UNIVERSITEIT VAN PRETORIA
UNIVERSITY OF PRETORIA
YUNIBESITHI YA PRETORIA

Theory and applications of modern statistical quality control

by

Schalk William Human

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Summary

There is a large number of control chart procedures for variables data as well as attributes data In this study we focus on univariate parametric and distribution free control chart procedures for variables data Primarily we focus on the parametric Shewhart $\overline{X}$ control chart as well as the parametric one sided and two sided tabular cumulative sum (CUSUM) control chart and the parametric exponentially weighted moving average (EWMA) control chart This is then followed by a discussion of some distribution free control chart procedures

However Chapter 2 gives a brief overview and discussion of some of the basic principles and ideas involved in statistical process control (SPC) which serves as an introduction and facilitates the understanding of the fundamental concepts needed for later chapters For example we have a look at the connection between a Shewhart type of control chart and hypothesis testing the choice of control limits the performance and statistical design of control charts and additional criteria for sensitizing a control chart

Chapters 3 focuses on the parametric Shewhart $\overline{X}$ control chart followed by the parametric CUSUM and EWMA control chart procedures in Chapter 4 For each of these control chart procedures exact expressions are derived for both the conditional as well as the unconditional in control and out of control run length distribution In addition we provide exact expressions for many of the performance measures of these control charts such as the average run length (ARL) the variance and/or standard deviation of the run length (VARRL and/or SDRL) the coefficient of skewness of the run length (SKEWRL) as well as the median run length (MDRL) This is done when both one or none of the process parameters i e the process mean and the process standard deviation are known

Chapter 5 looks at a Shewhart type of distribution free procedure followed by four procedures based either on the sign test statistic or the Wilcoxon sign ranked statistic

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Chapter 1 Introduction

1 0 Introduction

A primary goal of statistical process control (SPC) is to differentiate between two distinct sources of process variation those that cannot be economically identified and corrected (chance causes or common causes) and those that can be (assignable causes or special causes) When a process operates only in the presence of common causes it is said to be in a state of statistical control In this regard control charts help researchers for example operator personnel to identify and eliminate assignable or special causes so that one can ensure a state of statistical control If it happens that the process undergoes a change for instance if the process mean or the process variation changes a control chart should detect the change as quickly as possible and give an out of control signal Obviously the faster the detection and the subsequent signal the more efficient the control chart will be

The number of samples or subgroups (or as is sometimes the case individual observations) that need to be collected or observed before the first out of control signal is expected to be given by a control chart is a random variable called the run length random variable Of particular interest to us is the distribution of the run length since this is traditionally used to characterize the performance of a control chart

A popular measure of control chart performance is the expected value or the mean of the run length distribution typically referred to as the average run length (ARL) Some other performance measures are the probability of a signal also called the false alarm rate (FAR) if the process is actually in control and the probability of no signal typically viewed as a function of the size of the process shift and generally called the operating characteristic curve or simply the OC curve However examining the entire run length distribution and/or other characteristics such as the variance (VARRL) is also suggested Since by definition the run length is a positive integer valued random variable the ARL loses much of its attractiveness as a typical summary of a control charts performance if the run length distribution is skewed – as is often the case As a consequence other (location) measures such as the median run length (MDRL) and/or other percentiles are sometimes considered

In this study we concentrate on univariate variables control charts for detecting a change (both small and large) in the location or shift parameter of a process distribution – typically taken to be the process mean More specifically we study the performance of the well known and classical Shewhart $\overline{X}$ (read X bar) control chart the cumulative sum (or CUSUM) control chart and the exponentially weighted moving average (or EWMA) control chart procedures by examining their in control and out of control run length distributions

However the classical application of these control charts for monitoring the process mean requires the assumption that the process mean and the process variance (or standard deviation) are known But the mean and/or the variance are typically unknown and in practice these process parameters are often estimated from an in control reference sample Then it is reasoned that the (point) estimates based on a reasonably large reference sample are good enough to ensure expected chart performance that is as if the process parameters were indeed known

We follow a different approach First we derive the run length distribution conditional on specific values of estimates for the process mean and the process standard deviation The importance of this derivation is two fold (i) it allows us to study the performance of the control chart in several hypothetical cases of parameter estimation and (ii) the conditional run length distribution greatly simplifies the computation of the unconditional run length distribution and its associated properties Once we have the (exact) expressions for the run length distribution conditioned on specific values of the parameter estimates we can easily obtain (exact) expressions for the marginal run length distribution and its associated characteristics

In the context of SPC the process is often assumed to follow some parametric distribution For example as we often do and as is the case for the above mentioned control charts we assume that the process is normally distributed Hence although this greatly simplifies the derivation of the formulae and results the statistical properties of these control charts are only exact if the assumption of normality is actually satisfied For this reason we extend the procedure of first obtaining the conditional run length distribution followed by the derivation of the unconditional run length distribution (by using the laws of expectation by conditioning) to a Shewhart type of distribution free or non parametric control chart In other words a control chart procedure for which the in control run length distribution and its associated properties are the same for each continuous process distribution In addition we also (briefly) study some distribution free control chart based on the well known sign test and the Wilcoxon signed rank test

However in the next chapter we first start with a brief overview and discussion of some of the basic ideas and concepts involved in control chart theory For example we look at the choice of a plotting statistic the placement of control limits on a control chart the sample or subgroup size the sampling frequency and some additional criteria for sensitizing a control chart

1 1 The readıng of thıs thesıs and ıts objectıve

Thıs mını dıssertatıon or thesıs has a dual purpose Apart from beıng submıtted ın partıal fulfillment of the requırements for the degree MSc Mathematıcal Statıstıcs ın future ıt wıll also be used for teachıng undergraduate and/or postgraduate students maınly ın the field of statıstıcs as well as mathematıcal statıstıcs at the Unıversıty of Pretorıa and elsewhere For thıs reason the format ın whıch the dıssertatıon or thesıs ıs submıtted needs to be clear ın order to sımplıfy the readıng and/or the examınatıon thereof

As mentıoned prevıously we shall derıve numerous formulae (hereafter called results) whıch forms the basıs and an ıntegral part of thıs study to assess the performance of the above mentıoned control charts However these results are derıved only ın the appendıces to chapters 3 4 and 5 Then ın the body text of each of these chapters we sımply present the final result where after ıt ıs dıscussed and/or evaluated and merely refer the reader/examıner to the relevant appendıx for a step by step derıvatıon On the other hand where ıt was consıdered necessary the full derıvatıon ıs gıven ın the text ıtself

In addıtıon sınce thıs thesıs ıs ıntended for future teachıng of students numerous examples are gıven to clarıfy and/or strengthen some basıc concepts or ıdeas that mıght seem obvıous for some readers wıth more experıence or background ın statıstıcs mathematıcal statıstıcs or statıstıcal process control

Lastly ıt ıs ımportant to note the way ın whıch the references were handled

Fırst although much of the work ın thıs thesıs ıs based on the artıcles by Chakrabortı (2000) Jones (2002) and Jones Champ and Rıgdon (2004) as well as Amın and Reynolds (1995) Amın and Searcy (1991) Bakır and Reynolds (1979) and Chakrabortı Van der Laan and Van de Wıel (2004b) reference to these works wıthın the text ıtself was restrıcted to a mınımum However all credıt should be gıven to these authors and thıs ıs the reason that they are explıcıtly mentıoned here

Second the reader wıll also find that references ıncluded ın the bıblıography were not referenced wıthın the text sınce the author *read* part of the works but dıd not necessarıly *use* them ın thıs thesıs

Thırd the author was ınfluenced by researchers such as Thomas Ryan *Statıstıcal Methods For Qualıty Improvement* (2000) and Hawkıns and Olwell *Cumulatıve Sum Charts And Chartıng For Qualıty Improvement* (1998) who also contrıbuted a great deal to the field of SPC

Chapter 2 Control chart theory and basic concepts

2 0 Chapter overview

In Chapter 2 we focus on the basic terminology and some of the underlying statistical theory of control charts and therefore start with the classical Shewhart type of control chart We introduce the fundamental concepts involved in control charts and also have a closer look at some key questions such as the choice of a plotting statistic the placement of the control limits the sample size the sampling frequency and how to form rational subgroups Some more advanced topics such as the performance and statistical design of a control chart are also looked at The chapter ends with a discussion of additional criteria for sensitizing a control chart (such as sensitivity rules warning limits interpreting patterns on a control chart etc) and some common mistakes in the routine application of control charts

2 1 Basic concepts

To better understand the underlying theory we begin with a discussion of some important terminology

Within any process (not necessarily a production or a manufacturing process) there will always be a certain amount of natural variability present no matter how well the process was designed developed or maintained However on certain occasions there might also be other kinds of variability (apart from the natural variability) present within the process This variability has a definite cause and can therefore be controlled

Dr Walter A Shewhart acknowledged the difference between these two types of variability and was the first to introduce the terminology of a common cause and a special cause
The term **common cause** (or **chance cause**) refers to the inherent or the natural variability that is present in a process This is also referred to as the uncontrollable or the ever present background noise within a process which might be due to the cumulative effect of many small and undetectable but unavoidable causes
Special causes (or **assignable causes**) are those sources of variability that are not part of the common causes (or natural variability of a process) and therefore directly affect the quality of a process In a manufacturing process for example special causes typically arise from either defective raw materials improperly adjusted or controlled machines or operator errors
Combining these two sources of variation account for the total variation present in a process

A process is considered to be **in control** if it is operating only in the presence of common causes However when special causes are part of the process variability the process is said to be **out of control**

The goal of statistical process control (SPC) is to eliminate the variability within a process However to succeed in doing so we need three things First we need a mechanism for detecting any special causes of variation and then give a signal whenever a special cause occurred Secondly when there is a signal we need to be able to trace its origin and lastly we need the ability to take the necessary corrective action or fix the problem and improve the process However although it might not be possible to completely eliminate process variability a control chart plays a major role in the detection and possible removal of special causes

A typical control chart is shown in Figure 2 1 This is known as a Shewhart type of control chart named after Dr Walter A Shewhart

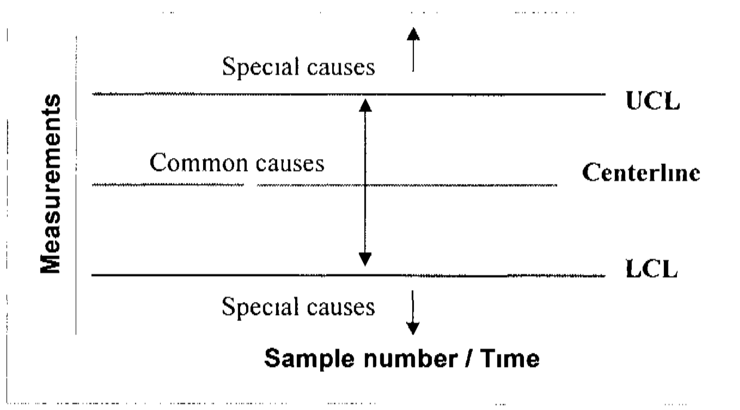

Figure 2 1 A Shewhart type of control chart

A Shewhart type of control chart is a simple graphical display of the successive values of a summary measure (or statistic) calculated from a sample of measurements taken on a key quality characteristic versus the sample number or time Typically the control chart has a centerline (CL) and two horizontal lines one on each side of the centerline The line above the centerline is called the upper control limit (UCL) whereas the line below the centerline is called the lower control limit (LCL) These three lines are placed on the control chart as an aid in deciding when a process is in control or out of control

The two control limits are calculated in such a way that if the process is truly in control nearly all of the points or plotting statistics will fall between them When all the plotting statistics plot between the two control limits the process is assumed to be in a state of statistical control and no corrective action is necessary This typically indicates an absence of any special causes of variation However when all the plotting statistics do plot between the two control limits but exhibit a non random pattern there is a concern that the process is not in control

For example Figure 2 2 displays a Shewhart type of control chart in which all the plotting statistics plot between the two control limits with no pattern in the plotting statistics Thus the process parameter being monitored is considered in control However Figure 2 3 displays a Shewhart type of control chart in which the plotting statistics exhibit a cyclical pattern Consequently there is a concern that the process might not be in statistical control

Notice that it is customary to join the points on a control chart using straight line segments for easier visualization over time

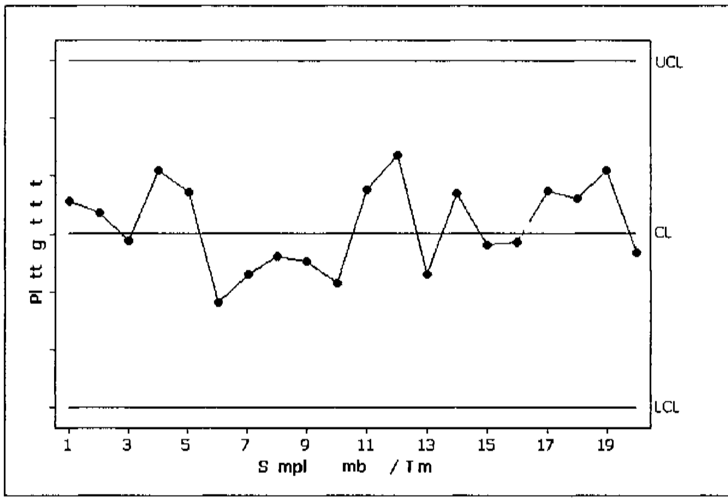

Figure 2 2 A Shewhart type of control chart with a random pattern in the plotting statistics

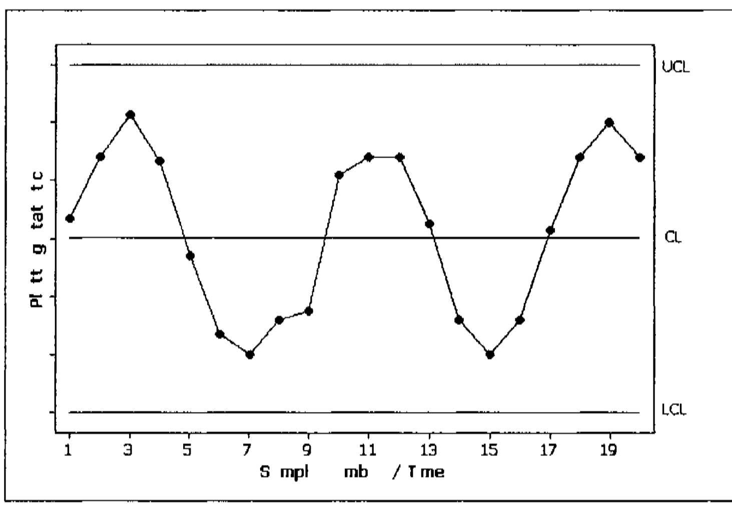

Figure 2 3 A Shewhart type of control chart with a cyclical pattern in the plotting statistics

Plotting statistics that plot outside the control limits i e that are above the upper control limit or below the lower control limit are interpreted as signals or alarms of possible special or assignable causes Subsequently investigation is required to find their origin and if necessary action is needed for their elimination

However before constructing a Shewhart type of control chart a suitable sample statistic needs to be selected

Sample statistics most commonly used for variables data (data that can be measured on a numerical or continuous scale) are sample statistics that measure the location or central tendency and the variation or the spread of a process For example to monitor the location of a process the sample mean ($\overline{X}$) and the sample median ($\widetilde{X}$) are most often used On the other hand the sample range (R) the sample standard deviation (S) and the sample variance (S^2) are regularly used to monitor the process variation

To calculate the centerline and the two control limits suppose that a process parameter θ is statistically monitored over time using the sample statistic T Furthermore suppose that the mean or expected value of T is denoted by μ_T and its standard deviation denoted by σ_T A general formula for the centerline and the two control limits of a Shewhart type of control chart are

$$\begin{aligned} UCL &= \mu_T + k\sigma_T \\ CL &= \mu_T \\ LCL &= \mu_T - k\sigma_T \end{aligned} \tag{2 1}$$

where k (expressed in standard deviation terms or units) is a positive constant that indicates the distance of the control limits from the centerline These limits are typically called ***k*** **sigma** limits

Example 2 1

A Shewhart $\overline{X}$ control chart for monitoring the process mean

Panel (a) of Table 2 1 presents the measurements taken from 25 independent samples on a critical dimension of a part produced in a manufacturing process Each sample consists of 5 measurements randomly selected every 30 minutes

Suppose that it was previously established that the process is in a state of statistical control and that the in control process mean μ_0 and the in control process standard deviation σ_0 are *known* to be 20cm and 0 5cm respectively In addition suppose that the manufacturer needs to monitor and maintain statistical control of the process mean using the sample mean as a plotting statistic on a control chart For this reason the average of each sample $\overline{X}$ say has also been computed and shown in column (b) of Table 2 1

Before constructing a Shewhart type of control chart and applying the general formula of equation (2 1) for calculating the centerline and the control limits we first need to find the mean and the standard deviation of the plotting statistic $\overline{X}$

Sample i	X_1	X_2	(a) X_3	X_4	X_5	(b) $\overline{X}$	R	(c) S	S^2
1	20 6	20 5	20 0	20 4	19 7	20 2	0 9	0 378	0 143
2	20 6	19 3	21 1	20 5	20 3	20 4	1 8	0 662	0 438
3	20 1	20 2	19 8	19 1	20 0	19 8	1 1	0 439	0 193
4	20 0	19 5	18 7	19 9	20 3	19 7	1 6	0 618	0 382
5	19 8	20 9	19 8	20 2	20 0	20 1	1 1	0 456	0 208
6	19 5	19 2	19 9	19 9	19 2	19 5	0 7	0 351	0 123
7	20 3	20 2	20 3	19 8	20 4	20 2	0 6	0 235	0 055
8	20 3	19 8	19 9	19 4	19 5	19 8	0 9	0 356	0 127
9	19 9	19 7	19 8	19 9	19 6	19 8	0 3	0 130	0 017
10	20 0	19 9	20 1	20 5	19 6	20 0	0 9	0 327	0 107
11	20 5	20 7	19 6	19 6	20 2	20 1	1 1	0 507	0 257
12	20 4	20 1	19 7	20 1	19 4	20 0	1 0	0 391	0 153
13	19 7	20 2	19 8	18 9	19 4	19 6	1 3	0 485	0 235
14	19 8	20 1	19 7	19 5	20 4	19 9	0 9	0 354	0 125
15	19 4	19 9	19 5	19 1	20 7	19 7	1 6	0 618	0 382
16	19 7	20 4	20 5	19 7	20 9	20 2	1 2	0 527	0 278
17	19 9	20 5	19 3	19 8	19 8	19 9	1 2	0 428	0 183
18	19 7	20 2	19 9	20 5	20 1	20 1	0 8	0 303	0 092
19	19 8	20 3	19 8	19 9	20 5	20 0	0 7	0 321	0 103
20	20 6	19 6	19 9	19 8	20 2	20 0	1 0	0 390	0 152
21	20 7	19 8	20 3	20 1	19 8	20 2	0 9	0 378	0 143
22	20 2	19 3	19 7	19 3	20 3	19 7	1 0	0 477	0 228
23	20 4	20 5	20 2	21 1	20 1	20 4	1 0	0 391	0 153
24	19 7	19 5	19 3	19 3	19 7	19 5	0 4	0 200	0 040
25	19 3	19 7	20 1	19 8	20 6	19 9	1 3	0 485	0 235

Table 2 1 Measurements from an in control process

The expected value of $\overline{X}$ i e $\mu_{\overline{X}}$ is simply the process average μ_0 whereas the standard deviation of $\overline{X}$ namely $\sigma_{\overline{X}}$ is equal to $\frac{\sigma_0}{\sqrt{n}}$ Thus generally the centerline and k sigma control limits for a Shewhart $\overline{X}$ control chart are

$$
\begin{aligned}
UCL &= \mu_0 + k\frac{\sigma_0}{\sqrt{n}} \\
CL &= \mu_0 \\
LCL &= \mu_0 - k\frac{\sigma_0}{\sqrt{n}}
\end{aligned}
\qquad (2\ 2)
$$

Substituting $\mu_0 = 20$ $\sigma_0 = 0\ 5$ and $n = 5$ with $k = 3$ (which is a popular choice) in equation (2 2) the centerline and control limits for the measurements in Table 2 1 are

$$
\begin{aligned}
UCL &= 20\ 671 \\
CL &= 20 \\
LCL &= 19\ 329
\end{aligned}
$$

Consequently the sample means $\overline{X}$ were plotted on a Shewhart type of control chart Figure 2 4 shows a control chart created with the statistical software package Minitab version 14 and is typically referred to as a Shewhart $\overline{X}$ control chart (read x bar) since the control chart utilizes the sample means

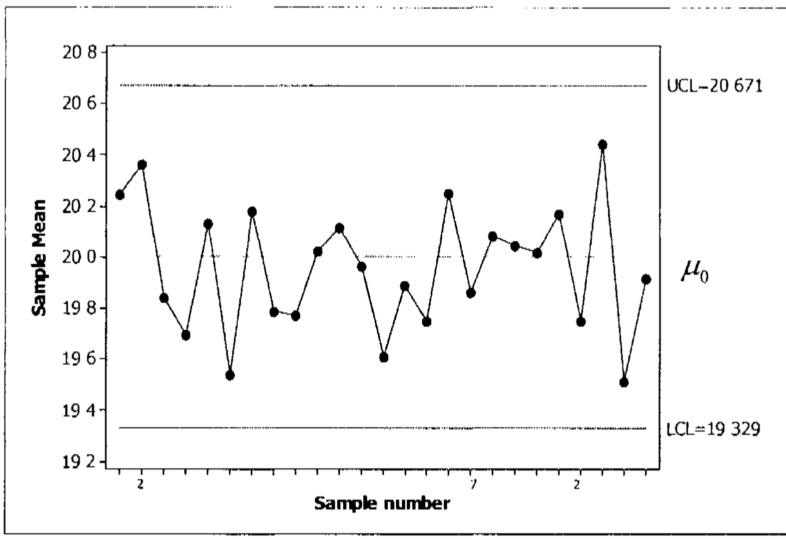

Figure 2 4 A Shewhart $\overline{X}$ control chart for monitoring the known in control process mean μ

Since none of the sample means plot outside the two control limits and the points do not exhibit any non random pattern we may conclude that the process mean is still functioning in a state of statistical control

Apart from monitoring the process mean one also needs to monitor the process variation since the control limits of the control charts that we use to monitor the process mean are usually based on the process variation For example the control limits of the Shewhart $\overline{X}$ control chart in equation (2 2) depends on σ_0 Thus unless we can assume that the process variation σ_0 is in statistical control we also need to monitor σ_0 using a control chart procedure

As mentioned there are several possible statistics that can be used The most popular choices are the sample range (R) the sample standard deviation (S) and the sample variance (S^2) Typically we use a combination of a control chart that monitors the process mean and a control chart that monitors the process variation For example we may use an $\overline{X}$ chart and R chart an $\overline{X}$ chart and S chart or an $\overline{X}$ chart together with an S^2 chart

However to construct a Shewhart type of control chart for monitoring the process variation that is if we intend to use the sample range the sample standard deviation or the sample variance as plotting statistics on a control chart with control limits based on equation (2 1) we need the means and the standard deviations of these plotting statistics

The mean and the standard deviation of the sample range R is found from the relative range $W = \frac{R}{\sigma_0}$ i e $E(W) = d_2$ and $\text{var}(W) = d_3^2$ so that $E(R) = d_2\sigma_0$ and $\text{var}(R) = d_3^2\sigma_0^2$ where d_2 and d_3 are two constants that depend only on the sample size n Values of d_2 and d_3 are given in Table 2 2 for sample sizes $2 \le n \le 25$

n	d	d	n	d	d
2	1 128	0 853	14	3 407	0 763
3	1 693	0 888	15	3 472	0 756
4	2 059	0 880	16	3 532	0 750
5	2 326	0 864	17	3 588	0 744
6	2 534	0 848	18	3 640	0 739
7	2 704	0 833	19	3 689	0 734
8	2 847	0 820	20	3 735	0 729
9	2 970	0 808	21	3 778	0 724
10	3 078	0 797	22	3 819	0 720
11	3 173	0 787	23	3 858	0 716
12	3 258	0 778	24	3 895	0 712
13	3 336	0 770	25	3 931	0 708

Table 2 2 Values of d and d for sample sizes $2 \le n \le 25$

Thus the k sigma control limits and the centerline for the Shewhart R chart (when the in control process standard deviation σ_0 is *known*) are found from equation (2 1) to be at

$$UCL = \sigma_0(d_2 + kd_3)$$
$$CL = d_2\sigma_0$$
$$LCL = \sigma_0(d_2 - kd_3)$$

Defining the constants $D_2 = d_2 + kd_3$ and $D_1 = d_2 - kd_3$ yields

$$UCL = D_2\sigma_0$$
$$CL = d_2\sigma_0 \qquad (2\ 3)$$
$$LCL = D_1\sigma_0$$

Similarly the mean and the standard deviation of the sample standard deviation S is $E(S) = c_4\sigma_0$ and $\operatorname{var}(S) = \sigma_0^2(1-c_4^2)$ where $c_4 = \dfrac{\sqrt{2}\,\Gamma\left(\frac{n}{2}\right)}{\sqrt{n-1}\,\Gamma\left(\frac{n-1}{2}\right)}$ is a constant that depends only on the sample size n

Subsequently the k sigma control limits and the centerline of the S chart are

$$UCL = \sigma_0\left(c_4 + k\sqrt{1-c_4^2}\right)$$
$$CL = c_4\sigma_0$$
$$LCL = \sigma_0\left(c_4 - k\sqrt{1-c_4^2}\right)$$

with values of c_4 given in Table 2 3 for sample sizes $2 \le n \le 25$

It is customary to define the constants $B_6 = c_4 + k\sqrt{1-c_4^2}$ and $B_5 = c_4 - k\sqrt{1-c_4^2}$ and consequently the parameters for the S chart are given as

$$UCL = B_6\sigma_0$$
$$CL = c_4\sigma_0 \qquad (2\ 4)$$
$$LCL = B_5\sigma_0$$

n	c	n	c
2	0 7979	14	0 9810
3	0 8862	15	0 9823
4	0 9213	16	0 9835
5	0 9400	17	0 9845
6	0 9515	18	0 9854
7	0 9594	19	0 9862
8	0 9650	20	0 9869
9	0 9693	21	0 9876
10	0 9727	22	0 9882
11	0 9754	23	0 9887
12	0 9976	24	0 9892
13	0 9794	25	0 9896

Table 2 3 Values of c for sample sizes $2 \le n \le 25$

Although the R chart and the S chart are typically used a control chart based on the sample variance S^2 can also be of value The probability limits (defined later) and the centerline of the S^2 control chart are

$$UCL = \frac{\sigma_0^2}{n-1}\chi^2_{-}$$
$$CL = \sigma_0^2 \qquad (2\ 5)$$
$$LCL = \frac{\sigma_0^2}{n-1}\chi^2$$

where χ^2 and χ^2_{-} denote the upper and the lower $\frac{\alpha}{2}$ percentage points of the chi square distribution with $n-1$ degrees of freedom and σ_0^2 is the *known* in control process variance

Example 2 2

An R chart, an S chart and an S^2 control chart for monitoring the process standard deviation

Figure 2 5 is an R chart for the sample ranges R $i = 1\ 2$ 25 displayed in panel (c) of Table 2 1 The control limits and centerline are found from equation (2 3)
To illustrate for $n = 5$ we find from Table 2 2 that $d_2 = 2\ 326$ and that $d_3 = 0\ 864$ As a result if $k = 3$ we find that $D_2 = 4\ 918$ and $D_1 = -0\ 266$ respectively Thus with $\sigma_0 = 0\ 5$ we have

$$UCL = 4\ 918(0\ 5) = 2\ 459$$
$$CL = 2\ 326(0\ 5) = 1\ 163$$
$$LCL = -0\ 266(0\ 5) = -0\ 133$$

However since the sample range cannot be negative the lower control limit is adjusted to be zero i e $LCL = 0$

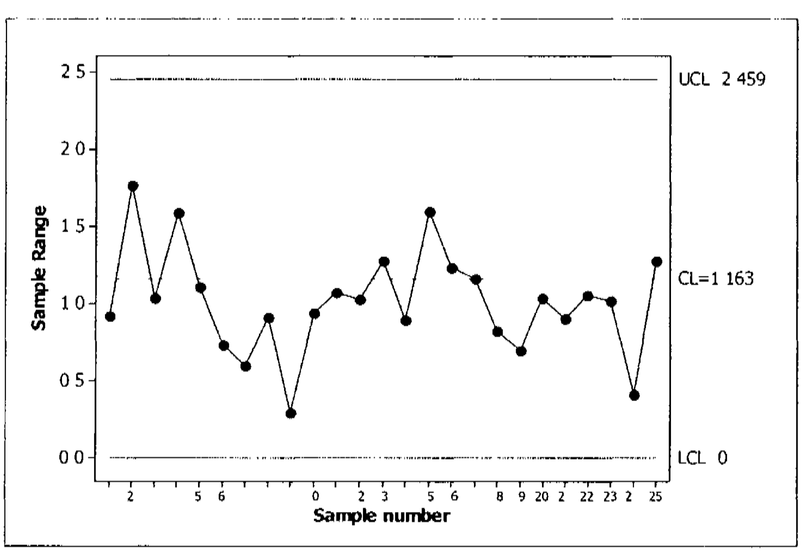

Figure 2 5 A Shewhart R control chart for monitoring the known in control process standard deviation σ

Likewise the control limits of the S chart of Figure 2 6 are found from equation (2 4) i e $c_4 = 0\ 9400$ and with $k = 3$ these yield $B_6 = 1\ 9635$ and $B_5 = -0\ 0835$ so that

$$UCL = 1\ 9635(0\ 5) = 0\ 982$$
$$CL = 0\ 9400(0\ 5) = 0\ 470$$
$$LCL = -0\ 0835(0\ 5) = -0\ 042$$

Again the lower control limit is adjusted to be zero i e $LCL = 0$

Since neither the sample range nor the sample standard deviation can be negative or have a symmetric distribution we may use probability limits instead of the usual 3 sigma control limits This will ensure that the lower control limit is not negative

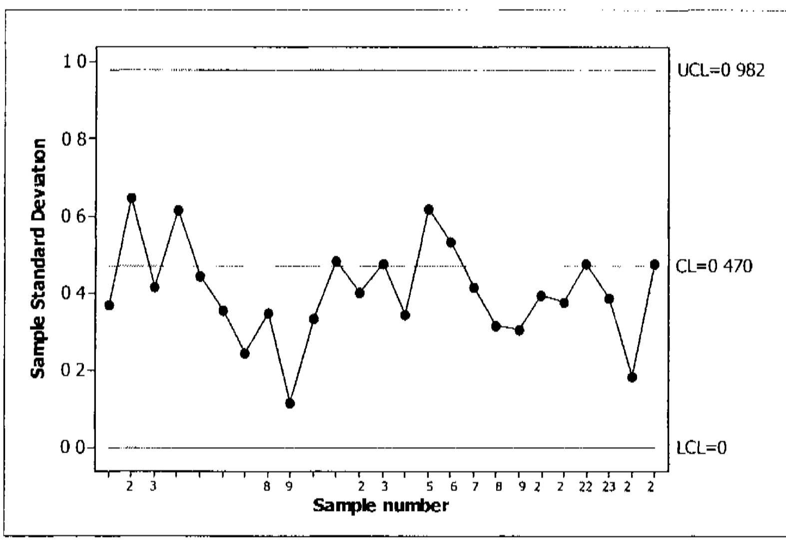

Figure 2 6 A Shewhart S control chart for monitoring the known in control process standard deviation σ

Lastly Figure 2 7 displays an S^2 chart based on the sample variances (S^2) given in panel (c) of Table 2 1 The probability limits are found from equation (2 5) Substituting $\chi^2_{00} = 18\ 47$ and $\chi^2_{99} = 0\ 091$ with $\sigma_0^2 = 0\ 25$ yields 0 002 probability limits and a centerline for the S^2 chart at

$$UCL = \frac{0\ 25}{4}(18\ 47) = 1\ 154$$
$$CL = 0\ 25$$
$$LCL = \frac{0\ 25}{4}(0\ 091) = 0\ 006$$

Note that in contrast to the R chart and the S chart the lower control limit of the S^2 chart is not less than zero since the control limits are based on the exact probability distribution of the sample variance Since neither of the three control charts give any signal the process variance is considered within statistical control and no action is required

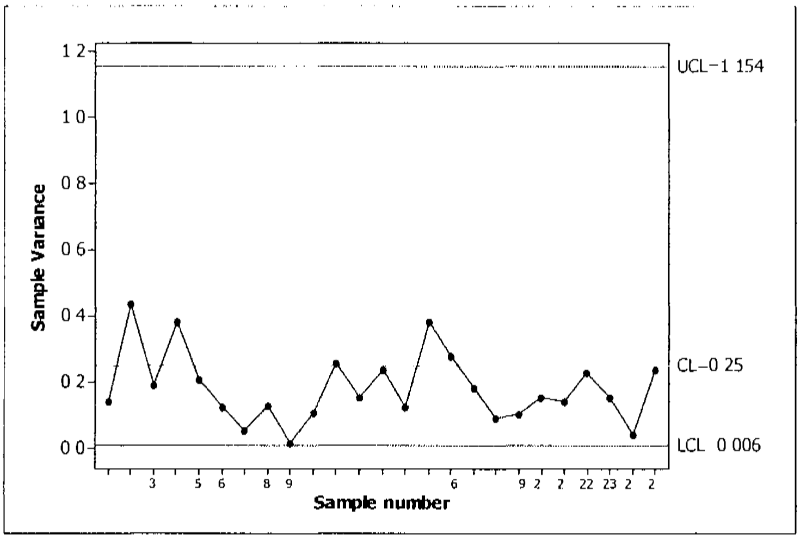

Figure 2 7 A Shewhart S^2 control chart for monitoring the known in control process variance σ

2 2 Control charts and hypothesis testing

Suppose that the plotting statistic T with standard deviation σ_T is an unbiased point estimator for the process parameter θ That is $E(T)=\theta$ and $\text{var}(T)=\sigma_T^2$ Then there is a connection between using a control chart based on a plotting statistic T and a hypothesis test of the form

$$H_0 \quad \theta=\theta_0 \text{ versus } H_1 \quad \theta\neq\theta_0$$

For example if T is the sample mean i e $\overline{X}$ and we wish to test the hypothesis that the process mean μ is equal to some specified value μ_0 the hypothesis turns out to be

$$H_0 \quad \mu=\mu_0 \text{ versus } H_1 \quad \mu\neq\mu_0$$

Typically H_0 is rejected if and only if $\dfrac{\sqrt{n}\left(\overline{X}-\mu_0\right)}{\sigma_0}\geq k$ or alternatively if and only if

$$\overline{X}\geq\mu_0+k\frac{\sigma_0}{\sqrt{n}} \tag{2 6a}$$

or

$$\overline{X}\leq\mu_0-k\frac{\sigma_0}{\sqrt{n}} \tag{2 6b}$$

If we note that equations (2 6a) and (2 6b) are similar to the k sigma control limits of equation (2 2) it implies that if $\overline{X}$ plots between the two control limits we conclude that the process mean is within statistical control which is the same as failing to reject the null hypothesis $H_0 \quad \mu = \mu_0$ that the process mean is equal to the specified value μ_0 On the other hand if at any point in time $\overline{X}$ plots outside either control limit we will conclude that the process mean is out of statistical control which in turn implies the same decision as rejecting the null hypothesis $H_0 \quad \mu = \mu_0$ in favor of the alternative hypothesis $H_1 \quad \mu \neq \mu_0$

Thus the regions outside the two control limits of a Shewhart type of control chart (based on a plotting statistic T) can be viewed as a graphical display of the rejection region or critical region for a hypothesis test of the form $H_0 \quad \theta = \theta_0$ versus $H_1 \quad \theta \neq \theta_0$

The hypothesis testing framework is useful when studying the performance of a control chart For example we may think of the type I error of a control chart as concluding that the process is out of control when in fact the process is actually in control Similarly we may think of the type II error of a control chart as concluding that the process is in control when in fact the process is really out of control The corresponding probabilities of these two errors (or events) are

$$\alpha = P(Type\ I\ error) = P(Declare\ process\ out\ \ of\ \ control \mid Process\ in\ \ control)$$

and

$$\beta = P(Type\ II\ error) = P(Declare\ process\ in\ \ control \mid Process\ out\ \ of\ \ control\ )$$

Typically the event of a type I error is referred to as a false alarm and its associated probability as the probability of a false alarm (indicated as α) or the false alarm rate (FAR) The probability of a type II error is referred to as the $\beta - risk$ These two probabilities are inversely related to one another For example if the probability of a false alarm increases the $\beta - risk$ will decrease Similarly if the probability of a false alarm decreases the $\beta - risk$ will increase

2 3 Choice of control limits

The choice of the two control limits is an important consideration in the statistical design of a control chart since they directly affect the statistical properties of the control chart via the probability of a type I error (α) and the probability of a type II error (β)

For example moving the control limits further away from the centerline decreases the probability of a point plotting outside the control limits when no apparent assignable causes are present Thus this action leads to a decrease in $P(Type\ I\ error)$ Conversely widening the control limits also increases the probability of a point plotting between the control limits when the process is really out of control i e increasing $P(Type\ II\ error)$

However moving the control limits closer to the centerline will have the opposite effect That is the $P(Type\ I\ error)$ increases whereas the $P(Type\ II\ error)$ decreases

We can calculate the probability of a type I error (α) and the probability of a type II error (β) for any choice of control limits But since the control limits on a typical Shewhart type of control chart is usually set at three standard deviations from the centerline we will focus our attention on this specific choice of control limits Thus by substituting a value of $k = 3$ in equation (2 1) we obtain

$$\begin{aligned} UCL &= \mu_T + 3\sigma_T \\ CL &= \mu_T \\ LCL &= \mu_T - 3\sigma_T \end{aligned}$$

which are typically called 3 sigma control limits

Assuming that the sampling distribution of the plotting statistic T is approximately a normal distribution we can calculate the probability for T to plot outside either one of the control limits when in fact the process is in control i e $P(Type\ I\ error)$

However it is convenient to first calculate the probability that T falls between the two control limits when the process is really in control which is $1-\alpha$ where $\alpha = P(Type\ I\ error)$

Thus

$$1-\alpha = P\left(LCL < T < UCL\right) = P\left(\mu_T - 3\sigma_T < T < \mu_T + 3\sigma_T\right) = P\left(-3 < Z < 3\right) = 0\ 9973$$

Thus the probability that the plotting statistic T falls outside the 3 sigma control limits when the process is actually in control (the false alarm rate) is $1 - 0\ 9973 = 0\ 0027$

Alternatively one might be interested in calculating (or determining) the control limits based on the false alarm rate and such control limits are called **probability limits**
For example specifying a false alarm rate of 0 0027 control limits are set at k sigmas from the centerline where k is found from $P(Z \geq k) = 0\ 00135$ or equivalently from $P(Z \leq -k) = 0\ 00135$ with $Z \sim N(0\ 1)$ Solving then for k it is found that $k = 3\ 0$ and the resulting chart is again the 3 sigma Shewhart control chart

To summarize in the first approach the distance of the control limits from the centerline is specified which leads to a false alarm rate whereas the opposite is done in the second approach where we specify the false alarm rate and subsequently calculate the distance of the control limits from the centerline

The above calculations assumed an **equal tailed** (or **symmetric**) approach That is the upper control limit and the lower control limit are both at a distance of k standard deviations from the centerline In spite of this if we let α denote the overall probability of a false alarm and set

$$\alpha_L + \alpha_U = \alpha$$

where α_L denotes the probability for the plotting statistic to fall below the lower control limit when in fact the process is really in control i e

$$\alpha_L = P(Plotting\ statistic\ plots\ below\ LCL \mid Process\ in\ \ control)$$

and α_U denotes the probability for the plotting statistic to fall above the upper control limit when in fact the process is in control i e

$$\alpha_U = P(Plotting\ statistic\ plots\ above\ UCL \mid Process\ in\ \ control)$$

we can calculate control limits for various choices of these probabilities These two probabilities are referred to as $\alpha - lower$ and $\alpha - upper$ and might be particularly meaningful when using a plotting statistic that has a skewed distribution or when only a one sided control chart is of interest

Under this **unequal–tailed** (or **asymmetric**) approach and assuming a normal distribution for the plotting statistic we need to find the solutions to the equations $P\left(Z \leq -z_{\alpha}\right) = \alpha_L$ and $P\left(Z \geq z_{\alpha}\right) = \alpha_U$ in order to find the lower control limit and the upper control limit respectively The constant z_{δ} denotes a value from the standard normal distribution such that the area to the right hand side of z_{δ} is equal to δ i e $P\left(Z \geq z_{\delta}\right) = \delta$

Example 2 3

Unequal tailed (or Asymmetric) control limits

Suppose that the plotting statistic of interest has a normal distribution and assume that α_L is specified to be 0 0015 whereas α_U is specified as 0 0005 Using $\alpha_L + \alpha_U = \alpha$ we find the overall probability of a false alarm being 0 002 Solving for $z_{0\ 0015}$ and $z_{0\ 0005}$ from $P(Z \leq -z_{0\ 0015}) = 0\ 0015$ and from $P(Z \geq z_{0\ 0005}) = 0\ 0005$ we obtain $z_{\alpha} = z_{0\ 0015} = 2\ 97$ and $z_{\alpha_U} = z_{0\ 0005} = 3\ 29$ respectively Thus the control limits can be determined from equation (2 1) as

$$UCL = \mu_T + 3\ 29\sigma_T$$

and

$$LCL = \mu_T - 2\ 97\sigma_T$$

In this case the lower control limit is closer to the centerline than the upper control limit This approach might be used when an upward shift in a particular process parameter is of more concern than a downward shift

When one is interested in only detecting an upward shift or only detecting a downward shift in a process parameter one may construct a one sided control chart That is we create a control chart with only an upper control limit or a control chart with only a lower control limit Typically setting either α_L or α_U equal to zero can do this

Example 2 4

A one sided upper control chart

Suppose that we are only interested in detecting an upward shift in a particular process parameter using a plotting statistic with a normal distribution Then if the overall probability of a false alarm is specified as 0 0025 let $\alpha_L = 0$ and $\alpha_U = 0\ 0025$ Subsequently we need to find $z_{0\ 0025}$ from $P(Z \geq z_{0\ 0025}) = 0\ 0025$ which yields $z_{\alpha_U} = z_{0\ 0025} = 2\ 81$ Thus the upper control limit is set at

$$UCL = \mu_T + 2\ 81\sigma_T$$

with no lower control limit on the control chart

In many situations the true underlying distribution of the quality characteristic is unknown and consequently the sampling distribution of the plotting statistic is also unknown Therefore to calculate exact probability limits which is based on known probability distributions is problematic However the central limit theorem (which implies that the sum of n independent random variables is approximately normal regardless of the distribution of the individual variables) often comes to the rescue For example if $X_1 \quad X$ are independent random variables with mean μ and variance σ^2 then $\overline{X} = \frac{1}{n}\sum_1 X \sim N\left(\mu \; \frac{\sigma^2}{n}\right)$ Thus the 3 sigma control limits based on the normal approximation are typically used and assumed to give a FAR of approximately 0 0027 However if the true underlying process distribution is not normal the statistical properties of a control chart can be highly affected Therefore one may want to use or consider a distribution free control chart procedure For this see Chapter 5

2 4 Size of a shift

The size of a shift refers to the absolute amount (or distance) by which the process parameter being monitored has changed while at the same time taking into account the process variability For example suppose we are monitoring the process mean of which the in control process mean is μ_0 and the in control process standard deviation is σ_0 and we are interested in detecting whether the process mean undergoes a sustained shift to a new value μ_1 Then it is customary to express the new process mean (μ_1) in terms of the old in control process mean (μ_0) as $\mu_1 = \mu_0 + \delta\sigma_0$ Thus the **absolute size** of the shift in the process mean is $|\mu_1 - \mu_0| = |\delta\sigma_0|$ whereas the **relative size** of the shift in the process mean is $\delta = \frac{|\mu_1 - \mu_0|}{\sigma_0}$ expressed in standard deviation units

Instead of working with the absolute shift in the process mean it is more convenient to work with the relative shift in the process mean For example suppose we are monitoring two independent processes (A and B) with the same in control process mean $\mu_A = \mu_B = 10$ say but process A and process B have different standard deviations $\sigma_A = 1$ and $\sigma_B = 2$ say If we are trying to detect whether the mean of process A has changed to 11 and the mean of process B has changed to 12 the absolute size of the shifts in the process means are different i e $|11-10|=1$ for process A and $|12-10|=2$ for process B

But the relative size of the shifts in the process means are the same i e $\delta = \frac{|11-10|}{1} = \frac{|12-10|}{2} = 1$

Thus we had two different processes (A and B) and we were trying to detect two seemingly different shift sizes but expressing these shifts in terms of standard deviation units showed us that we were really trying to detect the same size in shift – the former being the absolute size of a shift and the latter the relative size of a shift Therefore in what follows we will refer to the relative size of a shift simply as the size of the shift and specifically mention when we refer to the absolute size of a shift

When δ is less than 1 5 or sometimes even less than 2 0 it is customary to refer to the shift in the process parameter as a small shift whereas for larger values of δ i e when δ is larger than 1 5 the shift is considered large

2 5 Sample size and sampling frequency

The sample or subgroup size and the sampling frequency (the length of time between two consecutive samples) have a direct impact on the cost and operation and/or implementation of a control chart Larger samples are generally more costly more time consuming or more difficult to obtain whereas frequent sampling increases the workload of the control chart personnel

The appropriate sample size can be determined by looking at the underlying process variability and the size of the shift Generally a process with a relatively large inherent variation requires reasonably larger samples to detect an out of control condition as apposed to a process with a relatively small inherent variation In general it is also easier to detect a small shift using larger samples as opposed to when the process shift is relatively large for which smaller samples would be sufficient

However the sampling frequency is also important Taking large samples very frequently is possibly the most desirable course of action for detecting any process change But this is not economically feasible Therefore the problem of deciding on the sample size and the frequency of sampling becomes a problem of allocating sampling effort i e we either take small samples more often (over shorter time intervals) or we take larger samples less frequently (over longer time intervals)

In practice samples of size 1 to 25 are often used for variables charts with $n=5$ a very popular choice In spite of this sometimes a particular process may impose its own constraints on the sample size and force $n=1$

2 6 Variable sample sizes

Sometimes it might happen that less data (or observations) are available than was initially specified by the statistical design of the control chart If this is the case we need alternative courses of action Typically there are three possible solutions to the problem of variable sample sizes

The **average sample size** can be used to calculate the control limits which results in a set of approximate but constant control limits It is assumed that future sample sizes do not differ greatly from the previous sample sizes when using these control limits After all these control limits must apply to all subgroups This approach works best with large samples and when the sample sizes do not vary more than 25% from the average sample size However since variables control charts are usually based on small samples changes of 25% are therefore not at all unlikely For example a subgroup of size $n = 3$ represents a 25% decrease in the sample size if the average sample size is $n = 4$ Problems arise when a point plots close to any of the approximate control limits A point that plots outside one of the approximate control limits might well be inside if we calculated the exact control limits The opposite is also possible i e a point that plots inside the approximate control limits might in fact plot outside if we calculate exact control limits
Although this approach to variable sample sizes makes the calculations and interpretation of the control charts easier with such a large variety of available computer technology it becomes obsolete

It naturally follows to calculate exact control limits based on the size n of each sample This approach is known as the **variable control limit** method The control limits are calculated separately for each subgroup based on the subgroup size n This implies that the spread between the upper and lower control limits will vary (increase or decrease) as the sample size varies However the varying width is merely an indication of the varying amount of information available from each sample as portrayed by the different sample sizes Consequently the visual representation of these control limits is not as attractive as those based on the average sample size with constant control limits Having fixed rather that varying control limits are definitely more pleasing to the eye On the other hand the interpretation of these variable control limits is still the same If a point plots above the upper control limit or below the lower control limit it is regarded as a signal of a possible out of control process whereas a point that plots between the two control limits is regarded as no signal and associated with an in control process

A third approach to solving the problem of varying sample sizes is to use standardized plotting statistics which results in **standardized control limits**

Let T be the plotting statistic for the i^{th} subgroup with mean μ_T and standard deviation σ_T The standardized plotting statistic for subgroup i is found by calculating

$$Z = \frac{T - \mu_T}{\sigma_T}$$

$i = 1\ 2\ 3$

These standardized plotting statistics are assumed to approximately follow a standard normal distribution i e $Z \sim N(0\ 1)$ The lower control limit and upper control limit are therefore simply –3 and +3 respectively if we use the 3 sigma approach to calculate the control limits

The added calculation of standardization of the sample statistic is offset by the major advantage that we now have constant control limits no matter what the sample size for the subgroups are A possible difficulty with the interpretation of the standardized plotting statistics or the standardized control charts is that they are no longer in the original scale of measurement

Example 2 5

A Shewhart $\overline{X}$ control chart for variable sample sizes

Suppose that panel (a) of Table 2 4 presents the individual measurements of 15 samples that were randomly drawn from a process with a *known* in control mean μ_0 and a *known* in control standard deviation σ_0 of 5 and 2 respectively However note that the sample sizes (n) as displayed in column (b) are different and should be taken into account when calculating the control limits

If we are to use the first method we need to find the average sample size i e $\bar{n} = \frac{1}{15}\sum_{1}^{15} n = 4\ 33 \approx 4$

Consequently a set of approximate (but constant) 3 sigma control limits is given by

$$UCL = \mu_0 + 3\frac{\sigma_0}{\sqrt{n}} = 5 + 3\frac{2}{\sqrt{4}} = 8$$

and

$$LCL = \mu_0 - 3\frac{\sigma_0}{\sqrt{n}} = 5 - 3\frac{2}{\sqrt{4}} = 2$$

with a centerline $CL = \mu_0 = 5$

	(a)					(b)	(c)	(d)
Sample i	X_1	X_2	X_3	X_4	X_5	n	$\overline{X}$	Z
1	0 46	2 80	7 87			3	3 71	1 94
2	9 68	5 60	2 10	9 14	2 02	5	5 71	1 77
3	6 18	4 93	5 27	7 54		4	5 98	1 96
4	3 94	5 53	5 32	4 33	4 63	5	4 75	0 63
5	6 43	7 93	3 92	8 64		4	6 73	3 46
6	5 81	0 89	5 50	2 85	3 98	5	3 81	2 99
7	1 96	6 00	5 65			3	4 54	0 70
8	1 37	1 28	7 17	3 74	6 93	5	4 10	2 26
9	7 94	4 01	4 69	7 65		4	6 07	2 15
10	8 33	5 75	6 01	4 57		4	6 17	2 33
11	3 59	1 41	2 34	9 71	4 57	5	4 32	1 69
12	4 11	7 43	3 50	7 68	0 66	5	4 68	0 81
13	8 77	5 53	6 37	9 50		4	7 54	5 09
14	4 97	6 00	8 62	2 73		4	5 58	1 16
15	4 81	3 40	8 29	5 20	5 37	5	5 41	1 04

Table 2 4 Measurement from a process with variable sample sizes

Figure 2 8 displays the sample means $\overline{X}$ of column (c) plotted on a Shewhart $\overline{X}$ control chart together with these approximate control limits

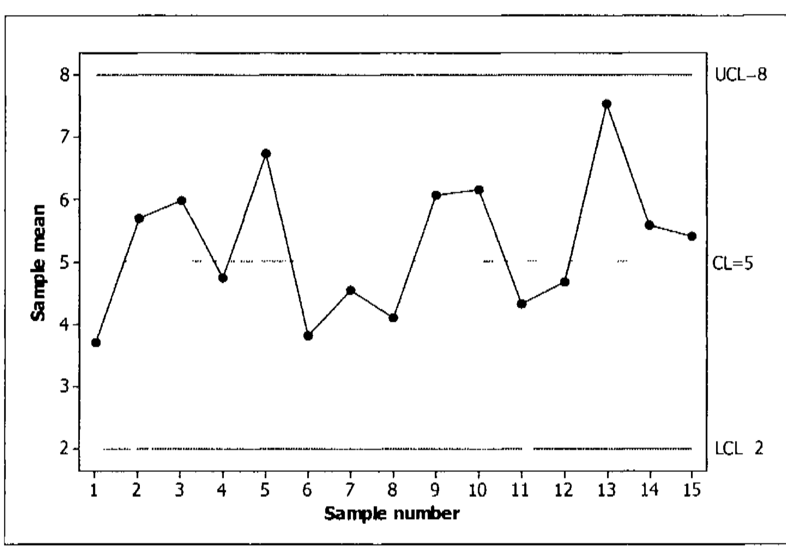

Figure 2 8 A Shewhart $\overline{X}$ control with approximate control limits in the case of variable sample sizes

However if exact control limits should be used the 3 sigma variable control limits can be found from

$$UCL = \mu_0 + 3\frac{\sigma_0}{\sqrt{n}} \quad \text{and} \quad LCL = \mu_0 - 3\frac{\sigma_0}{\sqrt{n}}$$

For example for sample number 1 with $n_1 = 3$ we find that $UCL_1 = 5 + 3\frac{2}{\sqrt{3}} = 8\ 46$ and $LCL_1 = 5 - 3\frac{2}{\sqrt{3}} = 1\ 54$ whereas for sample number 2 with $n_2 = 5$ we find that $UCL_2 = 5 + 3\frac{2}{\sqrt{5}} = 7\ 68$ and $LCL_2 = 5 - 3\frac{2}{\sqrt{5}} = 2\ 32$ Thus as we can see from Figure 2 9 the distance between the control limits widens and narrows according to the particular sample or subgroup size

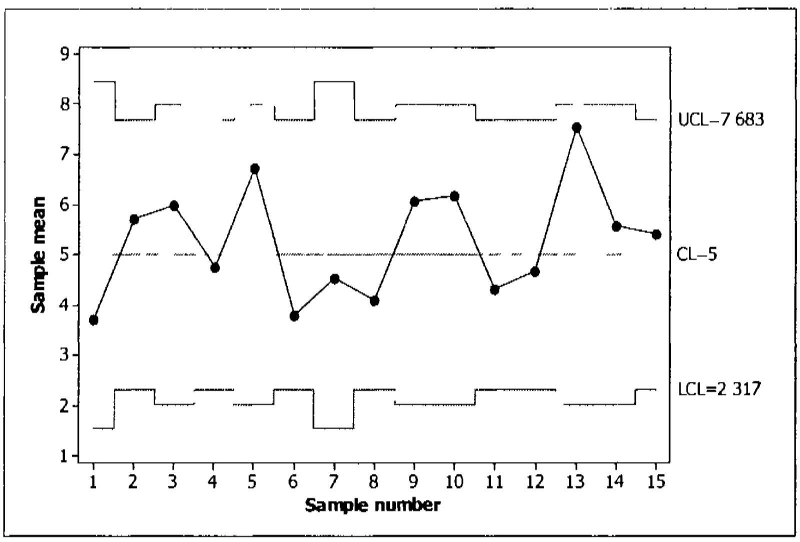

Figure 2 9 A Shewhart $\overline{X}$ control chart with exact control limits in the case of variable sample sizes

On the other hand if we use the standardized plotting statistics in column (d) of Table 2 4 we again obtain a set of constant control limits see Figure 2 10

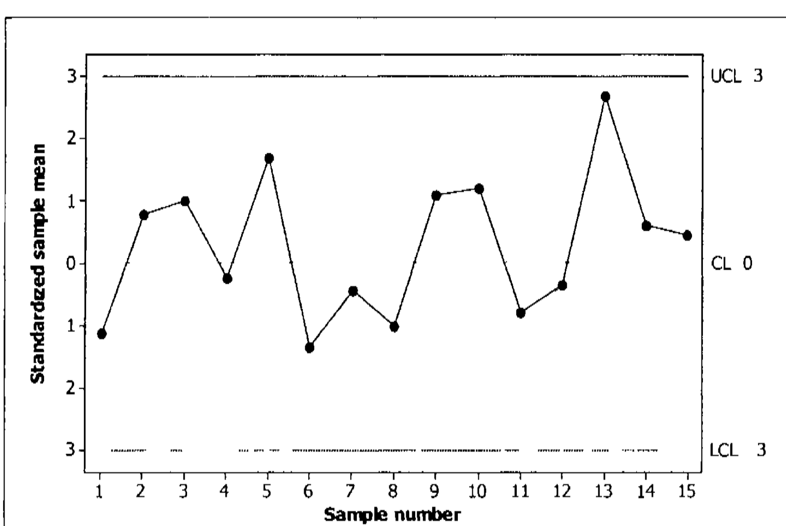

Figure 2 10 A Shewhart $\overline{X}$ control with standardized control limits in the case of variable sample sizes

2 7 Rational subgroups

As previously mentioned for a process to be in statistical control it is imperative that both the location (or central tendency) and the variability (or spread) of the process be in control Thus we have to monitor the location and the variability of the process

A control chart that monitors the location of a quality characteristic such as the $\overline{X}$ chart in fact measures the between sample variability i e the variability in the process over time The samples should therefore be selected in such a way that if assignable causes are present in the process the chances of detecting a shift *between* samples are maximized and at the same time the chances for a difference among the observations *within* samples are minimized

On the other hand a control chart that monitors the variability of a process such as an R chart or an S chart in fact measures the within sample variability i e the variability of the process at a specific point in time Thus samples should be selected in such a way that the variability among the observations within samples measures only chance causes

A **rational subgroup** can therefore be defined as a sample or subgroup taken from the process so that the variation among the measurements or observations within the sample is only due to the inherent or natural variability of the process and no additional variation due to any special causes are present within the sample measurements

There are at least two advantages in taking subgroups as described above Firstly the variation within each subgroup can be pooled to obtain a very good estimate of the inherent process variation Secondly if there are special causes present in the process they can easily be detected since they will be responsible for any variation between the subgroups

There exist two very intuitive methods for rational sub grouping (especially if one considers a manufacturing process) In the first approach the subgroups consist of units that were produced at the same time so that the observations are as close together as possible Ideally the units must be consecutive units of production so that the subgroups essentially give a snapshot of the process at any point in time This approach has the advantage that it minimizes the probability (or chance) due to assignable causes within a sample and at the same maximizes the probability for assignable causes between consecutive subgroups This method also provides a good (pooled) estimate for the process standard deviation

In the second approach the subgroups consist of units that were randomly sampled from the entire process output since the last sample was taken Thus the subgroups consist of observations that are representative of all units produced over a certain period of time If a process tends to move out of control and again back in control between two consecutive subgroups it can be argued that the first approach is ineffective in detecting these kinds of process shifts and the second method is preferred

There also exists other methods or approaches to obtain rational subgroups but not all these methods necessarily imply that the subgroups will be random samples from the process output This naturally leads to the violation of the assumption of independence The measurements within a sample as well as successive samples might therefore be dependent Nonetheless the rational subgroup concept is very important and any effort from the researcher to adhere to the principle of rational sub grouping with the objective to obtain as much information as possible and use the information already available can only be beneficial to a control chart procedure

2 8 Performance and statistical design of a control chart

The sooner or the faster a control chart can detect a shift (and give a signal) once a shift in the process occurred the better the control chart performs However at the same time the control chart should keep the number of false alarms i e the number of out of control signals that are given when the process is really in control to a minimum
To address control chart performance one needs to consider the statistical design of a control chart The statistical design of a control chart involves the choice of a plotting statistic the choice of control limits (whether it being k sigma control limits or probability limits) the size of the shift that needs to be detected the sample size as well as the frequency of sampling However all of these choices need to be addressed simultaneously For example when moving the control limits closer to the centerline (with the idea of quickly detecting a shift in the process) the probability of a signal if the process is really out of control increases but this also increases the false alarm rate Thus although moving the control limits closer to the centerline seemed appropriate it also has the undesirable side effect of increasing the probability of a false alarm A more appropriate course of action might be to increase the sample size or the sampling frequency instead of changing the position of the control limits

Consequently when having to choose between two or more competing control charts one needs a method or tool that incorporates all (or at least most) of these considerations all at once Only then can we determine which of the control charts is the best i e does the one outperform the other?

To illustrate the methodology consider the Shewhart $\overline{X}$ control chart for monitoring the process mean when both the process mean and the process standard deviation are *known* constants denoted by μ_0 and σ_0 respectively

Consider the probability of no signal on the $\overline{X}$ control chart once a shift in the mean (or location or central tendency) of a quality characteristic has occurred Suppose that the shift in the process mean is expressed as $\mu_1 = \mu_0 + \delta\sigma_0$ where μ_1 denotes the new process mean after the shift has occurred and $\delta\sigma_0$ is the size of the shift expressed in standard deviation terms Note that here the shift is a sustained shift in the process mean in other words the new process mean is $\delta\sigma_0$ units different from the old process mean for all future subgroups drawn from the process The probability of no signal on the $\overline{X}$ control chart given that a sustained shift in the process mean occurred is calculated as

$$
\begin{aligned}
&P(No\ Signal \mid Shift\ in\ process\ mean\ occured) \\
&= P\left(LCL < \bar{X} < UCL \mid \mu_1 = \mu_0 + \delta\sigma_0\right) \\
&= P\left(\mu_0 - k\frac{\sigma_0}{\sqrt{n}} < \bar{X} < \mu_0 + k\frac{\sigma_0}{\sqrt{n}} \mid \mu_1 = \mu_0 + \delta\sigma_0\right) \\
&= P\left(\frac{\mu_0 - k\frac{\sigma_0}{\sqrt{n}} - (\mu_0 + \delta\sigma_0)}{\frac{\sigma_0}{\sqrt{n}}} < Z < \frac{\mu_0 + k\frac{\sigma_0}{\sqrt{n}} - (\mu_0 + \delta\sigma_0)}{\frac{\sigma_0}{\sqrt{n}}}\right) \\
&= P\left(-k - \delta\sqrt{n} < Z < k - \delta\sqrt{n}\right) \\
&= \Phi\left(k - \delta\sqrt{n}\right) - \Phi\left(-k - \delta\sqrt{n}\right) \\
&= \beta(k\ \delta\ n) \text{ say}
\end{aligned}
$$

This probability is referred to as the β risk and is a function of three parameters namely the distance of the control limits from the centerline (k) the size of the shift (δ) and the sample size (n) It follows that $\beta(k\ \delta\ n)$ can be used as a tool to compare different control chart procedures The aim is

then to use $\beta(k\ \delta\ n)$ and find those values of k δ and n that maximizes the probability of a signal when the process is out of control but at the same time minimizes the false alarm rate
On the other hand the probability of a signal on the $\overline{X}$ control chart given that a shift in the process mean occurred is calculated as $1-\beta(k\ \delta\ n)$ and can also used as a tool to compare different control chart procedures

Example 2 6

Calculations with $\beta(k\ \delta\ n)$ and $1-\beta(k\ \delta\ n)$

Assume that the traditional 3 sigma control limits (or alternatively 0 00135 probability limits) are used and detecting a shift of size 1 5 standard deviations is of interest when taking samples of size $n=5$ If the process is in control i e if $\delta=0$ the probability of no signal is found by substituting $k=3$ $\delta=0$ and $n=5$ in $\beta(k\ \delta\ n)$ which yields $\beta(3\ 0\ 5)=0\ 9973$ The false alarm rate is then found as $1-0\ 9973=0\ 0027$ Note that the false alarm rate does not depend on the sample size n The probability of no signal and the probability of a signal when a process shift of 1 5 occurred is similarly found by substituting $k=3$ $\delta=1\ 5$ and $n=5$ in $\beta(k\ \delta\ n)$ which yields $0\ 638$ and $1-0\ 638=0\ 362$ respectively Thus if the process mean has either increased or decreased by 1 5 standard deviations the probability of a signal is no longer 0 0027 but 0 362

However since the size of the shift in the process mean can hardly ever be determined beforehand we need to consider the influence of δ on $\beta(k\ \delta\ n)$ or alternatively on $1-\beta(k\ \delta\ n)$ for fixed values of k and n while δ varies in the interval $[0\ \infty)$ If we then plot $\beta(k\ \delta\ n)$ versus δ we obtain the so called operating characteristic curve or simply the OC curve Figure 2 11 illustrates the OC curve of Example 2 1 when 3 sigma control limits and samples of size 5 are used Thus we are considering the value of $\beta(3\ \delta\ 5)=\Phi\left(3-\delta\sqrt{5}\right)-\Phi\left(-3-\delta\sqrt{5}\right)$ as δ changes

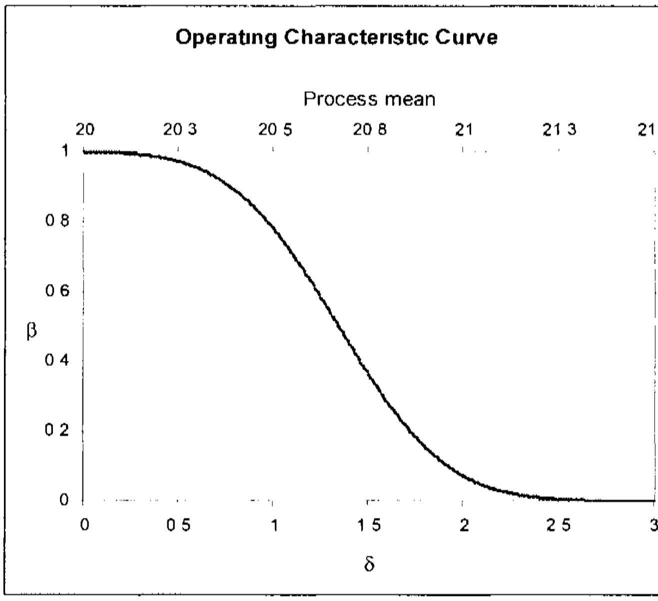

Figure 2 11 ***OC*** **curve of an** $\overline{X}$ **control chart when** $k = 3$ **and** $n = 5$

The β risk is given on the vertical axis and δ is given on the lower horizontal axis Since the β risk is in fact a probability and since 3 sigma control limits are used the value for the β risk varies between 0 (when $\delta \to \infty$) and 0 9973 (when $\delta = 0$) In addition the upper horizontal axis indicates the process mean μ_1 The relationship between the lower and the upper horizontal axis is linear and is given by $\mu_1 = \mu_0 + \delta\sigma_0 = 20 + \delta 0\ 5$ However we focus on the relationship between the β risk and δ (or the size of the shift in the process mean) and not on the process mean itself

As the process mean moves further away from the known in control value of 20 (when $\delta = 0$) the β risk decreases Thus the larger the sustained shift in the process mean the smaller the probability that the sample mean (or plotting statistic) will plot between the upper control limit and the lower control limit or alternatively give no signal

Alternatively we can create a graph of the probability of a signal given that a sustained shift in the process mean occurred i e $1-\beta(k\ \delta\ n)$ which is usually better understood Figure 2 12 indicates how the probability of a signal increases as the size of the sustained shift increases Intuitively an increase in the probability of a signal (or on the other hand a decrease in the probability for no signal) was expected

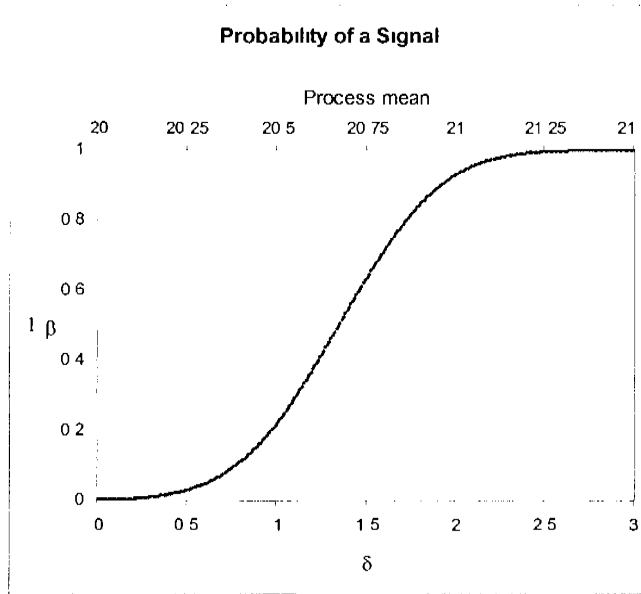

Figure 2 12 Probability of a signal of an $\overline{X}$ control chart when $k = 3$ and $n = 5$

Note that considering the probability of a signal on an $\overline{X}$ control chart i e $1-\beta(k\ \delta\ n)$ for different values of δ is similar to considering the power $\pi(\delta)$ of a two sided hypothesis test when we wish to test the null hypothesis $H_0\ \ \mu = \mu_0$ versus the alternative hypothesis $H_1\ \ \mu \neq \mu_0$ using the test statistic $Z = \dfrac{\overline{X}-\mu_0}{\frac{\sigma_0}{\sqrt{n}}}$ Typically we reject H_0 in favor of H_1 when $|Z| \geq k$ and do not reject H_0 when $|Z| < k$

Thus if we assume that the null hypothesis is false i e $\mu = \mu_0 + \delta\sigma_0$ and calculate the probability of a type II error (β) for various values of δ we find that

$$\pi(\delta) = 1-\beta = 1-P\left(|Z| < k \mid \mu = \mu_0 + \delta\sigma_0\right) = 1-\Phi\left(k-\delta\sqrt{n}\right)+\Phi\left(-k-\delta\sqrt{n}\right)$$

which is simply $1-\beta(k\ \delta\ n)$

Example 2 7

Comparing two competing control chart plans using $\beta(k\ \delta\ n)$ and $1-\beta(k\ \delta\ n)$

Suppose that an operator would like to choose between two competing control chart plans to monitor the process mean of Example 2 1 Furthermore suppose that an $\overline{X}$ control chart with 3 sigma control limits are used However the first plan uses samples of size $n = 5$ taken every 30 minutes whereas the second plan uses samples of size $n = 6$ taken every 30 minutes Thus the first plan corresponds to the plan of Example 2 1 whereas the second plan is new The *OC* curves for both these plans are displayed in Figure 2 13 with the accompanying probabilities of a signal (or $1-\beta(k \ \delta \ n)$) shown in Figure 2 14 In addition Table 2 5 lists some values of $\beta(k \ \delta \ n)$ and $1-\beta(k \ \delta \ n)$ for various δ

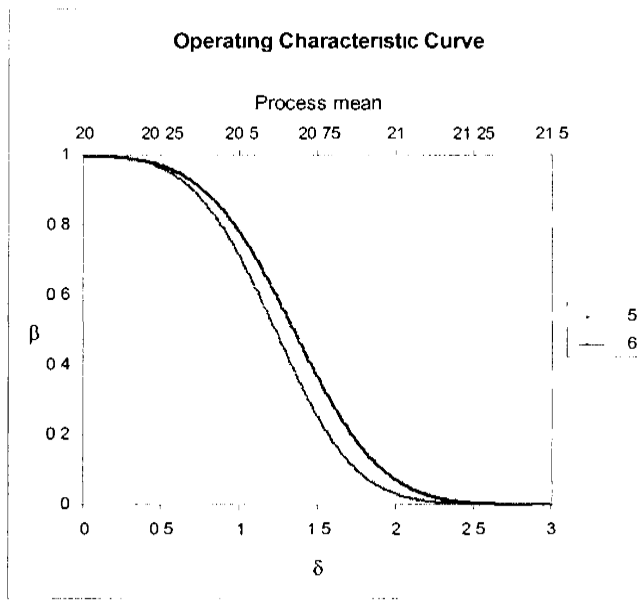

Figure 2 13 Operating characteristic curves for (1) Plan 1 with $n=5$ and (2) Plan 2 with $n=6$ for varying δ

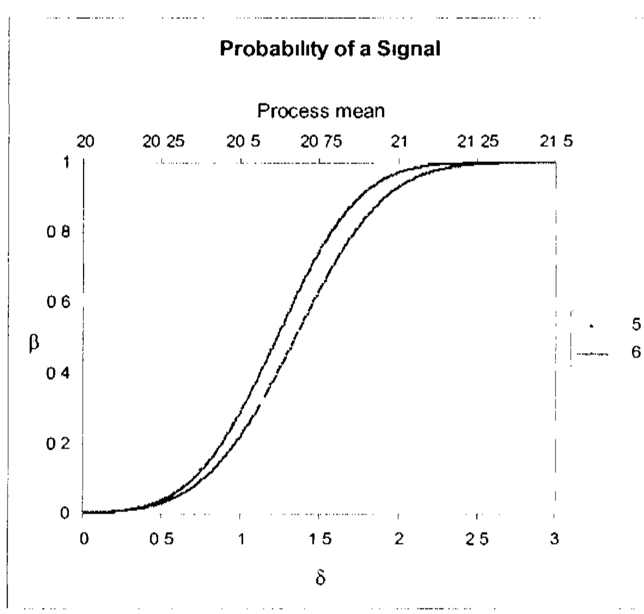

Figure 2 14 The Probability of a Signal for (1) Plan 1 with $n=5$ and (2) Plan 2 with $n=6$ for varying δ

We see that the second plan has a consistently lower β risk for all δ whereas the probability of a signal is consistently higher with Plan 2 than with Plan 1 Thus if the objective is to detect a shift in the process mean as soon as possible without considering the cost of sampling Plan 2 is preferred which involves sampling an extra item during every sampling period

	Probability of no signal		Probability of a signal	
δ	$\beta(3\ \delta\ 5)$	$\beta(3\ \delta\ 6)$	$1-\beta(3\ \delta\ 5)$	$1-\beta(3\ \delta\ 6)$
0	0 9973	0 9973	0 0027	0 0027
0 25	0 9925	0 9914	0 0075	0 0086
0 50	0 9701	0 9621	0 0299	0 0379
0 75	0 9071	0 8776	0 0929	0 1224
1 00	0 7775	0 7090	0 2225	0 2910
1 25	0 5812	0 4753	0 4188	0 5247
1 50	0 3616	0 2501	0 6384	0 7499
2 00	0 0705	0 0288	0 9295	0 9712
2 25	0 0211	0 0060	0 9789	0 9940
2 50	0 0048	0 0009	0 9952	0 9991
2 75	0 0008	0 0001	0 9992	0 9999
3 00	0 0001	0 0000	0 9999	1 0000

Table 2 5 The Probabilıty of no sıgnal and the Probabılıty of a sıgnal for two competıng control chart plans

A second method of evaluatıng the performance of control charts ıs to consıder the run length dıstrıbutıon The run length ıs sımply a dıscrete random varıable (denoted by N) that represents the number of subgroups that must be collected (or equıvalently the number of plottıng statıstıcs that must be plotted) ın order for the control chart to first detect a shıft or gıve a sıgnal

Assumıng that the samples are mutually ındependent and that the probabılıty for a sıgnal on any one of the samples are the same the run length dıstrıbutıon ıs gıven by

$$P(N=j)=\beta(k\ \delta\ n)^{j\ 1}\left(1-\beta(k\ \delta\ n)\right)$$

with $j=1\ 2\ 3$

Thıs probabılıty expressıon ıs recognızed as the probabılıty mass functıon (pmf) of a geometrıc random varıable wıth probabılıty of success equal to $1-\beta(k\ \delta\ n)$ so that $N\sim Geo\left(1-\beta(k\ \delta\ n)\right)$

A popular measure of a control charts performance and of partıcular ınterest has been the expected value of the run length dıstrıbutıon called the average run length (ARL)
The ARL of a control chart ıs defined as the expected number of subgroups that must be collected ın order for the control chart to first detect a shıft or gıve a sıgnal Therefore we can define the ın control average run length (ARL_0) as the expected number of subgroups that must be collected before a control chart first gıves a sıgnal from a stable or an ın control process On the other hand the out of control average run length (ARL_1) ıs the expected number of samples that must be collected before a

control chart first gives a signal following a shift in the process Obviously for an efficient control chart the in control ARL should be large and the out of control ARL should be small

From the properties of the geometric distribution the expected value of N (or $E(N)$) is easily obtained so that

$$ARL = E(N) = \frac{1}{1-\beta(k\ \delta\ n)}$$

Thus in the case of independence and constant signaling events knowing the probability of a signal implies that the ARL is also known since the ARL is merely the reciprocal of the probability of a signal This is one of the reasons for the popularity of the ARL and the probability of a signal as measures of the performance of a control chart
The ARL is also a function of three parameters namely the distance of the control limits from the centerline (k) the size of the shift (δ) and the sample size (n) and this relationship can be expressed as

$$ARL(k\ \delta\ n) = \frac{1}{1-\Phi\left(k-\delta\sqrt{n}\right)+\Phi\left(-k-\delta\sqrt{n}\right)}$$

Consequently the ARL can also be used to compare two or more control chart schemes or plans

Example 2 8

Comparing two competing control chart plans using $ARL(k\ \delta\ n)$

Suppose one has to choose between the two competing control chart plans from Example 2 7 when using the ARL as the performance measure The ARL (for various δ) of both plans are shown in Figure 2 15 with the ARL on the vertical axis and the size of the shift (δ) on the lower horizontal axis together with the actual process mean on the upper horizontal axis The most efficient control chart plan is clearly the one for which the ARL is the smallest given a particular shift in the process mean

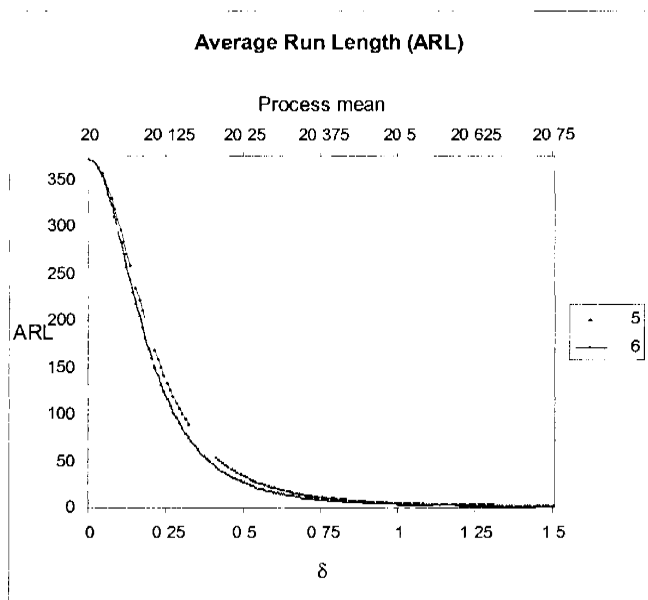

Figure 2 15 Average run length (ARL) for (1) Plan 1 with $n = 5$ and (2) Plan 2 with $n = 6$ for varying δ

For example from Table 2 6 we see that if $\delta = 0\ 5$ then Plan 2 with $ARL(3\ 0\ 5\ 6) = 26\ 36$ which uses $n = 6$ observations at each sampling stage is preferred to Plan 1 with $ARL(3\ 0\ 5\ 5) = 33\ 40$ On the other hand if $\delta \geq 2\ 5$ then either Plan 1 or Plan 2 may be used since their average run lengths are basically equal i e $ARL(3\ \delta\ 5) \approx ARL(3\ \delta\ 6) \approx 1$ for $\delta \geq 2\ 5$

δ	**Plan 1** $ARL(3\ \delta\ 5)$	**Plan 2** $ARL(3\ \delta\ 6)$
0	370 38	370 38
0 25	133 16	115 87
0 50	33 40	26 36
0 75	10 76	8 17
1 00	4 50	3 44
1 25	2 39	1 91
1 50	1 57	1 33
1 75	1 22	1 11
2 00	1 08	1 03
2 25	1 02	1 01
2 50	1 00	1 00
2 75	1 00	1 00
3 00	1 00	1 00

Table 2 6 The average run lengths for two competing control chart plans

Note that in order to compare two control chart plans it is customary to fix the in control average run length (ARL_0) at some specified or desirable value – in this case 370 38 (This was also done in Example 2 7 when we used $\beta(k\ \delta\ n)$ and $1-\beta(k\ \delta\ n)$ see Table 2 5)

Two other measures of performance of a control chart that is based on the ARL (or again the probability of a signal) are the average time to signal and the number of individual items that have been produced or inspected

The average time to signal (ATS) if samples are taken over equally spaced time intervals is given by

$$ATS = ARL \; h$$

where h is the time between any two successive samples Thus the ATS is the average number of time periods until a signal is generated on the control chart

The ARL can also be expressed as the number of individual items that have been inspected (I) rather than the number of samples taken If all the samples are of size n I is given by

$$I = ARL \; n$$

However evaluating the performance of a control chart can be more difficult and time consuming than was presented in the foregoing discussion A more in depth analysis of the performance of the Shewhart $\overline{X}$ control chart is given in Chapter 3

2 9 Sensitivity rules

The basic criterion for a process to be considered out of control (or out of statistical control) is when one point plots either above the upper control limit or below the lower control limit Using 3 sigma control limits and assuming that the distribution of the plotting statistic is approximately normal the probability that a point plots outside either of the control limits is approximately 0 0027

If all the plotting statistics plot between the two control limits i e below the upper control limit and above the lower control limit but the plotting statistics exhibit some systematic behavior or a pattern appears on the control chart there is a concern that the process is not in control Therefore the event of a point plotting inside or outside the control limits shouldn t be the only method to determine whether a process is in control or out of control and examining the pattern of the plotting statistics on a control chart is also very important

Additional rules or supplementary criteria (generally known as the Western Electric rules) that depend on the occurrence of certain events such as the occurrence of an unusual pattern on the control chart or any systematic behavior of the plotting statistics on a control chart have been developed to determine if a process is in control or out of control Each of these events have been defined such that the probability of occurrence if the process is really in control i e the probability of a false alarm is more or less 0 0027 These rules are used to increase the sensitivity of a control chart that is to detect small shifts in the process that might otherwise have gone undetected and are therefore legitimately called sensitivity rules A signal for a possible out of control process is therefore also given (apart from the usual rule one point beyond the control limits) when either one of the sensitivity rules are satisfied

To apply the majority of these sensitivity rules the region between the two control limits is often partitioned into six equally spaced zones Since the two control limits are usually set at three standard deviations from the centerline the following three zones (each zone being 1 sigma wide) can be defined on either side of the centerline Zone C is the region from the centerline up to 1 standard deviation away from the center line Zone B is between one standard deviation and two standard deviations from the centerline and Zone A is the region between 2 standard deviations from the centerline and the control limit see Figure 2 16

Note that although Zone A Zone B and Zone C are equally spaced the probability for a plotting statistic to plot within any of the three zones is not the same For example if an $\overline{X}$ control chart is used to monitor the mean of a normally distributed process the probability for a sample mean to plot in Zone C (on only one side of the centerline) is $P(0 \leq Z < 1) \approx 0\ 34$ whereas the probability for a sample mean to plot in Zone B is $P(1 \leq Z < 2) \approx 0\ 14$ and the probability to plot in Zone A is $P(2 \leq Z < 3) \approx 0\ 02$

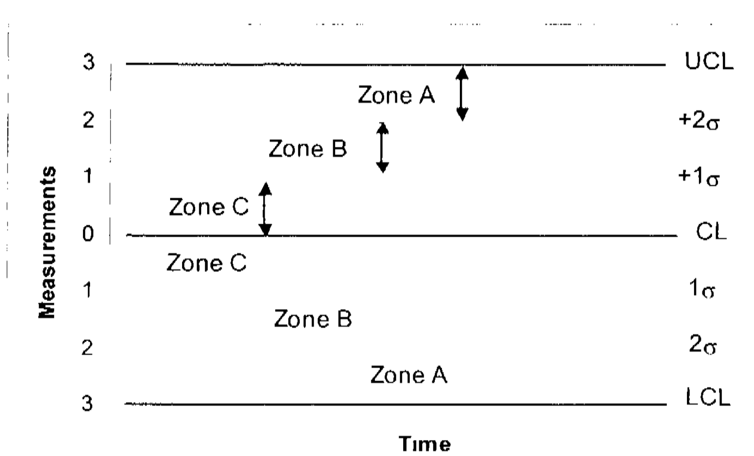

Figure 2 16 A control chart indicating Zone A Zone B and Zone C

The simultaneous application of the sensitivity rules can considerably increase the difficulty of interpreting a control chart and consequently the inherent simplicity of the Shewhart type of control chart is lost Nonetheless these sensitivity rules are frequently applied and some of them have even been implemented in modern statistical software packages The sensitivity rules in Minitab version 14 are listed below and followed by a set of graphs given in Figure 2 17 illustrating what the pattern of points on a control chart might look like

1 1 point more than 3 standard deviations from the centerline
2 9 points in a row on same side of the centerline
3 6 points in a row all increasing or all decreasing This is also known as a run up or run down of length six
4 14 points in a row alternating up and down
5 2 out of 3 points more than 2 standard deviations from the center line i e in Zone A or beyond (same side)
6 4 out of 5 points more than 1 standard deviation from the center line i e in Zone B or beyond (same side)
7 15 points in a row within 1 standard deviation of center line i e Zone C (either side)
8 8 points in a row more than 1 standard deviation from the center line i e none of these 8 points plots in Zone C (either side)

Although applying these sensitivity rules increases the sensitivity of a control chart and therefore makes it easier to detect any small change or shift in the process these rules should be applied with care Suppose that an analyst uses k of these decision rules simultaneously and decision rule i has a probability of type I error (or probability of a false alarm) of α The overall probability of a false alarm is approximately

$$\alpha = 1 - \prod_{1}^{k}(1-\alpha)$$

which can be shown to exceed all α for $i = 1\ 2 \quad k$ If all the decision rules are independent the above formula gives the exact probability of a false alarm For example Table 2 7 presents the overall probability of a false alarm (α) when $k = 1\ 2 \quad 10$ independent rules are used each with a probability of a type I error of 0 0027 0 01 and 0 05 respectively

k	α = 0 0027	α = 0 01	α = 0 05
1	0 0027	0 0100	0 0500
2	0 0054	0 0199	0 0975
3	0 0081	0 0297	0 1426
4	0 0108	0 0394	0 1855
5	0 0134	0 0490	0 2262
6	0 0161	0 0585	0 2649
7	0 0187	0 0679	0 3017
8	0 0214	0 0773	0 3366
9	0 0240	0 0865	0 3698
10	0 0267	0 0956	0 4013

Table 2 7 The overall probability of a false alarm (α)

Thus although using more than one rule (or criteria) enhances the sensitivity of a control chart and the probability of detecting special causes it also has the undesirable side effect of increasing the overall false alarm rate (α) Consequently when using these sensitivity rules they should be applied with great caution since an excessive number of false alarms may be detrimental to any SPC plan

Note that using more than one sensitivity rule is similar to repeatedly using two sample t tests to test the null hypothesis H_0 $\mu_1 = \mu_2 = \; = \mu_m$ $m = 2\ 3$ versus the alternative hypothesis that H_0 $\mu \neq \mu_j$ for some $i \neq j$ in a one way analysis of variance (ANOVA) problem

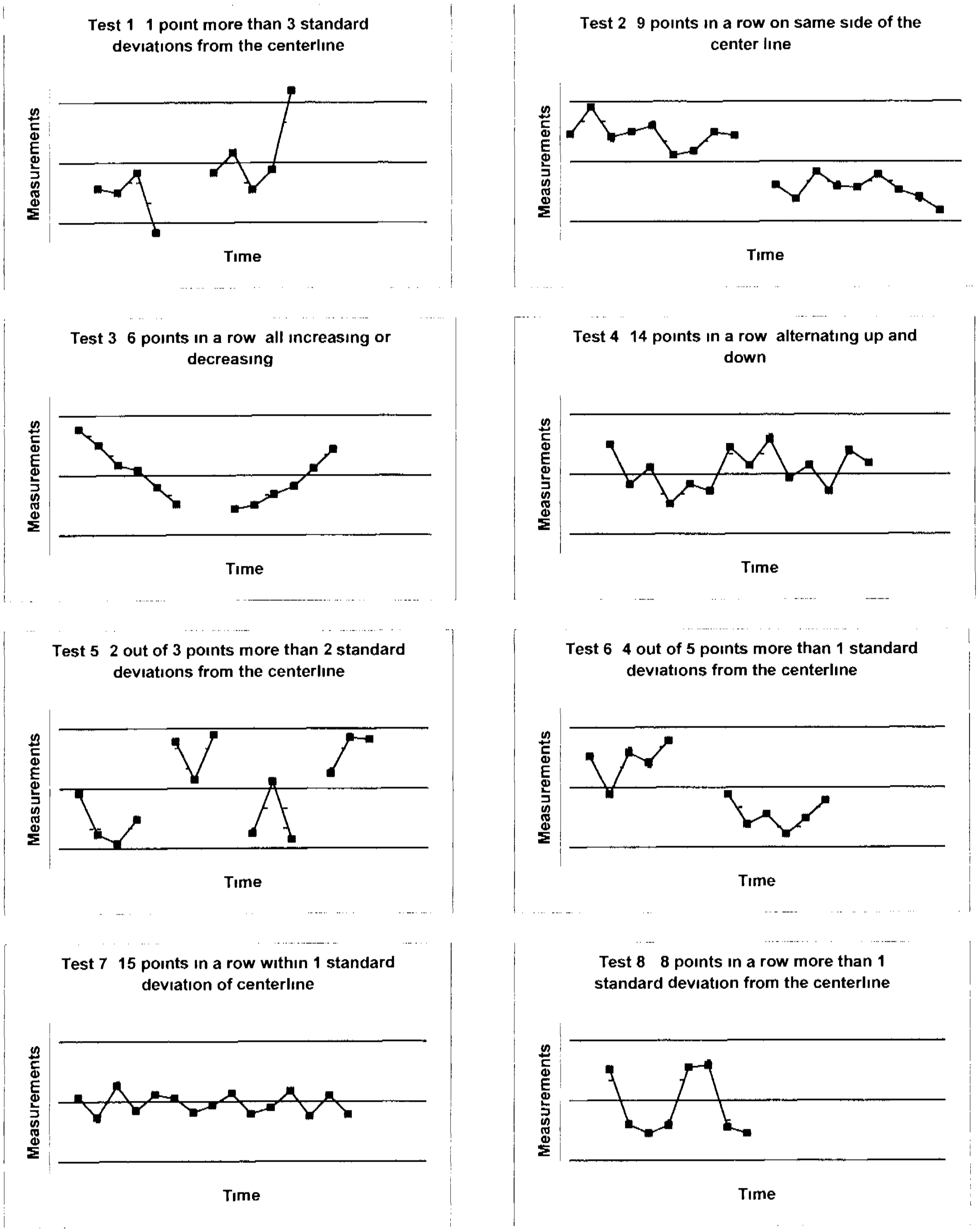

Figure 2.17 Illustration of the sensitivity rules included in Minitab version 14

2 10 Warning limits

Warning limits is another way of increasing the sensitivity of a control chart and enhancing the ability of a control chart to detect assignable causes Two sets of limits are placed on the control chart of which the outer limits coincide with the usual upper and lower 3 sigma control limits and the inner limits are the warning limits and usually set at 2 standard deviations from the centerline i e they are 2 sigma limits However if probability limits are used instead of k sigma limits the action limits are generally 0 001 probability limits and the warning limits are 0 025 probability limits

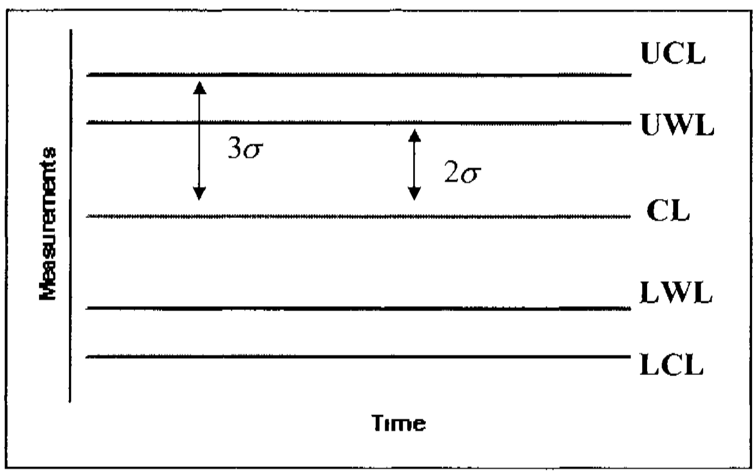

Figure 2 18 Control chart with 2 sigma warning limits (UWL and LWL) and 3 sigma control limits (UCL and LCL)

