Mar-85 to 17-Feb-11	6460	0.5153	0.5297
27-Mar-85 to 23-May-08	5758	0.5140	0.5170
23-May-08 to 17-Feb-11	687	0.5802	0.5325

Using absolute moments method **Aquarius Platinum Ltd** (Aquarius) had a $H \approx \frac{1}{2}$ on the whole interval, anti-persistent on the interval before the crash and persistent on the interval after the crash. Using the Higuchi method persistency was found on all three intervals, see table A.85. Using absolute moments method **Impala Platinum Holdings Limited** (Implats) stock had a $H \approx \frac{1}{2}$ on the whole interval and the interval before the crash while after the crash persistency was found. Using the Higuchi method persistency was found on the whole interval and the interval before the crash while after the crash there was a $H \approx \frac{1}{2}$, see table A.86. From sector, rail, road and freight, stock **Cargo Carriers Limited** (Cargo) was persistent for both methods except for one interval before the crash where a $H \approx$

Table A.84: Sector: Platinum, Stock: Anooraq Resources Corporation.

Anooraq		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
19-Dec-06 to 17-Feb-11	1042	0.5998	0.6228

Table A.85: Sector: Platinum, Stock: Aquarius Platinum Ltd.

Aquarius		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
08-Dec-04 to 17-Feb-11	1551	0.5448	0.6190
08-Dec-04 to 23-May-08	863	0.3383	0.5817
23-May-08 to 17-Feb-11	688	0.5546	0.5550

Table A.86: Sector: Platinum, Stock: Impala Platinum Holdings Limited.

Implats		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
27-Mar-85 to 17-Feb-11	6462	0.5151	0.5537
27-Mar-85 to 23-May-08	5774	0.5172	0.5517
23-May-08 to 17-Feb-11	688	0.5554	0.5062

$\frac{1}{2}$ was found, see table A.87. From sector real estate holdings and development, stock **Acucap Properties Limited** (Acucap) had a $H \approx \frac{1}{2}$ on the whole interval and the interval before the crash and anti-persistent after the crash using absolute moments method. While using the Higuchi method there was persistency on the whole interval and before the crash and anti-persistency after the crash, see table A.88. **Growthpoint Properties Limited** (Growpnt) was persistent on the whole interval and the interval before the crash and anti-persistent after the crash for both methods, see table A.89. **Hospitality Prop Fund A** (Hosp-a) had a $H \approx \frac{1}{2}$ on the

Table A.87: Sector: Rail, Road and Freight, Stock: Cargo Carriers Limited.

Cargo		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
05-Oct-87 to 17-Feb-11	5831	0.5802	0.5681
05-Oct-87 to 23-May-08	5143	0.5722	0.5432
23-May-08 to 17-Feb-11	688	0.7233	0.6321

Table A.88: Sector: Real Estate Holdings and Development, Stock: Acucap Properties Limited.

Acucap		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
27-Mar-02 to 17-Feb-11	2229	0.5055	0.6004
27-Mar-02 to 23-May-08	1540	0.5284	0.6185
23-May-08 to 17-Feb-11	687	0.2963	0.4295

Table A.89: Sector: Real Estate Holdings and Development, Stock: Growthpoint Properties Limited.

Growpnt		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
27-Nov-87 to 17-Feb-11	5793	0.5792	0.5967
27-Nov-87 to 23-May-08	5105	0.5851	0.5968
23-May-08 to 17-Feb-11	688	0.3097	0.4449

whole interval for both methods, see table A.90.

Using absolute moments method **Hyprop Investments Limited** (Hyprop) had a $H \approx \frac{1}{2}$ on the whole interval and the interval before the crash and after the crash anti-persistency was found. The Higuchi method showed persistency on the whole interval and the interval before the crash while $H \approx \frac{1}{2}$ on the interval after the crash, see table A.91. From sector, real estate investment trusts, stock **Capital Property Fund** (Capital) had a $H \approx \frac{1}{2}$ for both methods except for one interval after the

Table A.90: Sector: Real Estate Holdings and Development, Stock: Hospitality Prop Fund A.

Hosp-a		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
16-Feb-06 to 17-Feb-11	1251	0.4685	0.5247

Table A.91: Sector: Real Estate Holdings and Development, Stock: Hyprop Investments Limited.

Hyprop		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
24-Feb-88 to 17-Feb-11	5762	0.5423	0.5797
24-Feb-88 to 23-May-08	5074	0.5319	0.5640
23-May-08 to 17-Feb-11	688	0.3951	0.4685

Table A.92: Sector: Real Estate Investment Trusts, Stock: Capital Property Fund.

Capital		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
06-Sep-85 to 17-Feb-11	6352	0.4823	0.4887
06-Sep-85 to 23-May-08	5664	0.4912	0.4725
23-May-08 to 17-Feb-11	688	0.3351	0.4977

crash where anti-persistency was found, see table A.92. **Caital Shopping Centres Group Plc** (Capshop) showed persistency for both methods except for one interval before the crash where $H \approx \frac{1}{2}$ for the absolute moments method, see table A.93. Using one method **Emira Property Fund** (Emira) had a $H \approx \frac{1}{2}$ on the whole interval and the interval before the crash and anti-persistency after the crash using absolute moment, while the Higuchi method showed a $H \approx \frac{1}{2}$ on the whole interval, persistency on the interval before the crash and anti-persistency after the crash, see

Table A.93: Sector: Real Estate Investment Trusts, Stock: Caital Shopping Centres Group Plc.

Capshop		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
24-Jun-99 to 17-Feb-11	2913	0.5897	0.5842
24-Jun-99 to 23-May-08	2225	0.5104	0.5688
24-Jun-99 to 17-Feb-11	688	0.5805	0.6111

Table A.94: Sector: Real Estate Investment Trusts, Stock: Emira Property Fund.

Emira		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
28-Nov-03 to 17-Feb-11	1805	0.4640	0.5450
28-Nov-03 to 23-May-08	1117	0.5062	0.5602
23-May-08 to 17-Feb-11	688	0.2959	0.4356

Table A.95: Sector: Real Estate Investment Trusts, Stock: Fountainhead Prop Trust.

Fpt			
Period	N	Absolute Moment	Higuchi
04-Jun-07 to 17-Feb-11	931	0.4076	0.4450

table A.94. **Fountainhead Prop Trust** (Fpt) showed an anti-persistent behaviour on the whole interval for both methods, see table A.95. **Syfrets and Commercial Union Property Fund** (Sycom) had a $H \approx \frac{1}{2}$ on the whole interval and the interval before the crash and after the crash the stock showed an anti-persistent behaviour for both methods, see table A.96.

Sector, real estate investment and services, stock **Pangbourne Propertise Ltd** (Panprop) had a $H \approx \frac{1}{2}$ for both methods except for one interval where after the crash anti-persistency was found, see table A.97. Sector, restaurants and pubs, stock

Table A.96: Sector: Real Estate Investment Trusts, Stock: Syfrets and Commercial Union Property Fund.

Sycom		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
28-Apr-87 to 17-Feb-11	5972	0.5034	0.5153
28-Apr-87 to 23-May-08	5285	0.5279	0.5209
23-May-08 to 17-Feb-11	688	0.4253	0.3814

Table A.97: Sector: Real Estate Investment and Services, Stock: Pangbourne Propertise Ltd.

Panprop		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
28-Jul-87 to 17-Feb-11	5911	0.5039	0.5248
28-Jul-87 to 23-May-08	5223	0.4901	0.5120
23-May-08 to 17-Feb-11	688	0.4061	0.5037

Table A.98: Sector: Restaurants and Pubs, Stock: Famous Brands Ltd.

Fambrands		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
09-Nov-94 to 17-Feb-11	4057	0.5555	0.5705
09-Nov-94 to 23-May-08	3369	0.5518	0.5671
23-May-08 to 17-Feb-11	688	0.4685	0.5658

Famous Brands Ltd (Fambrands) showed persistency for both methods, except for one interval where the stock return had a $H \approx \frac{1}{2}$, see table A.98. Using one method **Spur Corporation Limited** (Spurcorp) had a $H \approx \frac{1}{2}$ on the whole interval and anti-persistency on the interval before and after the crash using absolute moments method. Using the Higuchi method persistency was found on the whole interval and a $H \approx \frac{1}{2}$ on the interval before and after the crash, see table A.99.

Table A.99: Sector: Restaurants and Pubs, Stock: Spur Corporation Limited.

Spurcorp		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
07-Dec-99 to 17-Feb-11	2797	0.5042	0.5734
07-Dec-99 to 23-May-08	2109	0.4251	0.5236
23-May-08 to 17-Feb-11	688	0.3740	0.4796

Appendix B

Hurst Tables: Part 2

From sector, retailers - multi department, stock, **Massmart Holdings Ltd** (Massmart) showed a $H \approx \frac{1}{2}$ on the whole interval and the interval before the crash and anti-persistency after the crash using absolute moments method, while using the Higuchi method persistency was found on the whole interval and the interval before the crash and after the crash a $H \approx \frac{1}{2}$ was found, see table B.1. **Nictus Beperk** (Nictus) showed persistency on the whole interval and the interval before the crash and a $H \approx \frac{1}{2}$ after the crash for both methods, see table B.2. **Verimark Holdings Ltd** (Verimark) showed persistency on the whole interval and the interval before the crash, while absolute moments method showed anti-persistency and the Higuchi method showed a $H \approx \frac{1}{2}$ after the crash, see table B.3. **Woolworths Holdings Limited** (Woolies) had a $H \approx \frac{1}{2}$ except for one interval, where the absolute moments method showed anti-persistent behaviour, see table B.4. absolute moments method showed that the stock for **Mr Price Group Limited** (Mrprice) had a $H \approx \frac{1}{2}$

Table B.1: Sector: Retailers - Multi Department, Stock: Massmart Holdings Ltd.

Massmart		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
04-Jul-00 to 17-Feb-11	2660	0.4870	0.5716
04-Jul-00 to 23-May-08	1972	0.5073	0.5942
23-May-08 to 17-Feb-11	688	0.3774	0.5024

Table B.2: Sector: Retailers - Multi Department, Stock: Nictus Beperk.

Nictus		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
26-Mar-87 to 17-Feb-11	5963	0.6433	0.6463
26-Mar-87 to 23-May-08	5275	0.6512	0.6613
23-May-08 to 17-Feb-11	688	0.5427	0.5276

Table B.3: Sector: Retailers - Multi Department, Stock: Verimark Holdings Ltd.

Verimark		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
12-Jul-05 to 17-Feb-11	1404	0.5860	0.6072
12-Jul-05 to 23-May-08	716	0.5813	0.5919
23-May-08 to 17-Feb-11	688	0.3840	0.4822

Table B.4: Sector: Retailers - Multi Department, Stock: Woolworths Holdings Limited.

Woolies		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
20-Oct-97 to 17-Feb-11	3329	0.5459	0.5487
20-Oct-97 to 23-May-08	2641	0.5463	0.5317
23-May-08 to 17-Feb-11	688	0.3938	0.5242

Table B.5: Sector: Retailers - Soft Goods, Stock: Mr Price Group Limited.

Mrprice		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
12-Jun-89 to 17-Feb-11	5411	0.5143	0.5964
12-Jun-89 to 23-May-08	4723	0.5206	0.5919
23-May-08 to 17-Feb-11	688	0.3082	0.5453

Table B.6: Sector: Retailers - Soft Goods, Stock: The Foschini Group Ltd.

Tfg		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
06-Sep-85 to 17-Feb-11	6351	0.5521	0.6171
06-Sep-85 to 23-May-08	5663	0.5624	0.6186
23-May-08 to 17-Feb-11	688	0.3687	0.5005

for the whole interval and the interval before the crash and anti-persistent after the crash. The Higuchi method showed persistency on the whole interval and the interval before the crash and a $H \approx \frac{1}{2}$ after the crash, see table B.5.

The Foschini Group Ltd (Tfg) showed persistency on the whole interval and the interval before the crash, after the crash absolute moments method showed an anti-persistent behaviour while the Higuchi method showed a $H \approx \frac{1}{2}$ behaviour, see table B.6. **Truworths International Limited** (Truwths) showed a $H \approx \frac{1}{2}$ for both methods, except for one interval and for absolute moments method where anti-persistency was found, see table B.7. **Ucs Group Limited** (Ucs) was persistent on the whole interval and the interval before the crash and after the crash the absolute moments method showed a $H \approx \frac{1}{2}$ while the Higuchi method showed an anti-persistent behaviour, see table B.8. From the sector, steel, stock **Arcelormittal** (Arcmittal) showed persistency for both methods except for one interval, where absolute moments method found a $H \approx \frac{1}{2}$, see table B.9. absolute moments method showed that **Evraz Highveld Steel and Van** (Ehsv) had a $H \approx \frac{1}{2}$ on the whole

Table B.7: Sector: Retailers - Soft Goods, Stock: Truworths International Limited.

Truwths		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
11-May-98 to 17-Feb-11	3194	0.5068	0.5226
11-May-98 to 23-May-08	2506	0.5166	0.5219
23-May-08 to 17-Feb-11	688	0.3480	0.5012

Table B.8: Sector: Software, Stock: Ucs Group Limited.

Ucs		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
09-Sep-98 to 17-Feb-11	3109	0.5763	0.5946
09-Sep-98 to 23-May-08	2421	0.5592	0.5756
23-May-08 to 17-Feb-11	688	0.5084	0.4490

interval and the interval before the crash, while after the crash persistency was found. The Higuchi method showed persistency on the whole interval and the interval after the crash while a $H \approx \frac{1}{2}$ was found on the interval before the crash, see table B.10. **Hulamin Limited** (Hulamin) showed a $H \approx \frac{1}{2}$ on the whole interval for both methods, see table B.11. **Kumba Iron Ore Ltd** (Kumbaio) showed a $H \approx \frac{1}{2}$ on the whole interval for both methods, see table B.12. From sector, telecommunications equipment, stock, **Vodacom Group Limited** (Vodacom) showed an anti-persistent behaviour on the whole interval for both methods, see table B.13.

Table B.9: Sector: Steel, Stock: Arcelormittal.

Arcmittal		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
08-Nov-89 to 17-Feb-11	5306	0.5668	0.5773
08-Nov-89 to 23-May-08	4618	0.5779	0.5701
23-May-08 to 17-Feb-11	688	0.5368	0.5668

Table B.10: Sector: Steel, Stock: Evraz Highveld Steel and Van.

Ehsv		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
06-Sep-85to 17-Feb-11	6350	0.5439	0.5539
06-Sep-85 to 23-May-08	5662	0.5401	0.5430
23-May-08 to 17-Feb-11	688	0.5784	0.5799

Table B.11: Sector: Steel, Stock: Hulamin Limited.

Hulamin		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
25-Jun-07 to 17-Feb-11	916	0.5096	0.5472

Table B.12: Sector: Steel, Stock: Kumba Iron Ore Ltd.

Kumbaio		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
20-Nov-06 to17-Feb-11	1093	0.5386	0.5323

Table B.13: Sector: Telecommunications Equipment, Stock: Vodacom Group Limited.

Vodacom		Hurst Parameter	
	N	Absolute Moment	Higuchi
18-May-09 to 17-Feb-11	443	0.2964	0.3253

Table B.14: Sector: Wireless Telecom Services, Stock: Allied Technologies Limited.

Altech		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
06-Sep-85 to 17-Feb-11	6352	0.5039	0.5386
06-Sep-85 to 23-May-08	5664	0.5322	0.5404
23-May-08 to 17-Feb-11	688	0.4967	0.5093

Table B.15: Sector: Wireless Telecom Services, Stock: Blue Label Telecoms Ltd.

Bluetel		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
14-Nov-07 to 17-Feb-11	816	0.5062	0.5399

From the sector, wireless telecom services, stock, **Allied Technologies Limited** (Altech) showed a $H \approx \frac{1}{2}$ on all the intervals for both methods, see table B.14. **Blue Label Telecoms Ltd** (Bluetel) showed a $H \approx \frac{1}{2}$ on all the intervals for both methods, see table B.15. absolute moments method showed that **MTN Group Limited** (Mtn) had a $H \approx \frac{1}{2}$ on all the intervals, while the Higuchi method showed a persistent stock on the whole interval and the interval before the crash and anti-persistent stock after the crash, see table B.16.

Table B.16: Sector: Wireless Telecom Services, Stock: MTN Group Limited.

Mtn		Hurst Parameter	
Period	N	Absolute Moment	Higuchi
15-Aug-95 to 17-Feb-11	3868	0.5454	0.5970
15-Aug-95 to 23-May-08	3180	0.5211	0.5997
23-May-08 to 17-Feb-11	688	0.4538	0.4136

Appendix C

Wick Calculus in Gaussian Spaces

C.1 Introduction

In this appendix we will be discussing white noise spaces and generalized processes given by the time derivative of a Wiener process, i.e. the velocity of the Wiener process. We mainly follow the presentation of Hida, Kuo, Potthoff and Striet, (1993). The generalized variables are in white noise spaces and we will consider stochastic analysis in Gaussian spaces. The Wiener Itô chaos decomposition theorem is used in the development of functions of white noise. We need to define a measure on a Gaussian space, which is the dual of a nuclear space. We cannot have a Lebesgue measure on an infinite dimensional vector space thus we will equip the space with a Gaussian measure. A measure is constructed in the space $\mathcal{S}'(\mathbb{R})$ of tempered distributions which is the dual of the Schwartz space of test functions $\mathcal{S}(\mathbb{R})$ .We will also define the Wick product of two distributions in terms of the $\mathcal{S}$ transform.

C.2 Preliminaries

Tables C.1 gives a list of the main spaces used in white noise analysis and some of the spaces will be defined in the next section. The spaces obey the following

$$\mathcal{N} \subset \mathcal{H} \subset \mathcal{N}^*$$

and

$$\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R}) \subset \mathcal{S}'(\mathbb{R})$$

and

$$(\mathcal{N})^1 \subset (\mathcal{N}) \subset L^2(\mu) \subset (\mathcal{N})^* \subset (\mathcal{N})^{-1}.$$

See below for the description of norm.$|\cdot|_n$. $|\cdot|$ is the norm induced by the scalar product on $\mathcal{N}$.

C.3 White Noise

To construct a countably Hilbert space we let $n \in \mathbb{N}$ and let $\mathcal{D}$ be a real vector space equipped with a sequence $\{(\cdot,\cdot)_n\}$ of scalar products which induce norms $|\cdot|_n$ on $\mathcal{D}$. Assume that the system of scalar products is compatible such that if $\{\xi_n\}$ is a sequence in $\mathcal{D}$ which converges to zero with respect to a norm $|\cdot|_n$ and is Cauchy with respect to $|\cdot|_m$, $m \in \mathbb{N}$, then it converges also to zero with respect to $|\cdot|_m$. Assume that the norms are increasing in the sense that for $n \leq m$ the norms obey $|\cdot|_n \leq |\cdot|_m$. Let $\mathcal{N}_n$ be the completion of $\mathcal{D}$ under the $|\cdot|_n$. Let $\mathcal{N}$ denote the intersection of the Hilbert spaces $\mathcal{N}_n$ as $\mathcal{N} = \cap_n \mathcal{N}_n$. $\mathcal{N}$ endowed with the projective limit topology τ_p is called a countably Hilbert space. Assume also that for every $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$, so that the injection ι_n^m from $\mathcal{N}_m$ into $\mathcal{N}_n$ is a Hilbert-Schmidt operator [1]. Let $\mathcal{N}_n^* = \mathcal{N}_{-n}$ with norm $|\cdot|_{-n}$ denote the dual of $\mathcal{N}_n$ and let $\mathcal{N}^*$ be the dual of $\mathcal{N}$. We have $\mathcal{N}^* = \cup_n \mathcal{N}_n^*$. The dual pairing between $\mathcal{N}^*$ and $\mathcal{N}$ is denoted by $\langle\cdot,\cdot\rangle$. Consider $\mathcal{N}^*$ as equipped with a weak topology σ. Let $\mathcal{B}$ be a Borel σ-algebra on $\mathcal{N}^*$ generated by cylinder sets, see Hida, Kuo, Potthoff and Striet (1993). $\mathcal{N}$ is a separable pre-Hilbert space with scalar product $(\cdot,\cdot)$, which is compatible with the topology of $\mathcal{N}$, (compatible with scalar products $(\cdot,\cdot)_n$, $n \in \mathbb{N}$).