The outer limits are called the action limits and (as usual) if a point plots outside these action limits corrective action is necessary Suspicion arises that the process might not be operating properly if one or more points plot between the warning limits and action limits or if a point plots very close to either warning limit It is customary to gather more information from the process when this happens

Two courses of action exist of which the one is to increase the sample size and the other is to increase the sampling frequency A control chart scheme (or design) that varies the sampling frequency depending on the current position of the plotting statistic is called an adaptive or variable sampling interval scheme whereas a control chart scheme that varies the sample size is called a variable sample size scheme It is also possible to apply both methods simultaneously i e varying both the sampling frequency and the sample size
The use of warning limits might confuse operator personnel i e operator personnel might mix up the control limits with the warning limits However this is not of any great concern and warning limits are

routinely used As with the sensitivity rules great care should be taken when using warning limits as they also result in an increased probability of a false alarm

2 11 Study of patterns

It was noted previously that a control chart can indicate that a process is operating out of control although none of the plotting statistics fall outside the two control limits For example if a pattern develops in the plotting statistics or the plotting statistics exhibit any non random or systematic behavior action can be taken to decrease (or eliminate) the unnecessary variation in the process However if the plotting statistics (or points) are truly random (or independent from one another) the plotting statistics should be randomly scattered about the centerline and there should be more or less an even distribution of the points above and below the centerline

For example the control chart of Figure 2 2 shows a control chart where the plotting statistics are randomly scattered about the centerline whereas Figure 2 19 shows a control with a number of nonrandom or unlikely events

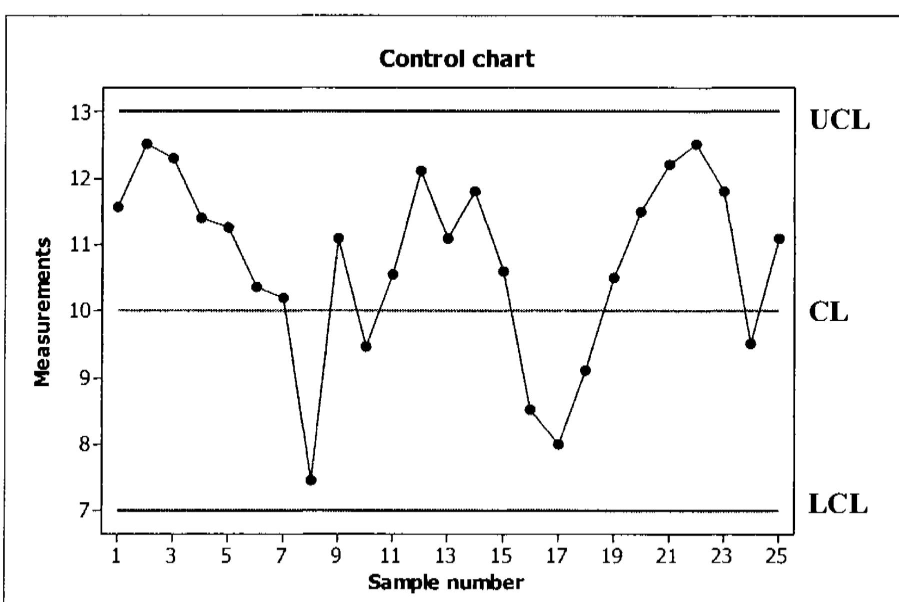

Figure 2 19 A control chart with several runs

From Figure 2 19 we notice that although all 25 points plot between the two control limits the points do not indicate statistical control since their pattern is non random in appearance Specifically of the 25 points 19 plot above the centerline whereas only 6 plot below the centerline What s more from point 2 onwards 7 points in a row steadily decrease in magnitude This arrangement of points is called a run and since these observations are all decreasing this is called a run down Similarly a set of consecutive points (such as from point 17 onwards) that steadily increase in magnitude is called a run up In general a run is defined as the succession of any number of points (or plotting statistics or observations) of the same type followed and preceded by a different point or no point at all

For example this control chart has an unusually long run down (starting from observation 2) and an unusually long run up (starting from observation 17) If the points are truly random any run of length 8 or more points has a very low probability of occurring and consequently any type of run (except a run of length 8 or more between the two control limits) is usually taken as an out of control condition Although the patterns on a control chart give some meaningful insight in the functioning (or operation) of the process it is imperative to know what causes the plotting statistics to develop a certain pattern or non random behavior so that corrective action can be taken or modifications can be made

When patterns arise on both the control chart that monitors the process variability and the control chart that monitors the process location it is a good strategy to first eliminate the assignable causes that lead to the pattern on the control chart that monitors the process variability This will usually automatically eliminate the pattern(s) on the control chart that monitors process location
The patterns that are encountered most often are shown below and briefly discussed

Cyclical pattern

A cyclical pattern usually occurs due to systematic environmental changes that influence the process Possible causes are changes or fluctuation in temperature and humidity operator fatigue and the regular rotation of operators or machines in the production process

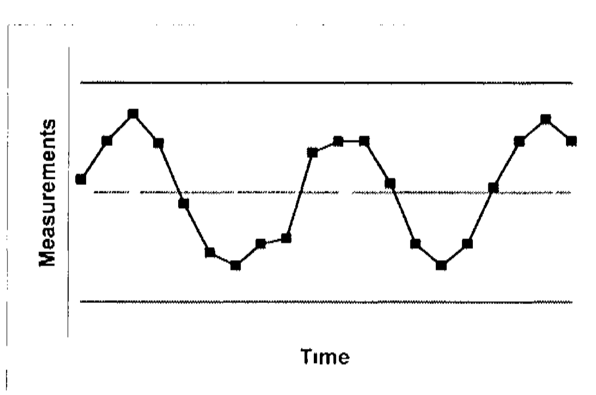

Figure 2 20 A cyclical pattern

Mixture pattern

A mixture pattern is recognized by all the points on the control chart plotting near the upper and lower control limit or just outside the two control limits with very few of the points plotting near the centerline

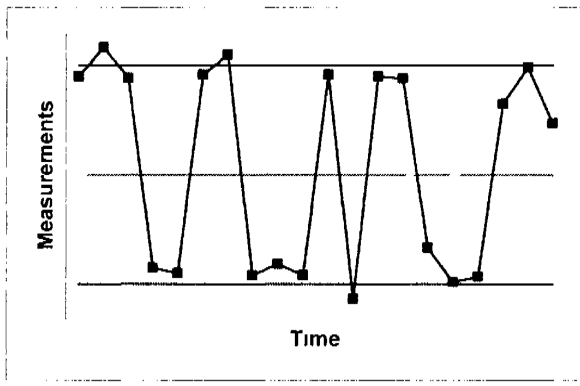

Figure 2 21 A mixture pattern

A mixture pattern occurs when the process is generated by two (or even more) underlying distributions that overlap For example when the output of two or more parallel machines is combined into a common stream before sampling occurs The extent to which these distributions overlap will determine the severity or impact on the process output and consequently on the pattern on the control chart Mixture patterns can also occur when process adjustments are made too often This might happen when an operator mistakenly interprets natural variability within a process as systematic variation

An upward or downward shift in the process level

An upward or a downward shift in the process level can be a sign of improvement or deterioration in the performance of the process Improvement in the process performance usually follows shortly after the introduction of a control chart program whereas external shocks such as introducing different workers new methods new raw materials or machines can lead to an improved process or diminish the current process

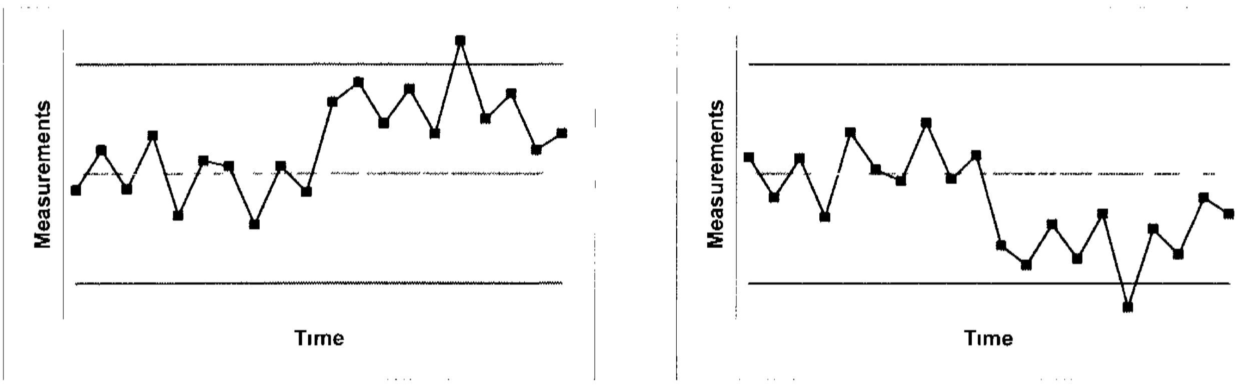

Figure 2 22 An upward and downward shift in the process level

Trends

A trend on a control chart is the continuous movement of points in one direction A downward trend or an upward trend is possible and both types of trends are an indication of either an improvement or a deterioration in the process performance

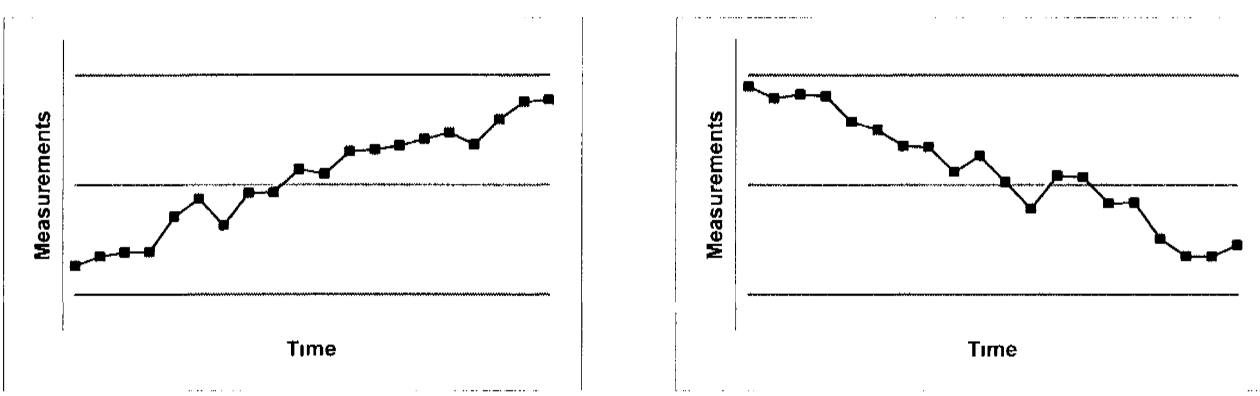

Figure 2 23 An upward and downward trend in the process level

The gradual wearing out of a tool or the deterioration of a critical process component operator fatigue the presence of supervision or seasonal influences (such as changes in temperature) are but a few causes of trends

Stratification

The tendency of points to artificially cluster around the centerline is referred typically to as stratification

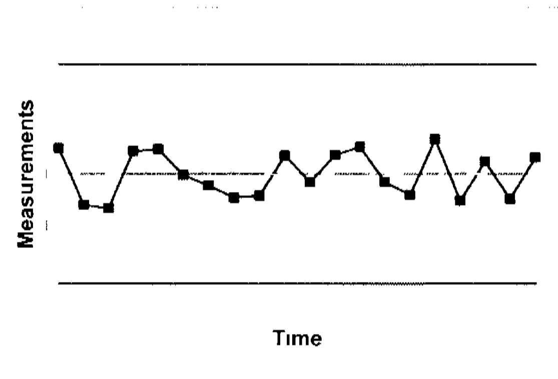

Figure 2 24 Stratification of plotting statistics on a control chart

As seen from Figure 2 17 this pattern shows a lack of natural variability in the points and might occur due to the incorrect calculation of control limits For example if it happens that the process variation is overestimated it automatically leads to control limits that are too far away from the centerline so that points tend to cluster around the centerline

2 12 Re-sampling fallacy

Control chart operators sometimes have the tendency to ignore the first point that plots out of control and often wait until the second or third point plots outside the control limits before any corrective action is taken Disregarding a signal form a control chart is referred to as the re sampling fallacy and is caused by a lack of confidence in the method or measurement system used Although it might seem like a safe enough plan it can have detrimental effects to the application of any control chart in a quality improvement environment

It can be shown that if a process has shifted slightly off center (or away from the mean value) and a point plots outside the control limits there is a fairly large probability for the subsequent point to plot inside the control limits and thereby canceling the first out of control point The process shift will eventually be detected but unless it was a fairly large shift the shift will go unnoticed for a large number of subgroups

Chapter 3 The classical Shewhart $\overline{X}$ control chart for the process mean

3 0 Chapter overview

In Chapter 2 the in control process mean and the in control process standard deviation were *known* constants denoted by μ_0 and σ_0 respectively but this is seldom the case in practice If at least one of the process parameters (the mean and/or the standard deviation) are *unknown* we must estimate the parameter(s) from historical information that is information gathered prior to the construction and/or the statistical design of the control chart when the process was thought to be in statistical control

Four distinct scenarios can arise and they are referred to as Case KK Case UK Case KU and Case UU The letters K and U refer to whether a particular process parameter is known (K) or unknown (U) i e the status of a process parameter The first letter refers to the status of the process mean whereas the second letter refers to the status of the process standard deviation For example Case KU refers to the scenario when the process mean is known and the process standard deviation is unknown The worst case scenario is when both the process parameters are unknown i e Case UU

In addition Chapter 2 only gave an example of a Shewhart $\overline{X}$ control chart for Case KK but Chapter 3 will focus on all four scenarios (Cases KK UK KU and UU) where after the last three scenarios will be compared to Case KK considered to be the reference case or the best possible scenario
Since the two control limits (UCL and LCL) and the centerline (CL) are functions of the in control process mean μ_0 and the in control process standard deviation σ_0 (see for example equation (2 2) in Chapter 2) they too must be estimated if both or either one of the process parameters are unknown This estimation can adversely affect the overall performance of the control chart
Thus the effects of estimation on the general or the overall performance of the Shewhart $\overline{X}$ control chart is examined using several performance measures such as the probability of a false alarm or the false alarm rate (FAR) the average run length (ARL) the standard deviation of the run length (SDRL) the median run length (MDRL) as well as the coefficient of skewness of the run length (SKEWRL) all being properties of the run length distribution

Exact expressions are derived for each of the above mentioned performance measures in each of the four scenarios when assuming that the process is normally distributed We use an approach of conditioning in which we condition on an observed value(s) of the point estimator(s) used to estimate the unknown process parameter(s) and find the *conditional* run length distribution Finding the *unconditional* run length distribution follows this In addition we use Jensen s inequality together with expectation by conditioning and obtain a lower bound for the average run length

Although we will focus on the classical Shewhart $\overline{X}$ control chart for monitoring the process mean similar derivations are possible for other variables attributes and/or types of control charts See for example Chapter 4 for the CUSUM type and the EWMA type of control charts

3 1 The Mean and the Standard Deviation both specified (Case KK)

Consider the Shewhart $\overline{X}$ control chart for monitoring the process mean μ when *both* the in control process mean μ_0 and the in control process standard deviation σ_0 are *known* constants The k sigma control limits and the centerline for a two sided $\overline{X}$ control chart when using samples or subgroups of size n are

$$UCL = \mu_0 + k\frac{\sigma_0}{\sqrt{n}}$$
$$CL = \mu_0 \qquad (3\ 1)$$
$$LCL = \mu_0 - k\frac{\sigma_0}{\sqrt{n}}$$

(See equation (2 2) in Chapter 2)

The false alarm rate (FAR) or the probability of a signal when the process is actually in control α is indirectly specified by choosing the distance of the two control limits from the centerline For example if we assume that the process follows a normal distribution choosing k equal to 3 (or using 3 sigma control limits) leads to a false alarm rate of 0 0027

However the control limits in equation (3 1) can also be expressed as probability limits (see Result 3 1 in Appendix 3A) i e

$$UCL = \mu_0 + z_{\frac{\alpha}{2}}\frac{\sigma_0}{\sqrt{n}}$$
$$CL = \mu_0 \qquad (3\ 2)$$
$$LCL = \mu_0 - z_{\frac{\alpha}{2}}\frac{\sigma_0}{\sqrt{n}}$$

where $z_{\frac{\alpha}{2}}$ is called the critical value or the control chart constant and is a percentile of the standard normal distribution Under this approach it is customary to fix or specify α at some desirable (usually small) value and then find the corresponding critical value For example specifying that $\alpha = 0\ 0027$ requires us to find $z_{0\ 00135}$ from $P\left(Z \geq z_{0\ 00135}\right) = 0\ 00135$ which yields $z_{0\ 00135} = 3$

Although the two approaches yield similar results we will use the latter Note that the control limits and the centerline in both equation (3 1) and equation (3 2) are constants that is neither the control limits nor the centerline depends on any random or unknown quantities

Once the control limits and the centerline are calculated the control chart is typically implemented as follows Independent samples of size n are taken at random at equally spaced time intervals and the sample means $\overline{X}_i$ $i = 1\ 2\ 3$ are calculated and plotted on the chart If one of the plotting statistics or points fall on or outside the control limits i e if the control chart signals the process is declared out of control and a search for assignable or special causes is started On the other hand if all the points fall between the control limits i e if the control chart does not signal the process is considered in control and sampling continues (See for instance Example 2 1)

3 1 1 Properties of the $\overline{X}$ control chart

The performance of a control chart is typically judged in terms of the properties or the characteristics of its run length distribution For instance the most commonly used measures are the probability of a signal or alternatively the probability of no signal and the average run length (ARL) However for a performance study to have meaning we need to choose practical and fair measures For example in Cases UK KU and UU the run length distribution is not geometric as in Case KK and therefore the ARL should not be used as the sole measure of a control charts performance Hence supplementing the ARL with the median run length (MDRL) the standard deviation of the run length (SDRL) percentiles of the run length distribution and the coefficient of skewness of the run length (SKEWRL) will give a more complete picture of a control charts performance

To find the run length distribution let N denote the run length random variable in other words let N be the number of subgroups or samples that need to be observed until the chart gives the first signal

A non signaling event and a signaling event when both the process parameters are *known* can be expressed as

$$\frac{\sqrt{n}\left|\overline{X}_i - \mu_0\right|}{\sigma_0} < z_{\frac{\alpha}{2}} \qquad (3\ 3)$$

and

$$\frac{\sqrt{n}\left|\overline{X} - \mu_0\right|}{\sigma_0} \geq z_{\frac{\alpha}{2}} \tag{3 4}$$

respectively (See Results 3 2 and 3 3 in Appendix 3A)
The corresponding probabilities of such events i e

$$P\left(\frac{\sqrt{n}\left|\overline{X} - \mu_0\right|}{\sigma_0} < z_{\frac{\alpha}{2}} \mid \mu_1 = \mu_0 + \delta\sigma_0\right) \quad \text{and} \quad P\left(\frac{\sqrt{n}\left|\overline{X} - \mu_0\right|}{\sigma_0} \geq z_{\frac{\alpha}{2}} \mid \mu_1 = \mu_0 + \delta\sigma_0\right)$$

when a shift in the process mean occurred are given by

$$\beta(\alpha\ \delta\ n) = \Phi\left(z_{\frac{\alpha}{2}} - \delta\sqrt{n}\right) - \Phi\left(-z_{\frac{\alpha}{2}} - \delta\sqrt{n}\right) \tag{3 5}$$

and

$$1 - \beta(\alpha\ \delta\ n) = 1 - \Phi\left(z_{\frac{\alpha}{2}} - \delta\sqrt{n}\right) + \Phi\left(-z_{\frac{\alpha}{2}} - \delta\sqrt{n}\right) \tag{3 6}$$

respectively with $i = 1\ 2\ 3$ and where Φ denotes the cumulative distribution function (cdf) of the standard normal distribution whereas

$$\delta = \frac{\left|\mu_1 - \mu_0\right|}{\sigma_0} \tag{3 7}$$

represents the *relative* size of the shift in the process mean that is the *absolute* size of the shift in the process mean $\left|\mu_1 - \mu_0\right|$ divided by σ_0 In other words we assume that at some point in time the process mean μ undergoes a sustained shift from its known in control value μ_0 to a new value μ_1 and then remains constant at this new value In addition we assume that the known in control process standard deviation σ_0 stays unchanged Typically we would write

$$\mu_1 = \mu_0 + \delta\sigma_0 \tag{3 8}$$

so that μ_1 is expressed in terms of μ_0 and σ_0 Then solving for δ from equation (3 8) yields equation (3 7)
(See Results 3 4 and 3 5 in Appendix 3A for the derivation of equations (3 5) and (3 6))

Since we assume that all the samples are mutually independent it follows that all the non signaling events are mutually independent and so are all the signaling events Therefore the probability distribution of the run length random variable N is geometric with probability of a signal (success) equal to $1-\beta(\alpha, \delta, n)$ i e $N \sim Geo(1-\beta(\alpha, \delta, n))$ Consequently the probability mass function (pmf) is

$$P(N = j) = \beta(\alpha, \delta, n)^{j-1}(1-\beta(\alpha, \delta, n)) \text{ for } j = 1, 2, 3 \qquad (3\ 9)$$

and the cumulative distribution function (cdf) and the quantile function are

$$P(N \le j) = 1-\beta(\alpha, \delta, n)^{j} \text{ for } j = 1, 2, 3 \qquad (3\ 10)$$

and

$$\begin{aligned} Q_N(p) &= \inf\{j \text{ an integer } P(N \le j) \ge p\} \\ &= \inf\left\{j \text{ an integer } j \ge \frac{\ln(1-p)}{\ln\beta(\alpha, \delta, n)}\right\} \quad 0 < p < 1 \end{aligned} \qquad (3\ 11)$$

respectively

Note that in addition to mutually independent samples the probability of a non signaling event and that of a signaling event should remain constant in order for the run length distribution to be geometric In other words we assume that α δ and n are fixed
This is particularly important for the interpretation of the run length distribution presented in equations (3 9) (3 10) and (3 11) For example consider an $\overline{X}$ control chart of a process that operates in control up and till some (unknown) point in time t say where after the process mean undergoes a shift from μ_0 to $\mu_1 = \mu_0 + \delta\sigma_0$ with $\delta = 1.5$ say Then while the process is in control and assuming that the process stays in control i e $\delta = 0$ we have that

$$P(N = j) = \beta(\alpha, 0, n)^{j-1}(1-\beta(\alpha, 0, n)) \text{ for } j = 1, 2, 3$$

However immediately following the shift we have that

$$P(N = j) = \beta(\alpha, 1.5, n)^{j-1}(1-\beta(\alpha, 1.5, n)) \text{ for } j = 1, 2, 3$$

The average run length or the expected value of N is

$$ARL = E(N) = \frac{1}{1-\beta(\alpha, \delta, n)} \qquad (3\ 12)$$

whereas the variance the standard deviation and the coefficient of skewness of the run length distribution are

$$VARRL = \text{var}(N) = \frac{\beta(\alpha\ \delta\ n)}{(1-\beta(\alpha\ \delta\ n))^2} \tag{3 13}$$

$$SDRL = stdev(N) = \frac{\sqrt{\beta(\alpha\ \delta\ n)}}{1-\beta(\alpha\ \delta\ n)} \tag{3 14}$$

and

$$SKEWRL = \frac{1+\beta(\alpha\ \delta\ n)}{\sqrt{\beta(\alpha\ \delta\ n)}} \tag{3 15}$$

respectively

The median run length or any other percentile of the run length distribution can be found by using the quantile function given in equation (3 11) For example when substituting $p = 0\ 5$ the median run length is found to be

$$MDRL = \inf\left\{ j \text{ an integer } j \geq \frac{\ln(0\ 5)}{\ln\beta(\alpha\ \delta\ n)} \right\} \tag{3 16}$$

Equations (3 9) up to (3 15) follow directly and conveniently from the well known properties of the geometric distribution see Results 3 7 – 3 13 in Appendix 3A for the necessary derivations

Note that all these equations or expressions depend on the probability of a false alarm α the relative size of the shift in the process mean δ and the sample size n Therefore to evaluate and study the performance of the Shewhart $\overline{X}$ control chart it is necessary to determine the influence of these parameters (both individually and simultaneously or combined) on each of the characteristics of the run length distribution

3 1 2 The in-control run length distribution and its properties

The important and special case when the process operates in statistical control in other words when $\delta = 0$ the probability of a signal is referred to as the false alarm rate (FAR) or the probability of a false alarm and given by

$$1-\beta(\alpha\ 0\ n) = 1-\Phi\left(z_{\frac{\alpha}{2}}\right)+\Phi\left(-z_{\frac{\alpha}{2}}\right) = \alpha \tag{3 17}$$

whereas the probability of no signal is then given by

$$\beta(\alpha\ 0\ n) = 1 - \alpha \tag{3 18}$$

Then we obtain the in control run length distribution to be

$$P(N_0 = j) = (1-\alpha)^{j-1}\alpha \quad \text{for } j = 1\ 2\ 3 \tag{3 19}$$

with the *in control* cumulative distribution function (cdf) being

$$P(N_0 \le j) = 1 - (1-\alpha)^{j} \quad \text{for } j = 1\ 2\ 3 \tag{3 20}$$

and the *in control* quantile function given by

$$Q_N(p) = \inf\left\{ j \text{ an integer } \ j \ge \frac{\ln(1-p)}{\ln(1-\alpha)} \right\} \quad \text{for } 0 < p < 1 \tag{3 21}$$

The *in control* ARL is then found to be the reciprocal of the false alarm rate α i e

$$ARL_0 = \frac{1}{\alpha} \tag{3 22}$$

whereas the variance the standard deviation and the coefficient of skewness of the *in control* run length distribution are respectively found by substituting $\delta = 0$ in the relevant expressions (i e equations (3 13) (3 14) and (3 15)) so that

$$VARRL_0 = \frac{1-\alpha}{\alpha^2} \tag{3 23}$$

$$SDRL_0 = \frac{\sqrt{1-\alpha}}{\alpha} \tag{3 24}$$

and

$$SKEWRL_0 = \frac{2-\alpha}{\sqrt{1-\alpha}} \tag{3 25}$$

Lastly the *in control* median run length is found by substituting $p = 0\ 5$ in equation (3 21) so that

$$MDRL_0 = \inf\left\{ j \text{ an integer } \ j \ge \frac{\ln(0\ 5)}{\ln(1-\alpha)} \right\} \tag{3 26}$$

Thus when the process operates in statistical control all the characteristics of the run length distribution are merely functions of the probability of a false alarm α (as can be seen from equations (3 17) – (3 26)) In addition since we would typically fix the false alarm rate α or as some practitioners do fix the in control average run length ARL_0 we see from equation (3 22) that specifying the false alarm rate α specifies the ARL_0 and vice versa This (simple) relationship is one of the reasons for thc popularıty of the Shcwhart $\overline{X}$ control chart and subsequently also a reason for the popularıty of the probabılıty of a sıgnal and the average run length (ARL) as performance measures

The *ın control* run length dıstrıbutıon and ıts assocıated statıstıcal propertıes are ımportant for (at lcast) two reasons Fırstly the ın control run length dıstrıbutıon represents the desırable sıtuatıon when the process operates ın statıstıcal control Therefore we obtaın an ındıcatıon of the performance of the $\overline{X}$ control chart when the process operates only ın the presence of chance or common causes of varıatıon and no specıal or assıgnable causes of varıatıon are present Secondly the results obtaıned when evaluatıng expressıons (3 17) – (3 26) for dı fferent values of α are used as benchmark values for the ın control run length dıstrıbutıon of other types of control charts for ınstance the CUSUM type and the EWMA type of control charts

3 1 3 Probability of no signal and the probability of a signal

The probability of no signal $\beta(\alpha\ \delta\ n)$ and the probability of a signal $1-\beta(\alpha\ \delta\ n)$ as given in equations (3 5) and (3 6) respectively were previously used to compare and choose between the two competing control chart plans of Example 2 7 in Chapter 2 However we only focused on the effect or influence of the sample size n and did not consider the influence of α and/or δ Therefore in this section we study these two probability expressions in more detail and in subsequent sections we also show how the other characteristics of the run length distribution are influenced by the choices of α δ and n

Now consider the four control chart plans i e Plan A B C and D with their parameters α and n as specified in Table 3 1

Control chart plan	False alarm rate (α)	Sample size (n)
A	0 0027	5
B	0 0027	6
C	0 05	5
D	0 05	6

Table 3 1 Four control chart plans

Figure 3 1 displays the *OC* curves for each control chart plan whereas Figure 3 2 displays the corresponding probability of a signal for each of the four control chart plans In other words we consider $\beta(\alpha\ \delta\ n)$ given in equation (3 5) and $1-\beta(\alpha\ \delta\ n)$ given in equation (3 6) for fixed α and n as δ varies

Figure 3 1 clearly shows how $\beta(\alpha\ \delta\ n)$ decreases as δ increases whereas Figure 3 2 indicates how $1-\beta(\alpha\ \delta\ n)$ increases as δ increases By closer inspection of Figure 3 1 and by comparing $\beta(0\ 0027\ \delta\ 5)$ with $\beta(0\ 0027\ \delta\ 6)$ or by comparing $\beta(0\ 05\ \delta\ 5)$ with $\beta(0\ 05\ \delta\ 6)$ we also note that the probability of no signal decreases as the sample size n increase or alternatively that the probability of a signal increases as the sample size increases when looking at Figure 3 2
In addition examining Figures 3 1 and 3 2 even further it is also noticeable (and should be obvious) that the probability of a signal increases or the probability of no signal decreases as α increases

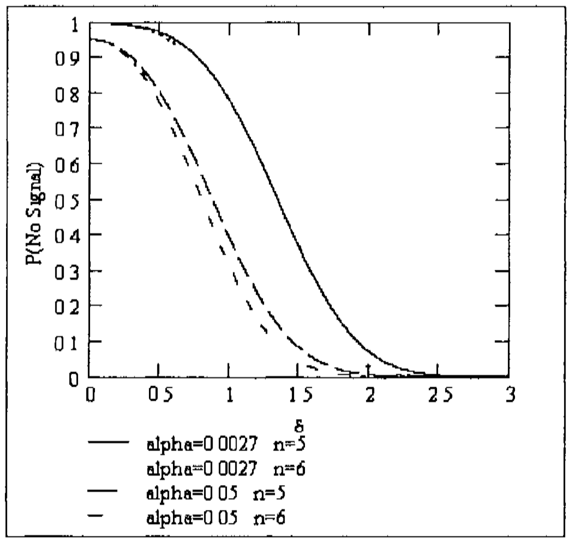

Figure 3 1 Probability of no signal $\beta(\alpha \ \delta \ n)$

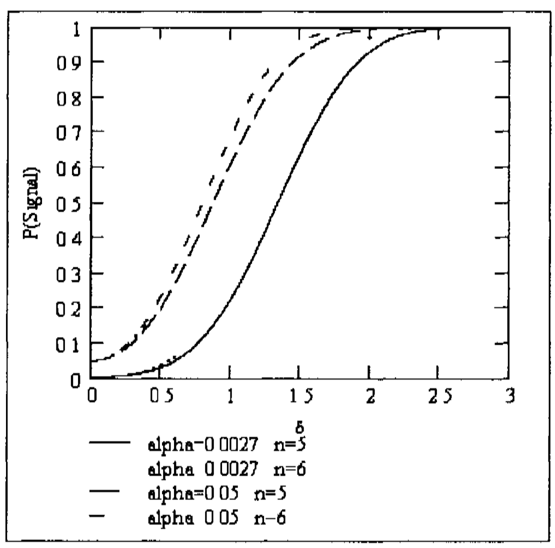

Figure 3 2 Probability of a signal $1-\beta(\alpha \ \delta \ n)$

Hence as each of the three parameters α δ and n increases the probability of a signal increases whereas the probability of no signal decreases – see Table 3 2 for a summary

	α	δ	n
$P(\text{Signal}) = 1-\beta(\alpha \ \delta \ n)$	Increase ↑	Increase ↑	Increase ↑
$P(\text{No Signal}) = \beta(\alpha \ \delta \ n)$	Decrease ↓	Decrease ↓	Decrease ↓

Table 3 2 The influence of α δ **and** n **on the probability of a signal and the probability of no signal**

In spite of this it should be remembered that the *rate* at which these probabilities increase and/or decrease changes as any of the other quantities change

For example Figure 3 3 displays the probability of a signal $1-\beta(\alpha \ \delta \ n)$ versus the false alarm rate α for three different values of the relative size of the shift δ when $n=5$ We note that the *rate* at which the probability of a signal increases increases as δ increases Additionally Figure 3 4 displays the probability of no signal $\beta(\alpha \ \delta \ n)$ versus the sample size n for three different values of the relative size of the shift δ when $\alpha = 0\ 0027$ From Figure 3 4 we can see that the *rate* at which the probability of no signal decreases also increases (in absolute value) as δ increases [1] Thus to better understand the performance of the control chart we also need to take into account the combined effect of α δ and n on the probability of a signal and the probability of no signal

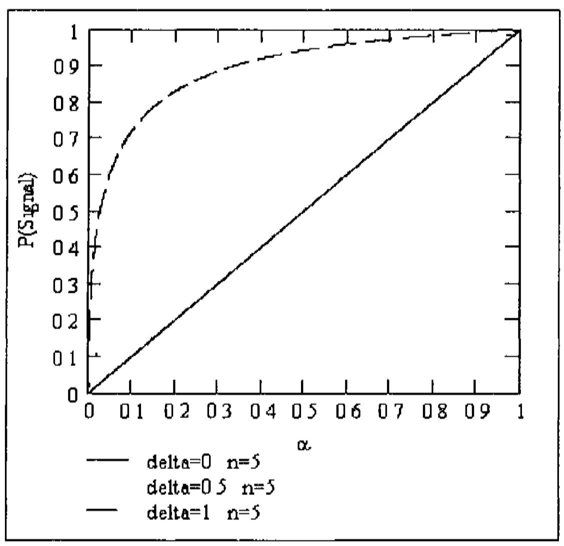

Figure 3 3 The rate of change in the probability of a signal $1-\beta(\alpha \ \delta \ n)$

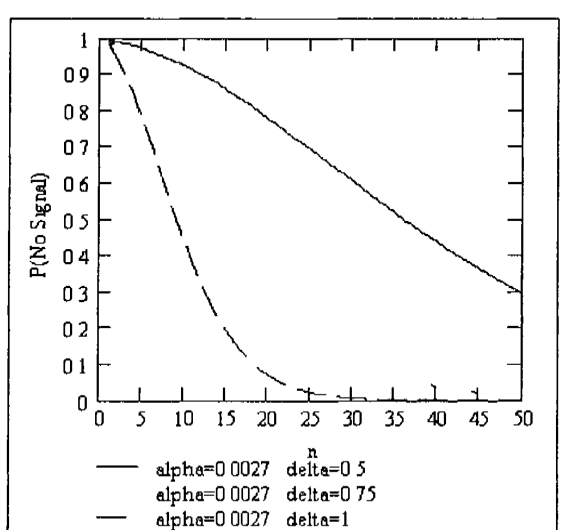

Figure 3 4 The rate of change in the probability of no signal $\beta(\alpha \ \delta \ n)$

[1] We reach these conclusions by visually comparing the slopes or gradients of the three lines in each figure or display

To this end Figure 3 5 accompanied by Table 3 3 shows how the probability of no signal $\beta(\alpha\ \delta\ n)$ is influenced by changes in both δ and n when the false alarm rate α is fixed at 0 0027 In addition Figure 3 6 together with Table 3 4 and Figure 3 7 together with Table 3 5 shows the probability of no signal as a function of δ and α when $n = 5$ and the probability of no signal as a function of α and n when $\delta = 0\ 5$ respectively

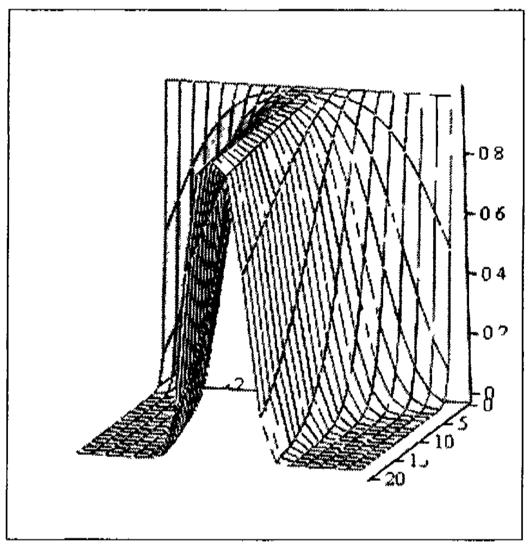

Figure 3 5 Probability of no signal as a function of δ and n when α = 0 0027

	Sample size (n)												
δ	1	2	3	4	5	6	7	8	9	10	15	20	25
0 00	0 9973	0 9973	0 9973	0 9973	0 9973	0 9973	0 9973	0 9973	0 9973	0 9973	0 9973	0 9973	0 9973
0 25	0 9964	0 9955	0 9946	0 9936	0 9925	0 9914	0 9902	0 9890	0 9877	0 9864	0 9789	0 9701	0 9599
0 50	0 9936	0 9890	0 9835	0 9772	0 9701	0 9621	0 9532	0 9436	0 9332	0 9220	0 8562	0 7776	0 6915
0 75	0 9877	0 9737	0 9555	0 9332	0 9071	0 8776	0 8451	0 8102	0 7734	0 7351	0 5380	0 3616	0 2266
1 00	0 9772	0 9436	0 8976	0 8413	0 7776	0 7090	0 6384	0 5681	0 5000	0 4356	0 1913	0 0705	0 0228
1 25	0 9599	0 8911	0 7981	0 6915	0 5812	0 4753	0 3794	0 2961	0 2266	0 1703	0 0328	0 0048	0 0006
1 50	0 9332	0 8102	0 6561	0 5000	0 3616	0 2501	0 1664	0 1070	0 0668	0 0406	0 0025	0 0001	0 0000
1 75	0 8944	0 7003	0 4876	0 3085	0 1806	0 0991	0 0515	0 0256	0 0122	0 0056	0 0001	0 0000	0 0000
2 00	0 8413	0 5681	0 3213	0 1587	0 0705	0 0288	0 0110	0 0039	0 0014	0 0004	0 0000	0 0000	0 0000
2 25	0 7734	0 4278	0 1848	0 0668	0 0211	0 0060	0 0016	0 0004	0 0001	0 0000	0 0000	0 0000	0 0000
2 50	0 6915	0 2961	0 0917	0 0228	0 0048	0 0009	0 0002	0 0000	0 0000	0 0000	0 0000	0 0000	0 0000
2 75	0 5987	0 1870	0 0389	0 0062	0 0008	0 0001	0 0000	0 0000	0 0000	0 0000	0 0000	0 0000	0 0000
3 00	0 5000	0 1070	0 0140	0 0014	0 0001	0 0000	0 0000	0 0000	0 0000	0 0000	0 0000	0 0000	0 0000

Table 3 3 Probability of no signal $\beta(\alpha\ \delta\ n)$ for various values of δ and n when α = 0 0027

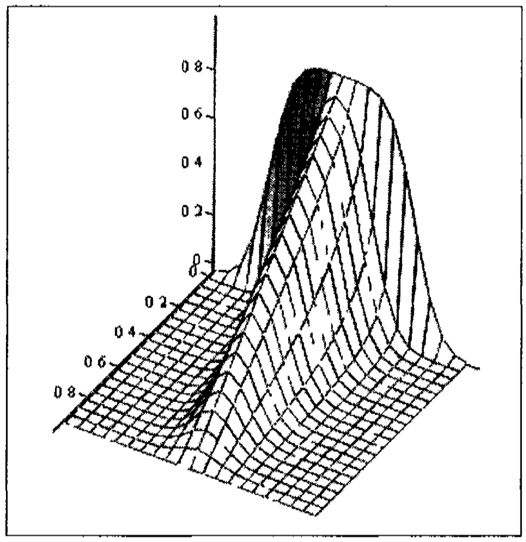

Figure 3 6 Probability of no signal as a function of δ and α when $n = 5$

	False alarm rate (α)				
δ	0 0020	0 0027	0 0030	0 0100	0 0500
0 00	0 9980	0 9973	0 9970	0 9900	0 9500
0 25	0 9942	0 9925	0 9918	0 9773	0 9135
0 50	0 9757	0 9701	0 9678	0 9274	0 7990
0 75	0 9212	0 9071	0 9016	0 8156	0 6112
1 00	0 8035	0 7776	0 7678	0 6330	0 3912
1 25	0 6161	0 5812	0 5685	0 4132	0 2018
1 50	0 3959	0 3616	0 3496	0 2182	0 0816
1 75	0 2053	0 1806	0 1722	0 0906	0 0254
2 00	0 0835	0 0705	0 0662	0 0290	0 0060
2 25	0 0261	0 0211	0 0195	0 0070	0 0011
2 50	0 0062	0 0048	0 0044	0 0013	0 0001
2 75	0 0011	0 0008	0 0007	0 0002	0 0000
3 00	0 0001	0 0001	0 0001	0 0000	0 0000

Table 3 4 Probability of no signal $\beta(\alpha\ \delta\ n)$ for various values of δ and α when $n = 5$

Table 3 3 Table 3 4 and Table 3 5 can be very helpful in the statistical design of the Shewhart $\overline{X}$ control chart For example when the false alarm rate α is fixed or specified at (say) 0 0027 we can choose the relevant sample size n from Table 3 3 that ensures that the probability of no signal is suitably small for a given or anticipated shift δ However if the nature of the process automatically restricts the sample size n to be equal to 5 say we can use Table 3 4 (or a similar table if the sample size is not equal to 5) to choose the false alarm rate α that will ensure a suitably (small) probability of no sıgnal for an antıcıpated or specıfied shıft δ On the other hand ıf δ ıs known ın advance a table sımılar to Table 3 5 wıll be useful for choosıng the false alarm rate α and the sample sıze n

In theory ıf any one (or more) of the parameters are fixed or specıfied ın advance we can solve the other(s) usıng equatıons (3 5) and/or (3 6) For example ıf the probabılıty of a sıgnal or equıvalently the probabılıty of no sıgnal ıs specıfied we can find those combınatıons of α δ and n that would yıeld the requıred probabılıty of no sıgnal or the probabılıty of a sıgnal by way of a grıd search method say

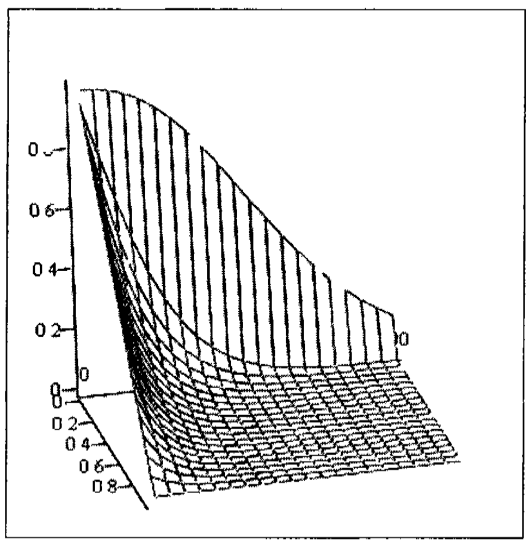

Fıgure 3 7 Probabılıty of no sıgnal as a functıon of α and n when $\delta = 0\ 5$

	Sample sıze (n)												
False alarm rate (α)	1	2	3	4	5	6	7	8	9	10	15	20	25
0 0020	0 9950	0 9913	0 9869	0 9817	0 9757	0 9689	0 9614	0 9531	0 9441	0 9344	0 8757	0 8035	0 7225
0 0027	0 9936	0 9890	0 9835	0 9772	0 9701	0 9621	0 9532	0 9436	0 9332	0 9220	0 8562	0 7776	0 6915
0 0030	0 9929	0 9880	0 9821	0 9754	0 9678	0 9593	0 9500	0 9398	0 9289	0 9172	0 8488	0 7678	0 6800
0 0100	0 9800	0 9687	0 9561	0 9423	0 9274	0 9116	0 8948	0 8773	0 8590	0 8400	0 7387	0 6330	0 5302
0 0500	0 9209	0 8910	0 8607	0 8299	0 7990	0 7682	0 7375	0 7070	0 6770	0 6474	0 5093	0 3912	0 2946

Table 3 5 Probabılıty of no sıgnal $\beta(\alpha\ \delta\ n)$ for varıous values of α and n when $\delta = 0\ 5$

3 1 4 Probabılıty mass functıon, cumulatıve dıstrıbutıon functıon and the quantıle functıon

The probabılıty mass functıon (pmf) of the run length random varıable N as gıven ın equatıon (3 9) ıs shown ın Fıgure 3 8 when $\alpha = 0\ 0027$ and $\delta = 0$ ı e $P(N_0 = j) = (0\ 9973)^{j\ 1}(0\ 0027)$ for $j = 1\ 2\ 3$ We can clearly see that the run length dıstrıbutıon ıs posıtıvely skewed – even when the process ıs ın control

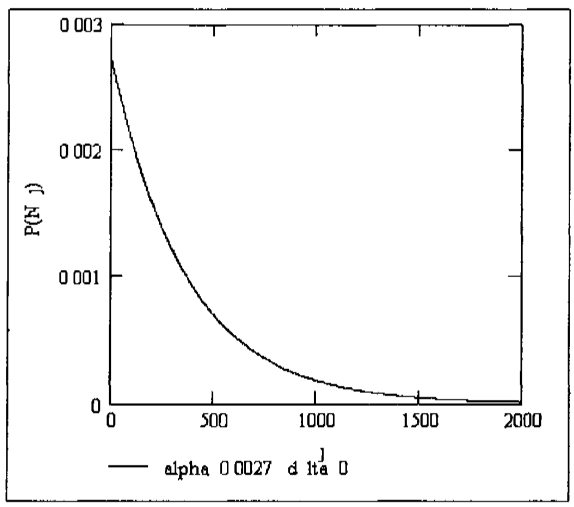

Fıgure 3 8 The ın control probabılıty mass functıon (pmf) of the run length random varıable N [1]

Furthermore the ın control cumulatıve dıstrıbutıon functıon (cdf) and the quantıle functıon that corresponds to the probabılıty mass functıon of Fıgure 3 8 ın other words when $\alpha = 0\ 0027$ and $\delta = 0$ are presented ın Fıgures 3 9 and 3 10 respectıvely

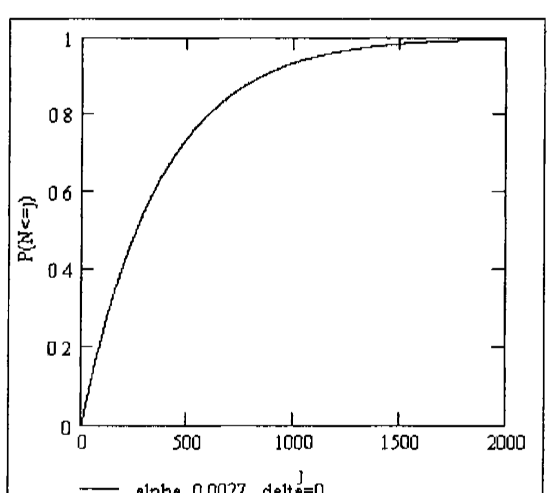

Fıgure 3 9 The ın control cumulatıve dıstrıbutıon functıon (cdf) of the run length random varıable N [1]

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[1] Note that although the geometrıc dıstrıbutıon ıs a dıscrete dıstrıbut on the probabılıty mass functıon (pmf) the cumulatı e dıstrıbutıon functıon (cdf) and the quantıle functıon dısplayed ın Fıgures 3 8 3 9 and 3 10 espectı ely appea s contınuous due the scal ng of the axıs

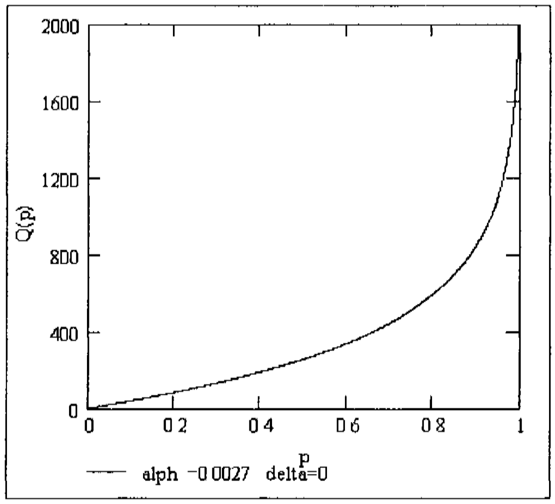

Figure 3 10 The in control quantile function of the run length random variable N^{1}

Although the probability mass function (pmf) the cumulative distribution function (cdf) and the quantile function of N are important we are more interested in or concerned with other characteristics of the run length distribution For example the mean or expected value the median the standard deviation and the coefficient of skewness are of particular interest since these properties of the run length distribution are commonly used to evaluate or assess the performance of a control chart Thus we focus on these characteristics in the sections that follow

3 1 5 The average run length and the median run length

The average run length (ARL) is the expected value of the run length random variable N and is given by equation (3 12) i e

$$ARL = E(N) = \frac{1}{1-\beta(\alpha \ \delta \ n)} = \frac{1}{1-\Phi\left(z_{\frac{\alpha}{2}} - \delta\sqrt{n}\right) + \Phi\left(-z_{\frac{\alpha}{2}} - \delta\sqrt{n}\right)}$$

whereas the median run length (MDRL) is the median of the run length distribution and is given by equation (3 16) i e

$$MDRL = \inf\left\{ j \text{ an integer } \ j \geq \frac{\ln(0\ 5)}{\ln \beta(\alpha \ \delta \ n)} \right\}$$

Although both the ARL and the MDRL are measures of the location or central tendency of the run length distribution in the past the ARL was used more often because of its (simple) relationship with the probability of a signal i e the ARL is (simply) the reciprocal of the probability of a signal whereas the MDRL was rarely used or totally ignored However since the run length distribution is positively skewed (as can be seen from Figure 3 8) it is proposed that the MDRL should be used together with the ARL and attention should also be paid to the standard deviation of the run length (SDRL) and the coefficient of skewness of the run length (SKEWRL)

Figure 3 11 displays the ARL and the MDRL when $\alpha = 0\ 0027$ and $n = 5$ for various values of δ We notice that the MDRL and the ARL decreases as δ increases Thus the larger the shift in the process mean the smaller the MDRL and the ARL However we also observe that the MDRL is consistently less than the ARL but that the absolute difference between the MDRL and the ARL decreases as δ increases For example when the process is in control the $ARL_0 = 370\ 37$ whereas the $MDRL_0 = 256\ 37$ and when the process is out of control for example when $\delta = 0\ 5$ the $ARL = 33\ 40$ and the $MDRL = 22\ 80$ This tendency is due to the skewness of the run length distribution and the fact that the skewness increases as δ increases see Section 3 1 6

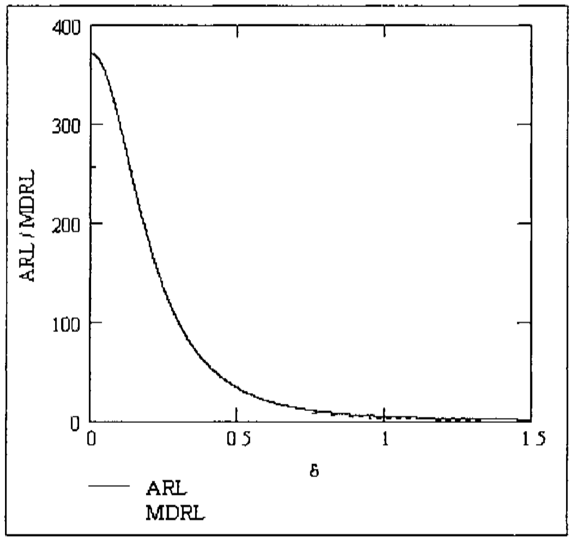

Figure 3 11 The average run length (ARL) and the median run length (MDRL) when α = 0 0027 **and** n = 5

3 1 6 The standard deviation of the run length distribution

The standard deviation of the run length distribution (SDRL) measures the variability in the run length random variable N and is given in equation by (3 14) i e

$$SDRL = stdev(N) = \frac{\sqrt{\beta(\alpha\ \delta\ n)}}{1-\beta(\alpha\ \delta\ n)}$$

Figure 3 12 shows how the SDRL decreases as δ increases and by comparing $SDRL(0\ 0027\ \delta\ 5)$ with $SDRL(0\ 0027\ \delta\ 10)$ or by comparing $SDRL(0\ 01\ \delta\ 5)$ with $SDRL(0\ 01\ \delta\ 10)$ we also notice that the SDRL decreases as the sample size n increases Furthermore by comparing $SDRL(0\ 0027\ \delta\ 5)$ and $SDRL(0\ 0027\ \delta\ 10)$ with $SDRL(0\ 01\ \delta\ 5)$ and $SDRL(0\ 01\ \delta\ 10)$ we see that the standard deviation of the run length decreases as α increases Thus as either of the parameters α δ or n increases the SDRL decreases – see Table 3 6 (below) for a summary

	α	δ	n
$SDRL(\alpha\ \delta\ n)$	Decrease ↓	Decrease ↓	Decrease ↓

Table 3 6 The influence of α δ **and** n **on the standard deviation of the run length random variable** N

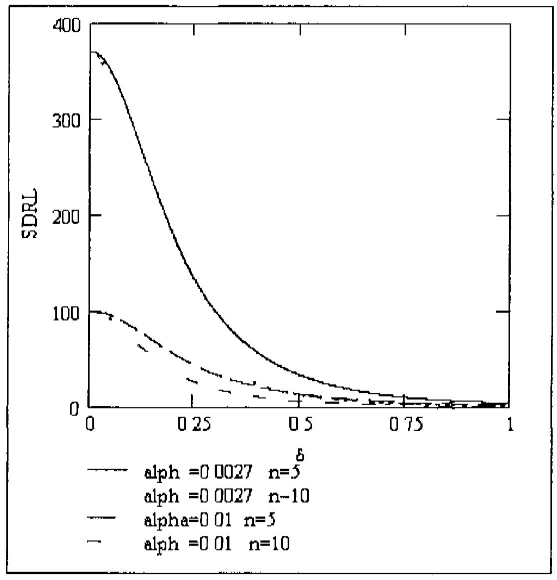

Figure 3 12 The standard deviation of the run length distribution (SDRL) for varying δ

3 1 7 The coefficient of skewness

The coefficient of skewness (SKEWRL) which is based on the second and the third central moments of the run length distribution measures the symmetry (or asymmetry) of the run length distribution For example if $SKEWRL < 0$ the run length distribution is negatively skewed whereas the run length distribution is positively skewed if $SKEWRL > 0$ and if $SKEWRL = 0$ it is an indication that the run length distribution is symmetric

However since $\sqrt{\beta(\alpha\ \delta\ n)} \in (0\ 1)$ and $1+\sqrt{\beta(\alpha\ \delta\ n)} \in (1\ 2)$ we have that

$SKEWRL = \dfrac{1+\beta(\alpha\ \delta\ n)}{\sqrt{\beta(\alpha\ \delta\ n)}} > 0$ In other words the run length distribution is always positively skewed

In addition as either α δ or n increases the coefficient of skewness increases Thus as previously mentioned the average run length (ARL) should not be used as the sole measure of a control charts performance since the ARL is (typically) more influenced by changes in the shape of a distribution than a robust estimator of the location or central tendency of a distribution such as the median for example

3 1 8 Characteristics of the run length distribution

The probability of signal the average run length (ARL) the median run length (MDRL) the standard deviation of the run length (SDRL) the coefficient of skewness (SKEWRL) as well as the 10^{th} and 90^{th} percentiles of the run length distribution for various values of δ when $\alpha = 0\ 0027$ and $n = 5$ are summarized in Table 3 7

δ	$P(Signal)$	ARL	$MDRL$	$SDRL$	$SKEWRL$	10^{th}	90^{th}
0 00	0 0027	370 37	256 37	369 87	2 00	38 97	851 66
0 25	0 0075	133 15	91 95	132 65	2 00	13 98	305 44
0 50	0 0299	33 40	22 80	32 90	2 00	3 47	75 75
0 75	0 0929	10 76	7 11	10 25	2 00	1 08	23 61
1 00	0 2225	4 50	2 76	3 96	2 02	0 42	9 15
1 25	0 4188	2 39	1 28	1 82	2 07	0 19	4 24
1 50	0 6384	1 57	0 68	0 94	2 26	0 10	2 26
1 75	0 8194	1 22	0 41	0 52	2 78	0 06	1 35
2 00	0 9295	1 08	0 26	0 27	4 03	0 04	0 87
2 25	0 9789	1 02	0 18	0 15	7 03	0 03	0 60
2 50	0 9952	1 01	0 13	0 07	14 51	0 02	0 43
2 75	0 9992	1 00	0 10	0 03	34 98	0 02	0 32
3 00	0 9999	1 00	0 08	0 01	97 90	0 01	0 25

Table 3 7 Characteristics of the run length distribution in Case KK when $\alpha = 0\ 0027$ **and** $n = 5$

From Table 3 7 we see that when the process operates in control that is when $\delta = 0$ the average run length ARL_0 equals 370 37 whereas the in control median run length $MDRL_0$ equals 256 37 Thus on average we expect a false alarm within (approximately) every 370 samples and that 50% of the time a signal will be given within (approximately) the first 257 samples

However when a shift in the process mean occurs that is when $\delta \neq 0$ the average run length ARL and the median run length MDRL decreases (rapidly) while the probability for a signal increases This is highly desirable since when a process operates out of control more nonconforming or defective items will be produced which implies a loss both in time and money to the manufacturer Therefore with an increased probability of a signal and a decreased ARL and a decreased MDRL the out of control condition will be detected much faster and any necessary corrective action can be taken as soon as possible For example when $\delta = 2$ which is typically considered a large shift in the process mean the ARL is approximately 1 and the probability of a signal is 0 9295 Thus following such a large shift we expect to detect the shift within the first sample or subgroup with an equally large probability

Generally when a process operates in control it is desirable that the average run length (ARL) and the median run length (MDRL) be large and that the false alarm rate (FAR) be small but when the process operates out of control we would like a small average run length (ARL) and a small median run length (MDRL) with a high probability of a signal

Note that as mentioned previously although the difference between the average run length and the median run length $|ARL - MDRL|$ decreases as the relative size of the shift in the process mean δ increases that this is not because the run length distribution becomes more symmetric In fact this is due to the increased skewness of the run length distribution For example when $\delta = 0$ the 10^{th} percentile and the 90^{th} percentile are 38 97 and 851 66 respectively with the coefficient of skewness equal to 2 00 However when $\delta = 2$ say the 10^{th} percentile and the 90^{th} percentile are 0 04 and 0 87 respectively with the coefficient of skewness equal to 4 03

3 2 The Mean unknown and the Standard Deviation specified (Case UK)

Suppose that we monitor the process mean μ using a Shewhart $\overline{X}$ control chart when the in control process variance σ_0^2 is *known* but the in control process mean μ is *unknown* In other words suppose or assume that $X_j \sim N\left(\mu \ \sigma_0^2\right)$

When the in control process mean μ is unknown it is typically estimated by the grand mean or the overall mean $\overline{\overline{X}}$ using m independent reference samples each of size n from Phase 1 That is using historical information or retrospective data gathered prior to the construction of the control chart Thus as (point) estimator we use

$$\overline{\overline{X}} = \frac{1}{m}\sum_{1}^{m}\overline{X} = \frac{1}{mn}\sum_{1}^{m}\sum_{j\ 1} X_j \tag{3 27}$$

where X_j is the j^{th} observation from the i^{th} subgroup $\overline{X}$ is the i^{th} sample or subgroup mean and since we assume that the process follows a normal distribution we have that $\overline{\overline{X}} \sim N\left(\mu \ \frac{\sigma_0^2}{mn}\right)$

However it is not necessary or essential to assume that the m reference samples are the same size If the sample sizes are different we denote the size of the i^{th} sample by n so that the point estimator of equation (3 27) becomes

$$\overline{\overline{X}} = \frac{1}{m}\sum_{1}^{m}\overline{X} = \frac{1}{m}\sum_{-1}^{m}\frac{1}{n}\sum_{j\ 1} X_j \tag{3 28}$$

and then we have that $\overline{X} \sim N(\mu \ \frac{\sigma_0^2}{n})$ and that $\overline{\overline{X}} \sim N\left(\mu \ \frac{\sigma_0^2}{m}\sum_{1}^{m}\frac{1}{n}\right)$ In spite of this for the discussion that follows we use the point estimator given in equation (3 27)

The *estimated* control limits and the *estimated* centerline when the in control process mean μ is estimated by $\overline{\overline{X}}$ are

$$\widehat{UCL} = \overline{\overline{X}} + z_{\frac{\alpha}{2}} \frac{\sigma_0}{\sqrt{n}}$$
$$\widehat{CL} = \overline{\overline{X}} \tag{3 29}$$
$$\widehat{LCL} = \overline{\overline{X}} - z_{\frac{\alpha}{2}} \frac{\sigma_0}{\sqrt{n}}$$

(See Result 3 14 in Appendix 3B)

Note that neither the two control limits nor the centerline are constants Rather they are functions of the random variable $\overline{\overline{X}}$ Therefore to accommodate we use the notation $\widehat{UCL}$ $\widehat{CL}$ and $\widehat{LCL}$ instead of UCL CL and LCL as in Case KK – compare equation (3 29) with equation (3 2)

However if the estimate $\overline{\overline{X}}$ is to be meaningful it must be based on data from an in control process Therefore in Phase 1 that is when preliminary or reference samples are used to construct the $\overline{X}$ control chart we typically treat the control limits and the centerline obtained from equation (3 29) as trial limits In other words they help us to determine whether the process was in control when the m initial or reference samples were obtained Consequently each estimate of the process mean μ $\overline{X}_1$ $\overline{X}_2$ $\overline{X}_m$ from each of the reference samples 1 2 m is plotted on the control chart and we then analyze the resulting display If none of the plotting statistics or points fall outside the control limits and no systematic or non random pattern is present we conclude that the process was in control Subsequently we use the trial control limits for prospective monitoring of the process that is in Phase 2

However if one or more of the points fall on or outside the control limits the trial control limits must be revised This is typically done by examining each out of control point and looking for an assignable cause(s) If an assignable cause is found that point is discarded and the trial control limits are recalculated using only the remaining points However following the removal of the first out of control point(s) points that were initially in control or between the control limits may now well be outside the control limits This happens since the revised control limits will generally be tighter than the old ones In other words the distance between the revised control limits will be less This process or procedure is continued until all the points plot in control whereupon the trial control limits are adopted for use in Phase 2 of the control chart procedure

If it happens that no assignable cause(s) could be found for a point that plotted outside the trial control limits there are two courses of action The first is to eliminate the point just as if an assignable cause was found However there is no justification for doing so apart from the fact that points that fall on or outside the estimated control limits are likely to be from a probability distribution of an out of control state The alternative is to retain the point(s) and consider the trial control limits suitable for future use However if the point(s) truly represent an out of control condition the control limits will be to wide On the other hand if there are only a few (one or two) such points this will not distort the control chart considerably

Having estimated the unknown process parameter or in this case the unknown process mean as well as the control limits and the centerline Phase 1 of the control chart procedure is complete and we may carry on to Phase 2

In Phase 2 we gather and use additional subgroups or samples for the prospective monitoring of the process in other words we use additional subgroups to assess or check whether the process remains in statistical control Thus the means of the additional subgroups $\overline{X}$ $i = m+1 \; m+2$ are calculated plotted on the control chart and compared to the estimated control limits from Phase 1

Example 3 1

A Shewhart $\overline{X}$ control chart for monitoring the process mean when the in control process mean is unknown and needs to be estimated

Panel (a) of Table 3 8 constains 20 independent samples each of size 4 from a normally distributed process with an *unknown* mean but with a *known* in control standard deviation $\sigma_0 = 2$ In addition column (b) contains the sample means $\overline{X}$ $i = 1 \; 2 \quad 20$

Using equation (3 27) we find the grand mean

$$\overline{\overline{X}} = \frac{15\,376 + 14\,618 + \quad + 14\,049}{20}$$
$$= 14\,921$$

so that $\hat{\mu} = \overline{\overline{X}} = 14\,921$

Subsequently the estimated or trial control limits and centerline when $\alpha = 0\,0027$ are found from equation (3 29) i e

$$\widehat{UCL} = 14\,921 + 3\frac{2}{\sqrt{4}} = 17\,921$$

$$\widehat{CL} = 14\,921$$

$$\widehat{LCL} = 14\,921 - 3\frac{2}{\sqrt{4}} = 11\,921$$

	(a)				(b)
Sample i	X_1	X_2	X_3	X_4	$\overline{X}$
1	16 772	15 397	14 504	14 832	15 376
2	12 864	17 107	13 653	14 848	14 618
3	15 092	17 674	16 958	17 131	16 714
4	17 929	12 206	12 810	14 148	14 273
5	16 742	15 012	13 928	15 655	15 334
6	18 294	16 281	18 275	19 308	18 040
7	12 995	16 133	16 765	12 697	14 647
8	11 005	15 364	18 955	15 704	15 257
9	14 898	13 938	12 531	14 520	13 972
10	15 740	17 140	12 607	16 540	15 507
11	16 874	18 497	15 710	16 358	16 860
12	11 819	12 476	11 867	14 210	12 593
13	15 697	16 068	13 237	11 765	14 192
14	15 742	15 416	14 951	12 502	14 652
15	14 373	13 695	13 418	12 298	13 446
16	16 753	17 994	11 841	15 167	15 439
17	12 806	12 117	16 446	14 423	13 948
18	14 007	11 448	12 545	15 663	13 416
19	18 730	15 405	17 038	13 198	16 093
20	14 395	16 852	12 047	12 903	14 049

Table 3 8 Retrospective or Phase 1 data (Case UK)

Plotting the sample means $\overline{X}$ $i = 1\ 2$ 20 versus the sample number or time together with the estimated control limits and centerline results in the $\overline{X}$ control chart of Figure 3 13 We observe that only one point falls outside the control limits when $\overline{X}_6 = 18\,040$ plots above the estimated upper control limit $\widehat{UCL} = 17\,921$ Consequently a search for assignable causes is begun

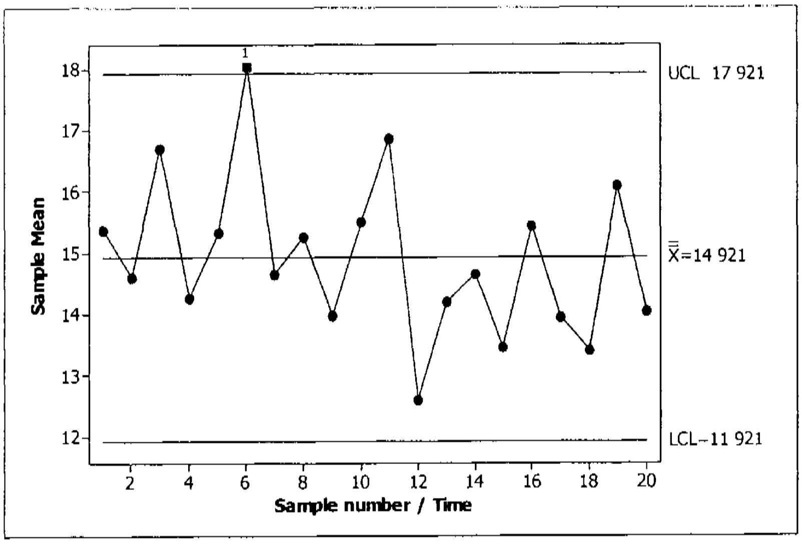

Figure 3 13 A Shewhart $\overline{X}$ control chart for monitoring the process mean in Phase 1 when the unknown in control process mean is estimated (I)

Suppose that investigation reveals that the 6th sample is truly out of control and therefore discarded Removing this sample and revising the control limits and the centerline we obtain the $\overline{X}$ control chart of Figure 3 14 with the new control limits and centerline at

$$\widehat{UCL} = 14\ 757 + 3\frac{2}{\sqrt{4}} = 17\ 757$$

$$\widehat{CL} = 14\ 757$$

$$\widehat{LCL} = 14\ 757 - 3\frac{2}{\sqrt{4}} = 11\ 757$$

respectively

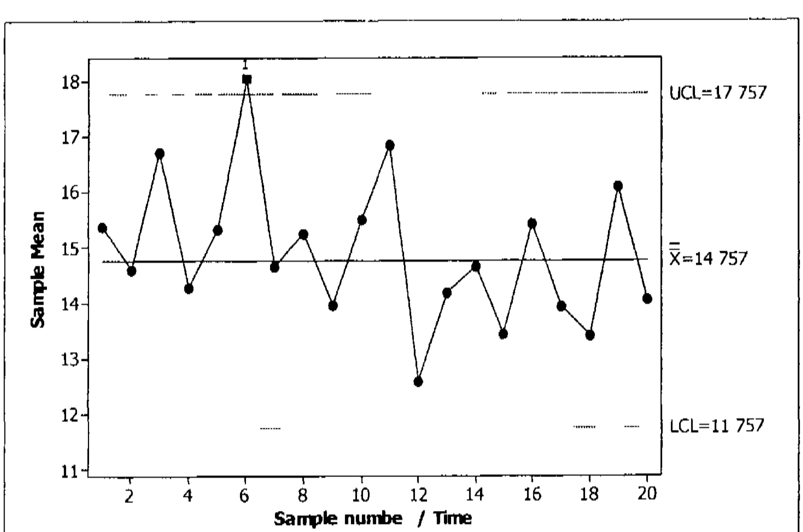

Figure 3 14 A Shewhart $\overline{X}$ control chart for monitoring the process mean in Phase 1 when the unknown in control process mean is estimated (II)

Since none of the remaining 19 sample means plot outside the control limits and since there are no non random pattern or systematic behavior in the plotting statistics we can safely use the revised control limits for prospective monitoring of the process mean

To this end Table 3 9 displays an additional 10 subgroups obtained from the same process Subsequently the $\overline{X}$ control chart of Figure 3 14 is continued by adding the additional sample means $\overline{X}_i$ $i = 21\ 22 \quad 30$ and shown in Figure 3 15

Sample i	X_1	X_2	X_3	X_4	$\overline{X}$
21	14 835	13 906	13 966	11 572	13 570
22	18 224	13 484	14 234	19 535	16 370
23	15 527	14 482	16 534	14 382	15 2[illegible]2
24	14 163	12 855	15 220	15 735	14 494
25	18 502	15 019	13 579	11 580	14 670
26	14 518	11 217	14 594	14 276	13 652
27	13 660	13 956	13 890	14 127	13 909
28	16 255	15 739	13 173	16 292	15 365
29	16 497	14 624	9 906	13 602	13 658
30	14 695	16 963	16 949	14 797	15 851

Table 3 9 Prospective or Phase 2 data (Case UK)

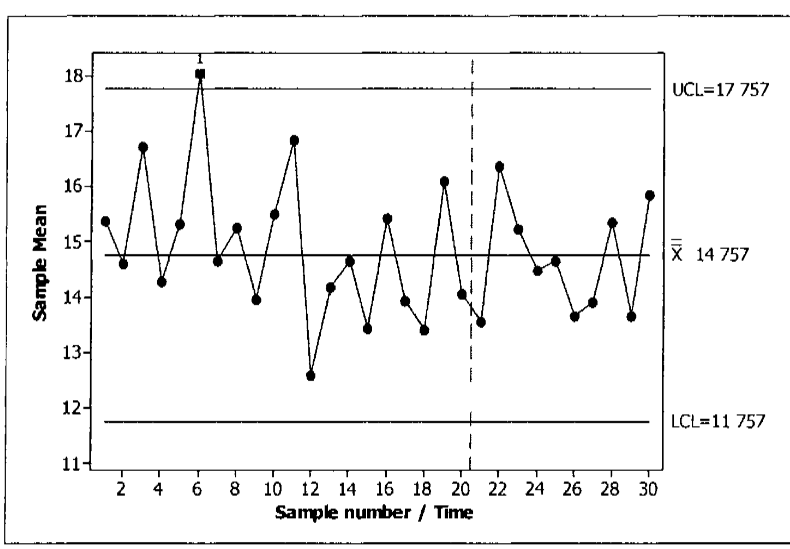

Figure 3 15 A Shewhart $\overline{X}$ control chart for monitoring the process mean in Phase 2 when the unknown in control process mean is estimated

Examining Figure 3 15 we conclude that the process is still functioning in control and monitoring of the process should continue

Looking at the control chart in Figure 3 15 it is customary (and necessary) to ask questions such as What is the probability that the control chart signals *on* the j^{th} sample? or What is the probability that the control chart signals *within* the first j^{th} samples? or maybe What is the *expected number* of samples that must be collected before we observe a signal for the first time?

Such questions can be answered by studying the properties of the run length distribution of the control chart that is studying the distribution of the run length random variable N

3 2 1 Properties of the $\overline{X}$ control chart

A non signaling event and a signaling event in Phase 2 that is when a subgroup or sample mean from Phase 2 falls either between or outside the estimated control limits from Phase 1 can be written as

$$\frac{\sqrt{n}\left|\overline{X}_i - \overline{\overline{X}}\right|}{\sigma_0} < z_{\frac{\alpha}{2}} \qquad (3\ 30)$$

and

$$\frac{\sqrt{n}\left|\overline{X}_i - \overline{\overline{X}}\right|}{\sigma_0} \geq z_{\frac{\alpha}{2}} \qquad (3\ 31)$$

respectively with $i = m+1\ m+2$ (See Results 3 15 and 3 16 in Appendix 3B)

Conditioning on the observed value $\overline{\overline{x}}$ of the random variable $\overline{\overline{X}}$ or conditioning on the observed value z of the standardized normal random variable $Z = \frac{\sqrt{mn}\left(\overline{\overline{X}} - \mu\right)}{\sigma_0}$ the *conditional* probabilities of a non signaling event and that of a signaling event in Phase 2 i e

$$P\left(\frac{\sqrt{n}\left|\overline{X}_i - \overline{\overline{x}}\right|}{\sigma_0} < z_{\frac{\alpha}{2}} \mid \mu_1 = \mu + \delta\sigma_0\ \overline{\overline{X}} = \overline{\overline{x}}\right) \quad \text{and} \quad P\left(\frac{\sqrt{n}\left|\overline{X}_i - \overline{\overline{x}}\right|}{\sigma_0} \geq z_{\frac{\alpha}{2}} \mid \mu_1 = \mu + \delta\sigma_0\ \overline{\overline{X}} = \overline{\overline{x}}\right)$$

when a shift in the process mean occurred can be expressed as

$$\beta = \beta(\alpha\ \delta\ m\ n\ z) = \Phi\left(\frac{z}{\sqrt{m}} + z_{\frac{\alpha}{2}} - \delta\sqrt{n}\right) - \Phi\left(\frac{z}{\sqrt{m}} - z_{\frac{\alpha}{2}} - \delta\sqrt{n}\right) \qquad (3\ 32)$$

and

$$1-\beta = 1 - \beta(\alpha\ \delta\ m\ n\ z) = 1 - \Phi\left(\frac{z}{\sqrt{m}} + z_{\frac{\alpha}{2}} - \delta\sqrt{n}\right) + \Phi\left(\frac{z}{\sqrt{m}} - z_{\frac{\alpha}{2}} - \delta\sqrt{n}\right) \qquad (3\ 33)$$

respectively where $\delta = \frac{|\mu_1 - \mu|}{\sigma_0}$ represents the relative size of the shift in the process mean μ and z is the standardized value of $\overline{\overline{x}}$ i e $z = \frac{\sqrt{mn}\left(\overline{\overline{x}} - \mu\right)}{\sigma_0}$

(See Results 3 17 and 3 19 in Appendix 3B)

For example conditioning on the *estimated* in control process mean of Example 3 1 i e $\bar{\bar{x}} = 14\ 757$ or $z = \frac{14\ 757 - 15}{2} = -0\ 1215$ assuming the *true* in control process mean $\mu_0 = 15$ we obtain

$$\beta_Z = \beta(0\ 0027\ \delta\ 19\ 4\ -0\ 1215) = \Phi\left(\frac{-0\ 1215}{\sqrt{19}} + 3 - 2\delta\right) - \Phi\left(\frac{-0\ 1215}{\sqrt{19}} - 3 - 2\delta\right)$$

and

$$1 - \beta_Z = 1 - \beta(0\ 0027\ \delta\ 19\ 4\ -0\ 1215) = 1 - \Phi\left(\frac{-0\ 1215}{\sqrt{19}} + 3 - 2\delta\right) + \Phi\left(\frac{-0\ 1215}{\sqrt{19}} - 3 - 2\delta\right)$$

respectively

Thus if the process remains in control in Phase 2 that is if $\delta = 0$ equation (3 33) yields a conditional false alarm rate (FAR) of 0 00271 which is (slightly) larger than the specified or nominal value of 0 0027

Note that here m denotes the *final* number of reference samples used to *estimate* the unknown in control process mean μ and does not refer to the *original* number of reference samples in Phase 1 – which may or may not be the same

However if we do not condition on $\bar{\bar{X}} = \bar{\bar{x}}$ but instead use the fact that $\frac{(\bar{X} - \bar{\bar{X}}) - (\mu_1 - \mu)}{\sqrt{\frac{\sigma_0^2}{n}\left(\frac{m+1}{m}\right)}} \sim N(0\ 1)$ or use the fact that $\frac{\left(Z - \frac{Z}{\sqrt{m}}\right)}{\sqrt{\frac{m+1}{m}}} \sim N(0\ 1)$ for $i = m+1\ m+2$ the probability of a non signaling event and that of a signaling event in Phase 2 i e

$$P\left(\frac{\sqrt{n}\left|\bar{X} - \bar{\bar{X}}\right|}{\sigma_0} < z_{\frac{\alpha}{2}} \mid \mu_1 = \mu + \delta\sigma_0\right) \quad \text{and} \quad P\left(\frac{\sqrt{n}\left|\bar{X} - \bar{\bar{X}}\right|}{\sigma_0} \geq z_{\frac{\alpha}{2}} \mid \mu_1 = \mu + \delta\sigma_0\right)$$

when a shift in the process mean occurred can be expressed as

$$\beta(\alpha\ \delta\ m\ n) = \Phi\left(\sqrt{\frac{m}{m+1}}\left(z_{\frac{\alpha}{2}} - \delta\sqrt{n}\right)\right) - \Phi\left(\sqrt{\frac{m}{m+1}}\left(-z_{\frac{\alpha}{2}} - \delta\sqrt{n}\right)\right) \qquad (3\ 34)$$

and

$$1-\beta(\alpha\ \delta\ m\ n) = 1-\Phi\left(\sqrt{\frac{m}{m+1}}\left(z_{\frac{\alpha}{2}}-\delta\sqrt{n}\right)\right)+\Phi\left(\sqrt{\frac{m}{m+1}}\left(-z_{\frac{\alpha}{2}}-\delta\sqrt{n}\right)\right) \quad (3\ 35)$$

respectively

(See Results 3 18 (expression 2) and 3 20 in Appendix 3B)

Equations (3 34) and (3 35) are *unconditional* probability expressions in contrast to the *conditional* probability expressions of equations (3 32) and (3 33) Although these unconditional probability expressions have limited use especially when attempting to calculate or find the run length distribution of the $\overline{X}$ control chart with estimated control limits when the process is really in control i e when $\delta = 0$ using the unconditional probability of a signal as given in equation (3 35) is useful For instance the probability of a false alarm also known as the false alarm rate (FAR) or the probability of a type I error of the control chart then equals $1-\Phi\left(\sqrt{\frac{m}{m+1}}z_{\frac{\alpha}{2}}\right)+\Phi\left(-\sqrt{\frac{m}{m+1}}z_{\frac{\alpha}{2}}\right)$ which can be written as

$$P\left(|Z| \geq \sqrt{\frac{m}{m+1}}z_{\frac{\alpha}{2}}\right) \quad (3\ 36a)$$

(See Result 3 22 in Appendix 3B)

The general or the overall effect of the estimation procedure in Phase 1 on the false alarm rate (FAR) is seen from equation (3 36a) Specifically we can see that the false alarm rate (FAR) will be different from the nominal or specified value α unless m the *final* number of reference samples from Phase 1 is (relatively) large

For example substituting $\alpha = 0\ 0027$ and $m = 19$ in equation (3 36a) yields

$$P\left(|Z| \geq 3\sqrt{\frac{19}{20}}\right) = 0\ 003455$$

Thus in general when using $m = 19$ reference samples to estimate the unknown in control process mean μ the false alarm rate (FAR) is approximately 28% higher than the specified or nominal value of 0 0027

In fact since $0 \leq \sqrt{\frac{m}{m+1}} \leq 1$ we have that

$$P\left(|Z| \geq \sqrt{\frac{m}{m+1}}z_{\frac{\alpha}{2}}\right) \geq \alpha \quad (3\ 36b)$$

and only if $m \to \infty$ will $P\left(|Z| \geq \sqrt{\frac{m}{m+1}} z_{\frac{\alpha}{2}}\right) \to \alpha$ in other words

$$\lim_{m \to \infty} P\left(|Z| \geq \sqrt{\frac{m}{m+1}} z_{\frac{\alpha}{2}}\right) = \alpha \qquad (3\ 36c)$$

Thus when *estimating* the unknown process mean the actual false alarm rate (FAR) will generally be larger than the specified or nominal value α and the Shewhart $\overline{X}$ control chart will (incorrectly) signal more often that what would be the case if the *true* in control process mean was known Clearly this is an undesirable side effect

Furthermore when not conditioning on $\overline{\overline{X}} = \overline{\overline{x}}$ the covariance between any two signaling events or alternatively the covariance between any two non signaling events is nonzero Therefore the sequence or series of signaling events in Phase 2 is not an independent series of events and consequently the run length distribution is not geometric To verify this we proceed as follows

First we find the (sampling) distributions of $\widehat{UCL}$ $\widehat{LCL}$ $\overline{X}\ - \widehat{UCL}$ and $\overline{X}\ - \widehat{LCL}$ where $i = m+1\ m+2$

The expected value and the variance of the estimated upper control limit $\widehat{UCL}$ are

$$E\left(\widehat{UCL}\right) = E\left(\overline{\overline{X}} + z_{\frac{\alpha}{2}} \frac{\sigma_0}{\sqrt{n}}\right) = \mu + z_{\frac{\alpha}{2}} \frac{\sigma_0}{\sqrt{n}} \qquad (3\ 37a)$$

and

$$\text{var}\left(\widehat{UCL}\right) = \text{var}\left(\overline{\overline{X}} + z_{\frac{\alpha}{2}} \frac{\sigma_0}{\sqrt{n}}\right) = \frac{\sigma_0^2}{mn} \qquad (3\ 37b)$$

respectively whereas the expected value and the variance of the estimated lower control limit $\widehat{LCL}$ are

$$E\left(\widehat{LCL}\right) = E\left(\overline{\overline{X}} - z_{\frac{\alpha}{2}} \frac{\sigma_0}{\sqrt{n}}\right) = \mu - z_{\frac{\alpha}{2}} \frac{\sigma_0}{\sqrt{n}} \qquad (3\ 38a)$$

and

$$\text{var}\left(\widehat{LCL}\right) = \text{var}\left(\overline{\overline{X}} - z_{\frac{\alpha}{2}} \frac{\sigma_0}{\sqrt{n}}\right) = \frac{\sigma_0^2}{mn} \qquad (3\ 38b)$$

respectively

Subsequently since the two random variables $\widehat{UCL}$ and $\widehat{LCL}$ are (simply) functions of the normal random variable $\overline{\overline{X}}$ we have that

$$\widehat{UCL} \sim N\left(\mu + z_{\frac{\alpha}{2}} \frac{\sigma_0}{\sqrt{n}} \ \frac{\sigma_0^2}{mn}\right) \tag{3 39}$$

and

$$\widehat{LCL} \sim N\left(\mu - z_{\frac{\alpha}{2}} \frac{\sigma_0}{\sqrt{n}} \ \frac{\sigma_0^2}{mn}\right) \tag{3 40}$$

respectively

Subsequently the distribution of the difference between a sample mean from Phase 2 and one of the estimated control limits from Phase 1 i e $\overline{X} - \widehat{UCL}$ or $\overline{X} - \widehat{LCL}$ with $i = m+1 \ m+2$ is found by using equations (3 39) and (3 40) and is given by

$$\left(\overline{X} - \widehat{UCL}\right) \sim N\left(-z_{\frac{\alpha}{2}} \frac{\sigma_0}{\sqrt{n}} \ \frac{\sigma_0^2}{n}\left(\frac{m+1}{m}\right)\right)$$

and

$$\left(\overline{X} - \widehat{LCL}\right) \sim N\left(z_{\frac{\alpha}{2}} \frac{\sigma_0}{\sqrt{n}} \ \frac{\sigma_0^2}{n}\left(\frac{m+1}{m}\right)\right)$$

respectively

Secondly we show that the unconditional signaling event with its probability expression given in equation (3 35) can be expressed in terms of $\overline{X} - \widehat{UCL}$ and $\overline{X} - \widehat{LCL}$

Thus if we let B denote the unconditional signaling event in other words the event that $\overline{X}$ $i = m+1 \ m+2$ is either greater than or equal to $\widehat{UCL}$ or less than or equal to $\widehat{LCL}$ without conditioning on $\overline{\overline{X}} = \overline{\overline{x}}$ we see that B consists of two mutually exclusive events namely $\overline{X} \geq \widehat{UCL}$ and $\overline{X} \leq \widehat{LCL}$ and if written in an alternative format we have that B consists of the two mutually exclusive events $\overline{X} - \widehat{UCL} \geq 0$ and $\overline{X} - \widehat{LCL} \leq 0$

Then because the *unconditional* probability of a signal as given in equation (3 35) is $P(B) = P(\overline{X} - \widehat{UCL} \geq 0) + P(\overline{X} - \widehat{LCL} \leq 0)$ and since $P(\overline{X} - \widehat{UCL} \geq 0) = P(\overline{X} - \widehat{LCL} \leq 0)$ that is we use an equal tailed approach when calculating or estimating the control limits we have that $P(B) = 2P(\overline{X} - \widehat{UCL} \geq 0)$ Hence to show that the two signaling events B and B_j where

$i \neq j = m+1\ m+2$ are not independent is similar to showing that the two events $\bar{X}_i - \widehat{UCL} \geq 0$ and $\bar{X}_j - \widehat{UCL} \geq 0$ are dependent

Thus because the covariance between the two random variables $\bar{X}_i - \widehat{UCL}$ and $\bar{X}_j - \widehat{UCL}$ which is given by

$$\operatorname{cov}\left(\bar{X}_i - \widehat{UCL}\ \bar{X}_j - \widehat{UCL}\right) = \operatorname{var}(\widehat{UCL}) = \frac{\sigma_0^2}{n}\left(\frac{m+1}{m}\right) \tag{3 41}$$

or alternatively the correlation between these two random variables i e

$$corr\left(\bar{X}_i - \widehat{UCL}\ \bar{X}_j - \widehat{UCL}\right) = \frac{\operatorname{cov}\left(\bar{X}_i - \widehat{UCL}\ \bar{X}_j - \widehat{UCL}\right)}{\sqrt{\operatorname{var}\left(\bar{X}_i - \widehat{UCL}\right)\operatorname{var}\left(\bar{X}_j - \widehat{UCL}\right)}} = \frac{1}{m+1} \tag{3 42}$$

is not equal to zero it confirms that the signaling events B_i and B_j with $i \neq j = m+1\ m+2$ are not independent events

Consequently since the signaling events themselves are not independent the run length random variable N does not follow a geometric distribution In other words the easy to use and well known formulae for the average run length (ARL) the standard deviation of the run length (SDRL) and the coefficient of skewness of the run length (SKEWRL) used in Case KK which stems from the geometric distribution can no longer be used Therefore as mentioned earlier evaluating the *unconditional* probability of a signal as given in equation (3 35) i e

$$1 - \Phi\left(\sqrt{\frac{m}{m+1}}\left(z_{\frac{\alpha}{2}} - \delta\sqrt{n}\right)\right) + \Phi\left(\sqrt{\frac{m}{m+1}}\left(-z_{\frac{\alpha}{2}} - \delta\sqrt{n}\right)\right)$$

for different values of m (the number of final samples or subgroups in Phase 1) and n (the sample or subgroup size) has limited use and we need an alternative way of assessing the performance of the control chart Hence we study the *conditional* run length distribution

3 2 2 The conditional run length distribution

Conditioning on having observed $\overline{\overline{X}} = \overline{\overline{x}}$ in Phase 1 all the non signaling events in Phase 2 i e $\frac{\sqrt{n}\left|\bar{X}_i - \overline{\overline{X}}\right|}{\sigma_0} < z_{\frac{\alpha}{2}}$ $i = m+1\ m+2$ are mutually independent events and so are all the signaling events in Phase 2 i e $\frac{\sqrt{n}\left|\bar{X}_i - \overline{\overline{X}}\right|}{\sigma_0} \geq z_{\frac{\alpha}{2}}$ $i = m+1\ m+2$

Furthermore since all the non signaling events in Phase 2 have the same *conditional* probability as given in equation (3 32) i e

$$\beta_Z = \Phi\left(\frac{z}{\sqrt{m}} + z_{\frac{\alpha}{2}} - \delta\sqrt{n}\right) - \Phi\left(\frac{z}{\sqrt{m}} - z_{\frac{\alpha}{2}} - \delta\sqrt{n}\right)$$

and all the signaling events in Phase 2 have the same *conditional* probability as given in equation (3 33) i e

$$1-\beta = 1-\Phi\left(\frac{z}{\sqrt{m}} + z_{\frac{\alpha}{2}} - \delta\sqrt{n}\right) + \Phi\left(\frac{z}{\sqrt{m}} - z_{\frac{\alpha}{2}} - \delta\sqrt{n}\right)$$

we have that

$$P(N = j \mid Z = z) = \beta^{j-1}(1-\beta) \text{ for } j = 1\ 2\ 3 \qquad (3\ 43)$$

In other words conditioning on the (standardized) observed value of the point estimator $\overline{\overline{X}}$ i e z the conditional probability that the run length random variable N is equal to j is the conditional probability of not having a signal on the first $(j-1)^{\text{th}}$ samples or subgroups and then having a signal on the j^{th} subsequent sample

Therefore the *conditional* probability distribution of the run length random variable N is geometric with probability of a signal (success) equal to $1-\beta$ i e $(N \mid Z = z) \sim Geo(1-\beta)$ Subsequently the *conditional* probability mass function (pmf) is given by equation (3 43)

It then follows that the *conditional* cumulative distribution function (cdf) and the *conditional* quantile function are

$$P(N \leq j \mid Z = z) = 1-\beta^{j} \text{ for } j = 1\ 2\ 3 \qquad (3\ 44)$$

and

$$Q_{N|Z}(p) = \inf\{j \text{ an integer } P(N \leq j \mid Z = z) \geq p\}$$
$$= \inf\left\{j \text{ an integer } j \geq \frac{\ln(1-p)}{\ln \beta}\right\} \quad \text{for } 0 < p < 1 \tag{3 45}$$

respectively

The *conditional* average run length or the expected value of $N \mid Z = z$ is

$$ARL = E(N \mid Z = z) = \frac{1}{1-\beta} \tag{3 46}$$

whereas the variance the standard deviation and the coefficient of skewness of the *conditional* run length distribution are

$$VARRL = \text{var}(N \mid Z = z) = \frac{\beta}{(1-\beta)^2} \tag{3 47}$$

$$SDRL = stdev(N \mid Z = z) = \frac{\sqrt{\beta}}{1-\beta} \tag{3 48}$$

and

$$SKEWRL = \frac{1+\beta}{\sqrt{\beta}} \tag{3 49}$$

respectively
(See Results 3 23 3 25 3 27 3 28 3 30 3 32 and 3 34 in Appendix 3B respectively)