[1] u is a Hilbert-Schmidt operator if u: $H_1 \to H_2$ which are Hilbert spaces such that u admits an orthonormal representation with the scalar coeficients in l_2 which is a Hilbert sequence space.
(See Diestel, Jarchowand and Tonge 1995, page 84)

Table C.1: Table of main spaces

Space	Description
$\mathcal{D}$	Linear space on $\mathbb{R}$ with sequence $(\cdot,\cdot)$.
$\mathcal{N}_n$	Completion of $\mathcal{D}$ under $\left\|\cdot\right\|_n$
$\mathcal{N}$	$=\cap_n\mathcal{N}_n$ Nuclear separable pre-Hilbert space
	with a scalar product compatible with the topology on $\mathcal{N}$.
$\mathcal{N}_n^*$	$=\mathcal{N}_{-n}$ Dual of $\mathcal{N}_n$.
$\mathcal{N}^*$	$=\cup_n\mathcal{N}_n^*$ Dual of $\mathcal{N}^*$.
$\mathcal{B}$	Borel σ-algebra on $\mathcal{N}^*$.
$\mathcal{S}(\mathbb{R})$	Choice for $\mathcal{N}$, a Schwartz space of test functions.
$\mathcal{S}'(\mathbb{R})$	Dual of $\mathcal{S}(\mathbb{R})$ a space of generalized functions.
$\mathcal{H}$	Completion of $\mathcal{N}$ under $\left\|\cdot\right\|$, can be $\mathcal{N}_n$ for some $n\in\mathbb{N}$.
$L^2\left(\mathbb{R}^d\right)$	Choice for $\mathcal{H}$.
$L^p\left(\mathcal{N}^*,\mathcal{B},\mu\right)$	$=(L^p)$ for $p\geq 1$.
$\mathcal{P}$	Algebra of polynomials generated by X_ξ, see below.
$\mathcal{R}$	Image of (L^2) under $\mathcal{T}$ transform.
$\mathcal{H}^{(n)}$	Closed linear span of $\{\mathcal{H}_\alpha,\alpha\in I^n\}$ the nth homogenous chaos
	where $\mathcal{H}_\alpha$ is a product of Hermite polynomials.
(L^2)	$=\oplus\mathcal{H}^{(n)}$ direct sum of $\mathcal{H}^{(n)}$.
$\mathcal{H}_{\mathbb{C}}^{\hat{\otimes}n}$	Tensor power of Hilbert space with dual $\mathcal{N}_{\mathbb{C}}^{*\hat{\otimes}n}$.
$\mathcal{N}^{\hat{\otimes}n}$	$=\text{prlim}_{p\in\mathbb{N}}\,\mathcal{H}_p^{\hat{\otimes}n}$.
$\mathcal{N}^{*\hat{\otimes}n}$	$=\text{indlim}_{p\in\mathbb{N}}\,\mathcal{H}_{-p}^{\hat{\otimes}n}$.
$L^1(\mu)$	Space of integrable functions with a Gaussian measure μ.
$L^2(\mu)$	Space of square integrable functions with a Gaussian measure μ.
$\mathcal{L}_n(\mu)$	Subspace of $L^2(\mu)$ consisting of $I_n\left(f^{(n)}\right)$ with $f^{(n)}\in\mathcal{H}_{\mathbb{C}}^{\hat{\otimes}n}$.
$\mathcal{P}(\mathcal{N}^*)$	Smooth polynomials on $\mathcal{N}^*$.
$(\mathcal{H}_p)_q^\beta$	Completion of $\mathcal{P}(\mathcal{N}^*)$.
$(\mathcal{N})^\beta$	$=\cap_{p,q\geq 0}(\mathcal{H}_p)_q^\beta$.
$(\mathcal{N})^{-\beta}$	$=\cup_{p,q\geq 0}(\mathcal{H}_{-p})_{-q}^{-\beta}$.

We will now state the Minlos Theorem.

Definition C.1 (*Hida, Kuo, Potthoff and Striet, 1993*). *We let C be a function on $\mathcal{N}$ such that C is continuous on $\mathcal{N}$, C is positive definite and $C(0) = 1$, then C is a characteristic function on $\mathcal{N}$. Then there exist a unique probability measure μ_C on $(\mathcal{N}^*, \mathcal{B})$ such that for all $\xi \in \mathcal{N}$ we have*

$$\int_{\mathcal{N}^*} \exp(i\langle x, \xi\rangle)\, d\mu_C(x) = C(\xi). \tag{C.1}$$

If C is a continuous function with respect to the norm $|\cdot|_p$ for $p \in \mathbb{N}$ and if $n\,(> p)$ is such that the injection $\iota_n^m : \mathcal{N}_n \to \mathcal{N}_p$ is of Hilbert-Schmidt type, then $\mu_C(\mathcal{N}_{-n}) = 1$.

The sample paths of white noise are not functions but generalized functions. The topology structure of $\mathcal{S}(\mathbb{R})$ is one of nuclear countably Hilbert spaces. Gaussian spaces are the topological dual of a countably Hilbert space equipped with a Gaussian measure. We choose $\mathcal{N}$ to be the Schwartz space $\mathcal{S}(\mathbb{R})$ of test functions. The norm induced by the scalar products is denoted by $|\cdot|$ and the completion of $\mathcal{N}$ with respect to this norm is denoted by $\mathcal{H}$. We can choose $\mathcal{H}$ to be one of the spaces $\mathcal{N}_n$ (Hida, Kuo, Potthoff and Striet, 1993). Consider the characteristic function C on $\mathcal{N}$

$$C(\xi) = \exp\left(-\frac{1}{2}|\xi|^2\right). \tag{C.2}$$

Consider the probability space $(\mathcal{N}^*, \mathcal{B}, \mu)$, where μ is the Gaussian measure given by equation C.1 and equation C.2 We call $(\mathcal{N}^*, \mathcal{B}, \mu)$ the Gaussian space associated with $(\mathcal{N}, |\cdot|)$ and choosing $\mathcal{N}^* = \mathcal{S}'(\mathbb{R})$ we get the space

$$(\mathcal{S}'(\mathbb{R}), \mathcal{B}, \mu) \tag{C.3}$$

which is called the white noise space. For $d \in \mathbb{N}$ we let $\mathcal{N} = \mathcal{S}(\mathbb{R}^d)$ be the Schwartz space of rapidly decreasing $C^\infty-$functions on $\mathbb{R}^d$, endowed with it's usual topology.

$\mathcal{S}\left(\mathbb{R}^d\right)$ is countably Hilbert and it is nuclear. Let $\mathcal{H} = L^2\left(\mathbb{R}^d\right)$ be real with a Lebesgue measure on it. Then the space $\left(\mathcal{S}'\left(\mathbb{R}^d\right), \mathcal{B}, \mu\right)$ is called white noise with a d-dimensional time parameter. If we let $d = 1$ then we have white noise.

For $\xi \in \mathcal{N}$, we denote the random variable $x \longmapsto \langle x, \xi \rangle$ by X_ξ. The associated coordinate process X over $\mathcal{N}$ is given by $X_\xi(x) = \langle x, \xi \rangle$ which we will call white noise. The mapping $\xi \longmapsto X_\xi$ is called the canonical coordinate process over $\mathcal{N}$ and we let $\mathcal{P}$ denote the algebra of polynomials generated by X_ξ which are dense in (L^p). For $f \in \mathcal{H}$ the connection between white noise and Brownian motion is the coordinate process on $\mathcal{N}$ which has an extension to $\mathcal{H}$ such that this mapping is linear from $\mathcal{H}$ into $\cap_{p \geq 1}(L^p)$ and is continuous in every $\|\cdot\|_p$ for $p \geq 1$ which is denoted by $X : f \mapsto X_f(\cdot) = \langle \cdot, f \rangle$. The process is centered Gaussian with covariance

$$\int_{\mathcal{N}^*} \langle x, f \rangle \langle x, g \rangle \, d\mu(x) = (f, g).$$

Let 1_A be the indicator of $A \in \mathcal{B}(\mathbb{R})$. Then we form random variables as follows $\left\{X_{1_{[0,t)}} = \left\langle \cdot, 1_{[0,t)} \right\rangle, \ t \in \mathbb{R}_+\right\}$, which are centered Gaussian with the following covariance structure

$$\int_{\mathcal{S}'\left(\mathbb{R}^d\right)} \left\langle \cdot, 1_{[0,t)} \right\rangle \left\langle \cdot, 1_{[0,s)} \right\rangle d\mu(x) = \left(1_{[0,t)}, 1_{[0,s)}\right) = t \wedge s.$$

At time zero this family of random variables is a one dimensional Brownian motion starting at the origin.

The time derivative of Brownian motion is a distribution. For time $t > 0$ we can write Brownian motion as $B(\cdot) = \left\langle \cdot, 1_{[0,t)} \right\rangle$, and if we let $x(t)$ be the time derivative of Brownian motion then Brownian motion is represented as

$$B(t, x) = \int_0^t x(s) \, ds.$$

This again is a family of random variables which is centered Gaussian with covariance under the measure μ which is given as

$$E_\mu [B(t) B(s)] = t \wedge s.$$

Then we can define $B(t)$ as a Brownian motion.

C.3.1 $\mathcal{S}$-Transform

Consider the structure $(L^2) = L^2(\mathcal{N}^*, \mathcal{B}, \mu)$ then the Wiener-Itô decomposition theorem states that (L^2) has a direct sum decomposition into homogeneous chaos. In order to do so we will introduce the $\mathcal{T}$ transform first, it's sort of like the Fourier transform but not quite. The second transform is the $\mathcal{S}$ transform and both transforms are represented as elements in (L^2) in terms of Wick powers of distributions. The details of the $\mathcal{T}$ transform deals with complex numbers and will not be discussed here and for further details refer to Hida, Kuo, Potthoff and Striet (1993).

Definition C.2 (*Hida, Kuo, Potthoff and Striet* (1993)). *For $\xi \in \mathcal{N}$ and $\varphi \in (L^2)$ the $\mathcal{T}$ transformation on (L^2) is given by*

$$\mathcal{T}\varphi(\xi) = \int \exp(i \langle x, \xi \rangle) \varphi(x) \, d\mu(x).$$

$\mathcal{T}$ transforms functionals on $\mathcal{N}^*$ into an $\mathcal{N}$-functional and is linear. The image of (L^2) under $\mathcal{T}$ is a vector space of complex valued functions on $\mathcal{N}$ denoted by $\mathcal{R}$. $\mathcal{T}$ is isomorphic from (L^2) onto $\mathcal{R}$. The mapping $\lambda \to \mathcal{T}\varphi(\lambda\xi)$ has the entire analytic extension.

Definition C.3 (*Hida, Kuo, Potthoff and Striet* (1993)). *For $\varphi \in (L^2)$, $\xi \in \mathcal{N}$ and $x \in \mathcal{N}^*$ we set the $\mathcal{S}$ transformation on (L^2) as*

$$\mathcal{S}\varphi(\xi) = C(\xi) \mathcal{T}\varphi(-i\xi)$$

or

$$\mathcal{S}\varphi(\xi) = \int \varphi(x) : \exp(\langle x, \xi \rangle) : d\mu(x)$$

with

$$: \exp(\langle x, \xi \rangle) := \exp\left(\langle x, \xi \rangle - \frac{1}{2}|\xi|^2\right). \tag{C.4}$$

Equation $(C.4)$ is often referred to as a Wick exponentials and is often represented with the Wick exponential symbol $^{\diamond}$.

C.3.2 Chaos Expansion

For $n \in \mathbb{N}$ we define Hermite polynomials with u of order n and parameter $\sigma > 0$ as

$$: u^n :_{\sigma^2} = \left(-\sigma^2\right)^n \exp\left(\frac{u^2}{2\sigma^2}\right) \frac{d^n}{du^n} \exp\left(-\frac{u^2}{2\sigma^2}\right).$$

For the construction of a basis of (L^2) we let $f \in \mathcal{H}$ and we form random variables as

$$: \langle \cdot, f \rangle^n :=: \langle \cdot, f \rangle^n :_{|f|^2},$$

and we note that $: \langle x, \lambda\xi \rangle^n :$ and $: \langle \lambda x, \xi \rangle^n :$ are not the same. Let I^n be a set of naturally ordered n-tuples α in $\mathbb{N}^n$, $n_k(\alpha)$ is a number of entries in α equal to $k \in \mathbb{N}$ and $n(\alpha)! = \prod_{k=1}^{\infty} n_k(\alpha)!$ with the empty product equal to one. Let $(e_k, k \in \mathbb{N})$ be an orthonormal basis of $\mathcal{H}$ and let $n \in \mathbb{N}$, $\alpha \in I^n$. Set $H_\alpha(x)! = \prod_{k=1}^{\infty} : \langle x, e_k \rangle^{n_k(\alpha)} :$ and the collection of $\{H_\alpha, \alpha \in I, n \in \mathbb{N}_0\}$ forms an orthogonal basis of (L^2) with norm $\|H_\alpha\|_2^2 = n(\alpha)!$. For $\varphi \in (L^2)$ we have $\varphi(x) = \sum_{n=0}^{\infty} \varphi^{(n)}(x)$ and the chaos decomposition follows as

$$\varphi^{(n)}(x) = \sum_{\alpha \in I^n} a_\alpha H_\alpha(x).$$

C.4 Generalized Functions

Let $\hat{\otimes}$ be the symmetric tensor product (Hida, Kuo, Potthoff and Striet (1993)). Then we define Wick powers by

$$: x^{\otimes 0} := 1$$

$$: x^{\otimes 1} := x$$

$$: x^{\otimes n} :=: x^{\otimes n-1} : \hat{\otimes} x - (n-1) : x^{\otimes n-2} \hat{\otimes} Tr$$

where $Tr = \int \delta_t^{\otimes 2} dt$ and is an element of $\mathcal{T}$ and a distribution on $\mathcal{S}(\mathbb{R}^2)$. If $f \in \mathcal{S}(\mathbb{R}^2)$, then $\langle Tr, f \rangle = \int_{-\infty}^{\infty} f(t,t)\, dt$ see Hida, Kuo, Potthoff and Striet (1993). For $\xi \in \mathcal{N}$ and $x \in \mathcal{N}^*$ we have a Wick exponential of the form

$$: \exp(\langle x, \xi \rangle) := \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle : x^{\otimes n} :, \xi^{\otimes n} \right\rangle$$

see Grothaus, Kondratiev and Us (1998). For $n \in \mathbb{N}_0$ we let $x \mapsto : x^{\otimes n} :$ be a map from $\mathcal{N}^*$ into $\mathcal{N}^{*\hat{\otimes} n}$. For $\varphi^{(n)} \in \mathcal{N}^{*\hat{\otimes} n}$ we define smooth Wick monomials of order n corresponding to the kernel $\varphi^{(n)}$ as follows $I_n\left(\varphi^{(n)}\right)(x) = \left\langle : x^{\otimes n} :, \varphi^{(n)} \right\rangle$. Smooth Wick monomials are orthogonal with respect to the inner product. Let $f^{(n)} \in \mathcal{N}_{\mathbb{C}}^{*\hat{\otimes} n}$ then we let $\mathcal{L}_n(\mu)$ be a subspace of $L^2(\mu)$ consisting of $I_n\left(f^{(n)}\right)(x)$. Let

$$\mathcal{P}(\mathcal{N}^*) = \left\{ \varphi | \varphi(x) = \sum_{n=0}^{N} \left\langle : x^{\otimes n} :, \varphi^{(n)} \right\rangle, \varphi^{(n)} \in \mathcal{N}_{\mathbb{C}}^{*\hat{\otimes} n},\ x \in \mathcal{N}^*, N \in \mathbb{N}_0 \right\}$$

be a space which is dense in $L^2(\mu)$. Grothaus, Kondratiev and Us (1998) infer that for any $\varphi \in \mathcal{P}(\mathcal{N}^*)$ we have

$$L^2(\mu) = \bigoplus_{n=0}^{\infty} \mathcal{L}_n(\mu).$$

Definition C.4 (*Grothaus, Kondratiev and Us* (1998)). *For any* $f \in L^2(\mu)$ *and* $f^{(n)} \in \mathcal{N}_{\mathbb{C}}^{*\hat{\otimes} n}$ *we have the Itô-Segal-Wiener chaos decomposition*

$$f(x) = \sum_{n=0}^{\infty} \left\langle : x^{\otimes n} :, f^{(n)} \right\rangle.$$

To get generalized functions we have to construct a space of test functions. For $x \in \mathcal{N}^*$ consider Wick polynomials $f(x) = \sum_{n=0}^{N} \left\langle : x^{\otimes n} :, f^{(n)} \right\rangle$, with norm $\|f\|_{p,q,\beta}^2 = \sum_{n=0}^{\infty} (n!)^{(1+\beta)} 2^{nq} \left| f^{(n)} \right|_p^2$ and then we define a Hilbert space as follows

$$(\mathcal{H}_p)_q^\beta = \left\{ f \in L^2(\mu) \,|\, f(x) = \sum_{n=0}^{N} \left\langle : x^{\otimes n} :, f^{(n)} \right\rangle, \|f\|_{p,q,\beta}^2 < \infty \right\}.$$

The space of test functions $(\mathcal{N})^\beta$ is defined as the projective limit of the spaces $(\mathcal{H}_p)_q^\beta$ and $(\mathcal{N})^\beta = \cap_{p,q \geq 0} (\mathcal{H}_p)_q^\beta$. Let $(\mathcal{H}_{-p})_{-q}^{-\beta}$ be the dual with respect to $L^2(\mu)$ and let $(\mathcal{N})^{-\beta} = \cup_{p,q \geq 0} (\mathcal{H}_{-p})_{-q}^{-\beta}$. The space of Hida distributions is $(\mathcal{N})^* = (\mathcal{N})^{-0}$. Let $(\mathcal{N})^1$ be the largest space of generalized stochastic functions and the dual paring between $(\mathcal{N})^1$ and $(\mathcal{N})^{-1}$ be denoted by $\langle\langle \cdot, \cdot \rangle\rangle$.

Let $\Phi \in (\mathcal{N})^{-1}$ be a distribution such that $E_\mu(\Phi) = \langle\langle \Phi, 1 \rangle\rangle$. For $\varphi \in (\mathcal{N})^1$ consider $\Phi^{(n)} \in \mathcal{N}_{\mathbb{C}}^{*\hat{\otimes} n}$ then there exists a distribution $I_n\left(\Phi^{(n)}\right)$ that acts as test functions in the sense that we have a pairing

$$\left\langle\left\langle I_n\left(\Phi^{(n)}\right), \varphi \right\rangle\right\rangle = n! \left\langle \Phi^{(n)}, \varphi^{(n)} \right\rangle$$

see Grothaus, Kondratiev and Us (1998). Denote $I_n\left(\Phi^{(n)}\right) = \left\langle : x^{\otimes n} :, \Phi^{(n)} \right\rangle$ as distributions then for any $\Phi \in (\mathcal{N})^1$ we have the unique decomposition

$$\Phi = \sum_{n=0}^{\infty} \left\langle : x^{\otimes n} :, \Phi^{(n)} \right\rangle$$

if it converges in $(\mathcal{N})^1$. The dual pairing is given by

$$\langle\langle \Phi, \varphi \rangle\rangle = \sum_{n=0}^{\infty} n! \left\langle \Phi^{(n)}, \varphi^{(n)} \right\rangle.$$

Let $x \in \mathcal{N}^*$ at $\xi \in \mathcal{N}$ then the $\mathcal{S}$ transform of a generalized function is defined as

$$\begin{aligned} \mathcal{S}\Phi(\xi) &= \langle\langle \Phi, : \exp(\langle \cdot, \xi \rangle) : \rangle\rangle \\ &= \sum_{n=0}^{\infty} \left\langle \Phi^{(n)}, \xi^{\otimes n} \right\rangle. \end{aligned}$$

If we let $\Phi, \Psi \in (\mathcal{N})^{-1}$, then we define the Wick product by

$$\Phi \diamond \Psi = \mathcal{S}^{-1} (\mathcal{S}\Phi \cdot \mathcal{S}\Psi)$$

which is a element in $(\mathcal{N})^{-1}$.

We refer readers to theorem 5.1 for the algebraic properties of the Wick product.

Appendix D

Malliavin Derivative

Consider a Banach space $\Omega = C_0([0,T])$, where $C_0([0,T])$ is a space of continuous, real functions ω on $[0,T]$ such that $\omega(0) = 0$, then

$$\|\omega\|_\infty = \sup_{t\in[0,T]} |\omega(t)|$$

is a norm on Ω. We call Ω a Wiener space. Let $L^2([0,T])$ be the space of deterministic square integrable functions with respect to the Lebesgue measure $\lambda(dt) = dt$ on $[0,T]$. Let $g \in L^2([0,T])$ be a deterministic function and we put

$$\gamma(t) = \int_0^t g(s)\, ds \tag{D.1}$$

the integral is an element of Ω (see Øksendal, 1997).

Definition D.1 *Øksendal (1997). Let $F : \Omega \to \mathbb{R}$ be a random variable then we define the directional derivative of F at the point $\omega \in \Omega$ in the direction of $\gamma \in \Omega$ by*

$$D_\gamma F(\omega) = \frac{d}{d\varepsilon}[F(\omega + \varepsilon\gamma)]_{\varepsilon=0}$$

if it exists in the (strong) sense that

$$\mathbf{D}_\gamma F(\omega) = \lim_{\varepsilon\to 0} \frac{F(\omega+\varepsilon\gamma) - F(\omega)}{\varepsilon}$$

exists in $L^2(\Omega)$.

Definition D.2 *Øksendal (1997). Assume that* $F : \Omega \to \mathbb{R}$ *has a directional derivative in all directions* γ *of the form* $(D.1)$ *and there also exists* $\psi(t,\omega) \in L^2([0,T] \times \Omega)$ *such that* $\mathbf{D}_\gamma F(\omega)$ *can be written in the following form*

$$\mathbf{D}_\gamma F(\omega) = \int_0^T \psi(t,\omega)\, g(t)\, dt.$$

Then we say F *is differentiable and the derivative of* F *is given as*

$$\mathbf{D}_t F(\omega) = \psi(t,\omega).$$

Let $\mathcal{D}_{1,2}$ be a set of all differentiable random variables.