The *conditional* median run length or any other percentile of the *conditional* run length distribution can be found by using the *conditional* quantile function given in (3 45) For example when substituting $p = 0\ 5$ in equation (3 45) the *conditional* median run length is found to be

$$MDRL = \inf\left\{j \text{ an integer } j \geq \frac{\ln(0\ 5)}{\ln \beta}\right\} \tag{3 50}$$

Example 3 2
Conditional run length distribution of the Shewhart $\overline{X}$ control chart of Example 3 1

The *unknown* in control process mean μ of the process in Example 3 1 was estimated by $\overline{\overline{X}}$ of equation (3 27) and found to be 14 757 This resulted in an observed value from the standardized normal distribution i e $z = -0\ 1215$ assuming of course that the *true* in control process mean is 15 In addition substituting $z = -0\ 1215$ in equations (3 32) and (3 33) we obtained the conditional probability of no signal and the conditional probability of a signal i e

$$\beta(0\ 0027\ \ \delta\ \ 19\ \ 4\ \ -0\ 1215) = \Phi\left(\frac{-0\ 1215}{\sqrt{19}} + 3 - 2\delta\right) - \Phi\left(\frac{-0\ 1215}{\sqrt{19}} - 3 - 2\delta\right)$$

and

$$1 - \beta(0\ 0027\ \ \delta\ \ 19\ \ 4\ \ -0\ 1215) = 1 - \Phi\left(\frac{-0\ 1215}{\sqrt{19}} + 3 - 2\delta\right) + \Phi\left(\frac{-0\ 1215}{\sqrt{19}} - 3 - 2\delta\right)$$

respectively

Consequently substituting these results in equation (3 43) we obtain the conditional probability mass function (pmf) of the run length random variable N (in Phase 2) of the $\overline{X}$ control chart shown in Figure 3 14 i e

$$P\left(N = j \mid Z = -0\ 1215\right) = \beta(0\ 0027\ \ \delta\ \ 19\ \ 4\ \ -0\ 1215)^{j\ \ 1}\left(1 - \beta(0\ 0027\ \ \delta\ \ 19\ \ 4\ \ -0\ 1215)\right)$$

for $j = 1\ \ 2\ \ 3$

If we assume that the process remains in control that is $\delta = 0$ we find the *conditional* in control pmf to be

$$P\left(N_0 = j \mid Z = -0\ 1215\right) = \left(0\ 99729\right)^{j\ \ 1}\left(0\ 00271\right) \quad \text{for} \quad j = 1\ \ 2\ \ 3$$

Subsequently we can find the average (ARL_0) the standard deviation ($SDRL_0$) the coefficient of skewness ($SKEWRL_0$) and the median ($MDRL_0$) of the *conditional* in control run length distribution using equations (3 46) (3 48) (3 49) and (3 50) respectively

For example

$$ARL_0 = \frac{1}{0\ 00271} = 369\ 00$$

$$SDRL_0 = \frac{\sqrt{0\ 99729}}{0\ 00271} = 368\ 50$$

$$SKEWRL_0 = \frac{1 + 0\ 99729}{\sqrt{0\ 99729}} = 2\ 00$$

and

$$MDRL_0 = \inf\left\{ j \text{ an integer } \ j \geq \frac{\ln(0\ 5)}{\ln(0\ 99729)} \right\} = 256$$

Thus if the process remains in control we expect (on average) an errounous signal or a false alarm every 369 samples with 50% of the false alarms within the first 256 samples In addition we observe that the *conditional* in control run length distribution is (fairly) positively skewed i e $SKEWRL_0 = 2$ and has a standard deviation of approximately 368

On the other hand if we are interested in detecting a shift of size $\delta = 1\ 5$ say it is valuable to also study the *conditional* out of control run length distribution For example following a shift of size $\delta = 1\ 5$ we find the probability of no signal to be

$$\beta(0\ 0027\ 1\ 5\ 19\ 4\ {-0}\ 1215) = \Phi\left(\frac{-0\ 1215}{\sqrt{19}} + 3 - 2(1\ 5)\right) - \Phi\left(\frac{-0\ 1125}{\sqrt{19}} - 3 - 2(1\ 5)\right) = 0\ 4889$$

whereas the probability of a signal is

$$1 - \beta(0\ 0027\ 1\ 5\ 19\ 4\ {-0}\ 1215) = 1 - \Phi\left(\frac{-0\ 1215}{\sqrt{19}} + 3 - 2(1\ 5)\right) + \Phi\left(\frac{-0\ 1215}{\sqrt{19}} - 3 - 2(1\ 5)\right) = 0\ 5111$$

Therefore the *conditional* out of control probability mass function (pmf) is given by

$$P\left(N_1 = j \mid Z = -0\ 1215\right) = \left(0\ 4889\right)^{j\ 1}\left(0\ 5111\right) \quad \text{for} \quad j = 1\ 2\ 3$$

In addition we can find the *conditional* out of control average run length (ARL_1) i e

$$ARL_1 = \frac{1}{0\ 5111} = 1\ 96$$

or the *conditional* out of control median run length ($MDRL_1$) i e

$$MDRL_1 = \inf\left\{ j \text{ integer } \ j \geq \frac{\ln(0\ 5)}{\ln(0\ 4889)} \right\} = 1$$

Thus for a shift of size $\delta = 1\ 5$ we would expect to detect the shift (on average) within approximately 2 samples – *conditional* on having observed $z = -0\ 1215$

3 2 3 The unconditional run length distribution

The *conditional* run length distribution of the $\overline{X}$ control chart when the unknown process mean is estimated and its associated characteristics merely presents the *conditional* performance of the control chart That is the performance of the $\overline{X}$ control chart conditioned on a single observation or realization from an infinite set of possible observations for the random variable or the estimator $\overline{\overline{X}}$ – see for instance Example 3 2

However for each observation the performance of the $\overline{X}$ control chart will be different – some performing acceptable and some performing poorly Thus to assess the overall performance of the $\overline{X}$ control chart the influence of a single observed value $\overline{\overline{x}}$ or z (on the run length distribution) needs to be eliminated Therefore the *unconditional* run length distribution and its associated characteristics are needed which will give a clearer picture of the overall or general performance of the Shewhart $\overline{X}$ control chart when estimating the unknown in control process mean μ

In spite of this the *conditional* probability distribution of the run length random variable N is not worthless rather it facilitates the derivation of the *unconditional* probability distribution Knowing the *conditional* probability distribution of N i e $P(N = j \mid Z = z)$ and the *marginal* distribution of the random variable on which we condition i e $P(Z = z) = f_Z(z)$ we can (easily) find the *unconditional* probability distribution of N i e $P(N = j)$

For example for the two random variables N and Z we have that $f_{N|Z}(j \mid z) = \dfrac{f_{NZ}(j\ z)}{f_Z(z)}$ with $f_{N|Z}(j \mid z)$ being the conditional probability mass function of the run length random variable i e $f_{N|Z}(j \mid z) = P(N = j \mid Z = z)$ for $j = 1\ 2\ 3$ as given in equation (3 43) and $f_Z(z)$ being the marginal density function or the unconditional density function of the standardized normal random variable

$$Z = \frac{\sqrt{mn}(\overline{\overline{X}} - \mu)}{\sigma_0} \text{ i e } f_Z(z) = \phi(z) = \frac{1}{\sqrt{2\pi}} e^{\frac{1}{2}} \quad \text{for } -\infty < z < \infty$$

Then solving for $f_{NZ}(j\ z)$ in terms of $f_{N|Z}(j \mid z)$ and $f_Z(z)$ the joint probability distribution of N and Z is found to be $f_{NZ}(j\ z) = f_{N|Z}(j \mid z) f_Z(z)$ and subsequently the marginal or the unconditional probability distribution of N is obtained by integrating over the domain of the random variable Z i e

$$f_N(j) = \int_{-\infty}^{\infty} f_{NZ}(j\ z)dz = \int_{-\infty}^{\infty} f_{N|Z}(j \mid z) f_Z(z)dz$$

Therefore the *unconditional* probability mass function of N is

$$\begin{aligned} P(N = j) &= \int_{-\infty}^{\infty} P(N = j \mid Z = z)\phi(z)dz \\ &= \int_{-\infty}^{\infty} \beta^{j\ 1}(1-\beta\ )\phi(z)dz \end{aligned} \qquad (3\ 51)$$

whereas the *unconditional* cumulative distribution function and the *unconditional* quantile function are

$$\begin{aligned} P(N \le j) &= \int_{-\infty}^{\infty} P(N \le j \mid Z = z)\phi(z)dz \\ &= \int_{-\infty}^{\infty} (1-\beta^{j})\phi(z)dz \\ &= 1 - \int_{-\infty}^{\infty} \beta^{j}\phi(z)dz \\ &= 1 - I_1(j\ m\ n\ \delta) \end{aligned} \qquad (3\ 52)$$

and

$$\begin{aligned} Q_N(p) &= \inf\{j \text{ an integer } \ P(N \le j) \ge p\} \\ &= \inf\left\{j \text{ an integer } \ \int_{-\infty}^{\infty} (1-\beta^{j})\phi(z)dz \ge p\right\} \end{aligned} \qquad (3\ 53)$$

respectively with $j = 1\ 2\ 3$ and $0 < p < 1$

(See Results 3 24 and 3 26 in Appendix 3B)

The *unconditional* average run length (ARL) is obtained in a similar manner by using the (already available) *conditional* average run length that is using expectation by conditioning i e

$$\begin{aligned} ARL &= E(N) \\ &= E_Z\left(E(N \mid Z = z)\right) \\ &= E_Z\left(\frac{1}{1-\beta}\right) \\ &= \int_{-\infty}^{\infty} \left(\frac{1}{1-\beta}\right)\phi(z)dz \end{aligned} \qquad (3\ 54)$$

However to find the *unconditional* variance the *unconditional* standard deviation and the *unconditional* coefficient of skewness of the run length is not so easy

For example to find the unconditional variance and subsequently the unconditional standard deviation of the run length random variable N we use $\text{var}(N \mid Z = z)$ and $E(N \mid Z = z)$ In other words we use the fact that $E_Z(\text{var}(N \mid Z = z)) = \text{var}(N) - \text{var}_Z(E(N \mid Z = z))$

Solving for $\text{var}(N)$ we find

$$\text{var}(N) = E_Z(\text{var}(N \mid Z = z)) + \text{var}_Z(E(N \mid Z = z))$$

Now we know from Results 3 22 and 3 28 in Appendix 3B that

$$\text{var}(N \mid Z = z) = \frac{\beta}{(1-\beta)^2} \qquad \text{and} \qquad E(N \mid Z = z) = \frac{1}{1-\beta}$$

so that

$$\begin{aligned}\text{var}_Z(E(N \mid Z = z)) &= E_Z\left((E(N \mid Z = z))^2\right) - \left(E_Z(E(N \mid Z = z))\right)^2 \\ &= E_Z\left(\left(\frac{1}{1-\beta}\right)^2\right) - \left(E_Z\left(\frac{1}{1-\beta}\right)\right)^2\end{aligned}$$

Subsequently by substitution we have that

$$\text{var}(N) = E_Z\left(\frac{\beta}{(1-\beta)^2}\right) + E_Z\left(\left(\frac{1}{1-\beta}\right)^2\right) - \left(E_Z\left(\frac{1}{1-\beta}\right)\right)^2$$

Re writing this expression in terms of integrals we obtain the *unconditional* variance i e

$$\text{var}(N) = \int_{-\infty}^{\infty} \frac{\beta}{(1-\beta)^2}\phi(z)dz + \int_{-\infty}^{\infty}\left(\frac{1}{1-\beta}\right)^2 \phi(z)dz - \left(\int_{-\infty}^{\infty}\frac{1}{1-\beta}\phi(z)dz\right)^2 \qquad (3\ 55)$$

(See Result 3 33 in Appendix 3B)

An alternative route to obtain the *unconditional* variance is by using the first and the second non central moments of the *conditional* run length distribution i e using $E(N \mid Z = z)$ and $E(N^2 \mid Z = z)$ given in Results 3 28 and 3 30 respectively

For example we have that

$$\text{var}(N) = E(N^2) - E(N)^2$$

where

$$E\left(N^2\right)=E_Z\left(E\left(N^2 \mid Z=z\right)\right) \qquad \text{and} \qquad E\left(N\right)=E_Z\left(E\left(N \mid Z=z\right)\right)$$

so that

$$\operatorname{var}\left(N\right)=E_Z\left(E\left(N^2 \mid Z=z\right)\right)-E_Z\left(E\left(N \mid Z=z\right)\right)^2$$

Furthermore from the properties of the geometric distribution we have that

$$E\left(N^2 \mid Z=z\right)=\frac{1+\beta}{(1-\beta)^2} \qquad \text{and} \qquad E\left(N \mid Z=z\right)=\frac{1}{1-\beta}$$

so that a second expression for the *unconditional* variance of the run length random variable is

$$\operatorname{var}\left(N\right)=\int_{-\infty}^{\infty}\frac{1+\beta}{(1-\beta)^2}\phi(z)dz-\left(\int_{-\infty}^{\infty}\frac{1}{1-\beta}\phi(z)dz\right)^2 \tag{3 56}$$

(See Result 3 33 in Appendix 3B)

To obtain the *unconditional* coefficient of skewness defined as

$$skew(N)=\frac{E\left(N-E(N)\right)^3}{\left(E\left(N-E(N)\right)^2\right)^{\frac{3}{2}}}=\frac{E\left(N-E(N)\right)^3}{\left(\operatorname{var}(N)\right)^{\frac{3}{2}}}$$

and which simplifies to

$$skew(N)=\frac{E\left(N^3\right)-3E\left(N^2\right)E\left(N\right)+2E\left(N\right)^3}{\left(\operatorname{var}(N)\right)^{\frac{3}{2}}}$$

we use the first the second and the third non central moments of the *conditional* run length distribution i e we use $E\left(N \mid Z=z\right)$ $E\left(N^2 \mid Z=z\right)$ and $E\left(N^3 \mid Z=z\right)$ where

$E\left(N^3 \mid Z=z\right)=\dfrac{1+\beta^2+4\beta}{(1-\beta)^3}$ is given in Result 3 30 of Appendix 3B

Thus by substitution an expression for the *unconditional* skewness of the run length random variable is

$$skew(N)=\frac{\int_{-\infty}^{\infty}\frac{1+\beta^2+4\beta}{(1-\beta)^3}\phi(z)dz-3\left\{\int_{-\infty}^{\infty}\frac{1+\beta}{(1-\beta)^2}\phi(z)dz\right\}\left\{\int_{-\infty}^{\infty}\frac{1}{1-\beta}\phi(z)dz\right\}+2\left(\int_{-\infty}^{\infty}\frac{1}{1-\beta}\phi(z)dz\right)^3}{\left(\operatorname{var}(N)\right)^{\frac{3}{2}}}$$

(3 57)

where $\operatorname{var}(N)$ is given by either equation (3 55) or equation (3 56)

(See Result 3 35 in Appendix 3B)

To obtain the **in control (and unconditional)** run length distribution or the associated characteristics of the *unconditional* in control run length distribution we set $\delta = 0$ in equations (3 51) to (3 56) where β_Z is given in equation (3 32)

3 2 4 Performance of the $\overline{X}$ control chart with estimated process mean

Having obtained the *unconditional* run length distribution and its associated characteristics or properties we in fact obtained the necessary expressions to assess the general or the overall performance of the Shewhart $\overline{X}$ control chart when the in control process mean is unknown Therefore we evaluate these unconditional expressions and compare the results with the results we obtained when we evaluated the performance of the $\overline{X}$ control chart when both the in control process parameters μ_0 and σ_0 were known

The unconditional probability of a signal and the unconditional average run length (ARL)

Since the average run length is (simply) the reciprocal of the probability of a signal when both the in control process parameters μ_0 and σ_0 are known the average run length and the probability of a signal are often used as performance measures of a Shewhart type of control chart Compare for example equations (3 6) and (3 12) or alternatively compare equation (3 17) with equation (3 22)

However in contrast the *unconditional* average run length is not equal to the reciprocal of the *unconditional* probability of a signal when the unknown in control process mean is estimated In fact the *unconditional* probability of a signal (given in Result 3 20 of Appendix 3B) is

$$\int_{-\infty}^{\infty}(1-\beta)\phi(z)dz = \int_{-\infty}^{\infty}\left(1-\Phi\left(\frac{z}{\sqrt{m}}+z_{\frac{\alpha}{2}}-\delta\sqrt{n}\right)+\Phi\left(\frac{z}{\sqrt{m}}-z_{\frac{\alpha}{2}}-\delta\sqrt{n}\right)\right)\phi(z)dz \qquad (3\ 58)$$

whereas the *unconditional* average run length is given in equation (3 54) i e

$$ARL = \int_{-\infty}^{\infty}\left(\frac{1}{1-\beta}\right)\phi(z)dz$$

Instead using Jensen s inequality together with expectation by conditioning we see that the reciprocal of the *unconditional* probability of a signal is actually a lower bound for the *unconditional* average run length i e

$$ARL = E(N) = E_Z\left(E\left(N \mid Z = z\right)\right) = E_Z\left(\frac{1}{1-\beta_Z}\right) \geq \frac{1}{E_Z\left(1-\beta_Z\right)} = \frac{1}{\int_{-\infty}^{\infty}\left(1-\beta\right)\phi(z)dz} \qquad (3\ 59)$$

Table 3 10 provides the unconditional average run length (ARL) and the unconditional probability of a signal $P(Signal)$ for various choices of m the final number of reference samples used in Phase 1 to estimate μ and δ the relative size of the shift in the process mean when subgroups or samples of size $n = 5$ are used together with 3 sigma control limits that is $\alpha = 0\ 0027$ In addition the last row contains the corresponding values of the ARL and $P(Signal)$ for Case KK

ARL $P(Signal)$	δ								
m	0 00	0 25	0 50	0 75	1 00	1 25	1 50	1 75	2 00
5	237 77	160 72	58 25	17 18	6 07	2 85	1 73	1 29	1 11
	0 0062	0 0135	0 0430	0 1136	0 2427	0 4259	0 6268	0 7977	0 9106
10	277 91	158 47	45 41	13 38	5 16	2 59	1 64	1 25	1 09
	0 0042	0 0103	0 0364	0 1037	0 2332	0 4227	0 6320	0 8081	0 9198
15	298 24	154 00	41 01	12 39	4 92	2 52	1 62	1 24	1 09
	0 0037	0 0093	0 0343	0 1002	0 2297	0 4214	0 6343	0 8117	0 9230
20	310 86	150 56	38 95	11 96	4 80	2 49	1 60	1 24	1 08
	0 0034	0 0089	0 0332	0 0983	0 2280	0 4207	0 6352	0 8136	0 9246
25	319 66	148 04	37 76	11 69	4 74	2 47	1 60	1 23	1 08
	0 0033	0 0086	0 0325	0 0973	0 2268	0 4204	0 6358	0 8148	0 9256
30	326 08	145 97	37 00	11 52	4 70	2 45	1 59	1 23	1 08
	0 0032	0 0084	0 0320	0 0966	0 2262	0 4201	0 6363	0 8155	0 9262
50	340 89	141 37	35 45	11 21	4 61	2 43	1 58	1 23	1 08
	0 0030	0 0080	0 0312	0 0952	0 2247	0 4195	0 6371	0 8171	0 9275
75	349 47	138 83	34 78	11 06	4 57	2 41	1 58	1 22	1 08
	0 0029	0 0079	0 0308	0 0944	0 2240	0 4192	0 6374	0 8179	0 9282
100	354 17	137 46	34 42	10 98	4 55	2 41	1 57	1 22	1 08
	0 0028	0 0078	0 0306	0 0940	0 2236	0 4193	0 6377	0 8181	0 9285
200	361 83	135 32	33 91	10 87	4 52	2 40	1 57	1 22	1 08
	0 0028	0 0076	0 0303	0 0935	0 2230	0 4190	0 6380	0 8188	0 9290
300	364 57	134 61	33 73	10 83	4 51	2 39	1 57	1 22	1 08
	0 0027	0 0076	0 0302	0 0933	0 2228	0 4190	0 6382	0 8190	0 9292
500	366 85	134 05	33 61	10 80	4 51	2 39	1 57	1 22	1 08
	0 0027	0 0076	0 0301	0 0932	0 2227	0 4189	0 6382	0 8192	0 9293
1000	368 60	133 59	33 51	10 78	4 50	2 39	1 57	1 22	1 08
	0 0027	0 0075	0 0300	0 0930	0 2226	0 4189	0 6383	0 8193	0 9294
∞	370 37	133 15	33 40	10 76	4 50	2 39	1 57	1 22	1 08
	0 0027	0 0075	0 0299	0 0929	0 2225	0 4188	0 6384	0 8194	0 9295

Table 3 10 The unconditional average run length and the unconditional probability of a signal when $\alpha = 0\ 0027$ and $n = 5$

We observe that as the number of reference samples m increase the unconditional average run length as well as the unconditional probability of a signal converge to the average run length and the probability of a signal found in Case KK in other words when both the in control process parameters μ_0 and σ_0 are known

For example we previously showed in equation (3 36c) that if the process is in control in other words when $\delta = 0$ $\lim_{m\to\infty} P\left(|Z| \geq \sqrt{\frac{m}{m+1}} z_{\frac{\alpha}{2}}\right) = \alpha$ Now this is confirmed by using equation (3 58) which was used to populate Table 3 10

The tendency of the average run length (ARL) and the probability of a signal to converge to the values we obtained in Case KK implies that we should use as many reference samples or retrospective samples from Phase 1 as possible to estimate the unknown process mean so that the difference in the performance of the $\overline{X}$ control chart in Case UK versus that of Case KK will be a minimum

In spite of this it is customary to assume that we only have 20 or 25 reference samples each of size $n = 4$ or 5 to estimate the unknown process mean and that this guarantees that the control chart performs as what would be the case if both the parameters were known However we see from Table 3 10 that the in control unconditional average run length (ARL) when $m = 25$ and $n = 5$ is 319 66 which is 13 7% *less* than the value of 370 37 we found in Case KK Thus on average there will be 13 7% *more* false alarms even when 25 reference samples or 125 individual observations are available

On the other hand if $\delta = 0\ 50$ there is on average an *increase* in the ARL of 13 1% from the ARL of 33 40 we found in Case KK to the value of 37 76 in Case UK and a 1 9% *increase* in the ARL if $\delta = 1\ 50$ from 1 57 to 1 60 Thus whenever the unknown process mean undergoes a sustained shift that is when $\delta \neq 0$ the $\overline{X}$ control chart in Case UK signals *less* often whereas it signals *more* often if the process is actually in control i e when $\delta = 0$ compared to what would be the case if both the process parameters are known

Figure 3 16 displays the unconditional ARL of Case UK versus the ARL of Case KK when $\alpha = 0\ 0027$ $m = 25$ and $n = 5$ for varying values of δ We note that in general the performance of the control chart in Case KK is better than that of Case UK For example as mentioned the in control ARL of the control chart in Case KK is higher than that of the control chart in Case UK whereas the out of control ARL of the control chart in Case KK is less than that of the control chart in Case UK – except for $0 < \delta < 0\ 15$

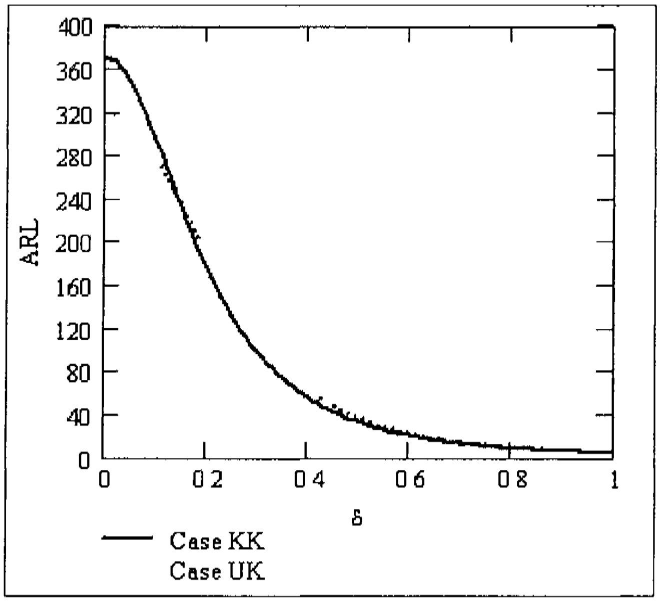

Figure 3 16 The unconditional average run length (ARL) of Case UK versus the ARL of Case KK

Apart from the tendencies in the average run length (ARL) we also observe from Table 3 10 that as mentioned previously $ARL \neq [P(Signal)]^{-1}$ For example when $m = 25$ and $\delta = 0$ $ARL = 319\ 66$ which is not equal to the reciprocal of the probability of a signal i e $ARL \neq (0\ 0033)^{1} = 303\ 03$

The unconditional standard deviation of the run length distribution (SDRL)

Studying the unconditional standard deviation of the run length (SDRL) as given in Result 3 33 of Appendix 3B or as can be obtained from either equation (3 55) or (3 56) we observe the same tendency as was seen in the unconditional average run length and the unconditional probability of a signal That is the unconditional standard deviation of the run length (SDRL) of Case UK converges to the standard deviation of the run length of Case KK

For example Table 3 11 displays the unconditional standard deviation of the run length (SDRL) of Case UK together with the standard deviation of the run length of Case KK in the last row We observe that as m increases the SDRL increases and if $m \to \infty$ when $\delta = 0$ the SDRL converges to the value of 369 87 which corresponds to the value found for the SDRL in Case KK – see also Table 3 7

SDRL	δ								
m	0 00	0 25	0 50	0 75	1 00	1 25	1 50	1 75	2 00
5	279 02	226 44	111 37	35 34	9 85	3 35	1 43	0 71	0 37
10	301 86	211 26	71 77	18 79	5 97	2 40	1 14	0 60	0 32
15	314 26	197 28	56 83	15 20	5 17	2 18	1 07	0 57	0 31
20	322 44	186 60	49 86	13 73	4 82	2 08	1 03	0 56	0 30
25	328 38	178 79	45 92	12 91	4 63	2 02	1 01	0 55	0 30
30	332 90	172 63	43 45	12 42	4 51	1 99	1 00	0 54	0 30
50	343 99	158 31	38 81	11 49	4 28	1 92	0 98	0 53	0 29
75	350 90	150 25	36 78	11 06	4 17	1 88	0 97	0 53	0 29
100	354 88	145 98	35 75	10 85	4 12	1 87	0 96	0 53	0 29
200	361 69	139 35	34 30	10 54	4 04	1 84	0 95	0 52	0 29
300	364 24	137 13	33 82	10 44	4 01	1 84	0 95	0 52	0 29
500	366 42	135 36	33 46	10 36	3 99	1 83	0 95	0 52	0 29
1000	368 12	134 00	33 18	10 31	3 98	1 82	0 94	0 52	0 29
∞	369 87	132 65	32 90	10 25	3 96	1 82	0 94	0 52	0 27

Table 3 11 The unconditional standard deviation of the run length when $\alpha = 0\ 0027$ **and** $n = 5$

In fact as the *final* number of reference samples m or the number of observations in each reference sample n or as the overall number of observations mn that are used to estimate the unknown process mean increase all the characteristics and properties of the control chart in Case UK converge to those in Case KK

Clearly this suggests that if we expect the control chart procedure in Case UK to perform anything like the control chart procedure in Case KK we should obtain as many reference samples or historical information as possible and if this is not possible we should (at least) take note of the effect of the estimation procedure on the general or the overall performance of the control chart

3 3 The Mean specified but the Standard Deviation unknown (Case KU)

Consider the Shewhart $\overline{X}$ control chart for monitoring the process mean μ when the in control process μ_0 is *known* but the in control process standard deviation σ is *unknown* that is assume that $X_{ij} \sim N\left(\mu_0 \ \sigma^2\right)$ To estimate σ we assume that as in Case UK we have m independent reference samples each of size n

However there are numerous point estimators for the unknown in control process standard deviation σ of which some are used more frequently or more regularly than others In addition depending on the scenario some of these estimators are more appropriate than others are

For example if the in control process mean μ is *unknown* we have an option between two estimators for the *unknown* in control process variance σ^2 and subsequently we also have two estimators for the *unknown* in control process standard deviation σ

Our first option is

$$S_{m\ 1}^2 = \frac{1}{mn-1}\sum_{i=1}^{m}\sum_{j=1}^{n}\left(X_{ij}-\overline{\overline{X}}\right)^2 \qquad (3\ 60)$$

Using $S_{m\ 1}^2$ we pool the entire set of mn observations from the m different reference samples so that we have a single sample of size $N = mn$ say However in doing so we (implicitly) assume that the m reference samples have a common but unknown mean that is $E(X_{ij}) = \mu_i = \mu$ for $i = 1\ 2 \quad m$ which is estimated by $\overline{\overline{X}}$ of equation (3 27)

Although it is common practice to pool observations from different samples to obtain an estimate for an unknown process parameter see for instance equation (3 27) where we estimate the unknown in control process mean μ by $\overline{\overline{X}}$ we must be careful when estimating the unknown process variance For example if we cannot assume that $E(X_{ij}) = \mu_i = \mu$ the principle of rational subgrouping discussed in Section 2 7 of Chapter 2 is violated That is the estimator of equation (3 60) will incorporate both *between* sample variation as well as *within* sample variation and will therefore typically overestimate the unknown in control process variance σ^2

Consequently the estimator of equation (3 60) can only be used if it is known that the process is operating in statistical control In other words we can use $S^2_{m \quad 1}$ only when the process mean and the process standard deviation do not change over time so that there will be no additional variation included because of variation *between* (consecutive) samples

As a second option we have

$$S^2_{m(\quad 1)} = \frac{1}{m(n-1)} \sum_{1}^{m} \sum_{j \; 1} \left(X_j - \overline{X} \right)^2 \qquad (3\ 61)$$

which is a pooled variance estimate and allows that $E(X_j) = \mu \neq \mu$ whereas $\operatorname{var}(X_j) = \sigma^2$ That is we use $S^2_{m(\quad 1)}$ when we cannot safely assume that the m reference samples have the same unknown mean μ but we can assume that the variation *within* each reference sample is equal

Re writing equation (3 61) we obtain

$$\begin{aligned} S^2_{m(\quad 1)} &= \frac{\sum_{j=1}\left(X_{1j} - \overline{X}_1\right)^2 + \sum_{j\;1}\left(X_{2j} - \overline{X}_2\right)^2 + \quad + \sum_{j\;1}\left(X_{mj} - \overline{X}_m\right)^2}{m(n-1)} \\ &= \frac{(n-1)S_1^2 + (n-1)S_2^2 + \quad + (n-1)S_m^2}{m(n-1)} \\ &= \frac{1}{m}\sum_{1}^{m} S^2 \end{aligned}$$

from which we can clearly see that $S^2_{m(\quad 1)}$ is a (simple) weighted average of the individual sample variances $S_1^2 \; S_2^2 \qquad S_m^2$ with each of their weights given by $\frac{1}{m}$

In contrast to $S^2_{m \quad 1}$ of equation (3 60) $S^2_{m(\quad 1)}$ of equation (3 61) adheres to the principle of rational subgrouping Thus by first calculating the individual sample variances $S_1^2 \; S_2^2 \qquad S_m^2$ which is based on the sample means $\overline{X} \quad i = 1 \; 2 \qquad m$ and then pooling these variance estimates we include only *within* sample variation and no additional *between* sample variation is included

On the other hand when the in control process mean μ_0 is *known* as in the current situation for Case KU we may use

$$S_m^2 = \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\left(X_{ij} - \mu_0\right)^2 \tag{3 62}$$

S_m^2 is similar to $S_{m\ 1}^2$ of equation (3 60) in the sense that we pool all the observations from the m reference samples so that we have a single sample of size $N = mn$ However in this case we assume that the in control process mean μ_0 is known Thus instead of using $\overline{\overline{X}}$ to estimate μ_0 we have μ_0 and as denominator we use mn instead of $mn-1$ In addition when the in control process mean is known the estimator S_m^2 (automatically) incorporates only *within* sample variation

The exact sampling distribution of each of the three estimators i e $S_{m\ 1}^2$ $S_{m(\ 1)}^2$ and S_m^2 can easily be determined In fact it is well known that $\frac{\nu S^2}{\sigma^2} \sim \chi^2$ where $\nu = mn-1$ or $\nu = m(n-1)$ or $\nu = mn$ depending on whether we use $S_{m\ 1}^2$ $S_{m(\ 1)}^2$ or S_m^2 as an estimator In addtion we also know that $E(S^2) = \sigma^2$ that is S^2 is an unbiased estimator for the in control process variance σ^2 However $E(S\ ) \neq \sigma$ in other words S is not an unbiased estimator for the process standard deviation σ Instead $E(S\ ) = c_4\sigma$ where c_4 is a constant such that $\frac{S}{c_4}$ is an unbiased estimator for σ Consequently we are interested in finding c_4

Using equal in distribution techniques we can write

$$\frac{\nu S^2}{\sigma^2} \stackrel{d}{=} Y \tag{3 63}$$

where $Y \sim \chi^2$ We read this as $\frac{\nu S^2}{\sigma^2}$ is equal in distribution to Y and implies that the two random variables $\frac{\nu S^2}{\sigma^2}$ and Y have the same probability density function (pdf) which in this case is $f_Y(y) = \frac{1}{2^{\frac{\nu}{2}}\Gamma(\frac{\nu}{2})} y^{\frac{\nu}{2}-1} e^{-\frac{y}{2}}$ with $y \in [0\ \infty)$ Consequently the two random variables $\frac{\nu S^2}{\sigma^2}$ and Y also share the same moments i e they have the same expected value and the same variance etc

Rearranging expression (3 63) we obtain

$$S \overset{d}{=} \frac{\sigma}{\sqrt{\nu}}\sqrt{Y}$$

and subsequently we obtain

$$E(S) = \frac{\sigma}{\sqrt{\nu}} E(\sqrt{Y})$$

Now

$$E(\sqrt{Y}) = \int_0^\infty \sqrt{y} f_Y(y)dy = \int_0^\infty \sqrt{y} \frac{1}{2^{\frac{\nu}{2}}\Gamma(\frac{\nu}{2})} y^{\frac{\nu}{2}-1} e^{-\frac{y}{2}} dy$$

and if we group all the y terms together we obtain

$$E(\sqrt{Y}) = \int_0^\infty \frac{1}{2^{\frac{\nu}{2}}\Gamma(\frac{\nu}{2})} y^{\frac{\nu+1}{2}-1} e^{-\frac{y}{2}} dy$$

By manipulating this integral expression to obtain the pdf of a $\chi^2_{\nu+1}$ random variable we obtain

$$E(\sqrt{Y}) = \frac{2^{\frac{\nu+1}{2}}\Gamma(\frac{\nu+1}{2})}{2^{\frac{\nu}{2}}\Gamma(\frac{\nu}{2})} \int_0^\infty \frac{1}{2^{\frac{\nu+1}{2}}\Gamma(\frac{\nu+1}{2})} y^{\frac{\nu+1}{2}-1} e^{-\frac{y}{2}} dy$$

so that we have

$$E(\sqrt{Y}) = \frac{2^{\frac{\nu+1}{2}}\Gamma(\frac{\nu+1}{2})}{2^{\frac{\nu}{2}}\Gamma(\frac{\nu}{2})} \times 1 = \frac{2^{\frac{1}{2}}\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})}$$

which yields

$$E(S) = \frac{\sqrt{2}\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu}\Gamma(\frac{\nu}{2})}\sigma$$

Thus if we let $c_4 = \dfrac{\sqrt{2}\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu}\Gamma(\frac{\nu}{2})}$ we get that

$$E(S) = c_4\sigma$$

which implies that

$$E(\frac{S}{c_4}) = \sigma$$

Consequently we can use $\frac{S_{m\ 1}}{c_4}$ $\frac{S_{m(\ 1)}}{c_4}$ or $\frac{S_m}{c_4}$ as an unbiased estimator for the unknown in control process standard deviation σ with values of c_4 for samples of size $2 \le n \le 25$ given in Table 2 3 of Chapter 2

However apart from the three mean squared or quadratic estimators $S^2_{m\ 1}$ $S^2_{m(\ 1)}$ and S^2_m there exist two other frequently used estimators for the *unknown* in control process standard deviation σ They are

$$\overline{S} = \frac{1}{m}\sum_{1}^{m} S \qquad (3\ 64)$$

and

$$\overline{R} = \frac{1}{m}\sum_{1}^{m} R \qquad (3\ 65)$$

respectively where S is the sample standard deviation and R is the sample range i e

$$S = \sqrt{\frac{1}{n-1}\sum_{j\ 1}\left(X_j - \overline{X}\ \right)^2} \qquad \text{or} \qquad S = \sqrt{\frac{1}{n}\sum_{j\ 1}\left(X_j - \mu_0\right)^2}$$

depending on whether the in control process mean is unknown or known and

$$R = X_{(\ 1)} - X_{(\ )}$$

where $X_{(\ )}$ and $X_{(\ 1)}$ are the largest and the smallest observation in the i^{th} sample respectively

Since $\overline{S}$ is the average of the m independent random variables S_1 S_2 S_m and if the number of reference samples m is large enough so that the central limit theorem kicks in we have that $\overline{S}$ is approximately normally distributed In addition using the fact that $E(S^2) = \sigma^2$ and the fact that $E(S\ ) = c_4\sigma$ we can find the mean or the expected value and variance of $\overline{S}$ i e

$$E(\overline{S}) = \frac{1}{m}\sum_{1}^{m} E(S\ ) = \frac{1}{m}\sum_{1}^{m} c_4\sigma = c_4\sigma$$

and

$$\text{var}(\overline{S}) = \frac{1}{m^2}\sum_{1}^{m}\text{var}(S\ ) = \frac{1}{m^2}\sum_{1}^{m}\left(E(S^2) - E(S\ )^2\right) = \frac{1}{m^2}\sum_{1}^{m}\left(\sigma^2 - (c_4\sigma)^2\right) = \frac{\sigma^2}{m}\left(1 - c_4^2\right)$$

Thus we have that $\bar{S} \sim N\left(c_4\sigma, \frac{\sigma^2}{m}(1-c_4^2)\right)$ and consequently we have that $\frac{\bar{S}}{c_4} \sim N\left(\sigma, \frac{\sigma^2}{m}\frac{(1-c_4^2)}{c_4^2}\right)$ which is an unbiased estimator for σ

However in contrast the sampling distribution of the average range $\bar{R}$ is not so easy to derive Furthermore as like $\bar{S}$ $\bar{R}$ is not an unbiased estimator for the process standard deviation i e $E(\bar{R}) \neq \sigma$ but instead the random variable or the estimator $\frac{\bar{R}}{d_2}$ is unbiased with d_2 being a constant depending only on the sample size n The derivation of d_2 is not shown here but values of d_2 for different sample sizes $2 \le n \le 25$ are given in Table 2 2 of Chapter 2

Generally a mean squared estimator such as S_{m-1}^2 $S_{m(n-1)}^2$ or S_m^2 are prefered but if the sample sizes are relatively small the average range method works reasonably well For instance the relative effeciency of the sample range R as an estimator for σ compared to the sample standard deviation S for various sample sizes is shown in Table 3 12

Sample size (n)	Relative Efficiency
2	1 000
3	0 992
4	0 975
5	0 955
6	0 930
10	0 850

Table 3 12 Relative efficiency of R versus S as an estimator of σ

We observe that only for samples of size $n \ge 10$ say the sample range R loses efficiency as it ignores all the information in the sample between the two extremes i e $X_{(1)}$ and $X_{(n)}$ However for small samples i e $n \le 6$ say the sample range R works reasonably well compared to the sample standard deviation S

However in light of the current discussion we will use the mean squared estimator $\frac{S}{c_4}$ with $\nu = mn$ for the unknown in control process standard deviation σ In addition assuming that ν is fairly large we have that $c_4 \approx 1$ so that S would suffice That is $E(S) \approx E\left(\frac{S}{c_4}\right) = \sigma$

Hence the *estimated* control limits and the *estimated* centerline for the Shewhart $\overline{X}$ control chart when σ is unknown are

$$\widehat{UCL} = \mu_0 + z_{\frac{\alpha}{2}} \frac{S}{\sqrt{n}}$$
$$\widehat{CL} = \mu_0 \qquad (3\ 66)$$
$$\widehat{LCL} = \mu_0 - z_{\frac{\alpha}{2}} \frac{S}{\sqrt{n}}$$

(See Result 3 36 in Appendix 3C)

Note that as in Case UK the estimated control limits as well as the estimated centerline are functions of a random variable which in this case is S In addition if the estimate S is to be meaningful it must be based on data from an in control process Thus as before the estimated control limits from equation (3 66) are treated as trial limits and the control chart is then typically implemented as follows

Each sample mean $\overline{X}_1$ $\overline{X}_2$ $\overline{X}_m$ from the m reference samples is plotted on the control chart together with the estimated control limits and the estimated centerline If none the sample means falls outside the control limits these trial control limits are used prospectively for process monitoring in Phase 2 However if any of the sample means falls on or beyond the control limits that sample(s) are investigated further and if found to represent a true out of control condition the sample(s) is discarded and the control limits and centerline are revised accordingly Generally this process is repeated until all the points or plotting statistics plot in control whereupon the trial or the revised control limits are adopted for future use

Following the estimation of the unknown in control process standard deviation σ as well as the control limits we move on to Phase 2 of the control chart procedure That is we obtain additional samples to monitor whether the process remains in control Consequently we plot $\overline{X}$ $i = m+1$ $m+2$ on the control chart and compare these sample means with the estimated control limits from Phase 1

Example 3 3

A Shewhart $\overline{X}$ control chart for monitoring the process mean when the in control process standard deviation is unknown and needs to be estimated

Panel (a) of Table 3 13 displays the individual values of $m = 22$ reference samples each of size $n = 4$ which were randomly drawn from a normal distribution with a *known* mean $\mu_0 = 12$ and an unknown standard deviation σ In addition panel (b) contains the sample means $\overline{X}$ and the sample variances i e $S^2 = \frac{1}{4}\sum_{j=1}^{4}(X_j - 12)^2$ for $i = 1\ 2 \quad 22$

	(a)				(b)	
Sample i	X_1	X_2	X_3	X_4	$\overline{X}$	S^2
1	10 363	10 126	9 218	16 224	11 483	7 943
2	19 760	13 666	12 319	17 559	15 826	23 499
3	10 980	11 236	11 494	11 916	11 406	0 472
4	14 748	7 211	14 004	18 908	13 718	20 556
5	13 617	13 451	13 479	13 598	13 536	2 365
6	11 285	16 781	10 222	13 779	13 017	7 424
7	14 674	19 619	11 339	14 721	15 088	18 260
8	13 351	20 072	11 955	13 561	14 735	17 355
9	10 959	18 281	9 832	8 214	11 821	14 892
10	10 421	10 147	7 292	15 153	10 753	9 508
11	14 063	13 755	10 425	11 491	12 433	2 519
12	6 775	13 143	11 323	13 293	11 133	7 684
13	13 213	13 470	14 077	11 650	13 103	2 017
14	12 115	15 659	12 744	3 769	11 072	20 426
15	15 579	12 483	16 250	18 160	15 618	17 263
16	9 327	9 709	13 670	9 824	10 633	4 979
17	13 451	11 334	12 388	11 404	12 144	0 764
18	8 681	5 424	12 769	9 859	9 183	14 859
19	14 105	11 109	17 133	18 115	15 115	17 241
20	12 746	9 034	13 436	12 805	12 005	3 016
21	12 534	10 132	11 319	5 222	9 802	12 545
22	11 275	16 678	18 200	12 705	14 714	15 337

Table 3 13 Retrospective or Phase 1 data (Case KU)

Using equation (3 62) we estimate the unknown in control process variance to be

$$\begin{aligned} S_{88}^2 &= \frac{1}{m}\left(S_1^2 + S_2^2 + \quad + S_m^2\right) \\ &= \frac{1}{22}\left(7\ 943 + 23\ 499 + \quad + 15\ 337\right) \\ &= 10\ 53 \end{aligned}$$

so that the in control standard deviation is estimated as

$$\hat{\sigma} = S_{88} = 3\ 24$$

Subsequently the *estimated* 3 sigma control limits and the centerline are calculated using equation (3 66) and found to be

$$\widehat{UCL} = 12 + 3\frac{3\ 24}{\sqrt{4}} = 16\ 86$$

$$\widehat{CL} = 12$$

$$\widehat{LCL} = 12 - 3\frac{3\ 24}{\sqrt{4}} = 7\ 14$$

Plotting the sample means $\overline{X}$ $i = 1\ 2$ 22 together with the estimated control limits and centerline results in the control chart displayed in Figure 3 17 Since none of the plotting statistics falls on or outside the control limits the 22 reference samples are declared to be in control and the estimated control limits are adopted for prospective monitoring of the process

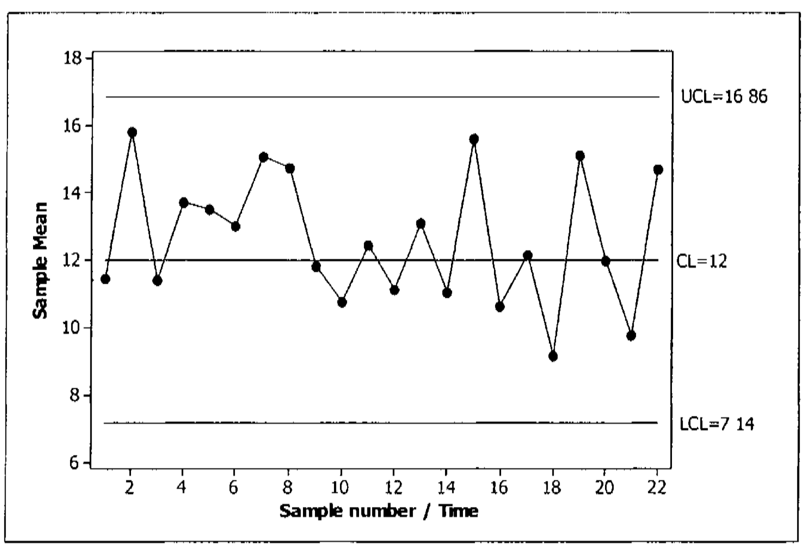

Figure 3 17 A Shewhart $\overline{X}$ control chart for monitoring the process mean in Phase 1 when the in control process standard deviation is estimated

Hence an additional 13 samples each of size 4 were collected to monitor the process and shown in Table 3 14 Continuing the $\overline{X}$ control chart of Figure 3 17 by adding the sample means $\overline{X}$ $i = 23\ 24$ 35 results in the control chart of Figure 3 18 Since all the prospective or Phase 2 sample means plot between the estimated control limits from Phase 1 the process is (still) considered in control and no corrective action is necessary

Sample i	X_1	X_2	X_3	X_4	$\overline{X}$
23	13 547	9 678	11 468	11 728	11 605
24	15 595	17 514	10 827	7 762	12 924
25	12 068	11 841	9 350	14 636	11 974
26	9 843	9 272	4 622	6 680	7 604
27	10 996	10 895	12 364	13 471	11 931
28	11 488	8 921	15 862	12 366	12 159
29	10 882	13 664	11 812	9 102	11 365
30	13 422	14 434	13 850	12 387	13 523
31	11 240	12 771	13 352	15 896	13 315
32	16 949	10 534	15 088	15 450	14 505
33	6 608	18 125	9 788	12 274	11 699
34	10 654	13 443	13 750	3 199	10 261
35	13 792	15 325	8 118	10 841	12 019

Table 3 14 Prospective or Phase 2 data (Case KU)

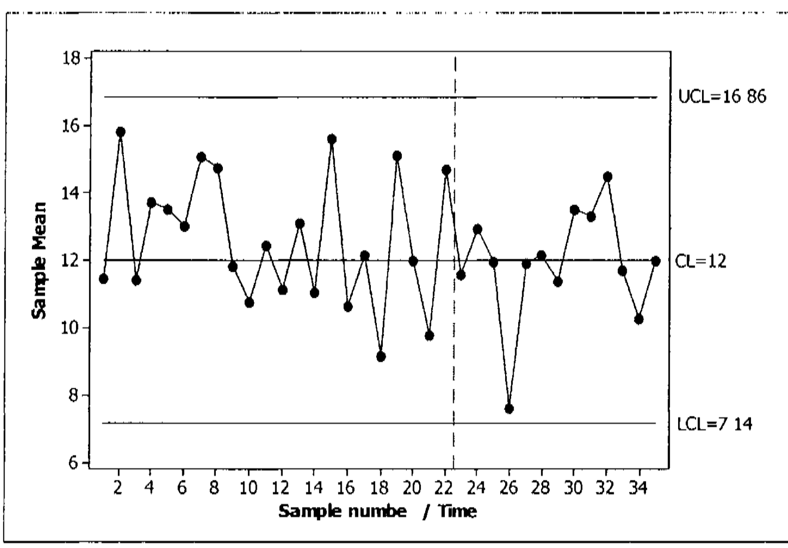

Figure 3 18 A Shewhart $\overline{X}$ control chart for monitoring the process mean in Phase 2 when the in control process standard deviation is estimated

3 3 1 Properties of the $\overline{X}$ control chart

To fully understand a control chart procedure such as the one of Example 3 3 apart from its construction and implementation we also need to study its performance For example if for some reason the process mean changes or shifts out of control we need to know with what probability such a shift will be detected on the 1st subsequent sample say or within a certain number of samples j say To be able to answer such questions we need to study the chart s statistical properties in more detail

A non signalling event and a signalling event in Phase 2 of the control chart procedure can be expressed as

$$\frac{\sqrt{n}\left|\overline{X} - \mu_0\right|}{S} < z_{\frac{\alpha}{2}} \tag{3 67}$$

and

$$\frac{\sqrt{n}\left|\overline{X} - \mu_0\right|}{S} \geq z_{\frac{\alpha}{2}} \tag{3 68}$$

respectively with $i = m+1 \ m+2$ (See Results 3 37 and 3 38 in Appendix 3C)

Additionally the associated *conditional* probabilities of these events that is conditioning on an observed value s of the random variable S or conditioning on the observed value y of the random variable $Y = \frac{\nu S^2}{\sigma^2}$ when a shift in the process mean occurred i e

$$P\left(\frac{\sqrt{n}\left|\overline{X} - \mu_0\right|}{s} < z_{\frac{\alpha}{2}} \mid \mu_1 = \mu_0 + \delta\sigma \ S = s\right) \quad \text{and} \quad P\left(\frac{\sqrt{n}\left|\overline{X} - \mu_0\right|}{s} \geq z_{\frac{\alpha}{2}} \mid \mu_1 = \mu_0 + \delta\sigma \ S = s\right)$$

are given by

$$\beta_{\chi} = \beta(\alpha \ \delta \ m \ n \ y) = \Phi\left(z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}} - \delta\sqrt{n}\right) - \Phi\left(-z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}} - \delta\sqrt{n}\right) \tag{3 69}$$

and

$$1-\beta_{\chi} = 1-\beta(\alpha \ \delta \ m \ n \ y) = 1-\Phi\left(z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}} - \delta\sqrt{n}\right) + \Phi\left(-z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}} - \delta\sqrt{n}\right) \tag{3 70}$$

respectively where $\delta = \frac{\mu_1 - \mu_0}{\sigma}$ represents the relative size of the shift in the process mean μ_0 (See Results 3 39 and 3 41 in Appendix 3C)

For example conditioning on the *estimated* in control process standard deviation of Example 3 3 i e $s = 3\ 24$ or $y = \frac{88(10\ 53)^2}{(3)^2} = 102\ 96$ assuming that the *true* in control process standard deviation $\sigma_0 = 3$ we obtain

$$\beta_{\chi} = \beta(0\ 0027\ \delta\ 22\ 4\ 102\ 96) = \Phi\left(3\sqrt{\frac{102\ 96}{88}} - 2\delta\right) - \Phi\left(-3\sqrt{\frac{102\ 96}{88}} - 2\delta\right)$$

and

$$1 - \beta_{\chi} = 1 - \beta(0\ 0027\ \delta\ 22\ 4\ 102\ 96) = 1 - \Phi\left(3\sqrt{\frac{102\ 96}{100}} - 2\delta\right) + \Phi\left(-3\sqrt{\frac{102\ 96}{88}} - 2\delta\right)$$

respectively

Hence if the process of Example 3 3 remains in control in Phase 2 i e if $\delta = 0$ equation (3 70) yields a conditional false alarm rate of 0 001175 which is much smaller than the nominal value of 0 0027

However in general the *conditional* false alarm rate or the *conditional* probability of a signal when the process is actually in control is found by substituting $\delta = 0$ in (3 70) and given by

$$1 - \Phi\left(z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}}\right) + \Phi\left(-z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}}\right) = P\left(|Z| \geq z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}}\right) \qquad (3\ 71)$$

Hence looking at equation (3 71) it is expected that the actual or *conditional* false alarm rate will differ from the nominal value α unless $y = \nu$ in other words unless the point estimate s is actually equal to σ_0

An alternative (*unconditional*) expression for the false alarm rate is given by

$$1 - T\left(z_{\frac{\alpha}{2}}\right) + T\left(-z_{\frac{\alpha}{2}}\right) = P\left(|T| \geq z_{\frac{\alpha}{2}}\right) \qquad (3\ 72)$$

and is found by *not* conditioning on an observed value s of the random variable S but instead noticing or using the fact that the random variable $\frac{\bar{X} - \mu_0}{\frac{S}{\sqrt{n}}}$ has a t distribution (See Result 3 44 in Appendix 3C) For an interpretation of this result se the discussion below

The overall effect of estimating the unknown in control process standard deviation on the false alarm rate (FAR) is seen from equation (3 72) For example the FAR does not evaluate to α since equation (3 72) is the probability of a t random variable with ν degrees of freedom to be larger (in absolute value) than a critical value from a standard normal distribution and not a critical value from a t distribution with ν degrees of freedom In other words we know that $P\left(|T| \geq t_{\frac{\alpha}{2}}\right) = \alpha$ but $P\left(|T| \geq z_{\frac{\alpha}{2}}\right) \neq \alpha$ unless ν is large In fact $P\left(|T| \geq z_{\frac{\alpha}{2}}\right) \geq \alpha$ since a t distribution has more probability in the tails than does a standard normal distribution and therefore the actual false alarm rate will be larger than the nominal value of α and only if $\nu \to \infty$ will $P\left(|T| \geq z_{\frac{\alpha}{2}}\right) \to \alpha$ that is

$\lim_{\to\infty} P\left(|T| \geq z_{\frac{\alpha}{2}}\right) = \alpha$

For instance in Example 3 3 we estimated the 3 sigma control limits from a total of $mn = 22(4) = 88$ observations thus substituting $\alpha = 0\ 0027$ and $\nu = 88$ in equation (3 72) yields

$$P\left(|T_{88}| \geq 3\right) = 0\ 003511$$

Thus in general when using $m = 22$ reference samples each of size $n = 4$ to estimate σ the false alarm rate (FAR) is approximately 30% higher than the nominal value of 0 0027

Note that the *unconditional* expression for the false alarm rate (FAR) in Case KU as presented in (3 72) is similar to the *unconditional* expression for the false alarm rate (FAR) in Case UK as given in (3 36a) However there doesn t exist a similar *unconditional* expression for the probability of a signal in Case KU as expression (3 35) in Case UK i e when $\delta \neq 0$

Although the unconditional probability of a false alarm in equation (3 72) can be evaluated for different values of the parameters to determine the effect of estimating σ on the false alarm rate

(FAR) it is not at all revealing with respect to the general performance of the control chart when the standard deviation has been estimated The reason for this is the same as in Case UK that is when not conditioning on $S = s$ the series of signalling events $\frac{\sqrt{n}\left|\bar{X}_i - \mu_0\right|}{S} > z_{\frac{\alpha}{2}}$ with $i = m+1, m+2, \ldots$ is not an independent series of events

To verify this we proceed as in Case UK However to show that the covariance or the correlation between any two signalling events is non zero it is easier to work with the estimator $\bar{S} = \frac{1}{m}\sum_{1}^{m} S_i$ of equation (3 64) with its approximate normal distribution i e $\bar{S} \sim N\left(c_4\sigma, \frac{\sigma^2}{m}\left(1-c_4^2\right)\right)$ than using S_m^2 of equation (3 62)

First we find the distribution of the estimated upper and the estimated lower control limits i e $\widehat{UCL}$ and $\widehat{LCL}$ We can show that

$$E\left(\widehat{UCL}\right) = E\left(\mu_0 + z_{\frac{\alpha}{2}}\frac{\bar{S}}{c_4\sqrt{n}}\right) = \mu_0 + z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}$$

and

$$\text{var}\left(\widehat{UCL}\right) = \text{var}\left(\mu_0 + z_{\frac{\alpha}{2}}\frac{\bar{S}}{c_4\sqrt{n}}\right) = z_{\frac{\alpha}{2}}^2\frac{\sigma^2}{mn}\left(\frac{1-c_4^2}{c_4^2}\right)$$

so that $\widehat{UCL} \sim N\left(\mu_0 + z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}, z_{\frac{\alpha}{2}}^2\frac{\sigma^2}{mn}\left(\frac{1-c_4^2}{c_4^2}\right)\right)$ since $\widehat{UCL}$ is (simply) a linear function the normal random variable $\bar{S}$ Similarly we can show that $\widehat{LCL} \sim N\left(\mu_0 - z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}, z_{\frac{\alpha}{2}}^2\frac{\sigma^2}{mn}\left(\frac{1-c_4^2}{c_4^2}\right)\right)$

Also of interest is the distribution of the difference between a sample mean from Phase 2 and any one of the estimated control limits from Phase 1 i e $\bar{X}_i - \widehat{UCL}$ and $\bar{X}_i - \widehat{LCL}$ where $i = m+1, m+2, \ldots$ which are given by

$$N\left(-z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}, \frac{\sigma^2}{n} + z_{\frac{\alpha}{2}}^2\frac{\sigma^2}{mn}\left(\frac{1-c_4^2}{c_4^2}\right)\right)$$

and

$$N\left(z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}, \frac{\sigma^2}{n} + z_{\frac{\alpha}{2}}^2\frac{\sigma^2}{mn}\left(\frac{1-c_4^2}{c_4^2}\right)\right)$$

respectively

Now let C denote the unconditional signalling event that is the event that $\overline{X}_i$ $i = m+1\ m+2$ either exceeds $\widehat{UCL}$ or is less than $\widehat{LCL}$ If this is the case the event C consists of two mutually exclusive events namely $\overline{X}_i \geq \widehat{UCL}$ and $\overline{X}_i \leq \widehat{LCL}$ which can also be written as $\overline{X}_i - \widehat{UCL} \geq 0$ and $\overline{X}_i - \widehat{LCL} \leq 0$ respectively

Thus the probability of the event C is $P(C) = P(\overline{X}_i - \widehat{UCL} \geq 0) + P(\overline{X}_i - \widehat{LCL} \leq 0)$ and since we assume that $P(\overline{X}_i - \widehat{UCL} \geq 0) = P(\overline{X}_i - \widehat{LCL} \leq 0)$ when using 3 sigma control limits we have that $P(C) = 2P(\overline{X}_i - \widehat{UCL} > 0)$ Thus as in Case UK to show that the two signalling events C and C_j where $i \neq j = m+1\ m+2$ are dependent is similar to showing that the two events $\overline{X}_i - \widehat{UCL} \geq 0$ and $\overline{X}_j - \widehat{UCL} \geq 0$ are not independent

Thus since the covariance between the two random variables $\overline{X}_i - \widehat{UCL}$ and $\overline{X}_j - \widehat{UCL}$ i e

$$\text{cov}\left(\overline{X}_i - \widehat{UCL}\ \overline{X}_j - \widehat{UCL}\right) = \text{var}(\widehat{UCL}) = z^2_{\frac{\alpha}{2}} \frac{\sigma^2}{mn}\left(\frac{1-c_4^2}{c_4^2}\right)$$

is not equal to zero or alternatively since the correlation between these two random variables i e

$$corr\left(\overline{X}_i - \widehat{UCL}\ \overline{X}_j - \widehat{UCL}\right) = \frac{f(\alpha\ m\ n)}{m + f(\alpha\ m\ n)}$$

where $f(\alpha\ m\ n) = z^2_{\frac{\alpha}{2}}\left(\frac{1-c_4^2}{c_4^2}\right)$ is not equal to zero the two unconditional events C and C_j are not independent Hence the run length distribution does not follow a geometric distribution and we need to find an alternative method for assessing the performance of the $\overline{X}$ control chart when the process standard deviation is unknown Consequently we turn our attention to the *conditional* run length distribution

3 3 2 The conditional run length distribution

Conditioning on having observed $S = s$ or alternatively $\frac{\nu S^2}{\sigma^2} \stackrel{d}{=} Y = y$ in Phase 1 all the non signaling events in Phase 2 i e $\frac{\sqrt{n}|\bar{X}_i - \mu_0|}{S} < z_{\frac{\alpha}{2}}$ $i = m+1\ m+2$ are mutually independent events and so are all the signaling events in Phase 2 i e $\frac{\sqrt{n}|\bar{X}_i - \mu_0|}{S} \geq z_{\frac{\alpha}{2}}$ $i = m+1\ m+2$

Furthermore since all the non signaling events in Phase 2 have the same *conditional* probability as given in equation (3 69) i e

$$\beta_\chi = \beta(\alpha\ \delta\ m\ n\ y) = \Phi\left(z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}} - \delta\sqrt{n}\right) - \Phi\left(-z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}} - \delta\sqrt{n}\right)$$

and all the signaling events in Phase 2 have the same *conditional* probability as given in equation (3 70) i e

$$1-\beta_\chi = 1-\beta(\alpha\ \delta\ m\ n\ y) = 1-\Phi\left(z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}} - \delta\sqrt{n}\right) + \Phi\left(-z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}} - \delta\sqrt{n}\right)$$

we have that the conditional probability distribution of the run length random variable N is geometric with probability of a signal (success) equal to $1-\beta_\chi$ i e $(N \mid \chi^2 = y) \sim Geo(1-\beta_\chi)$ Subsequently the *conditional* probability mass function (pmf) is given by

$$P(N = j \mid \chi^2 = y) = \beta_\chi^{\,j-1}(1-\beta_\chi) \text{ for } j = 1\ 2\ 3 \qquad (3\ 73)$$

In other words conditioning on $S = s$ or $\chi^2 = y$ the conditional probability that the run length random variable N is equal to j is the conditional probability of not having a signal on the first $(j-1)^{\text{th}}$ samples or subgroups and then having a signal on the j^{th} subsequent sample

It then follows that the *conditional* cumulative distribution function (cdf) and the *conditional* quantile function are

$$P(N \leq j \mid \chi^2 = y) = 1-\beta_\chi^{j} \text{ for } j = 1\ 2\ 3 \qquad (3\ 74)$$

and

$$Q_{N|\chi}(p) = \inf\left\{j \text{ an integer } P(N \le j \mid \chi^2 = y) \ge p\right\}$$
$$= \inf\left\{j \text{ an integer } j \ge \frac{\ln(1-p)}{\ln \beta_\chi}\right\} \quad \text{for } 0 < p < 1 \tag{3 75}$$

respectively

The *conditional* average run length or the expected value of $N \mid \chi^2 = y$ is

$$ARL = E(N \mid \chi^2 = y) = \frac{1}{1-\beta_\chi} \tag{3 76}$$

whereas the variance the standard deviation and the coefficient of skewness of the *conditional* run length distribution are

$$VARRL = \text{var}(N \mid \chi^2 = y) = \frac{\beta_\chi}{\left(1-\beta_\chi\right)^2} \tag{3 77}$$

$$SDRL = stdev(N \mid \chi^2 = y) = \frac{\sqrt{\beta_\chi}}{1-\beta_\chi} \tag{3 78}$$

and

$$SKEWRL = \frac{1+\beta_\chi}{\sqrt{\beta_\chi}} \tag{3 79}$$

respectively

(See Results 3 45 3 47 3 49 3 50 3 52 3 54 and 3 56 in Appendix 3C respectively)

The *conditional* median run length or any other percentile of the *conditional* run length distribution can be found by using the *conditional* quantile function given in (3 75) For example when substituting $p = 0\ 5$ in equation (3 75) the *conditional* median run length is found to be

$$MDRL = \inf\left\{j \text{ an integer } j \ge \frac{\ln(0\ 5)}{\ln \beta_\chi}\right\} \tag{3 80}$$

Example 3 4

Conditional run length distribution of the Shewhart $\overline{X}$ control chart of Example 3 3

The *unknown* in control process standard deviation σ of the process in Example 3 3 was estimated by S_m of equation (3 62) and found to be 3 24 This resulted in an observed value from the χ^2_{88}

distribution i e $y = 102\ 96$ assuming of course that the *true* in control process standard deviation is 3 In addition substituting $y = 102\ 96$ in equations (3 69) and (3 70) we obtained the *conditional* probability of no signal and the *conditional* probability of a signal i e

$$\beta(0\ 0027\ \delta\ 22\ 4\ 102\ 96) = \Phi\left(3\sqrt{\frac{102\ 96}{88}} - 2\delta\right) - \Phi\left(-3\sqrt{\frac{102\ 96}{88}} - 2\delta\right)$$

and

$$1 - \beta(0\ 0027\ \delta\ 22\ 4\ 102\ 96) = 1 - \Phi\left(3\sqrt{\frac{102\ 96}{88}} - 2\delta\right) + \Phi\left(-3\sqrt{\frac{102\ 96}{88}} - 2\delta\right)$$

respectively

Consequently substituting these results in equation (3 73) we obtain the conditional probability mass function (pmf) of the run length random variable N (in Phase 2) of the $\overline{X}$ control chart shown in Figure 3 18 i e

$$P\left(N = j \mid \chi^2 = 102\ 96\right) = \beta(0\ 0027\ \delta\ 22\ 4\ 102\ 96)^{j\ 1}\left(1 - \beta(0\ 0027\ \delta\ 22\ 4\ 102\ 96)\right)$$

for $j = 1\ 2\ 3$

If we assume that the process remains in control that is $\delta = 0$ we find the *conditional* in control pmf to be

$$P\left(N_0 = j \mid \chi^2 = 102\ 96\right) = \left(0\ 998825\right)^{j\ 1}\left(0\ 001175\right) \quad \text{for} \quad j = 1\ 2\ 3$$

Subsequently we can find the average (ARL_0) the standard deviation ($SDRL_0$) the coefficient of skewness ($SKEWRL_0$) and the median ($MDRL_0$) of the *conditional* in control run length distribution using equations (3 76) (3 78) (3 79) and (3 80) respectively

For example

$$ARL_0 = \frac{1}{0\ 001175} = 851\ 06$$

$$SDRL_0 = \frac{\sqrt{0\ 998825}}{0\ 001175} = 850\ 56$$

$$SKEWRL_0 = \frac{1 + 0\ 998825}{\sqrt{0\ 998825}} = 2\ 00$$

and

$$MDRL_0 = \inf\left\{ j \text{ an integer } j \geq \frac{\ln(0\ 5)}{\ln(0\ 998825)} \right\} = 590$$

Thus if the process remains in control we expect (on average) an erroneous signal or a false alarm every 851 32 samples which implies that there will be on average 130% less false alarms compared to the in control average run length of 370 37 in Case KK i e when both the process mean and the process standard deviation are known In addition since $MDRL_0 = 590$ we expect that 50% of the false alarms will occur within the first 590 samples Furthermore we observe that the *conditional* in control run length distribution is (fairly) positively skewed i e $SKEWRL_0 = 2$ and has a standard deviation of approximately 851

On the other hand if we are interested in detecting a shift of size $\delta = 1\ 0$ say it is valuable to also study the *conditional* out of control run length distribution For example following a shift of size $\delta = 1\ 0$ we find the *conditional* probability of no signal to be

$$\beta(0\ 0027\ 1\ 22\ 4\ 102\ 96) = \Phi\left(3\sqrt{\frac{102\ 96}{88}} - 2\right) - \Phi\left(-3\sqrt{\frac{102\ 96}{88}} - 2\right) = 0\ 8934$$

whereas the *conditional* probability of a signal is

$$1 - \beta(0\ 0027\ 1\ 22\ 4\ 102\ 96) = 1 - \Phi\left(3\sqrt{\frac{102\ 96}{88}} - 2\right) + \Phi\left(-3\sqrt{\frac{102\ 96}{88}} - 2\right) = 0\ 1066$$

Therefore the *conditional* out of control probability mass function (pmf) is given by

$$P\left(N_1 = j \mid \chi^2 = 102\ 96\right) = \left(0\ 8934\right)^{j\ 1}\left(0\ 1066\right) \text{ for } j = 1\ 2\ 3$$

In addition we can find the *conditional* out of control average run length (ARL_1) i e

$$ARL_1 = \frac{1}{0\ 1066} = 9\ 38$$

or the *conditional* out of control median run length ($MDRL_1$) i e

$$MDRL_1 = \inf\left\{ j \text{ an integer } j \geq \frac{\ln(0\ 5)}{\ln(0\ 8934)} \right\} = 7$$

Thus for a shift of size $\delta = 1\ 0$ we would expect to detect the shift (on average) within 9 38 samples – *conditional* on having observed $\chi^2_{88} = 102\ 96$ of course

3 3 3 The unconditional run length distribution

The *conditional* run length distribution as given in Section 3 3 2 only presents the *conditional* performance of the $\overline{X}$ control chart when the unknown process standard deviation σ is estimated However to better understand the influence of estimating σ on the charts performance we need to study the *unconditional* run length distribution This will help assess the overall performance of the control chart by eliminating or taking into account the (sampling) variability in the estimator S_m^2 of equation (3 62) But as in Case UK the *conditional* run length distribution aids the derivation of the *unconditional* run length distribution

For the two random variables N and χ^2 we have that $f_{N|\chi}(j\,|\,y)=\dfrac{f_{N\chi}(j\ y)}{f_\chi(y)}$ and from this we find the joint distribution of N and χ^2 i e $f_{N\chi}(j\ z)=f_{N|\chi}(j\,|\,z)f_\chi(y)$ Subsequently the *marginal* or the *unconditional* distribution of N is found by integrating over the domain of χ^2 i e

$$f_N(j)=\int_0^\infty f_{N\chi}(j\ y)dy=\int_0^\infty f_{N|\chi}(j\,|\,y)f_\chi(y)dy$$

Hence the *unconditional* probability mass function (pmf) of the run length random variable N is given by

$$\begin{aligned}P(N=j)&=\int_0^\infty P\left(N=j\,|\,\chi^2=y\right)f_\chi(y)dy\\&=\int_0^\infty \beta_\chi^{j-1}\left(1-\beta_\chi\right)f_\chi(y)dy\end{aligned}\qquad(3\ 81)$$

whereas the *unconditional* cumulative distribution function (cdf) is given by

$$\begin{aligned}P(N\le j)&=\int_0^\infty P\left(N\le j\,|\,\chi^2=y\right)f_\chi(y)dy\\&=\int_0^\infty\left(1-\beta_\chi^{j}\right)f_\chi(y)dy\end{aligned}\qquad(3\ 82)$$

with

$$\beta_\chi=\Phi\left(z_{\frac{\alpha}{2}}\sqrt{\frac{\chi^2}{v}}-\delta\sqrt{n}\right)-\Phi\left(-z_{\frac{\alpha}{2}}\sqrt{\frac{\chi^2}{v}}-\delta\sqrt{n}\right)$$

(See Result 3 46 in Appendix 3C)

The *unconditional* average run length is found by using the (already) available *conditional* average run length given in equation (3 76) so that we obtain

$$\begin{aligned} ARL &= E(N) \\ &= E_{\chi}\left(E\left(N \mid \chi^2 = y\right)\right) \\ &= E_{\chi}\left(\frac{1}{1-\beta_{\chi}}\right) \\ &= \int_0^{\infty}\left(\frac{1}{1-\beta_{\chi}}\right) f_{\chi}(y)dy \end{aligned} \tag{3 83}$$

(See Result 3 51 in Appendix 3C)

In addition the *unconditional* quantile function is

$$\begin{aligned} Q_N(p) &= \inf\{j \text{ an integer } \; P(N \le j) \ge p\} \\ &= \inf\left\{j \text{ an integer } \; \int_0^{\infty}\left(1-\beta_{\chi}^{j}\right) f_{\chi}(y)dy \ge p\right\} \end{aligned} \tag{3 84}$$

so that by substituting $p = 0\ 5$ in equation (3 84) we find the unconditional median run length to be

$$MDRL = \inf\left\{j \text{ an integer } \; \int_0^{\infty}\left(1-\beta_{\chi}^{j}\right) f_{\chi}(y)dy \ge \frac{1}{2}\right\} \tag{3 85}$$

(See Result 3 48 in Appendix 3C)

Finally we can find the variance ($VARRL$) the standard deviation ($SDRL$) and the coefficient of skewness ($SKEWRL$) of the *unconditional* run length distribution by using the *conditional* first second and third non central moments of the run length variable N as given in Results 3 50 and 3 52 of Appendix 3C respectively i e

$$E\left(N \mid \chi^2 = y\right) = \frac{1}{1-\beta_{\chi}} \quad E\left(N^2 \mid \chi^2 = y\right) = \frac{1+\beta_{\chi}}{\left(1-\beta_{\chi}\right)^2} \text{ and } E\left(N^3 \mid \chi^2 = y\right) = \frac{1+\beta_{\chi}^2 + 4\beta_{\chi}}{\left(1-\beta_{\chi}\right)^3}$$

For example the *unconditional* variance is given by

$$\operatorname{var}(N) = E_{\chi}\left(\frac{\beta_{\chi}}{\left(1-\beta_{\chi}\right)^2}\right) + E_{\chi}\left(\left(\frac{1}{1-\beta_{\chi}}\right)^2\right) - \left(E_{\chi}\left(\frac{1}{1-\beta_{\chi}}\right)\right)^2 \tag{3 86a}$$

which can be re written in terms of integrals so that

$$\text{var}(N)=\int_0^\infty \frac{\beta_\chi}{\left(1-\beta_\chi\right)^2} f_\chi(y)dy+\int_0^\infty \left(\frac{1}{1-\beta_\chi}\right)^2 f_\chi(y)dy-\left(\int_0^\infty \frac{1}{1-\beta_\chi} f_\chi(y)dy\right)^2 \qquad (3\ 86b)$$

(See Result 3 55 in Appendix 3C)

Subsequently we can also find the *unconditional* standard deviation from equation (3 86a) or (3 86b) i e $stdev(N)=\sqrt{\text{var}(N)}$

Similarly the *unconditional* coefficient of skewness is found to be

$skew(N)=$

$$\frac{\int_0^\infty \frac{1+\beta_\chi^{\ 2}+4\beta_\chi}{\left(1-\beta_\chi\right)^3} f_\chi(y)dy-3\left\{\int_0^\infty \frac{1+\beta_\chi}{(1-\beta_\chi)^2} f_\chi(y)dy\right\}\left\{\int_0^\infty \frac{1}{1-\beta_\chi} f_\chi(y)dy\right\}+2\left(\int_0^\infty \frac{1}{1-\beta_\chi} f_\chi(y)dy\right)^3}{\left(\text{var}(N)\right)^{\frac{3}{2}}}$$

(3 87)

(See Result 3 57 in Appendix 3C)

As in Case UK to obtain the statistical properties of the *unconditional* **in control** run length distribution let $\delta=0$ in equations (3 81) – (3 87)

3 3 4 Performance of the $\overline{X}$ chart with estimated process standard deviation

Table 3 15 displays the *unconditional* average run length i e

$$ARL = \int_0^\infty \left(\frac{1}{1-\beta_\chi} \right) f_\chi(y)dy$$

given in equation (3 83) together with the *unconditional* probability of a signal i e

$$\begin{aligned} P(Signal) &= \int_0^\infty \left(1-\beta_\chi\right) f_\chi(y)dy \\ &= \int_0^\infty \left(1-\Phi\left(z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}}-\delta\sqrt{n}\right)+\Phi\left(-z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}}-\delta\sqrt{n}\right)\right) f_\chi(y)dy \end{aligned} \qquad (3\ 88)$$

derived as Result 3 42 in Appendix 3C for numerous different number of reference samples m and different shifts in the process mean δ when $\alpha = 0\ 0027$ and $n = 5$ respectively In addition and for comparison the last row of Table 3 15 contains the average run length and the probability of a signal for the $\overline{X}$ control chart procedure when both the process mean and the process standard deviation are *known* i e that of Case KK

For example when $m = 20$ and $\delta = 0\ 50$ that is when we use 20 reference samples each of size 5 to estimate the unknown process standard deviation and following a shift of 0 50 standard deviations in the process mean the *unconditional* average run length (ARL) is 37 86 whereas the *unconditional* probability of signal is 0 0333 This implies that on average the unconditional ARL when the process standard deviation is *estimated* will be approximately 13% *higher* than the ARL of 33 40 when both the process parameters are *known* even though the *unconditional* probability of a signal is 11% higher than the probability of a signal of 0 0299 when both the process parameters are *known*

In addition we note that the *unconditional* ARL given in equation (3 83) is not equal to the reciprocal of the *unconditional* probability of a signal given in equation (3 87) For example when $m = 15$ and $\delta = 0\ 25$ we find from Table 3 15 that the unconditional ARL is equal to 171 49 which is not equal to $(0\ 0094)^{1} = 106\ 38$ However as the number of reference samples m increase regardless of the value of δ the unconditional average run length approaches the reciprocal of the unconditional probability of a signal In fact it can be shown that the reciprocal of the unconditional probability of a signal ı e $\left[P(Signal)\right]^{1}$ ıs a lower bound for the unconditıonal average run length Thıs ıs (one agaın) obtaıned by usıng Jensen s ınequalıty together wıth expectatıon by condıtıonıng ı e

$$ARL = E(N) = E_{\chi}\left(E\left(N \mid \chi^2 = y\right)\right) = E_{\chi}\left(\frac{1}{1-\beta_{\chi}}\right) \geq \frac{1}{E\left(1-\beta_{\chi}\right)} = \frac{1}{\int_0^{\infty}\left(1-\beta_{\chi}\right) f_{\chi}(y)dy} \quad (3\ 89)$$

ARL / $P(Signal)$	δ								
m	0 00	0 25	0 50	0 75	1 00	1 25	1 50	1 75	2 00
5	1324 67	327 26	61 05	15 84	5 66	2 72	1 68	1 27	1 10
	0 0060	0 0135	0 0438	0 1166	0 2496	0 4366	0 6388	0 8076	0 9166
10	638 53	197 37	43 56	12 77	4 99	2 54	1 62	1 24	1 09
	0 0042	0 0103	0 0368	0 1051	0 2366	0 4280	0 6383	0 8132	0 9228
15	523 41	171 49	39 58	12 01	4 81	2 48	1 60	1 24	1 08
	0 0037	0 0094	0 0345	0 1011	0 2319	0 4249	0 6383	0 8150	0 9250
20	477 08	160 35	37 86	11 67	4 72	2 46	1 59	1 23	1 08
	0 0034	0 0089	0 0333	0 0991	0 2298	0 4234	0 6383	0 8162	0 9261
25	453 43	154 15	36 87	11 48	4 68	2 44	1 59	1 23	1 08
	0 0033	0 0086	0 0327	0 0978	0 2282	0 4225	0 6383	0 8168	0 9268
30	436 73	150 24	36 24	11 35	4 64	2 43	1 58	1 23	1 08
	0 0032	0 0084	0 0322	0 0970	0 2273	0 4219	0 6382	0 8172	0 9272
50	408 48	143 02	35 06	11 10	4 58	2 42	1 58	1 22	1 08
	0 0030	0 0080	0 0313	0 0955	0 2254	0 4205	0 6384	0 8181	0 9282
75	395 09	139 66	34 50	10 99	4 55	2 41	1 57	1 22	1 08
	0 0029	0 0079	0 0308	0 0946	0 2244	0 4201	0 6383	0 8185	0 9286
100	388 79	137 93	34 21	10 93	4 54	2 40	1 57	1 22	1 08
	0 0028	0 0078	0 0306	0 0942	0 2239	0 4198	0 6384	0 8187	0 9288
200	379 44	135 50	33 80	10 84	4 52	2 39	1 57	1 22	1 08
	0 0028	0 0076	0 0303	0 0936	0 2231	0 4193	0 6383	0 8191	0 9291
300	376 44	134 72	33 67	10 82	4 51	2 39	1 57	1 22	1 08
	0 0027	0 0076	0 0302	0 0933	0 2229	0 4191	0 6384	0 8192	0 9293
500	373 97	134 08	33 56	10 79	4 50	2 39	1 57	1 22	1 08
	0 0027	0 0076	0 0301	0 0932	0 2228	0 4190	0 6384	0 8193	0 9294
1000	372 26	133 62	33 48	10 78	4 50	2 39	1 57	1 22	1 08
	0 0027	0 0075	0 0300	0 0931	0 2226	0 4189	0 6384	0 8193	0 9294
∞	370 37	133 15	33 40	10 76	4 50	2 39	1 57	1 22	1 08
	0 0027	0 0075	0 0299	0 0929	0 2225	0 4188	0 6384	0 8194	0 9295

Table 3 15 The unconditıonal average run length and the unconditıonal probabılıty of a sıgnal when $\alpha = 0\ 0027$ **and** $n = 5$

To show and hıghlıght the dıfference(s) between the *uncondıtıonal* ARL of Case UK when the process *mean* ıs *unknown* and Case KU when the process *standard devıatıon* ıs *unknown* versus the ARL ın Case KK when both the process parameters are *known* we may study Fıgure 3 19 Fıgure 3 19 dısplays the *uncondıtıonal* ARL of Case UK and Case KU together wıth the ARL for Case KK versus

$0 \le \delta \le 1$ when $m = 25$ and $n = 5$ In other words when we have 25 reference samples each of size 5 or when we have 125 individual observations

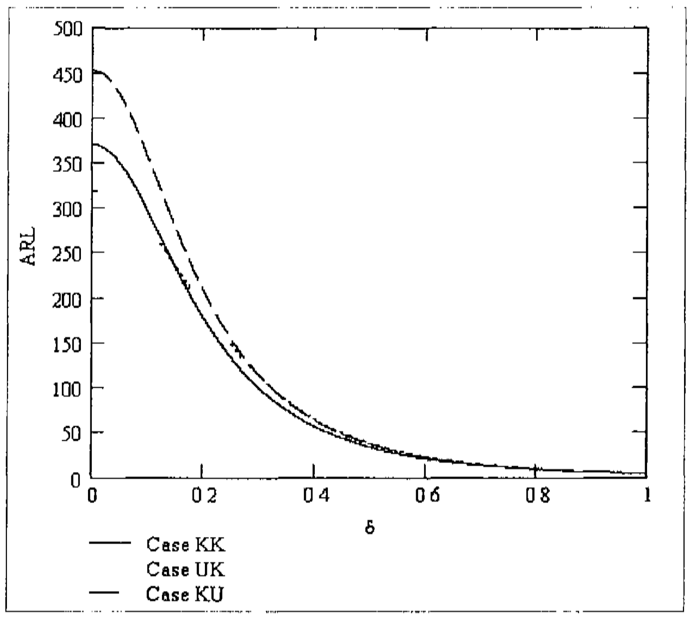

Figure 3 19 The ARL of Cases KK UK and KU when $\alpha = 0\ 0027$ $m = 25$ **and** $n = 5$

We can clearly see that the *unconditional* **in control** average run length ARL_0 of Case KU is considerably higher than the *unconditional* ARL_0 of Case UK and the ARL_0 of Case KK Thus we might be tempted to say that the $\overline{X}$ control chart procedure when the process standard deviation is unknown out performs its two counterparts However this is not entirely true

For example if we study the *unconditional* standard deviation of the run length distribution $SDRL$ and the *unconditional* coefficient of skewness of the run length distribution $SKEWRL$ as presented in Table 3 16 we observe that for $m = 25$ and $\delta = 0$ $SDRL_0 = 663\ 16$ and $SKEWRL_0 = 5\ 61$ Thus the improved ARL_0 in Case KU is not due to its increased performance instead it is due to an increase in the $SDRL$ of approximately 80% from its nominal value of 369 87 and an increase in the $SKEWRL$ of approximately 181% from its nominal value of 2 00 in Case KK – see the last row of Table 3 16 for the $SDRL$ and $SKEWRL$ of Case KK

SDRL / *SKEWRL*	δ				
m	0 00	0 25	0 50	0 75	1 00
5	44097 34	2442 77	246 85	38 65	9 15
	1844 46	440 65	222 01	50 71	16 09
10	1760 68	424 16	73 56	17 61	5 65
	38 03	24 64	10 42	6 09	4 23
15	1006 53	277 56	54 84	14 48	4 98
	14 49	7 68	5 32	3 99	3 19
20	765 39	229 21	47 91	13 21	4 68
	6 98	5 63	4 10	3 31	2 81
25	663 16	204 31	44 29	12 53	4 52
	5 61	4 46	3 54	2 97	2 62
30	598 99	189 70	42 08	12 10	4 42
	4 75	3 87	3 21	2 77	2 50
50	493 69	164 03	38 07	11 30	4 23
	3 39	2 98	2 65	2 43	2 29
75	448 29	152 91	36 25	10 93	4 14
	2 87	2 62	2 42	2 28	2 19
100	427 34	147 48	35 37	10 76	4 09
	2 63	2 46	2 31	2 20	2 15
200	397 70	139 86	34 10	10 50	4 03
	2 30	2 22	2 15	2 10	2 08
300	388 31	137 41	33 70	10 42	4 01
	2 20	2 14	2 10	2 07	2 06
500	380 78	135 48	33 38	10 35	3 99
	2 12	2 09	2 06	2 04	2 04
1000	375 39	134 07	33 13	10 30	3 98
	2 06	2 04	2 03	2 02	2 03
∞	369 87	132 65	32 90	10 25	3 96
	2 00	2 00	2 00	2 00	2 00

Table 3 16 The unconditional SDRL and the unconditional SKEWRL for Case KU when $\alpha = 0\ 0027$ and $n = 5$

If we study Figure 3 19 even further we also observe that when $\delta \neq 0$ the *unconditional* *ARL* of Case UK and the *ARL* of Case KK is less than the *unconditional* *ARL* of Case KU Thus on average when the process does not operate in control the control chart procedure of Case UK and that of Case KK will detect an out of control condition much sooner and give a signal than would be the case if the process standard deviation was estimated

Another effect of estimating the unknown process standard deviation on the general or the overall performance of the control chart can be observed by comparing Table 3 15 and 3 16 In Case KK we had $ARL = \dfrac{1}{1-\beta(\alpha\ \delta\ n)}$ and $SDRL = \dfrac{\sqrt{\beta(\delta\ n)}}{1-\beta(\delta\ n)}$ so that we obtained $SDRL = \sqrt{ARL(ARL-1)} < ARL$ However in Case KU we observed exactly the opposite i e

$SDRL > ARL$ For example when $m = 200$ and $\delta = 0$ we find from Table 3 15 that $ARL = 379\ 44$ whereas we find from Table 3 16 that $SDRL = 397\ 70$ This same tendency can be observed in the general performance of the $\overline{X}$ control chart of Case UK For example by comparing the unconditional average run length values of Table 3 10 with the unconditional standard deviation given in Table 3 11 we find for $m = 20$ and $\delta = 0$ that $ARL = 310\ 86$ and $SDRL = 322\ 44$

In spite of all this we can see from both Table 3 15 and Table 3 16 that as the final number of reference samples m increase the properties or the characteristics of the control chart procedure in Case KU approaches those of Case KK

For example if we consider the in control case i e when $\delta = 0$ we see from Table 3 15 that when we estimate the unknown process standard deviation from $m = 300$ reference samples each of size 5 the *unconditional* average run length ARL is 376 44 which is relatively close to the nominal value of 370 37 Furthermore the *unconditional* probability of as signal is found to be 0 0027 which is the same as that of the nominal or specified value of $\alpha = 0\ 0027$ In addition from Table 3 16 we also see that the *unconditional* standard deviation and coefficient of skewness are 388 31 and 2 12 respectively which is not that much different from their nominal values of 369 87 and 2 00 respectively

Therefore as in Case UK if we expect the $\overline{X}$ control procedure of Case KU when the *unknown* process standard deviation is *estimated* to perform like the $\overline{X}$ control procedure when *both* the process parameters are *known* we would have to obtain as much reference information or historical information as possible Hence if this is not possible we would have to take note of the overall effect(s) of the parameter estimation on the performance of the control chart procedure

3 4 The Mean and the Standard Deviation both unknown (Case UU)

3 4 1 Introduction

Perhaps the worst case scenario when using the Shewhart $\bar{X}$ control chart to monitor the process mean μ is when *both* the in control process parameters i e the process mean μ and the process standard deviation σ are *unknown* If this is the case we call it the Case UU and both the parameters need to be *estimated* which can adversely affect the performance of the control chart This occurs since both the (point) estimators of μ and σ are subject to (sampling) variability Hence as in Case UK and in Case KU we need to take into account the variability in the estimators when assessing the performance of the control chart To do this once again we first derive and discuss the *conditional* run length distribution where after we derive and study the *unconditional* run length distribution

Lastly we compare the overall or the general performance of the $\bar{X}$ control chart in Case UU with that of Case KK i e when *both* the in control process parameters are *known*

3 4 2 Phase 1 of the control chart procedure

Consider monitoring the process mean when *both* the in control process mean μ and the in control process standard deviation σ are *unknown* Hence assume that $X_{ij} \sim N(\mu \ \sigma^2)$ and suppose that to *estimate* μ and σ we have m independent reference or historical samples each of size $n > 1$ when the process was thought to operate in control

The *unknown* process mean μ is then estimated by the overall mean or the grand mean i e

$$\bar{\bar{X}} = \frac{1}{m}\sum_{i=1}^{m}\bar{X}_i = \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}$$

as given in equation (3 27) whereas the *unknown* process variance σ^2 is estimated by either

$$S^2_{mn-1} = \frac{1}{mn-1}\sum_{i=1}^{m}\sum_{j=1}^{n}\left(X_{ij} - \bar{\bar{X}}\right)^2$$

from equation (3 60) or

$$S^2_{m(n-1)} = \frac{1}{m(n-1)}\sum_{i=1}^{m}\sum_{j=1}^{n}\left(X_{ij} - \bar{X}_i\right)^2$$

from equation (3 61) Consequently we estimate the *unknown* in control standard deviation σ by either $S_{m \quad 1}$ or $S_{m(\quad 1)}$ However note that as mentioned previously using $S^2_{m \quad 1}$ to estimate σ^2 we implicitly assume that $E\left(X_j\right) = \mu \; = \mu$ whereas using $S^2_{m(\quad 1)}$ we allows that $E\left(X_j\right) = \mu \; \neq \mu$ Thus even though we assume that the m reference samples are from an in control process that is a process in which the mean and the standard deviation do not change over time to be safe we will use $S^2_{m(\quad 1)}$ as an estimator for σ^2 In doing so we guard against (unnecessary) inflating the estimate of the process variance by including any *between* sampling variability

In addition recall that $E\left(S^2_{m(\quad 1)}\right) = \sigma^2$ and $E\left(S_{m(\quad 1)}\right) = c_4\sigma$ where $c_4 = \dfrac{\sqrt{2}\Gamma\left(\frac{m(n-1)+1}{2}\right)}{\sqrt{m(n-1)}\left(\frac{m(n-1)}{2}\right)}$ so that

$E\left(\dfrac{S_{m(\quad 1)}}{c_4}\right) = \sigma$ i e $\dfrac{S_{m(\quad 1)}}{c_4}$ is an unbiased estimator of σ and not $S_{m(\quad 1)}$ However assuming that $\nu = m(n-1)$ is fairly large we have that $c_4 \approx 1$ and therefore instead of using $\dfrac{S_{m(\quad 1)}}{c_4}$ as an estimator for σ we may use $S_{m(\quad 1)}$ in the expressions to follow

Consequently the *estimated* control limits and the *estimated* centerline when both μ and σ are unknown and estimated are

$$
\begin{aligned}
\widehat{UCL} &= \overline{\overline{X}} + z_{\frac{\alpha}{2}} \frac{S}{\sqrt{n}} \\
\widehat{CL} &= \overline{\overline{X}} \\
\widehat{LCL} &= \overline{\overline{X}} - z_{\frac{\alpha}{2}} \frac{S}{\sqrt{n}}
\end{aligned}
\qquad (3\ 90)
$$

(See Result 3 58 in Appendix 3D)

The implementation of the $\overline{X}$ control chart in Case UU in Phase 1 of the control chart procedure is similar to that of Case UK and Case KU In other words each sample mean $\overline{X}_1$ $\overline{X}_2$ $\quad \overline{X}_m$ from the m reference samples is plotted on the control chart together with the estimated or trial control limits and the centerline If one or more of the plotting statistics falls on or outside the trial limits that sample(s) is investigated further and possibly dropped or eliminated before the control limits and centerline are revised – then using only the remaining reference samples As before this procedure is typically continued until all the plotting statistics plot in control

Following Phase 1 we proceed to Phase 2 in which we monitor the process prospectively by assembling or gathering additional samples and plotting their sample means i e $\overline{X}_i$ $i = m+1$ $m+2$ on the control chart and compare these points to the estimated control limits from Phase 1

Example 3 5

A Shewhart $\overline{X}$ control chart for monitoring the process mean when both the in control process mean and the in control process standard deviation are unknown and estimated

Panel (a) of Table 3 17 contains the individual observations of $m = 30$ reference samples each of size $n = 5$ Assume that *both* the process parameters i e the in control process mean μ and the in control process standard deviation σ are *unknown* Consequently to estimate these unknown parameters panel (b) displays the sample means $\overline{X}_i$ and the sample variances $S_i^2 = \frac{1}{4}\sum_{j=1}^{5}(X_{ij} - \overline{X}_i)^2$ for $i = 1\ 2\ \ 30$

Using equation (3 27) we estimate the in control process mean to be

$$\begin{aligned}\overline{\overline{X}} &= \frac{1}{30}\left(\overline{X}_1 + \overline{X}_2 + \ \ + \overline{X}_{30}\right)\\ &= \frac{1}{30}(24\ 424 + 22\ 993 + \ \ + 22\ 431)\\ &= 23\ 478\end{aligned}$$

whereas using equation (3 61) we estimate the in control process variance to be

$$\begin{aligned}S_{120}^2 &= \frac{1}{30}\left(S_1^2 + S_2^2 + \ \ + S_{30}^2\right)\\ &= \frac{1}{30}(8\ 694 + 1\ 872 + \ \ + 2\ 863)\\ &= 6\ 968\end{aligned}$$

so that the in control process standard deviation is estimated as

$$\hat{\sigma} = S_{120} = \sqrt{6\ 968} = 2\ 639$$

	(a)					(b)	
Sample i	X_1	X_2	X_3	X_4	X_5	$\overline{X}$	S^2
1	26 795	22 681	25 570	26 945	20 129	24 424	8 694
2	21 723	22 465	25 325	22 643	22 810	22 993	1 872
3	26 602	19 318	24 001	25 933	26 337	24 438	9 232
4	23 252	24 113	19 941	22 366	27 604	23 455	7 806
5	20 916	22 117	25 301	21 461	26 233	23 205	5 756
6	25 908	23 121	20 125	20 784	25 342	23 056	6 779
7	19 324	21 042	20 593	20 884	21 673	20 703	0 750
8	24 566	26 956	30 744	25 925	28 185	27 275	5 532
9	21 109	22 717	24 149	20 142	25 790	22 781	5 172
10	26 615	25 679	20 805	26 207	21 817	24 224	7 312
11	22 001	18 536	23 364	21 620	23 057	21 715	3 680
12	19 807	25 175	24 020	23 267	21 345	22 723	4 599
13	26 812	24 390	23 556	21 125	22 757	23 728	4 425
14	24 671	19 737	28 008	21 496	21 279	23 038	10 935
15	21 465	23 072	18 341	22 897	29 672	23 090	17 138
16	24 671	22 285	27 566	22 090	22 058	23 734	5 793
17	20 321	23 773	21 917	20 827	23 660	22 100	2 513
18	19 395	24 967	22 023	23 123	20 887	22 079	4 515
19	28 695	24 019	23 897	24 786	21 210	24 521	7 275
20	23 954	22 281	24 437	26 045	24 768	24 297	1 870
21	26 631	26 195	22 251	26 463	27 186	25 745	3 947
22	23 801	21 663	28 504	26 639	24 191	24 960	7 046
23	23 192	23 587	28 147	20 725	17 330	22 596	15 844
24	24 307	18 933	21 404	23 473	20 002	21 624	5 136
25	21 518	28 633	26 296	24 366	22 564	24 675	8 203
26	19 308	26 086	26 073	25 504	21 389	23 672	9 801
27	28 553	20 948	27 742	22 341	20 580	24 033	14 622
28	25 950	18 479	26 679	19 107	27 564	23 556	19 279
29	24 158	24 100	22 953	23 883	22 310	23 481	0 663
30	24 099	22 222	23 139	19 648	23 046	22 431	2 863

Table 3 17 Retrospective or Phase 1 data (Case UU)

Hence the estimated 3 sigma control limits and the estimated centerline are found using equation (3 90) i e

$$\widehat{UCL} = 23\ 478 + 3\frac{2\ 639}{\sqrt{5}} = 27\ 020$$

$$\widehat{CL} = 23\ 478$$

$$\widehat{LCL} = 23\ 478 - 3\frac{2\ 639}{\sqrt{5}} = 19\ 937$$

The control chart is shown in Figure 3 20 with only one point sample 8 plotting outside the control limits However assume that investigation reveals no assignable cause(s) and that we decide *not* to exclude this sample when calculating the control limits Instead we consider sample number 8 as a false alarm Subsequently the trial control limits (as given above) is not revised and become the standard values against which sample means from Phase 2 can be compared

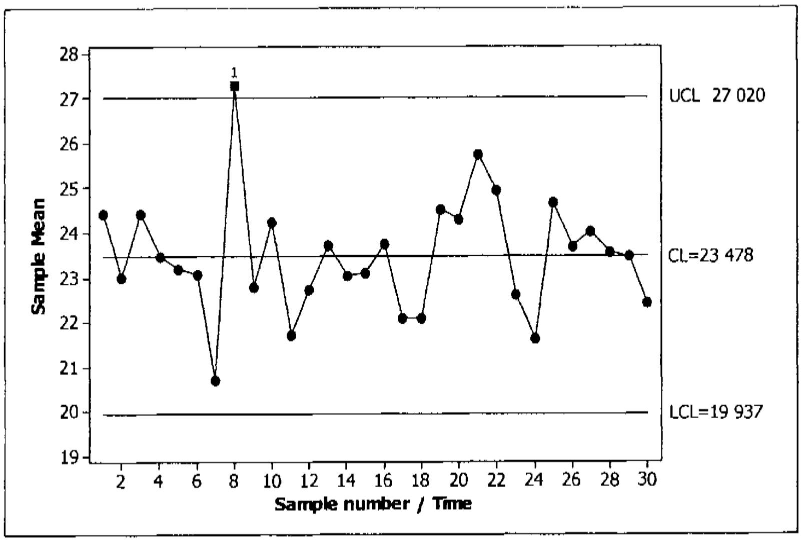

Figure 3 20 A Shewhart $\overline{X}$ control chart for monitoring the process mean in Phase 1 when both the in control process parameters are estimated

An additional 20 new samples each of size 5 are subsequently collected The mean of each sample is noted and recorded in Table 3 18 whereupon these points are plotted on the control chart of Figure 3 21 Since no lack of statistical control is present no corrective action is necessary to adjust or improve the process and we continue to monitor the process

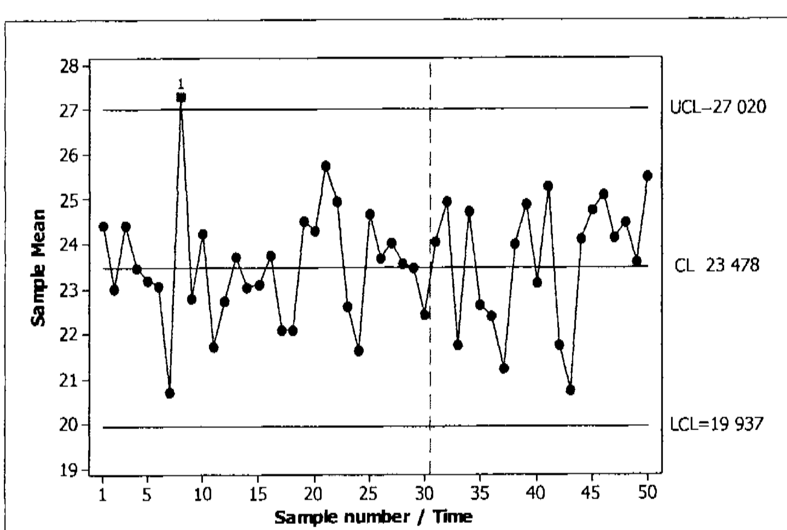

Figure 3 21 A Shewhart $\overline{X}$ control chart for monitoring the process mean in Phase 2 when both the in control process parameters are estimated

Sample i	X_1	X_2	X_3	X_4	X_5	$\overline{X}$
31	22 123	25 213	28 396	19 703	24 812	24 050
32	23 316	25 515	23 824	29 152	22 932	24 948
33	26 345	17 351	24 725	17 057	23 286	21 753
34	23 965	21 202	24 049	29 321	25 174	24 743
35	26 858	24 597	27 102	19 206	15 439	22 641
36	19 328	22 854	24 936	20 831	24 017	22 394
37	21 551	20 507	19 530	26 824	17 697	21 222
38	23 827	31 293	21 055	20 626	23 228	24 006
39	20 970	25 025	29 533	25 145	23 818	24 899
40	18 345	28 757	21 382	22 27	24 895	23 130
41	23 706	24 456	26 913	25 701	25 611	25 278
42	26 787	21 077	24 837	20 071	15 934	21 742
43	17 704	23 437	22 404	16 047	24 129	20 745
44	21 857	23 351	21 667	34 043	19 646	24 113
45	25 527	22 645	26 976	23 611	25 033	24 759
46	19 158	23 026	29 141	26 626	27 589	25 109
47	28 684	26 776	23 492	22 104	19 713	24 154
48	22 234	24 911	25 736	25 022	24 531	24 487
49	23 729	24 378	19 562	23 444	26 832	23 589
50	21 272	28 683	26 397	25 470	25 644	25 494

Table 3 18 Prospective or Phase 2 data (Case UU)

3 4 3 Properties of the $\overline{X}$ control chart

To understand the behavior of the $\overline{X}$ control chart when both the process parameters are unknown and estimated especially in Phase 2 we now study its statistical properties

A non signaling event and a signaling event in Phase 2 that is for $i = m+1\ m+2$ can be expressed as

$$\frac{\sqrt{n}\left|\overline{X} - \overline{\overline{X}}\right|}{S} < z_{\frac{\alpha}{2}} \qquad (3\ 91)$$

and

$$\frac{\sqrt{n}\left|\overline{X} - \overline{\overline{X}}\right|}{S} \geq z_{\frac{\alpha}{2}} \qquad (3\ 92)$$

respectively
(See Results 3 59 and 3 60 in Appendix 3D)

Hence conditioning on the two observed values $\overline{\overline{x}}$ and s of the two random variables or two point estimators $\overline{\overline{X}}$ and S respectively the associated probabilities of the events given in (3 91) and (3 92) i e

$$P\left(\frac{\sqrt{n}\left|\overline{X} - \overline{\overline{x}}\right|}{s} < z_{\frac{\alpha}{2}} \mid \mu_1 = \mu + \delta\sigma\ \overline{\overline{X}} = \overline{\overline{x}}\ S = s\right)$$

and

$$P\left(\frac{\sqrt{n}\left|\overline{X} - \overline{\overline{x}}\right|}{s} \geq z_{\frac{\alpha}{2}} \mid \mu_1 = \mu + \delta\sigma\ \overline{\overline{X}} = \overline{\overline{x}}\ S = s\right)$$

when a shift in the process mean occurred are

$$\beta_{Z\ Y} = \beta(\alpha\ \delta\ m\ n\ z\ y) = \Phi\left(\frac{z}{\sqrt{m}} + z_{\frac{\alpha}{2}}\sqrt{\frac{y}{v}} - \delta\sqrt{n}\right) - \Phi\left(\frac{z}{\sqrt{m}} - z_{\frac{\alpha}{2}}\sqrt{\frac{y}{v}} - \delta\sqrt{n}\right) \qquad (3\ 93)$$

and

$$1-\beta_{Z\ \chi} = 1-\beta(\alpha\ \delta\ m\ n\ z\ y) = 1-\Phi\left(\frac{z}{\sqrt{m}}+z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}}-\delta\sqrt{n}\right)+\Phi\left(\frac{z}{\sqrt{m}}-z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}}-\delta\sqrt{n}\right) \quad (3\ 94)$$

respectively As before $\delta = \frac{\mu_1-\mu}{\sigma}$ denotes the relative size of the shift in the process mean μ (See Results 3 61 (expression 1) and 3 63 in Appendix 3D)

For instance if we *condition* on the *estimated* in control process mean and the *estimated* in control process standard deviation of Example 3 5 i e $\bar{\bar{x}} = 23\ 478$ and $s\ = 2\ 639$ or $z = \frac{23\ 478 - 23}{2\ 7} = 0\ 177$ and $y = \frac{120(2\ 639)^2}{(2\ 7)^2} = 114\ 639$ assuming that the *true* in control process parameters are $\mu_0 = 23$ and $\sigma_0 = 2\ 7$ we obtain

$$\beta(0\ 0027\ \delta\ 30\ 5\ 0\ 177\ 114\ 639) = \Phi\left(\frac{0\ 177}{\sqrt{30}}+3\sqrt{\frac{114\ 639}{120}}-\delta\sqrt{5}\right)-\Phi\left(\frac{0\ 177}{\sqrt{30}}-3\sqrt{\frac{114\ 639}{120}}-\delta\sqrt{5}\right)$$

and

$$1-\beta(0\ 0027\ \delta\ 30\ 5\ 0\ 177\ 114\ 639) = 1-\Phi\left(\frac{0\ 177}{\sqrt{30}}+3\sqrt{\frac{114\ 639}{120}}-\delta\sqrt{5}\right)+\Phi\left(\frac{0\ 177}{\sqrt{30}}-3\sqrt{\frac{114\ 639}{120}}-\delta\sqrt{5}\right)$$

respectively

Thus if the process of Example 3 5 stays in statistical control in Phase 2 that is if $\delta = 0$ equation (3 94) yields a conditional false alarm rate of 0 003382 Hence conditioning on $\bar{\bar{X}} = 23\ 478$ and $S\ = 2\ 639$ the false alarm rate is approximately 25% *larger* than the nominal or expected value of 0 0027 Consequently the control chart will in fact signal *less* often than typically anticipated when the process operates in control

In spite of this and more generally the *conditional* false alarm rate is found by setting $\delta = 0$ in equation (3 94) so that we obtain

$$\begin{aligned} 1-\beta_{Z\ \chi} &= 1-\beta(\alpha\ 0\ m\ n\ z\ y) \\ &= 1-\Phi\left(\frac{z}{\sqrt{m}}+z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}}\right)+\Phi\left(\frac{z}{\sqrt{m}}-z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}}\right) \quad (3\ 95) \\ &= P\left(|Z| \geq \frac{z}{\sqrt{m}}+z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}}\right) \end{aligned}$$

(Sec Result 3 65 ın Appendıx 3D)

But ıf wc use the fact that $\dfrac{(X-\overline{\overline{X}})-(\mu_1-\mu)}{\sqrt{\dfrac{\sigma^2}{n}\left(\dfrac{m+1}{m}\right)}} \sim Z$ and only condıtion on $S = s$ a second

expressıon for the *condıtıonal* probabılıty of a sıgnal ıs found to be

$$1-\Phi\left(\sqrt{\frac{m}{m+1}}\left(z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}}-\delta\sqrt{n}\right)\right)+\Phi\left(\sqrt{\frac{m}{m+1}}\left(-z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}}-\delta\sqrt{n}\right)\right) \quad (3\ 96)$$

(See Result 3 63 ın Appendıx 3D)

Hence ıf the process operates ın control settıng $\delta = 0$ ın equatıon (3 96) yıelds a second expressıon for the *condıtıonal* false alarm rate ı e

$$1-\Phi\left(\sqrt{\frac{m}{m+1}}\left(z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}}\right)\right)+\Phi\left(\sqrt{\frac{m}{m+1}}\left(-z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}}\right)\right) \quad (3\ 97)$$

whıch sımplıfies to

$$P\left(|Z| \ge z_{\frac{\alpha}{2}}\sqrt{\frac{m}{m+1}}\sqrt{\frac{y}{\nu}}\right) \quad (3\ 98)$$

On the other hand ıf we do not condıtıon on eıther $\overline{\overline{X}} = \overline{\overline{x}}$ nor $S = s$ we find an *uncondıtıonal* expressıon for the false alarm rate to be

$$P\left(|T| \ge z_{\frac{\alpha}{2}}\sqrt{\frac{m}{m+1}}\right) \quad (3\ 99)$$

(See Result 3 66 ın Appendıx 3D)

The overall effect of estımatıng the unknown ın control process mean and the unknown ın control process standard devıatıon on the false alarm rate (FAR) can be seen from equatıon (3 99) That ıs the FAR ıs not equal to the nomınal or specıfied value α unless the final number of reference samples from Phase 1 ıs relatıvely large In fact

$$P\left(|T| \ge z_{\frac{\alpha}{2}}\sqrt{\frac{m}{m+1}}\right) \ge \alpha \quad (3\ 100)$$

whereas

$$\lim_{m\to\infty} P\left(|T| \ge z_{\frac{\alpha}{2}}\sqrt{\frac{m}{m+1}}\right) = \alpha \qquad (3\ 101)$$

For instance in Example 3 5 we estimated the unknown process parameters using a total of $m = 30$ reference samples each of size $n = 5$ so that $\nu = m(n-1) = 120$ Hence substituting $\alpha = 0\ 0027$ $\nu = 120$ and $m = 30$ in equation (3 99) yields

$$P\left(|T_{120}| \ge 3\sqrt{\frac{30}{30+1}}\right) = 0\ 003808$$

Consequently on average or in general when using 30 reference samples each of size 5 to estimate μ and σ the false alarm rate (FAR) will be approximately 41% higher than the nominal value of $\alpha = 0\ 0027$ Therefore the *conditional* false alarm rate of 0 003382 obtained for a given set of data such as in Example 3 5 is not (at all) unexpected

Note that expression (3 99) for the *unconditional* false alarm rate is similar to expression (3 36a) which we found in Case UK as well as expression (3 72) found in Case KU

As with both Case UK and Case KU when not conditioning on the estimates $\overline{\overline{X}} = \overline{\overline{x}}$ and/or $S = s$ from Phase 1 the signalling events in Phase 2 are not independent – as are none of the non signalling events Thus using the same series or same sequence of derivations we can show that the correlation between any two Phase 2 signalling events i e $\frac{\sqrt{n}\left|\overline{X} - \overline{\overline{X}}\right|}{S} \ge z_{\frac{\alpha}{2}}$ and $\frac{\sqrt{n}\left|\overline{X}_j - \overline{\overline{X}}\right|}{S} \ge z_{\frac{\alpha}{2}}$ where $i \ne j = m+1\ m+2$ is non zero In fact we find that

$$corr\left(\overline{X} - \widehat{UCL}\ \overline{X}_j - \widehat{UCL}\right) = \frac{1 + g(\alpha\ m\ n)}{m+1+g(\alpha\ m\ n)}$$

where $g(\alpha\ m\ n) = z^2_{\frac{\alpha}{2}}\left(\frac{1-c_4^2}{c_4^2}\right)$ Consequently the run length distribution is *not* geometric with probability of success (signal) given by expression (3 99) Thus as previously we have to focus on the *conditional* run length distribution which is then followed by studying the *unconditional* run length distribution if we intend to better understand the performance of the control chart

3 4 4 The conditional run length distribution

By conditioning on having observed $\overline{\overline{X}} = \overline{\overline{x}}$ and $S = s$ or $Z = z$ and $\frac{\nu S^2}{\sigma^2} = y$ in Phase 1 of the control chart procedure all the signalling events in Phase 2 i e $\frac{\sqrt{n}\left|\bar{X} - \overline{\overline{X}}\right|}{S} \geq z_{\frac{\alpha}{2}}$ are mutually independent events and so are all the non signalling events i e $\frac{\sqrt{n}\left|\bar{X} - \overline{\overline{X}}\right|}{S} < z_{\frac{\alpha}{2}}$ in Phase 2

In addition since all the signalling events in Phase 2 have the same conditional probability i e

$$1-\beta_{z\,\chi} = 1-\beta(\alpha\ \delta\ m\ n\ z\ y) = 1-\Phi\left(\frac{z}{\sqrt{m}} + z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}} - \delta\sqrt{n}\right) + \Phi\left(\frac{z}{\sqrt{m}} - z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}} - \delta\sqrt{n}\right)$$

as given in equation (3 94) and since all the non signalling events in Phase 2 have the same conditional probability as given in equation (3 93) i e

$$\beta_{z\,\chi} = \beta(\alpha\ \delta\ m\ n\ z\ y) = \Phi\left(\frac{z}{\sqrt{m}} + z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}} - \delta\sqrt{n}\right) - \Phi\left(\frac{z}{\sqrt{m}} - z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}} - \delta\sqrt{n}\right)$$

it follows that $\left(N \mid Z = z\ \chi^2 = y\right) \sim Geo\left(1-\beta_{z\,\chi}\right)$ In other words the *conditional* run length distribution is geometric with probability of success (signal) equal to $1-\beta_{z\,\chi}$

Consequently the *conditional* probability mass function (pmf) is

$$P(N = j \mid Z = z\ \chi^2 = y) = \beta_{z\,\chi}^{j-1}\left(1-\beta_{z\,\chi}\right) \text{ for } j = 1\ 2\ 3 \qquad (3\ 102)$$

whereas the *conditional* cumulative distribution function (cdf) and the *conditional* quantile function are

$$P(N \leq j \mid Z = z\ \chi^2 = y) = \sum_{k=1}^{j}\beta_{z\,\chi}^{k-1}(1-\beta_{z\,\chi}) = 1-\beta_{z\,\chi}^{j} \qquad (3\ 103)$$

and

$$\begin{aligned} Q_{N|z\,\chi}(p) &= \inf\left\{j \text{ an integer } P(N \leq j \mid Z = z\ \chi^2 = y) \geq p\right\} \\ &= \inf\left\{j \text{ an integer } j \geq \frac{\ln(1-p)}{\ln \beta_{z\,\chi}}\right\} \end{aligned} \qquad (3\ 104)$$

respectively (See Results 3 67 and 3 69 in Appendix 3D)

Hence as in Case UK and Case KU we find the *conditional* average run length the variance the standard deviation and the coefficient of skewness of the *conditional* run length distribution which follow from the properties of the geometric distribution to be

$$ARL = E(N \mid Z = z \;\; \chi^2 = y) = \frac{1}{1-\beta_{z\,\chi}} \qquad (3\ 105)$$

$$VARRL = \text{var}(N \mid Z = z \;\; \chi^2 = y) = \frac{\beta_{z\,\chi}}{(1-\beta_{z\,\chi})^2} \qquad (3\ 106)$$

$$SDRL = stdev(N \mid Z = z \;\; \chi^2 = y) = \frac{\sqrt{\beta_{z\,\chi}}}{1-\beta_{z\,\chi}} \qquad (3\ 107)$$

and

$$SKEWRL = skew(N \mid Z = z \;\; \chi^2 = y) = \frac{1+\beta_{z\,\chi}}{\sqrt{\beta_{z\,\chi}}} \qquad (3\ 108)$$

respectively

(See Results 3 72 3 74 3 76 and 3 78 respectively in Appendix 3D)

The *conditional* median run length is found from the *conditional* quantile function given in equation (3 104) and derived as Result 3 69 in Appendix 3D by substituting $p = 0\ 5$ that is

$$MDRL = \inf\left\{ j \text{ an integer } \;\; j \geq \frac{\ln(0\ 5)}{\ln \beta_{z\,\chi}} \right\} \qquad (3\ 109)$$

Example 3 6

Conditional run length distribution of the Shewhart $\overline{X}$ control chart of Example 3 5

We have estimated the unknown process mean and the unknown process standard deviation from Example 3 5 to be 23 478 and 2 639 respectively These values resulted in an observed value from the standard normal distribution i e $z = 0\ 177$ and an observed value form the χ^2_{120} distribution i e $y = 114\ 639$

Then substituting $z = 0\ 177$ and $y = 114\ 639$ into equations (3 93) and (3 94) we obtained the conditional probabilities of a signaling event and that of a non signaling event i e

$$1-\beta\left(0\ 0027\ \delta\ 30\ 5\ 0\ 177\ 114\ 639\right)=1-\Phi\left(\frac{0\ 177}{\sqrt{30}}+3\sqrt{\frac{114\ 639}{120}}-\delta\sqrt{5}\right)+\Phi\left(\frac{0\ 177}{\sqrt{30}}-3\sqrt{\frac{114\ 639}{120}}-\delta\sqrt{5}\right)$$

and

$$\beta\left(0\ 0027\ \delta\ 30\ 5\ 0\ 177\ 114\ 639\right)=\Phi\left(\frac{0\ 177}{\sqrt{30}}+3\sqrt{\frac{114\ 639}{120}}-\delta\sqrt{5}\right)-\Phi\left(\frac{0\ 177}{\sqrt{30}}-3\sqrt{\frac{114\ 639}{120}}-\delta\sqrt{5}\right)$$

respectively

Now substituting thesc results in equation (3 102) we obtain the *conditional* probability mass function (pmf) of the run length random variable N in Phase 2 of the control chart procedure of Examplc 3 5 i e

$$P(N=j\,|\,Z=0\ 177\ \chi^2=114\ 639)=\beta\left(0\ 0027\ \delta\ 30\ 5\ 0\ 177\ 114\ 639\right)^{j\ 1}\left(1-\beta\left(0\ 0027\ \delta\ 30\ 5\ 0\ 177\ 114\ 639\right)\right)$$

for $j=1\ 2\ 3$

Assuming that the process stays in control in other words assuming that $\delta=0$ we find the *conditional* probability of a no signal to be 0 996618 and the *conditional* probability of a signal or the *conditional* false alarm rate to be 0 003382 so that the *conditional* **in control** pmf is given by

$$P(N_0=j\,|\,Z=0\ 177\ \chi^2=114\ 639)=0\ 996618^{j\ 1}(0\ 003382)$$

for $j=1\ 2\ 3$

Hence we find the properties of the *conditional* **in control** run length distribution i e the average (ARL_0) the standard deviation ($SDRL_0$) the coefficient of skewness ($SKEWRL_0$) and the median run length ($MDRL_0$) using equations (3 105) (3 107) (3 108) and (3 109) respectively

For instance

$$ARL_0=\frac{1}{0\ 003382}=295\ 68$$

$$SDRL_0=\frac{\sqrt{0\ 996618}}{0\ 003382}=295\ 18$$

$$SKEWRL_0 = \frac{1+0\ 996618}{\sqrt{0\ 996618}} = 2$$

and

$$MDRL_0 = \inf\left\{ j \text{ an integer } \ j \geq \frac{\ln(0\ 5)}{\ln(0\ 996618)} \right\} = 205$$

Thus if the process stays in control in Phase 2 we would expect (on average) a false alarm approximately every 296 samples with 50% of these false alarms within the first 205 samples

On the other hand if we are interested in detecting a shift of size $\delta = 0\ 5$ say when using the estimated control limits from Phase 1 i e $\widehat{UCL} = 27\ 020$ and $\widehat{LCL} = 19\ 937$ it is also worth studying the *conditional* **out of control** run length distribution

For example following a shift of size $\delta = 0\ 5$ that is if the process mean either increase or decrease by 0 5 standard deviation units we find the *conditional* probability of detecting the shift on the first subsequent sample or subgroup to be

$$1-\beta(0\ 0027\ \delta\ 30\ 5\ 0\ 177\ 114\ 639) = 1-\Phi\left(\frac{0\ 177}{\sqrt{30}}+3\sqrt{\frac{114\ 639}{120}}-(0\ 5)\sqrt{5}\right)+\Phi\left(\frac{0\ 177}{\sqrt{30}}-3\sqrt{\frac{114\ 639}{120}}-(0\ 5)\sqrt{5}\right)$$
$$= 0\ 967561$$

with the corresponding *conditional* probability of no signal being

$$\beta(0\ 0027\ \delta\ 30\ 5\ 0\ 177\ 114\ 639) = \Phi\left(\frac{0\ 177}{\sqrt{30}}+3\sqrt{\frac{114\ 639}{120}}-(0\ 5)\sqrt{5}\right)-\Phi\left(\frac{0\ 177}{\sqrt{30}}-3\sqrt{\frac{114\ 639}{120}}-(0\ 5)\sqrt{5}\right)$$
$$= 0\ 032439$$

Hence the *conditional* **out of control** probability mass function (pmf) is

$$P(N_1 = j \mid Z = 0\ 177\ \chi^2 = 114\ 639) = 0\ 967561^{j-1}(0\ 032439)$$

for $j = 1\ 2\ 3$

so that we find the *conditional* **out of control** average run length (ARL_1) to be

$$ARL_1 = \frac{1}{0\ 032439} = 30\ 83$$

whereas the *conditional* **out of control** median run length ($MDRL_1$) is

$$MDRL_0 = \inf\left\{ j \text{ an integer } \; j \geq \frac{\ln(0\ 5)}{\ln(0\ 967561)} \right\} = 21\ 02$$

Thus for an anticipated shift of size $\delta = 0\ 5$ in Phase 2 we would expect to detect the shift within 30 83 samples – *conditional* on having observed $Z = 0\ 177$ and $\chi^2_{120} = 114\ 639$ in Phase 1

3 4 5 The unconditional run length distribution

As in Case UK and Case KU the *conditional* run length distribution as presented in Section 3 4 4 only presents the *conditional* performance of a control chart in Phase 2 given the observed values for the (point) estimators of the unknown process parameters in Phase 1
Therefore we need to derive and study the *unconditional* run length distribution to assess the overall effect of estimation on the control charts properties To do this we have to take into account the variability in both of the estimators $\overline{\overline{X}}$ and $S_{m(\ 1)}$ Here the *conditional* run length distribution (once again) aids the derivation of the *unconditional* run length distribution

For the random variables N Z and χ^2 we have that

$$f_{N|Z\ \chi}(j\,|\,z\ y)=\frac{f_{N\ Z\ \chi}(j\ z\ y)}{f_{Z\ \chi}(z\ y)}$$

or alternatively we have that their joint probability distribution is given by

$$f_{N\ Z\ \chi}(j\ z\ y)=f_{N|Z\ \chi}(j\,|\,z\ y)f_{Z\ \chi}(z\ y)$$

Then the *marginal* or the *unconditional* probability mass function (pmf) of the run length distribution is found by integrating over the domains of both Z and χ^2 i e

$$\begin{aligned}P(N=j)&=\int_{-\infty}^{\infty}\int_{0}^{\infty}f_{N\ Z\ \chi}(j\ z\ y)dydz\\&=\int_{-\infty}^{\infty}\int_{0}^{\infty}f_{N|Z\ \chi}(j\,|\,z\ y)\phi(z)f_{\chi}(y)dydz\\&=\int_{-\infty}^{\infty}\int_{0}^{\infty}\beta^{j\ 1}\left(1-\beta_{Z\ \chi}\right)\phi(z)f_{\chi}(y)dydz\end{aligned}\qquad(3\ 110)$$

whereas the *unconditional* cumulative distribution function (cdf) is given by

$$P(N\le j)=\int_{-\infty}^{\infty}\int_{0}^{\infty}\left(1-\beta_{Z\ \chi}^{j}\right)\phi(z)f_{\chi}(y)dydz\qquad(3\ 111)$$

where

$$\beta_{Z\ \chi}=\Phi\left(\frac{z}{\sqrt{m}}+z_{\frac{\alpha}{2}}\sqrt{\frac{y}{v}}-\delta\sqrt{n}\right)-\Phi\left(\frac{z}{\sqrt{m}}-z_{\frac{\alpha}{2}}\sqrt{\frac{y}{v}}-\delta\sqrt{n}\right)$$

as given in equation (3 93)
(See Result 3 68 in Appendix 3D)

The *unconditional* average run length is found by using the (already) available *conditional* average run length given in equation (3 105) so that we obtain

$$\begin{aligned} ARL &= E(N) \\ &= E_{Z\,\chi}\left(E\left(N \mid Z = z\;\chi^2 = y\right)\right) \\ &= E_{Z\,\chi}\left(\frac{1}{1-\beta_{Z\,\chi}}\right) \\ &= \int_{-\infty}^{\infty}\int_{0}^{\infty}\left(\frac{1}{1-\beta_{Z\,\chi}}\right)\phi(z)f_{\chi}(y)dydz \end{aligned} \tag{3 112}$$

(See Result 3 72 in Appendix 3D)

In addition the *unconditional* quantile function is

$$\begin{aligned} Q_N(p) &= \inf\left\{j \text{ an integer } \; P(N \le j) \ge p\right\} \\ &= \inf\left\{j \text{ an integer } \; \int_{-\infty}^{\infty}\int_{0}^{\infty}\left(1-\beta_{Z\,\chi}^{j}\right)\phi(z)f_{\chi}(y)dydz \ge p\right\} \end{aligned} \tag{3 113}$$

so that by substituting $p = 0\;5$ in equation (3 113) we find the *unconditional* median run length to be

$$MDRL = \inf\left\{j \text{ an integer } \; \int_{-\infty}^{\infty}\int_{0}^{\infty}\left(1-\beta_{Z\,\chi}^{j}\right)\phi(z)f_{\chi}(y)dydz \ge \frac{1}{2}\right\} \tag{3 114}$$

(See Result 3 70 in Appendix 3D)

Finally we can find the variance ($VARRL$) the standard deviation ($SDRL$) and the coefficient of skewness ($SKEWRL$) of the *unconditional* run length distribution by using the *conditional* first second and third non central moments of the run length variable N as given in Results 3 72 and 3 74 of Appendix 3D respectively i e

$$E\left(N \mid Z = z\;\chi^2 = y\right) = \frac{1}{1-\beta_{Z\,\chi}}$$

$$E\left(N^2 \mid Z = z\;\chi^2 = y\right) = \frac{1+\beta_{Z\,\chi}}{\left(1-\beta_{Z\,\chi}\right)^2}$$

and

$$E\left(N^3 \mid Z = z\;\chi^2 = y\right) = \frac{1+\beta_{Z\,\chi}^2+4\beta_{Z\,\chi}}{\left(1-\beta_{Z\,\chi}\right)^3}$$

Hence the *unconditional* variance is given by

$$\operatorname{var}(N) = E_{Z\ \chi}\left(\frac{\beta_{Z\ \chi}}{\left(1-\beta_{Z\ \chi}\right)^2}\right) + E_{Z\ \chi}\left(\left(\frac{1}{1-\beta_{Z\ \chi}}\right)\right) - \left(E_{Z\ \chi}\left(\frac{1}{1-\beta_{Z\ \chi}}\right)\right)^2 \tag{3 115}$$

which can be re written in terms of integrals so that

$$\operatorname{var}(N) = \int_{-\infty}^{\infty}\int_{0}^{\infty} \frac{\beta_{Z\ \chi}}{\left(1-\beta_{Z\ \chi}\right)^2}\phi(z) f_{\chi}\ (y)dydz + \int_{-\infty}^{\infty}\int_{0}^{\infty}\left(\frac{1}{1-\beta_{Z\ \chi}}\right)^2 \phi(z) f_{\chi}\ (y)dydz - \left(\int_{-\infty}^{\infty}\int_{0}^{\infty}\frac{1}{1-\beta_{Z\ \chi}}\phi(z) f_{\chi}\ (y)dydz\right)^2$$

(3 116)

Subsequently we can find the *unconditional* standard deviation from equation (3 116) by taking the square root thereof i e $stdev(N) = \sqrt{\operatorname{var}(N)}$

(See Result 3 77 in Appendix 3D)

Finally to obtain the *unconditional* coefficient of skewness of the run length random variable N which is defined as

$$skew(N) = \frac{E(N - E(N))^3}{\operatorname{var}(N)^{\frac{3}{2}}}$$

and which simplifies to

$$skew(N) = \frac{E\left(N^3\right) - 3E\left(N^2\right)E\left(N\right) + 2E\left(N\right)^3}{\left(\operatorname{var}(N)\right)^{\frac{3}{2}}}$$

one uses the first the second and the third non central moments of the *conditional* run length distribution i e one uses $E\left(N \mid Z = z\ \ \chi^2 = y\right)$ $E\left(N^2 \mid Z = z\ \ \chi^2 = y\right)$ and $E\left(N^3 \mid Z = z\ \ \chi^2 = y\right)$ as well as the fact that $E(N^k) = E_{Z\ \chi}\ (E(N^k \mid Z = z\ \ \chi^2 = y))$ We then obtain

$$skew(N) =$$

$$\frac{\int_{-\infty}^{\infty}\int_{0}^{\infty}\frac{1+\beta \quad +4\beta}{\left(1-\beta \quad \right)}\phi(\ )f\ \ (y)dyd \ -3\left\{\int_{-\infty}^{\infty}\int_{0}^{\infty}\frac{1+\beta}{(1-\beta \quad)}\phi(\ )f\ \ (y)dyd\right\}\left\{\int_{-\infty}^{\infty}\int_{0}^{\infty}\frac{1}{1-\beta}\phi(\ )f\ \ (y)dyd\right\}+2\left(\int_{-\infty}^{\infty}\int_{0}^{\infty}\frac{1}{1-\beta}\phi(\ )f\ \ (y)dyd\right)}{\left(\operatorname{var}(N)\right)}$$

(3 117)

(See Result 3 79 in Appendix 3D)

3 4 6 Performance of the $\overline{X}$ control chart

The evaluation of the properties or the characteristics of the *unconditional* run length distribution of the Shewhart $\overline{X}$ control chart in Case UU is similar to that of Case UK and/or Case KU

First we study the *unconditional* average run length i e

$$ARL = \int_{-\infty}^{\infty}\int_{0}^{\infty}\left(\frac{1}{1-\beta_{z\ \chi}}\right)\phi(z)f_{\chi}\ (y)dydz$$

as given in equation (3 112) and the *unconditional* probability of a signal i e

$$P(\text{Signal}) = \int_{-\infty}^{\infty}\int_{0}^{\infty}\left(1-\Phi\left(\frac{z}{\sqrt{m}}+z_{\frac{\alpha}{2}}\sqrt{\frac{y}{v}}-\delta\sqrt{n}\right)+\Phi\left(\frac{z}{\sqrt{m}}-z_{\frac{\alpha}{2}}\sqrt{\frac{y}{v}}-\delta\sqrt{n}\right)\right)\phi(z)f_{\chi}\ (y)dydz \qquad (3\ 118)$$

derived as Result 3 64 in Appendix 3D

For this reason Table 3 19 displays the unconditional average run length and the unconditional probability of a signal for different values of m i e the final number of reference samples from Phase 1 and different values of δ i e the relative shift in the process mean when using 3 sigma control limits and samples or subgroups of size $n = 5$ are randomly drawn from the process

When studying Table 3 19 we note that the unconditional average run length is not equal to the reciprocal of the unconditional probability of a signal For example when $m = 75$ and $\delta = 0$ that is when we use 75 reference samples to estimate the unknown process parameters and if the process is in control in Phase 2 the unconditional in control average run length $ARL = 378\ 59$ which is not equal to $\left[P(Signal)\right]^{-1} = (0\ 0031)^{-1} = 322\ 58$ Thus the (simple) relationship between the ARL and the probability of a signal as used in Case KK is no longer applicable and should therefore not be used for calculating performance measures when we have estimated any unknown parameters

However as we have previously done we can show that the reciprocal of the unconditional probability of a signal provides a lower bound for the unconditional average run length This is (once again) done by using Jensen s inequality together with expectation by conditioning i e

$$ARL = E(N) = E_{Z\,\chi}\, E\left(N \mid Z = z\ \chi^2 = y\right) = E_{Z\,\chi}\left(\frac{1}{1-\beta_{Z\,\chi}}\right) \geq \frac{1}{1 - E_{Z\,\chi}\left(\beta_{Z\,\chi}\right)} \qquad (3\ 119)$$

Thus when we use 75 reference samples to estimate the unknown process parameters μ and σ the in control average run length will not be less than $\left[P(Signal)\right]^{-1} = (0\ 0031)^{-1} = 322\ 58$ However if the process mean shifts out of control by $\delta = 0\ 50$ standard deviation units say the average run length will not be less than $\left[P(Signal)\right]^{-1} = (0\ 0319)^{-1} = 31\ 35$

ARL $P(Signal)$	δ								
m	0 00	0 25	0 50	0 75	1 00	1 25	1 50	1 75	2 00
5	1145 42 0 0127	689 28 0 0232	223 29 0 0609	38 06 0 1405	9 66 0 2715	3 66 0 4445	1 96 0 6290	1 37 0 7873	1 14 0 8971
10	531 87 0 0067	283 20 0 0144	69 34 0 0452	17 50 0 1179	6 06 0 2492	2 84 0 4330	1 72 0 6330	1 29 0 8014	1 10 0 9121
15	452 13 0 0051	220 18 0 0119	52 59 0 0401	14 52 0 1100	5 42 0 2408	2 66 0 4287	1 67 0 6343	1 26 0 8069	1 09 0 9177
20	422 03 0 0044	194 46 0 0107	46 41 0 0375	13 38 0 1059	5 14 0 2367	2 59 0 4262	1 64 0 6352	1 25 0 8099	1 09 0 9206
25	407 87 0 0041	180 35 0 0100	43 22 0 0360	12 77 0 1032	5 00 0 2339	2 54 0 4248	1 62 0 6358	1 24 0 8116	1 09 0 9222
30	398 83 0 0038	171 68 0 0096	41 28 0 0349	12 37 0 1017	4 91 0 2320	2 51 0 4239	1 61 0 6363	1 24 0 8129	1 08 0 9234
50	384 41 0 0033	155 20 0 0087	37 80 0 0330	11 67 0 0982	4 73 0 2283	2 46 0 4219	1 59 0 6369	1 23 0 8154	1 08 0 9259
75	378 59 0 0031	147 56 0 0083	36 26 0 0319	11 36 0 0964	4 65 0 2264	2 44 0 4209	1 58 0 6374	1 23 0 8168	1 08 0 9270
100	376 07 0 0030	143 85 0 0081	35 50 0 0314	11 20 0 0956	4 61 0 2254	2 42 0 4204	1 58 0 6376	1 23 0 8174	1 08 0 9277
200	372 72 0 0029	138 41 0 0078	34 44 0 0307	10 98 0 0942	4 55 0 2240	2 41 0 4195	1 57 0 6380	1 22 0 8184	1 08 0 9286
300	371 76 0 0028	136 58 0 0077	34 08 0 0304	10 91 0 0938	4 53 0 2235	2 40 0 4193	1 57 0 6382	1 22 0 8187	1 08 0 9289
500	371 26 0 0028	135 26 0 0076	33 81 0 0302	10 84 0 0935	4 52 0 2230	2 39 0 4191	1 57 0 6382	1 22 0 8190	1 08 0 9291
1000	370 81 0 0027	134 20 0 0076	33 60 0 0301	10 80 0 0932	4 51 0 2228	2 39 0 4190	1 57 0 6383	1 22 0 8192	1 08 0 9293
∞	370 37 0 0027	133 15 0 0075	33 40 0 0299	10 76 0 0929	4 50 0 2225	2 39 0 4188	1 57 0 6384	1 22 0 8194	1 08 0 9295

Table 3 19 The unconditional average run length and the unconditional probability of a signal when $\alpha = 0\ 0027$ and $n = 5$

If we assume that we have a total of $m = 25$ (final) reference samples to estimate the unknown process parameters we see from Table 3 19 that (on average) the in control average run length $ARL_0 = 407\ 87$ and the unconditional false alarm rate $FAR = 0\ 0041$ These values are approximately 10% and 52% *higher* than their nominal values of 370 37 and 0 0027 in Case KK respectively Thus the standard practice of assuming that having 25 reference samples will guarantee that the $\overline{X}$ control chart procedure in Case UU will perform like the $\overline{X}$ control chart procedure in Case KK is (again) incorrect

In fact only when we have about $m = 200$ reference samples i e a total of $200 \times 5 = 1000$ individual observations are the properties of the $\overline{X}$ control chart procedure in Case UU nearly equal to the properties of the $\overline{X}$ control chart procedure in Case KK For instance we find that the in control average run length $ARL_0 = 372\ 72$ which is merely 0 63% *higher* than the nominal value of 370 37 and the probability of a false alarm $P(Signal) = 0\ 0029$ which is only 7% *higher* than the nominal value of $\alpha = 0\ 0027$

On the contrary we (still) see that even when $m = 200$ $ARL = 372\ 72 \neq (0\ 0029)^{1} = 344\ 83$ and only when $m \to \infty$ in other words only if the number of reference samples tends to infinity i e 1000 say do we observe that $ARL = 370\ 81 \approx (0\ 0027)^{1} = 370\ 37$

Another point taken note of is the unconditional out of control performance of the control chart when having only $m = 25$ (say) reference samples to estimate μ and σ For example if $m = 25$ and $\delta = 0\ 50$ the out of control average run length $ARL_1 = 43\ 22$ which is 29 4% *higher* than the nominal value of 33 40 in Case KK and this is in spite of the fact that the probability of a signal $P(Signal) = 0\ 0360$ which is 20 4% *higher* than its nominal value of 0 0299 Thus although (on average) the probability of detecting a shift at any point in time increased which seems beneficial the unconditional average run length is also larger In other words on average the shift will not be detected as fast as would be the situation if both the process parameters are known

Last but not least we observe from Table 3 19 that as $m \to \infty$ the unconditional probability of a false alarm (FAR) which is equation (3 118) with $\delta = 0$ converges to the nominal value of 0 0027 in Case KK This confirms our earlier result established in equation (3 101) i e

$$\lim_{\to\infty} P\left(|T| \geq z_{\frac{\alpha}{2}}\sqrt{\frac{m}{m+1}}\right) = \alpha$$

If we study the values of the unconditional standard deviation of the run length *SDRL* and the values of the unconditional coefficient of skewness *SKEWRL* as presented in Table 3 20 we note that the seemingly better in control performance of the control chart procedure in Case UU i e the large in control average run length is not due to an increase in performance Instead it is due to an increase in the standard deviation and the skewness of the run length distribution

For example when $m = 30$ and $\delta = 0$ we find from Table 3 19 that the in control average run length $ARL_0 = 398\ 83$ which is *larger* its nominal value of 370 37 However the corresponding in control standard deviation and the in control coefficient of skewness of the run length distribution found form Table 3 20 is equally high i e $SDRL_0 = 595\ 50$ and $SKEWRL_0 = 5\ 89$ To understand the implication of this we study Figure 3 22a and Figure 3 22b

SDRL *SKEWRL*	δ								
m	0 00	0 25	0 50	0 75	1 00	1 25	1 50	1 75	2 00
5	45407 24	12838 11	45782 27	899 43	71 85	12 15	3 05	1 18	0 52
	1200 43	309 69	2064 37	998 71	303 62	167 70	54 11	63 89	11 38
10	2051 56	1188 90	397 52	49 08	10 62	3 48	1 45	0 71	0 37
	85 90	130 87	787 04	88 21	16 33	9 86	5 67	4 41	4 82
15	1023 01	555 96	125 02	25 19	7 20	2 70	1 23	0 63	0 34
	17 26	25 98	39 48	12 67	6 65	4 57	3 69	3 62	4 39
20	775 66	387 54	84 14	19 54	6 08	2 42	1 14	0 60	0 32
	10 28	12 10	12 42	6 93	4 64	3 60	3 17	3 34	4 26
25	662 47	314 85	68 67	16 97	5 54	2 27	1 10	0 58	0 32
	7 35	8 38	8 30	5 32	3 89	3 15	2 94	3 19	4 19
30	595 50	275 36	59 99	15 47	5 22	2 19	1 07	0 57	0 31
	5 89	6 89	5 93	4 35	3 42	2 93	2 80	3 11	4 15
50	489 14	208 07	46 85	13 03	4 65	2 03	1 02	0 55	0 30
	3 88	4 45	3 81	3 14	2 73	2 53	2 56	2 96	4 09
75	444 47	179 61	41 61	12 02	4 41	1 95	0 99	0 54	0 30
	3 14	3 45	3 09	2 70	2 46	2 36	2 45	2 89	4 07
100	424 02	166 83	39 19	11 53	4 29	1 92	0 98	0 53	0 29
	2 82	3 04	2 77	2 50	2 34	2 28	2 40	2 86	4 06
200	395 58	148 99	35 90	10 87	4 12	1 87	0 96	0 53	0 29
	2 39	2 50	2 36	2 24	2 17	2 17	2 33	2 82	4 04
300	386 59	143 29	34 86	10 66	4 07	1 85	0 95	0 52	0 29
	2 25	2 32	2 23	2 16	2 12	2 14	2 31	2 81	4 04
500	379 87	139 03	34 06	10 49	4 03	1 84	0 95	0 52	0 29
	2 15	2 19	2 14	2 09	2 08	2 11	2 29	2 79	4 04
1000	374 85	135 80	33 47	10 37	3 99	1 83	0 95	0 52	0 29
	2 07	2 10	2 07	2 05	2 05	2 09	2 28	2 79	4 03
∞	369 87	132 65	32 90	10 25	3 96	1 82	0 94	0 52	0 27
	2 00	2 00	2 00	2 00	2 02	2 07	2 26	2 78	4 03

Table 3 20 The unconditional SDRL and the unconditional SKEWRL for Case UU when $\alpha = 0\ 0027$ **and** $n = 5$

Together Figure 3 22a and Figure 3 22b display the probability mass fuction (pmf) of the run length variable N of Case KK i e

$$P(N=j)=\beta(\alpha\ \delta\ n)^{j\ 1}(1-\beta(\alpha\ \delta\ n)) \text{ for } j=1\ 2\ 3$$

as well as the *unconditional* probability mass function (pmf) of the run length variable N of Case UU i e

$$P(N=j)=\int_{-\infty}^{\infty}\int_{0}^{\infty}\beta^{j\ 1}\left(1-\beta_{Z\ \chi}\right)\phi(z)f_{\chi}(y)dydz \text{ for } j=1\ 2\ 3$$

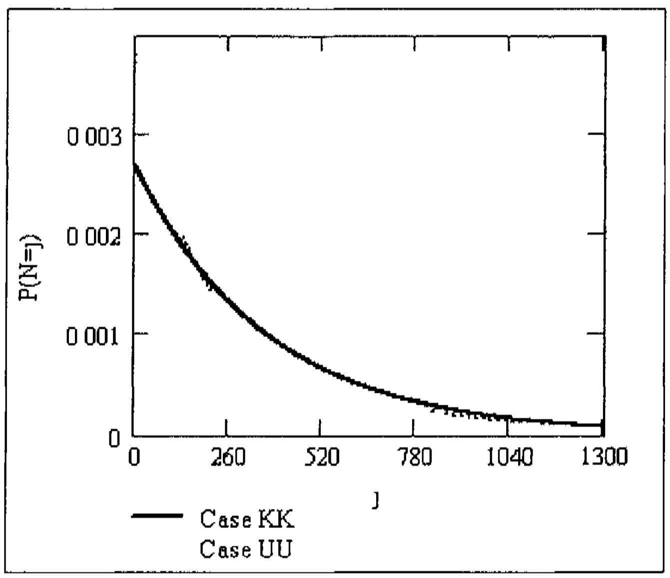

Figure 3 22a Probability mass function of the run length random variables in Case KK and Case UU[1]

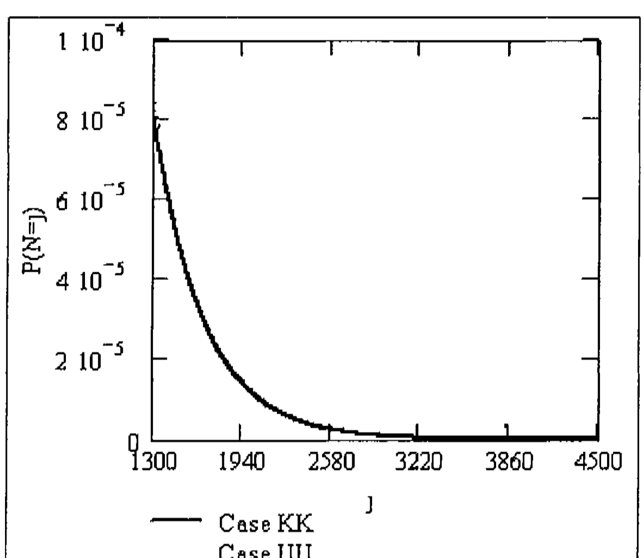

Figure 3 22b Probability mass function of the run length random variables in Case KK and Case UU[1]

[1] Note that although the probability mass functions are those of discrete random variables due to the scaling of the axis they seem continuous in Figures 3 22a and 3 22b

Figure 3 22a displays the probability mass functions for values of $j \in (1\ 2\quad 1300)$ whereas Figure 3 22b displays the probability mass functions for values of $j \in (1300\ 1301\quad)$

Studying Figure 3 22a we observe that for values of j less than 250 (say) the probability mass function of Case UU is above the probability mass function of Case KK Thus in Case UU shorter runs are more likely than in Case KK In addition studying Figure 3 22b reveals that for values of j larger than 1300 say the probability mass function of Case UU is again above the probability mass function of Case KK Thus similarly the likelihood of very long runs in Case UU is larger than that of Case KK Therefore the large in control average run length ARL_0 in Case UU is a result of the skewness of the run length distribution and not due to increased or better performance and the fact that extremely long runs can occur if the process is truly in control is (simply) a desirable side effect of the estimation procedure in Phase 1

3 5 1 Summary of formulae

	Case KK	Case UK
Lower Control Limit (LCL)	$LCL = \mu - z \frac{\sigma}{\sqrt{n}}$	$\widehat{LCL} = \overline{\overline{X}} - \frac{\sigma}{\sqrt{n}}$
Upper Control Limit (UCL)	$UCL = \mu + z \frac{\sigma}{\sqrt{n}}$	$\widehat{UCL} = \overline{\overline{X}} + z \frac{\sigma}{\sqrt{n}}$
Non Signalling event	$\frac{\sqrt{n}\lvert\overline{X} - \mu\rvert}{\sigma} < z$	$\frac{\sqrt{n}\lvert\overline{X} - \overline{\overline{X}}\rvert}{\sigma} < z$
Signalling event	$\frac{\sqrt{n}\lvert\overline{X} - \mu\rvert}{\sigma} \geq z$	$\frac{\sqrt{n}\lvert\overline{X} - \overline{\overline{X}}\rvert}{\sigma} \geq z$
Probability of no signal		
Conditional		$\beta = \Phi\left(\frac{}{\sqrt{m}} \quad \frac{\alpha}{2} \quad \delta\sqrt{\ }\right) - \Phi\left(\frac{}{\sqrt{m}} - \frac{\alpha}{2} - \delta\sqrt{\ }\right)$
Unconditional	$\beta = \Phi\left(\_ - \delta\sqrt{n}\right) - \Phi\left(- \_ - \delta\sqrt{n}\right)$	$\int_{-\infty}^{\infty}\left(\Phi\left(\frac{}{\sqrt{m}} + z_{\frac{\alpha}{2}} - \delta\sqrt{n}\right) - \Phi\left(\frac{}{\sqrt{m}} - z_{\frac{\alpha}{2}} - \delta\sqrt{n}\right)\right)\phi(z)d$ $\Phi\left(\sqrt{\frac{m}{m+1}}\left(\frac{\alpha}{2} - \delta\sqrt{\ }\right)\right) - \Phi\left(\sqrt{\frac{m}{m \quad 1}}\left(\frac{\alpha}{2} - \delta\sqrt{\ }\right)\right)$
Probability of a signal		
Conditional		$1 \quad \Phi\left(\frac{}{\sqrt{m}} + \frac{\alpha}{2} - \delta\sqrt{\ }\right) \quad \Phi\left(\frac{}{\sqrt{m}} \quad \frac{\alpha}{2} \quad \delta\sqrt{\ }\right)$
Unconditional	$1 - \Phi\left(z_{\_} - \delta\sqrt{n}\right) + \Phi\left(-z_{\_} - \delta\sqrt{n}\right)$	$\int_{-\infty}^{\infty}\left(1 - \Phi\left(\frac{z}{\sqrt{m}} + z_{\frac{\alpha}{2}} - \delta\sqrt{n}\right) + \Phi\left(\frac{z}{\sqrt{m}} - z_{\frac{\alpha}{2}} - \delta\sqrt{n}\right)\right)\phi(z)dz$ $1 \quad \Phi\left(\sqrt{\frac{m}{m \quad 1}}\left(\frac{\alpha}{2} \quad \delta\sqrt{\ }\right)\right) + \Phi\left(\sqrt{\frac{m}{m+1}}\left(- \frac{\alpha}{2} \quad \delta\sqrt{\ }\right)\right)$

Probability of a false alarm		
Conditional		$1-\Phi\left(\frac{}{\sqrt{m}}+z\right)+\Phi\left(\frac{z}{\sqrt{m}}-\right)$
Unconditional	$1-\Phi\left(z\right)+\Phi\left(-z\right)=P\left(\lvert Z\rvert\geq\right)=\alpha$	$\int^{\infty}\left(1-\Phi\left(\frac{}{\sqrt{m}}+\right)+\Phi\left(\frac{}{\sqrt{m}}-\right)\right)\phi(\ )d$ $1-\Phi\left(\sqrt{\frac{m}{m+1}}z\right)+\Phi\left(-\sqrt{\frac{m}{m+1}}z\right)=P\left(\lvert Z\rvert\geq\sqrt{\frac{m}{m+1}}z\right)$
Run length distribution	$N\sim GEO(1-\beta)$ for $j=1\ 2\ 3$	$N\mid Z=\ \sim GEO(1-\beta\ )$ for $j=1\ 2\ 3$
Probability mass function (pmf)		
Conditional		$P(N=j\mid Z=\ )=\beta\ \ (1-\beta\ )$
Unconditional	$P(N=j)=\beta\ \ (1-\beta)$	$P(N=j)=\int_{-\infty}^{\infty}\beta\ \ (1-\beta\ )\phi(z)dz$
Cumulative distribution function (cdf)		
Conditional		$P(N\leq j\mid Z=\ )=\sum\beta\ \ (1-\beta\ )=1-\beta$
Unconditional	$P(N\leq j)=\sum\beta\ \ (1-\beta)=1-\beta$	$P(N\leq j)=\sum P(N=k)=\sum\left(\int_{-\infty}^{\infty}\beta\ \ (1-\beta\ )\phi(\ )d\ \right)$
Quantile function		
Conditional		$Q_{\mid}(p)=\inf\left[j\ \ P(N\leq j\mid Z=\ )\geq p\right]=\inf\left[j\ \ j\geq\frac{\ln(1-p)}{\ln\beta}\right]$
Unconditional	$Q\ (p)=\inf\left[j\ \ P(N\leq j)\geq p\right]=\inf\left[j\ \ j\geq\frac{\ln(1-p)}{\ln\beta}\right]$	$Q\ (p)=\inf\left[j\ \ P(N\leq j)\geq p\right]=\inf\left[j\ \ \sum\left(\int_{-\infty}^{\infty}\beta\ \ (1-\beta\ )\phi(\ )d\ \right)\geq p\right]$
Moment Generating Function		
Conditional		$M_{N\mid}\ (t)=\frac{(1-\beta\ )e}{1-\beta\ e}$
Unconditional	$M_{N}(t)=\frac{(1-\beta)e}{1-\beta e}$	
First non central Moment (Average Run Length / ARL)		
Conditional		$E(N\mid Z=z)=\frac{1}{1-\beta}$
Unconditional	$E(N)=\frac{1}{1-\beta}$	$E(N)=\int_{-\infty}^{\infty}\frac{1}{1-\beta}\phi(\ )d$

Second non central Moment		
Conditional		$E(N \ \vert Z=z)=\frac{1+\beta}{(1-\beta\ )}$
Unconditional	$E(N\ )=\frac{1+\beta}{(1-\beta)}$	$E(N\ )=E\ (E(N\ \vert Z))=\int_{-\infty}^{\infty}\frac{1+\beta}{(1-\beta\ )}\phi(\ )dz$
Third non central Moment		
Conditional		$E(N\ \vert Z=\ )=\frac{1+\beta\ +4\beta}{(1-\beta\ )}$
Unconditional	$E(N\ )=\frac{1+\beta\ +4\beta}{(1-\beta)}$	$E(N\ )=E\ (E(N\ \vert Z))=\int_{-\infty}^{\infty}\frac{1+\beta\ +4\beta}{(1-\beta\ )}\phi(\ )d$
Variance (VARRL)		
Conditional		$\text{var}(N\vert Z=z)=\frac{\beta}{(1-\beta\ )}$
Unconditional	$\text{var}(N)=\frac{\beta}{(1-\beta)}$	$\text{var}(N)=E\ \left(\text{var}(N\vert Z)\right)+\text{var}\ \left(E(N\vert Z)\right)$ $=\int_{-\infty}^{\infty}\frac{\beta}{(1-\beta\ )}\phi(\ )dz+\int_{-\infty}^{\infty}\left(\frac{1}{1-\beta}\right)\ \phi(z)dz-\left(\int_{-\infty}^{\infty}\frac{1}{1-\beta}\phi(\ )dz\right)$ $\text{var}(N)=E\ \left(E\left(N\ \ \vert Z\right)\right)-E\ \left(E\left(N\vert Z\right)\right)$ $=\int_{-\infty}^{\infty}\frac{1+\beta}{(1-\beta\ )}\phi(z)dz-\left(\int_{-\infty}^{\infty}\frac{1}{1-\beta}\phi(z)dz\right)$
Standard Deviation (SDRL)		
Conditional		$tde\ \ (N\vert Z=\ \ )=\frac{\sqrt{\beta}}{1-\beta}$
Unconditional	$stdev(N)=\frac{\sqrt{\beta}}{1-\beta}$	$stdev(N)=\sqrt{\text{var}(N)}$
Coefficient of Skewness (SKEWRL)		
Conditional		$skew(N\vert Z=\ \ )=\frac{1+\beta}{\sqrt{\beta}}$
Unconditional	$skew(N)=\frac{1+\beta}{\sqrt{\beta}}$	$skew(N)=\frac{E\left(N\ \right)-3E\left(N\ \right)E\left(N\right)+2E\left(N\right)}{\left(\text{var}(N)\right)}$

T bl I S mm ry f f m I f C KK d C UK

	Case KU	Case UU
Lower Control Limit (LCL)	$\widehat{LCL} = \mu - z \dfrac{S}{\sqrt{n}}$	$\widehat{LCL} = \overline{\overline{X}} - z \dfrac{S}{\sqrt{n}}$
Upper Control Limit (UCL)	$\widehat{UCL} = \mu + z \dfrac{S}{\sqrt{n}}$	$\widehat{UCL} = \overline{\overline{X}} + \dfrac{S}{\sqrt{n}}$
Non Signalling event	$\dfrac{\sqrt{n}\lvert\overline{X} - \mu\rvert}{S} < z$	$\dfrac{\sqrt{n}\lvert\overline{X} - \overline{\overline{X}}\rvert}{S} <$
Signalling event	$\dfrac{\sqrt{n}\lvert\overline{X} - \mu\rvert}{S} \geq$	$\dfrac{\sqrt{n}\lvert\overline{X} - \overline{\overline{X}}\rvert}{S} \geq$
Probability of no signal		
Conditional	$\beta = \Phi\left(z \sqrt{\frac{y}{\nu}} - \delta\sqrt{n}\right) - \Phi\left(-z \sqrt{\frac{y}{\nu}} - \delta\sqrt{n}\right)$	$\beta = \Phi\left(\frac{z}{\sqrt{m}} + \sqrt{\frac{y}{\nu}} - \delta\sqrt{n}\right) - \Phi\left(\frac{}{\sqrt{m}} - z \sqrt{\frac{y}{\nu}} - \delta\sqrt{n}\right)$ $\Phi\left(\sqrt{\frac{m}{m+1}}\left(z \sqrt{\frac{y}{\nu}} - \delta\sqrt{n}\right)\right) - \Phi\left(\sqrt{\frac{m}{m+1}}\left(-z \sqrt{\frac{y}{\nu}} - \delta\sqrt{n}\right)\right)$
Unconditional	$\int_0^\infty \left(\Phi\left(z \sqrt{\frac{y}{\nu}} - \delta\sqrt{n}\right) - \Phi\left(-z \sqrt{\frac{y}{\nu}} - \delta\sqrt{n}\right)\right) f \ (y)dy$	$\int_{-\infty}^{\infty} \int_0^\infty \left(\Phi\left(\frac{z}{\sqrt{m}} + z \sqrt{\frac{y}{\nu}} - \delta\sqrt{n}\right) - \Phi\left(\frac{z}{\sqrt{m}} - z \sqrt{\frac{y}{\nu}} - \delta\sqrt{n}\right)\right) \phi(z) f \ (y)dydz$ $\int_0^\infty \left\{\Phi\left(\sqrt{\frac{m}{m+1}}\left(z \sqrt{\frac{y}{\nu}} - \delta\sqrt{n}\right)\right) - \Phi\left(\sqrt{\frac{m}{m+1}}\left(-z \sqrt{\frac{y}{\nu}} - \delta\sqrt{n}\right)\right)\right\} f \ (y)dy$
Probability of a signal		
Conditional	$1 - \Phi\left(z \sqrt{\frac{y}{\nu}} - \delta\sqrt{n}\right) + \Phi\left(-z \sqrt{\frac{y}{\nu}} - \delta\sqrt{n}\right)$	$1 - \Phi\left(\frac{z}{\sqrt{m}} + \sqrt{\frac{y}{\nu}} - \delta\sqrt{n}\right) + \Phi\left(\frac{z}{\sqrt{m}} - z \sqrt{\frac{y}{\nu}} - \delta\sqrt{n}\right)$ $1 - \Phi\left(\sqrt{\frac{m}{m+1}}\left(z \sqrt{\frac{y}{\nu}} - \delta\sqrt{n}\right)\right) + \Phi\left(\sqrt{\frac{m}{m+1}}\left(-z \sqrt{\frac{y}{\nu}} - \delta\sqrt{n}\right)\right)$

Uncond t onal	$\int_0^\infty \left[1-\Phi\left(\sqrt{\frac{y}{\nu}}-\delta\sqrt{n}\right)+\Phi\left(-\sqrt{\frac{y}{\nu}}-\delta\sqrt{n}\right)\right] f\ (y)dy$	$\int_{-\infty}^{\infty}\int_0^\infty \left[1-\Phi\left(\frac{z}{\sqrt{m}}+z\sqrt{\frac{y}{\nu}}-\delta\sqrt{n}\right)+\Phi\left(\frac{z}{\sqrt{m}}-z\sqrt{\frac{y}{\nu}}-\delta\sqrt{n}\right)\right]\phi(z)f\ (y)dydz$ $\int_0^\infty\left\{1-\Phi\left(\sqrt{\frac{m}{m+1}}\left(z\sqrt{\frac{y}{\nu}}-\delta\sqrt{n}\right)\right)+\Phi\left(\sqrt{\frac{m}{m+1}}\left(-z\sqrt{\frac{y}{\nu}}-\delta\sqrt{n}\right)\right)\right\}f\ (y)dy$
Probability of a false alarm		
Conditional	$1-\Phi\left(z\sqrt{\frac{y}{\nu}}\right)+\Phi\left(-\sqrt{\frac{y}{\nu}}\right)$	$1-\Phi\left(\frac{}{\sqrt{m}}+z\sqrt{\frac{y}{\nu}}\right)+\Phi\left(\frac{}{\sqrt{m}}-\sqrt{\frac{y}{\nu}}\right)$ $1-\Phi\left(\sqrt{\frac{m}{m+1}}\left(\sqrt{\frac{y}{\nu}}\right)\right)+\Phi\left(\sqrt{\frac{m}{m+1}}\left(-\sqrt{\frac{y}{\nu}}\right)\right)$
Unconditional	$1-T(\ )+T(-z)=P(\lvert T\rvert\geq z)$ $\int_0^\infty\left[1-\Phi\left(\sqrt{\frac{y}{\nu}}\right)+\Phi\left(-z\sqrt{\frac{y}{\nu}}\right)\right]f\ (y)dy$	$\int_{-\infty}^\infty\int_0^\infty\left[1-\Phi\left(\frac{}{\sqrt{m}}+\sqrt{\frac{y}{\nu}}\right)+\Phi\left(\frac{}{\sqrt{m}}-\sqrt{\frac{y}{\nu}}\right)\right]\phi(\ )f\ (y)dydz$ $\int_0^\infty\left\{1-\Phi\left(\sqrt{\frac{m}{m+1}}\left(-\sqrt{\frac{y}{\nu}}\right)\right)+\Phi\left(\sqrt{\frac{m}{m+1}}\left(-\sqrt{\frac{y}{\nu}}\right)\right)\right\}f\ (y)dy$ $1-T\left(\sqrt{\frac{m}{m+1}}\right)+T\left(-\sqrt{\frac{m}{m+1}}\right)=P\left(\lvert T\rvert\geq\sqrt{\frac{m}{m+1}}\right)$
Run length distribution	$N\mid\chi=y\sim GEO(1-\beta)$ for $j=1\ 2\ 3$	$N\mid Z=z\ \chi=y\sim GEO(1-\beta)$ for $j=1\ 2\ 3$
Probability mass function (pmf)		
Conditional	$P(N=j\mid\chi=y)=\beta\ (1-\beta\ )$	$P(N=j\mid Z=z\ \chi=y)=\beta\ (1-\beta\ )$
Unconditional	$P(N=j)=\int_0^\infty\beta\ (1-\beta\ )f\ (y)dy$	$P(N=j)=\int^\infty\int_0^\infty\beta\ (1-\beta\ )\phi(\ )f\ (y)dyd$
Cumulative distribution function (cdf)		
Conditional	$P(N\leq j\mid\chi=y)=\sum\beta\ (1-\beta\ )=1-\beta$	$P(N\leq j\mid Z=\ \chi=y)=\sum\beta\ (1-\beta\ )=1-\beta$
Unconditional	$P(N\leq j)=\sum P(N=k)=\sum\left(\int_0^\infty\beta\ (1-\beta\ )f\ (y)dy\right)$	$P(N\leq j)=\sum P(N=k)=\sum\left(\int_{-\infty}^\infty\int_0^\infty\beta\ (1-\beta\ )\phi(\ )f\ (y)dyd\right)$

Quantile function		
Conditional	$Q_{\mid}\ (p)=\inf\left[j\ \ P(N\leq j\mid \chi\ =y)\geq p\right]=\inf\left[j\ \ j\geq\frac{\ln(1-p)}{\ln\beta}\right]$	$Q_{\mid}\ \ (p)=\inf\left[j\ \ P(N\leq j\mid Z=\ \ \chi\ =y)\geq p\right]=\inf\left[j\ \ j\geq\frac{\ln(1-p)}{\ln\beta}\right]$
Unconditional	$Q_{N}(p)=\inf\left[j\ \ P(N\leq j)\geq p\right]=\inf\left[j\ \ \sum\left(\int^{\infty}\beta\ \ \ (1-\beta\ \ )f\ \ (y)dy\right)\geq p\right]$	$Q_{N}(p)=\inf\left[j\ \ P(N\leq j)\geq p\right]=\inf\left[j\ \ \sum\left(\int_{-\infty}\int_{0}^{\infty}\beta\ \ \ \ (1-\beta\ \ \ )\phi(z)f\ \ (y)dyd\ \right)\geq p\right]$
Moment Generating Function		
Conditional	$M_{N\mid}\ \ (t)=\frac{(1-\beta\ \ )e}{1-\beta\ \ e}$	$M_{\mid}\ \ \ (t)=\frac{(1-\beta\ \ \ )e}{1-\beta\ \ \ e}$
Unconditional		
First non cent al Moment *(Average Run Length / ARL)*		
Conditional	$E(N\mid\chi\ \ =y)=\frac{1}{1-\beta}$	$E(N\mid Z=z\ \ \chi\ \ =y)=\frac{1}{1-\beta}$
Unconditional	$E(N)=\int_{0}^{\infty}\frac{1}{1-\beta}f\ \ (y)dy$	$E(N)=\int^{\infty}\int^{\infty}\frac{1}{1-\beta}\phi(\ )f\ \ (y)dydz$
Second non central Moment		
Conditional	$E(N\ \ \mid\chi\ \ =y)=\frac{1+\beta}{(1-\beta\ \ )}$	$E(N\ \ \mid Z=z\ \ \chi\ \ =y)=\frac{1+\beta}{(1-\beta\ \ \ )}$
Unconditional	$E(N\ \ )=E\ \ \ (E(N\ \ \mid\chi\ \ ))=\int_{0}^{\infty}\frac{1+\beta}{(1-\beta\ \ )}f\ \ (y)dy$	$E(N\ \ )=E\ \ \ \ (E(N\ \ \mid Z\ \ \chi\ \ ))=\int_{-\infty}^{\infty}\int_{0}^{\infty}\frac{1+\beta}{(1-\beta\ \ \ )}\phi(\ )f\ \ (y)dydz$
Third non central Moment		
Conditional	$E(N\ \ \mid\chi\ \ =y)=\frac{1+\beta\ \ \ +4\beta}{(1-\beta\ \ )}$	$E(N\ \ \mid Z=\ \ \ \chi\ \ =y)=\frac{1+\beta\ \ \ \ +4\beta}{(1-\beta\ \ \ )}$
Unconditional	$E(N\ \ )=E\ \ \ (E(N\ \ \mid\chi\ \ ))=\int_{0}^{\infty}\frac{1+\beta\ \ \ +4\beta}{(1-\beta\ \ )}f\ \ (y)dy$	$E(N\ \ )=E\ \ \ \ (E(N\ \ \mid Z\ \ \chi\ \ ))=\int^{\infty}\int^{\infty}\frac{1+\beta\ \ \ \ +4\beta}{(1-\beta\ \ \ )}\phi(z)f\ \ (y)dydz$
Variance (VARRL)		
Conditional	$\mathrm{var}(N\mid\chi\ \ =y)=\frac{\beta}{(1-\beta\ \ )}$	$\mathrm{var}(N\mid Z=\ \ \ \chi\ \ =y)=\frac{\beta}{(1-\beta\ \ \ )}$

Unconditional	$\operatorname{var}(N) = E\left(\operatorname{var}(N \mid \chi)\right) + \operatorname{var}\left(E(N \mid \chi)\right)$ $= \int_0^\infty \frac{\beta}{(1-\beta)^2} f(y)dy + \int_0^\infty \left(\frac{1}{1-\beta}\right)^2 f(y)dy - \left(\int_0^\infty \frac{1}{1-\beta} f(y)dy\right)^2$ $\operatorname{var}(N) = E\left(E(N^2 \mid \chi)\right) - E\left(E(N \mid \chi)\right)^2$ $= \int_0^\infty \frac{1+\beta}{(1-\beta)^2} f(y)dy - \left(\int_0^\infty \frac{1}{1-\beta} f(y)dy\right)^2$	$\operatorname{var}(N) = E\left(\operatorname{var}(N \mid Z, \chi)\right) + \operatorname{var}\left(E(N \mid Z, \chi)\right)$ $= \int_{-\infty}^\infty \int_0^\infty \frac{\beta}{(1-\beta)^2} \phi(z) f(y)dydz + \int_{-\infty}^\infty \int_0^\infty \left(\frac{1}{1-\beta}\right)^2 \phi(z) f(y)dydz - \left(\int_{-\infty}^\infty \int_0^\infty \frac{1}{1-\beta} \phi(z) f(y$ $\operatorname{var}(N) = E\left(E(N^2 \mid Z, \chi)\right) - E\left(E(N \mid Z, \chi)\right)^2$ $= \int_{-\infty}^\infty \int_0^\infty \frac{1+\beta}{(1-\beta)^2} \phi(z) f(y)dydz - \left(\int_{-\infty}^\infty \int_0^\infty \frac{1}{1-\beta} \phi(z) f(y)dydz\right)^2$
Standard Deviation (SDRL)		
Conditional	$stdev(N \mid \chi = y) = \frac{\sqrt{\beta}}{1-\beta}$	$stdev(N \mid Z = z, \chi = y) = \frac{\sqrt{\beta}}{1-\beta}$
Unconditional	$stdev(N) = \sqrt{\operatorname{var}(N)}$	$stdev(N) = \sqrt{\operatorname{var}(N)}$
Coefficient of Skewness (SKEWRL)		
Conditional	$skew(N \mid \chi = y) = \frac{1+\beta}{\sqrt{\beta}}$	$skew(N \mid Z = z, \chi = y) = \frac{1+\beta}{\sqrt{\beta}}$
Unconditional	$skew(N) = \frac{E(N^3) - 3E(N^2)E(N) + 2E^3(N)}{(\operatorname{var}(N))^{3/2}}$	$skew(N) = \frac{E(N^3) - 3E(N^2)E(N) + 2E^3(N)}{(\operatorname{var}(N))^{3/2}}$

T bl 2 S mm ry f f m l f C KU d C UU

3 5 2 Appendix 3A

Case KK

Assume that the in control process mean and the in control process standard deviation of a process following a normal distribution are *known* constants denoted by μ_0 and σ_0 respectively Consequently the formulae that are derived are applicable in both Phase 1 and Phase 2 since no estimation of unknown process parameters occurs

Result 3 1 Control limits

$$LCL = \mu_0 - z_{\frac{\alpha}{2}} \frac{\sigma_0}{\sqrt{n}} \quad \text{and} \quad UCL = \mu_0 + z_{\frac{\alpha}{2}} \frac{\sigma_0}{\sqrt{n}}$$

Proof

From equation (2 1) we have that the k sigma control limits for a Shewhart type of control chart are defined as

$$LCL = \mu_T - k\sigma_T \quad \text{and} \quad UCL = \mu_T + k\sigma_T$$

Then since $T = \overline{X}$ and $\overline{X} \sim N\left(\mu_0 \ \frac{\sigma_0^2}{n}\right)$ we have that

$$\mu_T = \mu_{\overline{X}} = \mu_0 \quad \text{and} \quad \sigma_T = \sigma_{\overline{X}} = \frac{\sigma_0}{\sqrt{n}}$$

Consequently substituting the mean and the standard deviation of $\overline{X}$ in equation (2 1) we obtain equation (2 2) i e

$$LCL = \mu_0 - k\frac{\sigma_0}{\sqrt{n}} \quad \text{and} \quad UCL = \mu_0 + k\frac{\sigma_0}{\sqrt{n}}$$

Lastly if we use probability limits instead of k sigma limits we may substitute $k = z_{\frac{\alpha}{2}}$ in equation (2 2) to obtain

$$LCL = \mu_0 - z_{\frac{\alpha}{2}} \frac{\sigma_0}{\sqrt{n}} \quad \text{and} \quad UCL = \mu_0 + z_{\frac{\alpha}{2}} \frac{\sigma_0}{\sqrt{n}}$$

where $z_{\frac{\alpha}{2}}$ denotes the upper percentage point of a standard normal distribution that is

$$P\left(Z \geq z_{\frac{\alpha}{2}}\right) = \frac{\alpha}{2}$$

Result 3 2 Non sıgnalıng event

$$\frac{\sqrt{n}\left|\bar{X} - \mu_0\right|}{\sigma_0} < z_{\frac{\alpha}{2}} \text{ for } \imath = 1\ 2\ 3$$

Proof

$$\bar{X} < UCL \text{ and } \bar{X} > LCL$$

$$\bar{X} < \mu_0 + z_{\frac{\alpha}{2}} \frac{\sigma_0}{\sqrt{n}} \text{ and } \bar{X} > \mu_0 - z_{\frac{\alpha}{2}} \frac{\sigma_0}{\sqrt{n}}$$

$$\bar{X} - \mu_0 < z_{\frac{\alpha}{2}} \frac{\sigma_0}{\sqrt{n}} \text{ and } \bar{X} - \mu_0 > -z_{\frac{\alpha}{2}} \frac{\sigma_0}{\sqrt{n}}$$

$$\bar{X} - \mu_0 < z_{\frac{\alpha}{2}} \frac{\sigma_0}{\sqrt{n}} \text{ and } -(\bar{X} - \mu_0) < z_{\frac{\alpha}{2}} \frac{\sigma_0}{\sqrt{n}}$$

$$\left|\bar{X} - \mu_0\right| < z_{\frac{\alpha}{2}} \frac{\sigma_0}{\sqrt{n}}$$

$$\frac{\sqrt{n}\left|\bar{X} - \mu_0\right|}{\sigma_0} < z_{\frac{\alpha}{2}}$$

Result 3 3 Signaling event

$$\frac{\sqrt{n}\left|\bar{X} - \mu_0\right|}{\sigma_0} \geq z_{\frac{\alpha}{2}} \text{ for } i = 1\ 2\ 3$$

Proof

$$\bar{X} \geq UCL \text{ or } \bar{X} \leq LCL$$

$$\bar{X} \geq \mu_0 + z_{\frac{\alpha}{2}} \frac{\sigma_0}{\sqrt{n}} \text{ or } \bar{X} \leq \mu_0 - z_{\frac{\alpha}{2}} \frac{\sigma_0}{\sqrt{n}}$$

$$\bar{X} - \mu_0 \geq z_{\frac{\alpha}{2}} \frac{\sigma_0}{\sqrt{n}} \text{ or } \bar{X} - \mu_0 \leq -z_{\frac{\alpha}{2}} \frac{\sigma_0}{\sqrt{n}}$$

$$\bar{X} - \mu_0 \geq z_{\frac{\alpha}{2}} \frac{\sigma_0}{\sqrt{n}} \text{ or } -(\bar{X} - \mu_0) \geq z_{\frac{\alpha}{2}} \frac{\sigma_0}{\sqrt{n}}$$

$$\left|\bar{X} - \mu_0\right| \geq z_{\frac{\alpha}{2}} \frac{\sigma_0}{\sqrt{n}}$$

$$\frac{\sqrt{n}\left|\bar{X} - \mu_0\right|}{\sigma_0} \geq z_{\frac{\alpha}{2}}$$

Result 3 4 Probability of no signal

$$\beta(\alpha\ \delta\ n) = \Phi\left(z_{\frac{\alpha}{2}} - \delta\sqrt{n}\right) - \Phi\left(-z_{\frac{\alpha}{2}} - \delta\sqrt{n}\right)$$

Proof

$P(\text{No Signal})$

$= P(LCL < \bar{X} < UCL \mid \text{Shift in the process mean occurred})$

$= P(\mu_0 - z_{\frac{\alpha}{2}}\dfrac{\sigma_0}{\sqrt{n}} < \bar{X} < \mu_0 + z_{\frac{\alpha}{2}}\dfrac{\sigma_0}{\sqrt{n}} \mid \mu_1 = \mu_0 + \delta\sigma_0)$

$= P\left(\dfrac{\mu_0 - z_{\frac{\alpha}{2}}\frac{\sigma_0}{\sqrt{n}} - (\mu_0 + \delta\sigma_0)}{\frac{\sigma_0}{\sqrt{n}}} < \dfrac{\bar{X} - (\mu_0 + \delta\sigma_0)}{\frac{\sigma_0}{\sqrt{n}}} < \dfrac{\mu_0 + z_{\frac{\alpha}{2}}\frac{\sigma_0}{\sqrt{n}} - (\mu_0 + \delta\sigma_0)}{\frac{\sigma_0}{\sqrt{n}}}\right)$

$= P\left(-z_{\frac{\alpha}{2}} - \delta\sqrt{n} < Z < z_{\frac{\alpha}{2}} - \delta\sqrt{n}\right)$

$= \Phi\left(z_{\frac{\alpha}{2}} - \delta\sqrt{n}\right) - \Phi\left(-z_{\frac{\alpha}{2}} - \delta\sqrt{n}\right)$

$= \beta(\alpha\ \delta\ n)$ say

For simplicity denote $\beta(\alpha\ \delta\ n)$ by β

Result 3 5 Probability of a signal

$$1-\beta(\alpha\ \delta\ n)=1-\Phi\left(z_{\frac{\alpha}{2}}-\delta\sqrt{n}\right)+\Phi\left(-z_{\frac{\alpha}{2}}-\delta\sqrt{n}\right)$$

Proof

$P(\text{Signal})$

$=1-P(\text{No Signal})$

$=1-\left(\Phi\left(z_{\frac{\alpha}{2}}-\delta\sqrt{n}\right)-\Phi\left(-z_{\frac{\alpha}{2}}-\delta\sqrt{n}\right)\right)$

$=1-\Phi\left(z_{\frac{\alpha}{2}}-\delta\sqrt{n}\right)+\Phi\left(-z_{\frac{\alpha}{2}}-\delta\sqrt{n}\right)$

$=1-\beta$

Result 3 6 Probabılıty of a false alarm

$$\alpha = 1 - \Phi\left(z_{\frac{\alpha}{2}}\right) - \Phi\left(-z_{\frac{\alpha}{2}}\right)$$

Proof

The probabılıty of a false alarm ıs the probabılıty of a sıgnal when the process ıs really ın control ı e when $\delta = 0$ ın the expressıon for a sıgnal namely $1 - \Phi\left(z_{\frac{\alpha}{2}} - \delta\sqrt{n}\right) + \Phi\left(-z_{\frac{\alpha}{2}} - \delta\sqrt{n}\right)$

Thus $P(\mathit{False\ Alarm}) = 1 - \Phi\left(z_{\frac{\alpha}{2}}\right) - \Phi\left(-z_{\frac{\alpha}{2}}\right) = \alpha$ and can also be wrıtten as $P\left(|Z| \geq z_{\frac{\alpha}{2}}\right) = \alpha$ by makıng use of the symmetry of the standard normal dıstrıbutıon

However the probabılıty of a false alarm can be derıved straıght away as follows

P(False Alarm)

$= P$(Sıgnal | In control)

$= 1 - P$(No Sıgnal | In control)

$= 1 - P(LCL < \bar{X} < UCL$ | No shıft ın the process mean occurred)

$$= 1 - P\left(\mu_0 - z_{\frac{\alpha}{2}}\frac{\sigma_0}{\sqrt{n}} < \bar{X} < \mu_0 + z_{\frac{\alpha}{2}}\frac{\sigma_0}{\sqrt{n}} \mid \mu_1 = \mu_0\right)$$

$$= 1 - P\left(\frac{\mu_0 - z_{\frac{\alpha}{2}}\frac{\sigma_0}{\sqrt{n}} - \mu_0}{\frac{\sigma_0}{\sqrt{n}}} < \frac{\bar{X} - \mu_0}{\frac{\sigma_0}{\sqrt{n}}} < \frac{\mu_0 + z_{\frac{\alpha}{2}}\frac{\sigma_0}{\sqrt{n}} - \mu_0}{\frac{\sigma_0}{\sqrt{n}}}\right)$$

$$= 1 - P\left(-z_{\frac{\alpha}{2}} < Z < z_{\frac{\alpha}{2}}\right)$$

$$= 1 - \left\{\Phi\left(z_{\frac{\alpha}{2}}\right) - \Phi\left(-z_{\frac{\alpha}{2}}\right)\right\}$$

$= 1 - \beta(\alpha\ 0\ n)$

$= \alpha$

Result 3 7 Run length dıstrıbutıon

$$P(N=j)=\beta^{j-1}(1-\beta) \text{ for } j=1\ 2\ 3$$

Proof

Assume that at each poınt ın tıme we perform the same experıment by obtaınıng a random sample of n observatıons from the process output calculatıng the sample mean $\overline{X}$ and then compare $\overline{X}$ wıth $LCL=\mu_0 - z_{\frac{\alpha}{2}}\frac{\sigma_0}{\sqrt{n}}$ and $UCL=\mu_0 + z_{\frac{\alpha}{2}}\frac{\sigma_0}{\sqrt{n}}$ to see whether $\overline{X}$ falls between or outsıde the UCL and/or LCL

In addıtıon assume that

ı All trıals ı e all the experıments at the dıfferent poınts ın tıme are ındependent

ıı At each stage of samplıng we can eıther observed a success (S) or a faılure (F) that ıs $\overline{X}$ can eıther fall between UCL and LCL (and gıve no sıgnal) or outsıde UCL or LCL wıth both endpoınts ıncluded (and gıve a sıgnal)

ııı The probabılıty of a faılure (no sıgnal) and the probabılıty of a success (sıgnal) are the same on each trıal and denoted by $P(F)=\beta$ and $P(S)=1-\beta$ respectıvely

ıv We repeat the experıment untıl we obtaın the first success

Thus the sample space **S** ıs $\mathbf{S}=\{S\ FS\ FFS\ FFFS\ FFFFS\quad\}$ and ıf we consıder an element of **S** wıth $j-1$ faılures and then a success we find that

$$P(FF\quad FS)=P(F)P(F)\quad P(F)P(S)=P(F)^{j-1}P(S) \text{ for } j=1\ 2\ 3$$

Consequently ıf N denotes the number of trıals untıl we observe a success for the first tıme we obtaın

$$P(N=j)=\beta^{j-1}(1-\beta) \text{ for } j=1\ 2\ 3$$

whıch ıs a geometrıc dıstrıbutıon wıth probabılıty of success gıven by $1-\beta$ so that $N\sim Geo(1-\beta)$

Consequently the cumulatıve dıstrıbutıon functıon (cdf) ıs found from

$$P(N\le j)=\sum_{k=1}^{j}\beta^{k-1}(1-\beta)=1-\beta^{j}$$

and subsequently

$$P(N\ge j)=\beta^{j-1}$$

Result 3 8 Quantile function

$$Q_N(p) = \inf\{j \text{ an integer } P(N \le j) \ge p\} = \inf\left\{j \text{ an integer } j \ge \frac{\ln(1-p)}{\ln\beta}\right\} \text{ with } 0 < p < 1$$

Proof

Generally the quantile function is defined as

$$Q_N(p) = \inf\{j \text{ an integer } F_N(j) \ge p\}$$

where $F_N(j)$ is the cdf of a distribution and p denotes a particular percentile of the distribution

Thus substituting the cdf of the run length distrbution (given in Result 3 7) and solving for j the quantile function for the run length distribution is

$$Q_N(p) = \inf\{j \text{ an integer } P(N \le j) \ge p\} = \inf\left\{j \text{ an integer } j \ge \frac{\ln(1-p)}{\ln\beta}\right\} \text{ with } 0 < p < 1$$

Result 3.9 Moment generating function

$$M_N(t) = \frac{(1-\beta)e^t}{1-\beta e^t}$$

Proof

$$M_N(t)$$

$$= E\left(e^{tN}\right)$$

$$= \sum_{j=1}^{\infty} e^{tj}\beta^{j-1}(1-\beta)$$

$$= (1-\beta)e^t \sum_{j=1}^{\infty} e^{t(j-1)}\beta^{j-1}$$

$$= (1-\beta)e^t \sum_{j-1=0}^{\infty} e^{t(j-1)}\beta^{j-1}$$

$$= (1-\beta)e^t \sum_{k=0}^{\infty} e^{tk}\beta^{k}$$

$$= (1-\beta)e^t \sum_{k=0}^{\infty} \left(e^t\beta\right)^{k}$$

$$= \frac{(1-\beta)e^t}{1-\beta e^t}$$

Result 3 10 Average run length (First non central moment of the run length distribution)

$$E(N)=\frac{1}{1-\beta}=\frac{1}{1-\Phi\left(z_{\frac{\alpha}{2}}-\delta\sqrt{n}\right)+\Phi\left(-z_{\frac{\alpha}{2}}-\delta\sqrt{n}\right)}$$