Example D.1 *Øksendal (1997). Let* Ω *be a Wiener space and consider a Wiener process (Brownian motion)* $W(t,\omega)$, *then for* $\omega \in C_0([0,T])$ *we set*

$$W(t,\omega) = \omega(t).$$

For $f(s) \in L^2([0,T])$ *let*

$$F(\omega) = \int_0^T f(s)\, dW(s,\omega) = \int_0^T f(s)\, d\omega(s)$$

be a stochastic process. For $\gamma \in \Omega$ *if*

$$\begin{aligned} \gamma(t) &= \int_0^t g(s)\, ds \\ &\Rightarrow d\gamma(t) = g(t)\, dt \end{aligned}$$

we have

$$\begin{aligned} F(\omega + \varepsilon\gamma) &= \int_0^T f(t)\,(d\omega(t) + \varepsilon d\gamma(t)) \\ &= \int_0^T f(t)\, d\omega(t) + \varepsilon \int_0^T f(t)\, d\gamma(t) \\ &= \int_0^T f(t)\, d\omega(t) + \varepsilon \int_0^T f(t)\, g(t)\, dt \end{aligned}$$

it follows that

$$\begin{aligned}\mathbf{D}_{\gamma}F(\omega) &= \lim_{\varepsilon\to 0}\frac{F(\omega+\varepsilon\gamma)-F(\omega)}{\varepsilon} \\ &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(\int_0^T f(t)\,d\omega(t)+\varepsilon\int_0^T f(t)\,g(t)\,dt-\int_0^T f(t)\,d\omega(t)\right) \\ &= \int_0^T f(t)\,g(t)\,dt.\end{aligned}$$

Then F *is differentiable and* $F\in\mathcal{D}_{1,2}$. *For* $t\in[0,T]$ *and* $\omega\in\Omega$ *the derivative is*

$$\mathbf{D}_t F(\omega)=f(t).$$

Let $\chi_{[0,t]}(s)$ *be a piecewise defined function of the form*

$$\chi_{[0,t]}(s)=\begin{cases}1 & \text{if } s\in[0,t]\\ 0 & \text{if } s\notin[0,t]\end{cases}.$$

Suppose $t_1\in[0,t]$ *we let*

$$f(t)=\chi_{[0,t_1]}(t)$$

then

$$\begin{aligned}F(\omega) &= \int_0^T \chi_{[0,t_1]}(s)\,dW(s) \\ &= \int_0^{t_1} 1dW(s)+\int_{t_1}^T 0dW(s) \\ &= W(t_1,\omega).\end{aligned}$$

The derivative of the Wiener process follows as

$$\mathbf{D}_t W(t_1,\omega)=\chi_{[0,t_1]}(t).$$

Let $\mathbb{P}$ denote the family of all random variables of the form

$$F(\omega)=\varphi(\theta_1,..,\theta_n)$$

where $\varphi(x_1,..,x_n)=\sum_{\alpha}a_{\alpha}x^{\alpha}$ is a polynomial in n variables $x_1,..,x_n$ and $\alpha=(\alpha_1,..,\alpha_n)$ a multi-index and $\theta_i=\int_0^T f_i(t)\,dW(t)$ for some deterministic $f_i\in$

$L^2([0,T])$. We call such random variables Wiener polynomials, $\mathbb{P}$ is dense in $L^2(\Omega)$ and $\mathbb{P} \subset \mathcal{D}_{1,2}$ for more details refer to Øksendal (1997). For $F \in \mathcal{D}_{1,2}$ we introduce a norm $\|F\|_{1,2} = \|F\|_{L^2(\Omega)} + \|\mathbf{D}_t F\|_{L^2([0,T]\times\Omega)}$ on $\mathcal{D}_{1,2}$.

Definition D.3 *Øksendal (1997). Define $\mathbb{D}_{1,2}$ to be the closure of the family $\mathbb{P}$ with respect to the norm $\|\cdot\|_{1,2}$.*

Definition D.4 *Øksendal (1997). Now we let $F \in \mathbb{D}_{1,2}$ and suppose there exists a sequence $\{F_n\} \subset \mathbb{P}$ such that $\{F_n\} \to F$ in $L^2(\Omega)$ and $\{\mathbf{D}_t F_n\}_{n=1}^{\infty}$ is convergent in $L^2([0,T]\times\Omega)$. Then we define*

$$D_t F = \lim_{n\to\infty} \mathbf{D}_t F_n$$

and

$$D_\gamma F = \int_0^T D_t F \cdot g(t)\, dt$$

for all

$$\gamma(t) = \int_0^t g(s)\, ds$$

where $g(s) \in L^2([0,T])$. We call $D_t F$ the Malliavin derivative of F.

Appendix E

Optimization Algorithm

Algorithm E.1 *(Rardin, 2000). Simulated Annealing Search.*

Step 0: Initialization. Choose any starting feasible solution $x^{(0)}$, an iteration limit $t_{\max}$, and a relatively large initial temperature $q > 0$. Then set incumbent solution $\hat{x} \leftarrow x^{(0)}$ and solution index $t \leftarrow 0$.

Step 1: Stopping. If no move Δx in move set $\mathcal{M}$ leads to a feasible neighbor of current solution $x^{(t)}$, or if $t = t_{\max}$, then stop. Incumbent solution $\hat{x}$ is an approximate optimum.

Step 2: Provisional Move. Randomly choose a feasible move $\Delta x \in \mathcal{M}$ as a provisional $\Delta x^{(t+1)}$, and compute (possibly negative) net objective function improvement Δobj for moving from $x^{(t)}$ to $\left(x^{(t)} + \Delta x^{(t+1)}\right)$ (increase for a maximize, decrease for a minimize).

Step 3: Acceptance. If $\Delta x^{(t+1)}$ improves, or with probability $e^{\Delta obj/q}$ if $\Delta obj \leq 0$, accept $\Delta x^{(t+1)}$ and update

$$x^{(t+1)} \leftarrow x^{(t)} + \Delta x^{(t+1)}.$$

Otherwise return to step 2.

Step 4: Incumbent Solution. If the objective function value of $x^{(t+1)}$ is superior to that of incumbent solution $\hat{x}$, replace $\hat{x} \leftarrow x^{(t+1)}$.

Step 5: Temperature Reduction. If a sufficient number of iterations have passed since the last temperature change, reduce temperature q.

Step 6: Increment. Increment $t \leftarrow t + 1$, and return to step 1.

Appendix F

MATLAB Code

F.1 Hurst Parameter

F.1.1 Log Returns of a Stock Prices

The following function calculates the log returns of stock prices.

```
function [w]=logreturns(x)
[q,l]=size(x);
y=zeros(q,1);
rt=zeros(q-1,1);
y=x;
 for i=1:q-1
 rt(i)=log(y(i+1))-log(y(i));        %log returns of stock prices
 end
w=rt;
```

F.1.2 Aggregated Variance Method

The following function estimates the Hurst parameter using the aggregated variance method.

```
function []=AggregatedVarianceMethod(stockdata)
x=logreturns(stockdata);
```

```
N=length(x);
varall=[];
nall=[];
prev=999;
for n=2:floor(N/5)        % n number of entries in each block
 b=floor(N/n);        % b number of blocks
 if prev~=b
 prev=b;
 nobs=b*n ;        % Number of observations
 Z=reshape(x(1:nobs),n,b) ;        % Block matrix
 aveZ=mean(Z);        % Average of the mean of each block
 aveR=mean(aveZ);        % Mean of the average series
 varR=var(aveZ);        % Variance
 varall=[varall varR];
 nall=[nall n];
 end
end
X=nall;
Y=varall;
logX=log(X);
logY=log(Y);
p=polyfit(logX,logY,1); %Regression
```

F.1.3 Absolute Moments Method

The following function estimates the Hurst parameter absolute moments method.

```
function []=AbsoluteMomentsMethod(stockdata)
```

```
x=logreturns(stockdata);
moment=1;
N=length(x);
Aall=[];
nall=[];
prev=999;
for n=2:floor(N/5);     %n number of entries in each block
 b=floor(N/n);     % b number of blocks
 if prev~=b
 prev=b;
 x=x-mean(x);
 nobs=b*n ;     % number of observations
 Z=reshape(x(1:nobs),n,b);     % block matrix
 aveZ=mean(Z);     % average of the mean of each block
 A=sum((abs(aveZ)).^moment)/b;
 Aall=[Aall A];
 nall=[nall n];
 end
end
X=nall;
 Y=Aall;
logX=log(X);
logY=log(Y);
p=polyfit(logX,logY,1);
HurstAbsoluteMoments=p(1)/moment+1
```

F.1.4 Higuchi Method

The following function estimates the Hurst parameter Higuchi method.

```
function []=HiguchiMethod(stockdata)
x=logreturns(stockdata);
N=length(x)      %N length of the series
Lall=[];
nall=[];
prev=999;
x=cumsum(x);
for n=2:floor(N/5)      % n number of entries in each block
 numb=floor(N/n);      % numb is the number of blocks
 if prev~=numb
 prev=numb;
 b=floor((N-n)/n);
 A=zeros(n,b);      % temp length
 for i=1:n
 for k=1:b
 A(i,k)=abs(x(i+k*n)-x(i+(k-1)*n)); % second sum of L(n)
 end
 end
 avgA=mean(A,2) ;      % mean of A in colunm vector form
 L=sum(avgA)*((N-1)/n^3);
 Lall=[Lall L];
 nall=[nall n];
 end
end
```

```
X=nall;
Y=Lall;
logX=log(X);
logY=log(Y);
p=polyfit(logX,logY,1);
HurstHiguchi=2+p(1)
```

F.1.5 Rescaled Range Method

The following function estimates the Hurst parameter rescaled range method.

```
function []=RescaledRangeMethod(stockdata)
x=logreturns(stockdata);
N=length(x);        %N length of the series
RSavgall=[];
nall=[];
prev=999;
for n=5:floor(N/5)       % n number of entries in each block
 b=floor(N/n) ;       % b number of blocks
 if prev~=b
 prev=b;
 nobs=b*n ;       % number of observations
 Z=reshape(x(1:nobs),n,b);       % block matrix
 aveZ=mean(Z);       % average of the mean of each block
 W=Z-ones(n,1)*aveZ;       % mean adjusted series W
 V=cumsum(W) ;       % series V is cumulative deviation from the mean
 R=max(V)-min(V);       % Range
 S=std(Z);       % The standard deviation
```

```
 RS=R./S ;       %Rescaled Range estimate
 RSavg=mean(RS);        % mean of R/S
 RSavgall=[RSavgall RSavg];
 nall=[nall n];
 end
end
X=nall;
Y=RSavgall;
logX=log(X);
logY=log(Y);
p=polyfit(logX,logY,1);
HurstRescaledRange=p(1)
```