Proof

In general the k^{th} non central moment $E\left(N^k\right)$ is the k^{th} derivative of the moment generating function $M_N(t)$ evaluated in the point zero i e $E\left(N^k\right)=\frac{d^k}{dt^k}M_N(t)|_{t=0}$

Thus since the average run length is the first non central moment of the run length distribution we need $E(N)=\frac{d}{dt}M_N(t)|_{t=0}$

Therefore

$$E(N)$$

$$=\frac{d}{dt}M_N(t)|_{t=0}$$

$$=\frac{-(-1-\beta)e^t}{\left(-1+\beta e^t\right)^2}|_{t=0}$$

$$=\frac{1}{1-\beta}$$

and it follows that

$$ARL=E(N)=\frac{1}{1-\beta}=\frac{1}{1-\Phi\left(z_{\frac{\alpha}{2}}-\delta\sqrt{n}\right)+\Phi\left(-z_{\frac{\alpha}{2}}-\delta\sqrt{n}\right)}$$

Result 3 11 Second non central moment and third non central moment of the run length distribution

$$E\left(N^2\right)=\frac{1+\beta}{(1-\beta)^2} \quad \text{and} \quad E(N^3)=\frac{1+\beta^2+4\beta}{(1-\beta)^3}$$

Proof

In general the k^{th} non central moment $E\left(N^k\right)$ is the k^{th} derivative of the moment generating function $M_N(t)$ evaluated in the point zero i e $E\left(N^k\right)=\frac{d^k}{dt^k}M_N(t)|_{t=0}$

Therefore

$$E\left(N^2\right)$$

$$=\frac{\partial^2}{\partial t^2}M_N(t)_{|t=0}$$

$$=\frac{(-1-\beta)e^t\left(1+\beta e^t\right)}{\left(-1+\beta e^t\right)^3}|_{t=0}$$

$$=\frac{1+\beta}{(1-\beta)^2}$$

and

$$E(N^3)$$

$$=\frac{\partial^3}{\partial t^3}M_N(t)_{|t=0}$$

$$=\frac{(-1-\beta)e^t\left(1+\beta^2e^{2t}+4\beta e^t\right)}{\left(-1+\beta e^t\right)^4}|_{t=0}$$

$$=\frac{1+\beta^2+4\beta}{(1-\beta)^3}$$

Result 3 12 Variance and Standard deviation of the run length distribution

$$\text{var}(N) = \frac{\beta}{(1-\beta)^2} \quad \text{and} \quad stdev(N) = \frac{\sqrt{\beta}}{1-\beta}$$

Proof

The variance of the run length distribution and the standard deviation of the run length distribution are obtained from their definitions i e

$$\text{var}(N) = E(N - E(N))^2 = E(N^2) - E(N)^2$$

and

$$stdev(N) = \sqrt{\text{var}(N)}$$

Thus substituting the expressions for the first non central moment $E(N)$ and the second non central moment $E(N^2)$ we obtain

$$\text{var}(N) = \frac{1+\beta}{(1-\beta)^2} - \frac{1}{(1-\beta)^2} = \frac{\beta}{(1-\beta)^2}$$

and subsequently

$$stdev(N) = \frac{\sqrt{\beta}}{1-\beta}$$

Result 3 13 Coefficient of skewness of the run length distribution

$$skew(N) = \frac{1+\beta}{\sqrt{\beta}}$$

Proof

The coefficient of skewness is defined as

$$skew(N) = \frac{E(N - E(N))^3}{\operatorname{var}(N)^{\frac{3}{2}}}$$

which can be simplified to

$$skew(N) = \frac{E\left(N^3\right) - 3E\left(N^2\right)E\left(N\right) + 2E\left(N\right)^3}{\left(\operatorname{var}(N)\right)^{\frac{3}{2}}}$$

Substituting the expressions for the first non central moment $E\left(N\right)$ the second non central moment $E\left(N^2\right)$ and the third non central moment $E\left(N^3\right)$ as well as the expression for the variance $\operatorname{var}(N)$ the expression for the coefficient of skewness simplifies to

$$skew(N) = \frac{1+\beta}{\sqrt{\beta}}$$

3 5 3 Appendix 3B

Case UK

Assume that the in control process mean is *unknown* whereas the in control process standard deviation is *known* and denoted by μ and σ_0 respectively Furthermore assume that the process follows a normal distribution so that $X_{ij} \sim N(\mu, \sigma_0^2)$ for $i = 1, 2, 3, \ldots$ and $j = 1, 2, 3, \ldots, n$ In addition assume that we estimate the unknown in control process mean using m reference samples each of size n from Phase 1 and that the point estimator used is the overall mean or the grand mean i e

$$\bar{\bar{X}} = \frac{1}{m}\sum_{i=1}^{m}\bar{X}_i = \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}X_{ij}$$

with an observed value of $\bar{\bar{X}}$ denoted by $\bar{\bar{x}}$ which is an observation from a $N(\mu, \frac{\sigma_0^2}{mn})$ distribution

Since the estimation of the unknown process parameter (which in this case is the in control process mean) occurs in Phase 1 the formulae and/or expressions that are derived are only applicable in Phase 2

Result 3 14 Control lımıts

$$\widehat{LCL} = \overline{\overline{X}} - z_{\frac{\alpha}{2}} \frac{\sigma_0}{\sqrt{n}} \quad \text{and} \quad \widehat{UCL} = \overline{\overline{X}} + z_{\frac{\alpha}{2}} \frac{\sigma_0}{\sqrt{n}}$$

Proof

From Result 3 1 ın Appendıx 3A we have that

$$LCL = \mu_0 - z_{\frac{\alpha}{2}} \frac{\sigma_0}{\sqrt{n}} \quad \text{and} \quad UCL = \mu_0 + z_{\frac{\alpha}{2}} \frac{\sigma_0}{\sqrt{n}}$$

However sınce the ın control process mean ıs unknown we subtıtute μ_0 wıth ıts poınt estımator $\overline{\overline{X}}$ and obtaın

$$\widehat{LCL} = \overline{\overline{X}} - z_{\frac{\alpha}{2}} \frac{\sigma_0}{\sqrt{n}} \quad \text{and} \quad \widehat{UCL} = \overline{\overline{X}} + z_{\frac{\alpha}{2}} \frac{\sigma_0}{\sqrt{n}}$$

Result 3 15 Non signalling event

$$\frac{\sqrt{n}\left|\bar{X} - \bar{\bar{X}}\right|}{\sigma_0} < z_{\frac{\alpha}{2}} \quad \text{for } i = m+1 \;\; m+2 \;\; m+3$$

Proof

$$\bar{X} < \widehat{UCL} \text{ and } \bar{X} > \widehat{LCL}$$

$$\bar{X} < \bar{\bar{X}} + z_{\frac{\alpha}{2}} \frac{\sigma_0}{\sqrt{n}} \text{ and } \bar{X} > \bar{\bar{X}} - z_{\frac{\alpha}{2}} \frac{\sigma_0}{\sqrt{n}}$$

$$\bar{X} - \bar{\bar{X}} < z_{\frac{\alpha}{2}} \frac{\sigma_0}{\sqrt{n}} \text{ and } \bar{X} - \bar{\bar{X}} > -z_{\frac{\alpha}{2}} \frac{\sigma_0}{\sqrt{n}}$$

$$\bar{X} - \bar{\bar{X}} < z_{\frac{\alpha}{2}} \frac{\sigma_0}{\sqrt{n}} \text{ and } -(\bar{X} - \bar{\bar{X}}) < z_{\frac{\alpha}{2}} \frac{\sigma_0}{\sqrt{n}}$$

$$\left|\bar{X} - \bar{\bar{X}}\right| < z_{\frac{\alpha}{2}} \frac{\sigma_0}{\sqrt{n}}$$

$$\frac{\sqrt{n}\left|\bar{X} - \bar{\bar{X}}\right|}{\sigma_0} < z_{\frac{\alpha}{2}}$$

Result 3 16 Signaling event

$$\frac{\sqrt{n}\left|\bar{X} - \bar{\bar{X}}\right|}{\sigma_0} \geq z_{\frac{\alpha}{2}} \text{ for } i = m+1 \;\; m+2 \;\; m+3$$

Proof

$$\bar{X} \geq \widehat{UCL} \text{ or } \bar{X} \leq \widehat{LCL}$$

$$\bar{X} \geq \bar{\bar{X}} + z_{\frac{\alpha}{2}} \frac{\sigma_0}{\sqrt{n}} \text{ or } \bar{X} \leq \bar{\bar{X}} - z_{\frac{\alpha}{2}} \frac{\sigma_0}{\sqrt{n}}$$

$$\bar{X} - \bar{\bar{X}} \geq z_{\frac{\alpha}{2}} \frac{\sigma_0}{\sqrt{n}} \text{ or } \bar{X} - \bar{\bar{X}} \leq -z_{\frac{\alpha}{2}} \frac{\sigma_0}{\sqrt{n}}$$

$$\bar{X} - \bar{\bar{X}} \geq z_{\frac{\alpha}{2}} \frac{\sigma_0}{\sqrt{n}} \text{ or } -(\bar{X} - \bar{\bar{X}}) \geq z_{\frac{\alpha}{2}} \frac{\sigma_0}{\sqrt{n}}$$

$$\left|\bar{X} - \bar{\bar{X}}\right| \geq z_{\frac{\alpha}{2}} \frac{\sigma_0}{\sqrt{n}}$$

$$\frac{\sqrt{n}\left|\bar{X} - \bar{\bar{X}}\right|}{\sigma_0} \geq z_{\frac{\alpha}{2}}$$

Result 3 17 Probability of no signal – Conditional

$$\beta(\alpha\ \delta\ m\ n\ z) = \Phi\left(\frac{z}{\sqrt{m}} + z_{\frac{\alpha}{2}} - \delta\sqrt{n}\right) - \Phi\left(\frac{z}{\sqrt{m}} - z_{\frac{\alpha}{2}} - \delta\sqrt{n}\right)$$

Proof

Let $P_C(\,)$ denote a conditional probability that is we condition on a sustained shift in the process mean as well as on an observed value of the random variable $\overline{\overline{X}}$ i e we condition on the value $\overline{\overline{x}}$ Thus

$$P_C(\text{No Signal})$$

$$= P(\widehat{LCL} < \overline{X} < \widehat{UCL} \mid \text{A shift in the process mean occured } \overline{\overline{X}} = \overline{\overline{x}})$$

$$= P\left(\overline{\overline{x}} - z_{\frac{\alpha}{2}}\frac{\sigma_0}{\sqrt{n}} < \overline{X} < \overline{\overline{x}} + z_{\frac{\alpha}{2}}\frac{\sigma_0}{\sqrt{n}} \mid \mu_1 = \mu + \delta\sigma_0\ \overline{\overline{X}} = \overline{\overline{x}}\right)$$

$$= P\left(\frac{\overline{\overline{x}} - z_{\frac{\alpha}{2}}\frac{\sigma_0}{\sqrt{n}} - (\mu + \delta\sigma_0)}{\frac{\sigma_0}{\sqrt{n}}} < \frac{\overline{X} - (\mu + \delta\sigma_0)}{\frac{\sigma_0}{\sqrt{n}}} < \frac{\overline{\overline{x}} + z_{\frac{\alpha}{2}}\frac{\sigma_0}{\sqrt{n}} - (\mu + \delta\sigma_0)}{\frac{\sigma_0}{\sqrt{n}}} \mid \overline{\overline{X}} = \overline{\overline{x}}\right)$$

$$= P\left(\left(\frac{\overline{\overline{x}} - \mu}{\frac{\sigma_0}{\sqrt{mn}}}\right)\frac{1}{\sqrt{m}} - z_{\frac{\alpha}{2}} - \delta\sqrt{n} < Z < \left(\frac{\overline{\overline{x}} - \mu}{\frac{\sigma_0}{\sqrt{mn}}}\right)\frac{1}{\sqrt{m}} + z_{\frac{\alpha}{2}} - \delta\sqrt{n} \mid \overline{\overline{X}} = \overline{\overline{x}}\right)$$

$$= P\left(\frac{z}{\sqrt{m}} - z_{\frac{\alpha}{2}} - \delta\sqrt{n} < Z < \frac{z}{\sqrt{m}} + z_{\frac{\alpha}{2}} - \delta\sqrt{n} \mid Z = z\right)$$

$$= \Phi\left(\frac{z}{\sqrt{m}} + z_{\frac{\alpha}{2}} - \delta\sqrt{n}\right) - \Phi\left(\frac{z}{\sqrt{m}} - z_{\frac{\alpha}{2}} - \delta\sqrt{n}\right)$$

$$= \beta(\alpha\ \delta\ m\ n\ z) \quad \text{say}$$

For simplicity denote $\beta(\alpha\ \delta\ m\ n\ z)$ by β_Z In this case the subscript Z indicates that the expression is a conditional expression i e conditional on an observed value of a standard normal random variable

$$z = \frac{\overline{\overline{x}} - \mu}{\frac{\sigma_0}{\sqrt{mn}}}$$

Result 3 18 Probability of no signal – Unconditional

$$P(\text{No Signal}) = \int_{-\infty}^{\infty}\left(\Phi\left(\frac{z}{\sqrt{m}}+z_{\frac{\alpha}{2}}-\delta\sqrt{n}\right)-\Phi\left(\frac{z}{\sqrt{m}}-z_{\frac{\alpha}{2}}-\delta\sqrt{n}\right)\right)\phi(z)dz$$

or

$$P(\text{No Signal}) = \Phi\left(\sqrt{\frac{m}{m+1}}\left(z_{\frac{\alpha}{2}}-\delta\sqrt{n}\right)\right)-\Phi\left(\sqrt{\frac{m}{m+1}}\left(-z_{\frac{\alpha}{2}}-\delta\sqrt{n}\right)\right)$$

Proof

Expression 1

$P(\text{No Signal})$

$$= \int_{-\infty}^{\infty} P_C(\text{No Signal})\phi(z)dz$$

$$= \int_{-\infty}^{\infty}\left(\Phi\left(\frac{z}{\sqrt{m}}+z_{\frac{\alpha}{2}}-\delta\sqrt{n}\right)-\Phi\left(\frac{z}{\sqrt{m}}-z_{\frac{\alpha}{2}}-\delta\sqrt{n}\right)\right)\phi(z)dz$$

Expression 2

$P(\text{No Signal})$

$= P(\widehat{LCL} < \bar{X} < \widehat{UCL} \mid \text{A shift in the process mean occured})$

$$= P\left(\bar{\bar{X}} - z_{\frac{\alpha}{2}}\frac{\sigma_0}{\sqrt{n}} < \bar{X} < \bar{\bar{X}} + z_{\frac{\alpha}{2}}\frac{\sigma_0}{\sqrt{n}} \mid \mu_1 = \mu + \delta\sigma_0\right)$$

$$= P\left(\frac{-z_{\frac{\alpha}{2}}\frac{\sigma_0}{\sqrt{n}} - \delta\sigma_0}{\sqrt{\frac{\sigma_0^2}{n}\left(\frac{m+1}{m}\right)}} < \frac{(\bar{X}-\bar{\bar{X}})-\delta\sigma_0}{\sqrt{\frac{\sigma_0^2}{n}\left(\frac{m+1}{m}\right)}} < \frac{z_{\frac{\alpha}{2}}\frac{\sigma_0}{\sqrt{n}}\mu - \delta\sigma_0}{\sqrt{\frac{\sigma_0^2}{n}\left(\frac{m+1}{m}\right)}}\right)$$

$$= P\left(\sqrt{\frac{m}{m+1}}\left(-z_{\frac{\alpha}{2}}-\delta\sqrt{n}\right) < Z < \sqrt{\frac{m}{m+1}}\left(z_{\frac{\alpha}{2}}-\delta\sqrt{n}\right)\right)$$

$$= \Phi\left(\sqrt{\frac{m}{m+1}}\left(z_{\frac{\alpha}{2}}-\delta\sqrt{n}\right)\right)-\Phi\left(\sqrt{\frac{m}{m+1}}\left(-z_{\frac{\alpha}{2}}-\delta\sqrt{n}\right)\right)$$

Result 3 19 Probability of a signal – Conditional

$$1-\beta(\alpha\ \delta\ m\ n\ z)=1-\Phi\left(\frac{z}{\sqrt{m}}+z_{\frac{\alpha}{2}}-\delta\sqrt{n}\right)+\Phi\left(\frac{z}{\sqrt{m}}-z_{\frac{\alpha}{2}}-\delta\sqrt{n}\right)$$

Proof

P_C(Signal)

$=1-P_C$(No Signal)

$$=1-\left(\Phi\left(\frac{z}{\sqrt{m}}+z_{\frac{\alpha}{2}}-\delta\sqrt{n}\right)-\Phi\left(\frac{z}{\sqrt{m}}-z_{\frac{\alpha}{2}}-\delta\sqrt{n}\right)\right)$$

$$=1-\Phi\left(\frac{z}{\sqrt{m}}+z_{\frac{\alpha}{2}}-\delta\sqrt{n}\right)+\Phi\left(\frac{z}{\sqrt{m}}-z_{\frac{\alpha}{2}}-\delta\sqrt{n}\right)$$

$=1-\beta_Z$

Result 3 20 Probability of a signal – Unconditional

$$P(\text{Signal}) = \int_{-\infty}^{\infty}\left(1-\Phi\left(\frac{z}{\sqrt{m}}+z_{\frac{\alpha}{2}}-\delta\sqrt{n}\right)+\Phi\left(\frac{z}{\sqrt{m}}-z_{\frac{\alpha}{2}}-\delta\sqrt{n}\right)\right)\phi(z)dz$$

or

$$P(\text{Signal}) = 1-\Phi\left(\sqrt{\frac{m}{m+1}}\left(z_{\frac{\alpha}{2}}-\delta\sqrt{n}\right)\right)+\Phi\left(\sqrt{\frac{m}{m+1}}\left(-z_{\frac{\alpha}{2}}-\delta\sqrt{n}\right)\right)$$

Proof

Expression 1

$$P(\text{Signal})$$

$$= \int_{-\infty}^{\infty} P_C(\text{Signal})\phi(z)dz$$

$$= \int_{-\infty}^{\infty}\left(1-\Phi\left(\frac{z}{\sqrt{m}}+z_{\frac{\alpha}{2}}-\delta\sqrt{n}\right)+\Phi\left(\frac{z}{\sqrt{m}}-z_{\frac{\alpha}{2}}-\delta\sqrt{n}\right)\right)\phi(z)dz$$

Expression 2

A second expression for the unconditional probability of a signal can be derived in a similar way as expression number two was obtained for the unconditional probability of no signal in Result 3 18 i e

$$\Phi\left(\sqrt{\frac{m}{m+1}}\left(z_{\frac{\alpha}{2}}-\delta\sqrt{n}\right)\right)-\Phi\left(\sqrt{\frac{m}{m+1}}\left(-z_{\frac{\alpha}{2}}-\delta\sqrt{n}\right)\right)$$

We have that $P(\text{Signal}) = 1 - P(\text{No Signal})$ since a non signaling event and a signaling event are two complementary events and therefore the second expression for the unconditional probability of a signal is

$$P(\text{Signal}) = 1-\Phi\left(\sqrt{\frac{m}{m+1}}\left(z_{\frac{\alpha}{2}}-\delta\sqrt{n}\right)\right)+\Phi\left(\sqrt{\frac{m}{m+1}}\left(-z_{\frac{\alpha}{2}}-\delta\sqrt{n}\right)\right)$$

Result 3 21 Probability of a false alarm – Conditional

$$P_C(\text{False Alarm}) = 1-\Phi\left(\frac{z}{\sqrt{m}}+z_{\frac{\alpha}{2}}\right)+\Phi\left(\frac{z}{\sqrt{m}}-z_{\frac{\alpha}{2}}\right)$$

Proof

$$
\begin{aligned}
&P_C(\text{False Alarm})\\
&= P_C\left(\text{Signal} \mid \text{In control}\right)\\
&= 1-P_C\left(\text{No Signal} \mid \text{In control}\right)\\
&= 1-P_C\left(\widehat{LCL} < \bar{X} < \widehat{UCL} \mid \text{In control}\right)\\
&= 1-P_C\left(\widehat{LCL} < \bar{X} < \widehat{UCL} \mid \mu_1 = \mu \;\; \bar{\bar{X}} = \bar{\bar{x}}\right)\\
&= 1-P\left(\bar{\bar{x}} - z_{\frac{\alpha}{2}}\frac{\sigma_0}{\sqrt{n}} < \bar{X} < \bar{\bar{x}} + z_{\frac{\alpha}{2}}\frac{\sigma_0}{\sqrt{n}} \mid \mu_1 = \mu \;\; \bar{\bar{X}} = \bar{\bar{x}}\right)\\
&= 1-P\left(\frac{\bar{\bar{x}} - z_{\frac{\alpha}{2}}\frac{\sigma_0}{\sqrt{n}} - \mu}{\frac{\sigma_0}{\sqrt{n}}} < \frac{\bar{X}-\mu}{\frac{\sigma_0}{\sqrt{n}}} < \frac{\bar{\bar{x}} + z_{\frac{\alpha}{2}}\frac{\sigma_0}{\sqrt{n}} - \mu}{\frac{\sigma_0}{\sqrt{n}}} \mid \bar{\bar{X}} = \bar{\bar{x}}\right)\\
&= 1-P\left(\left(\frac{\bar{\bar{x}}-\mu}{\frac{\sigma_0}{\sqrt{mn}}}\right)\frac{1}{\sqrt{m}} - z_{\frac{\alpha}{2}} < Z < \left(\frac{\bar{\bar{x}}-\mu}{\frac{\sigma_0}{\sqrt{mn}}}\right)\frac{1}{\sqrt{m}} + z_{\frac{\alpha}{2}} \mid \bar{\bar{X}} = \bar{\bar{x}}\right)\\
&= 1-P\left(\frac{z}{\sqrt{m}} - z_{\frac{\alpha}{2}} < Z < \frac{z}{\sqrt{m}} + z_{\frac{\alpha}{2}} \mid Z = z\right)\\
&= 1-\left(\Phi\left(\frac{z}{\sqrt{m}}+z_{\frac{\alpha}{2}}\right)-\Phi\left(\frac{z}{\sqrt{m}}-z_{\frac{\alpha}{2}}\right)\right)\\
&= 1-\Phi\left(\frac{z}{\sqrt{m}}+z_{\frac{\alpha}{2}}\right)+\Phi\left(\frac{z}{\sqrt{m}}-z_{\frac{\alpha}{2}}\right)\\
&= 1-\beta(\alpha \;\; 0 \;\; m \;\; n \;\; z)
\end{aligned}
$$

This expression could also have been obtained by simply substituting $\delta = 0$ in the expression for the conditional probability of a signal in Result 3 19 given that a shift in the process mean occurred

Therefore Result 3 21 is the same as Result 3 19 i e $1-\Phi\left(\frac{z}{\sqrt{m}}+z_{\frac{\alpha}{2}}-\delta\sqrt{n}\right)+\Phi\left(\frac{z}{\sqrt{m}}-z_{\frac{\alpha}{2}}-\delta\sqrt{n}\right)$ with $\delta = 0$

Result 3 22 Probability of a false alarm – Unconditional

$$P(\text{Signal} \mid \text{In control}) = \int_{-\infty}^{\infty}\left(1-\Phi\left(\frac{z}{\sqrt{m}}+z_{\frac{\alpha}{2}}\right)+\Phi\left(\frac{z}{\sqrt{m}}-z_{\frac{\alpha}{2}}\right)\right)\phi(z)dz$$

or

$$P(\text{False Alarm}) = 1-\Phi\left(\sqrt{\frac{m}{m+1}}z_{\frac{\alpha}{2}}\right)+\Phi\left(-\sqrt{\frac{m}{m+1}}z_{\frac{\alpha}{2}}\right)$$

Proof

Expression 1

$P(\text{Signal} \mid \text{In control})$

$= \int_{-\infty}^{\infty} P_C(\text{Signal} \mid \text{In control})\phi(z)dz$

$= \int_{-\infty}^{\infty}\left(1-\Phi\left(\frac{z}{\sqrt{m}}+z_{\frac{\alpha}{2}}\right)+\Phi\left(\frac{z}{\sqrt{m}}-z_{\frac{\alpha}{2}}\right)\right)\phi(z)dz$

Expression 2

A second expression for the unconditional probability of a false alarm can be obtained from the second expression for the unconditional probability of a signal given in Result 3 20 i e

$1-\Phi\left(\sqrt{\frac{m}{m+1}}\left(z_{\frac{\alpha}{2}}-\delta\sqrt{n}\right)\right)+\Phi\left(\sqrt{\frac{m}{m+1}}\left(-z_{\frac{\alpha}{2}}-\delta\sqrt{n}\right)\right)$ by setting $\delta = 0$

Thus

$$P(\text{False Alarm}) = 1-\Phi\left(\sqrt{\frac{m}{m+1}}z_{\frac{\alpha}{2}}\right)+\Phi\left(-\sqrt{\frac{m}{m+1}}z_{\frac{\alpha}{2}}\right)$$

Result 3 23 Run length dıstrıbutıon – Condıtıonal

$$P(N = j \mid Z = z) = \beta_Z^{j-1}(1-\beta_Z) \text{ for } j = 1\ 2\ 3$$

Proof

Assume that at each poınt ın tıme we perform the same experıment by obtaınıng a random sample of n observatıons from the process output calculatıng the sample mean $\overline{X}$ and then compare $\overline{X}$ wıth $\widehat{LCL} = \overline{\overline{X}} - z_{\frac{\alpha}{2}}\frac{\sigma_0}{\sqrt{n}}$ and $\widehat{UCL} = \overline{\overline{X}} + z_{\frac{\alpha}{2}}\frac{\sigma_0}{\sqrt{n}}$ to see whether $\overline{X}$ falls between or outsıde $\widehat{UCL}$ and/or $\widehat{LCL}$

In addıtıon assume that we condıtıon on an observed value of $\overline{\overline{X}}$ denoted $\overline{\overline{x}}$ (or z ıf $\overline{\overline{X}}$ ıs wrıtten ın ıts canonıcal form namely Z) so that we actually compare $\overline{X}$ wıth $\widehat{LCL} = \overline{\overline{x}} - z_{\frac{\alpha}{2}}\frac{\sigma_0}{\sqrt{n}}$ and $\widehat{UCL} = \overline{\overline{x}} + z_{\frac{\alpha}{2}}\frac{\sigma_0}{\sqrt{n}}$ that ıs both control lımıts are constants and then have that

ı All trıals ı e all the experıments at the dıfferent poınts ın tıme are ındependent

ıı At each stage of samplıng we can eıther observed a success (S) or a faılure (F) that ıs $\overline{X}$ can eıther fall between $\widehat{UCL}$ and $\widehat{LCL}$ (and gıve no sıgnal) or outsıde $\widehat{UCL}$ or $\widehat{LCL}$ wıth both endpoınts ıncluded (and gıve a sıgnal)

ııı The condıtıonal probabılıty of a faılure (no sıgnal) and the condıtıonal probabılıty of a success (sıgnal) are the same on each trıal and denoted by $P(F) = \beta_Z$ and $P(S) = 1-\beta_Z$ respectıvely

ıv We repeat the experıment untıl we obtaın the first success

Thus the sample space **S** ıs **S** $= \{S\ FS\ FFS\ FFFS\ FFFFS\ \ \}$ and ıf we consıder an element of **S** wıth $j-1$ faılures and then a success we find that

$$P(FF\ \ FS) = P(F)P(F)\ \ P(F)P(S) = P(F)^{j-1}P(S) \text{ for } j = 1\ 2\ 3$$

Consequently ıf N denotes the number of trıals untıl we observe a success for the first tıme we obtaın

$$P(N = j \mid Z = z) = \beta_Z^{j-1}(1-\beta_Z) \text{ for } j = 1\ 2\ 3$$

which is a geometric distribution with probability of success given by $1-\beta_Z$ so that

$N \mid Z = z \sim Geo\left(1-\beta_Z\right)$

Consequently the cumulative distribution function (cdf) is found from

$$P(N \leq j \mid Z = z) = \sum_{k=1}^{j} \beta_Z^{k-1}(1-\beta_Z) = 1-\beta_Z^{j}$$

and subsequently

$$P(N \geq j \mid Z = z) = \beta_Z^{j-1}$$

Result 3 24 Run length distribution – Unconditional

$$P(N = j) = \int_{-\infty}^{\infty} \beta_Z^{j-1}(1-\beta_Z)\phi(z)dz \text{ for } j = 1\ 2\ 3$$

Proof

The conditional run length distribution (given in Result 2 23) is denoted by $P(N = j \mid Z = z)$ whereas the joint distribution of the run length random variable (N) and the random variable Z is denoted by $P(N = j\ Z = z)$ The marginal distributions of these two random variables are denoted by $P(N = j)$ and $P(Z = z)$ respectively

The unconditional (or marginal) run length distribution of N i e $P(N = j)$ given the fact that the conditional run length distribution i e $P(N = j \mid Z = z)$ and the marginal distribution of the random variable Z are known can be found by using the fact that $P(N = j \mid Z = z) = \dfrac{P(N = j\ Z = z)}{P(Z = z)}$ or alternatively the fact that $P(N = j\ Z = z) = P(N = j \mid Z = z)P(Z = z)$

Therefore to find the marginal distribution of N (also called the unconditional run length distribution) requires integrating the joint distribution of N and Z over the domain of the random variable Z i e

$$P(N = j) = \int_{-\infty}^{\infty} P(N = j\ Z = z)dz = \int_{-\infty}^{\infty} P(N = j \mid Z = z)P(Z = z)dz$$

Thus the unconditional probability mass function (pmf) is

$$P(N = j)$$

$$= \int_{-\infty}^{\infty} P(N = j \mid Z = z)\phi(z)dz$$

$$= \int_{-\infty}^{\infty} \beta_Z^{j-1}(1-\beta_Z)\phi(z)dz \text{ for } j = 1\ 2\ 3$$

whereas the unconditional cumulative distribution function (cdf) is

$$P(N \le j) = \sum_{k=1}^{j} P(N = k) = \sum_{k=1}^{j}\left(\int_{-\infty}^{\infty} \beta_Z^{k-1}(1-\beta_Z)\phi(z)dz\right) = \int_{-\infty}^{\infty}\left(1-\beta_Z^{j}\right)\phi(z)dz$$

Another useful way of writing the unconditional run length distribution is as

$P(N \ge j) = \int_{-\infty}^{\infty} P(N \ge j \mid Z = z)\phi(z)dz = \int_{-\infty}^{\infty} \beta_Z^{j-1}\phi(z)dz$ and this expression can conveniently be written as $I_1(j-1\ m\ n\ \delta\ \alpha)$

Result 3 25 Quantile function – Conditional

$$Q_{N|Z}(p) = \inf\{j \text{ an integer } P(N \le j \mid Z = z) \ge p\} = \inf\left\{j \text{ an integer } j \ge \frac{\ln(1-p)}{\ln \beta_Z}\right\}$$

Proof

Generally the quantile function is defined as

$$Q_N(p) = \inf\{j \text{ an integer } F_N(j) \ge p\}$$

where $F_N(j)$ is the cdf of a distribution and p denotes a particular percentile of the distribution

Thus substituting the conditional cdf of the run length distribution (given in Result 3 23) and solving for j the quantile function for the conditional run length distribution is

$$Q_{N|Z}(p) = \inf\{j \text{ an integer } P(N \le j \mid Z = z) \ge p\} = \inf\left\{j \text{ an integer } j \ge \frac{\ln(1-p)}{\ln \beta_Z}\right\} \text{ with } 0 < p < 1$$

Result 3 26 Quantile function – Unconditional

$$Q_N(p) = \inf\{j \text{ an integer } P(N \le j) \ge p\} = \inf\left\{j \text{ an integer } \sum_{k=1}^{j}\left(\int_{-\infty}^{\infty}\beta_Z^{k-1}(1-\beta_Z)\phi(z)dz\right) \ge p\right\}$$

Proof

The unconditional quantile function when the unknown in control process mean is estimated is

$$Q_N(p) = \inf\{j \text{ an integer } P(N \le j) \ge p\}$$
$$= \inf\left\{j \text{ an integer } \sum_{k=1}^{j} P(N = k) \ge p\right\} \quad \text{with } 0 < p < 1$$
$$= \inf\left\{j \text{ an integer } \sum_{k=1}^{j}\left(\int_{-\infty}^{\infty}\beta_Z^{k-1}(1-\beta_Z)\phi(z)dz\right) \ge p\right\}$$

$P(N \le j)$ denoting the uncondtional cumulative distribution function (given in Result 3 24) and $P(N = j)$ denoting the uncondtional probability mass function (also given in Result 3 24) This formula can not be simplified any further that is we cannot find a closed form expression for j

Result 3 27 Moment generating function – Conditional

$$M_{N|Z}(t) = \frac{(1-\beta_Z)e^t}{1-\beta_Z e^t}$$

Proof

$M_{N|Z}(t)$

$= \sum_{j=1}^{\infty} e^{tj}\beta_Z^{j-1}(1-\beta_Z)$

$= (1-\beta_Z)e^t \sum_{j=1}^{\infty} e^{t(j-1)}\beta_Z^{j-1}$

$= (1-\beta_Z)e^t \sum_{j-1=0}^{\infty} e^{t(j-1)}\beta_Z^{j-1}$

$= (1-\beta_Z)e^t \sum_{k=0}^{\infty} e^{tk}\beta_Z^{k}$

$= (1-\beta_Z)e^t \sum_{k=0}^{\infty} (e^t\beta_Z)^{k}$

$= \frac{(1-\beta_Z)e^t}{1-\beta_Z e^t}$ provided $e^t\beta_Z < 1$

Result 3 28 Average run length (First non central moment of the run length distribution) – Conditional

$$E(N \mid Z = z) = \frac{1}{1-\beta_Z} = \frac{1}{1-\Phi\left(\frac{z}{\sqrt{m}} + z_{\frac{\alpha}{2}} - \delta\sqrt{n}\right) + \Phi\left(\frac{z}{\sqrt{m}} - z_{\frac{\alpha}{2}} - \delta\sqrt{n}\right)}$$

Proof

In general the k^{th} conditional non central moment $E\left(N^k \mid Z = z\right)$ is the k^{th} derivative of the conditional moment generating function $M_{N|Z}(t)$ evaluated in the point zero i e

$$E\left(N^k \mid Z = z\right) = \frac{d^k}{dt^k} M_{N|Z}(t)|_{t=0}$$

Thus since the average run length is the first non central moment of the run length distribution we need $E\left(N \mid Z = z\right) = \frac{d}{dt} M_{N|Z}(t)|_{t=0}$

Therefore

$$E\left(N \mid Z = z\right)$$

$$= \frac{d}{dt} M_{N|Z}(t)|_{t=0}$$

$$= \frac{-\left(-1-\beta_Z\right)e^t}{\left(-1+\beta_Z e^t\right)^2}|_{t=0}$$

$$= \frac{1}{1-\beta_Z}$$

and it follows that

$$ARL = E(N \mid Z = z) = \frac{1}{1-\beta_Z} = \frac{1}{1-\Phi\left(\frac{z}{\sqrt{m}} + z_{\frac{\alpha}{2}} - \delta\sqrt{n}\right) + \Phi\left(\frac{z}{\sqrt{m}} - z_{\frac{\alpha}{2}} - \delta\sqrt{n}\right)}$$

Result 3 29 Average run length – Unconditional

$$E(N) = \int_{-\infty}^{\infty} \frac{1}{1-\beta_Z}\phi(z)dz$$

Proof

In general the k^{th} unconditional non central moment can be obtained from the k^{th} conditional non central moment i e $E\left(N^k\right) = E_Z\left(E\left(N^k \mid Z = z\right)\right)$

Thus since the unconditional average run length is the first non central moment of the unconditional run length distribution we need to find $E\left(N\right) = E_Z\left(E\left(N \mid Z = z\right)\right)$

Therefore

$E(N)$

$= \int_{-\infty}^{\infty} E(N \mid Z = z)\phi(z)dz$

$= \int_{-\infty}^{\infty} \frac{1}{1-\beta_Z}\phi(z)dz$

Result 3 30 Second non central moment and third non central moment – Conditional

$$E(N^2 \mid Z=z)=\frac{1+\beta_Z}{(1-\beta_Z)^2} \quad \text{and} \quad E(N^3 \mid Z=z)=\frac{1+\beta_Z^2+4\beta_Z}{(1-\beta_Z)^3}$$

Proof

In general the k^{th} conditional non central moment $E\left(N^k \mid Z=z\right)$ is the k^{th} derivative of the conditional moment generating function $M_{N|Z}(t)$ evaluated in the point zero i e

$$E\left(N^k \mid Z=z\right)=\frac{d^k}{dt^k}M_{N|Z}(t)|_{t=0}$$

Therefore

$$E\left(N^2 \mid Z=z\right)$$

$$=\frac{\partial^2}{\partial t^2}M_{N|Z}(t)_{|t=0}$$

$$=\frac{\left(-1-\beta_Z\right)e^t\left(1+\beta_Z e^t\right)}{\left(-1+\beta_Z e^t\right)^3}|_{t=0}$$

$$=\frac{1+\beta_Z}{\left(1-\beta_Z\right)^2}$$

and

$$E(N^3 \mid Z=z)$$

$$=\frac{\partial^3}{\partial t^3}M_{N|Z}(t)_{|t=0}$$

$$=\frac{\left(-1-\beta_Z\right)e^t\left(1+\beta_Z^2 e^{2t}+4\beta_Z e^t\right)}{\left(-1+\beta_Z e^t\right)^4}|_{t=0}$$

$$=\frac{1+\beta_Z^2+4\beta_Z}{(1-\beta_Z)^3}$$

Result 3 31 Second non central moment and Third non central moment – Unconditional

$$E(N^2) = \int_{-\infty}^{\infty} \frac{1+\beta_Z}{(1-\beta_Z)^2}\phi(z)dz \quad \text{and} \quad E(N^3) = \int_{-\infty}^{\infty} \frac{1+\beta_Z^2+4\beta_Z}{(1-\beta_Z)^3}\phi(z)dz$$

Proof

In general the k^{th} unconditional non central moment can be obtained from the k^{th} conditional non central moment i e $E\left(N^k\right) = E_Z\left(E\left(N^k \mid Z = z\right)\right)$

Therefore

$E(N^2)$

$= E_Z(E(N^2 \mid Z = z))$

$= \int_{-\infty}^{\infty} \frac{1+\beta_Z}{(1-\beta_Z)^2}\phi(z)dz$

and

$E(N^3)$

$= E_Z(E(N^3 \mid Z = z))$

$= \int_{-\infty}^{\infty} \frac{1+\beta_Z^2+4\beta_Z}{(1-\beta_Z)^3}\phi(z)dz$

Result 3 32 Variance and Standard deviation of the run length distribution – Conditional

$$\text{var}(N \mid Z = z) = \frac{\beta_Z}{(1-\beta_Z)^2} \quad \text{and} \quad stdev(N \mid Z = z) = \frac{\sqrt{\beta_Z}}{1-\beta_Z}$$

Proof

The variance of the conditional run length distribution and the standard deviation of the conditional run length distribution are obtained from their definitions i e

$$\text{var}(N \mid Z = z) = E(N - E(N \mid Z = z) \mid Z = z)^2 = E(N^2 \mid Z = z) - E(N \mid Z = z)^2$$

and

$$stdev(N \mid Z = z) = \sqrt{\text{var}(N \mid Z = z)}$$

Thus substituting the espressions for the first non central moment $E(N \mid Z = z)$ and the second non central moment $E(N^2 \mid Z = z)$ of the conditional run length distribution we obtain

$$\text{var}(N \mid Z = z) = \frac{1+\beta_Z}{(1-\beta_Z)^2} - \frac{1}{(1-\beta_Z)^2} = \frac{\beta_Z}{(1-\beta_Z)^2}$$

and

$$stdev(N \mid Z = z) = \frac{\sqrt{\beta_Z}}{1-\beta_Z}$$

Result 3 33 Variance and Standard deviation of the run length distribution – Unconditional

$$\text{var}(N) = \int_{-\infty}^{\infty} \frac{\beta}{(1-\beta)^2}\phi(z)dz + \int_{-\infty}^{\infty}\left(\frac{1}{1-\beta}\right)^2 \phi(z)dz - \left(\int_{-\infty}^{\infty}\frac{1}{1-\beta}\phi(z)dz\right)^2$$

or

$$\text{var}(N) = \int_{-\infty}^{\infty} \frac{1+\beta}{(1-\beta)^2}\phi(z)dz - \left(\int_{-\infty}^{\infty}\frac{1}{1-\beta}\phi(z)dz\right)^2$$

Proof

Expression 1

Using the fact that

$$E_Z\left(\text{var}(N \mid Z=z)\right) = \text{var}(N) - \text{var}_Z\left(E(N \mid Z=z)\right)$$

we can find the unconditional variance i e var(N) and subsequently the unconditional standard deviation i e *stdev*(N) of the run length random variable by using the conditional expected value and the conditional variance of N given $Z=z$ i e using $\text{var}(N \mid Z=z)$ and $E(N \mid Z=z)$

Thus solving for $\text{var}(N)$ in the above expression we have

$$\text{var}(N) = E_Z\left(\text{var}(N \mid Z=z)\right) + \text{var}_Z\left(E(N \mid Z=z)\right)$$

Now

$$\text{var}(N \mid Z=z) = \frac{\beta}{(1-\beta)^2} \qquad \text{and} \qquad E(N \mid Z=z) = \frac{1}{1-\beta}$$

so that

$$\begin{aligned}\text{var}_Z\left(E(N \mid Z=z)\right) &= E_Z\left(\left(E(N \mid Z=z)\right)^2\right) - \left(E_Z\left(E(N \mid Z=z)\right)\right)^2 \\ &= E_Z\left(\left(\frac{1}{1-\beta}\right)^2\right) - \left(E_Z\left(\frac{1}{1-\beta}\right)\right)^2\end{aligned}$$

and by substitution we have that

$$\text{var}(N) = E_Z\left(\frac{\beta}{(1-\beta)^2}\right) + E_Z\left(\left(\frac{1}{1-\beta}\right)^2\right) - \left(E_Z\left(\frac{1}{1-\beta}\right)\right)^2$$

Re writing this expression in terms of integrals we obtain the unconditional variance

$$\text{var}(N) = \int_{-\infty}^{\infty} \frac{\beta}{(1-\beta)^2}\phi(z)dz + \int_{-\infty}^{\infty}\left(\frac{1}{1-\beta}\right)^2 \phi(z)dz - \left(\int_{-\infty}^{\infty}\frac{1}{1-\beta}\phi(z)dz\right)^2$$

Expression 2

Another route in obtaining the unconditional variance of the run length random variable is by making use of the first and second non central moments of the conditional run length distribution i e using $E(N \mid Z = z)$ and $E(N^2 \mid Z = z)$

We have that

$$\text{var}(N) = E(N^2) - E(N)^2$$

but

$$E(N^2) = E_Z\left(E(N^2 \mid Z = z)\right) \quad \text{and} \quad E(N) = E_Z\left(E(N \mid Z = z)\right)$$

so that

$$\text{var}(N) = E_Z\left(E(N^2 \mid Z = z)\right) - E_Z\left(E(N \mid Z = z)\right)^2$$

Furthermore from the properties of the geometric distribution we have that

$$E(N^2 \mid Z = z) = \frac{1+\beta}{(1-\beta)^2} \quad \text{and} \quad E(N \mid Z = z) = \frac{1}{1-\beta}$$

so that a second expression for the unconditional variance of the run length random variable is given by

$$\text{var}(N) = \int_{-\infty}^{\infty}\frac{1+\beta}{(1-\beta)^2}\phi(z)dz - \left(\int_{-\infty}^{\infty}\frac{1}{1-\beta}\phi(z)dz\right)^2$$

Result 3 34 Coefficient of skewness of the run length distribution – Conditional

$$skew(N \mid Z = z) = \frac{1+\beta_Z}{\sqrt{\beta_Z}}$$

Proof

The conditional coefficient of skewness is defined as

$$skew(N \mid Z = z) = \frac{E(N - E(N \mid Z = z) \mid Z = z)^3}{\operatorname{var}(N \mid Z = z)^{\frac{3}{2}}}$$

which can be simplified to

$$skew(N \mid Z = z) = \frac{E\left(N^3 \mid Z = z\right) - 3E\left(N^2 \mid Z = z\right)E\left(N \mid Z = z\right) + 2E\left(N \mid Z = z\right)^3}{\left(\operatorname{var}(N \mid Z = z)\right)^{\frac{3}{2}}}$$

Substituting the expressions for the first non central moment $E(N \mid Z = z)$ the second non central moment $E(N^2 \mid Z = z)$ and the third non central moment $E(N^3 \mid Z = z)$ of the conditional run length distribution as well as the expression for the conditional variance $\operatorname{var}(N \mid Z = z)$ the expression for the conditional coefficient of skewness simplifies to

$$skew(N \mid Z = z) = \frac{1+\beta_Z}{\sqrt{\beta_Z}}$$

Result 3 35 Coefficient of skewness of the run length distribution – Unconditional

$$skew(N)=\frac{\int_{-\infty}^{\infty}\frac{1+\beta^{2}+4\beta}{(1-\beta)^{3}}\phi(z)dz-3\left\{\int_{-\infty}^{\infty}\frac{1+\beta}{(1-\beta)^{2}}\phi(z)dz\right\}\left\{\int_{-\infty}^{\infty}\frac{1}{1-\beta}\phi(z)dz\right\}+2\left(\int_{-\infty}^{\infty}\frac{1}{1-\beta}\phi(z)dz\right)^{3}}{\left(\mathrm{var}(N)\right)^{\frac{3}{2}}}$$

Proof

To obtain the unconditional coefficient of skewness of the run length random variable i e

$$skew(N)=\frac{E(N-E(N))^{3}}{\mathrm{var}(N)^{\frac{3}{2}}}=\frac{E\left(N^{3}\right)-3E\left(N^{2}\right)E\left(N\right)+2E\left(N\right)^{3}}{\left(\mathrm{var}(N)\right)^{\frac{3}{2}}}$$

one uses the first the second and the third non central moments of the conditional run length distribution i e one uses $E\left(N\,|\,Z=z\right)$ $E\left(N^{2}\,|\,Z=z\right)$ and $E\left(N^{3}\,|\,Z=z\right)$ as well as the fact that $E(N^{k})=E_{Z}(E(N^{k}\,|\,Z=z))$

Thus an expression for the unconditional coefficient of skewness of the run length random variable is given by

$$skew(N)=\frac{\int_{-\infty}^{\infty}\frac{1+\beta^{2}+4\beta}{(1-\beta)^{3}}\phi(z)dz-3\left\{\int_{-\infty}^{\infty}\frac{1+\beta}{(1-\beta)^{2}}\phi(z)dz\right\}\left\{\int_{-\infty}^{\infty}\frac{1}{1-\beta}\phi(z)dz\right\}+2\left(\int_{-\infty}^{\infty}\frac{1}{1-\beta}\phi(z)dz\right)^{3}}{\left(\mathrm{var}(N)\right)^{\frac{3}{2}}}$$

3 5 4 Appendix 3C

Case KU

Assume that the in control process mean is *known* whereas the in control process standard deviation is *unknown* and denoted by μ_0 and σ respectively Furthermore assume that the process follows a normal distribution so that $X_{ij} \sim N\left(\mu_0 \; \sigma^2\right)$ for $i = 1\ 2\ 3$ and $j = 1\ 2\ 3$ n In addition assume that we estimate the unknown in control process standard deviation using m reference samples (each of size n) from Phase 1 and that the point estimator used is S with an observed value of S denoted by s

Since the estimation of the unknown process parameter (which in this case is the in control process standard deviation) occurs in Phase 1 the formulae and/or expressions that are derived are only applicable in Phase 2

Result 3 36 Control limits

$$\widehat{LCL} = \mu_0 - z_{\frac{\alpha}{2}} \frac{S}{\sqrt{n}} \quad \text{and} \quad \widehat{UCL} = \mu_0 + z_{\frac{\alpha}{2}} \frac{S}{\sqrt{n}}$$

Proof

From Result 3 1 (in Case KK) we have that

$$LCL = \mu_0 - z_{\frac{\alpha}{2}} \frac{\sigma_0}{\sqrt{n}} \quad \text{and} \quad UCL = \mu_0 + z_{\frac{\alpha}{2}} \frac{\sigma_0}{\sqrt{n}}$$

However since the in control process standard deviation is unknown we subtitute σ_0 with its point estimator S and obtain

$$\widehat{LCL} = \mu_0 - z_{\frac{\alpha}{2}} \frac{S}{\sqrt{n}} \quad \text{and} \quad \widehat{UCL} = \mu_0 + z_{\frac{\alpha}{2}} \frac{S}{\sqrt{n}}$$

Result 3 37 Non sıgnallıng event

$$\frac{\sqrt{n}\left|\bar{X} - \mu_0\right|}{S} < z_{\frac{\alpha}{2}} \text{ for } \imath = m+1 \; m+2 \; m+3$$

Proof

$$\bar{X} < \widehat{UCL} \text{ and } \bar{X} > \widehat{LCL}$$

$$\bar{X} < \mu_0 + z_{\frac{\alpha}{2}} \frac{S}{\sqrt{n}} \text{ and } \bar{X} > \mu_0 - z_{\frac{\alpha}{2}} \frac{S}{\sqrt{n}}$$

$$\bar{X} - \mu_0 < z_{\frac{\alpha}{2}} \frac{S}{\sqrt{n}} \text{ and } \bar{X} - \mu_0 > -z_{\frac{\alpha}{2}} \frac{S}{\sqrt{n}}$$

$$\bar{X} - \mu_0 < z_{\frac{\alpha}{2}} \frac{S}{\sqrt{n}} \text{ and } -(\bar{X} - \mu_0) < z_{\frac{\alpha}{2}} \frac{S}{\sqrt{n}}$$

$$\left|\bar{X} - \mu_0\right| < z_{\frac{\alpha}{2}} \frac{S}{\sqrt{n}}$$

$$\frac{\sqrt{n}\left|\bar{X} - \mu_0\right|}{S} < z_{\frac{\alpha}{2}}$$

Result 3 38 Sıgnalıng event

$$\frac{\sqrt{n}\left|\bar{X} - \mu_0\right|}{S} \geq z_{\frac{\alpha}{2}} \text{ for } \imath = m+1\ m+2\ m+3$$

Proof

$$\bar{X} \geq \widehat{UCL} \text{ or } \bar{X} \leq \widehat{LCL}$$

$$\bar{X} \geq \mu_0 + z_{\frac{\alpha}{2}}\frac{S}{\sqrt{n}} \text{ or } \bar{X} \leq \mu_0 - z_{\frac{\alpha}{2}}\frac{S}{\sqrt{n}}$$

$$\bar{X} - \mu_0 \geq z_{\frac{\alpha}{2}}\frac{S}{\sqrt{n}} \text{ or } \bar{X} - \mu_0 \leq -z_{\frac{\alpha}{2}}\frac{S}{\sqrt{n}}$$

$$\bar{X} - \mu_0 \geq z_{\frac{\alpha}{2}}\frac{S}{\sqrt{n}} \text{ or } -(\bar{X} - \mu_0) \geq z_{\frac{\alpha}{2}}\frac{S}{\sqrt{n}}$$

$$\left|\bar{X} - \mu_0\right| \geq z_{\frac{\alpha}{2}}\frac{S}{\sqrt{n}}$$

$$\frac{\sqrt{n}\left|\bar{X} - \mu_0\right|}{S} \geq z_{\frac{\alpha}{2}}$$

Result 3 39 Probability of no signal – Conditional

$$\beta(\alpha\ \delta\ m\ n\ y) = \Phi\left(z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}} - \delta\sqrt{n}\right) - \Phi\left(-z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}} - \delta\sqrt{n}\right)$$

Proof

Let $P_C(\,)$ denote a conditional probability that is we condition on a sustained shift in the process mean as well as on an observed value of the random variable S i e we condition on the value s Thus

P_C(No Signal)

$= P(\widehat{LCL} < \bar{X} < \widehat{UCL} \mid \text{A shift in the process mean occured } S = s)$

$$= P\left(\mu_0 - z_{\frac{\alpha}{2}}\frac{s}{\sqrt{n}} < \bar{X} < \mu_0 + z_{\frac{\alpha}{2}}\frac{s}{\sqrt{n}} \mid \mu_1 = \mu_0 + \delta\sigma\ S = s\right)$$

$$= P\left(\frac{\mu_0 - z_{\frac{\alpha}{2}}\frac{s}{\sqrt{n}} - (\mu_0 + \delta\sigma)}{\frac{\sigma}{\sqrt{n}}} < \frac{\bar{X} - (\mu_0 + \delta\sigma)}{\frac{\sigma}{\sqrt{n}}} < \frac{\mu_0 + z_{\frac{\alpha}{2}}\frac{s}{\sqrt{n}} - (\mu_0 + \delta\sigma)}{\frac{\sigma}{\sqrt{n}}} \mid S = s\right)$$

$$= P\left(-z_{\frac{\alpha}{2}}\frac{s}{\sigma} - \delta\sqrt{n} < Z < z_{\frac{\alpha}{2}}\frac{s}{\sigma} - \delta\sqrt{n} \mid S = s\right)$$

$$= P\left(-z_{\frac{\alpha}{2}}\frac{1}{\sqrt{\nu}}\sqrt{\frac{\nu s^2}{\sigma^2}} - \delta\sqrt{n} < Z < z_{\frac{\alpha}{2}}\frac{1}{\sqrt{\nu}}\sqrt{\frac{\nu s^2}{\sigma^2}} - \delta\sqrt{n} \mid S = s\right)$$

$$= P\left(-z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}} - \delta\sqrt{n} < Z < z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}} - \delta\sqrt{n} \mid \chi^2 = y\right)$$

$$= \Phi\left(z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}} - \delta\sqrt{n}\right) - \Phi\left(-z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}} - \delta\sqrt{n}\right)$$

$= \beta(\alpha\ \delta\ m\ n\ y)$ say

For simplicity denote $\beta(\alpha\ \delta\ m\ n\ y)$ by β_χ In this case the subscript χ^2 indicates that the expression is a conditional expression i e conditional on an observed value of a chi square random variable $Y = \chi^2 = \frac{\nu s^2}{\sigma^2}$

Result 3 40 Probability of no signal – Unconditional

$$P(\text{No Signal}) = \int_0^{\infty}\left(\Phi\left(z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}} - \delta\sqrt{n}\right) - \Phi\left(-z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}} - \delta\sqrt{n}\right)\right) f_{\chi}(y)dy$$

Proof

$$P(\text{No Signal})$$

$$= \int_0^{\infty} P_C(\text{No Signal}) f_{\chi}(y)dy$$

$$= \int_0^{\infty}\left(\Phi\left(z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}} - \delta\sqrt{n}\right) - \Phi\left(-z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}} - \delta\sqrt{n}\right)\right) f_{\chi}(y)dy$$

Result 3 41 Probabılıty of a sıgnal – Condıtıonal

$$1-\beta(\alpha\ \delta\ m\ n\ y)=1-\Phi\left(z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}}-\delta\sqrt{n}\right)+\Phi\left(-z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}}-\delta\sqrt{n}\right)$$

Proof

P_C(Sıgnal)

$=1-P_C$(No Sıgnal)

$$=1-\left(\Phi\left(z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}}-\delta\sqrt{n}\right)-\Phi\left(-z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}}-\delta\sqrt{n}\right)\right)$$

$$=1-\Phi\left(z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}}-\delta\sqrt{n}\right)+\Phi\left(-z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}}-\delta\sqrt{n}\right)$$

$=1-\beta_{\chi}$

Result 3 42 Probability of a signal – Unconditional

$$P(\text{Signal}) = \int_0^\infty \left(1 - \Phi\left(z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}} - \delta\sqrt{n}\right) + \Phi\left(-z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}} - \delta\sqrt{n}\right)\right) f_\chi(y)dy$$

Proof

$P(\text{Signal})$

$$= \int_0^\infty P_C(\text{Signal}) f_\chi(y)dy$$

$$= \int_0^\infty \left(1 - \Phi\left(z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}} - \delta\sqrt{n}\right) + \Phi\left(-z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}} - \delta\sqrt{n}\right)\right) f_\chi(y)dy$$

Result 3 43 Probabılıty of a false alarm – Condıtıonal

$$P_C(\text{False Alarm}) = 1 - \Phi\left(z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}}\right) + \Phi\left(-z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}}\right)$$

Proof

$$P_C\left(\text{Sıgnal} \mid \text{In control}\right)$$

$$= 1 - P_C\left(\text{No Sıgnal} \mid \text{In control}\right)$$

$$= 1 - P_C\left(\widehat{LCL} < \bar{X} < \widehat{UCL} \mid \text{In control}\right)$$

$$= 1 - P_C\left(\widehat{LCL} < \bar{X} < \widehat{UCL} \mid \mu_1 = \mu_0 \; S = s\right)$$

$$= 1 - P\left(\mu_0 - z_{\frac{\alpha}{2}}\frac{s}{\sqrt{n}} < \bar{X} < \mu_0 + z_{\frac{\alpha}{2}}\frac{s}{\sqrt{n}} \mid \mu_1 = \mu_0 \; S = s\right)$$

$$= 1 - P\left(\frac{\mu_0 - z_{\frac{\alpha}{2}}\frac{s}{\sqrt{n}} - \mu_0}{\frac{\sigma}{\sqrt{n}}} < \frac{\bar{X} - \mu_0}{\frac{\sigma}{\sqrt{n}}} < \frac{\mu_0 + z_{\frac{\alpha}{2}}\frac{s}{\sqrt{n}} - \mu_0}{\frac{\sigma}{\sqrt{n}}} \mid S = s\right)$$

$$= 1 - P\left(-z_{\frac{\alpha}{2}}\frac{s}{\sqrt{\sigma}} < Z < z_{\frac{\alpha}{2}}\frac{s}{\sqrt{\sigma}} \mid S = s\right)$$

$$= 1 - P\left(-z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}} < Z < z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}} \mid \chi^2 = y\right)$$

$$= 1 - \left(\Phi\left(z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}}\right) - \Phi\left(-z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}}\right)\right)$$

$$= 1 - \Phi\left(z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}}\right) + \Phi\left(-z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}}\right)$$

$$= 1 - \beta(\alpha \; 0 \; m \; n \; y) \text{ say}$$

This expression could also have been obtained by simply setting $\delta = 0$ in the expression for the conditional probability of a signal in Result 3 41 given that a shift in the process mean occurred Therefore Result 3 43 is the same as Result 3 41 i e

$$1-\Phi\left(z_{\frac{\alpha}{2}}\sqrt{\frac{y}{v}}-\delta\sqrt{n}\right)+\Phi\left(-z_{\frac{\alpha}{2}}\sqrt{\frac{y}{v}}-\delta\sqrt{n}\right) \text{ with } \delta = 0$$

Result 3 44 Probability of a false alarm – Unconditional

$$P(\text{Signal} \mid \text{In control}) = \int_0^\infty \left(1 - \Phi\left(z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}}\right) + \Phi\left(-z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}}\right)\right) f_{\chi}(y)dy$$

or

$$P(\text{False Alarm}) = 1 - T\left(z_{\frac{\alpha}{2}}\right) + T\left(-z_{\frac{\alpha}{2}}\right)$$

Proof

Expression 1

$P(\text{Signal} \mid \text{In control})$

$= \int_0^\infty P_C(\text{Signal} \mid \text{In control}) f_{\chi}(y)dy$

$= \int_0^\infty \left(1 - \Phi\left(z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}}\right) + \Phi\left(-z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}}\right)\right) f_{\chi}(y)dy$

Expression 2

$P(\text{No Signal} \mid \text{In control})$

$= P(\widehat{LCL} < \bar{X} < \widehat{UCL} \mid \text{No shift in the process mean occured})$

$= P\left(\mu_0 - z_{\frac{\alpha}{2}}\frac{S}{\sqrt{n}} < \bar{X} < \mu_0 + z_{\frac{\alpha}{2}}\frac{S}{\sqrt{n}} \mid \mu_1 = \mu_0\right)$

$= P\left(\frac{\mu_0 - z_{\frac{\alpha}{2}}\frac{S}{\sqrt{n}} - \mu_0}{\frac{S}{\sqrt{n}}} < \frac{\bar{X} - \mu_0}{\frac{S}{\sqrt{n}}} < \frac{\mu_0 + z_{\frac{\alpha}{2}}\frac{S}{\sqrt{n}} - \mu_0}{\frac{S}{\sqrt{n}}}\right)$

$= P\left(-z_{\frac{\alpha}{2}} < t < z_{\frac{\alpha}{2}}\right)$ where t denotes a Students t – random variable with ν degrees of freedom

$= T\left(z_{\frac{\alpha}{2}}\right) - T\left(-z_{\frac{\alpha}{2}}\right)$ where T denotes the c d f of a Students t – random variable with ν degrees of freedom

Therefore the second unconditional expression for the probability of a false alarm is

$$P(\text{False Alarm}) = 1 - T\left(z_{\frac{\alpha}{2}}\right) + T\left(-z_{\frac{\alpha}{2}}\right)$$

Result 3 45 Run length distribution – Conditional

$$P(N=j\mid\chi^2=y)=\beta_\chi^{j-1}(1-\beta_\chi) \text{ for } j=1\ 2\ 3$$

Proof

Assume that at each point in time we perform the same experiment by obtaining a random sample of n observations from the process output calculating the sample mean $\overline{X}$ and then compare $\overline{X}$ with $\widehat{LCL}=\mu_0-z_{\frac{\alpha}{2}}\frac{S}{\sqrt{n}}$ and $\widehat{UCL}=\mu_0+z_{\frac{\alpha}{2}}\frac{S}{\sqrt{n}}$ to see whether $\overline{X}$ falls between or outside $\widehat{UCL}$ and/or $\widehat{LCL}$

In addition assume that we condition on an observed value of S denoted s (or y if S is written in its canonical form namely χ^2) so that we actually compare $\overline{X}$ with $\widehat{LCL}=\mu_0-z_{\frac{\alpha}{2}}\frac{s}{\sqrt{n}}$ and $\widehat{UCL}=\mu_0+z_{\frac{\alpha}{2}}\frac{s}{\sqrt{n}}$ that is both control limits are constants and then have that

i All trials i e all the experiments at the different points in time are independent

ii At each stage of sampling we can either observed a success (S) or a failure (F) that is $\overline{X}$ can either fall between $\widehat{UCL}$ and $\widehat{LCL}$ (and give no signal) or outside $\widehat{UCL}$ or $\widehat{LCL}$ with both endpoints included (and give a signal)

iii The conditional probability of a failure (no signal) and the conditional probability of a success (signal) are the same on each trial and denoted by $P(F)=\beta_\chi$ and $P(S)=1-\beta_\chi$ respectively

iv We repeat the experiment until we obtain the first success

Thus the sample space **S** is **S** $=\{S\ FS\ FFS\ FFFS\ FFFFS\quad\}$ and if we consider an element of **S** with $j-1$ failures and then a success we find that

$$P(FF\ \ FS)=P(F)P(F)\ \ P(F)P(S)=P(F)^{j-1}P(S) \text{ for } j=1\ 2\ 3$$

Consequently if N denotes the number of trials until we observe a success for the first time we obtain

$$P(N=j\mid\chi^2=y)=\beta_\chi^{j-1}(1-\beta_\chi) \text{ for } j=1\ 2\ 3$$

which is a geometric distribution with probability of success given by $1-\beta_\chi$ so that

$N \mid \chi^2 = y \sim Geo\left(1-\beta_\chi\right)$

Consequently the cumulative distribution function (cdf) is found from

$$P(N \leq j \mid \chi^2 = y) = \sum_{k=1}^{j} \beta_\chi^{k-1}(1-\beta_\chi) = 1-\beta_\chi^{j}$$

and subsequently

$$P(N \geq j \mid \chi^2 = y) = \beta_\chi^{j-1}$$

Result 3 46 Run length distribution – Unconditional

$$P(N = j) = \int_0^\infty \beta_\chi^{j-1}(1-\beta_\chi) f_\chi(y)dy \quad \text{for } j = 1\ 2\ 3$$

Proof

The conditional run length distribution (given in Result 3 45) is denoted by $P(N = j \mid \chi^2 = y)$ whereas the joint distribution of the run length random variable N and the random variable χ^2 is denoted by $P(N = j\ \chi^2 = y)$ The marginal distributions of these two random variables are denoted by $P(N = j)$ and $P(\chi^2 = y)$ respectively

The unconditional (or marginal) run length distribution of N i e $P(N = j)$ given the fact that the conditional run length distribution i e $P(N = j \mid \chi^2 = y)$ and the marginal distribution of the random variable χ^2 are known can be found by using the fact that $P(N = j \mid \chi^2 = y) = \dfrac{P(N = j\ \chi^2 = y)}{P(\chi^2 = y)}$ or alternatively the fact that $P(N = j\ \chi^2 = y) = P(N = j \mid \chi^2 = y)P(\chi^2 = y)$

Therefore to find the marginal distribution of N (also called the unconditional run length distribution) requires integrating the joint distribution of N and χ^2 over the domain of the random variable χ^2 i e

$$P(N = j) = \int_0^\infty P(N = j\ \chi^2 = y)dz = \int_0^\infty P(N = j \mid \chi^2 = y)P(\chi^2 = y)dz$$

Thus the unconditional probability mass function (pmf) is

$$P(N = j)$$

$$= \int_0^\infty P(N = j \mid \chi^2 = y) f_\chi(y)dy$$

$$= \int_0^\infty \beta_\chi^{j-1}(1-\beta_\chi) f_\chi(y)dy \quad \text{for } j = 1\ 2\ 3$$

whereas the unconditional cumulative distribution function (cdf) is

$$P(N \le j) = \sum_{k=1}^{j} P(N = k) = \sum_{k=1}^{j}\left(\int_0^\infty \beta_\chi^{j-1}(1-\beta_\chi) f_\chi(y)dy\right) = \int_0^\infty \left(1-\beta_\chi^{j}\right) f_\chi(y)dy$$

Another useful way of writing the unconditional run length distribution is as

$P(N \ge j) = \int_0^\infty P(N \ge j \mid \chi^2 = y) f_\chi(y)dy = \int_0^\infty \beta_\chi^{j-1} f_\chi(y)dy$ and this expression can conveniently be written as $I_2(j-1\ m\ n\ \delta\ \alpha)$

Result 3 47 Quantile function – Conditional

$$Q_{N|\chi}(p) = \inf\left\{ j \text{ an integer } P(N \leq j \mid \chi^2 = y) \geq p \right\} = \inf\left\{ j \text{ an integer } j \geq \frac{\ln(1-p)}{\ln \beta_\chi} \right\}$$

Proof

Generally the quantile function is defined as

$$Q_N(p) = \inf\left\{ j \text{ an integer } F_N(j) \geq p \right\}$$

where $F_N(j)$ is the cdf of a distribution and p denotes a particular percentile of the distribution

Thus substituting the conditional cdf of the run length distrbution (given in Result 3 45) and solving for j the quantile function for the conditional run length distribution is

$$Q_{N|\chi}(p) = \inf\left\{ j \text{ an integer } P(N \leq j \mid \chi^2 = y) \geq p \right\} = \inf\left\{ j \text{ an integer } j \geq \frac{\ln(1-p)}{\ln \beta_\chi} \right\} \text{ with } 0 < p < 1$$

Result 3 48 Quantile function – Unconditional

$$Q_N(p) = \inf\{j \text{ an integer } P(N \le j) \ge p\} = \inf\left\{j \text{ an integer } \sum_{k=1}^{j}\left(\int_0^\infty \beta_\chi^{k-1}(1-\beta_\chi)f_\chi(y)dy\right) \ge p\right\}$$

Proof

The unconditional quantile function when the unknown in control process standard deviation is estimated is

$$\begin{aligned} Q_N(p) &= \inf\{j \text{ an integer } P(N \le j) \ge p\} \\ &= \inf\left\{j \text{ an integer } \sum_{k=1}^{j} P(N=k) \ge p\right\} \qquad \text{with } 0 < p < 1 \\ &= \inf\left\{j \text{ an integer } \sum_{k=1}^{j}\left(\int_0^\infty \beta_\chi^{k-1}(1-\beta_\chi)f_\chi(y)dy\right) \ge p\right\} \end{aligned}$$

$P(N \le j)$ denoting the uncondtional cumulative distribution function (given in Result 3 46) and $P(N = j)$ denoting the uncondtional probability mass function (also given in Result 3 46) This formula can not be simplified any further

Result 3 49 Moment generating function – Conditional

$$M_{N|\chi}(t) = \frac{(1-\beta_\chi)e^t}{1-\beta_\chi e^t}$$

Proof

$M_{N|\chi}(t)$

$$=\sum_{j=1}^{\infty} e^{tj}\beta_\chi^{j-1}\left(1-\beta_\chi\right)$$

$$=\left(1-\beta_Z\right)e^t\sum_{j=1}^{\infty} e^{t(j-1)}\beta_\chi^{j-1}$$

$$=\left(1-\beta_\chi\right)e^t\sum_{j-1=0}^{\infty} e^{t(j-1)}\beta_\chi^{j-1}$$

$$=\left(1-\beta_\chi\right)e^t\sum_{k=0}^{\infty} e^{tk}\beta_\chi^{k}$$

$$=\left(1-\beta_\chi\right)e^t\sum_{k=0}^{\infty}\left(e^t\beta_\chi\right)^k$$

$$=\frac{\left(1-\beta_\chi\right)e^t}{1-\beta_\chi e^t} \text{ provided } e^t\beta_\chi < 1$$

Result 3 50 Average run length (First non central moment of the run length distribution) – Conditional

$$E(N \mid \chi^2 = y) = \frac{1}{1-\beta_\chi} = \frac{1}{1-\Phi\left(z_{\frac{\alpha}{2}}\sqrt{\frac{y}{v}}-\delta\sqrt{n}\right)+\Phi\left(-z_{\frac{\alpha}{2}}\sqrt{\frac{y}{v}}-\delta\sqrt{n}\right)}$$

Proof

In general the k^{th} conditional non central moment $E\left(N^k \mid \chi^2 = y\right)$ is the k^{th} derivative of the conditional moment generating function $M_{N|\chi}(t)$ evaluated in the point zero i e

$$E\left(N^k \mid \chi^2 = y\right) = \frac{d^k}{dt^k} M_{N|\chi}(t)|_{t\ 0}$$

Thus since the average run length is the first non central moment of the run length distribution we need $E\left(N \mid \chi^2 = y\right) = \frac{d}{dt} M_{N|\chi}(t)|_{t\ 0}$

Therefore

$$E\left(N \mid \chi^2 = y\right)$$

$$= \frac{d}{dt} M_{N|\chi}(t)|_{t\ 0}$$

$$= \frac{-\left(-1-\beta_\chi\right)e^t}{\left(-1+\beta_\chi e^t\right)^2}|_{t\ 0}$$

$$= \frac{1}{1-\beta_\chi}$$

and it follows that

$$E(N \mid \chi^2 = y) = E\left(N \mid \chi^2 = y\right) = \frac{1}{1-\beta_\chi} = \frac{1}{1-\Phi\left(z_{\frac{\alpha}{2}}\sqrt{\frac{y}{v}}-\delta\sqrt{n}\right)+\Phi\left(-z_{\frac{\alpha}{2}}\sqrt{\frac{y}{v}}-\delta\sqrt{n}\right)}$$

Result 3 51 Average run length – Unconditional

$$E(N) = \int_0^\infty \frac{1}{1-\beta_\chi} f_\chi(y)dy$$

Proof

In general the k^{th} unconditional non central moment can be obtained from the k^{th} conditional non central moment i e $E\left(N^k\right) = E_\chi\left(E\left(N^k \mid \chi^2 = y\right)\right)$

Thus since the unconditional average run length is the first non central moment of the unconditional run length distribution we need to find $E\left(N\right) = E_\chi\left(E\left(N \mid \chi^2 = y\right)\right)$

Therefore

$E(N)$

$= \int_0^\infty E(N \mid \chi^2 = y) f_\chi(y)dy$

$= \int_0^\infty \frac{1}{1-\beta_\chi} f_\chi(y)dy$

Result 3 52 Second non central moment and third non central moment – Conditional

$$E(N^2 \mid \chi^2 = y) = \frac{1+\beta_\chi}{(1-\beta_\chi)^2} \quad \text{and} \quad E(N^3 \mid \chi^2 = y) = \frac{1+\beta_\chi^2 + 4\beta_\chi}{(1-\beta_\chi)^3}$$

Proof

In general the k^{th} conditional non central moment $E\left(N^k \mid \chi^2 = y\right)$ is the k^{th} derivative of the conditional moment generating function $M_{N|\chi}(t)$ evaluated in the point zero i e

$$E\left(N^k \mid \chi^2 = y\right) = \frac{d^k}{dt^k} M_{N|\chi}(t)|_{t=0}$$

Therefore

$$E\left(N^2 \mid \chi^2 = y\right)$$

$$= \frac{\partial^2}{\partial t^2} M_{N|\chi}(t)_{|t=0}$$

$$= \frac{\left(-1-\beta_\chi\right)e^t\left(1+\beta_\chi e^t\right)}{\left(-1+\beta_\chi e^t\right)^3}|_{t=0}$$

$$= \frac{1+\beta_\chi}{\left(1-\beta_\chi\right)^2}$$

and

$$E(N^3 \mid \chi^2 = y)$$

$$= \frac{\partial^3}{\partial t^3} M_{N|\chi}(t)_{|t=0}$$

$$= \frac{\left(-1-\beta_\chi\right)e^t\left(1+\beta_\chi^2 e^{2t} + 4\beta_\chi e^t\right)}{\left(-1+\beta_\chi e^t\right)^4}|_{t=0}$$

$$= \frac{1+\beta_\chi^2 + 4\beta_\chi}{(1-\beta_\chi)^3}$$

Result 3 53 Second non central moment and Third non central moment – Unconditional

$$E(N^2) = \int_0^\infty \frac{1+\beta_\chi}{(1-\beta_\chi)^2} f_\chi(y)dy \quad \text{and} \quad E(N^3) = \int_0^\infty \frac{1+\beta_\chi^2+4\beta_\chi}{(1-\beta_\chi)^3} f_\chi(y)dy$$

Proof

In general the k^{th} unconditional non central moment can be obtained from the k^{th} conditional non central moment i e $E\left(N^k\right) = E_\chi\left(E\left(N^k \mid \chi^2 = y\right)\right)$

Therefore

$$E(N^2)$$

$$= E_\chi(E(N^2 \mid \chi^2 = y))$$

$$= \int_0^\infty \frac{1+\beta_\chi}{(1-\beta_\chi)^2} f_\chi(y)dy$$

and

$$E(N^3)$$

$$= E_\chi(E(N^3 \mid \chi^2 = y))$$

$$= \int_0^\infty \frac{1+\beta_\chi^2+4\beta_\chi}{(1-\beta_\chi)^3} f_\chi(y)dy$$

Result 3 54 Variance and Standard deviation of the run length distribution – Conditional

$$\mathrm{var}(N \mid \chi^2 = y) = \frac{\beta_\chi}{(1-\beta_\chi)^2} \quad \text{and} \quad stdev(N \mid \chi^2 = y) = \frac{\sqrt{\beta_\chi}}{1-\beta_\chi}$$

Proof

The variance of the conditional run length distribution and the standard deviation of the conditional run length distribution are obtained from their definitions i e

$$\mathrm{var}(N \mid \chi^2 = y) = E(N - E(N \mid \chi^2 = y) \mid \chi^2 = y)^2 = E(N^2 \mid \chi^2 = y) - E(N \mid \chi^2 = y)^2$$

and

$$stdev(N \mid \chi^2 = y) = \sqrt{\mathrm{var}(N \mid \chi^2 = y)}$$

Thus substituting the expressions for the first non central moment $E(N \mid \chi^2 = y)$ and the second non central moment $E(N^2 \mid \chi^2 = y)$ of the conditional run length distribution we obtain

$$\mathrm{var}(N \mid \chi^2 = y) = \frac{1+\beta_\chi}{\left(1-\beta_\chi\right)^2} - \frac{1}{\left(1-\beta_\chi\right)^2} = \frac{\beta_\chi}{(1-\beta_\chi)^2}$$

and

$$stdev(N \mid \chi^2 = y) = \frac{\sqrt{\beta_\chi}}{1-\beta_\chi}$$

Result 3 55 Variance and Standard deviation of the run length distribution – Unconditional

$$\text{var}(N) = \int_0^\infty \frac{\beta_\chi}{\left(1-\beta_\chi\right)^2} f_\chi(y)dy + \int_0^\infty \left(\frac{1}{1-\beta_\chi}\right)^2 f_\chi(y)dy - \left(\int_0^\infty \frac{1}{1-\beta_\chi} f_\chi(y)dy\right)^2$$

or

$$\text{var}(N) = \int_0^\infty \frac{1+\beta_\chi}{(1-\beta_\chi)^2} f_\chi(y)dy - \left(\int_0^\infty \frac{1}{1-\beta_\chi} f_\chi(y)dy\right)^2$$

Proof

Expression 1

Using the fact that

$$E_\chi\left(\text{var}\left(N \mid \chi^2 = y\right)\right) = \text{var}(N) - \text{var}_\chi\left(E\left(N \mid \chi^2 = y\right)\right)$$

we can find the unconditional variance i e $\text{var}(N)$ and subsequently the unconditional standard deviation i e $stdev(N)$ of the run length random variable by using the conditional expected value and the conditional variance of N given $\chi^2 = y$ i e using $\text{var}\left(N \mid \chi^2 = y\right)$ and $E\left(N \mid \chi^2 = y\right)$

Thus solving for $\text{var}(N)$ in the above expression we have

$$\text{var}(N) = E_\chi\left(\text{var}\left(N \mid \chi^2 = y\right)\right) + \text{var}_\chi\left(E\left(N \mid \chi^2 = y\right)\right)$$

Now

$$\text{var}\left(N \mid \chi^2 = y\right) = \frac{\beta_\chi}{\left(1-\beta_\chi\right)^2} \qquad \text{and} \qquad E\left(N \mid \chi^2 = y\right) = \frac{1}{1-\beta_\chi}$$

so that

$$\begin{aligned}\text{var}_\chi\left(E\left(N \mid \chi^2 = y\right)\right) &= E_\chi\left(\left(E\left(N \mid \chi^2 = y\right)\right)^2\right) - \left(E_\chi\left(E\left(N \mid \chi^2 = y\right)\right)\right)^2 \\ &= E_\chi\left(\left(\frac{1}{1-\beta_\chi}\right)^2\right) - \left(E_\chi\left(\frac{1}{1-\beta_\chi}\right)\right)^2\end{aligned}$$

and by substitution we have that

$$\text{var}(N) = E_{\chi}\left(\frac{\beta_{\chi}}{\left(1-\beta_{\chi}\right)^{2}}\right) + E_{\chi}\left(\left(\frac{1}{1-\beta_{\chi}}\right)\right) - \left(E_{\chi}\left(\frac{1}{1-\beta_{\chi}}\right)\right)^{2}$$

Re writing this expression in terms of integrals we obtain the unconditional variance

$$\text{var}(N) = \int_0^\infty \frac{\beta_{\chi}}{\left(1-\beta_{\chi}\right)^{2}} f_{\chi}(y)dy + \int_0^\infty \left(\frac{1}{1-\beta_{\chi}}\right)^{2} f_{\chi}(y)dy - \left(\int_0^\infty \frac{1}{1-\beta_{\chi}} f_{\chi}(y)dy\right)^{2}$$

Expression 2

Another route in obtaining the unconditional variance of the run length random variable is by making use of the first and second non central moments of the conditional run length distribution i e using $E\left(N \mid \chi^2 = y\right)$ and $E\left(N^2 \mid \chi^2 = y\right)$

We have that

$$\text{var}(N) = E\left(N^2\right) - E(N)^2$$

but

$$E\left(N^2\right) = E_{\chi}\left(E\left(N^2 \mid \chi^2 = y\right)\right) \quad \text{and} \quad E(N) = E_{\chi}\left(E\left(N \mid \chi^2 = y\right)\right)$$

so that

$$\text{var}(N) = E_{\chi}\left(E\left(N^2 \mid \chi^2 = y\right)\right) - E_{\chi}\left(E\left(N \mid \chi^2 = y\right)\right)^2$$

Furthermore from the properties of the geometric distribution we have that

$$E\left(N^2 \mid \chi^2 = y\right) = \frac{1+\beta_{\chi}}{(1-\beta_{\chi})^2} \quad \text{and} \quad E\left(N \mid \chi^2 = y\right) = \frac{1}{1-\beta_{\chi}}$$

so that a second expression for the unconditional variance of the run length random variable is given by

$$\text{var}(N) = \int_0^\infty \frac{1+\beta_{\chi}}{(1-\beta_{\chi})^2} f_{\chi}(y)dy - \left(\int_0^\infty \frac{1}{1-\beta_{\chi}} f_{\chi}(y)dy\right)^2$$

Result 3 56 Coefficient of skewness of the run length distribution – Conditional

$$skew(N \mid \chi^2 = y) = \frac{1+\beta_\chi}{\sqrt{\beta_\chi}}$$

Proof

The conditional coefficient of skewness is defined as

$$skew(N \mid \chi^2 = y) = \frac{E(N - E(N \mid \chi^2 = y) \mid \chi^2 = y)^3}{\operatorname{var}(N \mid \chi^2 = y)^{\frac{3}{2}}}$$

which can be simplified to

$$skew(N \mid \chi^2 = y) = \frac{E\left(N^3 \mid \chi^2 = y\right) - 3E\left(N^2 \mid \chi^2 = y\right)E\left(N \mid \chi^2 = y\right) + 2E\left(N \mid \chi^2 = y\right)^3}{\left(\operatorname{var}(N \mid \chi^2 = y)\right)^{\frac{3}{2}}}$$

Substituting the expressions for the first non central moment $E(N \mid \chi \;\; = y)$ the second non central moment $E(N^2 \mid \chi^2 = y)$ and the third non central moment $E(N^3 \mid \chi^2 = y)$ of the conditional run length distribution as well as the expression for the conditional variance $\operatorname{var}(N \mid \chi^2 = y)$ the expression for the conditional coefficient of skewness simplifies to

$$skew(N \mid \chi^2 = y) = \frac{1+\beta_\chi}{\sqrt{\beta_\chi}}$$

Result 3 57 Coefficient of skewness of the run length distribution – Unconditional

$$skew(N) = \frac{\int_0^\infty \frac{1+\beta_\chi^{\ 2}+4\beta_\chi}{\left(1-\beta_\chi\right)^3} f_\chi(y)dy - 3\left\{\int_0^\infty \frac{1+\beta_\chi}{(1-\beta_\chi)^2} f_\chi(y)dy\right\}\left\{\int_0^\infty \frac{1}{1-\beta_\chi} f_\chi(y)dy\right\} + 2\left(\int_0^\infty \frac{1}{1-\beta_\chi} f_\chi(y)dy\right)^3}{\left(\mathrm{var}(N)\right)^{\frac{3}{2}}}$$

Proof

To obtain the unconditional coefficient of skewness of the run length random variable i e

$$skew(N) = \frac{E(N-E(N))^3}{\mathrm{var}(N)^{\frac{3}{2}}} = \frac{E\left(N^3\right)-3E\left(N^2\right)E\left(N\right)+2E\left(N\right)^3}{\left(\mathrm{var}(N)\right)^{\frac{3}{2}}}$$

one uses the first the second and the third non central moments of the conditional run length distribution i e one uses $E\left(N \mid \chi^2 = y\right)$ $E\left(N^2 \mid \chi^2 = y\right)$ and $E\left(N^3 \mid \chi^2 = y\right)$ as well as the fact that

$E(N^k) = E_\chi\left(E(N^k \mid \chi^2 = y)\right)$

Thus an expression for the unconditional coefficient of skewness of the run length random variable is given by

$$skew(N) = \frac{\int_0^\infty \frac{1+\beta_\chi^{\ 2}+4\beta_\chi}{\left(1-\beta_\chi\right)^3} f_\chi(y)dy - 3\left\{\int_0^\infty \frac{1+\beta_\chi}{(1-\beta_\chi)^2} f_\chi(y)dy\right\}\left\{\int_0^\infty \frac{1}{1-\beta_\chi} f_\chi(y)dy\right\} + 2\left(\int_0^\infty \frac{1}{1-\beta_\chi} f_\chi(y)dy\right)^3}{\left(\mathrm{var}(N)\right)^{\frac{3}{2}}}$$

3 5 5 Appendix 3D

Case UU

Assume that both the in control process mean and the in control process standard deviation are *unknown* and denoted by μ and σ respectively Furthermore assume that the process follows a normal distribution so that $X_{ij} \sim N\left(\mu \ \sigma^2\right)$ for $i = 1 \ 2 \ 3$ and $j = 1 \ 2 \ 3 \quad n$ In addition assume that we estimate the unknown in control process mean using m reference samples (each of size n) from Phase 1 and that the point estimator used is the overall mean (or grand mean) i e

$$\bar{\bar{X}} = \frac{1}{m}\sum_{i=1}^{m}\bar{X}_i = \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}X_{ij}$$

with an observed value of $\bar{\bar{X}}$ denoted by $\bar{\bar{x}}$ which is an observation from a $N(\mu \ \frac{\sigma_0^2}{mn})$ distribution

As an estimator for the unknown in control process standard deviation we use S with an observed value of S denoted by s where $\frac{\nu S^2}{\sigma^2} \sim \chi^2$

Since the estimation of the unknown process parameter (which in this case is the in control process mean) occurs in Phase 1 the formulae and/or expressions that are derived are only applicable in Phase 2

Result 3 58 Control lımıts

$$\widehat{LCL} = \overline{\overline{X}} - z_{\frac{\alpha}{2}} \frac{S}{\sqrt{n}} \quad \text{and} \quad \widehat{UCL} = \overline{\overline{X}} + z_{\frac{\alpha}{2}} \frac{S}{\sqrt{n}}$$

Proof

From Result 3 1 ın Appendıx 3A we have that

$$LCL = \mu_0 - z_{\frac{\alpha}{2}} \frac{\sigma_0}{\sqrt{n}} \quad \text{and} \quad UCL = \mu_0 + z_{\frac{\alpha}{2}} \frac{\sigma_0}{\sqrt{n}}$$

However sınce both the ın control process mean and the ın control process standard devıatıon are unknown we subtıtute μ_0 wıth ıts poınt estımator $\overline{\overline{X}}$ and substıtute σ_0 wıth ıts poınt estımator S and obtaın

$$\widehat{LCL} = \overline{\overline{X}} - z_{\frac{\alpha}{2}} \frac{S}{\sqrt{n}} \quad \text{and} \quad \widehat{UCL} = \overline{\overline{X}} + z_{\frac{\alpha}{2}} \frac{S}{\sqrt{n}}$$

Result 3 59 Non signalling event

$$\frac{\sqrt{n}\left|\bar{X} - \bar{\bar{X}}\right|}{S} < z_{\frac{\alpha}{2}} \text{ for } i = m+1 \; m+2 \; m+3$$

Proof

$$\bar{X} < \widehat{UCL} \text{ and } \bar{X} > \widehat{LCL}$$

$$\bar{X} < \bar{\bar{X}} + z_{\frac{\alpha}{2}} \frac{S}{\sqrt{n}} \text{ and } \bar{X} > \bar{\bar{X}} - z_{\frac{\alpha}{2}} \frac{S}{\sqrt{n}}$$

$$\bar{X} - \bar{\bar{X}} < z_{\frac{\alpha}{2}} \frac{S}{\sqrt{n}} \text{ and } \bar{X} - \bar{\bar{X}} > -z_{\frac{\alpha}{2}} \frac{S}{\sqrt{n}}$$

$$\bar{X} - \bar{\bar{X}} < z_{\frac{\alpha}{2}} \frac{S}{\sqrt{n}} \text{ and } -(\bar{X} - \bar{\bar{X}}) < z_{\frac{\alpha}{2}} \frac{S}{\sqrt{n}}$$

$$\left|\bar{X} - \bar{\bar{X}}\right| < z_{\frac{\alpha}{2}} \frac{S}{\sqrt{n}}$$

$$\frac{\sqrt{n}\left|\bar{X} - \bar{\bar{X}}\right|}{S} < z_{\frac{\alpha}{2}}$$

Result 3 60 Signaling event

$$\frac{\sqrt{n}\left|\bar{X}_i - \bar{\bar{X}}\right|}{S} \geq z_{\frac{\alpha}{2}} \text{ for } i = m+1,\ m+2,\ m+3$$

Proof

$$\bar{X}_i \geq \widehat{UCL} \text{ or } \bar{X}_i \leq \widehat{LCL}$$

$$\bar{X}_i \geq \bar{\bar{X}} + z_{\frac{\alpha}{2}} \frac{S}{\sqrt{n}} \text{ or } \bar{X}_i \leq \bar{\bar{X}} - z_{\frac{\alpha}{2}} \frac{S}{\sqrt{n}}$$

$$\bar{X}_i - \bar{\bar{X}} \geq z_{\frac{\alpha}{2}} \frac{S}{\sqrt{n}} \text{ or } \bar{X}_i - \bar{\bar{X}} \leq -z_{\frac{\alpha}{2}} \frac{S}{\sqrt{n}}$$

$$\bar{X}_i - \bar{\bar{X}} \geq z_{\frac{\alpha}{2}} \frac{S}{\sqrt{n}} \text{ or } -(\bar{X}_i - \bar{\bar{X}}) \geq z_{\frac{\alpha}{2}} \frac{S}{\sqrt{n}}$$

$$\left|\bar{X}_i - \bar{\bar{X}}\right| \geq z_{\frac{\alpha}{2}} \frac{S}{\sqrt{n}}$$

$$\frac{\sqrt{n}\left|\bar{X}_i - \bar{\bar{X}}\right|}{S} \geq z_{\frac{\alpha}{2}}$$

Result 3 61 Probability of no signal – Conditional

$$\beta(\alpha\ \delta\ m\ n\ z\ y)=\Phi\left(\frac{z}{\sqrt{m}}+z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}}-\delta\sqrt{n}\right)-\Phi\left(\frac{z}{\sqrt{m}}-z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}}-\delta\sqrt{n}\right)$$

or

$$\beta(\alpha\ \delta\ m\ n\ z\ y)=\Phi\left(\sqrt{\frac{m}{m+1}}\left(z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}}-\delta\sqrt{n}\right)\right)-\Phi\left(\sqrt{\frac{m}{m+1}}\left(-z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}}-\delta\sqrt{n}\right)\right)$$

Proof

Expression 1

Let $P_C(\,)$ denote a conditional probability that is we condition on a sustained shift in the process mean as well as on the observed values of the random variables $\overline{\overline{X}}$ and S i e we condition on the values $\overline{\overline{x}}$ and s Thus

P_C(No Signal)