F.2 Fractional Brownian Motion

The following function generates a fractional Brownian motion using equation (4.2) .

```
function []=FractionalBrownianMotion(H)
b=1000;        % b Lower bound for the 1st intergral
N=1000;        %N Sample number
b1=randn(b+1,1);
b2=randn(N+1,1);
ch=sqrt((2*H*gamma(3/2-H))/(gamma(H+1/2)*gamma(2-2*H)));   %Normal-
izing constant
bh=zeros(N,1);
for n=1:N
 x1=0;
```

```
for k=-b:-1
x1=x1+((n-k)^(H-0.5)-(-k)^(H-0.5))*b1(k+b+1);
end
x2=0;
for k=0:n-1
x2=x2+(n-k)^(H-0.5)*b2(k+1);
end
ch;
bh(n)=ch*(x1+x2);
end
bh;
plot (n),hold;
hold on;
plot(bh,'-r');
axis([0, N, -100, 100])
xlabel('Time')
ylabel('Fractional Brownian Motion')
title('Plot Fractional Brownian Motion');
end
```

F.3 Fractional Black-Scholes Functions

F.3.1 Fractional Call Price vs Spot

The following function calculates the fractional Black-Scholes European call price for Rostek and Schöbel's model for varying spot and plots the resulting values. Similar codes were written for Hu and Øksendal's and Necula's models.

```
function []=CallvstockRostek()
K=100;
r=0.02;
sigma=0.2;
T=2;
t=0;
H1=0.2
H2=0.8
c1all=[];
Sall=[];
c2all=[];
c3all=[]
for S=K-20:K+20
ph1=((sin(pi*(H1-0.5)))/(pi*(H1-0.5)))*((gamma((3/2-H1)^2))/(gamma(2-
2*H1)));
dH11=(log(S/K)+r*(T-t)+((ph1*sigma^2)/2)*(T-t)^(2*H1))/(sqrt(ph1)*sigma*(T-
t)^H1);
dH21=dH11-(sqrt(ph1)*sigma*(T-t)^H1);
ndH11=normcdf(dH11);
ndH21=normcdf(dH21);
c1=S*ndH11-K*exp(-r*(T-t))*ndH21;
ph2=((sin(pi*(H2-0.5)))/(pi*(H2-0.5)))*((gamma((3/2-H2)^2))/(gamma(2-
2*H2)));
dH12=(log(S/K)+r*(T-t)+((ph2*sigma^2)/2)*(T-t)^(2*H2))/(sqrt(ph2)*sigma*(T-
t)^H2);
dH22=dH12-(sqrt(ph2)*sigma*(T-t)^H2);
ndH12=normcdf(dH12);
```

```
ndH22=normcdf(dH22);
c2=S*ndH12-K*exp(-r*(T-t))*ndH22;
% calculates the black scholes call option price
d1=(log(S/K)+(r-sigma^2/2)*(T))/(sigma*sqrt(T)); %calculates the d1 value
d2=d1-sigma*sqrt(T);
nd1=normcdf(d1);
nd2=normcdf(d2); % normcdf is a built in function that calculates
% the value from the cumulative normal distribution
c3=S*nd1-K*exp(-r*(T))*nd2; % computes the call black scholes call price
c1all=[c1all c1];
c2all=[c2all c2];
c3all=[c3all c3];
Sall=[Sall S];
end
c1all;
c2all;
c3all;
Sall;
plot(Sall,c1all,'-y','LineWidth',2);
hold on
plot(Sall,c2all,'-g','LineWidth',2)
hold on
plot(Sall,c3all,'-r','LineWidth',2)
xlabel('Spot Price S '), ylabel('Call Price'), title('Price of Rostek European call
and T=2 and t=0');
```

F.3.2 Fractional Call Price vs Hurst

The following function calculates Necula's price of European call with varying Hurst and plots the resulting values. Similar codes were written for Hu and Øksendal's and Rostek and Schöbel's models.

```
function []=CallPricevsHurstNecula()
S=100;
K=100;
r=0.02;
sigma=0.2;
T=5
t1=1
pall1=[];
Hall1=[];
for H=1:1000
H=H/1000;
D1=((log(S/K)+(r)*(T-t1))+(sigma^2/2)*(T^(2*H)-t1^(2*H)))/(sigma*sqrt(T^(2*H)-
t1^(2*H)));
D2=((log(S/K)+(r)*(T-t1))-(sigma^2/2)*(T^(2*H)-t1^(2*H)))/(sigma*sqrt(T^(2*H)-
t1^(2*H)));
nD1=normcdf(D1);
nD2=normcdf(D2);
p=S*nD1-K*exp(-r*(T-t1))*nD2;
pall1=[pall1 p];
Hall1=[Hall1 H];
end
```

```
t2=2.5
pall2=[];
Hall2=[];
for H=1:1000
H=H/1000;
D1=((log(S/K)+(r)*(T-t2))+(sigma^2/2)*(T^(2*H)-t2^(2*H)))/(sigma*sqrt(T^(2*H)-
t2^(2*H)));
D2=((log(S/K)+(r)*(T-t2))-(sigma^2/2)*(T^(2*H)-t2^(2*H)))/(sigma*sqrt(T^(2*H)-
t2^(2*H)));
nD1=normcdf(D1);
nD2=normcdf(D2);
p=S*nD1-K*exp(-r*(T-t2))*nD2;
pall2=[pall2 p];
Hall2=[Hall2 H];
end
t3=4
pall3=[];
Hall3=[];
for H=1:1000
H=H/1000;
D1=((log(S/K)+(r)*(T-t3))+(sigma^2/2)*(T^(2*H)-t3^(2*H)))/(sigma*sqrt(T^(2*H)-
t3^(2*H)));
D2=((log(S/K)+(r)*(T-t3))-(sigma^2/2)*(T^(2*H)-t3^(2*H)))/(sigma*sqrt(T^(2*H)-
t3^(2*H)));
nD1=normcdf(D1);
nD2=normcdf(D2);
p=S*nD1-K*exp(-r*(T-t3))*nD2;
```

```
pall3=[pall3 p];
Hall3=[Hall3 H];
end
plot(Hall1,pall1,'-b');
hold on;
plot(Hall2,pall2,'-r');
hold on;
plot(Hall3,pall3,'-g');
xlabel('Hurst Parameter H'),ylabel('Price Fractional European Call Necula'),title('Call Price for fixed T=5 and t=1, t=2.5 and t=4')
```

F.3.3 Fractional Call Price vs Hurst vs Volatility

The following function calculates the conjectured Hu and Øksendal's fractional Black price for varying Hurst and varying volatility and plots the 3D graph. Similar codes were written for Necula's and Rostek and Schöbel's fractional Black models.

```
% On main screen
[X,Y] = meshgrid([0.05:0.01:0.9],[0.51:0.01:1]);
Z=CallpricevsHurstvsVolatilityHuandQksendal(110,100,100,1,0.25,X,Y);
mesh(X,Y,Z)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%


function [c]=CallpricevsHurstvsVolatilityHuandQksendal(Ft,St,K,T,t,sigma,H)
format long g
r=log(Ft/St)/(T-t);
out1=[];
```

```
[m1,n1] = size(H);
c=ones(m1,n1)*999;
for i=1:m1
for j=1:n1
dH1=(log(Ft/K)+(sigma(i,j)^2/2)*(T-t)^(2*H(i,j)))/(sigma(i,j)*sqrt((T-t)^(2*H(i,j))));
dH2=(log(Ft/K)-(sigma(i,j)^2/2)*(T-t)^(2*H(i,j)))/(sigma(i,j)*sqrt((T-t)^(2*H(i,j))));
ndH1=normcdf(dH1);
ndH2=normcdf(dH2);
c(i,j)=exp(-r*(T-t))*(Ft*ndH1-K*ndH2); %Fractional Future Call
out=[sigma(i,j), H(i,j), c(i,j)];
out1 = (cat(1,out1, out));
end
end
```

F.4 Implied volatility

F.4.1 Black-Scholes Formula

The following function calculates the price of the classical Black-Scholes option price.

```
function [x]= BScallputOption(callput,St,K,r,sigma,t,T)
%call=1 and put=0
T1=(T-t);
if callput
 d1=(log(St/K)+(r+sigma^2/2)*(T1))/(sigma*sqrt(T1));
 d2=d1-sigma*sqrt(T1);
 nd1=normcdf(d1);
 nd2=normcdf(d2) ;
 x=St*nd1-K*exp(-r*(T1))*nd2;
```

```
end
if ~callput
 d1=(log(St/K)+(r+sigma^2/2)*(T1))/(sigma*sqrt(T1));
 d2=d1-sigma*sqrt(T1);
 nd1=normcdf(-d1);
 nd2=normcdf(-d2);
 x=K*exp(-r*(T1))*nd2-St*nd1;
end
end
```

F.4.2 Backing Out Implied Volatility

The following function obtains the implied volatilities for individual options given all the other inputs using the method in Benninga (2000).

```
function []=callvolatilitytarget()
format long
callput=1;
K=28500.00;
T=40970.00/365;
St=[27753
 27180
 26554
 ];
r=[5.400550548
 5.40206233
 5.398904459
 ];
```

```
t=[40787.00
 40788.00
 40791.00
 ]/365;
target=[2050
 1700
 1450
 ];
[n m]=size(target);
error=999*ones(n,1);
callvolatility=999*ones(n,1);
for i=1:n
 high=1;
 low=0;
 while (high-low)>0.0001
 if BScallputOption(callput,St(i),K,r(i)/100,((high+low)/2),t(i),T)>target(i)
 high=(high+low)/2;
 else
 low=(high+low)/2;
 end
 end
 callvolatility(i)=(high+low)/2;
 error(i)=BScallputOption(callput,St(i),K,r(i)/100,callvolatility(i),t(i),T)-target(i);
end
callvolatility
error
end
```

F.5 Implied Fractional Volatilities

The following code uses the simulated annealing algorithm by Vandekerckhove's (2006) to obtain the fractional implied volatility for a collection of options given all the other inputs. The Black model for Rostek and Schöbel code is presented. Similar codes are written for the classical Black-Scholes, Black, Hu and Øksendal and Necula's models.

F.5.1 External Hurst Loop

```
numberofdays=3;
    numiter=3;
    options.Verbosity=0;
    error=@BlackRostekErrorFunction;
    ErrorRostek=99*ones(numberofdays,7);
    ImpliedRostek=99*ones(numberofdays,7);
    for k=1:numberofdays
     for j=1:7
     loss=@(p)error(p(1),j,k);
     x1=99*ones(numiter,1);
     f1=99*ones(numiter,1);
     for i=1:numiter;
     vol(i)=rand()+0.005;
     [x f]=anneal(loss,[vol(i)],options);
     x1(i,1)=x;
     f1(i)=f;
     end
```

```
 time=k
 C = cat(2,f1,x1);
 C1= sort(C,1);
 Error=C1(1,1);
 ImpliedVol=C1(1,2);
 ErrorRostek(k,j)=Error;
 ImpliedRostek(k,j)=ImpliedVol;
 end
end
ErrorRostek1= ErrorRostek(:,1:3)
ImpliedRostek1=ImpliedRostek(:,1:3)
ErrorRostek2= ErrorRostek(:,4:6)
ImpliedRostek2=ImpliedRostek(:,4:6)
ErrorRostek3= ErrorRostek(:,7)
ImpliedRostek3=ImpliedRostek(:,7)
```

F.5.2 Error Function

```
function e2=BlackRostekErrorFunction(vol,index,time)
    AllHurst=[0.5043
     0.51426
     0.54736
     0.6
     0.7
     0.8
     0.9];
    Hurst=AllHurst(index);
```

```
Ft=[29653
 29629
 29628];
St=[29457
 29457
 29402];
t=[40654.00
 40659.00
 40661.00]/365;
T=40709.00/365;
Opt=[1
 1
 1
 ];
sigma=[ 0.2544 0.2256 0.2201
 0.2544 0.2256 0.2201
 0.2543 0.2256 0.2201
 ];
K=[25000
 27000
 27400];
r(time)=log(Ft(time)/St(time))/(T-t(time));
[n n2]=size(K);
if vol <= 0
 e2=10^15;
else
 error = 0;
```

```
vol;
for i =1:n
d1=(log(Ft(time)/K(i))+(sigma(time,i)^2/2)*(T-t(time)))/(sigma(time,i)*sqrt(T-t(time))) ;
d2=d1-sigma(time,i)*sqrt(T-t(time));
nd1=normcdf(d1);
nd2=normcdf(d2);
nd1p=normcdf(-d1);
nd2p=normcdf(-d2);
ph=((sin(pi*(Hurst-0.5)))/(pi*(Hurst-0.5)))*(gamma((3/2-Hurst)^2))/(gamma(2-2*Hurst));
dH21=(log(Ft(time)/K(i))+((ph*vol^2)/2)*(T-t(time))^(2*Hurst))/(sqrt(ph)*vol*(T-t(time))^Hurst);
dH22=dH21-(sqrt((ph))*vol*(T-t(time))^Hurst);
ndH21=normcdf(dH21);
ndH22=normcdf(dH22);
ndH21p=normcdf(-dH21);
ndH22p=normcdf(-dH22);
if Opt(i)
black(i)=exp(-r(time)*(T-t(time)))*(Ft(time)*nd1-K(i)*nd2);
x2(i)=exp(-r(time)*(T-t(time)))*(Ft(time)*ndH21-K(i)*ndH22);
end;
if ~Opt(i)
black(i)=exp(-r(time)*(T-t(time)))*(K(i)*nd2p-Ft(time)*nd1p);
x2(i)=exp(-r(time)*(T-t(time)))*(K(i)*ndH22p-Ft(time)*ndH21p);
end;
d2=x2(i)-black(i);
```

```
 error=error+(x2(i) - black(i))^2;
 end
 e2=error;
end
end
```

F.6 Out-of-Sample Pricing

The following code calculates the percentage pricing errors and the absolute pricing errors for a collection of ALSI calls on futures.

```
function []=OutofsampleALSIcollection()
format long g
t0=40634.00/365;
T=40709.00/365; % Date of expiration 15 June 2011
time =[ 40637.00      40638.00      40639.00]/365;
T=T-t0;
time=time-t0;
Ft = [29531
29706
29882
]; % Future vector
St=[29235
29422
29655
]; % Spot Prices
K =[25000
```

```
27000
27400
]; %Strike Price
Opt=[1
1
1]; % Option type: 1=Call, 0=Put
sigma=[0.2714    0.2607    0.2606
0.2433    0.2326    0.2326
0.238    0.2273    0.2273]; %Black Volatility Starting date 04 April 2011
impliedsigma=[0.236001725
0.237491498
0.226252265
]; %Black Volatility Starting date 01 April 2011
Hurst=0.5043; % Hurst Parameter
impliedsigma1=[0.2376
0.2392
0.2278 ]; % Implied Volatility Hu and Qksendal Starting date 01 April 2011
impliedsigma2=[0.237550633
0.239013246
0.227646677
]; % Implied Volatility Necula Starting date 01 April 2011
impliedsigma3=[0.23761491
0.239190639
0.227796687
]; % Implied Volatility Rostek and Schobel Starting date 01 April 2011
%Initializing%%%%%%%%%%%%%%%%%%%%%%%%%%%%
[n n1]=size(Opt) ; % n number of options,
```

```
[t1 t]=size(time) ; % number of time step
for j=1:t
r(j)=log(Ft(j)/St(j))/(T-time(j)); % vector of interest rates
end
Black=999*ones(n,t);
modelHu1=999*ones(n,t);
modelNecula1=999*ones(n,t);
modelRostek1=999*ones(n,t);
impliedBlack=999*ones(n,t);
error = 999*ones(n,t);
errorpercent=999*ones(n,t);
error1=999*ones(n,t);
error2=999*ones(n,t);
error3=999*ones(n,t);
errorpercent1=999*ones(n,t);
errorpercent2=999*ones(n,t);
errorpercent3=999*ones(n,t);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Black
function [P] = modelBlack(Ft1,r1,K1,T1,time1,sigma1,Opt1)
d1=(log(Ft1/K1)+(sigma1^2/2)*(T1-time1))/(sigma1*sqrt(T1-time1));
d2=(log(Ft1/K1)-(sigma1^2/2)*(T1-time1))/(sigma1*sqrt(T1-time1));
nd1c=normcdf(d1);
nd2c=normcdf(d2);
nd1p=normcdf(-d1);
nd2p=normcdf(-d2);
if Opt1
```

```
P=exp(-r1*(T1-time1))*(Ft1*nd1c-K1*nd2c);
end
if ~Opt1
P=exp(-r1*(T1-time1))*(K1*nd2p-Ft1*nd1p);
end
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Hu and Qksendal
function [P] = modelHu(Ft1,r1,K1,T1,time1,sigma1,H1,Opt1)
dH11=(log(Ft1/K1)+(sigma1^2/2)*(T1-time1)^(2*H1))/(sigma1*sqrt(T1-time1)^(2*H1));
dH12=(log(Ft1/K1)-(sigma1^2/2)*(T1-time1)^(2*H1))/(sigma1*sqrt(T1-time1)^(2*H1));
ndH11c=normcdf(dH11);
ndH12c=normcdf(dH12);
ndH11p=normcdf(-dH11);
ndH12p=normcdf(-dH12);
if Opt1
P=exp(-r1*(T1-time1))*(Ft1*ndH11c-K1*ndH12c);
end
if ~Opt1
P=exp(-r1*(T1-time1))*(K1*ndH12p-Ft1*ndH11p);
end
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Necula
function [P] = modelNecula(Ft1,r1,K1,T1,time1,sigma1,H1,Opt1)
dH11=(log(Ft1/K1)+(sigma1^2/2)*(T1^(2*H1)-time1^(2*H1)))/(sigma1*sqrt(T1^(2*H1)-
time1^(2*H1)));
```

```
dH12=(log(Ft1/K1)-(sigma1^2/2)*(T1^(2*H1)-time1^(2*H1)))/(sigma1*sqrt(T1^(2*H1)-
time1^(2*H1)));
ndH11c=normcdf(dH11);
ndH12c=normcdf(dH12);
ndH11p=normcdf(-dH11);
ndH12p=normcdf(-dH12);
if Opt1
P=exp(-r1*(T1-time1))*(Ft1*ndH11c-K1*ndH12c);
end
if ~Opt1
P=exp(-r1*(T1-time1))*(K1*ndH12p-Ft1*ndH11p);
end
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Rostek
function [P] = modelRostek(Ft1,r1,K1,T1,time1,sigma1,H1,Opt1)
ph=((sin(pi*(H1-0.5)))/(pi*(H1-0.5)))*(gamma((3/2-H1)^2))/(gamma(2-2*H1));
dH21=(log(Ft1/K1)+((ph*sigma1^2)/2)*(T1-time1)^(2*H1))/(sqrt(ph)*sigma1*(T1-
time1)^H1);
dH22=(log(Ft1/K1)-((ph*sigma1^2)/2)*(T1-time1)^(2*H1))/(sqrt(ph)*sigma1*(T1-
time1)^H1);
ndH21c=normcdf(dH21);
ndH22c=normcdf(dH22);
ndH21p=normcdf(-dH21);
ndH22p=normcdf(-dH22);
if Opt1
P=exp(-r1*(T1-time1))*(Ft1*ndH21c-K1*ndH22c);
```

```
end
if ~Opt1
P=exp(-r1*(T1-time1))*(K1*ndH22p-Ft1*ndH21p);
end;
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Black
for i=1:n
for j=1:t
Black(i,j)=modelBlack(Ft(j),r(j),K(i),T,time(j),sigma(i,j),Opt(i));
impliedBlack(i,j)=modelBlack(Ft(j),r(j),K(i),T,time(j),impliedsigma(j),Opt(i));
modelHu1(i,j)=modelHu(Ft(j),r(j),K(i),T,time(j),impliedsigma1(j),Hurst,Opt(i));
modelNecula1(i,j)=modelNecula(Ft(j),r(j),K(i),T,time(j),impliedsigma2(j),Hurst,Opt(i));
modelRostek1(i,j)=modelRostek(Ft(j),r(j),K(i),T,time(j),impliedsigma3(j),Hurst,Opt(i));
error(i,j)=abs(Black(i,j)-impliedBlack(i,j));
errorpercent(i,j)=(Black(i,j)-impliedBlack(i,j))/Black(i,j);
error1(i,j)=abs(Black(i,j)-modelHu1(i,j));
errorpercent1(i,j)=(Black(i,j)-modelHu1(i,j))/Black(i,j);
error2(i,j)=abs(Black(i,j)-modelNecula1(i,j));
errorpercent2(i,j)=(Black(i,j)-modelNecula1(i,j))/Black(i,j);
error3(i,j)=abs(Black(i,j)-modelRostek1(i,j));
errorpercent3(i,j)=(Black(i,j)-modelRostek1(i,j))/Black(i,j);
end
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
time=t
numberofoptions=n
```

```
errorbyoption = 1/t*sum(error,2)
errorbyoptionpercent = 1/t*sum(errorpercent,2)*100
errorbyday = 1/n*sum(error,1)'
errorbydaypercent = 1/n*sum(errorpercent,1)'*100
aveabserror=1/t*1/n*sum(sum(error,2))
avepercenterror=1/t*1/n*sum(sum(errorpercent,2))*100
errorbyoption1 = 1/t*sum(error1,2)
errorbyoptionpercent1 = 1/t*sum(errorpercent1,2)*100
errorbyday1 = 1/n*sum(error1,1)'
errorbydaypercent1 = 1/n*sum(errorpercent1,1)'*100
aveabserror1=1/t*1/n*sum(sum(error1,2))
avepercenterror1=1/t*1/n*sum(sum(errorpercent1,2))*100
errorbyoption2 = 1/t*sum(error2,2)
errorbyoptionpercent2 = 1/t*sum(errorpercent2,2)*100
errorbyday2 = 1/n*sum(error2,1)'
errorbydaypercent2 = 1/n*sum(errorpercent2,1)'*100
aveabserror2=1/t*1/n*sum(sum(error2,2))
avepercenterror2=1/t*1/n*sum(sum(errorpercent2,2))*100
errorbyoption3 = 1/t*sum(error3,2)
errorbyoptionpercent3 = 1/t*sum(errorpercent3,2)*100
errorbyday3 = 1/n*sum(error3,1)'
errorbydaypercent3 = 1/n*sum(errorpercent3,1)'*100
aveabserror3=1/t*1/n*sum(sum(error3,2))
avepercenterror3=1/t*1/n*sum(sum(errorpercent3,2))*100
Black=Black'
ImpliedBlack=impliedBlack'
HuandQksendal=modelHu1'
```

```
Necula=modelNecula1'
Rostek=modelRostek1'
allaveabserror=[aveabserror;aveabserror1;aveabserror2;aveabserror3]
allavepercenterror=[avepercenterror;avepercenterror1;avepercenterror2;avepercenterror3]
end
```