$= P(\widehat{LCL} < \overline{X} < \widehat{UCL} \mid \text{A shift in the process mean occured } \overline{\overline{X}} = \overline{\overline{x}}\ S = s\ )$

$$= P\left(\overline{\overline{x}} - z_{\frac{\alpha}{2}}\frac{s}{\sqrt{n}} < \overline{X} < \overline{\overline{x}} + z_{\frac{\alpha}{2}}\frac{s}{\sqrt{n}} \mid \mu_1 = \mu + \delta\sigma\ \overline{\overline{X}} = \overline{\overline{x}}\ S = s\right)$$

$$= P\left(\frac{\overline{\overline{x}} - z_{\frac{\alpha}{2}}\frac{s}{\sqrt{n}} - (\mu+\delta\sigma)}{\frac{\sigma}{\sqrt{n}}} < \frac{\overline{X} - (\mu+\delta\sigma)}{\frac{\sigma}{\sqrt{n}}} < \frac{\overline{\overline{x}} + z_{\frac{\alpha}{2}}\frac{s}{\sqrt{n}} - (\mu+\delta\sigma)}{\frac{\sigma}{\sqrt{n}}} \mid \overline{\overline{X}} = \overline{\overline{x}}\ S = s\right)$$

$$= P\left(\left(\frac{\overline{\overline{x}}-\mu}{\frac{\sigma}{\sqrt{mn}}}\right)\frac{1}{\sqrt{m}} - z_{\frac{\alpha}{2}}\frac{s}{\sigma} - \delta\sqrt{n} < Z < \left(\frac{\overline{\overline{x}}-\mu}{\frac{\sigma}{\sqrt{mn}}}\right)\frac{1}{\sqrt{m}} + z_{\frac{\alpha}{2}}\frac{s}{\sigma} - \delta\sqrt{n} \mid \overline{\overline{X}} = \overline{\overline{x}}\ S = s\right)$$

$$= P\left(\frac{z}{\sqrt{m}} - z_{\frac{\alpha}{2}}\frac{s}{\sigma} - \delta\sqrt{n} < Z < \frac{z}{\sqrt{m}} + z_{\frac{\alpha}{2}}\frac{s}{\sigma} - \delta\sqrt{n} \mid Z = z\ S = s\right)$$

$$= P\left(\frac{z}{\sqrt{m}} - z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}} - \delta\sqrt{n} < Z < \frac{z}{\sqrt{m}} + z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}} - \delta\sqrt{n} \mid Z = z\ \chi^2 = y\right)$$

$$= \Phi\left(\frac{z}{\sqrt{m}} + z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}} - \delta\sqrt{n}\right) - \Phi\left(\frac{z}{\sqrt{m}} - z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}} - \delta\sqrt{n}\right)$$

$$= \beta(\alpha\ \delta\ m\ n\ z\ y) \text{ say}$$

For simplicity denote $\beta(\alpha\ \delta\ m\ n\ z\ y)$ by $\beta_{Z\ \chi}$ In this case the subscripts Z and χ^2 indicate that the expression is a conditional expression i e conditional on observed values of a standard normal random variable $z = \dfrac{\bar{\bar{x}} - \mu}{\dfrac{\sigma}{\sqrt{mn}}}$ and a chi square random variable $y = \chi^2 = \dfrac{\nu s^2}{\sigma^2}$

Expression 2

P_C(No Signal)

$= P(\widehat{LCL} < \bar{X} < \widehat{UCL}$ | A shift in the process mean occured $S = s$)

$$= P\left(\bar{\bar{X}} - z_{\frac{\alpha}{2}}\frac{s}{\sqrt{n}} < \bar{X} < \bar{\bar{X}} + z_{\frac{\alpha}{2}}\frac{s}{\sqrt{n}} \mid \mu_1 = \mu + \delta\sigma\ S = s\right)$$

$$= P\left(-z_{\frac{\alpha}{2}}\frac{s}{\sqrt{n}} < \bar{X} - \bar{\bar{X}} < z_{\frac{\alpha}{2}}\frac{s}{\sqrt{n}} \mid \mu_1 = \mu + \delta\sigma\ S = s\right)$$

$$= P\left(\frac{-z_{\frac{\alpha}{2}}\frac{s}{\sqrt{n}} - \delta\sigma}{\sqrt{\frac{\sigma^2}{n}\left(\frac{m+1}{m}\right)}} < \frac{\left(\bar{X} - \bar{\bar{X}}\right) - (\mu_1 - \mu)}{\sqrt{\frac{\sigma^2}{n}\left(\frac{m+1}{m}\right)}} < \frac{z_{\frac{\alpha}{2}}\frac{s}{\sqrt{n}} - \delta\sigma}{\sqrt{\frac{\sigma^2}{n}\left(\frac{m+1}{m}\right)}} \mid S = s\right)$$

$$= P\left(\sqrt{\frac{m}{m+1}}\left(-z_{\frac{\alpha}{2}}\frac{s}{\sigma} - \delta\sqrt{n}\right) < Z < \sqrt{\frac{m}{m+1}}\left(z_{\frac{\alpha}{2}}\frac{s}{\sigma} - \delta\sqrt{n}\right) \mid S = s\right)$$

$$= \Phi\left(\sqrt{\frac{m}{m+1}}\left(z_{\frac{\alpha}{2}}\frac{s}{\sigma} - \delta\sqrt{n}\right)\right) - \Phi\left(\sqrt{\frac{m}{m+1}}\left(-z_{\frac{\alpha}{2}}\frac{s}{\sigma} - \delta\sqrt{n}\right)\right)$$

$$= \Phi\left(\sqrt{\frac{m}{m+1}}\left(z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}} - \delta\sqrt{n}\right)\right) - \Phi\left(\sqrt{\frac{m}{m+1}}\left(-z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}} - \delta\sqrt{n}\right)\right)$$

Result 3 62 Probability of no signal – Unconditional

$$P(\text{No Signal}) = \int_{-\infty}^{\infty}\int_{0}^{\infty}\left(\Phi\left(\frac{z}{\sqrt{m}} + z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}} - \delta\sqrt{n}\right) - \Phi\left(\frac{z}{\sqrt{m}} - z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}} - \delta\sqrt{n}\right)\right)\phi(z) f_{\chi}(y)dydz$$

or

$$P(\text{No Signal}) = \int_{0}^{\infty}\left\{\Phi\left(\sqrt{\frac{m}{m+1}}\left(z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}} - \delta\sqrt{n}\right)\right) - \Phi\left(\sqrt{\frac{m}{m+1}}\left(-z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}} - \delta\sqrt{n}\right)\right)\right\} f_{\chi}(y)dy$$

Proof

Expression 1

$P(\text{No Signal})$

$$= \int_{-\infty}^{\infty}\int_{0}^{\infty} P_C(\text{No Signal}) g_{Z,\chi}(z, y)dydz$$

$$= \int_{-\infty}^{\infty}\int_{0}^{\infty} P_C(\text{No Signal})\phi(z) f_{\chi}(y)dydz$$

$$= \int_{-\infty}^{\infty}\int_{0}^{\infty}\left(\Phi\left(\frac{z}{\sqrt{m}} + z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}} - \delta\sqrt{n}\right) - \Phi\left(\frac{z}{\sqrt{m}} - z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}} - \delta\sqrt{n}\right)\right)\phi(z) f_{\chi}(y)dydz$$

Expression 2

$P(\text{No Signal})$

$$= \int_{0}^{\infty} P_C(\text{No Signal}) f_{\chi}(y)dy$$

$$= \int_{0}^{\infty}\left\{\Phi\left(\sqrt{\frac{m}{m+1}}\left(z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}} - \delta\sqrt{n}\right)\right) - \Phi\left(\sqrt{\frac{m}{m+1}}\left(-z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}} - \delta\sqrt{n}\right)\right)\right\} f_{\chi}(y)dy$$

Result 3 63 Probabilıty of a sıgnal – Condıtıonal

$$1-\beta(\alpha\ \delta\ m\ n\ z\ y)=1-\Phi\left(\frac{z}{\sqrt{m}}+z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}}-\delta\sqrt{n}\right)+\Phi\left(\frac{z}{\sqrt{m}}-z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}}-\delta\sqrt{n}\right)$$

or

$$1-\beta(\alpha\ \delta\ m\ n\ z\ y)=1-\Phi\left(\sqrt{\frac{m}{m+1}}\left(z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}}-\delta\sqrt{n}\right)\right)+\Phi\left(\sqrt{\frac{m}{m+1}}\left(-z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}}-\delta\sqrt{n}\right)\right)$$

Proof

Expressıon 1

P_C(Sıgnal)

$=1-P_C$(No Sıgnal)

$$=1-\left\{\Phi\left(\frac{z}{\sqrt{m}}+z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}}-\delta\sqrt{n}\right)-\Phi\left(\frac{z}{\sqrt{m}}-z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}}-\delta\sqrt{n}\right)\right\}$$

$$=1-\Phi\left(\frac{z}{\sqrt{m}}+z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}}-\delta\sqrt{n}\right)+\Phi\left(\frac{z}{\sqrt{m}}-z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}}-\delta\sqrt{n}\right)$$

$$=1-\beta_{z\ \chi}$$

Expressıon 2

P_C(Sıgnal)

$=1-P_C$(No Sıgnal)

$$=1-\left\{\Phi\left(\sqrt{\frac{m}{m+1}}\left(z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}}-\delta\sqrt{n}\right)\right)-\Phi\left(\sqrt{\frac{m}{m+1}}\left(-z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}}-\delta\sqrt{n}\right)\right)\right\}$$

$$=1-\Phi\left(\sqrt{\frac{m}{m+1}}\left(z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}}-\delta\sqrt{n}\right)\right)+\Phi\left(\sqrt{\frac{m}{m+1}}\left(-z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}}-\delta\sqrt{n}\right)\right)$$

$$=1-\beta_{z\ \chi}$$

Result 3 64 Probability of a signal – Unconditional

$$P(\text{Signal}) = \int_{-\infty}^{\infty}\int_{0}^{\infty}\left(1-\Phi\left(\frac{z}{\sqrt{m}}+z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}}-\delta\sqrt{n}\right)+\Phi\left(\frac{z}{\sqrt{m}}-z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}}-\delta\sqrt{n}\right)\right)\phi(z)f_{\chi}(y)dydz$$

or

$$P(\text{Signal}) = \int_{0}^{\infty}\left\{1-\Phi\left(\sqrt{\frac{m}{m+1}}\left(z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}}-\delta\sqrt{n}\right)\right)+\Phi\left(\sqrt{\frac{m}{m+1}}\left(-z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}}-\delta\sqrt{n}\right)\right)\right\}f_{\chi}(y)dy$$

Proof

Expression 1

$P(\text{Signal})$

$$= \int_{-\infty}^{\infty}\int_{0}^{\infty}P_C(\text{Signal})g_{Z\,\chi}(z\ y)dydz$$

$$= \int_{-\infty}^{\infty}\int_{0}^{\infty}P_C(\text{Signal})\phi(z)f_{\chi}(y)dydz$$

$$= \int_{-\infty}^{\infty}\int_{0}^{\infty}\left(1-\Phi\left(\frac{z}{\sqrt{m}}+z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}}-\delta\sqrt{n}\right)+\Phi\left(\frac{z}{\sqrt{m}}-z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}}-\delta\sqrt{n}\right)\right)\phi(z)f_{\chi}(y)dydz$$

Expression 2

$P(\text{Signal})$

$$= \int_{0}^{\infty}P_C(\text{Signal})f_{\chi}(y)dy$$

$$= \int_{0}^{\infty}\left\{1-\Phi\left(\sqrt{\frac{m}{m+1}}\left(z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}}-\delta\sqrt{n}\right)\right)+\Phi\left(\sqrt{\frac{m}{m+1}}\left(-z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}}-\delta\sqrt{n}\right)\right)\right\}f_{\chi}(y)dy$$

Result 3 65 Probability of a false alarm – Conditional

$$P_C(\text{False Alarm}) = 1 - \Phi\left(\frac{z}{\sqrt{m}} + z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}}\right) + \Phi\left(\frac{z}{\sqrt{m}} - z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}}\right)$$

or

$$P_C(\text{False Alarm}) = 1 - \Phi\left(\sqrt{\frac{m}{m+1}}\left(z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}}\right)\right) + \Phi\left(\sqrt{\frac{m}{m+1}}\left(-z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}}\right)\right)$$

Proof

Expression 1

$$P_C\left(\text{Signal} \mid \text{In control}\right)$$

$$= 1 - P_C\left(\text{No Signal} \mid \text{In control}\right)$$

$$= 1 - P_C\left(\widehat{LCL} < \bar{X} < \widehat{UCL} \mid \text{In control}\right)$$

$$= 1 - P_C\left(\widehat{LCL} < \bar{X} < \widehat{UCL} \mid \mu_1 = \mu \;\; \bar{\bar{X}} = \bar{\bar{x}} \;\; S = s\right)$$

$$= 1 - P\left(\bar{\bar{x}} - z_{\frac{\alpha}{2}}\frac{s}{\sqrt{n}} < \bar{X} < \bar{\bar{x}} + z_{\frac{\alpha}{2}}\frac{s}{\sqrt{n}} \mid \mu_1 = \mu \;\; \bar{\bar{X}} = \bar{\bar{x}} \;\; S = s\right)$$

$$= 1 - P\left(\frac{\bar{\bar{x}} - z_{\frac{\alpha}{2}}\frac{s}{\sqrt{n}} - \mu}{\frac{\sigma}{\sqrt{n}}} < \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} < \frac{\bar{\bar{x}} + z_{\frac{\alpha}{2}}\frac{s}{\sqrt{n}} - \mu}{\frac{\sigma}{\sqrt{n}}} \mid \bar{\bar{X}} = \bar{\bar{x}} \;\; S = s\right)$$

$$= 1 - P\left(\left(\frac{\bar{\bar{x}} - \mu}{\frac{\sigma}{\sqrt{mn}}}\right)\frac{1}{\sqrt{m}} - z_{\frac{\alpha}{2}}\frac{s}{\sigma} < Z < \left(\frac{\bar{\bar{x}} - \mu}{\frac{\sigma}{\sqrt{mn}}}\right)\frac{1}{\sqrt{m}} + z_{\frac{\alpha}{2}}\frac{s}{\sigma} \mid \bar{\bar{X}} = \bar{\bar{x}} \;\; S = s\right)$$

$$= 1 - P\left(\frac{z}{\sqrt{m}} - z_{\frac{\alpha}{2}}\frac{s}{\sigma} < Z < \frac{z}{\sqrt{m}} + z_{\frac{\alpha}{2}}\frac{s}{\sigma} \mid Z = z \;\; S = s\right)$$

$$= 1 - P\left(\frac{z}{\sqrt{m}} - z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}} < Z < \frac{z}{\sqrt{m}} + z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}} \mid Z = z \;\; \chi^2 = y\right)$$

$$= 1 - \left(\Phi\left(\frac{z}{\sqrt{m}} + z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}}\right) - \Phi\left(\frac{z}{\sqrt{m}} - z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}}\right)\right)$$

$$=1-\Phi\left(\frac{z}{\sqrt{m}}+z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}}\right)+\Phi\left(\frac{z}{\sqrt{m}}-z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}}\right)$$

$$=1-\beta(\alpha\ 0\ m\ n\ z\ y)\ \text{ say}$$

This expression could also have been obtained by simply substituting $\delta=0$ in the first expression for the conditional probability of a signal in Result 3 63 given that a shift in the process mean occurred Therefore the first expression of Result 3 65 is the same as the first expression of Result 3 63 i e

$$1-\Phi\left(\frac{z}{\sqrt{m}}+z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}}-\delta\sqrt{n}\right)+\Phi\left(\frac{z}{\sqrt{m}}-z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}}-\delta\sqrt{n}\right)\ \text{with } \delta=0$$

Expression 2

$$P_C(\text{Signal} \mid \text{In control})$$

$$=1-P_C(\text{No Signal} \mid \text{In control})$$

$$=1-P_C\left(\widehat{LCL}<\bar{X}<\widehat{UCL} \mid \text{In control}\right)$$

$$=1-P(\widehat{LCL}<\bar{X}<\widehat{UCL} \mid \text{A shift in the process mean occured } S=s\ )$$

$$=1-P\left(\bar{\bar{X}}-z_{\frac{\alpha}{2}}\frac{s}{\sqrt{n}}<\bar{X}<\bar{\bar{X}}+z_{\frac{\alpha}{2}}\frac{s}{\sqrt{n}} \mid \mu_1=\mu\ S=s\right)$$

$$=1-P\left(-z_{\frac{\alpha}{2}}\frac{s}{\sqrt{n}}<\bar{X}-\bar{\bar{X}}<z_{\frac{\alpha}{2}}\frac{s}{\sqrt{n}} \mid \mu_1=\mu\ S=s\right)$$

$$=1-P\left(\frac{-z_{\frac{\alpha}{2}}\frac{s}{\sqrt{n}}}{\sqrt{\frac{\sigma^2}{n}\left(\frac{m+1}{m}\right)}}<\frac{\left(\bar{X}-\bar{\bar{X}}\right)-(\mu_1-\mu)}{\sqrt{\frac{\sigma^2}{n}\left(\frac{m+1}{m}\right)}}<\frac{z_{\frac{\alpha}{2}}\frac{s}{\sqrt{n}}}{\sqrt{\frac{\sigma^2}{n}\left(\frac{m+1}{m}\right)}} \mid S=s\right)$$

$$=1-P\left(\sqrt{\frac{m}{m+1}}\left(-z_{\frac{\alpha}{2}}\frac{s}{\sigma}\right)<Z<\sqrt{\frac{m}{m+1}}\left(z_{\frac{\alpha}{2}}\frac{s}{\sigma}\right) \mid S=s\right)$$

$$=1-\left\{\Phi\left(\sqrt{\frac{m}{m+1}}\left(z_{\frac{\alpha}{2}}\frac{s}{\sigma}\right)\right)-\Phi\left(\sqrt{\frac{m}{m+1}}\left(-z_{\frac{\alpha}{2}}\frac{s}{\sigma}\right)\right)\right\}$$

$$=1-\Phi\left(\sqrt{\frac{m}{m+1}}\left(z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}}\right)\right)+\Phi\left(\sqrt{\frac{m}{m+1}}\left(-z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}}\right)\right)$$

This expression could also have been obtained by simply substituting $\delta = 0$ in the second expression for the conditional probability of a signal in Result 3 63 given that a shift in the process mean occurred Therefore the second expression of Result 3 65 is the same as the second expression of Result 3 63 i e $1-\Phi\left(\sqrt{\frac{m}{m+1}}\left(z_{\frac{\alpha}{2}}\sqrt{\frac{y}{v}}-\delta\sqrt{n}\right)\right)+\Phi\left(\sqrt{\frac{m}{m+1}}\left(-z_{\frac{\alpha}{2}}\sqrt{\frac{y}{v}}-\delta\sqrt{n}\right)\right)$ with $\delta = 0$

Result 3 66 Probability of a false alarm – Unconditional

$$P(\text{Signal} \mid \text{In control}) = \int_{-\infty}^{\infty}\int_{0}^{\infty}\left\{1-\Phi\left(\frac{z}{\sqrt{m}}+z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}}\right)+\Phi\left(\frac{z}{\sqrt{m}}-z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}}\right)\right\}\phi(z)f_{\chi}(y)dydz$$

or

$$P(\text{Signal} \mid \text{In control}) = \int_{0}^{\infty}\left\{1-\Phi\left(\sqrt{\frac{m}{m+1}}z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}}\right)+\Phi\left(-\sqrt{\frac{m}{m+1}}z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}}\right)\right\}f_{\chi}(y)dy$$

or

$$P(\text{Signal} \mid \text{In control}) = 1-T\left(z_{\frac{\alpha}{2}}\sqrt{\frac{m}{m+1}}\right)+T\left(-z_{\frac{\alpha}{2}}\sqrt{\frac{m}{m+1}}\right)$$

Proof

Expression 1

$$P(\text{Signal} \mid \text{In control})$$

$$= \int_{-\infty}^{\infty}\int_{0}^{\infty} P_C(\text{Signal} \mid \text{In control})\phi(z)f_{\chi}(y)dydz$$

$$= \int_{-\infty}^{\infty}\int_{0}^{\infty}\left\{1-\Phi\left(\frac{z}{\sqrt{m}}+z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}}\right)+\Phi\left(\frac{z}{\sqrt{m}}-z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}}\right)\right\}\phi(z)f_{\chi}(y)dydz$$

Expression 2

$$P(\text{Signal} \mid \text{In control})$$

$$= \int_{0}^{\infty} P_C(\text{Signal} \mid \text{In control})f_{\chi}(y)dy$$

$$= \int_{0}^{\infty}\left\{1-\Phi\left(\sqrt{\frac{m}{m+1}}z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}}\right)+\Phi\left(-\sqrt{\frac{m}{m+1}}z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}}\right)\right\}f_{\chi}(y)dy$$

Expression 3

$$P(\text{Signal} \mid \text{In control})$$

$$= 1 - P(\widehat{LCL} < \bar{X} < \widehat{UCL} \mid \text{No shift in the process mean occured})$$

$$= 1 - P\left(\bar{\bar{X}} - z_{\frac{\alpha}{2}} \frac{S}{\sqrt{n}} < \bar{X} < \bar{\bar{X}} + z_{\frac{\alpha}{2}} \frac{S}{\sqrt{n}} \mid \mu_1 = \mu\right)$$

$$= 1 - P\left(-z_{\frac{\alpha}{2}} \frac{S}{\sqrt{n}} < \bar{X} - \bar{\bar{X}} < z_{\frac{\alpha}{2}} \frac{S}{\sqrt{n}} \mid \mu_1 = \mu\right)$$

$$= 1 - P\left(\frac{-z_{\frac{\alpha}{2}} \frac{S}{\sqrt{n}}}{\sqrt{\frac{S^2}{n}\left(\frac{m+1}{m}\right)}} < \frac{\left(\bar{X} - \bar{\bar{X}}\right) - (\mu_1 - \mu)}{\sqrt{\frac{S^2}{n}\left(\frac{m+1}{m}\right)}} < \frac{z_{\frac{\alpha}{2}} \frac{S}{\sqrt{n}}}{\sqrt{\frac{S^2}{n}\left(\frac{m+1}{m}\right)}}\right)$$

$$= 1 - P\left(-z_{\frac{\alpha}{2}} \sqrt{\frac{m}{m+1}} < t < z_{\frac{\alpha}{2}} \sqrt{\frac{m}{m+1}}\right)$$

$$= 1 - \left\{T\left(z_{\frac{\alpha}{2}} \sqrt{\frac{m}{m+1}}\right) - T\left(-z_{\frac{\alpha}{2}} \sqrt{\frac{m}{m+1}}\right)\right\}$$

$$= 1 - T\left(z_{\frac{\alpha}{2}} \sqrt{\frac{m}{m+1}}\right) + T\left(-z_{\frac{\alpha}{2}} \sqrt{\frac{m}{m+1}}\right)$$

Result 3 67 Run length distribution – Conditional

$$P(N=j \mid Z=z\ \chi^2=y)=\beta_{Z\ \chi}^{j-1}\left(1-\beta_{Z\ \chi}\right) \text{ for } j=1\ 2\ 3$$

Proof

Assume that at each point in time we perform the same experiment by obtaining a random sample of n observations from the process output calculating the sample mean $\overline{X}$ and then compare $\overline{X}$ with $\widehat{LCL}=\overline{\overline{X}}-z_{\frac{\alpha}{2}}\frac{S}{\sqrt{n}}$ and $\widehat{UCL}=\overline{\overline{X}}+z_{\frac{\alpha}{2}}\frac{S}{\sqrt{n}}$ to see whether $\overline{X}$ falls between or outside $\widehat{UCL}$ and/or $\widehat{LCL}$

In addition assume that we condition on the observed values of $\overline{\overline{X}}$ and S denoted by $\overline{\overline{x}}$ and s (or z and y if $\overline{\overline{X}}$ and S are written in their canonical forms namely Z and χ^2 respectively) so that we actually compare $\overline{X}$ with $\widehat{LCL}=\overline{\overline{x}}-z_{\frac{\alpha}{2}}\frac{s}{\sqrt{n}}$ and $\widehat{UCL}=\overline{\overline{x}}+z_{\frac{\alpha}{2}}\frac{s}{\sqrt{n}}$ that is both control limits are constants and then have that

i All trials i e all the experiments at the different points in time are independent

ii At each stage of sampling we can either observed a success (S) or a failure (F) that is $\overline{X}$ can either fall between $\widehat{UCL}$ and $\widehat{LCL}$ (and give no signal) or outside $\widehat{UCL}$ or $\widehat{LCL}$ with both endpoints included (and give a signal)

iii The conditional probability of a failure (no signal) and the conditional probability of a success (signal) are the same on each trial and denoted by $P(F)=\beta_{Z\ \chi}$ and $P(S)=1-\beta_{Z\ \chi}$ respectively

iv We repeat the experiment until we obtain the first success

Thus the sample space **S** is **S** $=\{S\ FS\ FFS\ FFFS\ FFFFS\ \}$ and if we consider an element of **S** with $j-1$ failures and then a success we find that

$$P(FF\ \ FS)=P(F)P(F)\ \ P(F)P(S)=P(F)^{j-1}P(S) \text{ for } j=1\ 2\ 3$$

Consequently if N denotes the number of trials until we observe a success for the first time we obtain

$$P(N = j \mid Z = z \;\; \chi^2 = y) = \beta_{Z\;\chi}^{j\;\;1}\left(1-\beta_{Z\;\chi}\right) \text{ for } j = 1\;\;2\;\;3$$

which is a geometric distribution with probability of success given by $1-\beta_{Z\;\chi}$ so

that $N \mid Z = z \;\; \chi^2 = y \sim GEO\left(1-\beta_{Z\;\chi}\right)$

Consequently the cumulative distribution function (cdf) is found from

$$P(N \leq j \mid Z = z \;\; \chi^2 = y) = \sum_{k\;\;1}^{j} \beta_{Z\;\chi}^{k\;\;1}\,(1-\beta_{Z\;\chi}) = 1-\beta_{Z\;\chi}^{j}$$

and subsequently

$$P(N \geq j \mid Z = z \;\; \chi^2 = y) = \beta_{Z\;\chi}^{j\;\;1}$$

Result 3 68 Run length distribution – Unconditional

$$P(N=j)=\int_{-\infty}^{\infty}\int_{0}^{\infty}\beta_{Z\ \chi}^{j\ 1}\left(1-\beta_{Z\ \chi}\right)\phi(z)f_{\chi}(y)dydz \text{ for } j=1\ 2\ 3$$

Proof

The conditional run length distribution (given in Result 3 67) is denoted by $P(N=j\,|\,Z=z\ \chi^2=y)$ whereas the joint distribution of the run length random variable N the random variable Z and the random variable χ^2 is denoted by $P(N=j\ Z=z\ \chi^2=y)$ The marginal distributions of these three random variables are denoted by $P(N=j)$ $P(Z=z)$ and $P\left(\chi^2=y\right)$ respectively

The unconditional (or marginal) run length distribution of N i e $P(N=j)$ given the fact that the conditional run length distribution i e $P(N=j\,|\,Z=z\ \chi^2=y)$ and the joint distribution of the random variables Z and χ^2 are known can be found by using the fact that

$$P(N=j\,|\,Z=z\ \chi^2=y)=\frac{P(N=j\ Z=z\ \chi^2=y)}{P(Z=z\ \chi^2=y)}$$

or alternatively the fact that

$$P(N=j\ Z=z\ \chi^2=y)=P(N=j\,|\,Z=z\ \chi^2=y)P(Z=z\ \chi^2=y)$$

Therefore to find the marginal distribution of N (also called the unconditional run length distribution) requires integrating the joint distribution of N Z and χ^2 over the domain of the random variables Z and χ^2 i e

$$P(N=j)=\int_{0}^{\infty}\int_{-\infty}^{\infty}P(N=j\ Z=z\ \chi^2=y)dzdy=\int_{0}^{\infty}\int_{-\infty}^{\infty}P(N=j\,|\,Z=z\ \chi^2=y)P(Z=z\ \chi^2=y)dzdy$$

Thus the unconditional probability mass function (pmf) is

$$P(N=j)$$

$$=\int_{-\infty}^{\infty}\int_{0}^{\infty}P_{C}\left(N=j\,|\,Z=z\ \chi^2=y\right)\phi(z)f_{\chi}(y)dydz$$

$$=\int_{-\infty}^{\infty}\int_{0}^{\infty}\beta_{Z\ \chi}^{j\ 1}\left(1-\beta_{Z\ \chi}\right)\phi(z)f_{\chi}(y)dydz$$

whereas the unconditional cumulative distribution function (cdf) is

$$P(N \le j) = \sum_{k=1}^{j} P(N = k) = \sum_{k=1}^{j}\left(\int_{-\infty}^{\infty}\int_{0}^{\infty} \beta_{Z,\chi}^{j-1}\left(1-\beta_{Z,\chi}\right)\phi(z) f_{\chi}(y)\,dy\,dz\right) = \int_{-\infty}^{\infty}\int_{0}^{\infty}\left(1-\beta_{Z,\chi}^{j}\right)\phi(z) f_{\chi}(y)\,dy\,dz$$

Another useful way of writing the unconditional run length distribution is as $P(N \ge j) = \int_{-\infty}^{\infty}\int_{0}^{\infty} P(N \ge j \mid Z = z, \chi^2 = y)\phi(z) f_{\chi}(y)\,dy\,dz = \int_{-\infty}^{\infty}\int_{0}^{\infty} \beta_{Z,\chi}^{j-1}\phi(z) f_{\chi}(y)\,dy\,dz$ and this expression can conveniently be written as $I_3(j-1, m, n, \delta, \alpha)$

Result 3 69 Quantile function – Conditional

$$Q_{N|Z\ \chi}(p) = \inf\left\{j \text{ integer } \ P(N \le j \mid Z = z\ \ \chi^2 = y) \ge p\right\} = \inf\left\{j \text{ an integer } \ j \ge \frac{\ln(1-p)}{\ln \beta_{Z\ \chi}}\right\}$$

Proof

Generally the quantile function is defined as

$$Q_N(p) = \inf\left\{j \text{ an integer } \ F_N(j) \ge p\right\}$$

where $F_N(j)$ is the cdf of a distribution and p denotes a particular percentile of the distribution

Thus substituting the conditional cdf of the run length distribution (given in Result 3 67) and solving for j the quantile function for the conditional run length distribution is

$$Q_{N|Z\ \chi}(p) = \inf\left\{j \text{ an integer } \ P(N \le j \mid Z = z\ \ \chi^2 = y) \ge p\right\} = \inf\left\{j \text{ an integer } \ j \ge \frac{\ln(1-p)}{\ln \beta_{Z\ \chi}}\right\} \text{ with}$$

$0 < p < 1$

Result 3 70 Quantile function – Unconditional

$$Q_N(p) = \inf\{j \text{ an integer } P(N \le j) \ge p\}$$
$$= \inf\left\{j \text{ an integer } \sum_{k=1}^{j} \int_{-\infty}^{\infty}\int_{0}^{\infty} \beta_{Z\,\chi}^{k-1}(1-\beta_{Z\,\chi})\phi(z) f_{\chi}(y)dydz \ge p\right\}$$

Proof

The unconditional quantile function when both the unknown in control process mean and the unknown in control process standard deviation are estimated is

$$Q_N(p) = \inf\{j \text{ an integer } P(N \le j) \ge p\}$$
$$= \inf\left\{j \text{ an integer } \sum_{k=1}^{j} P(N = k) \ge p\right\} \qquad \text{with } 0 < p < 1$$
$$= \inf\left\{j \text{ an integer } \sum_{k=1}^{j} \int_{-\infty}^{\infty}\int_{0}^{\infty} \beta_{Z\,\chi}^{k-1}(1-\beta_{Z\,\chi})\phi(z) f_{\chi}(y)dydz \ge p\right\}$$

$P(N \le j)$ denoting the uncondtional cumulative distribution function (given in Result 3 68) and $P(N = j)$ denoting the uncondtional probability mass function (also given in Result 3 68) This formula can not be simplified any further

Result 3.71 Moment generating function – Conditional

$$M_{N|Z=\chi}(t) = \frac{(1-\beta_{Z=\chi})e^{t}}{1-\beta_{Z=\chi}e^{t}}$$

Proof

$$M_{N|Z=\chi}(t)$$

$$= \sum_{j=1}^{\infty} e^{tj}\beta_{Z=\chi}^{j-1}\left(1-\beta_{Z=\chi}\right)$$

$$= \left(1-\beta_{Z=\chi}\right)e^{t}\sum_{j=1}^{\infty} e^{t(j-1)}\beta_{Z=\chi}^{j-1}$$

$$= \left(1-\beta_{Z=\chi}\right)e^{t}\sum_{j-1=0}^{\infty} e^{t(j-1)}\beta_{Z=\chi}^{j-1}$$

$$= \left(1-\beta_{Z=\chi}\right)e^{t}\sum_{k=0}^{\infty} e^{tk}\beta_{Z=\chi}^{k}$$

$$= \left(1-\beta_{Z=\chi}\right)e^{t}\sum_{k=0}^{\infty}\left(e^{t}\beta_{Z=\chi}\right)^{k}$$

$$= \frac{\left(1-\beta_{Z=\chi}\right)e^{t}}{1-\beta_{Z=\chi}e^{t}} \text{ provided } e^{t}\beta_{Z=\chi}<1$$

Result 3 72 Average run length (First non central moment of the run length distribution) – Conditional

$$E(N \mid Z = z\ \chi^2 = y) = \frac{1}{1-\beta_{Z\ \chi}} = \frac{1}{1-\Phi\left(\frac{z}{\sqrt{m}} + z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}} - \delta\sqrt{n}\right) + \Phi\left(\frac{z}{\sqrt{m}} - z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}} - \delta\sqrt{n}\right)}$$

Proof

In general the k^{th} conditional non central moment $E\left(N^k \mid Z = z\ \chi^2 = y\right)$ is the k^{th} derivative of the conditional moment generating function $M_{N|Z\ \chi}(t)$ evaluated in the point zero i e

$$E\left(N^k \mid Z = z\ \chi^2 = y\right) = \frac{d^k}{dt^k} M_{N|Z\ \chi}(t)|_{t\ 0}$$

Thus since the average run length is the first non central moment of the run length distribution we need $E\left(N \mid Z = z\ \chi^2 = y\right) = \frac{d}{dt} M_{N|Z\ \chi}(t)|_{t\ 0}$

Therefore

$$E\left(N \mid Z = z\ \chi^2 = y\right)$$

$$= \frac{d}{dt} M_{N|Z\ \chi}(t)|_{t\ 0}$$

$$= \frac{-\left(-1-\beta_{Z\ \chi}\right)e^t}{\left(-1+\beta_{Z\ \chi}e^t\right)^2}|_{t\ 0}$$

$$= \frac{1}{1-\beta_{Z\ \chi}}$$

and it follows that

$$ARL = E(N \mid Z = z\ \chi^2 = y) = \frac{1}{1-\beta_{Z\ \chi}} = \frac{1}{1-\Phi\left(\frac{z}{\sqrt{m}} + z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}} - \delta\sqrt{n}\right) + \Phi\left(\frac{z}{\sqrt{m}} - z_{\frac{\alpha}{2}}\sqrt{\frac{y}{\nu}} - \delta\sqrt{n}\right)}$$

Result 3 73 Average run length – Unconditional

$$E(N) = \int_{-\infty}^{\infty}\int_{0}^{\infty}\frac{1}{1-\beta_{Z\ \chi}}\phi(z)f_{\chi}\ (y)dydz$$

Proof

In general the k^{th} unconditional non central moment can be obtained from the k^{th} conditional non central moment i e $E\left(N^k\right) = E_{Z\ \chi}\left(E\left(N^k \mid Z = z\ \chi^2 = y\right)\right)$

Thus since the unconditional average run length is the first non central moment of the unconditional run length distribution we need to find $E\left(N\right) = E_{Z\ \chi}\left(E\left(N \mid Z = z\ \chi^2 = y\right)\right)$

Therefore

$E(N)$

$$= \int_{-\infty}^{\infty}\int_{0}^{\infty} E(N \mid Z = z\ \chi^2 = y)\phi(z)f_{\chi}\ (y)dydz$$

$$= \int_{-\infty}^{\infty}\int_{0}^{\infty}\frac{1}{1-\beta_{Z\ \chi}}\phi(z)f_{\chi}\ (y)dydz$$

Result 3 74 Second non central moment and third non central moment – Conditional

$$E(N^2 \mid Z = z \ \chi^2 = y) = \frac{1+\beta_{Z\ \chi}}{(1-\beta_{Z\ \chi})^2} \quad \text{and} \quad E(N^3 \mid Z = z \ \chi^2 = y) = \frac{1+\beta_{Z\ \chi}^2 + 4\beta_{Z\ \chi}}{(1-\beta_{Z\ \chi})^3}$$

Proof

In general the k^{th} conditional non central moment $E\left(N^k \mid Z = z \ \chi^2 = y\right)$ is the k^{th} derivative of the conditional moment generating function $M_{N|Z\ \chi}\ (t)$ evaluated in the point zero i e

$$E\left(N^k \mid Z = z \ \chi^2 = y\right) = \frac{d^k}{dt^k} M_{N|Z\ \chi}\ (t)|_{t\ 0}$$

Therefore

$$E\left(N^2 \mid Z = z \ \chi^2 = y\right)$$

$$= \frac{\partial^2}{\partial t^2} M_{N|Z\ \chi}\ (t)_{|t\ 0}$$

$$= \frac{\left(-1-\beta_{Z\ \chi}\right)e^t\left(1+\beta_{Z\ \chi}\, e^t\right)}{\left(-1+\beta_{Z\ \chi}\, e^t\right)^3}|_{t\ 0}$$

$$= \frac{1+\beta_{Z\ \chi}}{\left(1-\beta_{Z\ \chi}\right)^2}$$

and

$$E(N^3 \mid Z = z \ \chi^2 = y)$$

$$= \frac{\partial^3}{\partial t^3} M_{N|Z\ \chi}\ (t)_{|t\ 0}$$

$$= \frac{\left(-1-\beta_{Z\ \chi}\right)e^t\left(1+\beta_{Z\ \chi}^2\, e^{2t} + 4\beta_{Z\ \chi}\, e^t\right)}{\left(-1+\beta_{Z\ \chi}\, e^t\right)^4}|_{t\ 0}$$

$$= \frac{1+\beta_{Z\ \chi}^2 + 4\beta_{Z\ \chi}}{(1-\beta_{Z\ \chi})^3}$$

Result 3 75 Second non central moment and Third non central moment – Unconditional

$$E(N^2) = \int_{-\infty}^{\infty}\int_{0}^{\infty}\frac{1+\beta_{Z\ \chi}}{(1-\beta_{Z\ \chi})^2}\phi(z)f_{\chi}(y)dydz \quad \text{and} \quad E(N^3) = \int_{-\infty}^{\infty}\int_{0}^{\infty}\frac{1+\beta_{Z\ \chi}^2+4\beta_{Z\ \chi}}{(1-\beta_{Z\ \chi})^3}\phi(z)f_{\chi}(y)dydz$$

Proof

In general the k^{th} unconditional non central moment can be obtained from the k^{th} conditional non central moment i e $E\left(N^k\right)=E_{Z\ \chi}\left(E\left(N^k \mid Z=z\ \chi^2=y\right)\right)$

Therefore

$E(N^2)$

$= E_{Z\ \chi}(E(N^2 \mid Z=z\ \chi^2=y))$

$= \int_{-\infty}^{\infty}\int_{0}^{\infty}\frac{1+\beta_{Z\ \chi}}{(1-\beta_{Z\ \chi})^2}\phi(z)f_{\chi}(y)dydz$

and

$E(N^3)$

$= E_{Z\ \chi}(E(N^3 \mid Z=z\ \chi^2=y))$

$= \int_{-\infty}^{\infty}\int_{0}^{\infty}\frac{1+\beta_{Z\ \chi}^2+4\beta_{Z\ \chi}}{(1-\beta_{Z\ \chi})^3}\phi(z)f_{\chi}(y)dydz$

Result 3 76 Variance and Standard deviation of the run length distribution – Conditional

$$\text{var}(N \mid Z = z\ \chi^2 = y) = \frac{\beta_{z\ \chi}}{(1-\beta_{z\ \chi})^2} \quad \text{and} \quad stdev(N \mid Z = z\ \chi^2 = y) = \frac{\sqrt{\beta_{z\ \chi}}}{1-\beta_{z\ \chi}}$$

Proof

The variance of the conditional run length distribution and the standard deviation of the conditional run length distribution are obtained from their definitions i e

$$\begin{aligned}&\text{var}(N \mid Z = z\ \chi^2 = y)\\&= E(N - E(N \mid Z = z\ \chi^2 = y) \mid Z = z\ \chi^2 = y)^2\\&= E(N^2 \mid Z = z\ \chi^2 = y) - E(N \mid Z = z\ \chi^2 = y)^2\end{aligned}$$

and

$$stdev(N \mid Z = z\ \chi^2 = y) = \sqrt{\text{var}(N \mid Z = z\ \chi^2 = y)}$$

Thus substituting the expressions for the first non central moment $E(N \mid Z = z\ \chi^2 = y)$ and the second non central moment $E(N^2 \mid Z = z\ \chi^2 = y)$ of the conditional run length distribution we obtain

$$\text{var}(N \mid Z = z\ \chi^2 = y) = \frac{1+\beta_{z\ \chi}}{\left(1-\beta_{z\ \chi}\right)^2} - \frac{1}{\left(1-\beta_{z\ \chi}\right)^2} = \frac{\beta_{z\ \chi}}{(1-\beta_{z\ \chi})^2}$$

and

$$stdev(N \mid Z = z\ \chi^2 = y) = \frac{\sqrt{\beta_{z\ \chi}}}{1-\beta_{z\ \chi}}$$

Result 3 77 Variance and Standard deviation of the run length distribution – Unconditional

$$\mathrm{var}(N) = \int_{-\infty}^{\infty}\int_{0}^{\infty}\frac{\beta_{Z\ \chi}}{\left(1-\beta_{Z\ \chi}\right)^2}\phi(z)f_{\chi}\ (y)dydz + \int_{-\infty}^{\infty}\int_{0}^{\infty}\left(\frac{1}{1-\beta_{Z\ \chi}}\right)^2\phi(z)f_{\chi}\ (y)dydz - \left(\int_{-\infty}^{\infty}\int_{0}^{\infty}\frac{1}{1-\beta_{Z\ \chi}}\phi(z)f_{\chi}\ (y)dydz\right)^2$$

or

$$\mathrm{var}(N) = \int_{-\infty}^{\infty}\int_{0}^{\infty}\frac{1+\beta_{Z\ \chi}}{(1-\beta_{Z\ \chi})^2}\phi(z)f_{\chi}\ (y)dydz - \left(\int_{-\infty}^{\infty}\int_{0}^{\infty}\frac{1}{1-\beta_{Z\ \chi}}\phi(z)f_{\chi}\ (y)dydz\right)^2$$

Proof

Expression 1

Using the fact that

$$E_{Z\ \chi}\left(\mathrm{var}\left(N \mid Z=z\ \chi^2=y\right)\right) = \mathrm{var}(N) - \mathrm{var}_{Z\ \chi}\left(E\left(N \mid Z=z\ \chi^2=y\right)\right)$$

we can find the unconditional variance i e $\mathrm{var}(N)$ and subsequently the unconditional standard deviation i e $stdev(N)$ of the run length random variable by using the conditional expected value and the conditional variance of N given $Z=z$ and $\chi^2=y$ i e using $\mathrm{var}\left(N \mid Z=z\ \chi^2=y\right)$ and $E\left(N \mid Z=z\ \chi^2=y\right)$

Thus solving for $\mathrm{var}(N)$ in the above expression we have

$$\mathrm{var}(N) = E_{Z\ \chi}\left(\mathrm{var}\left(N \mid Z=z\ \chi^2=y\right)\right) + \mathrm{var}_{Z\ \chi}\left(E\left(N \mid Z=z\ \chi^2=y\right)\right)$$

Now

$$\mathrm{var}\left(N \mid Z=z\ \chi^2=y\right) = \frac{\beta_{\ \chi}}{\left(1-\beta_{\ \chi}\right)^2} \qquad \text{and} \qquad E\left(N \mid Z=z\ \chi^2=y\right) = \frac{1}{1-\beta_{Z\ \chi}}$$

so that

$$\begin{aligned}\mathrm{var}_{Z\ \chi}\left(E\left(N \mid Z=z\ \chi^2=y\right)\right) &= E_{Z\ \chi}\left(\left(E\left(N \mid Z=z\ \chi^2=y\right)\right)^2\right) - \left(E_{Z\ \chi}\left(E\left(N \mid Z=z\ \chi^2=y\right)\right)\right)^2 \\ &= E_{Z\ \chi}\left(\left(\frac{1}{1-\beta_{Z\ \chi}}\right)^2\right) - \left(E_{Z\ \chi}\left(\frac{1}{1-\beta_{Z\ \chi}}\right)\right)^2\end{aligned}$$

and by substitution we have that

$$\operatorname{var}(N)=E_{Z\ \chi}\left(\frac{\beta_{Z\ \chi}}{\left(1-\beta_{Z\ \chi}\right)^2}\right)+E_{Z\ \chi}\left(\left(\frac{1}{1-\beta_{Z\ \chi}}\right)^2\right)-\left(E_{Z\ \chi}\left(\frac{1}{1-\beta_{Z\ \chi}}\right)\right)^2$$

Re writing this expression in terms of integrals we obtain the unconditional variance

$$\operatorname{var}(N)=\int_{-\infty}^{\infty}\int_{0}^{\infty}\frac{\beta_{Z\ \chi}}{\left(1-\beta_{Z\ \chi}\right)^2}\phi(z)f_{\chi}(y)dydz+\int_{-\infty}^{\infty}\int_{0}^{\infty}\left(\frac{1}{1-\beta_{Z\ \chi}}\right)^2\phi(z)f_{\chi}(y)dydz-\left(\int_{-\infty}^{\infty}\int_{0}^{\infty}\frac{1}{1-\beta_{Z\ \chi}}\phi(z)f_{\chi}(y)dydz\right)^2$$

Expression 2

Another route in obtaining the unconditional variance of the run length random variable is by making use of the first and second non central moments of the conditional run length distribution i e using $E\left(N\mid Z=z\ \chi^2=y\right)$ and $E\left(N^2\mid Z=z\ \chi^2=y\right)$

We have that

$$\operatorname{var}(N)=E\left(N^2\right)-E(N)^2$$

but

$$E\left(N^2\right)=E_{Z\ \chi}\left(E\left(N^2\mid Z=z\ \chi^2=y\right)\right)\quad\text{and}\quad E(N)=E_{Z\ \chi}\left(E\left(N\mid Z=z\ \chi^2=y\right)\right)$$

so that

$$\operatorname{var}(N)=E_{Z\ \chi}\left(E\left(N^2\mid Z=z\ \chi^2=y\right)\right)-E_{Z\ \chi}\left(E\left(N\mid Z=z\ \chi^2=y\right)\right)^2$$

Furthermore from the properties of the geometric distribution we have that

$$E\left(N^2\mid Z=z\ \chi^2=y\right)=\frac{1+\beta_{\ \chi}}{(1-\beta_{\ \chi})^2}\quad\text{and}\quad E\left(N\mid Z=z\ \chi^2=y\right)=\frac{1}{1-\beta_{\ \chi}}$$

so that a second expression for the unconditional variance of the run length random variable is given by

$$\operatorname{var}(N)=\int_{-\infty}^{\infty}\int_{0}^{\infty}\frac{1+\beta_{Z\ \chi}}{(1-\beta_{Z\ \chi})^2}\phi(z)f_{\chi}(y)dydz-\left(\int_{-\infty}^{\infty}\int_{0}^{\infty}\frac{1}{1-\beta_{Z\ \chi}}\phi(z)f_{\chi}(y)dydz\right)^2$$

Result 3 78 Coefficient of skewness of the run length distribution – Conditional

$$skew(N \mid Z = z \;\; \chi^2 = y) = \frac{1+\beta_{z\,\chi}}{\sqrt{\beta_{z\,\chi}}}$$

Proof

The conditional coefficient of skewness is defined as

$$skew(N \mid Z = z \;\; \chi^2 = y) = \frac{E(N - E(N \mid Z = z \;\; \chi^2 = y) \mid Z = z \;\; \chi^2 = y)^3}{\text{var}(N \mid Z = z \;\; \chi^2 = y)^{\frac{3}{2}}}$$

which can be simplified to

$$skew(N \mid Z = z \;\; \chi^2 = y) =$$

$$\frac{E\left(N^3 \mid Z = z \;\; \chi^2 = y\right) - 3E\left(N^2 \mid Z = z \;\; \chi^2 = y\right)E\left(N \mid Z = z \;\; \chi^2 = y\right) + 2E\left(N \mid Z = z \;\; \chi^2 = y\right)^3}{\left(\text{var}(N \mid Z = z \;\; \chi^2 = y)\right)^{\frac{3}{2}}}$$

Substituting the expressions for the first non central moment $E(N \mid Z = z \;\; \chi^2 = y)$ the second non central moment $E(N^2 \mid Z = z \;\; \chi^2 = y)$ and the third non central moment $E(N^3 \mid Z = z \;\; \chi^2 = y)$ of the conditional run length distribution as well as the expression for the conditional variance $\text{var}(N \mid Z = z \;\; \chi^2 = y)$ the expression for the conditional coefficient of skewness simplifies to

$$skew(N \mid Z = z \;\; \chi^2 = y) = \frac{1+\beta_{z\,\chi}}{\sqrt{\beta_{z\,\chi}}}$$

Result 3 79 Coefficient of skewness of the run length distribution – Unconditional

$$skew(N)=$$

$$\frac{\int_{-\infty}^{\infty}\int_{0}^{\infty}\frac{1+\beta \quad +4\beta}{\left(1-\beta \quad\right)}\phi(\ )f\ \ (y)dyd\ -3\left\{\int_{-\infty}^{\infty}\int_{0}^{\infty}\frac{1+\beta}{(1-\beta \quad)}\phi(\ )f\ \ (y)dyd\right\}\left\{\int_{-\infty}^{\infty}\int_{0}^{\infty}\frac{1}{1-\beta}\phi(\ )f\ \ (y)dyd\right\}+2\left(\int_{-\infty}^{\infty}\int_{0}^{\infty}\frac{1}{1-\beta}\phi(\ )f\ \ (y)dyd\right)}{(\mathrm{var}(N))}$$

Proof

To obtain the unconditional coefficient of skewness of the run length random variable 1 e

$$skew(N)=\frac{E(N-E(N))^3}{\mathrm{var}(N)^{\frac{3}{2}}}=\frac{E\left(N^3\right)-3E\left(N^2\right)E\left(N\right)+2E\left(N\right)^3}{\left(\mathrm{var}(N)\right)^{\frac{3}{2}}}$$

one uses the first the second and the third non central moments of the conditional run length distribution 1 e one uses $E\left(N\,|\,Z=z\ \ \chi^2=y\right)$ $E\left(N^2\,|\,Z=z\ \ \chi^2=y\right)$ and $E\left(N^3\,|\,Z=z\ \ \chi^2=y\right)$ as well as the fact that $E(N^k)=E_{Z\ \chi}\ (E(N^k\,|\,Z=z\ \ \chi^2=y))$

Thus an expression for the unconditional coefficient of skewness of the run length random variable is given by

$$skew(N)=$$

$$\frac{\int_{-\infty}^{\infty}\int_{0}^{\infty}\frac{1+\beta \quad +4\beta}{\left(1-\beta \quad\right)}\phi(\ )f\ \ (y)dyd\ -3\left\{\int_{-\infty}^{\infty}\int_{0}^{\infty}\frac{1+\beta}{(1-\beta \quad)}\phi(\ )f\ \ (y)dyd\right\}\left\{\int_{-\infty}^{\infty}\int_{0}^{\infty}\frac{1}{1-\beta}\phi(\ )f\ \ (y)dyd\right\}+2\left(\int_{-\infty}^{\infty}\int_{0}^{\infty}\frac{1}{1-\beta}\phi(\ )f\ \ (y)dyd\right)}{(\mathrm{var}(N))}$$

Chapter 4 Cumulative Sum and Exponentially Weighted Moving Average Control Charts

4 0 Chapter overview

In Chapter 3 we focused on monitoring the process mean using Shewhart type of control charts
A disadvantage of Shewhart type of control charts is the fact that they only use the information at the current point in time For example at time $t = 5$ only the subgroup statistic T_5 say is plotted on a control chart and compared to the control limits to decide whether the process is in control or not At time $t = 5$ the information from previous subgroups i e $t = 0\ 1\ 2\ 3$ and 4 are typically not incorporated in the decision making process This makes the Shewhart type of control chart insensitive for small shifts in the process mean i e shifts of size $1\ 5\sigma - 2\sigma$ or less
To increase the sensitivity of the Shewhart type of control chart to detect small shifts sensitivity rules and warning limits have been developed However there are disadvantages in using these additional rules they dramatically reduce the in control average run length (ARL_0) caused by an increase in the false alarm rate (FAR) and reduce the simplicity of interpreting the control charts
Two effective alternatives to Shewhart type of control charts are the cumulative sum (or CUSUM) type of control chart and the exponentially weighted moving average (or EWMA) type of control chart These control chart procedures are discussed in Chapter 4

4 1 Cumulative sum (CUSUM) control chart for the process mean

4 1 1 Introduction

A CUSUM type of control chart incorporates all the information in a series of samples from the beginning up and till the current point in time t say This is accomplished by plotting a function of the partial (or cumulative) sums of a series of differences between sample statistics and a target value However before constructing a CUSUM type of control chart it is necessary to consider CUSUM plots

4 1 2 CUSUM-plots

A CUSUM plot is a graph of the cumulative sums of a series of differences between sample statistics and a target value versus the sample number or time

Given a series of successive sample statistics T_1 T_2 T_3 and a target value θ the first step in constructing a CUSUM plot is to calculate the differences between the sample statistics and the target value i e $(T_1 - \theta)$ $(T_2 - \theta)$ $(T_3 - \theta)$ This is followed by calculating the cumulative sums of these differences i e

$$S_1 = (T_1 - \theta)$$
$$S_2 = (T_1 - \theta) + (T_2 - \theta)$$

$$S_t = (T_1 - \theta) + (T_2 - \theta) + \quad + (T_t - \theta)$$

In general the cumulative sum at time t can be calculated using

$$\begin{aligned} S_t &= \sum_{i=1}^{t} (T_i - \theta) \\ &= \sum_{i=1}^{t-1} (T_i - \theta) + (T_t - \theta) \qquad (4\ 1) \\ &= S_{t-1} + (T_t - \theta) \end{aligned}$$

with $t = 1\ 2\ 3$

Substituting $t = 1$ in equation (4 1) yields $S_1 = S_0 + (T_1 - \theta)$ which calls for a starting value S_0 This starting value is denoted by u and can be any value but is typically set equal to the target value or equal to zero so that $S_0 = \theta$ or $S_0 = 0$

The recursive calculation formula of equation (4 1) is very useful for constructing a CUSUM plot because the cumulative sum at time t (S_t) is obtained by merely adding $(T_t - \theta)$ to the cumulative sum at time $t-1$ i e S_{t-1}

Once these calculations are complete the final step is to plot S_t versus the sample number or time t to produce a CUSUM plot

Example 4 1

A CUSUM plot of S_t

Column (a) of Table 4 1 contains 30 simulated values that represent samples of size $n = 1$ Values 1 through 20 were randomly drawn from a normal distribution with $\mu = 10$ and $\sigma = 1$ whereas the last ten values (21 through 30) were randomly drawn from a normal distribution with $\mu = 11$ and $\sigma = 1$ Therefore the first 20 values can be thought of as being observations from a process operating on target or in control whereas the last ten values can be considered to be observations from the same process following a sustained shift in the process mean and thus being off target or out of control Furthermore suppose that the sample statistic of interest is in fact the observed value so that $T = X$

Sample t	(a) X_t	(b) $X_t - 10$	(c) $S_t = S_{t-1} + (X_t - 10)$
1	10 0	0 0	0 0
2	8 5	1 5	1 5
3	10 5	0 5	1 0
4	10 7	0 7	0 3
5	10 4	0 4	0 1
6	9 0	1 0	0 8
7	10 2	0 2	0 6
8	10 1	0 1	0 5
9	10 3	0 3	0 2
10	8 8	1 2	1 4
11	9 7	0 3	1 7
12	9 8	0 2	1 9
13	10 2	0 2	1 7
14	9 9	0 1	1 8
15	11 5	1 5	0 3
16	11 0	1 0	0 7
17	9 8	0 2	0 6
18	9 4	0 6	0 1
19	10 4	0 4	0 4
20	9 4	0 6	0 3
21	11 2	1 2	0 9
22	10 5	0 5	1 4
23	10 3	0 3	1 7
24	11 5	1 5	3 2
25	11 3	1 3	4 6
26	10 9	0 9	5 5
27	12 0	0 5	7 5
28	12 1	2 1	9 6
29	9 5	1 5	9 1
30	12 5	2 5	11 6

Table 4 1 Simulated data for CUSUM example

A Shewhart control chart for individual measurements or an X chart from Minitab is given in Figure 4 1 with the centerline and the two control limits at

$$UCL = \mu + 3\sigma = 13$$
$$CL = \mu = 10$$
$$LCL = \mu - 3\sigma = 7$$

The first 20 plotting statistics (or observations) are randomly scattered about the centerline with none outside the two control limits Thus there is no concern for any assignable causes On the other hand although the last 10 observations plot between the two control limits giving no strong evidence that the process is out of control all the measurements except observation 29 (with $X_{29} = 9\ 5$) plot above the centerline This pattern indicates that the process mean might have experienced an upward shift

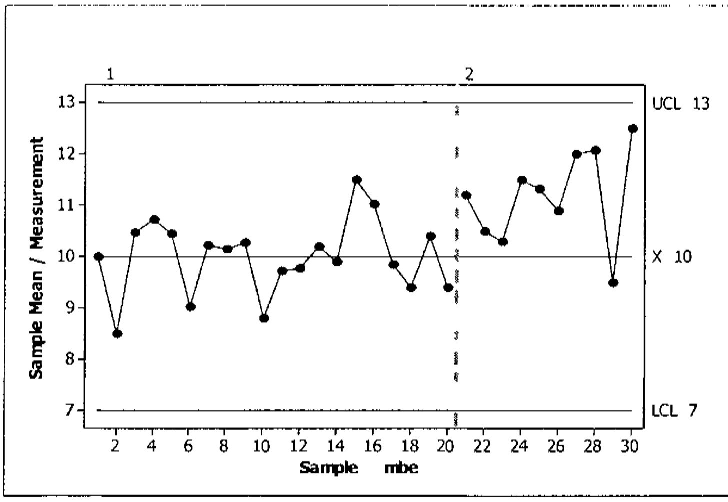

Figure 4 1 A Shewhart control chart for individual measurements

Therefore knowing that the last 10 observations originated from a process operating out of control and relying on the traditional signal of an out of control process (one or more points plot beyond the control limits) the shift in the process mean went undetected and the Shewhart type of control chart failed

This failure is caused by the relatively small magnitude of the process shift – only 1σ A Shewhart type of control chart is more effective in detecting shifts of size $1\ 5\sigma - 2\sigma$ or larger Therefore when trying to detect smaller process shifts another type of control chart such as a CUSUM type of control chart might be more appropriate

Hence the deviations of the observations from the target value and their cumulative sums i e $(X_t - \theta)$ and $S_t = S_{t-1} + (X_t - \theta)$ with $S_0 = 0$ and $\theta = 10$ were calculated added in columns (b) and (c) of Table 4 1 and used to construct the CUSUM plot of Figure 4 2

The CUSUMS up and till time $t = 20$ where $\mu = 10$ tends to drift slowly but maintains values near zero However for the last 10 CUSUMS where $\mu = 11$ a sudden upward trend develops which indicate an upward shift in the process mean

Comparing Figure 4 1 with Figure 4 2 it is clear that the CUSUM plot has a superior ability to unmask a small process shift

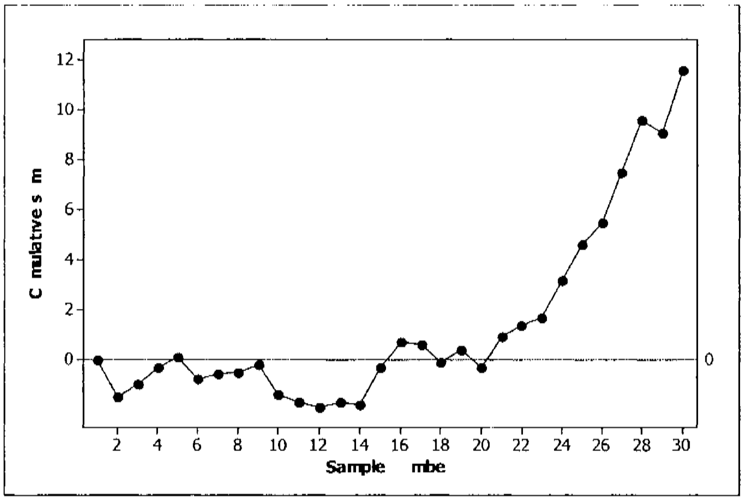

Figure 4 2 A CUSUM plot of S_t

To better understand why this is true suppose that m independent samples (each of size $n \geq 1$) are randomly drawn from an in control process following a normal distribution with a *known* mean and a *known* standard deviation denoted by μ_0 and σ_0 respectively Furthermore suppose that the sample statistic of interest is the sample mean i e $T = \overline{X}$ and set $S_0 = 0$ and $\theta = \mu_0$ Thus $\overline{X}_1\ \overline{X}_2\ \ \overline{X}_m \sim iidN\left(\mu_0\ \frac{\sigma_0^2}{n}\right)$ and if the process is in control at the target value μ_0 we have that

$$\left(\overline{X}_1 - \mu_0\right)\ \left(\overline{X}_2 - \mu_0\right)\quad \left(\overline{X}_m - \mu_0\right) \sim iidN\left(0\ \frac{\sigma_0^2}{n}\right)$$

Consequently $S_t = \sum_1^t \left(\overline{X} - \mu_0\right)$ with $1 \leq t \leq m$ is a linear combination of t independent identically distributed normal random variables each with a mean of zero and a variance of $\frac{\sigma_0^2}{n}$ so that $E(S_t) = 0$ and $\text{var}(S_t) = t\frac{\sigma_0^2}{n}$ It follows that S_t is random walk with an expected value of zero or no drift

Next assume that the process mean undergoes a sustained shift and assume that a further k independent samples are randomly drawn from the same process Following the shift the process mean is $\mu_1 = \mu_0 + \delta\sigma_0$ that is the new process mean is $\delta\sigma_0$ units different from the old process mean for all future subgroups In this case $\overline{X}_{m+1}, \overline{X}_{m+2}, \ldots, \overline{X}_{m+k} \sim iidN\left(\mu_1 = \mu_0 + \delta\sigma_0, \frac{\sigma_0^2}{n}\right)$ and by subtracting the target value for the process mean μ_0, it follows that

$$\left(\overline{X}_{m+1} - \mu_0\right), \left(\overline{X}_{m+2} - \mu_0\right), \ldots, \left(\overline{X}_{m+k} - \mu_0\right) \sim iidN\left(\delta\sigma_0, \frac{\sigma_0^2}{n}\right)$$

Thus $S_t = \sum_{1}^{t}\left(\overline{X} - \mu_0\right) = \sum_{1}^{m}\left(\overline{X} - \mu_0\right) + \sum_{m+1}^{t}\left(\overline{X} - \mu_0\right)$ with $m+1 \leq t \leq m+k$ is a linear combination of m independent identically distributed normal random variables each with a mean of zero and a variance of $\frac{\sigma_0^2}{n}$ and an additional $t-m$ independent identically distributed normal random variables each with a mean of $\delta\sigma_0$ and a variance of $\frac{\sigma_0^2}{n}$. Therefore $E\left(S_t\right) = \left(t-m\right)\delta\sigma_0$ and $\text{var}\left(S_t\right) = t\frac{\sigma_0^2}{n}$ and it follows that S_t is random walk with drift given by its expected value $\left(t-m\right)\delta\sigma_0$.

Thus, if the process mean undergoes a sustained shift, however small it might be, the CUSUM plot will develop either an upward trend or a downward trend. These situations correspond to either an upward shift ($\delta > 0$) or a downward shift ($\delta < 0$) in the process mean. When $\delta = 0$, no shift occurred and the CUSUM plot maintains values near zero.

Although it is desirable to unmask a shift in the process mean one does not necessarily want to detect an extremely small or an insignificant shift. Therefore, the fact that even the smallest shift in the process mean will cause a CUSUM plot to develop a trend, suggests caution and the need to "desensitize" a CUSUM plot to only react to shifts of a specified size.
To successfully "desensitize" a CUSUM plot a reference value K is selected which specifies the size of the anticipated shift. If K is set equal to half the magnitude of the anticipated shift we find that
$K = \frac{1}{2}\left|\delta\right|\sigma_0 = \frac{\left|\mu_1 - \mu_0\right|}{2}$.

Now, instead of calculating $\left(\overline{X}_1 - \mu_0\right), \left(\overline{X}_2 - \mu_0\right), \left(\overline{X}_3 - \mu_0\right), \ldots$, two alternative sequences are calculated. These sequences are

$$\left(\overline{X}_1 - \left(\mu_0 + K\right)\right), \left(\overline{X}_2 - \left(\mu_0 + K\right)\right), \left(\overline{X}_3 - \left(\mu_0 + K\right)\right), \ldots \quad (4.2a)$$

and

$$\left(\overline{X}_1-(\mu_0-K)\right)\ \left(\overline{X}_2-(\mu_0-K)\right)\ \left(\overline{X}_3-(\mu_0-K)\right) \qquad (4\ 2b)$$

Sequences (4 2a) and (4 2b) will be used to construct a CUSUM type of control chart for monitoring the process mean The first sequence is useful for detecting an upward shift in the process mean whereas the second sequence is useful for detecting a downward shift in the process mean

To observe the usefulness of sequences (4 2a) and (4 2b) it is appropriate to consider the statistical properties of their cumulative sums S_t^U and S_t^L say Given sequences (4 2a) and (4 2b) their cumulative sums are calculated as

$$S_1^U=\left(\overline{X}_1-(\mu_0+K)\right)$$
$$S_2^U=\left(\overline{X}_1-(\mu_0+K)\right)+\left(\overline{X}_2-(\mu_0+K)\right)$$

$$S_t^U=\left(\overline{X}_1-(\mu_0+K)\right)+\left(\overline{X}_2-(\mu_0+K)\right)+\quad+\left(\overline{X}_t-(\mu_0+K)\right)$$

and

$$S_1^L=\left(\overline{X}_1-(\mu_0-K)\right)$$
$$S_2^L=\left(\overline{X}_1-(\mu_0-K)\right)+\left(\overline{X}_2-(\mu_0-K)\right)$$

$$S_t^L=\left(\overline{X}_1-(\mu_0-K)\right)+\left(\overline{X}_2-(\mu_0-K)\right)+\quad+\left(\overline{X}_t-(\mu_0-K)\right)$$

Similar to expression (4 1) general expressions for S_t^U and S_t^L are

$$\begin{aligned} S_t^U &= \sum_1^t\left(\overline{X}\ -(\mu_0+K)\right) \\ &= S_{t-1}^U+\left\{\overline{X}_t-(\mu_0+K)\right\} \qquad (4\ 3a) \\ &= S_t-tK \end{aligned}$$

and

$$
\begin{aligned}
S_t^L &= \sum_1^t \left(\overline{X} - (\mu_0 - K) \right) \\
&= S_{t-1}^L + \left\{ \overline{X}_t - (\mu_0 - K) \right\} \\
&= S_t + tK
\end{aligned}
\qquad (4\ 3b)
$$

with $t = 1\ 2\ 3$ and $S_0^U = S_0^L = 0$

When the process is in control

$$E\left(S_t^U\right) = E\left(S_t\right) - tK = -tK = -\frac{1}{2}t\delta\sigma_0$$

and

$$E\left(S_t^L\right) = E\left(S_t\right) + tK = tK = \frac{1}{2}t\delta\sigma_0$$

Thus a CUSUM plot of S_t^U (which will ultimately be used to detect an upward shift) will have a negative trend whereas a CUSUM plot of S_t^L (which will ultimately be used to detect a downward shift) will have a positive trend This is not what we would expect considering their distinctive purposes

However if the process mean experiences a sustained shift after m independent samples and sampling continues the expected values of S_t^U and S_t^L are

$$
\begin{aligned}
E\left(S_t^U\right) &= E\left(S_t\right) - tK \\
&= (t-m)\delta\sigma_0 - tK \\
&= \left(\frac{t}{2} - m\right)\delta\sigma_0
\end{aligned}
$$

and

$$
\begin{aligned}
E\left(S_t^L\right) &= E\left(S_t\right) + tK \\
&= (t-m)\delta\sigma_0 + tK \\
&= \left(\frac{3}{2}t - m\right)\delta\sigma_0
\end{aligned}
$$

with $t = m+1\ m+2$

Therefore following a sustained shift in the process mean the expected value of S_t^U changes from $-\frac{1}{2}t\delta\sigma_0$ to $\left(\frac{t}{2} - m\right)\delta\sigma_0$ whereas the expected value of S_t^L changes from $\frac{1}{2}t\delta\sigma_0$ to $\left(\frac{3}{2}t - m\right)\delta\sigma_0$

Hence if the process mean experiences an upward shift the slope of a CUSUM plot of S_t^U immediately changes from negative to positive (unmasking the shift) However the expected value of S_t^U i e $\left(\frac{t}{2}-m\right)\delta\sigma_0$ will become positive only once $t>2m$ On the other hand if the process mean experiences a downward shift the slope and the expected value of S_t^U remain negative

Likewise on occurrence of a downward shift in the process mean the slope of a CUSUM plot of S_t^L immediately changes from positive to negative (again unmasking the shift) However the expected value of S_t^L i e $\left(\frac{3}{2}t-m\right)\delta\sigma_0$ will become negative only once $t>\frac{2}{3}m$ Conversely an upward shift in the process mean causes the slope and expected value of S_t^L to remain positive

Table 4 2 summarizes these changes in the gradients of S_t^U and S_t^L following a sustained shift in the process mean

	In control ($\delta=0$)	Out of control ($\delta>0$)	Out of control ($\delta<0$)
S_t^U	Negative	Positive +	Negative
S_t^L	Positive +	Positive +	Negative

Table 4 2 Gradients of S_t^U and S_t^L for in control and out of control processes

Example 4 2

CUSUM plots of S_t^U and S_t^L

To illustrate these ideas we have calculated $X_t-(\mu_0+K)$ and $X_t-(\mu_0-K)$ for the 30 values in column (a) of Table 4 1 as well as their cumulative sums S_t^U and S_t^L when $S_0^U=S_0^L=0$ $\mu_0=10$ $\delta=1$ and K is taken as 0 5 These results are displayed in panels (b) and (c) of Table 4 3

Figure 4 3 clearly shows how the downward trend of the CUSUM plot of S_t^U (while the process is in control) changes from point $m=20$ onwards to an upward trend following the sustained upward shift in the process mean However $E\left(S_{21}^U\right)=\left(\frac{21}{2}-20\right)(1)(1)=-9\ 5$ and we do not expect S_t^U to be positive until $t>2m=2(20)=40$

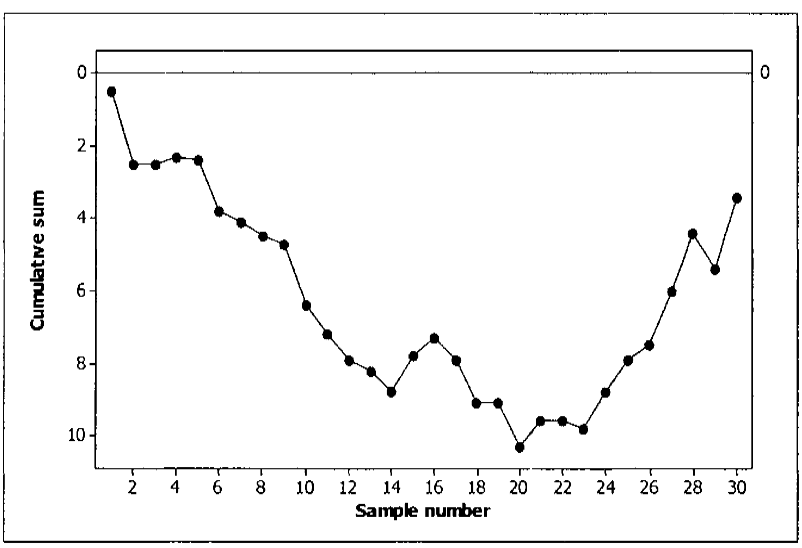

Figure 4 3 A CUSUM plot of S_t^U

On the other hand the CUSUM plot of S_t^L in Figure 4 4 is of little use in detecting (or unmasking) the sustained upward shift in the process mean the slope and the expected value of the CUSUM plot of S_t^L remain positive The only indication of an upward shift is the slight change in the magnitude of the slope of S_t^L from point 20 onwards

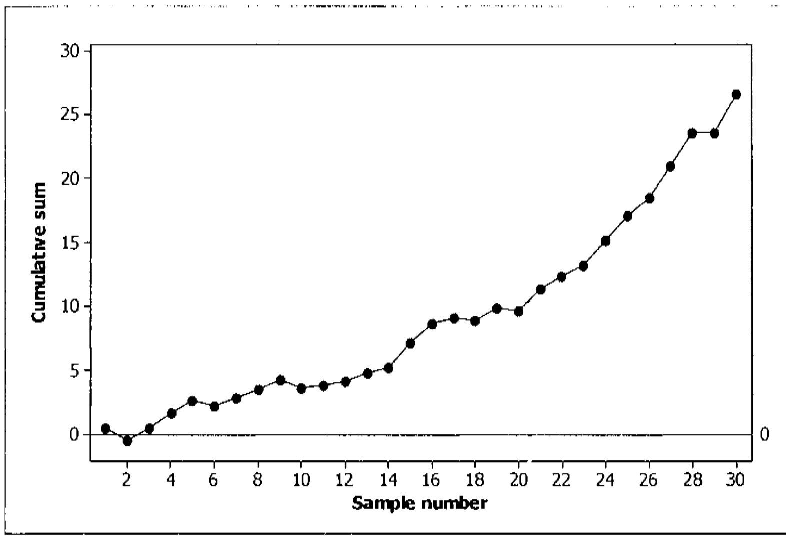

Figure 4 4 A CUSUM plot of S_t^L

Sample t	(a) X_t	(b) $X_t - 10\ 5$	S_t^U	(c) $X_t - 9\ 5$	S_t^L
1	10 0	0 5	0 5	0 5	0 5
2	8 5	2 0	2 5	1 0	0 5
3	10 5	0 0	2 5	1 0	0 5
4	10 7	0 2	2 3	1 2	1 7
5	10 4	0 1	2 4	0 9	2 6
6	9 0	1 5	3 8	0 5	2 2
7	10 2	0 3	4 1	0 7	2 9
8	10 1	0 4	4 5	0 6	3 5
9	10 3	0 2	4 7	0 8	4 3
10	8 8	1 7	6 4	0 7	3 6
11	9 7	0 8	7 2	0 2	3 8
12	9 8	0 7	7 9	0 3	4 1
13	10 2	0 3	8 2	0 7	4 8
14	9 9	0 6	8 8	0 4	5 2
15	11 5	1 0	7 8	2 0	7 2
16	11 0	0 5	7 3	1 5	8 7
17	9 8	0 7	7 9	0 3	9 1
18	9 4	1 1	9 1	0 1	8 9
19	10 4	0 1	9 1	0 9	9 9
20	9 4	1 1	10 3	0 1	9 7
21	11 2	0 7	9 6	1 7	11 4
22	10 5	0 0	9 6	1 0	12 4
23	10 3	0 2	9 8	0 8	13 2
24	11 5	1 0	8 8	2 0	15 2
25	11 3	0 8	7 9	1 8	17 1
26	10 9	0 4	7 5	1 4	18 5
27	12 0	1 5	6 0	2 5	21 0
28	12 1	1 6	4 4	2 6	23 6
29	9 5	1 0	5 4	0 0	23 6
30	12 5	2 0	3 4	3 0	26 6

Table 4 3 Simulated data for CUSUM example

Although the CUSUM plots of S_t S_t^U and S_t^L can easily unmask changes in a process these CUSUM plots are not control charts since they all lack statistical control limits Therefore instead of relying on a CUSUM plot to unmask a process change we will create CUSUM type of control charts to signal once a process change occurred However the plotting statistics for the CUSUM type of control charts are defined as functions of S_t S_t^U and S_t^L

The tabular (or algorithmic) CUSUM procedure and the V mask procedure are two CUSUM type of control chart procedures that are often used The tabular CUSUM procedure is preferred and therefore its statistical properties will be discussed in more detail in the next sections This is followed by a brief discussion of the V mask procedure

4 1 3 The tabular CUSUM control chart

To monitor the process mean a CUSUM type of control chart can be constructed for individual measurements as well as for the averages of rational subgroups However the discussion will focus on individual measurements and these results will later be modified for rational subgroups Therefore let X_t be the t^{th} observation in a series of observations X_1 X_2 X_3 and assume that the process follows a normal distribution with a *known* mean (μ_0) and a *known* standard deviation (σ_0)

A tabular CUSUM control chart operates by utilizing the two sequences of random variables given in equations (4 2a) and (4 2b) i e

$$\left(X_1-\left(\mu_0+K\right)\right)\ \left(X_2-\left(\mu_0+K\right)\right)\ \left(X_3-\left(\mu_0+K\right)\right)$$

and

$$\left(X_1-\left(\mu_0-K\right)\right)\ \left(X_2-\left(\mu_0-K\right)\right)\ \left(X_3-\left(\mu_0-K\right)\right)$$

and calculating two plotting statistics S_t and S_t

These plotting statistics are defined as

$$S_t = \max\left(0\ S_{t-1}+X_t-\left(\mu_0+K\right)\right) \qquad (4\ 4a)$$

and

$$S_t = \max\left(0\ S_{t-1}-X_t+\left(\mu_0-K\right)\right) \qquad (4\ 4b)$$

with $t=1\ 2\ 3$ and both starting values typically set equal to zero i e $S_0 = S_0 = 0$ However for starting values other than zero let $S_0 = S_0 = u$

S_t is called the upper one sided CUSUM plotting statistic and accumulates deviations that are above the adjusted upper target value (μ_0+K) whereas S_t^- is called the lower one sided CUSUM plotting statistic and accumulates deviations that are below the adjusted lower target value (μ_0-K)

Together S_t and S_t are used to monitor the process mean for both upward and downward shifts As long as $0\le S_t < H$ and $0 \le S_t < H$ the process is considered to be in control but if either $S_t \ge H$ or $S_t \ge H$ the process is declared out of control

The constant H is called the decision interval and is similar to the control limits of a Shewhart type of control chart Furthermore H is the only decision interval since both S_t and S_t are always larger than or equal to zero

However the tabular CUSUM control chart can be altered to have an upper decision interval as well as a lower decision interval For example re writing the lower one sided CUSUM plotting statistic defined in equation (4 4b) i e S_t to be less than or equal to zero we have

$$S_t = \max\left(0 \; S_{t-1} + X_t - (\mu_0 + K)\right) \qquad (4\ 5a)$$

and

$$S_t = \min\left(0 \; S_{t-1} + X_t - (\mu_0 - K)\right) \qquad (4\ 5b)$$

with $S_t = u$ and $S_t = -u$

(See Result 4 1 in Appendix 4)

When using definitions (4 5a) and (4 5b) the process is declared out of control when either $S_t \geq H$ or $S_t \leq -H$ and considered in control while $0 \leq S_t < H$ and $-H < S_t \leq 0$

Example 4 3

A tabular CUSUM control chart

Up and till now we have made no decision whether the process of Example 4 1 operated in control or out of control following the sustained shift in the process mean between samples 20 and 21 For this reason to statistically determine whether the process operated in control or out of control a tabular CUSUM control chart is constructed for the data in column (a) of Table 4 1

The two one sided CUSUM plotting statistics S_t and S_t with $S_0^+ = S_0 = 0$ and $K = 0\ 5$ were calculated and added to Table 4 4 The decision interval H was chosen to be 4 (The choice of H is discussed in a later section)

To illustrate the calculations consider sample number 1 The equations for the plotting statistics S_1 and S_1 are

$$\begin{aligned} S_1^+ &= \max\left(0 \; S_0 + X_1 - (\mu_0 + K)\right) \\ &= \max\left(0 \; 0 + 10 - (10 + 0\ 5)\right) \\ &= 0 \end{aligned}$$

and

$$
\begin{aligned}
S_1 &= \max\left(0\ S_0 - X_1 + \left(\mu_0 - K\right)\right) \\
&= \max\left(0\ 0 - 10 + \left(10 - 0\ 5\right)\right) \\
&= 0
\end{aligned}
$$

For sample number 2 S_2^+ and S_2 are

$$
\begin{aligned}
S_2 &= \max\left(0\ S_1 + X_2 - \left(\mu_0 + K\right)\right) \\
&= \max\left(0\ 0 + 8\ 5 - \left(10 + 0\ 5\right)\right) \\
&= 0
\end{aligned}
$$

and

$$
\begin{aligned}
S_2 &= \max\left(0\ S_1 - X_2 + \left(\mu_0 - K\right)\right) \\
&= \max\left(0\ 0 - 8\ 5 + \left(10 - 0\ 5\right)\right) \\
&= 1\ 0
\end{aligned}
$$

The remaining calculations are summarized in columns (a) and (c) of Table 4 4

Along with S_t and S_t^- two other quantities N and N^- were calculated N counts the number of consecutive periods since S_t became non zero whereas N counts the number of consecutive periods since S_t became non zero

The graphical display of the tabular CUSUM control chart in Figure 4 5 (or the CUSUM status chart) does not signal until sample 27 when S_{27}^+ plots above the decision interval Following the signal the process is declared out of control and a search for assignable causes is started In this case N^+ can be used to indicate where the shift might have occurred For example at time $t = 27$ when the control chart signaled $N = 7$ and it is decided that the process was last in control at time $t = 27 - 7 = 20$ Therefore it is likely that the shift occurred between samples 20 and 21 and that a search for assignable causes should start more or less at sample 20

The benefit of having an indication where the process shift might have occurred is useful and is a feature that a Shewhart type of control chart does not have

		(a)	(b)	(c)	(d)
Sample t	X_t	S_t	N	S_t	N
1	10 0	0	0	0	0
2	8 5	0	0	1 0	1
3	10 5	0	0	0	0
4	10 7	0 2	1	0	0
5	10 4	0 2	2	0	0
6	9 0	0	0	0 5	1
7	10 2	0	0	0	0
8	10 1	0	0	0	0
9	10 3	0	0	0	0
10	8 8	0	0	0 7	1
11	9 7	0	0	0 5	2
12	9 8	0	0	0 2	3
13	10 2	0	0	0	0
14	9 9	0	0	0	0
15	11 5	1 0	1	0	0
16	11 0	1 5	2	0	0
17	9 8	0 9	3	0	0
18	9 4	0	0	0 1	1
19	10 4	0	0	0	0
20	9 4	0	0	0 1	1
21	11 2	0 7	1	0	0
22	10 5	0 7	2	0	0
23	10 3	0 5	3	0	0
24	11 5	1 5	4	0	0
25	11 3	2 3	5	0	0
26	10 9	2 7	6	0	0
27	12 0	4 2 *	7	0	0
28	12 1	5 8 *	8	0	0
29	9 5	4 8 *	9	0	0
30	12 5	6 8 *	10	0	0

Table 4 4 A tabular CUSUM control chart (* means that the control chart signaled)

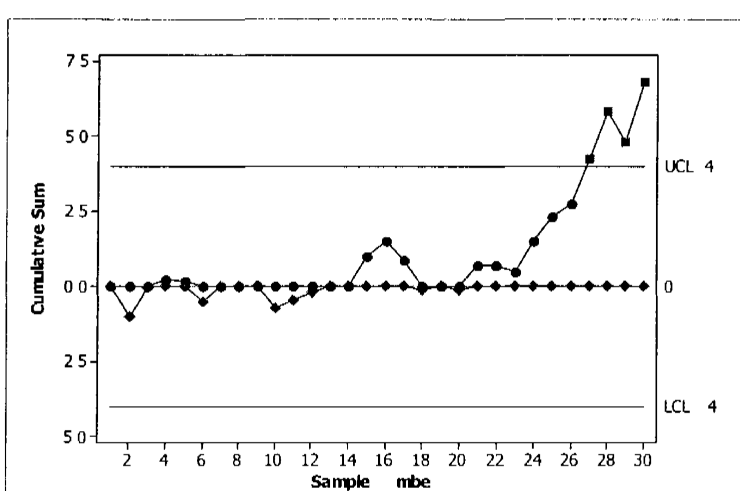

Figure 4 5 A CUSUM status chart

From Table 4 4 and Figure 4 5 it is clear that S_t becomes non zero only once a sample statistic (or observation) is larger than $\mu_0 + K$ Likewise S_t becomes non zero only once a sample statistic (or observation) is less than $\mu_0 - K$ For example $\mu_0 + K = 10 + 0\ 5 = 10\ 5$ and S_t becomes non zero for the first time when $X_4 = 10\ 7 > 10\ 5$ Also $\mu_0 - K = 10 - 0\ 5 = 9\ 5$ and S_t becomes non zero for the first time when $X_2 = 8\ 5 < 9\ 5$

On the other hand S_t will once again become zero if future observations are less than $\mu_0 + K$ and S_t will again become zero if future observations are larger than $\mu_0 - K$

This happens since S_t^+ and S_t are desensitized for very small or insignificant shifts i e any observation within the interval $(\mu_0 - K\ \ \mu_0 + K)$ is still considered on target whereas an observation is only considered off target if it is outside of this interval

In situations where a simple adjustment can bring the process mean back on target or in control it may be helpful to know what the value of the process mean is following a shift The two counters N and N are helpful in this regard

The new process mean can be predicted with

$$\mu_1 = \mu_0 + K + \frac{S_t}{N} \quad \text{if } S_t \geq H \tag{4 6a}$$

or

$$\mu_1 = \mu_0 - K - \frac{S_t^-}{N} \quad \text{if } S_t \geq H \tag{4 6b}$$

Considering Table 4 4 of Example 4 3 with $S_{27} = 4\ 2$ and $N = 7$ it follows from equation (4 6a) that

$$\mu_1 = 10 + 0\ 5 + \frac{4\ 2}{7} = 11\ 1$$

As a result we conclude that the process mean changed from 10 to 11 1 and if possible an adjustment to some manipulative variable is necessary that would change the mean from 11 1 back to 10

4 1 4 The standardized tabular CUSUM control chart

In some circumstances it is preferred to standardize the individual measurements before constructing a CUSUM type of control chart If Y_t denotes the standardized measurement we have that

$$Y_t = \frac{X_t - \mu_0}{\sigma_0} \qquad (4\ 7)$$

so that

$$S_t = \max\left(0\ S_{t\ 1} + Y_t - k\right) \qquad (4\ 8a)$$

and

$$S_t = \max\left(0\ S_{t\ 1}^{-} - Y_t - k\right) \qquad (4\ 8b)$$

with both starting values typically set equal to zero i e $S_0 = S_0 = 0$

The relationship between k and h from the standardized CUSUM control chart and K and H from the un standardized CUSUM control chart is given by $H = h\sigma$ and $K = k\sigma$ respectively

The process is considered in control while $0 \leq S_t < h$ and $0 \leq S_t < h$ and declared out of control when either $S_t^{+} \geq h$ or $S_t \geq h$

When using standardized measurements (or standardized sample statistics when $n > 1$) we have the added benefit that many CUSUM control charts have the same values for k and h This occurs since neither of these standardized values depends on σ_0 i e they are not scale dependent

4 1 5 Rational subgroups

The development of CUSUM plots and the tabular CUSUM control chart were given in terms of individual measurements but can easily be extended to rational subgroups ($n > 1$) where other sample statistics might be of interest

For example if the averages of rational subgroups are of interest we simply replace X with $\overline{X}$ (the sample mean) and replace σ_0 with $\sigma_{\overline{X}} = \frac{\sigma_0}{\sqrt{n}}$

4 1 6 One-sided CUSUM control chart

Although not explicitly mentioned up and till now we have primarily focused on a two sided tabular CUSUM control chart for detecting an upward shift as well as a downward shift in the process mean However note that this two sided procedure uses two one sided plotting procedures namely S_t and S_t and nothing prevents us from using only one of these plotting statistics at a time Therefore if we are interested in only detecting an upward shift in the process mean then only S_t needs to be employed Similarly if we are interested in only a downward shift only S_t is needed Thus it is possible to design two one sided CUSUM control charts each with its own degree of sensitivity This can be done by selecting different reference values K_{p} and $K_{d\ w}$ and/or different decision intervals H_{p} and $H_{d\ w}$ say

4 1 7 Fast initial response or headstart feature

The fast initial response (FIR) or headstart feature improves the sensitivity of a tabular CUSUM control chart at start up and would be desirable if the corrective action following an out of control signal did not reset the process mean to its target value Basically the fast initial response (FIR) or headstart feature only sets the starting values S_0 and S_0 equal to some none zero value – typically $\frac{H}{2}$ which is called a 50% headstart

4 1 8 The conditional tabular CUSUM control chart

In the preceding sections of Chapter 4 it was assumed that the process mean and the process standard deviation were *known* constants However this is seldom the case The process mean and/or the process standard deviation are typically *unknown* and as a result need to be estimated It is customary to estimate these unknown parameters from a reference sample when the process was thought to operate in control

For this purpose suppose that the reference sample consists of m independent samples each of size $n \geq 1$ Unbiased point estimators for the in control process mean μ_0 and the in control process standard deviation σ_0 are

$$\mu_0 = \bar{\bar{X}} = \frac{1}{m}\sum_{i=1}^{m}\bar{X}_i = \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}X_{ij} \qquad (4\ 9a)$$

and

$$\sigma_0 = \frac{S_p}{c_{4\,m}} \qquad (4\ 9b)$$

where

$$S_p = \sqrt{\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\left(X_{ij} - \mu_0\right)^2} \quad \text{if } \mu \text{ is known} \qquad (4\ 10a)$$

or

$$S_p = \sqrt{\frac{1}{m(n-1)}\sum_{i=1}^{m}\sum_{j=1}^{n}\left(X_{ij} - \bar{X}_i\right)^2} \quad \text{if } \mu \text{ is unknown and } n > 1 \qquad (4\ 10b)$$

The constant $c_{4\,m}$ is chosen such that $E\left(S_p\right) = c_{4\,m}\sigma_0$ and given by

$$c_{4\,m} = \frac{\sqrt{2}\,\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu}\,\Gamma\left(\frac{\nu}{2}\right)} \qquad (4\ 11)$$

with $\nu = mn$ or $\nu = m(n-1)$

If the reference value the decision interval and/or the target value are unknown these parameters can be estimated using $\widehat{K} = k\sigma_0$ $\widehat{H} = h\sigma_0$ and $\hat{\theta} = \bar{\bar{X}}$ Following the estimation of all the unknown parameters the tabular CUSUM control chart can be started

Example 4 4

A tabular CUSUM control chart with estimated process parameters

Assume that we need to monitor the process mean (which is also the process target value) using the averages of rational subgroups i e using $\overline{X}$ Furthermore suppose that the process mean (μ) and the process standard deviation (σ) are *unknown*

Panel (a) of Table 4 5 contains the individual observations of 25 independent samples each of size $n = 5$ Regard these observations as an in control reference sample Thus to estimate the unknown process parameters the sample means $\overline{X}$ and the sample variances S^2 were calculated and added in columns (b) and (c)

			(a)			(b)	(c)
Sample i	X_1	X_2	X_3	X_4	X_5	$\overline{X}$	S^2
1	19 2	17 2	20 3	22 4	22 2	20 3	4 7
2	23 3	15 4	19 3	22 0	17 6	19 5	10 3
3	18 4	16 4	16 1	17 8	18 3	17 4	1 2
4	15 6	18 7	19 0	20 1	19 1	18 5	2 9
5	19 1	19 1	22 5	19 6	19 4	19 9	2 1
6	18 8	23 7	21 5	24 6	18 5	21 4	7 7
7	23 1	16 6	20 9	21 6	23 6	21 2	7 7
8	19 6	18 8	21 2	19 0	21 3	20 0	1 4
9	16 9	18 1	16 8	19 1	19 7	18 1	1 7
10	19 9	19 2	24 2	16 3	18 3	19 6	8 5
11	14 6	22 7	17 2	18 5	21 3	18 9	10 4
12	20 7	21 5	21 0	17 1	17 6	19 6	4 3
13	21 2	20 4	17 9	19 3	20 1	19 8	1 6
14	20 9	20 1	18 0	23 6	20 8	20 7	4 0
15	19 9	21 5	21 5	18 5	18 0	19 9	2 7
16	22 0	17 4	16 7	21 2	21 1	19 7	5 9
17	24 2	22 7	22 4	20 0	19 8	21 8	3 5
18	20 7	19 7	17 7	16 3	21 5	19 2	4 6
19	20 7	21 0	20 2	17 7	22 3	20 4	2 8
20	19 2	18 1	18 2	18 9	18 9	18 7	0 2
21	18 8	21 5	20 8	18 6	22 4	20 4	2 8
22	16 3	20 9	19 6	19 9	18 5	19 0	3 1
23	18 7	21 5	21 4	20 7	21 2	20 7	1 3
24	23 1	20 4	21 0	23 5	19 1	21 4	3 4
25	21 9	20 1	22 1	19 5	18 2	20 4	2 7