Appendix G

ALSI Calls on Futures

The following tables give the pricing errors and percentage pricing errors by option and by day for ALSI calls on futures.

Table G.1: ALSI calls on futures. Pricing errors and percentage pricing errors by option.

	ALSI Futures Calls Expiring 2011/06/15			
	Pricing Errors using all Calls			
	Hurst	0.54736		
	Number of Options	21	Number of Days	38
	Implied Black	**Hu and Øksendal**	**Necula**	**Rostek and Schöbel**
Strike	**Error by Option**			
25000	36.01267	36.11486	35.91045	36.11514
27000	64.95573	65.63267	64.48794	65.62453
27400	63.62436	64.49102	63.05681	64.47999
28000	52.71764	53.86132	52.01139	53.84689
28200	46.66848	47.89045	45.92311	47.87538
28500	36.04474	37.26003	35.29596	37.25149
28700	29.12576	30.25905	28.37196	30.25621
28750	27.53241	28.59683	26.81399	28.59288
29000	21.68534	22.28707	21.12443	22.29079
29100	19.91391	20.3756	19.41607	20.37941
29250	18.89785	18.73814	18.52373	18.7682
29500	18.26538	17.98747	17.90207	18.01847
30000	29.66293	28.56175	30.40122	28.56626
30500	42.68532	41.68379	43.43167	41.68722
30600	44.45484	43.49494	45.18513	43.49763
30850	47.73538	46.87893	48.42272	46.87994
31000	48.9562	48.16019	49.61632	48.16033
31500	49.33508	48.72377	49.90046	48.72172
31550	49.11158	48.51701	49.66736	48.51481
31600	48.88394	48.30578	49.43014	48.30344
31700	48.26111	47.71474	48.78821	47.71215
Strike	**Error by Option Percent**			
25000	0.859319	0.861879	0.856939	0.861876
27000	2.830676	2.863563	2.811298	2.862936
27400	3.249189	3.29965	3.221788	3.298608
28000	3.503195	3.594615	3.458867	3.592596
28200	3.393217	3.503388	3.341647	3.500921
28500	2.919394	3.063804	2.85513	3.060516
28700	2.340962	2.512999	2.266834	2.509044
28750	2.159108	2.338741	2.082318	2.334601
29000	0.928315	1.150765	0.836847	1.145575
29100	0.280381	0.522479	0.182351	0.5168
29250	-0.907062	-0.63232	-1.015804	-0.638823
29500	-3.553078	-3.213689	-3.682372	-3.221869
30000	-12.41393	-11.88965	-12.59818	-11.90296
30500	-29.27329	-28.42667	-29.54399	-28.44991
30600	-34.23136	-33.29079	-34.52562	-33.31716
30850	-50.20576	-48.95951	-50.57345	-48.99681
31000	-63.16245	-61.66309	-63.5879	-61.71015
31500	-143.4532	-140.3041	-144.2216	-140.4287
31550	-157.0701	-153.6307	-157.8955	-153.7711
31600	-172.6236	-168.8491	-173.5144	-169.0082
31700	-209.3078	-204.7288	-210.3539	-204.9352

Table G.2: ALSI calls on futures. Pricing errors by day.

	Pricing Errors using all Calls			
	Hurst	0.54736		
	Number of Options	21	Number of Days	38
	Implied Black	**Hu and Øksendal**	**Necula**	**Rostek and Schöbel**
Date	**Error by Day**			
2011/04/04	68.77634	68.97466	68.18412	68.97545
2011/04/05	72.59072	72.35284	73.0144	72.34428
2011/04/06	65.5511	65.56487	65.47979	65.55926
2011/04/07	64.76699	64.82036	64.68712	64.849
2011/04/08	63.91944	63.9451	63.86537	63.93905
2011/04/11	62.06702	62.14758	61.92232	62.14111
2011/04/12	61.90386	61.99182	61.82954	62.03203
2011/04/13	60.74477	60.83851	60.62661	60.82633
2011/04/14	60.38515	60.46626	60.31019	60.50393
2011/04/15	58.25169	58.35069	58.15867	58.33713
2011/04/18	61.6008	62.42296	60.96875	62.39794
2011/04/19	55.18862	55.02577	55.2784	55.00793
2011/04/20	71.2158	70.82573	71.57203	70.80743
2011/04/21	53.75487	53.83852	53.72615	53.82646
2011/04/26	50.15676	50.45536	49.96142	50.46031
2011/04/28	48.25558	48.37009	48.18401	48.37489
2011/04/29	47.90571	47.94815	47.87586	47.94902
2011/05/03	44.57582	45.11029	44.29834	45.11946
2011/05/04	36.499	36.57359	36.45632	36.5623
2011/05/05	32.75306	32.71164	32.7794	32.7085
2011/05/06	35.82445	35.68014	35.86799	35.70513
2011/05/09	30.01239	30.37821	29.87623	30.35883
2011/05/10	33.59632	33.24219	33.7049	33.25505
2011/05/11	27.503	27.55267	27.49064	27.56062
2011/05/12	24.33054	24.45423	24.28939	24.42849
2011/05/13	25.55922	25.44037	25.58326	25.45029
2011/05/16	24.36696	24.2906	24.41548	24.28954
2011/05/17	21.31996	21.47541	21.31624	21.46202
2011/05/19	20.37751	20.21946	20.41893	20.21345
2011/05/20	18.53237	18.59048	18.51425	18.57441
2011/05/23	13.09957	13.3509	13.10546	13.34022
2011/05/24	25.27752	24.91135	25.35555	24.95867
2011/05/25	15.80214	15.53892	15.81181	15.55249
2011/05/26	16.15567	15.96067	16.18269	15.95701
2011/05/27	17.1595	17.32434	17.15279	17.31871
2011/05/30	13.30812	13.492	13.28332	13.49334
2011/05/31	12.86447	12.8712	12.87165	12.87065
2011/06/01	12.24551	12.50851	12.24176	12.51924

Table G.3: ALSI calls on futures. Percentage pricing errors by day.

	ALSI Futures Calls Expiring 2011/06/15			
	Pricing Errors using all Calls			
	Hurst	0.54736		
	Number of Options	21	Number of Days	38
	Implied Black	**Hu and Øksendal**	**Necula**	**Rostek and Schöbel**
Date	**Error by Day Percent**			
2011/04/04	-6.363026	-6.115722	-7.10233	-6.114736
2011/04/05	-14.16887	-14.07097	-14.3433	-14.06745
2011/04/06	-6.344509	-6.309183	-6.527173	-6.32359
2011/04/07	-7.53055	-7.449654	-7.651648	-7.406233
2011/04/08	-6.66437	-6.600046	-6.799719	-6.61522
2011/04/11	-6.169359	-5.960521	-6.543416	-5.977311
2011/04/12	-9.570245	-9.45739	-9.665658	-9.405825
2011/04/13	-10.31622	-10.18965	-10.47586	-10.20609
2011/04/14	-9.840376	-9.725023	-9.947035	-9.671475
2011/04/15	-12.12046	-11.99959	-12.28027	-12.01614
2011/04/18	-8.194898	-7.54692	-8.694988	-7.566596
2011/04/19	-24.31735	-24.09557	-24.4396	-24.07127
2011/04/20	-24.38584	-24.25311	-24.5071	-24.24688
2011/04/21	-10.98798	-10.80627	-11.05038	-10.83245
2011/04/26	-10.44823	-9.779205	-10.88681	-9.768119
2011/04/28	-10.89241	-10.6266	-11.05871	-10.61548
2011/04/29	-9.971308	-9.84968	-10.05687	-9.847195
2011/05/03	-12.84156	-12.02261	-13.26892	-12.00862
2011/05/04	-28.98271	-28.71955	-29.19918	-28.74646
2011/05/05	-42.35852	-41.94292	-42.62219	-41.91141
2011/05/06	-35.15258	-34.69008	-35.29209	-34.77018
2011/05/09	-40.22352	-38.62156	-40.82586	-38.70585
2011/05/10	-33.79558	-33.38618	-33.92121	-33.40104
2011/05/11	-26.60513	-26.22398	-26.70034	-26.16326
2011/05/12	-33.66914	-33.12522	-33.85077	-33.23816
2011/05/13	-34.08493	-33.50803	-34.20164	-33.5562
2011/05/16	-28.42085	-27.05834	-28.81667	-27.07094
2011/05/17	-34.64547	-33.87117	-34.66408	-33.93764
2011/05/19	-43.01899	-41.69013	-43.36693	-41.63956
2011/05/20	-40.54431	-39.75266	-40.84919	-39.84308
2011/05/23	-86.27485	-80.3793	-87.39906	-80.60212
2011/05/24	-408.2131	-401.9121	-409.5614	-402.7239
2011/05/25	-114.5225	-111.9713	-114.6165	-112.1024
2011/05/26	-92.61958	-90.79988	-92.87243	-90.76587
2011/05/27	-42.88075	-41.74368	-42.92728	-41.78237
2011/05/30	-60.26737	-55.70863	-60.89846	-55.67618
2011/05/31	-57.79806	-55.97946	-57.89294	-56.12331
2011/06/01	-59.65338	-57.35839	-59.68639	-57.26548

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