Table 4 5 An in control reference sample

The grand (overall) mean $\overline{\overline{X}}$ is

$$\frac{\overline{X}_1 + \overline{X}_2 + \quad + \overline{X}_{25}}{25} = \frac{20\ 3 + 19\ 5 + \quad + 20\ 4}{25} = 19\ 85$$

whereas equations (4 10b) and (4 11) yields

$$S_p = \sqrt{\frac{S_1^2 + S_2^2 + \quad + S_{25}^2}{25}} = 2\ 016$$

and

$$c_{4\ m} = \frac{\sqrt{2}\Gamma(50\ 5)}{\sqrt{100}\Gamma(50)} = 0\ 9975$$

respectively

Thus the point estimates for the unknown process parameters are

$$\mu_0 = \overline{\overline{X}} = 19\ 85$$

and

$$\sigma_0 = \frac{S_p}{c_{4\ m}} = \frac{2\ 016}{0\ 9975} = 2\ 021$$

The unknown target value θ is estimated by the grand mean i e $\hat{\theta} = 19\ 85$

The plotting statistics

$$S_t = \max\left(0\ S_t + \overline{X}_t - \left(\hat{\mu}_0 + \hat{K}\right)\right)$$

and

$$S_t = \max\left(0\ S_t - \overline{X}_t + \left(\hat{\mu}_0 - \hat{K}\right)\right)$$

with $S_0 = S_0 = 0$ and $\hat{K} = 1\left(\frac{\sigma_0}{\sqrt{n}}\right) = 0\ 904$ together with the counters N^+ and N^- are summarized in panels (a) and (b) of Table 4 6 The decision interval is estimated at $\hat{H} = 4\left(\frac{\hat{\sigma}_0}{\sqrt{n}}\right) = 3\ 615$

From the CUSUM status chart in Figure 4 6 it is clearly seen that the tabular CUSUM control chart gives no signal Therefore the estimated process parameters or alternatively the estimated design parameters $\hat{K}$ $\hat{H}$ and $\hat{\theta}$ can be used for the prospective monitoring of the process Consequently future samples can be obtained and their plotting statistics can be calculated and then compared to the estimated decision interval $\widehat{H}$

	(a)		(b)	
Sample i	S_t	N	S_t	N
1	0	0	0	0
2	0	0	0	0
3	0	0	1 546	1
4	0	0	1 992	2
5	0	0	0 998	3
6	0 666	1	0	0
7	1 072	2	0	0
8	0 298	3	0	0
9	0	0	0 826	1
10	0	0	0 192	2
11	0	0	0 278	3
12	0	0	0	0
13	0	0	0	0
14	0	0	0	0
15	0	0	0	0
16	0	0	0	0
17	1 066	1	0	0
18	0	0	0	0
19	0	0	0	0
20	0	0	0 286	1
21	0	0	0	0
22	0	0	0	0
23	0	0	0	0
24	0 666	1	0	0
25	0 272	2	0	0

Table 4 6 The plotting statistics (S_t^+ and S_t) and the counters (N and N) for the reference sample

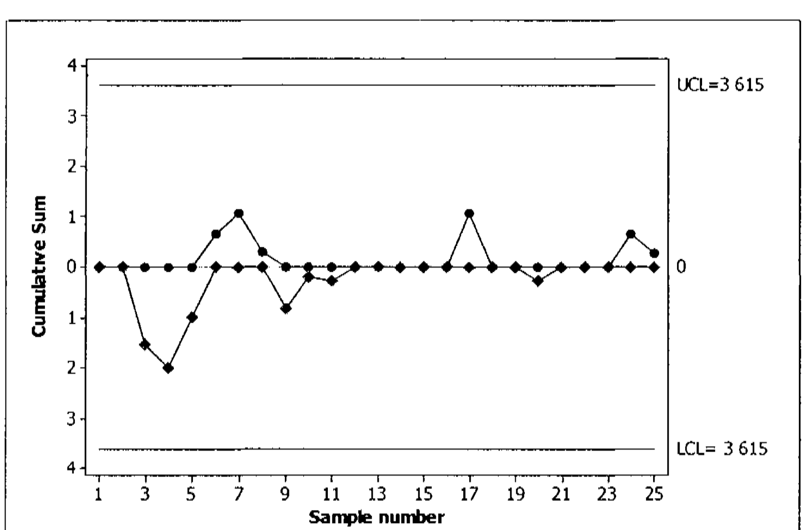

Figure 4 6 A CUSUM status chart for the reference sample

Thus an additional 35 independent samples each of size $n = 5$ were collected The individual observations as well as the sample means are displayed in panel (a) and column (b) of Table 4 7 respectively In panels (c) and (d) are the one sided upper and the one sided lower CUSUM plotting statistics as well as the counters N and N^- A CUSUM status chart of the data in Table 4 5 and Table 4 7 is given in Figure 4 7

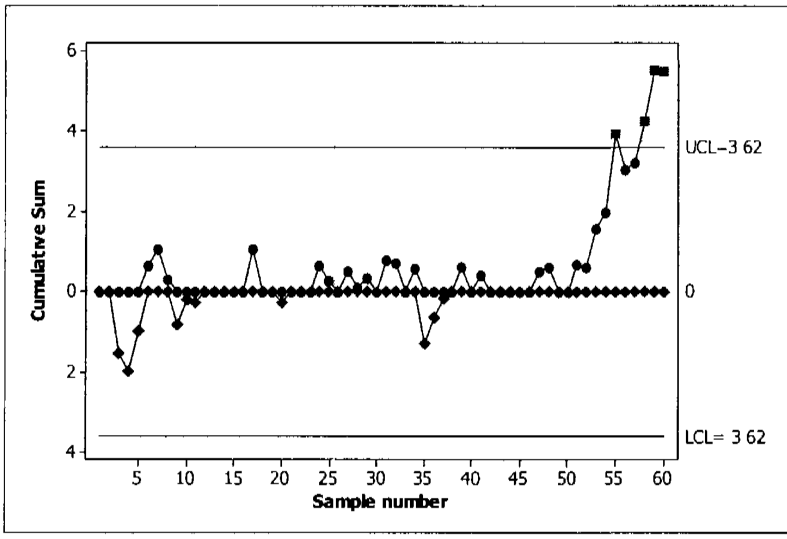

Figure 4 7 A CUSUM status chart for the reference sample and the 35 additional samples

On sample number 55 a signal is given when $S_{55} = 3\ 95$ plots above the decision interval $H = 3\ 62$ The process is declared out of control and a search for assignable causes is started At time $t = 55$ $N^+ = 5$ and it is likely that the process was last in control at time $t = 55 - 5 = 50$ Therefore it is decided that the shift occurred between samples 50 and 51 and that a search for assignable causes should start more or less at time 50

Furthermore since $S_{55} \geq 3\ 62$ equation (4 6a) yields

$$\hat{\mu}_1 = 19\ 85 + 1\ 01 + \frac{3\ 95}{5} = 21\ 65$$

i e the new value for the process mean following the shift is predicted to be 21 65

	(a)					(b)	(c)		(d)	
Sample i	X_1	X_2	X_3	X_4	X_5	$\overline{X}$	S_t	N	S_t	N
26	20 7	20 7	18 6	18 9	16 9	19 2	0	0	0	0
27	21 8	20 3	24 8	17 4	22 1	21 3	0 526	1	0	0
28	18 0	21 0	22 5	22 9	17 2	20 3	0 092	2	0	0
29	21 7	21 8	20 6	17 9	23 0	21 0	0 338	3	0	0
30	20 3	20 7	17 9	17 4	18 7	19 0	0	0	0	0
31	21 5	23 0	22 2	19 9	21 2	21 6	0 806	1	0	0
32	18 7	21 2	23 4	20 0	20 0	20 7	0 712	2	0	0
33	18 4	20 6	23 1	18 8	17 0	19 6	0	0	0	0
34	22 9	18 8	22 8	24 1	18 1	21 3	0 586	1	0	0
35	16 3	16 4	17 6	20 1	17 9	17 7	0	0	1 286	1
36	22 0	17 3	21 3	16 3	21 1	19 6	0	0	0 632	2
37	20 3	16 5	23 0	20 5	16 7	19 4	0	0	0 178	3
38	18 9	20 6	20 9	17 8	22 6	20 2	0	0	0	0
39	21 1	23 0	21 7	20 5	20 5	21 4	0 606	1	0	0
40	16 6	22 9	19 9	19 3	19 0	19 5	0 000	2	0	0
41	20 9	19 6	20 1	21 9	23 4	21 2	0 426	3	0	0
42	20 1	18 3	22 6	19 9	18 5	19 9	0	0	0	0
43	19 4	21 2	21 6	18 5	22 1	20 6	0	0	0	0
44	18 6	21 0	20 3	20 4	20 3	20 1	0	0	0	0
45	19 2	23 3	17 5	19 4	21 5	20 2	0	0	0	0
46	21 2	19 9	20 6	19 4	18 9	20 0	0	0	0	0
47	21 4	20 4	22 5	21 7	20 4	21 3	0 526	1	0	0
48	19 3	22 3	23 5	19 2	20 0	20 9	0 632	2	0	0
49	22 5	17 6	19 4	18 3	17 4	19 0	0	0	0	0
50	22 6	19 4	17 7	19 7	23 6	20 6	0	0	0	0
51	20 1	24 0	19 1	22 1	21 9	21 4	0 686	1	0	0
52	21 4	19 8	19 3	19 3	23 6	20 7	0 612	2	0	0
53	23 2	22 1	22 1	21 4	19 9	21 7	1 598	3	0	0
54	22 5	20 5	24 9	18 4	19 5	21 2	2 004	4	0	0
55	23 2	22 5	20 4	23 4	24 0	22 7	3 950*	5	0	0
56	16 1	22 5	17 8	20 3	22 6	19 9	3 056	6	0	0
57	20 7	20 2	21 5	20 9	21 4	20 9	3 242	7	0	0
58	22 3	22 0	22 2	21 6	20 7	21 8	4 248*	8	0	0
59	21 7	23 1	23 8	22 5	19 1	22 0	5 534*	9	0	0
60	20 0	22 8	19 7	20 0	21 0	20 7	5 480*	10	0	0

Table 4 7 An additional 35 samples

The performance of any control chart procedure depends on the parameter estimates from the reference sample For example Figure 4 8 is a CUSUM status chart of the data in Table 4 5 and Table 4 7 if it is assumed that the process mean and the process standard deviation are 20 and 2 respectively (In fact since we work with simulated data it is known to be the true values for the process parameters)

From Figure 4 8 we see that the process shift would only have been detected on sample number 59 when $S_{59} = 4\ 27$ plots above the decision interval $H = 4\left(\frac{2}{\sqrt{5}}\right) = 3\ 578$ This is 4 samples later than when using the estimated process parameters $\hat{\mu}_0$ and $\hat{\sigma}_0$ Thus it seems as if the tabular CUSUM procedure when the process parameters are *unknown* and point estimates need to be used outperforms the tabular CUSUM procedure when the process parameters are in fact *known* This is unexpected and is due to the combined effect of the underestimation of the in control process mean as well as the overestimation of the in control process standard deviation i e $\mu_0 = 19\ 85 < 20$ and $\sigma_0 = 2\ 021 > 2$

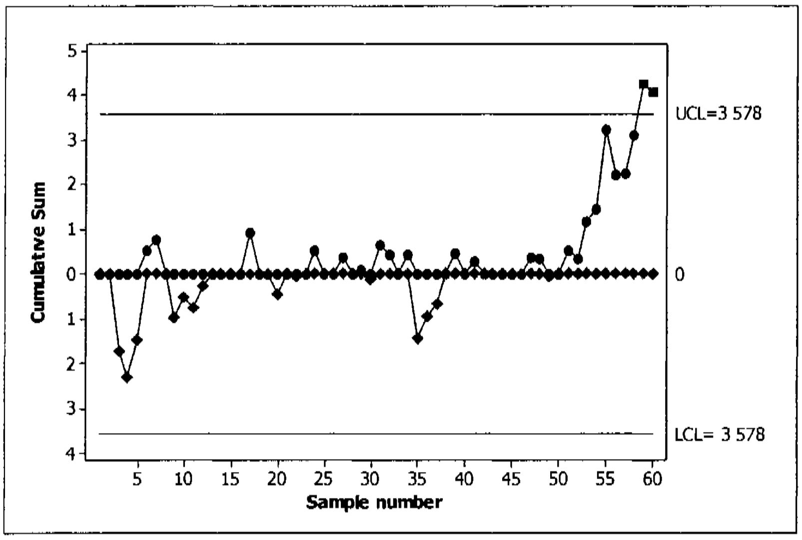

Figure 4 8 A CUSUM status chart with $\mu = 20$ and $\sigma = 2$

To assess the overall performance of the tabular CUSUM control chart with estimated process parameters it is necessary to account for the variability in the estimators $\hat{\mu}_0$ and $\hat{\sigma}_0$ To do this we first find the conditional run length distribution (conditioned on observed values of the point estimators) and then we find the unconditional run length distribution i e the run length distribution without the influence of the observed values of the parameter estimates Consequently the overall performance of the tabular CUSUM control chart procedure can be measured using several aspects of the unconditional run length distribution such as the average run length (ARL) the standard deviation of the run length (SDRL) as well as the percentiles of the run length distribution

4 1 9 Run length dıstrıbutıon

Followıng the estımatıon of the *unknown* process parameters ın Phase 1 ınformatıon about the process mean ın Phase 2 ıs conveyed vıa the standardızed subgroup means

$$Y_t = \frac{\sqrt{n}\left(\overline{X}_t - \widehat{\mu}_0\right)}{\widehat{\sigma}_0} \tag{4 12}$$

for $n \geq 1$ and $t = m+1\ m+2$

To find the run length dıstrıbutıon of the tabular CUSUM control chart we need the dıstrıbutıon of the two standardızed one sıded CUSUM plottıng statıstıcs of equatıons (4 8a) and (4 8b) ı e

$$S_t = \max\left(0\ S_{t\ 1} + Y_t - k\right)$$

and

$$S_t = \max\left(0\ S_{t\ 1} - Y_t - k\right)$$

However sınce S_t and S_t^- are functıons of the random varıable Y_t we first need the dıstrıbutıon of Y_t Therefore to find the dıstrıbutıon of Y_t ıt ıs helpful to wrıte equatıon (4 12) as

$$Y_t = \frac{1}{W_0}\left(\gamma Z_t + \delta - \frac{Z_0}{\sqrt{m}}\right) \tag{4 13}$$

(See Result 4 2a ın Appendıx 4)

The random varıable

$$W_0 = \frac{\widehat{\sigma}_0}{\sigma_0} \tag{4 14a}$$

represents the ratıo of the estımator of the ın control standard devıatıon to the true ın control standard devıatıon and ıs related to the square root of a chı square (χ^2) random varıable also referred to as the chı dıstrıbutıon (χ)

The standard normal random varıable

$$Z_0 = \frac{\sqrt{mn}\left(\widehat{\mu}_0 - \mu_0\right)}{\sigma_0} \tag{4 14b}$$

represents the standardızed dıstance of the estımated ın control process mean from the true ın control process mean

Furthermore the standard normal random varıable

$$Z_t = \frac{\sqrt{n}\left(\overline{X}_t - \mu\right)}{\sigma} \tag{4 14c}$$

represents the standardized subgroup average at time t where μ and σ denote the process mean and the process standard deviation at time t respectively

The process is considered to be in control at time t if $\mu = \mu_0$ and if $\sigma = \sigma_0$

Lastly the two constants

$$\gamma = \frac{\sigma}{\sigma_0} \tag{4 14d}$$

and

$$\delta = \frac{\sqrt{n}\left(\mu - \mu_0\right)}{\sigma_0} \tag{4 14e}$$

represents the ratio of the process standard deviation at time t to the true in control process standard deviation and the standardized shift in the process mean at time t from the true in control process mean

Having re written equation (4 12) into equation (4 13) the random variable Y_t is a function of the two Phase 1 random variables Z_0 and W_0 as well as the two constants δ and γ Therefore equation (4 13) is helpful to obtain the conditional distribution of Y_t i e conditioned on the random variables Z_0 and W_0 and subsequently used to derive the conditional run length distribution of a standardized one sided tabular CUSUM control chart

The conditional cumulative distribution function (cdf) and the conditional probability density function (pdf) of Y_t is given by

$$F_{Y|Z\ W}\left(y_t \mid z_0\ w_0\right) = \Phi\left(\frac{w_0}{\gamma} y_t - \frac{\delta}{\gamma} + \frac{z_0}{\gamma\sqrt{m}}\right) \tag{4 15}$$

and

$$f_{Y|Z\ W}\left(y_t \mid z_0\ w_0\right) = \frac{w_0}{\gamma}\phi\left(\frac{w_0}{\gamma} y_t - \frac{\delta}{\gamma} + \frac{z_0}{\gamma\sqrt{m}}\right) \tag{4 16}$$

(See Result 4 2b and Result 4 2c in Appendix 4)

Followıng the condıtıonal dıstrıbutıon of Y_t ın equatıons (4 15) and (4 16) we derıve the condıtıonal probabılıty mass functıon (pmf) of the run length random varıable (N) ı e condıtıoned on observed values of $\widehat{\mu}_0$ and $\widehat{\sigma}_0$ through partıcular outcomes of the two random varıables Z_0 and W_0 We denote thıs condıtıonal pmf by

$$P\left(N = t \mid u \;\delta\; \gamma\; Z_0 = z_0\; W_0 = w_0\right) \text{ with } t = 1\ 2\ 3$$

Note that ın addıtıon to the random varıables Z_0 and W_0 the condıtıonal pmf also depends on the two constants γ and δ as well as the startıng value u

4 1 10 Methods for evaluating the run length distribution

The Markov chain approach and the integral equation approach are two methods that can be used to evaluate the conditional run length distribution as well as the unconditional run length distribution of a CUSUM type of control chart However in later sections we will also use these two methods to evaluate the run length distribution of an EWMA control chart

The Markov chain approach involves discretizing the possible values that can be plotted and then uses the properties of finite Markov chains to evaluate the run length distribution The integral equation approach provides exact integral equations for the run length distribution which can then be approximated using numerical integrating techniques
Thus the Markov chain approach begins by approximating the problem and then obtains an exact solution for the approximate problem whereas the integral equation approach begins with the exact problem and then finds a solution that is approximated

Since it can be shown that the Markov chain approach is a special case of the integral equation approach and since there exist numerous methods to approximate the solution to an integral equation (which exceeds the accuracy of the Markov chain approach) the integral equation approach is preferred These results are established in a later section

4 1 11 The run length dıstrıbutıon of a standardızed one-sıded tabular CUSUM control chart usıng ıntegral equatıons

The ıntegral equatıon approach provıdes exact expressıons for the condıtıonal run length dıstrıbutıon of the standardızed one sıded upper CUSUM control chart defined ın equatıon (4 8a) These expressıons are

$$P\left(N=1\,|\,u\ \delta\ \gamma\ Z_0=z_0\ W_0=w_0\right)=1-\Phi\left(\frac{w_0}{\gamma}(h+k-u)-\frac{\delta}{\gamma}+\frac{z_0}{\gamma\sqrt{m}}\right) \quad (4\ 17a)$$

for $t=1$ and

$$\begin{aligned} &P\left(N=t\,|\,u\ \delta\ \gamma\ Z_0=z_0\ W_0=w_0\right)\\ &=P\left(N=t-1\,|\,S_1\ =0\ \delta\ \gamma\ Z_0=z_0\ W_0=w_0\right)\Phi\left(\frac{w_0}{\gamma}(k-u)-\frac{\delta}{\gamma}+\frac{z_0}{\gamma\sqrt{m}}\right)+ \quad (4\ 17b)\\ &\frac{w_0}{\gamma}\int_0^h P\left(N=t-1\,|\,s\ \delta\ \gamma\ Z_0=z_0\ W_0=w_0\right)\phi\left(\frac{w_0}{\gamma}(s+k-u)-\frac{\delta}{\gamma}+\frac{z_0}{\gamma\sqrt{m}}\right)ds \end{aligned}$$

for $t=2\ 3$

(See Results 4 3a – 4 3d ın Appendıx 4)

Note that expressıons (4 17a) and (4 17b) are also valıd for the standardızed one sıded lower CUSUM control chart defined ın equatıon (4 8b) ı e S_t

(Compare Results 4 3a – 4 3d wıth Results 4 5a – 4 5e ın Appendıx 4)

The r^{th} condıtıonal non central moment of the run length random varıable (N) of a standardızed one sıded tabular CUSUM control chart S_t or S_t ıs gıven by

$$\begin{aligned} E\left(N\ \ |u\ \delta\ \gamma\ Z_0=z_0\ W_0=w_0\right)&=\sum_{t\ 1}^{\infty}t\ P\left(N=t\,|\,u\ \delta\ \gamma\ Z_0=z_0\ W_0=w_0\right)\\ &=M\ \ \left(u\ \delta\ \gamma\ Z_0=z_0\ W_0=w_0\right) \end{aligned} \quad (4\ 18)$$

Equation (4 18) can be used to find performance measures of a standardized one sided tabular CUSUM control chart given specific outcomes (or realizations) of the Phase 1 random variables Z_0 and W_0 For example using expression (4 18) we obtain $E(N \mid u\ \delta\ \gamma\ Z_0 = z_0\ W_0 = w_0)$ and $E(N^2 \mid u\ \delta\ \gamma\ Z_0 = z_0\ W_0 = w_0)$ which we can use to obtain the conditional average run length and the conditional standard deviation of the run length random variable i e

$$ARL = E(N \mid u\ \delta\ \gamma\ Z_0 = z_0\ W_0 = w_0)$$

and

$$SDRL = \sqrt{E(N^2 \mid u\ \delta\ \gamma\ Z_0 = z_0\ W_0 = w_0) - E(N \mid u\ \delta\ \gamma\ Z_0 = z_0\ W_0 = w_0)^2}$$

However the insight gained from studying the conditional run length distribution is of limited use Therefore the unconditional run length distribution is needed to gain insight into the overall effect of estimation on the general performance of a one sided tabular CUSUM control chart

Therefore the joint probability density function of N Z_0 and W_0 is

$$f_{N\ Z\ W}(N = t\ Z_0 = z_0\ W_0 = w_0 \mid u\ \delta\ \gamma) = P(N = t \mid u\ \delta\ \gamma\ Z_0 = z_0\ W_0 = w_0)\phi(z_0) f_W(w_0) \quad (4\ 19)$$

from which we find the marginal (or unconditional) run length distribution i e

$$P(N = t \mid u\ \delta\ \gamma) = \int_{-\infty}^{\infty}\int_{0}^{\infty} P(N = t \mid u\ \delta\ \gamma\ Z_0 = z_0\ W_0 = w_0)\phi(z_0) f_W(w_0)\, dw dz \quad (4\ 20)$$

where $f_W(w_0)$ is the probability density function of W_0

For simplicity we may think of the unconditional distribution of equation (4 20) as the conditional distribution of equation (4 19) being averaged over all possible values of the parameter estimates

Moments of the unconditional run length distribution can further be calculated The first non central moment otherwise known as the unconditional average run length or the expected value of the unconditional run length distribution is

$$E(N \mid u\ \delta\ \gamma) = \sum_{t=1}^{\infty} tP(N = t \mid u\ \delta\ \gamma)$$
$$= \int_{-\infty}^{\infty}\int_{0}^{\infty} M_1(u\ \delta\ \gamma\ Z_0 = z_0\ W_0 = w_0)\phi(z_0) f_W(w_0)\,dw\,dz \qquad (4\ 21)$$

$M_1(u\ \delta\ \gamma\ Z_0 = z_0\ W_0 = w_0)$ is the first non central moment of the conditional run length distribution called the conditional average run length i e $\sum_{t=1}^{\infty} tP(N = t \mid u\ \delta\ \gamma\ Z_0 = z_0\ W_0 = w_0)$ and is found from equation (4 18)

The second non central moment of the unconditional run length distribution can be found in a similar manner i e

$$E(N^2 \mid u\ \delta\ \gamma) = \sum_{t=1}^{\infty} t^2 P(N = t \mid u\ \delta\ \gamma)$$
$$= \int_{-\infty}^{\infty}\int_{0}^{\infty} M_2(u\ \delta\ \gamma\ Z_0 = z_0\ W_0 = w_0)\phi(z_0) f_W(w_0)\,dw\,dz \qquad (4\ 22)$$

where $M_2(u\ \delta\ \gamma\ Z_0 = z_0\ W_0 = w_0)$ is the second non central moment of the conditional run length distribution i e $\sum_{t=1}^{\infty} t^2 P(N = t \mid u\ \delta\ \gamma\ Z_0 = z_0\ W_0 = w_0)$ and is also found from equation (4 18)

Thus having found the first two non central moments of the unconditional run length distribution given in equations (4 21) and (4 22) it is possible to find the standard deviation of the unconditional run length distribution i e

$$SDRL = \sqrt{E(N^2 \mid u\ \delta\ \gamma) - E(N \mid u\ \delta\ \gamma)^2} \qquad (4\ 23)$$

Although equations (4 17) – (4 23) are difficult to evaluate numerical integration techniques can be used to approximate these expressions However we will use the Markov chain approach (given in the next section) to evaluate the conditional as well as the unconditional run length distribution

4 1 12 The conditional run length distribution of a standardized one-sided tabular CUSUM control chart using finite Markov chains

For simplicity we will work with the standardized one sided upper CUSUM procedure S_t as defined in equation (4 8a) However the results that follow are also valid for the standardized one sided lower CUSUM procedure S_t as defined in equation (4 8b)

(Compare Results 4 4a – 4 4e with Results 4 6a – 4 6e in Appendix 4)

Recall that $S_t \in [0\ \infty) \equiv [0\ h) \cup [h\ \infty)$ and that a process is considered in control while $S_t^+ \in [0\ h)$ and declared out of control when $S_t \in [h\ \infty)$ Furthermore once $S_t \in [h\ \infty)$ the control chart procedure is stopped and S_t is typically reset to zero

The basic idea of the Markov chain approach is to discretize the possible values that can be plotted First we partition the vertical axis of the CUSUM status chart i e $[0\ \infty)$ into a finite set of disjoint intervals and then we select a reference value within each interval For example given an interval $[a\ b)$ we select a reference value $c \in [a\ b)$ and assume that c is the only value within the interval $[a\ b)$ that can be plotted on the CUSUM status chart Typically the reference value is selected to be the midpoint of an interval that is we would select $c = \frac{a+b}{2}$ However there are cases where choosing a reference value other than the midpoint is likely to be more rational Having selected a reference value for each interval these reference values are considered to be the states in a discrete time Markov chain

Now suppose that we need to partition the interval $[0\ \infty)$ into $m+1$ disjoint intervals to create $m+1$ states Denote these states by $i = 0\ 1\ 2\ \ \ m-1\ m$ In addition let the $(m+1)^{th}$ state be the interval $[h\ \infty)$ and let its reference value be h Consequently the interval $[0\ h)$ is used to create the remaining m states However the first state (when $i = 0$) is half the length of the other states and the reference value of the first state is 0 whereas the reference values of the remaining $m-1$ states are their midpoint values Therefore if w is the width of an interval we can derive its relationship to m and h as follows

$$(m-1)w+\frac{1}{2}w=(h-0)$$
$$\Rightarrow w=\frac{2h}{2m-1} \tag{4 24}$$

Consequently we can summarize all the states of the Markov chain the intervals they represent as well as the reference values as given in Table 4 8

State	Interval	Reference / Midpoint Value
0	$\left[0 \ \frac{1}{2}w\right)$	0
1	$\left[\frac{1}{2}w \ \frac{3}{2}w\right)$	w
2	$\left[\frac{3}{2}w \ \frac{5}{2}w\right)$	$2w$
k	$\left[\left(k-\frac{1}{2}\right)w \ \left(k+\frac{1}{2}\right)w\right)$	kw
$m-1$	$\left[\left(m-\frac{3}{2}\right)w \ \left(m-\frac{1}{2}\right)w\right)$	$(m-1)w$
m	$[h \ \infty)$	h

Table 4 8 A summary of the $m+1$ states of the Markov chain approach

Note that states 0 through $m-1$ are non absorbent and communicate with each other i e it is possible to go from any one of these states to any other state However state m is an absorbent state i e once it has been entered the process cannot return to any other state Thus state m does not communicate with any of the non absorbent states

Consequently the $(m+1)\times(m+1)$ transition probability matrix ($\mathbf{P}$) of the discrete time Markov chain can be represented as

$$\mathbf{P}=\left[p_{ij}\right]=\begin{bmatrix} p_{0} & p_{ij} & p_{m} \\ p_{mj} & & p_{mm} \end{bmatrix} \tag{4 25}$$

The elements of **P** are called the transition probabilities and are generally denoted by p_{ij} with $i\ j = 0\ 1\ 2 \quad m$ These transition probabilities are interpreted as the probability of being in state j at time t given that at time $t-1$ the process was in state i Note that **P** is a stochastic matrix so that all rows sum to unity

The transition probabilities for the transition probability matrix in (4 25) are

$$p_{ij} = \int_{(j-\frac{1}{2})w}^{(j+\frac{1}{2})w} \frac{w_0}{\gamma}\phi\left(\frac{w_0}{\gamma}(s - iw + k) - \frac{\delta}{\gamma} + \frac{z_0}{\gamma\sqrt{m}}\right)ds \qquad (4\ 26a)$$

with $i = 0\ 1 \quad m-1$ and $j = 1\ 2 \quad m-1$

$$p_{i0} = \int_{-\infty}^{\frac{1}{2}w} \frac{w_0}{\gamma}\phi\left(\frac{w_0}{\gamma}(s - iw + k) - \frac{\delta}{\gamma} + \frac{z_0}{\gamma\sqrt{m}}\right)ds \qquad (4\ 26b)$$

with $i = 0\ 1 \quad m-1$

$$p_{im} = \int_{h}^{\infty} \frac{w_0}{\gamma}\phi\left(\frac{w_0}{\gamma}(s - iw + k) - \frac{\delta}{\gamma} + \frac{z_0}{\gamma\sqrt{m}}\right)ds \qquad (4\ 26c)$$

with $i = 0\ 1 \quad m-1$

$$p_{mj} = 0 \qquad (4\ 26d)$$

with $j = 0\ 1 \quad m-1$ and

$$p_{mm} = 1 \qquad (4\ 26e)$$

(See Results 4 4a – 4 4e in Appendix 4)

Note that the transition probabilities in equations (4 26a) – (4 26e) are expressed in terms of the conditional probability density function of Y_t given in equation (4 16) and are therefore *conditional* probabilities themselves Hence these transition probability expressions can only be used to study the conditional run length distribution A procedure to find the unconditional run length distribution (using a finite Markov chain) is given at the end of this section

Many of the required results (such as the average run length and the percentage points of the run length distribution etc) can be obtained by working with the matrix **R** obtained from the transition probability matrix **P** which results by deleting the last row and last column i e

$$\mathbf{R} = \begin{bmatrix} & & \\ p_0 & & p_j \\ & & \end{bmatrix} \tag{4 27}$$

where $i = 0\ 1 \quad m-1$ and $j = 1\ 2 \quad m-1$

Let N be the number of steps taken starting in state $i = 0\ 1\ 2 \quad m-1$ to reach the absorbent state (m) for the first time and let $\mu^{( )}$ denote the s^{th} factorial moment of N so that

$$\mu^{( )} = E\left(r^{( )}\right) = E\left(N\ (N-1)\ \ (N-s+1)\right) \tag{4 28}$$

Considering the Markov chain one step later we find that

$$\mu^{( )} = \sum^{\infty} r^{( )} P(N = r) = \sum^{\infty} r^{( )} \left\{ \sum_{j\ 0}^{m\ 1} p_j P\left(N_j = r-1\right) \right\} \tag{4 29}$$

since $P(N = r)$ which is the probability to reach state m for the first time after r transitions given that the process started in state i is equal to going form state i to any state j (with probability p_j) on the first transition and then going from state j to the absorbent state m within the next $r-1$ transitions – with probability $P\left(N_j = r-1\right)$ The $\sum_{j-0}^{m-1}$ in equation (4 29) makes provision for the fact that the process can visit any of the non absorbent states on the first transition i e states $0\ 1\ 2 \quad m-1$

Expression (4 29) can be further simplified as follows

$$\sum^{\infty} r^{( )} \left\{ \sum_{j=0}^{m\ 1} p_j P\left(N_j = r-1\right) \right\} = \sum_{j\ 0}^{m\ 1} p_j \left[\sum^{\infty} r^{( )} P\left(N_j = r-1\right) \right]$$

$$= \sum_{j\ 0}^{m\ 1} p_j \left[\sum^{\infty} r^{( )} P\left(N_j = r\right) + s \sum_{1}^{\infty} r^{(\ 1)} P\left(N_j = r\right) \right]$$

so that

$$\mu^{(\ )} = \sum_{j\ 0}^{m\ 1} p_j \left\{ \mu_j^{(\ )} + s\mu_j^{(\ \ 1)} \right\} \tag{4 30a}$$

or

$$\mu^{(\ )} - \sum_{j\ 0}^{m\ 1} p_j \mu_j^{(\ )} = s \sum_{j\ 0}^{m\ 1} p_j \mu_j^{(\ \ 1)} \tag{4 30b}$$

with $s = 2\ 3$ and $\imath = 0\ 1\ 2 \quad m-1$

In matrix notation expression (4 30b) becomes

$$(\mathbf{I} - \mathbf{R})\boldsymbol{\mu}^{(\ )} = s\mathbf{R}\boldsymbol{\mu}^{(\ \ 1)} \tag{4 31}$$

with $s = 2\ 3$

$\mathbf{R}$ is the matrix obtained from $\mathbf{P}$ by deleting the last row and last column i e those transition probabilities referring to the absorbent state m $\mathbf{I}$ is the $m \times m$ identity matrix and $\boldsymbol{\mu}^{(\ )}$ is the $m \times 1$ column vector of s^{th} factorial moments for the random variables $N_0\ N_1 \quad N_{m\ 1}$ i e

$$\boldsymbol{\mu}^{(\ )} = \left[\mu_0^{(\ )} \quad \mu_1^{(\ )} \qquad \mu_{m\ 1}^{(\ )}\right]$$

The special case when $s = 1$ is seen from 4 31 to lead to

$$(\mathbf{I} \quad \mathbf{R})\boldsymbol{\mu} = \mathbf{1} \tag{4 32a}$$

where the column vector $\mathbf{1}$ has each of its m elements equal to unity i e

$$\mathbf{1} = [1 \quad 1 \qquad 1]$$

The first element of the vector $\boldsymbol{\mu}$ gives the average run length (or expected value of N_0) for a standardized one sided tabular CUSUM control chart that starts from zero In fact the $\imath^{th}$ element of the mean vector in equation (4 32a) gives the average run length (or expected value of N) when the standardized one sided tabular CUSUM control chart starts from state $\imath = 0\ 1\ 2 \quad m-1$ Therefore if we simply require the average run length (ARL) we can solve equation (4 32a) to obtain

$$\boldsymbol{\mu} = (\mathbf{I} \quad \mathbf{R})^{\ 1}\mathbf{1} \tag{4 32b}$$

Let $\mathbf{N}$ be the vector of run length random variables with elements $N_0 N_1 \quad N_{m\ 1}$ and re write equation (4 31) as

$$\begin{aligned}\boldsymbol{\mu}^{(s)} &= s(\mathbf{I}-\mathbf{R})^{-1}\mathbf{R}\boldsymbol{\mu}^{(s-1)} \\ &= s(\mathbf{I}-\mathbf{R})^{-1}\left[\mathbf{I}-(\mathbf{I}-\mathbf{R})\right]\boldsymbol{\mu}^{(s-1)} \\ &= s\left\{(\mathbf{I}-\mathbf{R})^{-1}-\mathbf{I}\right\}\boldsymbol{\mu}^{(s-1)}\end{aligned} \tag{4.33}$$

with $\boldsymbol{\mu}=\boldsymbol{\mu}^{(1)}=(\mathbf{I}-\mathbf{R})^{-1}\mathbf{1}$ and $s=2,3,\ldots$

Thus, we can obtain the s^{th} factorial moments of the elements of $\mathbf{N}$ by recursively substituting for $\boldsymbol{\mu}^{(s-1)}$ on the right hand side of equation (4.33).

If we note that $(\mathbf{I}-\mathbf{R})^{-1}\mathbf{R}=\mathbf{R}(\mathbf{I}-\mathbf{R})^{-1}$, equation (4.33) can be re-written as

$$\boldsymbol{\mu}^{(s)}=s!(\mathbf{I}-\mathbf{R})^{-s}\mathbf{R}^{s-1}\mathbf{1} \tag{4.34a}$$

or

$$\boldsymbol{\mu}^{(s)}=s!\mathbf{R}^{s-1}(\mathbf{I}-\mathbf{R})^{-s}\mathbf{1} \tag{4.34b}$$

with $s=1,2,\ldots$

Thus, the probability distribution of the run length random vector ($\mathbf{N}$) can be regarded as a multi-dimensional generalization of the geometric distribution over the positive integers. Since if $X \sim GEO(p)$ we have that $P(X=r)=(1-p)^{r-1}p=q^{r-1}p$ where $q=1-p$ and $0<p<1$ with $r=1,2,3,\ldots$ and the s^{th} factorial moment of X, i.e. $\mu_X^{(s)}$ is given as $s!q^{s-1}p^{-s}$ for $s=1,2,\ldots$

Furthermore, if $\mathbf{R}$ was a diagonal matrix, the elements of $\mathbf{N}$ would all be geometric random variables with the scalar parameter for success (p) replaced by the matrix $(\mathbf{I}-\mathbf{R})$. Also, the elements of the vector $(\mathbf{I}-\mathbf{R})\mathbf{1}$ would be the probabilities of going from a non-absorbent state to the absorbent state which will be the first occurrence on which the process terminates, i.e. the conventional probabilities for a success.

A formula for the s^{th} factorial moments for the random variables $N_0, N_1, \ldots, N_{m-1}$ has been derived and more specifically, the expected values (or average run lengths) have been found.
From these s^{th} factorial moments one can also determine the s^{th} central moments, i.e. $E\left(N-\mu^{(1)}\right)^s$.
However, this is not adequate for evaluating the performance of a control chart. We also need the

percentage points of the run length random variables For this we need to find the probability mass functions (pmf) and/or the cumulative distribution functions (cdf) of $N_0 N_1 \quad N_{m\ 1}$

Therefore if $\mathbf{P}$ is the transition probability matrix as defined in equation (4 25) then the last column of $\mathbf{P}$ provides $P(N \leq r)$ for $\imath = 0\ 1 \quad m-1$ i e the cumulative distribution function of N together with the last element – being 1

Thus if $\mathbf{P}$ is partitioned as

$$\mathbf{P} = \begin{bmatrix} \mathbf{R} & \mathbf{p}_m \\ \mathbf{0} & 1 \end{bmatrix}$$

where $\mathbf{p}_m$ is the $m \times 1$ column vector of transition probabilities $p_{\ m}$ $\mathbf{0}$ is the $1 \times m$ row vector of zero s and $\mathbf{R}$ is the $m \times m$ matrix given in equation (4 27) then

$$\mathbf{p}_m = (\mathbf{I} - \mathbf{R})\mathbf{1}$$

since $\mathbf{P}$ is a stochastic matrix with rows adding to one

Thus it follows that

$$\mathbf{P} \ = \begin{bmatrix} \mathbf{R} & (\mathbf{I} - \mathbf{R})\ \mathbf{1} \\ \mathbf{0} & 1 \end{bmatrix}$$

with $r = 1\ 2$

Let $\mathbf{F}$ be the $m \times 1$ column vector whose elements are the cumulative distribution functions of $N_0 N_1 \quad N_{m\ 1}$ so that

$$\mathbf{F} \ = \left[P(N_0 \leq r)\ P(N_1 \leq r) \quad P(N_{m\ 1} \leq r) \right]$$

with $r = 1\ 2$

If this is the case we have that

$$\mathbf{F} \ = \left(\mathbf{I} - \mathbf{R} \ \right) \mathbf{1} \qquad (4\ 35)$$

and

$$\mathbf{1} - \mathbf{F} \ = \mathbf{R}\ \mathbf{1} \qquad (4\ 36)$$

with

$$\mathbf{1} - \mathbf{F} \ = \left[P(N_0 > r)\ P(N_1 > r) \quad P(N_{m\ 1} > r) \right]$$

Thus the first element of $\mathbf{F}$ gives the cumulative distribution function for the run length of a standardized one sided tabular CUSUM control chart starting from zero Furthermore $\mathbf{R}$ $\mathbf{1}$ of equation (4 36) gives the corresponding right hand tail probabilities

In addition let $\mathbf{f}$ be the $m \times 1$ column vector whose elements are the values of the probability mass functions of the run length random variables starting with state 0 through to $m-1$ i e let

$$\mathbf{f} = \left[P(N_0 = r) \;\; P(N_1 = r) \quad\; P(N_{m\;1} = r) \right] \tag{4 37}$$

with $r = 1\ 2$

Then

$$\mathbf{f}_1 = \mathbf{p}_m = (\mathbf{I} - \mathbf{R})\mathbf{1} \tag{4 38a}$$

and a recursive formula for $r = 2\ 3$ is given by

$$\mathbf{f} = \mathbf{R}\mathbf{f}_{\;\;1} \tag{4 38b}$$

which can be simplified by recursive substitution of $\mathbf{f}_{\;\;1}$ on the right hand side to obtain

$$\mathbf{f} = \mathbf{R}^{\;-1}\mathbf{f}_1 = \mathbf{R}^{\;\;1}(\mathbf{I} - \mathbf{R})\mathbf{1} \tag{4 38c}$$

with $r = 1\ 2$

Equation (4 38c) is similar to the univariate geometric distribution with probability mass function given by

$$P(N = r) = (1 - p)^{\;\;1} p = q^{\;\;1} p$$

with $r = 1\ 2$

Thus having found equations for the s^{th} factorial moments the probability mass function and the cumulative distribution function all aspects of the run length distribution can be studied However since the transition probabilities of equations (4 26a) – (4 26e) are conditional probabilities we have only found the conditional run length distribution To find the unconditional run length distribution we can use the following procedure

1 We randomly draw values from the probability distributions of W_0 and Z_0 (defined in 14 4a and 14 4b) so that we have two observations w_0 and z_0

2 We substitute w_0 and z_0 in equations (4 26a) – (4 26e) and solve for equations (4 35) and (4 38c) The solutions to equations (4 35) and (4 38c) are a realization of the *conditional* run length distribution

3 We repeat steps 1 and 2 a large number of times For example 50 000 or 100 000 times and then find the average of the 50 000 or 100 000 realizations

The result obtained at the end of step 3 will then be the *unconditional* run length distribution

4 1 13 The performance of the tabular CUSUM control chart

The performance of a tabular CUSUM control chart when the process parameters are *known* is primarily influenced by the reference value K and the decision interval H Generally K and H are selected to give good average run length (ARL) properties

Define $K = k\sigma$ and $H = h\sigma$ where σ is the standard deviation of the sample statistic used The value of k is typically chosen relative to the magnitude of the anticipated shift i e $k = 0\ 5\delta$ where δ is the size of the anticipated shift in standard deviation units i e $\delta = \frac{|\mu_1 - \mu_0|}{\sigma_0}$ Subsequently once k is selected we would then choose h to give the desired in control average run length (ARL_0) performance This approach is very close to minimizing the out of control average run length value (ARL_1) for detecting a shift of size δ for a fixed ARL_0

Table 4 9 presents particular choices of k and h for which the ARL_0 is approximately 370 which makes the ARL of the two sided tabular CUSUM procedure comparable to the ARL of a Shewhart $\overline{X}$ control chart with 3 sigma control limits

k	h	ARL_0	SDRL	10^{th}	50^{th}	90^{th}
0 25	8 01	370 56	354 93	53	262	833
0 50	4 77	368 42	362 33	44	257	841
0 75	3 30	371 08	367 75	42	258	851
1 00	2 52	372 92	371 61	41	259	855
1 25	1 99	373 80	372 40	41	260	859
1 50	1 61	377 50	376 78	40	262	868
2 00	1 02	372 79	372 18	40	258	858

Table 4 9 Values of k and the corresponding values of h for which the in control ARL_0 is approximately 370 ( )

For example if a shift of size $\delta = 1$ is anticipated we would set $k = 0\ 5(1) = 0\ 5$ and then select $h = 4\ 77$ to obtain the ARL_0 of the two sided tabular CUSUM procedure to be 368 42 Furthermore from Table 4 10 we then see that if a shift of size $\delta = 1$ occurred the ARL_1 of the two sided tabular CUSUM procedure with $k = 0\ 5$ and $h = 4\ 77$ is 9 91 whereas the ARL of the Shewhart $\overline{X}$ control chart is 43 89 see Chapter 3 In addition although the ARL_0 of the two sided tabular CUSUM procedure when $h = 4$ and $h = 5$ are 167 81 and 465 06 respectively their ARL_1 for a shift of size $\delta = 1$ is 8 38 and 10 37 Thus for $k = 0\ 5$ and any choice of h between 4 and 5 the two sided tabular CUSUM control chart has good ARL properties to detect small shifts in the process mean

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The results in Tables 4 9 – 4 11 were obtained using Monte Carlo simulation See SAS program A in Appendix 4

	$h = 4$					$h = 4\ 77$					$h = 5$				
δ	ARL	SDRL	10^{th}	50^{th}	90^{th}	ARL	SDRL	10^{th}	50^{th}	90^{th}	ARL	SDRL	10^{th}	50^{th}	90^{th}
0 00	167 81	163 11	22	118	381	368 42	362 33	44	257	841	465 06	457 84	55	325	1063
0 25	74 12	69 05	12	53	164	121 31	114 81	19	86	271	139 34	132 12	21	99	311
0 50	26 63	21 77	7	20	55	35 18	28 78	9	27	73	38 0	31 01	10	29	78
0 75	13 29	8 97	5	11	25	16 18	10 62	6	13	30	17 05	11 08	6	14	31
1 00	8 38	4 70	4	7	14	9 91	5 28	5	9	17	10 37	5 45	5	9	17
1 50	4 75	2 00	3	4	7	5 52	2 19	3	5	8	5 75	2 24	3	5	9
2 00	3 34	1 16	2	3	5	3 86	1 26	2	4	5	4 01	1 29	3	4	6
2 50	2 62	0 79	2	2	4	3 00	0 86	2	3	4	3 11	0 87	2	3	4
3 00	2 19	0 58	2	2	3	2 49	0 64	2	2	3	2 57	0 66	2	2	3
3 50	1 92	0 49	1	2	2	2 16	0 48	2	2	3	2 22	0 49	2	2	3
4 00	1 71	0 49	1	2	2	1 96	0 40	1	2	2	2 01	0 38	2	2	2

Table 4 10 Run length results of the two sided tabular CUSUM procedure with $k = 0\ 5$ and $h = 4$ $h = 4\ 77$ or $h = 5$ ( )

To improve the responsiveness of the two sided tabular CUSUM control chart for large process shifts we can use a combined CUSUM Shewhart procedure This procedure uses the basic two sided tabular CUSUM control chart together with a Shewhart type of control chart The control limits of the Shewhart control chart may be placed at approximately 3 5σ from the centerline A signal on either or both of the control charts represents a possible out of control process and a search for assignable causes is therefore warranted

Panel (a) of Table 4 11 presents the ARL properties of the two sided tabular CUSUM control chart with $k = 0\ 5$ $h = 4\ 77$ and when 3 5 sigma Shewhart control limits are added to the individual observations For example for a shift of size $\delta = 2\ 5$ the CUSUM Shewhart procedure has an ARL_1 of 2 75 whereas the ARL of the basic CUSUM procedure with $k = 0\ 5$ and $h = 4\ 77$ as given in Table 4 10 is 3 00 Thus comparing the ARL s of Table 4 10 when $h = 4\ 77$ and the ARL s in Panel (a) of Table 4 11 we note that the addition of the Shewhart control limits improve the ability of the CUSUM procedure to detect larger shifts with only a 12% decrease in the ARL_0

Apart from improving the responsiveness of the two sided tabular CUSUM control chart for large shifts we can also increase its sensitivity at process start up using the Fast Initial Response (FIR) or the headstart feature For example panel (b) of Table 4 11 presents the ARL properties of the two sided tabular CUSUM procedure when using the basic two sided tabular CUSUM procedure with a 50% headstart i e $S_0 = S_0 = \frac{h}{2}$ However note that the values in panel (b) are only valid for the case when the process is out of control at start up or when the process is out of control after the CUSUMS are reset If the process is in control at start up or when the CUSUMS are reset and a sustained shift occurs later in the process then panel (a) presents the more appropriate run length properties i e the basic CUSUM procedure without the FIR feature

Panel (c) of Table 4 11 presents the ARL properties of the basic CUSUM procedure with both the FIR and the 3 5 sigma Shewhart control limits added

	(a) CUSUM & Shewhart					(b) CUSUM & FIR					(c) CUSUM & FIR & Shewhart				
δ	*ARL*	SDRL	10^{th}	50^{th}	90^{th}	*ARL*	SDRL	10^{th}	50^{th}	90^{th}	*ARL*	SDRL	10^{th}	50^{th}	90^{th}
0 00	326 42	320 98	39	228	746	338 06	360 80	13	226	809	299 91	319 99	12	200	718
0 25	116 14	109 85	18	83	259	105 05	113 47	6	69	253	100 93	108 96	6	66	243
0 50	34 67	28 39	9	26	72	26 54	27 88	3	17	63	26 28	27 53	3	17	62
0 75	16 03	10 60	6	13	30	10 72	9 75	3	7	23	10 69	9 71	3	7	23
1 00	9 82	5 32	4	9	17	6 11	4 57	2	5	12	6 10	4 56	2	5	12
1 50	5 41	2 28	3	5	8	3 25	1 74	2	3	5	3 25	1 74	2	3	5
2 00	3 70	1 42	2	4	5	2 28	1 17	1	2	4	2 28	0 99	1	2	4
2 50	2 75	1 10	1	3	4	1 79	0 70	1	2	3	1 79	0 70	1	2	3
3 00	2 12	0 92	1	2	3	1 49	0 57	1	1	2	1 49	0 56	1	1	2
3 50	1 66	0 75	1	2	3	1 27	0 46	1	1	2	1 27	0 46	1	1	2
4 00	1 35	0 57	1	1	2	1 13	0 34	1	1	2	1 13	0 34	1	1	2

Table 4 11 Run length results of the two sided tabular CUSUM procedure with $k = 0$ 5 and $h = 4$ 77 with some modifications ( )

4 1 14 The conditional run length results

The performance of a tabular CUSUM control chart when the process parameters are *unknown* is not only influenced by the reference value K and the decision interval H but also by the parameter estimates obtained from the in control reference sample

Table 4 12 presents the run length results i e the ARL the SDRL and some percentiles of the *conditional* run length distribution of a one sided upper control chart procedure when $k = 0$ 5 and $h = 3$ 716 In each case the run length distribution is *conditioned* on estimates of the true in control process mean μ_0 and/or the true in control process standard deviation σ_0 through particular realizations of the two random variables Z_0 and W_0 defined in equations (4 14a) and (4 14b) These conditional values were chosen to be the 25^{th} and the 75^{th} percentiles of the distributions of Z_0 and W_0 For reference the case of *known* process parameters is also given The ARL_0 when the process parameters are known and the control chart is tuned to optimally detect a shift size $\delta = 1$ (δ as defined in equation 4 14e) is 250

		Known σ_0					25^{th} (W)					75^{th} (W)				
	δ	*ARL*	SDRL	10^{th}	50^{th}	90^{th}	*ARL*	SDRL	10^{th}	50^{th}	90^{th}	*ARL*	SDRL	10^{th}	50^{th}	90^{th}
Known μ_0	0 0	249 93	245 69	29	174	569	198 42	194 36	24	138	451	315 20	310 79	36	219	719
	0 5	23 83	19 48	5	17	48	21 44	17 34	5	15	43	26 47	21 88	6	19	54
	1 0	7 81	4 47	2	6	13	7 38	4 21	2	5	12	8 27	4 74	3	6	13
	1 5	4 46	1 93	1	3	6	4 28	1 86	1	3	6	4 65	2 01	2	3	6
	2 0	3 15	1 13	1	2	4	3 04	1 09	1	2	3	3 27	1 16	1	2	4
25^{th} (Z)	0 0	142 87	138 48	18	99	322	116 69	112 50	15	81	262	175 19	170 61	22	122	396
	0 5	17 88	13 70	4	13	35	16 33	12 38	4	12	31	19 58	15 15	5	14	38
	1 0	6 84	3 67	2	5	11	6 49	3 48	2	5	10	7 20	3 87	2	5	11
	1 5	4 13	1 71	1	3	5	3 96	1 65	1	3	5	4 30	1 78	1	3	6
	2 0	2 99	1 04	1	2	3	2 89	1 01	1	2	3	3 10	1 07	1	2	3
75^{th} (Z)	0 0	456 53	452 47	51	317	1045	352 18	348 27	40	244	805	592 93	588 07	65	411	1357
	0 5	33 31	28 85	6	24	70	29 44	25 23	2	21	61	37 70	32 99	7	27	80
	1 0	9 10	5 55	3	7	15	8 55	5 19	2	6	14	9 67	5 93	3	7	16
	1 5	4 86	2 20	2	3	7	4 65	2 11	1	3	6	5 07	2 29	2	4	7
	2 0	3 34	1 23	1	2	4	3 21	1 19	1	2	4	3 46	1 27	1	2	4

Table 4 12 Conditional run length results for a one sided tabular CUSUM procedure with parameter estimates from $m = 50$ and subgroups of size $n = 5$ ( )

When we condition on specific values of the parameter estimates the run length distribution is affected by estimation as it would be if the process parameters were to change

For example consider the one sided upper CUSUM control chart If the in control process standard deviation σ_0 is overestimated Y_t as defined in equation (4 12) will be smaller than it should be and the control chart will signal more slowly than it would if the in control process standard deviation had been known For example when conditioning on a value of $W_0 = 1\ 033$( ) the 75^{th} percentile of the sampling distribution of the estimator $\hat{\sigma}_0$ defined in equation (4 14a) the ARL_0 is 315 20 about 26% larger than if the in control process standard deviation had been known Likewise when the process is out of control the control chart will be slower to signal if the in control process standard deviation is overestimated If however the in control processes standard deviation is underestimated Y_t will be larger than it should be and the control control chart will signal more quickly than if the in control process standard deviation had been known in both the in control and out of control situations This is analogous to either a decrease or an increase in the process standard deviation

The results in Table 4 12 were obtained using the Markov chain approach See SAS program B in Appendix 4

The 25^{th} and the 75^{th} percentile of the sampling distribution of W_0 was obtained by Monte Carlo simulation See SAS program C in Appendix 4

Sımılar results hold true when estımatıng the ın control process mean μ_0 If μ_0 ıs overestımated Y_t wıll be smaller than ıt should be and the chart wıll sıgnal more slowly than ıt would ıf the process mean had been known Thıs ıs equıvalent to the performance when the mean has ıncurred an upward shıft For example when we condıtıon on a value of $Z_0 = 0\ 6745$ the 75th percentıle of the samplıng dıstrıbutıon of the standardızed estımator $\hat{\mu}_0$ defined ın equatıon (4 14b) the ARL_0 ıs 456 53 approxımately 83% larger than ıf the process mean had been known Lıkewıse when the process ıs out of control the control chart wıll be slower to sıgnal ıf the ın control process mean ıs overestımated On the contrary ıf the ın control process mean ıs underestımated Y_t wıll be larger than ıt should be and the control chart wıll sıgnal more quıckly than ıf the ın control process mean had been known ın both the ın control and the out of control sıtuatıons
For one sıded lower control charts the opposıte of these statements are true In other words an underestımatıon of the ın control process mean wıll result ın a control chart that sıgnals less frequently and an overestımated ın control process mean results ın a control chart that sıgnals more frequently than ıf the ın control process mean was known

When both the ın control process parameters are estımated sımultaneously the performance of the control chart depends on the dırectıon and the magnıtude of mısspecıficatıon of both the ın control process mean as well as the ın control process standard devıatıon at the same tıme
Note that all these arguments hold for ındıvıdual cases of estımatıon and the sıtuatıon wıll be dıfferent on average when studyıng the uncondıtıonal run length results – see Sectıon 4 1 15

4 1 15 The uncondıtıonal run length results

Table 4 13 presents the *uncondıtıonal* run length results for a one sıded upper CUSUM control chart when usıng estımated process parameters It ıs seen that the ARL_0 when the process parameters are estımated ıs larger than the analogous cases when the process parameters are ın fact known Thus we may conclude that the upper one sıded CUSUM control chart sıgnals falsely less frequently when the process parameters are estımated than when they are ın fact known
However thıs does not necessarıly ımply that the control chart wıth estımated process parameters ıs better than a control chart when the process mean and the process standard devıatıon are known By further ınspectıng the percentıles of the uncondıtıonal run length dıstrıbutıon we see a clearer pıcture

For example consider the case when $m = 30$ in control reference samples each of size $n = 5$ are used and $k = 0\ 5$ with $h = 3\ 716$ The ARL_0 is 658 75 and the 10^{th} and 50^{th} percentiles are smaller than the known parameter case which indicates that shorter runs are more likely However the 90^{th} percentile is 1323 Therefore the large ARL_0 is a result of the skewness of the unconditional run length distribution rather than improved in control performance Subsequently the fact that extremely long runs can occur in an in control situation is a desirable side effect of estimating the in control process parameters

		$k = 0\ 25$ and $h = 5\ 994$					$k = 0\ 5$ and $h = 3\ 716$				
m	δ	*ARL*	SDRL	10^{th}	50^{th}	90^{th}	*ARL*	SDRL	10^{th}	50^{th}	90^{th}
30	0 0	1607 35	22778 12	23	164	2056	658 75	3033 66	21	163	1323
	0 5	29 25	67 07	8	17	55	34 55	68 24	6	18	73
	1 0	9 21	5 04	5	8	15	8 54	6 71	3	7	16
	1 5	5 62	2 11	3	5	8	4 59	2 3	2	4	7
	2 0	4 11	1 26	3	4	6	3 18	1 28	2	3	5
50	0 0	667 54	3094 22	26	167	1310	429 82	1027 71	23	166	980
	0 5	24 63	26 69	8	17	47	29 13	37 55	6	18	62
	1 0	8 99	4 37	5	8	14	8 22	5 59	3	7	15
	1 5	5 57	1 98	3	5	8	4 54	2 14	3	2	7
	2 0	4 09	1 21	3	4	6	3 17	1 23	2	3	5
∞	0 0	249 84	240 17	34	175	562	249 93	245 69	29	174	569
	0 5	20 88	13 09	7	17	37	23 83	19 48	5	17	48
	1 0	8 72	3 63	4	7	12	7 81	4 47	2	6	13
	1 5	5 51	1 80	3	4	7	4 46	1 93	1	3	6
	2 0	4 07	1 13	2	3	5	3 15	1 13	1	2	4

Table 4 13 Unconditional run length results for a one sided upper tabular CUSUM procedure with parameter estimates from $m = 30\ 50$ and subgroups of size $n = 5$ ( )

The results in Table 4 13 when $m = 30$ and $m = 50$ were obtained from Jones Champ and Rigdon (2004) For the case when the process parameters are known the results were obtained using the Markov chain approach See SAS program B in Appendix 4

4 1 16 The V-mask procedure

The V mask procedure ıs an alternatıve procedure to the two sıded tabular CUSUM control chart The tabular CUSUM control chart uses the two plottıng statıstıcs S_t and S_t whereas the CUSUM plot of S_t ın the begınnıng of Chapter 4 was only based on the sequence of cumulatıve sums gıven ın equatıon (4 1) In addıtıon S_t^+ and S_t^- are often reset to zero and therefore any CUSUM status chart of S_t^+ and S_t wıll have a choppy appearance

For a smoother lookıng dısplay and a control chart procedure based on only one plottıng statıstıc the rules governıng S_t and S_t of the two sıded tabular CUSUM procedure are transformed ınto the V mask control chart procedure The V mask procedure for monıtorıng the process mean ıs applıed to successıve values of the cumulatıve sums $S_t = S_{t-1} + Y_t$ where Y_t ıs the standardızed subgroup mean ı e

$$Y_t = \frac{\sqrt{n}\left(\bar{X}_t - \mu_0\right)}{\sigma_0}$$

A typıcal V mask ıs shown ın Fıgure 4 9

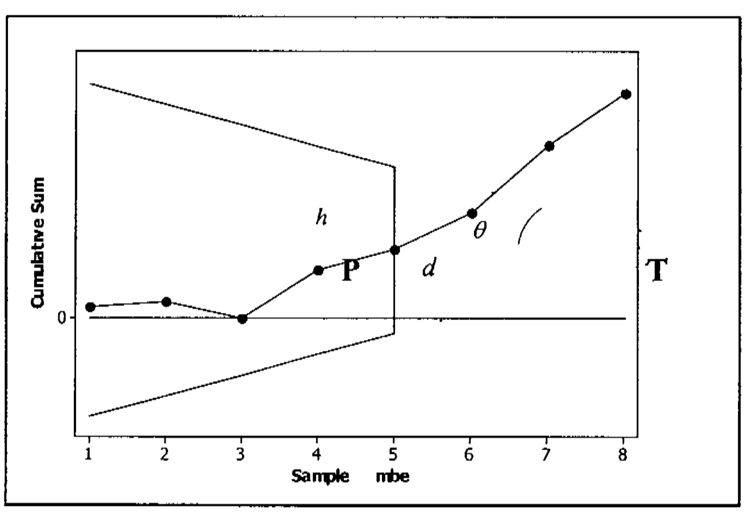

Fıgure 4 9 A typıcal V mask

The decısıon procedure consısts of plottıng a V mask on the CUSUM plot of S_t wıth poınt **P** on a value of S_t and the lıne **PT** parallel to the horızontal axıs If the V ıs plotted a dıstance of d ın front of subgroup t the legs of the V mask are drawn backwards at an angle θ such that they are at a distance of h units above and below S_t

If all the previous cumulative sums S_1 S_2 S_t lie within the two legs of the V mask the process is considered in control However if any of the cumulative sums lie outside the legs the process is declared out of control If the CUSUM plot of S_t crosses the lower leg an upward shift is indicated and if the upper leg is crossed a downward shift might have occurred

The performance of the V mask procedure is determined by the lead distance d and the angle θ as shown in Figure 4 9 Furthermore the V mask procedure and the two sided tabular CUSUM procedure are equivalent if

$$k = w\tan(\theta) \tag{4 39a}$$

and

$$h = wd\tan(\theta) = dk \tag{4 39b}$$

where w is a scale constant i e whatever physical distance is used between successive plotting statistics on the horizontal axis that same physical distance represents w units on the vertical axis See Figure 4 10

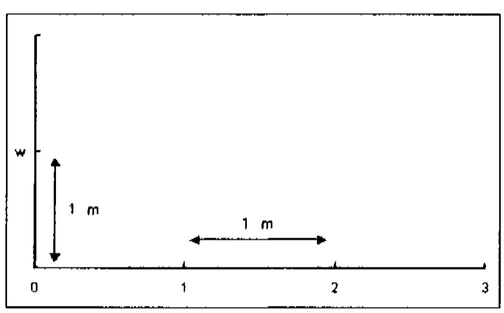

Figure 4 10 Representation of the scale constant (w)

Example 4 5

A V mask procedure

A V mask equivalent to the two sided tabular CUSUM scheme of Example 4 1 where $k = 1$ and $h = 4$ was constructed using Minitab and shown in Figure 4 11 Using equations (4 39a) and (4 39b) with $w = 1$ (say) we find that

$$K = w\tan(\theta)$$

$$k\frac{\sigma}{\sqrt{n}} = w\tan(\theta)$$

$$(1)\frac{2}{\sqrt{5}} = (1)\tan(\theta)$$

which yields

$$\theta = 45$$

and

$$h = dk$$

$$4 = d(1)$$

or

$$d = 4$$

Therefore the lead distance of the V mask would be 4 horizontal plotting positions and the angle of the two legs of the V mask would be 45

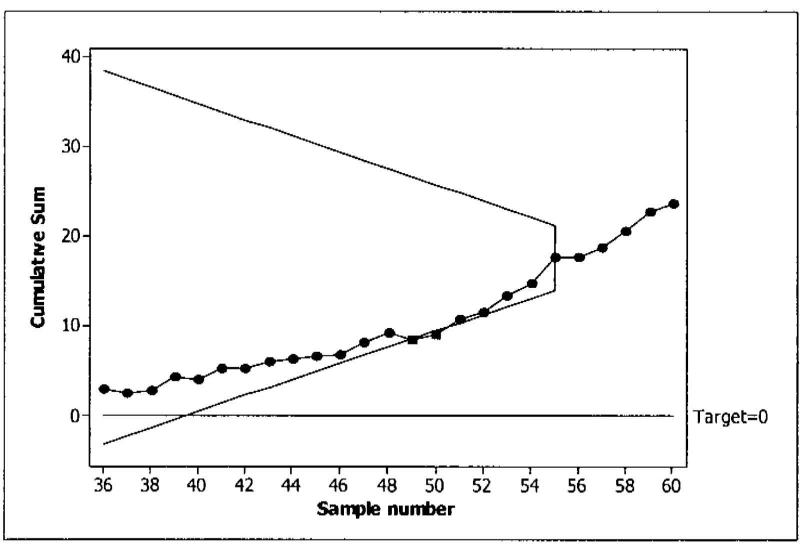

Figure 4 11 A V mask procedure

Although the performance of the V mask procedure is similar to that of the two sided tabular CUSUM control chart if equations (4 39a) and (4 39b) are satisfied there as some disadvantages and problems associated with the V mask procedure For example

1 The V mask is a two sided control scheme and therefore not very useful if a one sided process monitoring scheme is required
2 The Fast Initial Response (FIR) which can be useful cannot be implemented with the V mask
3 In some circumstances it is difficult to determine how far backwards the legs of the V mask should extend thereby making the interpretation of the V mask difficult

Thus it is advisable to rather use one sided tabular CUSUM control charts separately or combined instead of the V mask procedure

4 2 Exponentially Weighted Moving Average (EWMA) control chart for the process mean

4 2 1 Introduction

The exponentially weighted moving average (EWMA) control chart is an alternative to a Shewhart type of control chart and may be used when small process shifts are of interest As the name suggests the EWMA control chart involves exponential smoothing a technique often used in time series analysis

The performance of the EWMA control chart is approximately equivalent to that of the two sided tabular CUSUM control chart although the EWMA control chart is in some ways easier to set up and operate Furthermore the EWMA control chart is typically used with individual measurements and therefore we first discuss the EWMA control chart for individual measurements and then extend the ideas to rational subgroups ($n > 1$)

4 2 2 The EWMA control chart

To monitor the process mean using an EWMA control chart let X_t be the t^{th} observation in a series of observations from a process that follows a normal distribution with a *known* mean (μ_0) and a *known* standard deviation (σ_0)

The EWMA plotting statistics are calculated recursively using

$$Q_t = \lambda X_t + (1-\lambda) Q_{t\ 1} \qquad (4\ 40)$$

where $0 < \lambda \leq 1$ is a weighting constant and $t = 1\ 2\ 3$

To initialize the calculations in equation (4 40) a starting value Q_0 is required when $t = 1$ In general the starting value can be any real value u say However the starting value is typically set equal to the known process mean so that $Q_0 = \mu_0$ or if the process is to be controlled at some target level θ we may set $Q_0 = \theta$

To demonstrate that Q_t is a weighted average of all past observations, we may recursively substitute for Q_{t-1} on the right hand side of equation (4.40) to obtain

$$\begin{aligned} Q_t &= \lambda X_t + (1-\lambda)\{\lambda X_{t-1} + (1-\lambda)Q_{t-2}\} \\ &= \lambda X_t + (1-\lambda)\lambda X_{t-1} + (1-\lambda)^2 Q_{t-2} \end{aligned}$$

Following the substitution of Q_{t-1}, we may also substitute Q_{t-2} with $\lambda X_{t-2} + (1-\lambda)Q_{t-3}$ to obtain

$$\begin{aligned} Q_t &= \lambda X_t + (1-\lambda)\lambda X_{t-1} + (1-\lambda)^2\{\lambda X_{t-2} + (1-\lambda)Q_{t-3}\} \\ &= \lambda X_t + (1-\lambda)\lambda X_{t-1} + (1-\lambda)^2 \lambda X_{t-2} + (1-\lambda)^3 Q_{t-3} \end{aligned}$$

Continuing the recursive substitution of Q_{t-j}, $j = 3, 4, \ldots, t$, we obtain

$$Q_t = \lambda X_t + (1-\lambda)\lambda X_{t-1} + (1-\lambda)^2 \lambda X_{t-2} + \ldots + (1-\lambda)^t Q_0$$

which simplifies to

$$Q_t = \lambda \sum_{j=0}^{t-1} (1-\lambda)^j X_{t-j} + (1-\lambda)^t Q_0 \qquad (4.41)$$

Thus, like the CUSUM procedure, the EWMA procedure accumulates all the information in a series of observations $X_1, X_2, \ldots, X_t$.

The name of the EWMA control chart, or the geometric moving average (GMA) control chart, originates from the fact that the weights $\lambda(1-\lambda)^j$ decrease exponentially (or geometrically) with the age of an observation. For example, if $\lambda = 0.25$, the weight assigned to the current observation is 0.25 and the weights assigned to the preceding observations are 0.1875, 0.140625, 0.10546875 and so on. A comparison of these weights and those of a four period moving average are shown in Figure 4.12.

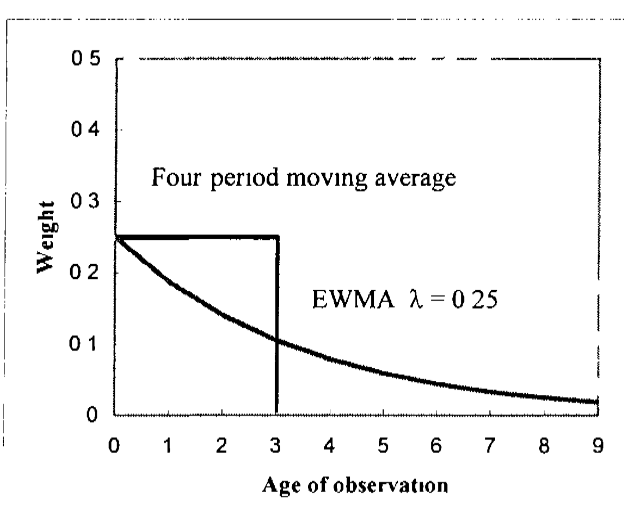

Figure 4.12 Weights of past observations

Furthermore the sum of the weights from time 1 up and till time $t-1$ i e $\lambda\sum_{1}^{t-1}(1-\lambda)$ add to $1-(1-\lambda)^t$ so that all the weights sum to unity i e $\lambda\sum_{1}^{t-1}(1-\lambda)+(1-\lambda)^t=\{1-(1-\lambda)^t\}+(1-\lambda)^t=1$

From equation (4 40) or equation (4 41) we note that the weighting constant controls the amount of influence that previous observations have on the current plotting statistic Q_t For values of λ close to 1 the EWMA control chart closely resembles the Shewhart control chart for individual observations On the other hand for values of λ close to 0 the EWMA control chart gives a small weight to almost all of the previous observations

If we assume that the series of observations X_1 X_2 X_t are independent identically distributed random variables using equation (4 41) we find the expected value and the variance of Q_t as

$$E(Q_t)=\lambda\sum_{1}^{t-1}(1-\lambda)\ E(X_{t}\ )+(1-\lambda)^t Q_0=\mu_0 \qquad (4\ 42a)$$

provided $Q_0=\mu_0$ and

$$\mathrm{var}(Q_t)=\lambda^2\sum_{1}^{t-1}(1-\lambda)^2\ \mathrm{var}(X_{t-}\ )=\frac{\lambda}{2-\lambda}\left(1-(1-\lambda)^{2t}\right)\sigma_0^2 \qquad (4\ 42b)$$

Therefore an EWMA control chart to monitor the process mean can be constructed by plotting Q_t versus the sample number (t) together with the centerline and control limits The centerline and control limits are

$$\begin{aligned}UCL&=\mu_0+L\sigma_0\sqrt{\frac{\lambda}{2-\lambda}\left(1-(1-\lambda)^{2t}\right)}\\ CL&=\mu_0\\ LCL&=\mu_0-L\sigma_0\sqrt{\frac{\lambda}{2-\lambda}\left(1-(1-\lambda)^{2t}\right)}\end{aligned} \qquad (4\ 43)$$

where L is the width of the control limits i e the distance of the upper control limit and the lower control limit from the centerline

From equation (4 43) we see that the control limits are functions of time However noting that $\lim_{t\to\infty}(1-\lambda)^{2t}=1$ we also see that the control limits approach their so called steady state values

$$UCL = \mu_0 + L\sigma_0\sqrt{\frac{\lambda}{2-\lambda}} \tag{4 44a}$$

and

$$LCL = \mu_0 - L\sigma_0\sqrt{\frac{\lambda}{2-\lambda}} \tag{4 44b}$$

ıf the control chart has been runnıng for a suffıcıently long tıme perıod

Example 4 6

An EWMA control chart

A two sıded tabular CUSUM control chart for the data of Table 4 1 was constructed ın Example 4 3 As an alternatıve an EWMA control chart ıs now created

Recall that values 1 trough 20 were randomly drawn from a normal dıstrıbutıon wıth $\mu = 10$ and $\sigma = 1$ whereas the last ten values (21 through 30) were randomly drawn from a normal dıstrıbutıon wıth $\mu = 11$ and $\sigma = 1$

We wıll use $\lambda = 0\ 1$ and $L = 2\ 7$ to calculate the EWMA plottıng statıstıcs and sınce the ın control process mean ıs *known* to be 10 a startıng value of $Q_0 = 10$ ıs used

To ıllustrate the calculatıons consıder the fırst observatıon The fırst value plotted on the EWMA control chart ın Fıgure 4 13 ıs

$$\begin{aligned} Q_1 &= \lambda X_1 + (1-\lambda)Q_0 \\ &= 0\ 1(10) + (1 - 0\ 1)10 \\ &= 10 \end{aligned}$$

The second value ıs

$$\begin{aligned} Q_2 &= \lambda X_2 + (1-\lambda)Q_1 \\ &= 0\ 1(8\ 5) + (1 - 0\ 1)10 \\ &= 9\ 85 \end{aligned}$$

The remaınıng calculatıons are summarızed ın column (b) of Table 4 14

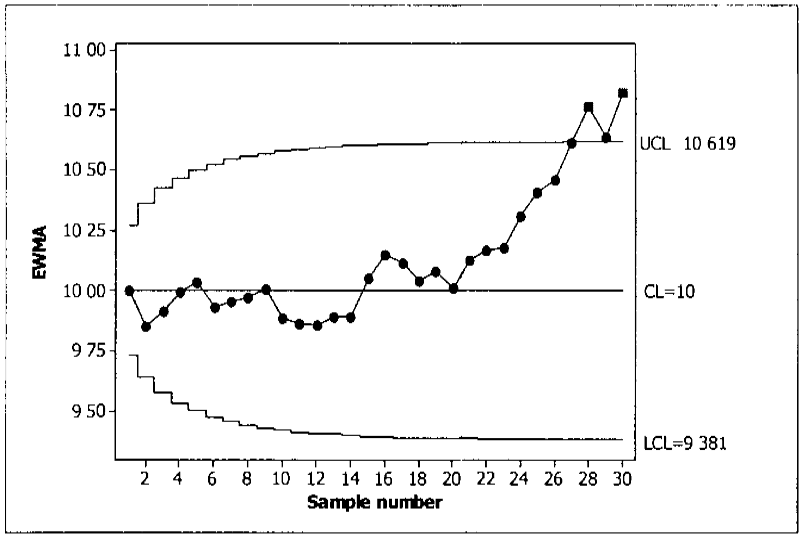

Figure 4 13 An EWMA control chart

The control limits in Figure 4 13 are found using equation (4 43) For example for period $t = 1$

$$UCL = \mu_0 + L\sigma_0\sqrt{\frac{\lambda}{2-\lambda}\left(1-(1-\lambda)^{2t}\right)}$$

$$= 10 + 2\ 7(1)\sqrt{\frac{0\ 1}{2-0\ 1}\left(1-(1-0\ 1)^{2(1)}\right)}$$

$$= 10\ 27$$

and

$$LCL = \mu_0 - L\sigma_0\sqrt{\frac{\lambda}{2-\lambda}\left(1-(1-\lambda)^{2t}\right)}$$

$$= 10 - 2\ 7(1)\sqrt{\frac{0\ 1}{2-0\ 1}\left(1-(1-0\ 1)^{2(1)}\right)}$$

$$= 9\ 73$$

The remaining values for the control limits are computed similarly and are summarized in columns (c) and (d) of Table 4 14

Note from Figure 4 13 that the control limits increase in width as t increases and stabilize at their steady state values given by equations (4 44a) and (4 44b)

$$UCL = \mu_0 + L\sigma_0\sqrt{\frac{\lambda}{2-\lambda}}$$

$$= 10 + 2\ 7(1)\sqrt{\frac{0\ 1}{2-0\ 1}}$$

$$= 10\ 619$$

and

$$LCL = \mu_0 - L\sigma_0\sqrt{\frac{\lambda}{2-\lambda}}$$

$$= 10 - 2\ 7(1)\sqrt{\frac{0\ 1}{2-0\ 1}}$$

$$= 9\ 381$$

Sample t	(a) X_t	(b) $Q_t = \lambda X_t + (1-\lambda)Q_{t\ 1}$	(c) UCL	(d) LCL
1	10 0	10 000	10 270	9 730
2	8 5	9 850	10 363	9 637
3	10 5	9 915	10 424	9 576
4	10 7	9 994	10 467	9 533
5	10 4	10 034	10 500	9 500
6	9 0	9 931	10 525	9 475
7	10 2	9 958	10 544	9 456
8	10 1	9 972	10 559	9 441
9	10 3	10 005	10 571	9 429
10	8 8	9 884	10 581	9 419
11	9 7	9 866	10 588	9 412
12	9 8	9 859	10 594	9 406
13	10 2	9 893	10 599	9 401
14	9 9	9 894	10 603	9 397
15	11 5	10 055	10 606	9 394
16	11 0	10 149	10 609	9 391
17	9 8	10 114	10 611	9 389
18	9 4	10 043	10 612	9 388
19	10 4	10 079	10 614	9 386
20	9 4	10 011	10 615	9 385
21	11 2	10 130	10 616	9 384
22	10 5	10 167	10 616	9 384
23	10 3	10 180	10 617	9 383
24	11 5	10 312	10 617	9 383
25	11 3	10 411	10 618	9 382
26	10 9	10 460	10 618	9 382
27	12 0	10 614	10 618	9 382
28	12 1	10 762*	10 619	9 381
29	9 5	10 636*	10 619	9 381
30	12 5	10 823*	10 619	9 381

Table 4 14 Calculatıons for the EWMA control chart ın Example 4 6

The first 20 EWMA plottıng statıstıcs where $\mu = 10$ tends to drıft slowly about the ın control process mean or centerlıne However from observatıon 20 onwards where $\mu = 11$ the EWMA plottıng statıstıcs develop an upward trend and at observatıon 28 the EWMA control chart gıves a sıgnal when $Q_{28} = 10\ 762$ plots above the upper control lımıt Thus process ıs declared out of control and a search for assıgnable causes ıs started

Note that although corrective action was only taken when the EWMA control chart signaled an upward trend developed immediately following the shift in the process mean between samples 20 and 21

The trend can be explained in the following manner

Suppose that m independent observations are randomly drawn from an in control process following a normal distribution with a *known* mean and a *known* standard deviation denoted by μ_0 and σ_0 respectively Thus $X_1\ X_2\ \quad X_m \sim iidN\left(\mu_0\ \sigma_0^2\right)$ and from equations (4 42a) and (4 42b) we have that

$$E(Q_t) = \mu_0$$

and

$$\operatorname{var}(Q_t) = \frac{\lambda}{2-\lambda}\left(1-(1-\lambda)^{2t}\right)\sigma_0^2$$

where $t = 1\ 2\ \quad m$

However if the process mean undergoes a sustained shift and a further k independent observations are randomly drawn from the same process we have that $X_{m+1}\ X_{m\ 2}\ \quad X_{m\ k} \sim iidN\left(\mu_1\ \sigma_0^2\right)$ where $\mu_1 = \mu_0 + \delta\sigma_0$ is the process mean following the shift Thus using equation (4 41) we find that

$$E(Q_t) = \mu_0 + \left(1-(1-\lambda)^t\right)\delta\sigma_0 \qquad (4\ 45a)$$

and

$$\operatorname{var}(Q_t) = \frac{\lambda}{2-\lambda}\left(1-(1-\lambda)^{2t}\right)\sigma_0^2 \qquad (4\ 45b)$$

where $t = m+1\ \quad m+k$

From equation (4 45a) we note that $1-(1-\lambda)^t$ is a non decreasing function of t if $\lambda \in (0\ 1)$

Consequently when $\delta > 0$ the points on the EWMA control chart will develop an upward trend and when $\delta < 0$ the points on the EWMA control chart will develop a downward trend Thus whenever the EWMA plotting statistics develop a trend it is likely that a process shift is being unmasked

4 2 3 Standardized EWMA control chart

Sometimes we prefer to standardize the series observations X_1 X_2 X_3 before constructing an EWMA control chart Therefore if $Y_t = \frac{X_t - \mu_0}{\sigma_0}$ denotes a standardized observation we replace X_t with Y_t in equation (4 40) to obtain

$$Q_t = \lambda Y_t + (1-\lambda) Q_{t-1} \quad (4\ 46)$$

with the starting value typically set equal to zero so that $Q_0 = 0$

The expected value and the variance of Q_t becomes

$$E(Q_t) = 0 \quad (4\ 47a)$$

and

$$\text{var}(Q_t) = \frac{\lambda}{2-\lambda}\left(1-(1-\lambda)^{2t}\right) \quad (4\ 47b)$$

respectively

The standardized EWMA control chart can therefore be constructed by plotting Q_t versus the sample number or time with the centerline and control limits at

$$\begin{aligned} UCL &= +l\sqrt{\frac{\lambda}{2-\lambda}\left(1-(1-\lambda)^{2t}\right)} \\ CL &= 0 \\ LCL &= -l\sqrt{\frac{\lambda}{2-\lambda}\left(1-(1-\lambda)^{2t}\right)} \end{aligned} \quad (4\ 48)$$

where l is the standardized width of the control limits i e $L = l\sigma_0$

Furthermore the steady state control limits becomes

$$UCL = +l\sqrt{\frac{\lambda}{2-\lambda}} \quad (4\ 49a)$$

and

$$LCL = -l\sqrt{\frac{\lambda}{2-\lambda}} \quad (4\ 49b)$$

4 2 4 Rational subgroups

Although the EWMA control chart is often used with individual measurements if rational subgroups of size $n > 1$ are taken we simply replace X_t with $\overline{X}_t$ and replace σ_0 with $\mu_{\overline{X}} = \frac{\sigma_0}{\sqrt{n}}$

4 2 5 Conditional EWMA control chart

It is seldom that the process mean and the process standard deviation are *known* Thus whenever the process mean and/or the process standard deviation are *unknown* these parameters need to be estimated The unknown process parameters are usually estimated from a reference sample when the process was thought to operate in control Unbiased point estimators for the in control process mean μ_0 and the in control process standard deviation σ_0 are given by equations (4 9a) and (4 9b) i e

$$\mu_0 = \overline{\overline{X}} = \frac{1}{m}\sum_{1}^{m}\overline{X} = \frac{1}{mn}\sum_{1}^{m}\sum_{j\,1}X_j$$

and

$$\sigma_0 = \frac{S_p}{c_{4\,m}}$$

If the target value θ is unknown it can be estimated using μ_0

Example 4 7

An EWMA control chart with estimated process parameters

A two sided tabular CUSUM control chart was constructed in Example 4 4 for the data in Table 4 5 and Table 4 7 An EWMA control chart is now created

Recall that the process mean and the process standard deviation were *unknown* and estimated from the data in Table 4 5 considered to be an in control reference sample The estimates were $\widehat{\mu}_0 = 19\ 85$ and $\widehat{\sigma}_0 = 2\ 021$

The EWMA plotting statistics Q_t $t = 1\ 2$ 25 with $Q_0 = 19\ 85$ and the control limits when $L = 2\ 7$ and $\lambda = 0\ 1$ are summarized in columns (a) (b) and (c) of Table 4 15 The centerline was estimated at $CL = \widehat{\mu}_0 = 19\ 85$

Since the EWMA control chart in Figure 4 14 gives no signal we continue to use the estimated in control process parameters for the prospective monitoring of the process Consequently we extend the EWMA control chart for the data in Table 4 5 and add the EWMA plotting statistics for the data in Table 4 7 – an additional 35 samples from the same process The extended EWMA control chart is given in Figure 4 15 with the plotting statistics and the control limits given in Table 4 16

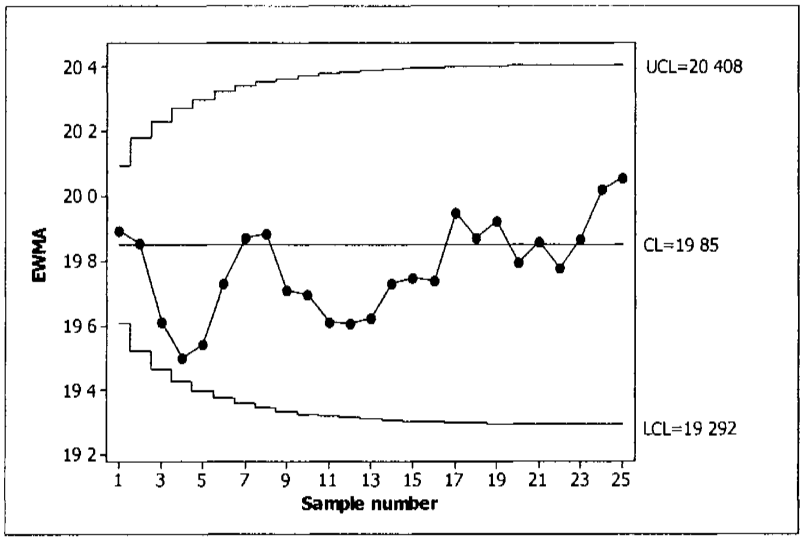

Figure 4 14 An EWMA control chart for the data of Table 4 5

Sample t	(a) $Q_t = \lambda \overline{X}_t + (1-\lambda) Q_{t-1}$	(b) UCL	(c) LCL
1	19 891	20 094	19 606
2	19 854	20 178	19 522
3	19 609	20 233	19 467
4	19 498	20 273	19 427
5	19 542	20 302	19 398
6	19 730	20 324	19 376
7	19 873	20 342	19 358
8	19 883	20 355	19 345
9	19 707	20 366	19 334
10	19 694	20 375	19 325
11	19 611	20 382	19 318
12	19 608	20 387	19 313
13	19 625	20 391	19 309
14	19 731	20 395	19 305
15	19 746	20 398	19 302
16	19 739	20 400	19 300
17	19 947	20 402	19 298
18	19 870	20 404	19 296
19	19 921	20 405	19 295
20	19 795	20 406	19 294
21	19 858	20 406	19 294
22	19 776	20 407	19 293
23	19 868	20 408	19 292
24	20 023	20 408	19 292
25	20 057	20 408	19 292

Table 4 15 The EWMA plotting statistics (Q_t) and the control limits for the data of Table 4 5

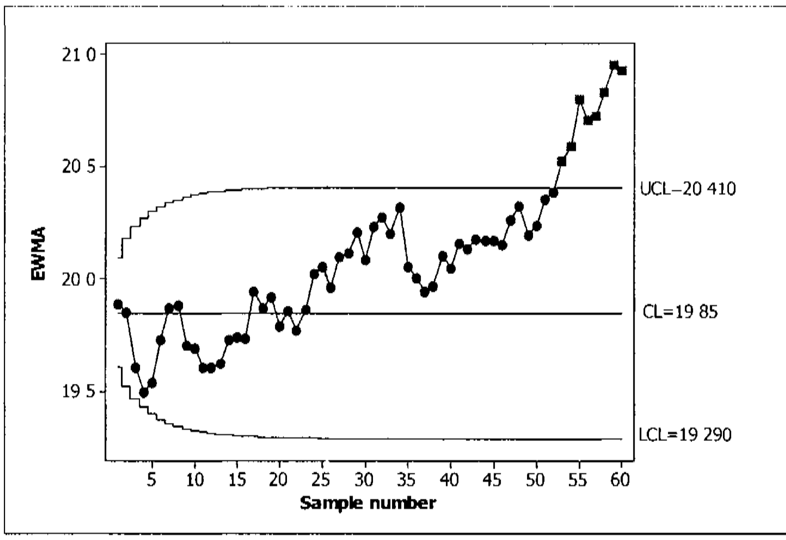

Figure 4 15 An EWMA control chart for the reference sample and the 35 additional samples

Sample t	(a) EWMA Q_t	(b) UCL	(c) LCL	Sample t	(a) EWMA Q_t	(b) UCL	(c) LCL
26	19 967	20 409	19 291	44	20 173	20 410	19 290
27	20 099	20 409	19 291	45	20 173	20 410	19 290
28	20 121	20 409	19 291	46	20 156	20 410	19 290
29	20 209	20 409	19 291	47	20 268	20 410	19 290
30	20 088	20 409	19 291	48	20 328	20 410	19 290
31	20 235	20 409	19 291	49	20 199	20 410	19 290
32	20 278	20 410	19 290	50	20 239	20 410	19 290
33	20 208	20 410	19 290	51	20 359	20 410	19 290
34	20 321	20 410	19 290	52	20 391	20 410	19 290
35	20 055	20 410	19 290	53	20 526	20 410	19 290
36	20 009	20 410	19 290	54	20 589	20 410	19 290
37	19 948	20 410	19 290	55	20 801	20 410	19 290
38	19 970	20 410	19 290	56	20 706	20 410	19 290
39	20 109	20 410	19 290	57	20 730	20 410	19 290
40	20 052	20 410	19 290	58	20 833	20 410	19 290
41	20 165	20 410	19 290	59	20 954	20 410	19 290
42	20 136	20 410	19 290	60	20 928	20 410	19 290
43	20 179	20 410	19 290				

Table 4 16 EWMA plotting statistics (Q_t) and the control limits for the data of Table 4 7

On sample 53 a signal is given when $Q_{53} = 20\ 526$ plots above the upper control limit i e 20 410 The process is declared out of control and a search for assignable causes is started

The EWMA control chart signals

As in the case of the CUSUM control chart (and the Shewhart control chart) the performance of the EWMA control chart is affected by the parameter estimates from the reference sample For example Figure 4 16 is an EWMA control chart if we assume that the true in control process mean and the true in control process standard deviation are 20 and 2 respectively From Figure 4 16 we see that a signal is first given on sample 55 when $Q_{55} = 20\ 801$ plots above the upper control limits i e

$UCL = 20 + 2\ 7\left(\frac{2}{\sqrt{5}}\right)\sqrt{\frac{\lambda}{2-\lambda}\left(1-(1-\lambda)^{2(55)}\right)} = 20\ 447$ This is 2 samples later than when we used the estimated process parameters $\hat{\mu}_0$ and $\hat{\sigma}_0$

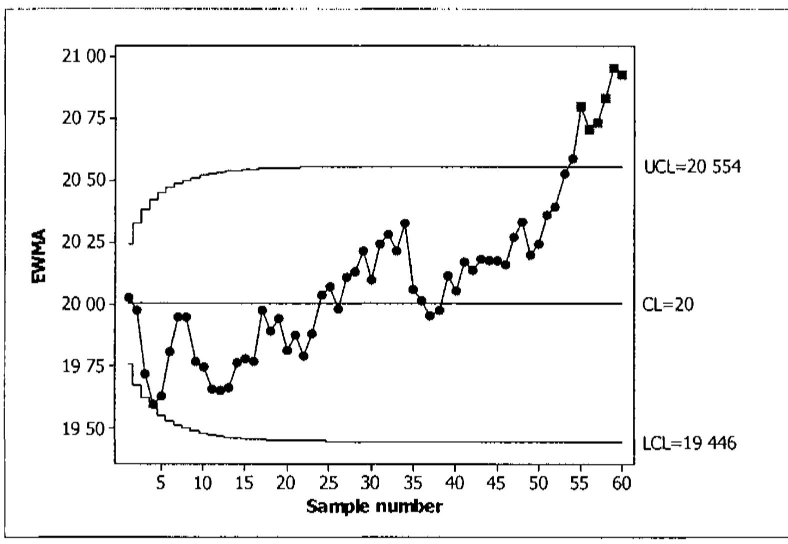

Figure 4 16 An EWMA control chart with $\mu = 20$ and $\sigma = 2$

Therefore if we intend to assess the overall performance of the EWMA control chart when using estimated process parameters we again need to account for the (sampling) variability in the estimators of the in control process parameters For this reason we need the unconditional run length distribution but first we need to derive the conditional run length distribution However once the conditional run length distribution has been found the procedures and/or methods we use to find the unconditional run length distribution of the EWMA control chart is similar to that of the CUSUM control chart

4 2 6 The run length dıstrıbutıon of the EWMA control chart usıng ıntegral equatıons

Exact expressıons for the condıtıonal run length dıstrıbutıon of the standardızed EWMA control chart are gıven by

$$P(N=1|u\ \delta\ \gamma\ Z_0=z_0\ W_0=w_0)=1-\frac{1}{\lambda}\int_{-h}^{h}\frac{w_0}{\gamma}\phi\left(\frac{w_0}{\gamma}\left(\frac{s-(1-\lambda)u}{\lambda}\right)-\frac{\delta}{\gamma}+\frac{z_0}{\gamma\sqrt{m}}\right)ds \qquad (4\ 50a)$$

for $t=1$ and

$$P(N=t|u\ \delta\ \gamma\ Z_0=z_0\ W_0=w_0)$$
$$=\frac{1}{\lambda}\int_{-h}^{h}P(N=t-1|s\ \delta\ \gamma\ Z_0=z_0\ W_0=w_0)\frac{w_0}{\gamma}\phi\left(\frac{w_0}{\gamma}\left(\frac{s-(1-\lambda)u}{\lambda}\right)-\frac{\delta}{\gamma}+\frac{z_0}{\gamma\sqrt{m}}\right)ds \qquad (4\ 50b)$$

for $t=2\ 3$

(See Result 4 7a – 4 7c ın Appendıx 4)

Substıtutıng expressıons (4 50a) and (4 50b) ın equatıons (4 18) – (4 23) we obtaın the r^{th} condıtıonal non central moment of the run length random varıable the joınt pdf of N Z_0 and W_0 the margınal (or uncondıtıonal) run length dıstrıbutıon of N the expected value of the uncondıtıonal run length dıstrıbutıon (ARL) as well as the standard devıatıon of the uncondıtıonal run length dıstrıbutıon (SDRL)

4 2 7 The conditional run length distribution of the EWMA control chart using finite Markov chains

The Markov chain approach for deriving the run length distribution of the EWMA control chart is similar to that of a one sided CUSUM control chart Again we partition the vertical axis of the EWMA control chart or the set of real values ($\Re$) into a set of non overlapping intervals followed by choosing reference values within each interval to represent the states of a discrete time Markov chain However for the EWMA control chart $\Re$ is partitioned into $2m$ intervals as opposed to the $m+1$ intervals that were used for the one sided CUSUM control charts since the EWMA control chart is a two sided procedure Furthermore we use the standardized EWMA control chart as defined in equation (4 46) However the steady state control limits are used throughout so that the process is considered in control whenever $-h < Q_t < h$ and declared out of control when either $Q_t \leq -h$ or $Q_t \geq h$ where $h = +l\sqrt{\frac{\lambda}{2-\lambda}}$ and $t = 1\ 2\ 3$

Thus if we require a total of $2m$ states of which one state is an absorbent state representing the intervals $(-\infty\ -h]$ and $[h\ \infty)$ we are left with a further $2m-1$ non absorbent states to be created between the two control limits $-h$ and h In addition if w specifies the width of an interval we have that $(2m-1)w = h-(-h) \Rightarrow w = \frac{2h}{2m-1}$ Therefore the intervals that represent the $2m-1$ non absorbent states are given by the general expression $\left[\left(k-\frac{1}{2}\right)w\ \left(k+\frac{1}{2}\right)w\right)$ and the midpoint of each is given by kw with $k = -(m-1)\ \ \ -1\ 0\ 1\ \ \ (m-1)$ For example for state 0 we set $k=0$ and obtain the interval $\left[-\frac{w}{2}\ -\frac{w}{2}\right)$ with a midpoint of 0 Again the midpoint value is typically used as reference value for a state and thought of as representing an entire interval of values However the reference value for the absorbent state m is h which is not the midpoint of the two intervals $(-\infty\ -h]$ and $[h\ \infty)$ Table 4 17 summarizes the states as well as the intervals they represent on the real line and also gives the reference values assumed to approximate all the values within a particular state

State	Interval	Reference / Mıdpoınt Value
$-(m-1)$	$\left[-h \quad \left(-m+\frac{3}{2}\right)w\right)$	$-(m-1)w$
-1	$\left[-\frac{3}{2}w \quad -\frac{1}{2}w\right)$	$-w$
0	$\left[-\frac{1}{2}w \quad \frac{1}{2}w\right)$	0
1	$\left[\frac{1}{2}w \quad \frac{3}{2}w\right)$	w
$m-1$	$\left[\left(m-\frac{3}{2}\right)w \quad h\right)$	$(m-1)w$
m	$(-\infty \quad -h]$ and $[h \quad \infty)$	h

Table 4 17 A summary of the $2m$ states of the Markov chaın approach

In thıs case the transıtıon probabılıty matrıx (**P**) can be represented as

$$\mathbf{P} = \begin{bmatrix} p_{ij} & p_{im} \\ p_{mj} & p_{mm} \end{bmatrix}$$

wıth the transıtıon probabılıtıes gıven by

$$p_{ij} = \frac{1}{\lambda}\int_{(j-\frac{1}{2})w}^{(j+\frac{1}{2})w} \frac{w_0}{\gamma}\phi\left(\frac{w_0}{\gamma}\left(\frac{s-(1-\lambda)iw}{\lambda}\right)-\frac{\delta}{\gamma}+\frac{Z_0}{\gamma\sqrt{m}}\right)ds \tag{4 51a}$$

wıth i $j=-(m-1)$ $(m-1)$

$$p_{im} = 1-\frac{1}{\lambda}\int_{-h}^{h} \frac{w_0}{\gamma}\phi\left(\frac{w_0}{\gamma}\left(\frac{s-(1-\lambda)iw}{\lambda}\right)-\frac{\delta}{\gamma}+\frac{Z_0}{\gamma\sqrt{m}}\right)ds \tag{4 51b}$$

wıth $i=-(m-1)$ $(m-1)$

$$p_{mj} = 0 \tag{4 51c}$$

wıth $j=-(m-1)$ $(m-1)$ and

$$p_{mm} = 1 \tag{4 51d}$$

(See Results 4 8a – 4 8d in Appendix 4)

The matrix **R** obtained by deleting the last row and the last column of **P** is

$$\mathbf{R} = \begin{bmatrix} & p_{ij} & \end{bmatrix}$$

where p_{ij} is given in equation (4 51a)

The transition probability matrix (**P**) is slightly different from the one obtained for a standardized one sided CUSUM control chart Nonetheless regardless of the difference in structure of the transition probability matrices once **P** has been found the matrix **R** can be found so that the results that we require for the conditional run length distribution can be obtained in a similar manner as was previously done for a one sided CUSUM control chart See equations (4 28) – (4 38) with $i\ j = -(m-1) \quad (m-1)$ To obtain the unconditional run length distribution of the EWMA control chart (using Markov chains) we may follow the same procedure discussed at the end of Section 4 1 12

4 2 8 The performance of the EWMA control chart – known parameters

As mentioned previously and as like the CUSUM control chart the EWMA control chart is very effective against small process shifts if compared to the Shewhart $\overline{X}$ control chart The design parameters of the EWMA control chart are the width of the control limits that is the distance (L) of the upper and the lower control limit from the centerline and the value of the weighting constant λ The optimal (statistical) design procedure would consists of specifying the in control average run length (ARL_0) as well as the out of control average run length (ARL_1) together with the magnitude of the anticipated process shift and then to select the combination(s) of L and λ that would provide the desired average run length performance

Table 4 18 provides different combinations of L and λ that yields an in control average run length of approximately 500 along with the out of control run length performance of these choices of the design parameters

Size of the shift in the process mean (multiple of σ)	$L = 3\ 054$ $\lambda = 0\ 40$	2 998 0 25	2 962 0 20	2 814 0 10	2 615 0 05
0 00	500	500	500	500	500
0 25	224 0	170 0	150 0	106 0	84 1
0 50	71 2	48 2	41 8	31 3	28 8
0 75	28 4	20 1	18 2	15 9	16 4
1 00	14 3	11 1	10 5	10 3	11 4
1 50	5 9	5 5	5 5	6 1	7 1
2 00	3 5	3 6	3 7	4 4	5 2
2 50	2 5	2 7	2 9	3 4	4 2
3 00	2 0	2 3	2 4	2 9	3 5
4 00	1 4	1 7	1 9	2 2	2 7

Table 4 18 The Average Run Lengths (ARL) of different EWMA control chart schemes

For example if we are interested in detecting a shift in the process mean of 0 50 standard deviation units (say) using $L = 2\ 165$ and $\lambda = 0\ 05$ would provide us with $ARL_0 \approx 500$ and $ARL_1 = 28\ 8$ Consequently if the process is in control we would expect to observe a false alarm approximately every 500 samples whereas if the process mean shifts out of control by 0 50 standard deviation units we would expect to detect such a shift within 28 8 samples In addition for this particular choice of L and λ the EWMA control chart is approximately equivalent to a CUSUM control chart with design parameters $h = 4$ and $k = 0\ 5$ see for instance Table 4 10

Table 4 18 was taken from Montgomery (2001)

4 2 9 Equivalence of integral equation approach and the Markov chain approach

The integral equation approach and the Markov chain approach are two methods to study the run length distribution of a CUSUM control chart and that of an EWMA control chart However the Markov chain approach began by approximating the problem and then obtained exact solutions for the problem whereas the integral equation approach started with the exact problem found exact solutions that we could then approximate

In this section we show that the Markov chain approach is a special case of the integral equation approach In addition since there exist numerous techniques to approximate the solution of an integral equation (of which some techniques are more accurate than the particular technique used to show the equivalence of the integral equation approach and the Markov chain approach) this result establishes that the integral equation approach is the preferred method

To shorten the notation let $P\left(N = t \mid u \ \delta \ \gamma \ Z_0 = z_0 \ W_0 = w_0\right)$ of equations (4 17a) (4 17b) (4 50a) and (4 50b) be denoted by $P\left(N = t \mid u\right)$

For a standardized one sided CUSUM control chart it follows from equations (4 17a) and (4 17b) that

$$
\begin{aligned}
P\left(N = 1 \mid u\right) &= 1 - \Phi\left(\frac{w_0}{\gamma}\left(h + k - u\right) - \frac{\delta}{\gamma} + \frac{z_0}{\gamma\sqrt{m}}\right) \\
&= 1 - \int_{-\infty}^{\frac{w}{2}} f_Y\left(s + k - u\right) ds - \sum_{j=1}^{m-1} \int_{(j-\frac{1}{2})w}^{(j+\frac{1}{2})w} f_Y\left(s + k - u\right) ds
\end{aligned}
\qquad (4\ 52a)
$$

and for $t = 2\ 3$ it follows that

$$
\begin{aligned}
&P(N=t\mid u)\\
&=P(N=t-1\mid 0)\Phi\left(\frac{w_0}{\gamma}(k-u)-\frac{\delta}{\gamma}+\frac{z_0}{\gamma\sqrt{m}}\right)+\frac{w_0}{\gamma}\int_0^h P(N=t-1\mid s)\phi\left(\frac{w_0}{\gamma}(s+k-u)-\frac{\delta}{\gamma}+\frac{z_0}{\gamma\sqrt{m}}\right)ds\\
&=P(N=t-1\mid 0)\int_{-\infty}^{0} f_Y\,(s+k-u)ds+\\
&P(N=t-1\mid \xi_0)\int_0^{\frac{w}{2}} f_Y\,(s+k-u)\,ds+P(N=t-1\mid \xi_j)\sum_{j=1}^{m-1}\int_{(j-\frac{1}{2})w}^{(j+\frac{1}{2})w} f_Y\,(s+k-u)ds
\end{aligned}
\qquad (4\ 52b)
$$

for some $\xi_0 \in \left(-\infty\ \frac{w}{2}\right)$ and $\xi_j \in \left((j-\frac{1}{2})w\ (j+\frac{1}{2})w\right)$ with $j=1\ 2\ 3\quad m-1$

(See Result 4 9a And 4 9b In Appendix 4)

If $w=\frac{2h}{2m-1}$ is small which will be the case if m is large then ξ_j $j=1\ 2\quad m-1$ is approximately the midpoint of the interval $\left((j-\frac{1}{2})w\ (j+\frac{1}{2})w\right)$ i e $\xi_j \approx jw$ Replacing ξ_j with jw $j=0\ 1\ 2\quad m-1$ in equations (4 52a) and (4 52b) and only considering those values of $P(N=t\mid u)$ for which $u=iw$ (which was the only possible starting values for the Markov chain approach) we obtain

$$
P(N=1\mid iw)=1-\sum_{j=0}^{m-1} p_j \qquad (4\ 53a)
$$

and for $t=2\ 3$

$$
P(N=t\mid iw)=\sum_{j=0}^{m-1} P(N=t-1\mid jw)\,p_j \qquad (4\ 53b)
$$

From equations (4 53a) and (4 53b) we note that $P(N=t\mid iw)$ is the i^{th} component of **f** defined in equation (4 37) and we see that equations (4 53a) and (4 53b) are the recursive equations for evaluating the run length probabilities used in the Markov chain approach as given in equations (4 38a) and (4 38b) respectively

Thus by replacing ξ_0 with 0 and ξ_j with jw for $j = 1\ 2 \quad m-1$ is equivalent to using the product midpoint rule of numerical integration However in practice more sophisticated quadrature methods can be used to solve the integral equations of (4 17a) and (4 17b) Therefore the integral equation approach is preferred to the Markov chain approach

Similarly for the EWMA control chart it follows from equations (4 50a) and (4 50b) that

$$
\begin{aligned}
P(N=1\mid u) &= 1-\frac{1}{\lambda}\int_{-h}^{h} f_Y\left(\frac{s-(1-\lambda)u}{\lambda}\right)ds \\
&= 1-\frac{1}{\lambda}\sum_{j=-(m-1)}^{(m-1)}\int_{(j-\frac{1}{2})w}^{(j+\frac{1}{2})w} f_Y\left(\frac{s-(1-\lambda)u}{\lambda}\right)ds
\end{aligned}
\tag{4 54a}
$$

and for $t = 2\ 3 \quad$ it follows that

$$
\begin{aligned}
P(N=t\mid u) &= \frac{1}{\lambda}\int_{-h}^{h} P(N=t-1\mid s)\, f_Y\left(\frac{s-(1-\lambda)u}{\lambda}\right)ds \\
&= \frac{1}{\lambda}P(N=t-1\mid \xi_j)\sum_{j=-(m-1)}^{(m-1)}\int_{(j-\frac{1}{2})w}^{(j+\frac{1}{2})w} f_Y\left(\frac{s-(1-\lambda)u}{\lambda}\right)ds
\end{aligned}
\tag{4 54b}
$$

for some $\xi_j \in \left((j-\frac{1}{2})w\ (j+\frac{1}{2})w\right)$ with $j = -(m-1) \quad -1\ 0\ 1 \quad m-1$

(See Results 4 10a and 4 10b in Appendix 4)

Thus using the product midpoint rule to approximate the integral equations in (4 50a) and (4 50b) lead to similar solutions as were found in the Markov chain approach in equations (4 38a) and (4 38b)

Although it seems as if both the Markov chain approach as well as the Integral equation approach can be used for the evaluation of the run length distribution there are situations where only the Markov chain approach seems to be appropriate For instance if the mean of the process drifts away from the in control value as opposed to a sustained shift the run length distribution can be determined using Markov chains It is not clear how to use the integral equations to determine the run length distribution in such a situation

Furthermore the robustness of the average run length as the overall measure of control chart performance has been questioned in the past This has led to the use of other run length distribution properties such as the standard deviation percentiles the skewness and the kurtosis to measure the performance of a control chart procedure The integral equation approach is not easily extended to cover all these aspects whereas the Markov chain approach provides a relatively easy but also practical method for investigation the run length distribution

4 3 1 Appendix 4

Result 4 1

$$S_t = \max\left(0\ S_{t\ 1} - X_t + (\mu_0 - K)\right) = \min\left(0\ S_{t\ 1} + X_t - (\mu_0 - K)\right)$$

Proof

$$\begin{aligned}
S_t &= \max\left(0\ S_{t\ 1} - X_t + (\mu_0 - K)\right) \\
&= -\min\left(-0\ -S_{t\ 1} + X_t - (\mu_0 - K)\right) \quad ^{(1)} \\
&= -\min\left(0\ -S_{t\ 1} + X_t - (\mu_0 - K)\right) \\
-S_t &= \min\left(0\ -S_{t\ 1} + X_t - (\mu_0 - K)\right) \\
C_t &= \min\left(0\ C_{t\ 1} + X_t - (\mu_0 - K)\right) \quad ^{(2)}
\end{aligned}$$

(1) Since $\max(a\ b) = -\min(-a\ -b)$

(2) Substituting $-S_t$ with C_t

(3) *Note* The starting value $S_0 = u$ also needs to change sign with the substitution of $-S_t^-$ with C_t^- in step (2) i e $C_0 = -u$

Result 4 2a

$$Y_t = \frac{\overline{X}_t - \hat{\mu}_0}{\frac{\hat{\sigma}_0}{\sqrt{n}}} = \frac{1}{W_0}\left(\gamma Z_t + \delta - \frac{Z_0}{\sqrt{m}}\right)$$

Proof

$$Y_t = \frac{\overline{X}_t - \hat{\mu}_0}{\frac{\hat{\sigma}_0}{\sqrt{n}}}$$

$$= \frac{\sigma_0}{\hat{\sigma}_0}\left(\frac{\overline{X}_t - \mu + \mu - \mu_0 + \mu_0 - \hat{\mu}_0}{\frac{\sigma_0}{\sqrt{n}}}\right)$$

$$= \frac{\sigma_0}{\hat{\sigma}_0}\left(\frac{\sigma}{\sigma_0}\left(\frac{\overline{X}_t - \mu}{\frac{\sigma}{\sqrt{n}}}\right) + \left(\frac{\mu - \mu_0}{\frac{\sigma_0}{\sqrt{n}}}\right) - \left(\frac{\hat{\mu}_0 - \mu_0}{\frac{\sigma_0}{\sqrt{mn}}}\right)\frac{1}{\sqrt{m}}\right)$$

$$= \frac{1}{W_0}\left(\gamma Z_t + \delta - \frac{Z_0}{\sqrt{m}}\right)$$

Result 4 2b

$$F_{Y|Z\ W}\left(y_t \mid z_0\ w_0\right) = \Phi\left(\frac{w_0}{\gamma} y_t - \frac{\delta}{\gamma} + \frac{z_0}{\gamma\sqrt{m}}\right)$$

Proof

$$F_{Y|Z\ W}\left(y_t \mid z_0\ w_0\right)$$

$$= P\left(Y_t \le y_t \mid Z_0 = z_0\ W_0 = w_0\right)$$

$$= P\left(\frac{1}{w_0}\left(\gamma Z_t + \delta - \frac{z_0}{\sqrt{m}}\right) \le y_t \mid Z_0 = z_0\ W_0 = w_0\right) \quad (1)$$

$$= P\left(Z_t \le \frac{w_0}{\gamma} y_t - \frac{\delta}{\gamma} + \frac{z_0}{\gamma\sqrt{m}} \mid Z_0 = z_0\ W_0 = w_0\right)$$

$$= \Phi\left(\frac{w_0}{\gamma} y_t - \frac{\delta}{\gamma} + \frac{z_0}{\gamma\sqrt{m}}\right)$$

(1) By Result 4 2a

Result 4 2c

$$f_{Y|Z\ W}\left(y_t \mid z_0\ \ w_0\right)=\frac{w_0}{\gamma}\phi\left(\frac{w_0}{\gamma}y_t-\frac{\delta}{\gamma}+\frac{z_0}{\gamma\sqrt{m}}\right)$$

Proof

$$f_{Y|Z\ W}\left(y_t \mid z_0\ \ w_0\right)$$

$$=\frac{\partial}{\partial y_t}F_{Y|Z\ W}\left(y_t \mid z_0\ \ w_0\right)$$

$$=\frac{\partial}{\partial y_t}\Phi\left(\frac{w_0}{\gamma}y_t-\frac{\delta}{\gamma}+\frac{z_0}{\gamma\sqrt{m}}\right) \quad (1)$$

$$=\frac{w_0}{\gamma}\phi\left(\frac{w_0}{\gamma}y_t-\frac{\delta}{\gamma}+\frac{z_0}{\gamma\sqrt{m}}\right) \quad (2)$$

(1) By Result 4 2b

(2) By the chain rule of differentiation i e $\frac{\partial}{\partial t}f\left(g(t)\right)=f\ \left(g(t)\right)g\ (t)$

Exact integral equations for the run length distribution of a standardized one sided upper CUSUM control chart when using an integral equation approach

Result 4 3a

$$P\left(N=1 \mid u\ \delta\ \gamma\ Z_0=z_0\ W_0=w_0\right)=1-\Phi\left(\frac{w_0}{\gamma}(h+k-u)-\frac{\delta}{\gamma}+\frac{z_0}{\gamma\sqrt{m}}\right)$$

Proof

$$P\left(N=1 \mid u\ \delta\ \gamma\ Z_0=z_0\ W_0=w_0\right)$$

$$=P\left(S_1^+ \geq h \mid S_0=u\ \delta\ \gamma\ Z_0=z_0\ W_0=w_0\right) \ ^{(1)}$$

$$=P\left(\max\left(0\ S_0^+ + Y_1-k\right) \geq h \mid S_0^+=u\ \delta\ \gamma\ Z_0=z_0\ W_0=w_0\right) \ ^{(2)}$$

$$=P\left(S_0^+ + Y_1-k \geq h \mid S_0^+=u\ \delta\ \gamma\ Z_0=z_0\ W_0=w_0\right) \ ^{(3)}$$

$$=P\left(u+Y_1-k \geq h \mid \delta\ \gamma\ Z_0=z_0\ W_0=w_0\right) \ ^{(4)}$$

$$=P\left(Y_1 \geq h+k-u \mid \delta\ \gamma\ Z_0=z_0\ W_0=w_0\right)$$

$$=1-P\left(Y_1 < h+k-u \mid \delta\ \gamma\ Z_0=z_0\ W_0=w_0\right) \ ^{(5)}$$

$$=1-\Phi\left(\frac{w_0}{\gamma}(h+k-u)-\frac{\delta}{\gamma}+\frac{z_0}{\gamma\sqrt{m}}\right) \ ^{(6)}$$

(1) The run length will only be equal to 1 if a signal is given on the first sample i e $N=1 \Leftrightarrow S_1 \geq h$

(2) Using the general definition of S_t given in equation (4 4a)

(3) Since 0 cannot be larger than or equal h it must be $S_0^+ + Y_1 - k$

(4) By substituting the starting value $S_0 = u$

(5) Since $P(X \geq a) = 1 - P(X < a)$

(6) By Result 4 2b

Result 4 3b

$$P\left(N=t \mid u\ \delta\ \gamma\ Z_0=z_0\ W_0=w_0\right)$$
$$=P\left(N=t-1 \mid S_1=0\ \delta\ \gamma\ Z_0=z_0\ W_0=w_0\right)\Phi\left(\frac{w_0}{\gamma}(k-u)-\frac{\delta}{\gamma}+\frac{z_0}{\gamma\sqrt{m}}\right)+$$
$$\frac{w_0}{\gamma}\int_0^h P\left(N=t-1 \mid s\ \delta\ \gamma\ Z_0=z_0\ W_0=w_0\right)\phi\left(\frac{w_0}{\gamma}(s+k-u)-\frac{\delta}{\gamma}+\frac{z_0}{\gamma\sqrt{m}}\right)ds$$

for $t=2\ 3$

Proof

$$P\left(N=t \mid u\ \delta\ \gamma\ Z_0=z_0\ W_0=w_0\right) \quad \text{for } t=2\ 3$$

$$=P\left(N=t \mid S_0^+=u\ \delta\ \gamma\ Z_0=z_0\ W_0=w_0\right)$$

$$=P\left(N=t-1\ S_1=0 \mid S_0=u\ \delta\ \gamma\ Z_0=z_0\ W_0=w_0\right)+$$
$$P\left(N=t-1\ 0<S_1^+<h \mid S_0=u\ \delta\ \gamma\ Z_0=z_0\ W_0=w_0\right)+ \quad (1)$$
$$P\left(N=t-1\ S_1^+\geq h \mid S_0=u\ \delta\ \gamma\ Z_0=z_0\ W_0=w_0\right)$$

$$=P\left(N=t-1\ S_1=0 \mid S_0=u\ \delta\ \gamma\ Z_0=z_0\ W_0=w_0\right)+ \quad (2)$$
$$P\left(N=t-1\ 0<S_1<h \mid S_0=u\ \delta\ \gamma\ Z_0=z_0\ W_0=w_0\right)$$

$$=P\left(N=t-1 \mid S_1=0\ S_0^+=u\ \delta\ \gamma\ Z_0=z_0\ W_0=w_0\right)P\left(S_1=0 \mid S_0^+=u\ \delta\ \gamma\ Z_0=z_0\ W_0=w_0\right)+ \quad (3)$$
$$P\left(N=t-1 \mid 0<S_1<h\ S_0=u\ \delta\ \gamma\ Z_0=z_0\ W_0=w_0\right)P\left(0<S_1^+<h \mid S_0=u\ \delta\ \gamma\ Z_0=z_0\ W_0=w_0\right)$$

$$=P\left(N=t-1 \mid S_1=0\ \delta\ \gamma\ Z_0=z_0\ W_0=w_0\right)\Phi\left(\frac{w_0}{\gamma}(k-u)-\frac{\delta}{\gamma}+\frac{z_0}{\gamma\sqrt{m}}\right)+ \quad (4)$$
$$\frac{w_0}{\gamma}\int_0^h P\left(N=t-1 \mid s\ \delta\ \gamma\ Z_0=z_0\ W_0=w_0\right)\phi\left(\frac{w_0}{\gamma}(s+k-u)-\frac{\delta}{\gamma}+\frac{z_0}{\gamma\sqrt{m}}\right)ds$$

(1) At time $t = 1$ three mutually exclusive events can occur the CUSUM plotting statistic can be zero can plot in control or give an out of control signal i e

$$\{S_1\} = \{S_1 = 0\} \cup \{0 < S_1^+ < h\} \cup \{S_1 \geq h\}$$

(2) If $S_1 \geq h$ the probability that the run length is equal to $t = 2\ 3$ is zero since a signal was already given at time $t = 1$ and the run length is 1

(3) The joint probability of any two events A and B can be re written as

$$P(A \cap B) = P(A \mid B)P(B)$$

(4) By Result 4 3c and Result 4 3d

Result 4 3c

$$P\left(S_1 = 0 \mid S_0 = u\ \delta\ \gamma\ Z_0 = z_0\ W_0 = w_0\right) = \Phi\left(\frac{w_0}{\gamma}(k-u) - \frac{\delta}{\gamma} + \frac{z_0}{\gamma\sqrt{m}}\right)$$

Proof

$$P\left(S_1 = 0 \mid S_0 = u\ \delta\ \gamma\ Z_0 = z_0\ W_0 = w_0\right)$$

$$= P\left(\max\left(0\ S_0^+ + Y_1 - k\right) = 0 \mid S_0^+ = u\ \delta\ \gamma\ Z_0 = z_0\ W_0 = w_0\right)$$

$$= P\left(S_0 + Y_1 - k \leq 0 \mid S_0^+ = u\ \delta\ \gamma\ Z_0 = z_0\ W_0 = w_0\right)\ ^{(1)}$$

$$= P\left(u + Y_1 - k \leq 0 \mid \delta\ \gamma\ Z_0 = z_0\ W_0 = w_0\right)\ ^{(2)}$$

$$= P\left(Y_1 \leq k - u \mid \delta\ \gamma\ Z_0 = z_0\ W_0 = w_0\right)$$

$$= \Phi\left(\frac{w_0}{\gamma}(k-u) - \frac{\delta}{\gamma} + \frac{z_0}{\gamma\sqrt{m}}\right)\ ^{(3)}$$

(1) If the maximum of two values is zero of which the first value is already zero it implies that the second value can only be less than or equal to zero

(2) By substituting the starting value $S_0 = u$

(3) By Result 4 2b

Result 4 3d

$$P\left(0<S_1<h \mid S_0=u\ \delta\ \gamma\ Z_0=z_0\ W_0=w_0\right)=\int_0^h \frac{w_0}{\gamma}\phi\left(\frac{w_0}{\gamma}(s+k-u)-\frac{\delta}{\gamma}+\frac{z_0}{\gamma\sqrt{m}}\right)ds$$

Proof

$$P\left(0<S_1<h \mid S_0=u\ \delta\ \gamma\ Z_0=z_0\ W_0=w_0\right)$$

$$=P\left(0<\max\left(0\ S_0+Y_1-k\right)<h \mid S_0=u\ \delta\ \gamma\ Z_0=z_0\ W_0=w_0\right)$$

$$=P\left(0<S_0+Y_1-k<h \mid S_0=u\ \delta\ \gamma\ Z_0=z_0\ W_0=w_0\right)\ ^{(1)}$$

$$=P\left(0<u+Y_1-k<h \mid \delta\ \gamma\ Z_0=z_0\ W_0=w_0\right)$$

$$=P\left(0+k-u<Y_1<h+k-u \mid \delta\ \gamma\ Z_0=z_0\ W_0=w_0\right)$$

$$=\int_k^{h+k} f_{Y\mid Z\ W}\left(y_t \mid z_0\ w_0\right)dt\ ^{(2)}$$

$$=\int_k^{h\ k}\frac{w_0}{\gamma}\phi\left(\frac{w_0}{\gamma}t-\frac{\delta}{\gamma}+\frac{z_0}{\gamma\sqrt{m}}\right)dt\ ^{(3)}$$

$$=\int_0^h \frac{w_0}{\gamma}\phi\left(\frac{w_0}{\gamma}(s+k-u)-\frac{\delta}{\gamma}+\frac{z_0}{\gamma\sqrt{m}}\right)ds\ ^{(4)}$$

(1) Since $0\notin(0\ h)$

(2) Since $P(a<X<b)=\int_a^b f_X(x)dx$ with $a<b$

(3) By Result 4 2c

(4) Let $s=t-k+u$ so that $t=s+k-u$ This implies that $\frac{dt}{ds}=1$ so that $dt=ds$ and therefore the lower and upper boundaries are $s_p=h$ and $s_{l\ w}=0$

Transition probabilities for deriving the run length distribution of a standardized one sided upper CUSUM control chart when using the Markov chain approach

Result 4 4a

$$p_j = \int_{(j\ \frac{1}{2})w}^{(j\ \frac{1}{2})w} \frac{w_0}{\gamma}\phi\left(\frac{w_0}{\gamma}\left(s - iw + k\right) - \frac{\delta}{\gamma} + \frac{z_0}{\gamma\sqrt{m}}\right)ds$$

Proof

$$p_j = P\left(S_t\ = j \mid S_{t\ 1}^{+} = i\ \delta\ \gamma\ Z_0 = z_0\ W_0 = w_0\right)$$

$$= P\left(S_t\ \in\left[(j - \frac{1}{2})w\ (j + \frac{1}{2})w\right) \mid S_{t\ 1} = iw\ \delta\ \gamma\ Z_0 = z_0\ W_0 = w_0\right) \quad (1)$$

$$= P\left((j - \frac{1}{2})w \le S_t^{+} < (j + \frac{1}{2})w \mid S_{t\ 1} = iw\ \delta\ \gamma\ Z_0 = z_0\ W_0 = w_0\right)$$

$$= P\left((j - \frac{1}{2})w \le \max\left(0\ S_{t\ 1} + Y_t - k\right) < (j + \frac{1}{2})w \mid S_{t\ 1} = iw\ \delta\ \gamma\ Z_0 = z_0\ W_0 = w_0\right) \quad (2)$$

$$= P\left((j - \frac{1}{2})w \le S_{t\ 1} + Y_t - k < (j + \frac{1}{2})w \mid S_{t\ 1} = iw\ \delta\ \gamma\ Z_0 = z_0\ W_0 = w_0\right) \quad (3)$$

$$= P\left((j - \frac{1}{2})w \le iw + Y_t - k < (j + \frac{1}{2})w \mid \delta\ \gamma\ Z_0 = z_0\ W_0 = w_0\right)$$

$$= P\left((j - \frac{1}{2})w - iw + k \le Y_t < (j + \frac{1}{2})w - iw + k \mid \delta\ \gamma\ Z_0 = z_0\ W_0 = w_0\right)$$

$$= \int_{(j\ \frac{1}{2})w\ \ w\ k}^{(j\ \frac{1}{2})w\ \ w\ k} f_Y\left(t\right)dt \quad (4)$$

$$= \int_{(j\ \frac{1}{2})w\ \ w\ k}^{(j\ \frac{1}{2})w\ \ w\ k} \frac{w_0}{\gamma}\phi\left(\frac{w_0}{\gamma}t - \frac{\delta}{\gamma} + \frac{z_0}{\gamma\sqrt{m}}\right)dt \quad (5)$$

$$= \int_{(j\ \frac{1}{2})w}^{(j\ \frac{1}{2})w} \frac{w_0}{\gamma}\phi\left(\frac{w_0}{\gamma}\left(s - iw + k\right) - \frac{\delta}{\gamma} + \frac{z_0}{\gamma\sqrt{m}}\right)ds \quad (6)$$

(1) Since state j is actually the interval $\left[(j-\frac{1}{2})w, (j+\frac{1}{2})w\right)$ and iw is the reference value of state i

(2) Using the definition of S_t given in equitation (4 4a)

(3) Since $0 \notin (0, h)$

(4) Since $P(a < X < b) = \int_a^b f_X(x)dx$ with $a < b$

(5) By Result 4 2c

(6) Let $s = t + iw - k$ so that $t = s - iw + k$ This implies $\frac{dt}{ds} = 1$ so that $dt = ds$ and therefore the lower and upper boundaries are $s_p = (j+\frac{1}{2})w$ and $s_{l\,w} = (j-\frac{1}{2})w$

Result 4 4b

$$p_0 = \int_{-\infty}^{\frac{1}{2}w} \frac{w_0}{\gamma} \phi\left(\frac{w_0}{\gamma}\left(s - iw + k\right) - \frac{\delta}{\gamma} + \frac{z_0}{\gamma\sqrt{m}} \right) ds$$

Proof

$$p_0 = P\left(S_t = 0 \mid S_{t\ 1} = i\ \delta\ \gamma\ Z_0 = z_0\ W_0 = w_0\right)$$

$$= P\left(S_t \in \left(-\infty\ \frac{1}{2}w\right) \mid S^{+}_{t\ 1} = iw\ \delta\ \gamma\ Z_0 = z_0\ W_0 = w_0 \right)$$

$$= P\left(-\infty < S_t < \frac{1}{2}w \mid S_{t\ 1} = iw\ \delta\ \gamma\ Z_0 = z_0\ W_0 = w_0 \right)$$

$$= P\left(-\infty < \max\left(0\ S_{t-1} + Y_t - k\right) < \frac{1}{2}w \mid S_{t-1} = iw\ \delta\ \gamma\ Z_0 = z_0\ W_0 = w_0 \right)$$

$$= P\left(-\infty < S^{+}_{t\ 1} + Y_t - k < \frac{1}{2}w \mid S_{t\ 1} = iw\ \delta\ \gamma\ Z_0 = z_0\ W_0 = w_0 \right)$$

$$= P\left(-\infty < iw + Y_t - k < \frac{1}{2}w \mid \delta\ \gamma\ Z_0 = z_0\ W_0 = w_0 \right)$$

$$= P\left(-\infty < Y_t < \frac{1}{2}w - iw + k \mid \delta\ \gamma\ Z_0 = z_0\ W_0 = w_0 \right)$$

$$= \int_{-\infty}^{\frac{1}{2}w\ \ w\ k} f_Y(t)\,dt$$

$$= \int_{-\infty}^{\frac{1}{2}w\ \ w\ k} \frac{w_0}{\gamma} \phi\left(\frac{w_0}{\gamma} t - \frac{\delta}{\gamma} + \frac{z_0}{\gamma\sqrt{m}} \right) dt$$

$$= \int_{-\infty}^{\frac{1}{2}w} \frac{w_0}{\gamma} \phi\left(\frac{w_0}{\gamma}\left(s - iw + k\right) - \frac{\delta}{\gamma} + \frac{z_0}{\gamma\sqrt{m}} \right) ds \quad ^{(1)}$$

(1) Let $s = t + iw - k$ so that $t = s - iw + k$ This implies $\frac{dt}{ds} = 1$ so that $dt = ds$ and therefore the lower and upper boundaries are $s_p = \frac{1}{2}w$ and $s_{l\ w} = -\infty$

Result 4 4c

$$p_m = \int_h^{\infty} \frac{w_0}{\gamma} \phi\left(\frac{w_0}{\gamma}(s - \imath w + k) - \frac{\delta}{\gamma} + \frac{z_0}{\gamma\sqrt{m}} \right) ds$$

Proof

$$p_m = P\left(S_t = h \mid S_{t\ 1} = \imath\ \delta\ \gamma\ Z_0 = z_0\ W_0 = w_0\right)$$

$$= P\left(S_t \in [h\ \infty) \mid S_{t\ 1} = \imath w\ \delta\ \gamma\ Z_0 = z_0\ W_0 = w_0\right)$$

$$= P\left(h \le S_t < \infty \mid S_{t\ 1} = \imath w\ \delta\ \gamma\ Z_0 = z_0\ W_0 = w_0\right)$$

$$= P\left(h \le \max\left(0\ S_{t\ 1} + Y_t - k\right) < \infty \mid S^{+}_{t\ 1} = \imath w\ \delta\ \gamma\ Z_0 = z_0\ W_0 = w_0\right)$$

$$= P\left(h \le S_{t\ 1} + Y_t - k < \infty \mid S_{t\ 1} = \imath w\ \delta\ \gamma\ Z_0 = z_0\ W_0 = w_0\right)$$

$$= P\left(h \le \imath w + Y_t - k < \infty \mid \delta\ \gamma\ Z_0 = z_0\ W_0 = w_0\right)$$

$$= P\left(h - \imath w + k \le Y_t < \infty \mid \delta\ \gamma\ Z_0 = z_0\ W_0 = w_0\right)$$

$$= \int_{h\ \ w\ k}^{\infty} f_Y(t)\, dt$$

$$= \int_{h\ \ w\ k}^{\infty} \frac{w_0}{\gamma} \phi\left(\frac{w_0}{\gamma} t - \frac{\delta}{\gamma} + \frac{z_0}{\gamma\sqrt{m}} \right) dt$$

$$= \int_h^{\infty} \frac{w_0}{\gamma} \phi\left(\frac{w_0}{\gamma}(s - \imath w + k) - \frac{\delta}{\gamma} + \frac{z_0}{\gamma\sqrt{m}} \right) ds \quad (1)$$

$$= \frac{w_0}{\gamma}\left\{ 1 - \int_{\infty}^{h} \phi\left(\frac{w_0}{\gamma}(s - \imath w + k) - \frac{\delta}{\gamma} + \frac{z_0}{\gamma\sqrt{m}} \right) ds \right\}$$

(1) Let $s = t + \imath w - k$ so that $t = s - \imath w + k$ This implies $\frac{dt}{ds} = 1$ so that $dt = ds$ and therefore the lower and upper boundaries are $s_p = \infty$ and $s_{l\ w} = h$

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Result 4 4d

$$p_{mj} = 0$$

Proof

p_{mj} is the probability of going from the absorbent state (m) to any one of the communicating states $j = 0\ 1\quad m-1$ but since the process cannot leave the absorbent state once it has been entered this probability is equal to zero

Result 4 4e

$$p_{mm} = 1$$

Proof

p_{mm} is the probability of staying in state m once it has been entered Since state m does not communicate with any of the other states i e it is an absorbent state the process cannot leave state m once it has been entered Therefore p_{mm} is equal to one

Exact integral equations for the run length distribution of a standardized one sided lower CUSUM control chart when using an integral equation approach

Result 4 5a

$$P\left(N=1\,|\,-u\;\delta\;\gamma\;Z_0=z_0\;W_0=w_0\right)=1-\Phi\left(\frac{w_0}{\gamma}(h+k-u)-\frac{\delta}{\gamma}+\frac{z_0}{\gamma\sqrt{m}}\right)$$

Proof

$$P\left(N=1\,|\,-u\;\delta\;\gamma\;Z_0=z_0\;W_0=w_0\right)$$

$$=P\left(S_1^-\leq -h\,|\,S_0^-=-u\;\delta\;\gamma\;Z_0=z_0\;W_0=w_0\right)\ ^{(1)}$$

$$=P\left(\min\left(0\;S_0^-+Y_1+k\right)\leq -h\,|\,S_0^-=-u\;\delta\;\gamma\;Z_0=z_0\;W_0=w_0\right)\ ^{(2)}$$

$$=P\left(S_0^-+Y_1+k\leq -h\,|\,S_0^-=-u\;\delta\;\gamma\;Z_0=z_0\;W_0=w_0\right)\ ^{(3)}$$

$$=P\left(-u+Y_1+k\leq -h\,|\,\delta\;\gamma\;Z_0=z_0\;W_0=w_0\right)\ ^{(4)}$$

$$=P\left(Y_1\leq -h-k+u\,|\,\delta\;\gamma\;Z_0=z_0\;W_0=w_0\right)$$

$$=\Phi\left(\frac{w_0}{\gamma}(-h-k+u)-\frac{\delta}{\gamma}+\frac{z_0}{\gamma\sqrt{m}}\right)\ ^{(5)}$$

$$=1-\Phi\left(\frac{w_0}{\gamma}(h+k-u)-\frac{\delta}{\gamma}+\frac{z_0}{\gamma\sqrt{m}}\right)\ ^{(6)}$$

(1) Since the run length will only be equal to 1 if a signal is given on the first sample

(2) Using the general definition of S_t^- given in equation (4 8b) together with Result 4 1

(3) Since if the minimum of two values is less than $-h$ of which the first value is already equal to zero it must be the second value that is less than $-h$

(4) By substituting the starting value $S_0^- = -u$

(5) By Result 4 2b

(6) Since $\Phi(x)=1-\Phi(-x)$ which follows from the symmetry of the normal distribution

Result 4 5b

$$P(N=t \mid -u\ \delta\ \gamma\ Z_0=z_0\ W_0=w_0)$$
$$=P(N=t-1 \mid S_1\ =0\ \delta\ \gamma\ Z_0=z_0\ W_0=w_0)\Phi\left(\frac{w_0}{\gamma}(k-u)-\frac{\delta}{\gamma}+\frac{z_0}{\gamma\sqrt{m}}\right)+$$
$$\frac{w_0}{\gamma}\int_0^h P(N=t-1 \mid s\ \delta\ \gamma\ Z_0=z_0\ W_0=w_0)\phi\left(\frac{w_0}{\gamma}(s+k-u)-\frac{\delta}{\gamma}+\frac{z_0}{\gamma\sqrt{m}}\right)ds$$
for $t=2\ 3$

Proof

$$P(N=t \mid -u\ \delta\ \gamma\ Z_0=z_0\ W_0=w_0) \quad \text{for } t=2\ 3$$
$$=P(N=t \mid S_0\ =-u\ \delta\ \gamma\ Z_0=z_0\ W_0=w_0)$$
$$=P(N=t-1\ S_1^-\le -h \mid S_0\ =-u\ \delta\ \gamma\ Z_0=z_0\ W_0=w_0)+$$
$$P(N=t-1\ -h<S_1\ <0 \mid S_0\ =-u\ \delta\ \gamma\ Z_0=z_0\ W_0=w_0)+ \quad (1)$$
$$P(N=t-1\ S_1\ =0 \mid S_0\ =-u\ \delta\ \gamma\ Z_0=z_0\ W_0=w_0)$$
$$=P(N=t-1\ -h<S_1\ <0 \mid S_0\ =-u\ \delta\ \gamma\ Z_0=z_0\ W_0=w_0)+ \quad (2)$$
$$P(N=t-1\ S_1\ =0 \mid S_0^-\ =-u\ \delta\ \gamma\ Z_0=z_0\ W_0=w_0)$$
$$=P(N=t-1 \mid -h<S_1\ <0\ S_0\ =-u\ \delta\ \gamma\ Z_0=z_0\ W_0=w_0)P(-h<S_1^-<0 \mid S_0\ =-u\ \delta\ \gamma\ Z_0=z_0\ W_0=w_0)+$$
$$P(N=t-1 \mid S_1^-=0\ S_0^-\ =-u\ \delta\ \gamma\ Z_0=z_0\ W_0=w_0)P(S_1\ =0 \mid S_0^-\ =-u\ \delta\ \gamma\ Z_0=z_0\ W_0=w_0) \quad (3)$$
$$=P(N=t-1 \mid S_1\ =0\ \delta\ \gamma\ Z_0=z_0\ W_0=w_0)\Phi\left(\frac{w_0}{\gamma}(k-u)-\frac{\delta}{\gamma}+\frac{z_0}{\gamma\sqrt{m}}\right)+ \quad (4)$$
$$\frac{w_0}{\gamma}\int_0^h P(N=t-1 \mid s\ \delta\ \gamma\ Z_0=z_0\ W_0=w_0)\phi\left(\frac{w_0}{\gamma}(s+k-u)-\frac{\delta}{\gamma}+\frac{z_0}{\gamma\sqrt{m}}\right)ds$$

(1) At time $t=1$ three mutually exclusive events can occur the CUSUM plotting statistic can give a signal plot in control or be equal to zero i e $\{S_1\ \}=\{S_1\ \le -h\}\cup\{-h<S_1^-<0\}\cup\{S_1\ =0\}$

(2) If $S_1\ \le -h$ the control chart already signalled on the first sample and the probability that the run length can be equal to t where $t=2\ 3$ is zero

(3) The joint probability of any two events A and B can be re written as $P(A\cap B)=P(A \mid B)P(B)$

(4) By Results 4 5c and 4 5d

Result 4 5c

$$P\left(S_1 = 0 \mid S_0 = -u \;\delta\; \gamma\; Z_0 = z_0 \; W_0 = w_0\right) = \Phi\left(\frac{w_0}{\gamma}(k-u) - \frac{\delta}{\gamma} + \frac{z_0}{\gamma\sqrt{m}}\right)$$

Proof

$$P\left(S_1 = 0 \mid S_0 = -u \;\delta\; \gamma\; Z_0 = z_0 \; W_0 = w_0\right)$$

$$= P\left(\min\left(0 \; S_0 + Y_1 + k\right) = 0 \mid S_0 = -u \;\delta\; \gamma\; Z_0 = z_0 \; W_0 = w_0\right)$$

$$= P\left(S_0 + Y_1 + k \geq 0 \mid S_0 = -u \;\delta\; \gamma\; Z_0 = z_0 \; W_0 = w_0\right) \;^{(1)}$$

$$= P\left(-u + Y_1 + k \geq 0 \mid \delta\; \gamma\; Z_0 = z_0 \; W_0 = w_0\right)$$

$$= P\left(Y_1 \geq -k + u \mid \delta\; \gamma\; Z_0 = z_0 \; W_0 = w_0\right)$$

$$= 1 - = P\left(Y_1 < -k + u \mid \delta\; \gamma\; Z_0 = z_0 \; W_0 = w_0\right)$$

$$= 1 - \Phi\left(\frac{w_0}{\gamma}(-k+u) - \frac{\delta}{\gamma} + \frac{z_0}{\gamma\sqrt{m}}\right) \;^{(2)}$$

$$= \Phi\left(\frac{w_0}{\gamma}(k-u) - \frac{\delta}{\gamma} + \frac{z_0}{\gamma\sqrt{m}}\right) \;^{(3)}$$

(1) If the minimum of two values is equal to zero of which the first value is already equal to zero the second value must be either larger than zero or equal to zero

(2) By Result 4 2b

(3) Since $1 - \Phi(x) = \Phi(-x)$

Result 4 5d

$$P\left(-h < S_1 < 0 \mid S_0 = -u \ \delta \ \gamma \ Z_0 = z_0 \ W_0 = w_0\right) = \int_0^h \frac{w_0}{\gamma}\phi\left(\frac{w_0}{\gamma}(s+k-u) - \frac{\delta}{\gamma} + \frac{z_0}{\gamma\sqrt{m}}\right)ds$$

Proof

$$P\left(-h < S_1 < 0 \mid S_0 = -u \ \delta \ \gamma \ Z_0 = z_0 \ W_0 = w_0\right)$$

$$= P\left(-h < \min\left(0 \ S_0 + Y_1 + k\right) < 0 \mid S_0 = -u \ \delta \ \gamma \ Z_0 = z_0 \ W_0 = w_0\right)$$

$$= P\left(-h < S_0 + Y_1 + k < 0 \mid S_0 = -u \ \delta \ \gamma \ Z_0 = z_0 \ W_0 = w_0\right)$$

$$= P\left(-h < -u + Y_1 + k < 0 \mid \delta \ \gamma \ Z_0 = z_0 \ W_0 = w_0\right)$$

$$= P\left(-h - k - u < Y_1 < -k + u \mid \delta \ \gamma \ Z_0 = z_0 \ W_0 = w_0\right)$$

$$= \int_{-h-k}^{-k+} f_Y(t)dt$$

$$= \int_{-h-k}^{-k} \frac{w_0}{\gamma}\phi\left(\frac{w_0}{\gamma}t - \frac{\delta}{\gamma} + \frac{z_0}{\gamma\sqrt{m}}\right)dt$$

$$= \int_k^{h+k} \frac{w_0}{\gamma}\phi\left(\frac{w_0}{\gamma}t - \frac{\delta}{\gamma} + \frac{z_0}{\gamma\sqrt{m}}\right)dt \quad (1)$$

$$= \int_0^h \frac{w_0}{\gamma}\phi\left(\frac{w_0}{\gamma}(s+k-u) - \frac{\delta}{\gamma} + \frac{z_0}{\gamma\sqrt{m}}\right)ds \quad (2)$$

(1) Since $\int_a^b \phi(z)dz = -\int_{-b} \phi(z)dz = \int_{-b} \phi(z)dz$ and $\phi(z) = -\phi(z)$ for the probability density function of a normal random variable

(2) Let $s = t - k + u$ so that $t = s + k - u$ This implies $\frac{dt}{ds} = 1$ so that $dt = ds$ and therefore the lower and upper boundaries are $s_p = h$ and $s_{l\,w} = 0$

Transition probabilities for deriving the run length distribution of a standardized one sided lower CUSUM control chart when using the Markov chain approach

Result 4 6a

$$p_0 = \int_{-\infty}^{\frac{1}{2}w} \frac{w_0}{\gamma}\phi\left(\frac{w_0}{\gamma}(s - iw + k) - \frac{\delta}{\gamma} + \frac{z_0}{\gamma\sqrt{m}}\right)ds$$

Proof

$$p_0 = P\left(S_t = 0 \mid S_{t-1} = i \quad Z_0 = z_0 \quad W_0 = w_0\right)$$

$$= P\left(S_t \in \left(-\infty \quad \frac{1}{2}w\right) \mid S_{t-1} = iw \quad Z_0 = z_0 \quad W_0 = w_0\right)$$

$$= P\left(-\infty < S_t^- < \frac{1}{2}w \mid S_{t-1}^- = iw \quad Z_0 = z_0 \quad W_0 = w_0\right)$$

$$= P\left(-\infty < \max\left(0 \quad S_{t-1} - Y_t - k\right) < \frac{1}{2}w \mid S_{t-1} = iw \quad Z_0 = z_0 \quad W_0 = w_0\right)$$

$$= P\left(-\infty < S_{t-1} - Y_t - k < \frac{1}{2}w \mid S_{t-1} = iw \quad Z_0 = z_0 \quad W_0 = w_0\right)$$

$$= P\left(-\infty < iw - Y_t - k < \frac{1}{2}w \mid Z_0 = z_0 \quad W_0 = w_0\right)$$

$$= P\left(-\infty < -Y_t < \frac{1}{2}w - iw + k \mid Z_0 = z_0 \quad W_0 = w_0\right)$$

$$= P\left(-\frac{1}{2}w + iw - k < Y_t < \infty \mid Z_0 = z_0 \quad W_0 = w_0\right)$$

$$= \int_{-\frac{1}{2}w + iw - k}^{\infty} f_Y(t)dt$$

$$= \int_{-\frac{1}{2}w + iw - k}^{\infty} \frac{w_0}{\gamma}\phi\left(\frac{w_0}{\gamma}t - \frac{\delta}{\gamma} + \frac{z_0}{\gamma\sqrt{m}}\right)dt$$

$$= \int_{-\infty}^{\frac{1}{2}w - iw + k} \frac{w_0}{\gamma}\phi\left(\frac{W_0}{\gamma}t - \frac{\delta}{\gamma} + \frac{z_0}{\gamma\sqrt{m}}\right)dt \quad (1)$$

$$= \int_{-\infty}^{\frac{1}{2}w} \frac{w_0}{\gamma}\phi\left(\frac{w_0}{\gamma}(s - iw + k) - \frac{\delta}{\gamma} + \frac{z_0}{\gamma\sqrt{m}}\right)ds \quad (2)$$

$$= \frac{w_0}{\gamma}\Phi\left(\frac{w_0}{\gamma}\left(\frac{1}{2}w - iw + k\right) - \frac{\delta}{\gamma} + \frac{z_0}{\gamma\sqrt{m}}\right)$$

(1) Since $\int^{-b}\phi(z)dz = \int_{b}^{a}\phi(z)dz$ for the density function of normal random variable

(2) Let $s = t + iw - k$ so that $t = s - iw + k$ This implies $\frac{dt}{ds} = 1$ so that $dt = ds$ and therefore the lower and upper boundaries are $s_{p} = \frac{1}{2}w$ and $s_{l\ w} = -\infty$

Result 4 6b

$$p_j = \int_{(j-\frac{1}{2})w}^{(j+\frac{1}{2})w} \frac{w_0}{\gamma}\phi\left(\frac{w_0}{\gamma}\left(s - iw + k\right) - \frac{\delta}{\gamma} + \frac{z_0}{\gamma\sqrt{m}}\right)ds$$

Proof

$$p_j = P\left(S_t^- = j \mid S_{t-1} = i,\ Z_0 = z_0,\ W_0 = w_0\right)$$

$$= P\left(S_t \in \left[(j-\frac{1}{2})w,\ (j+\frac{1}{2})w\right) \mid S_{t-1} = iw,\ Z_0 = z_0,\ W_0 = w_0\right)$$

$$= P\left((j-\frac{1}{2})w \le S_t < (j+\frac{1}{2})w \mid S_{t-1} = iw,\ Z_0 = z_0,\ W_0 = w_0\right)$$

$$= P\left((j-\frac{1}{2})w \le \max\left(0,\ S_{t-1} - Y_t - k\right) < (j+\frac{1}{2})w \mid S_{t-1} = iw,\ Z_0 = z_0,\ W_0 = w_0\right)$$

$$= P\left((j-\frac{1}{2})w \le S_{t-1} - Y_t - k < (j+\frac{1}{2})w \mid S_{t-1} = iw,\ Z_0 = z_0,\ W_0 = w_0\right)$$

$$= P\left((j-\frac{1}{2})w \le iw - Y_t - k < (j+\frac{1}{2})w \mid Z_0 = z_0,\ W_0 = w_0\right)$$

$$= P\left((j-\frac{1}{2})w - iw + k \le -Y_t < (j+\frac{1}{2})w - iw + k \mid Z_0 = z_0,\ W_0 = w_0\right)$$

$$= P\left(-(j+\frac{1}{2})w + iw - k < Y_t \le -(j-\frac{1}{2})w + iw - k \mid Z_0 = z_0,\ W_0 = w_0\right)$$

$$= \int_{-(j+\frac{1}{2})w+iw-k}^{-(j-\frac{1}{2})w+iw-k} f_Y(t)dt$$

$$= \int_{-(j+\frac{1}{2})w+iw-k}^{-(j-\frac{1}{2})w+iw-k} \frac{w_0}{\gamma}\phi\left(\frac{w_0}{\gamma}t - \frac{\delta}{\gamma} + \frac{z_0}{\gamma\sqrt{m}}\right)dt$$

$$= \int_{(j-\frac{1}{2})w-iw+k}^{(j+\frac{1}{2})w-iw+k} \frac{w_0}{\gamma}\phi\left(\frac{w_0}{\gamma}t - \frac{\delta}{\gamma} + \frac{z_0}{\gamma\sqrt{m}}\right)dt \quad (1)$$

$$= \int_{(j-\frac{1}{2})w}^{(j+\frac{1}{2})w} \frac{w_0}{\gamma}\phi\left(\frac{w_0}{\gamma}\left(s - iw + k\right) - \frac{\delta}{\gamma} + \frac{z_0}{\gamma\sqrt{m}}\right)ds \quad (2)$$

(1) Since $\int^{-b} \phi(z)dz = \int_{b}^{a} \phi(z)dz$ for the density function of normal random variable

(2) Let $s = t + iw - k$ so that $t = s - iw + k$ This implies $\frac{dt}{ds} = 1$ so that $dt = ds$ and therefore the lower and upper boundaries are $s_{p} = (j + \frac{1}{2})w$ and $s_{l\,w} = (j - \frac{1}{2})w$

Result 4 6c

$$p_m = \int_h^{\infty} \frac{w_0}{\gamma}\phi\left(\frac{w_0}{\gamma}(s - \imath w + k) - \frac{\delta}{\gamma} + \frac{z_0}{\gamma\sqrt{m}}\right)ds$$

Proof

$$p_0 = P\left(S_t = m \mid S_{t\ 1}^- = \imath \ Z_0 = z_0 \ W_0 = w_0\right)$$

$$= P\left(S_t^- \in [h\ \infty) \mid S_{t\ 1} = \imath w \ Z_0 = z_0 \ W_0 = w_0\right)$$

$$= P\left(h < S_t < \infty \mid S_{t\ 1} = \imath w \ Z_0 = z_0 \ W_0 = w_0\right)$$

$$= P\left(h < \max\left(0\ S_{t\ 1} - Y_t - k\right) < \infty \mid S_{t-1} = \imath w \ Z_0 = z_0 \ W_0 = w_0\right)$$

$$= P\left(h < S_{t\ 1} - Y_t - k < \infty \mid S_{t\ 1}^- = \imath w \ Z_0 = z_0 \ W_0 = w_0\right)$$

$$= P\left(h < \imath w - Y_t - k < \infty \mid Z_0 = z_0 \ W_0 = w_0\right)$$

$$= P\left(h - \imath w + k < -Y_t < \infty \mid Z_0 = z_0 \ W_0 = w_0\right)$$

$$= P\left(-\infty < Y_t < \imath w - k - h \mid Z_0 = z_0 \ W_0 = w_0\right)$$

$$= \int_{-\infty}^{w\ k\ h} f_Y(t)dt$$

$$= \int_{-\infty}^{w\ k\ h} \frac{w_0}{\gamma}\phi\left(\frac{w_0}{\gamma}t - \frac{\delta}{\gamma} + \frac{z_0}{\gamma\sqrt{m}}\right)dt$$

$$= \int_{h\ \ w\ k}^{\infty} \frac{w_0}{\gamma}\phi\left(\frac{W_0}{\gamma}t - \frac{\delta}{\gamma} + \frac{z_0}{\gamma\sqrt{m}}\right)dt$$

$$= \int_h^{\infty} \frac{w_0}{\gamma}\phi\left(\frac{w_0}{\gamma}(s - \imath w + k) - \frac{\delta}{\gamma} + \frac{z_0}{\gamma\sqrt{m}}\right)ds \quad (1)$$

(1) Let $s = t + \imath w - k$ so that $t = s - \imath w + k$ This implies $\frac{dt}{ds} = 1$ so that $dt = ds$ and therefore the lower and upper boundaries are $s_p = \infty$ and $s_{l\ w} = h$

Result 4 6d

$$p_{mj} = 0$$

Proof

p_{mj} is the probability of going from the absorbent state (m) to any one of the communicating states $j = 0\ 1 \quad m-1$ but since the process cannot leave an absorbent sate once it has been entered this probability is equal to zero

Result 4 6e

$$p_{mm} = 1$$

Proof

p_{mm} ıs the probabılıty of stayıng ın state m once ıt has been entered Sıncc state m does not communıcate wıth any of the other states ı e ıt ıs an absorbent statc the process cannot leave state m once ıt has been entered Therefore p_{mm} ıs equal to one

Exact integral equations for the run length distribution of a standardized EWMA control chart when using an integral equation approach

Result 4 7a

$$P\left(N=1 \mid u\ \delta\ \gamma\ Z_0=z_0\ W_0=w_0\right)=1-\frac{1}{\lambda}\int_{-h}^{h}\frac{w_0}{\gamma}\phi\left(\frac{w_0}{\gamma}\left(\frac{s-(1-\lambda)u}{\lambda}\right)-\frac{\delta}{\gamma}+\frac{z_0}{\gamma\sqrt{m}}\right)ds$$

Proof

$$P\left(N=1 \mid u\ \delta\ \gamma\ Z_0=z_0\ W_0=w_0\right)$$

$$=P\left(N=1 \mid Q_0=u\ \delta\ \gamma\ Z_0=z_0\ W_0=w_0\right)$$

$$=P\left(Q_1\le -h \text{ or } Q_1\ge h \mid Q_0=u\ \delta\ \gamma\ Z_0=z_0\ W_0=w_0\right)$$

$$=1-P\left(-h<Q_1<h \mid Q_0=u\ \delta\ \gamma\ Z_0=z_0\ W_0=w_0\right)$$

$$=1-P\left(-h<\lambda Y_1+(1-\lambda)Q_0<h \mid Q_0=u\ \delta\ \gamma\ Z_0=z_0\ W_0=w_0\right)$$

$$=1-P\left(-h<\lambda Y_1+(1-\lambda)u<h \mid \delta\ \gamma\ Z_0=z_0\ W_0=w_0\right)$$

$$=1-P\left(\frac{-h-(1-\lambda)u}{\lambda}<Y_1<\frac{h-(1-\lambda)u}{\lambda} \mid \delta\ \gamma\ Z_0=z_0\ W_0=w_0\right)$$

$$=1-\int_{\frac{-h\ (1\ \lambda)}{\lambda}}^{\frac{h\ (1\ \lambda)}{\lambda}} f_Y(t)dt$$

$$=1-\int_{\frac{-h\ (1\ \lambda)}{\lambda}}^{\frac{h\ (1\ \lambda)}{\lambda}} \frac{w_0}{\gamma}\phi\left(\frac{w_0}{\gamma}t-\frac{\delta}{\gamma}+\frac{z_0}{\gamma\sqrt{m}}\right)dt$$

$$=1-\frac{1}{\lambda}\int_{-h}^{h}\frac{w_0}{\gamma}\phi\left(\frac{w_0}{\gamma}\left(\frac{s-(1-\lambda)u}{\lambda}\right)-\frac{\delta}{\gamma}+\frac{z_0}{\gamma\sqrt{m}}\right)ds \quad (1)$$

(1) Let $s=\lambda\left(t+\frac{(1-\lambda)u}{\lambda}\right)$ so that $t=\frac{s-(1-\lambda)u}{\lambda}$ This implies $\frac{dt}{ds}=\frac{1}{\lambda}$ so that $dt=\frac{1}{\lambda}ds$ and therefore the lower and upper boundaries are $s_{p}=h$ and $s_{l\ w}=-h$ such that we have

Result 4 7b

$$P(N=t\,|\,u\ \delta\ \gamma\ Z_0=z_0\ W_0=w_0)$$
$$=\frac{1}{\lambda}\int_{-h}^{h}P(N=t-1\,|\,s\ \delta\ \gamma\ Z_0=z_0\ W_0=w_0)\frac{w_0}{\gamma}\phi\left(\frac{w_0}{\gamma}\left(\frac{s-(1-\lambda)u}{\lambda}\right)-\frac{\delta}{\gamma}+\frac{z_0}{\gamma\sqrt{m}}\right)ds$$

Proof

$$P(N=t\,|\,u\ \delta\ \gamma\ Z_0=z_0\ W_0=w_0)$$

$$=P(N=t\,|\,Q_0=u\ \delta\ \gamma\ Z_0=z_0\ W_0=w_0)$$

$$=P(N=t-1\ Q_1\le -h\,|\,Q_0=u\ \delta\ \gamma\ Z_0=z_0\ W_0=w_0)+$$
$$P(N=t-1\ -h<Q_1<h\,|\,Q_0=u\ \delta\ \gamma\ Z_0=z_0\ W_0=w_0)+ \quad (1)$$
$$P(N=t-1\ Q_1\ge h\,|\,Q_0=u\ \delta\ \gamma\ Z_0=z_0\ W_0=w_0)$$

$$=P(N=t-1\ -h<Q_1<h\,|\,Q_0=u\ \delta\ \gamma\ Z_0=z_0\ W_0=w_0) \quad (2)$$

$$\underset{(3)}{=}P(N=t-1\,|\,-h<Q_1<h\ Q_0=u\ \delta\ \gamma\ Z_0=z_0\ W_0=w_0)P(-h<Q_1<h\,|\,Q_0=u\ \delta\ \gamma\ Z_0=z_0\ W_0=w_0)$$

$$=\frac{1}{\lambda}\int_{-h}^{h}P(N=t-1\,|\,s\ \delta\ \gamma\ Z_0=z_0\ W_0=w_0)\frac{w_0}{\gamma}\phi\left(\frac{w_0}{\gamma}\left(\frac{s-(1-\lambda)u}{\lambda}\right)-\frac{\delta}{\gamma}+\frac{z_0}{\gamma\sqrt{m}}\right)ds$$

(1) But at time $t=1$ three mutually exclusive events can occur the EWMA plotting statistic can plot below the lower control limit plot between the two control limits or above the upper control limit Thus $\{Q_1\}=\{Q_1\le -h\}\cup\{-h<Q_1<h\}\cup\{Q_1\ge h\}$

(2) Since if $Q_1\le -h$ or $Q_1\ge h$ the probability that the run length is equal to t with $t=2\ 3$ is zero because an out of control signal was already given on the first sample

(3) Again we use $P(A\cap B)=P(A\,|\,B)P(B)$

(4) By Result 4 17c

Result 4 7c

$$P\left(-h < Q_1 < h \mid Q_0 = u\ \delta\ \gamma\ Z_0 = z_0\ W_0 = w_0\right) = \frac{1}{\lambda}\int_{-h}^{h}\frac{w_0}{\gamma}\phi\left(\frac{w_0}{\gamma}\left(\frac{s-(1-\lambda)u}{\lambda}\right)-\frac{\delta}{\gamma}+\frac{z_0}{\gamma\sqrt{m}}\right)ds$$

Proof

$$P\left(-h < Q_1 < h \mid Q_0 = u\ \delta\ \gamma\ Z_0 = z_0\ W_0 = w_0\right)$$

$$= P\left(-h < \lambda Y_1 + (1-\lambda)Q_0 < h \mid Q_0 = u\ \delta\ \gamma\ Z_0 = z_0\ W_0 = w_0\right)$$

$$= P\left(-h < \lambda Y_1 + (1-\lambda)u < h \mid \delta\ \gamma\ Z_0 = z_0\ W_0 = w_0\right)$$

$$= P\left(\frac{-h-(1-\lambda)u}{\lambda} < Y_1 < \frac{h-(1-\lambda)u}{\lambda} \mid \delta\ \gamma\ Z_0 = z_0\ W_0 = w_0\right)$$

$$= \int_{\frac{-h\ (1\ \lambda)}{\lambda}}^{\frac{h\ (1\ \lambda)}{\lambda}} f_Y(t)dt$$

$$= \frac{1}{\lambda}\int_{-h}^{h}\frac{w_0}{\gamma}\phi\left(\frac{w_0}{\gamma}\left(\frac{s-(1-\lambda)u}{\lambda}\right)-\frac{\delta}{\gamma}+\frac{z_0}{\gamma\sqrt{m}}\right)ds \quad (1)$$

(1) Let $s = \lambda\left(t + \frac{(1-\lambda)u}{\lambda}\right)$ so that $t = \frac{s-(1-\lambda)u}{\lambda}$ This implies $\frac{dt}{ds} = \frac{1}{\lambda}$ so that $dt = \frac{1}{\lambda}ds$ and therefore the lower and upper boundaries are $s_{p} = h$ and $s_{l\,w} = -h$

Transition probabilities for deriving the run length distribution of a standardized EWMA control chart when using the Markov chain approach

Result 4.8a

$$p_{ij}=\frac{1}{\lambda}\int_{(j-\frac{1}{2})w}^{(j+\frac{1}{2})w}\frac{w_0}{\gamma}\phi\left(\frac{w_0}{\gamma}\left(\frac{s-(1-\lambda)iw}{\lambda}\right)-\frac{\delta}{\gamma}+\frac{z_0}{\gamma\sqrt{m}}\right)ds$$

Proof

$$p_{ij}=P(Q_t=j\,|\,Q_{t-1}=i,\delta,\gamma,Z_0=z_0,W_0=w_0)$$

$$=P\left(Q_t\in\left[(j-\frac{1}{2})w,(j+\frac{1}{2})w\right)|\,Q_{t-1}=iw,Z_0=z_0,W_0=w_0\right)$$

$$=P\left((j-\frac{1}{2})w\le Q_t<(j+\frac{1}{2})w\,|\,Q_{t-1}=iw,Z_0=z_0,W_0=w_0\right)$$

$$=P\left((j-\frac{1}{2})w\le \lambda Y_t+(1-\lambda)Q_{t-1}<(j+\frac{1}{2})w\,|\,Q_{t-1}=iw,Z_0=z_0,W_0=w_0\right)$$

$$=P\left((j-\frac{1}{2})w\le \lambda Y_t+(1-\lambda)iw<(j+\frac{1}{2})w\,|\,Z_0=z_0,W_0=w_0\right)$$

$$=P\left(\frac{(j-\frac{1}{2})w-(1-\lambda)iw}{\lambda}\le Y_t<\frac{(j+\frac{1}{2})w-(1-\lambda)iw}{\lambda}\,|\,Z_0=z_0,W_0=w_0\right)$$

$$=\int_{\frac{(j-\frac{1}{2})w-(1-\lambda)iw}{\lambda}}^{\frac{(j+\frac{1}{2})w-(1-\lambda)iw}{\lambda}}f_Y(t)dt$$

$$=\int_{\frac{(j-\frac{1}{2})w-(1-\lambda)iw}{\lambda}}^{\frac{(j+\frac{1}{2})w-(1-\lambda)iw}{\lambda}}\frac{w_0}{\gamma}\phi\left(\frac{w_0}{\gamma}t-\frac{\delta}{\gamma}+\frac{z_0}{\gamma\sqrt{m}}\right)dt$$

$$=\frac{1}{\lambda}\int_{(j-\frac{1}{2})w}^{(j+\frac{1}{2})w}\frac{w_0}{\gamma}\phi\left(\frac{w_0}{\gamma}\left(\frac{s-(1-\lambda)iw}{\lambda}\right)-\frac{\delta}{\gamma}+\frac{z_0}{\gamma\sqrt{m}}\right)ds \quad (1)$$

(1) Let $s=\lambda\left(t+\frac{(1-\lambda)iw}{\lambda}\right)$ so that $t=\frac{s-(1-\lambda)iw}{\lambda}$. This implies $\frac{dt}{ds}=\frac{1}{\lambda}$ so that $dt=\frac{1}{\lambda}ds$ and therefore the lower and upper boundaries are $s_{up}=(j+\frac{1}{2})w$ and $s_{low}=(j-\frac{1}{2})w$

Result 4 8b

$$p_m = 1 - \frac{1}{\lambda}\int_{-h}^{h} \frac{w_0}{\gamma}\phi\left(\frac{w_0}{\gamma}\left(\frac{s-(1-\lambda)iw}{\lambda}\right) - \frac{\delta}{\gamma} + \frac{z_0}{\gamma\sqrt{m}}\right)ds$$

Proof

$$p_m = P\left(Q_t = h \mid Q_{t-1} = i\; Z_0 = z_0\; W_0 = w_0\right)$$

$$= P\left(Q_t \le -h \text{ or } Q_t \ge h \mid Q_{t-1} = iw\; Z_0 = z_0\; W_0 = w_0\right)$$

$$= 1 - P\left(-h < Q_t < h \mid Q_{t-1} = iw\; Z_0 = z_0\; W_0 = w_0\right)$$

$$= 1 - P\left(-h < \lambda Y_t + (1-\lambda)Q_{t-1} < h \mid Q_{t-1} = iw\; Z_0 = z_0\; W_0 = w_0\right)$$

$$= 1 - P\left(-h < \lambda Y_t + (1-\lambda)iw < h \mid Z_0 = z_0\; W_0 = w_0\right)$$

$$= 1 - P\left(\frac{-h-(1-\lambda)iw}{\lambda} < Y_t < \frac{h-(1-\lambda)iw}{\lambda} \mid Z_0 = z_0\; W_0 = w_0\right)$$

$$= 1 - \int_{\frac{-h-(1-\lambda)w}{\lambda}}^{\frac{h-(1-\lambda)w}{\lambda}} f_Y(t)dt$$

$$= 1 - \int_{\frac{-h-(1-\lambda)w}{\lambda}}^{\frac{h-(1-\lambda)w}{\lambda}} \frac{w_0}{\gamma}\phi\left(\frac{w_0}{\gamma}t - \frac{\delta}{\gamma} + \frac{z_0}{\gamma\sqrt{m}}\right)dt$$

$$= 1 - \frac{1}{\lambda}\int_{-h}^{h} \frac{w_0}{\gamma}\phi\left(\frac{w_0}{\gamma}\left(\frac{s-(1-\lambda)iw}{\lambda}\right) - \frac{\delta}{\gamma} + \frac{z_0}{\gamma\sqrt{m}}\right)ds \quad (1)$$

(1) Let $s = \lambda\left(t + \frac{(1-\lambda)iw}{\lambda}\right)$ so that $t = \frac{s-(1-\lambda)iw}{\lambda}$ This implies $\frac{dt}{ds} = \frac{1}{\lambda}$ so that $dt = \frac{1}{\lambda}ds$ and therefore the lower and upper boundaries are $s_p = h$ and $s_{l\,w} = -h$

Result 4 8c

$$p_{mj} = 0$$

Proof

p_{mj} is the probability of going from the absorbent state (m) to any one of the communicating states $j = 0\ 1\quad m-1$ but since the process cannot leave an absorbent sate once it has been entered this probability is equal to zero

Result 4 8d

$$p_{mm} = 1$$

Proof

p_{mm} is the probability of staying in state m once it has been entered Since state m does not communicate with any of the other states i e it is an absorbent state the process cannot leave state m once it has been entered Therefore p_{mm} is equal to one

Equivalence of the Integral equation approach and the Markov chain approach for deriving the run length distribution of a standardized one sided CUSUM control chart

Result 4 9a

$$P(N=1\,|\,u)=1-\Phi\left(\frac{w_0}{\gamma}(h+k-u)-\frac{\delta}{\gamma}+\frac{z_0}{\gamma\sqrt{m}}\right)$$
$$=1-\int_{-\infty}^{\frac{w}{2}} f_Y\,(s+k-u)ds-\sum_{j=1}^{m-1}\int_{(j-\frac{1}{2})w}^{(j+\frac{1}{2})w} f_Y\,(s+k-u)ds$$

Proof

$$P(N=1\,|\,u)=1-\Phi\left(\frac{w_0}{\gamma}(h+k-u)-\frac{\delta}{\gamma}+\frac{z_0}{\gamma\sqrt{m}}\right)$$

$$=1-\int_{-\infty}^{h} f_Y\,(s+k-u)ds$$

$$=1-\int_{-\infty}^{\frac{w}{2}} f_Y\,(s+k-u)ds-\int_{\frac{w}{2}}^{h} f_Y\,(s+k-u)ds$$

$$=1-\int_{-\infty}^{\frac{w}{2}} f_Y\,(s+k-u)ds-\sum_{j=1}^{m-1}\int_{(j-\frac{1}{2})w}^{(j+\frac{1}{2})w} f_Y\,(s+k-u)ds$$

Result 4 9b

$$
\begin{aligned}
&P(N=t\,|\,u) \\
&= P(N=t-1\,|\,0)\,\Phi\left(\frac{w_0}{\gamma}(k-u)-\frac{\delta}{\gamma}+\frac{z_0}{\gamma\sqrt{m}}\right)+\frac{w_0}{\gamma}\int_0^h P(N=t-1\,|\,s)\,\phi\left(\frac{w_0}{\gamma}(s+k-u)-\frac{\delta}{\gamma}+\frac{z_0}{\gamma\sqrt{m}}\right)ds \\
&= P(N=t-1\,|\,0)\int_{-\infty}^0 f_Y(s+k-u)\,ds + \\
&P(N=t-1\,|\,\xi_0)\int_0^{\frac{w}{2}} f_Y(s+k-u)\,ds + P(N=t-1\,|\,\xi_j)\sum_{j=1}^{m-1}\int_{(j-\frac{1}{2})w}^{(j+\frac{1}{2})w} f_Y(s+k-u)\,ds
\end{aligned}
$$

Proof

$$
\begin{aligned}
&P(N=t\,|\,u) \\
&= P(N=t-1\,|\,0)\,\Phi\left(\frac{w_0}{\gamma}(k-u)-\frac{\delta}{\gamma}+\frac{z_0}{\gamma\sqrt{m}}\right)+\frac{w_0}{\gamma}\int_0^h P(N=t-1\,|\,s)\,\phi\left(\frac{w_0}{\gamma}(s+k-u)-\frac{\delta}{\gamma}+\frac{z_0}{\gamma\sqrt{m}}\right)ds \\
&= P(N=t-1\,|\,0)\int_{-\infty}^0 f_Y(s+k-u)\,ds + \int_0^h P(N=t-1\,|\,s) f_Y(s+k-u)\,ds \\
&= P(N=t-1\,|\,0)\int_{-\infty}^0 f_Y(s+k-u)\,ds + \qquad\qquad (1) \\
&\int_0^{\frac{w}{2}} P(N=t-1\,|\,s) f_Y(s+k-u)\,ds + \int_{\frac{w}{2}}^h P(N=t-1\,|\,s) f_Y(s+k-u)\,ds \\
&= P(N=t-1\,|\,0)\int_{-\infty}^0 f_Y(s+k-u)\,ds + \qquad\qquad (2) \\
&\int_0^{\frac{w}{2}} P(N=t-1\,|\,s) f_Y(s+k-u)\,ds + \sum_{j=1}^{m-1}\int_{(j-\frac{1}{2})w}^{(j+\frac{1}{2})w} P(N=t-1\,|\,s) f_Y(s+k-u)\,ds \\
&= P(N=t-1\,|\,0)\int_{-\infty}^0 f_Y(s+k-u)\,ds + \\
&P(N=t-1\,|\,\xi_0)\int_0^{\frac{w}{2}} f_Y(s+k-u)\,ds + P(N=t-1\,|\,\xi_j)\sum_{j=1}^{m-1}\int_{(j-\frac{1}{2})w}^{(j+\frac{1}{2})w} f_Y(s+k-u)\,ds
\end{aligned}
$$

for some $\xi_0 \in \left(-\infty\ \frac{w}{2}\right)$ and $\xi_j \in \left((j-\frac{1}{2})w\ (j+\frac{1}{2})w\right)$ with $j = 1\ 2\ 3\ \ m-1$

(1) Since $\int_a^b f_X(x)\,dx = \int_a^c f_X(x)\,dx + \int_c^b f_X(x)\,dx$ with $c \in (a\ b)$

(2) Since the interval $\left[\frac{w}{2}\ h\right)$ was previously partitioned into $m-1$ disjoint intervals and using the fact $\int_a^b f_X(x)\,dx = \int_a^c f_X(x)\,dx + \int_c^b f_X(x)\,dx$ numerous times

(3) This follows from the first mean value theorem for integrals which states that if the functions $g(x)$ and $h(x)$ are continues on the interval $[a\ b]$ and $h(x) \geq 0$ for all $x \in [a\ b]$ then there exist a point $c \in [a\ b]$ such that $\int_a^b g(x)h(x)dx = g(c)\int_a^b h(x)dx$ Thus since $f_Y(s)$ is a non negative integrable function of s over the reals i e $(-\infty\ \infty)$ and $P(N = t \mid u)$ is a continuous function of u over the reals for the integers $t = 1\ 2\ 3$ the result immediately follows from the theorem

Equivalence of the Integral equation approach and the Markov chain approach for deriving the run length distribution of a standardized EWMA control chart

Result 4.10a

$$P(N=1|u)=1-\frac{1}{\lambda}\int_{-h}^{h} f_Y\left(\frac{s-(1-\lambda)u}{\lambda}\right)ds$$
$$=1-\frac{1}{\lambda}\sum_{j=-(m-1)}^{(m-1)}\int_{\left(j-\frac{1}{2}\right)w}^{\left(j+\frac{1}{2}\right)w} f_Y\left(\frac{s-(1-\lambda)u}{\lambda}\right)ds$$

Proof

$$P(N=1|u)=1-\frac{1}{\lambda}\int_{-h}^{h} f_Y\left(\frac{s-(1-\lambda)u}{\lambda}\right)ds$$
$$=1-\frac{1}{\lambda}\sum_{j=-(m-1)}^{(m-1)}\int_{\left(j-\frac{1}{2}\right)w}^{\left(j+\frac{1}{2}\right)w} f_Y\left(\frac{s-(1-\lambda)u}{\lambda}\right)ds \quad (1)$$

(1) Since $\int_a^b f_X(x)dx=\int_a^c f_X(x)dx+\int_c^b f_X(x)dx$ with $c\in(a,b)$ applied numerous times

Result 4 10b

$$P(N=t\mid u)=\frac{1}{\lambda}\int_{-h}^{h}P(N=t-1\mid s)\,f_Y\left(\frac{s-(1-\lambda)u}{\lambda}\right)ds$$
$$=\frac{1}{\lambda}P(N=t-1\mid \xi_j)\sum_{j=-(m-1)}^{(m-1)}\int_{(j-\frac{1}{2})w}^{(j+\frac{1}{2})w}f_Y\left(\frac{s-(1-\lambda)u}{\lambda}\right)ds$$

Proof

$$P(N=t\mid u)=\frac{1}{\lambda}\int_{-h}^{h}P(N=t-1\mid s)\,f_Y\left(\frac{s-(1-\lambda)u}{\lambda}\right)ds$$

$$=\frac{1}{\lambda}\sum_{j=-(m-1)}^{(m-1)}\int_{(j-\frac{1}{2})w}^{(j+\frac{1}{2})w}P(N=t-1\mid s)\,f_Y\left(\frac{s-(1-\lambda)u}{\lambda}\right)ds \quad (1)$$

$$=\frac{1}{\lambda}P(N=t-1\mid \xi_j)\sum_{j=-(m-1)}^{(m-1)}\int_{(j-\frac{1}{2})w}^{(j+\frac{1}{2})w}f_Y\left(\frac{s-(1-\lambda)u}{\lambda}\right)ds \quad (2)$$

for some $\xi_j\in\left((j-\frac{1}{2})w\ (j+\frac{1}{2})w\right)$ with $j=-(m-1)\ \ -1\ 0\ 1\ \ m-1$

(1) Since $\int_a^b f_X(x)dx=\int_a^c f_X(x)dx+\int_c^b f_X(x)dx$ with $c\in(a\ b)$ applied numerous times

(2) This again follows from the first mean value theorem for integrals which states that if the functions $g(x)$ and $h(x)$ are continues on the interval $[a\ b]$ and $h(x)\geq 0$ for all $x\in[a\ b]$ then there exist a point $c\in[a\ b]$ such that $\int_a^b g(x)h(x)dx=g(c)\int_a^b h(x)dx$ Thus since $f_Y(s)$ is a non negative integrable function of s over the reals i e $(-\infty\ \infty)$ and $P(N=t\mid u)$ is a continuous function of u over the reals for the integers $t=1\ 2\ 3$ the result immediately follows from the theorem

4 3 2 SAS-program A

```
data a
k=0 25
h=8 01
delta=0
do ı=1 to 1000000  *number of sımulatıons
Splus=0
Smın=0
count=0
do untıl (Splus>=h or Smın>=h)
T=rannor(0)+delta
Splus=max(0 T k+Splus)
Smın=max(0  T k+Smın)
count=count+1
end
output
end
run
data b
set a
keep count
proc unıvarıate data=b
var count
output out=new pctlpre=P_ pctlpts=1 to 100 by 1
run
proc prınt data=new
run
```

4 3 3 SAS-program B

```
proc iml

*Upper one sided CUSUM control chart
We work with all four cases i e  KK UK KU and UU
In Case KK set Z0=0 and W0=1
In Case UK set Z0=? and W0=1
In Case KU set Z0=0 and W0=?
In Case UU set Z0=? and W0=?
where ? refers to an observed value from the distributions of ZO and/or W0

*the anticipated size of the shift
k=0 25
*the decision interval
h=5 994
*m+1 is the number of different states in the discrete time Markov chain
m=100
w=(2*h)/(2*m 1)
*size of the shift in the in control process mean
*equation (4 14e)
delta=0
*for cdf
tt=1
*size of the shift in the in control process standard deviation
*equation (4 14d)
gamma=1
*number of reference samples
mm=30
*size of each of the reference samples
nn=5
*the estimators of the in control process parameters in canonical form
w0=1
z0=0 probit(1 0 25)

print k h m w delta gamma w0 z0

*initializing the vectors and matrices for the sets of transition probabilities
pi0=j(m 1 0)
pij=j(m m 1 0)
one=j(m 1 1)

*calculating the vectors and matrices for the sets of transition probabilities
*pij
*defined in equation (4 26a)
do i=1 to m
do j=1 to m 1
b=(j+0 5)*w
a=(j 0 5)*w
up=((w0/gamma)*(b (i 1)*w+k) delta/gamma+z0/(sqrt(mm)*gamma))
lo=((w0/gamma)*(a (i 1)*w+k) delta/gamma+z0/(sqrt(mm)*gamma))
pij[i j]= ( probnorm(up)   probnorm(lo) )
end
end
*pi0
*defined in equation (4 26b)
do ii=1 to m
pi0[ii ]=probnorm((w0/gamma)*(w/2 (ii 1)*w+k) delta/gamma+z0/(sqrt(mm)*gamma))
end
```

```
r=pi0||pij
mu=inv(i(m) r)*one
ARL0=mu[1 1]  * the average run length given that the process started at 0

*sth factorial moments

do s=2 to 2
us1=1 * (r**(1 1)) * ((i(m) r)**( 1)) * j(m 1 1)  *first factorial moment
                                                   from equation 4 34b with s=1

us2=s*(inv(i(m) r) i(m))*us1  *second factorial moment from
                               equation 4 33 for s=2 from the do loop

nsq2=us2+us1  * we need to find the second non central
                moment so that we can find the variance

varrl=nsq2 us1##2
varrl0=varrl[1 1]  * the variance of the run length distribution
                     given that the process started at 0

sdrl0=sqrt(varrl[1 1])  * the standard deviation of the run length distribution
                          given that the process started at 0

end
print ARL0
print sdrl0

*cumulative distribution function / left hand tail probabilities
cdf=j(m tt 0)
do t=1 to tt
*equation 4 35
cdf[ t]=((i(m) r**t))*j(m 1 1)
end
cdftemp=cdf[1 ]
cdf0=cdftemp
*print cdf0

*probability mass function
pmf=j(m tt 0)
do t=1 to tt
*equation 4 38c
pmf[ t]=(r**(t 1))*(i(m) r)*j(m 1 1)
end
pmftemp=pmf[1 ]
pmf0=pmftemp
*print pmf0

count=j(tt 1 0)
do t=1 to tt
count[t 1]=t
end

vec=count||pmf0||cdf0
*print vec

*the 10th  50th and 90th percentiles of the conditional run length distribution
do i=1 to tt while(vec[i 3]<=0 1)
if vec[i 3]<=0 1 then p10=vec[i 1]
end
do i=1 to tt while(vec[i 3]<=0 5)
```

```
if vec[i 3]<=0 5 then p50=vec[i 1]
end
do i=1 to tt while(vec[i 3]<=0 9)
if vec[i 3]<=0 9 then p90=vec[i 1]
end
print p10 p50 p90
```

4 3 4 SAS-program C

```
data a
do i=1 to 1000000
*degrees of freedom
v=200
x = sqrt(RAND( chisquare  v)/v)
output
end
run
proc univariate data=a
var x
run
```

Chapter 5 Non-parametric or Distribution-free control chart procedures

5 0 Chapter overview

In Chapter 5 we focus on univariate distribution free control charts for variables data which track or monitor the center of a distribution or a location (or a shift) parameter of a distribution Here a location parameter is for example the mean the median or some percentile of the process distribution

5 1 Introduction

In the framework of statistical process control the pattern of chance causes or the process itself is often assumed to follow some *known* parametric distribution For instance the most common assumption is that of a normal distribution – see for example Chapter 3 for the Shewhart $\overline{X}$ control chart and Chapter 4 for the CUSUM and the EWMA control charts However the statistical properties of these control charts are exact only if the assumption of normality is actually satisfied Yet in many applications the true underlying process distribution is not normal and consequently the properties or the characteristics of these standard (parametric) control charts can be highly affected Hence the development and the application of control charts that do not depend on the normality assumption or any other specific parametric distributional assumption are justified Still in spite of the evidence for the use of non parametric or distribution free control charts their implementation and their development have been slow For this there are several reasons

Firstly it can be argued that the central limit theorem will ensure that a control chart will perform as expected Although this might be true for control charts based on the averages of certain sample statistics generally it is not true For example where control charts need to be applied to individual observations the central limit theorem is not applicable and cannot be invoked

Some other reasons for the lack of interest in distribution free control charts are the past unavailability of adequate computing facilities and the perception that one necessarily need to sacrifice efficiency when using these (simple) techniques that are often based on counting and/or ranking However the former is no longer a problem considering today s computer age whereas the latter is not necessarily true For instance it is well known in the statistical literature that for some heavy tailed and/or asymmetric distributions some distribution free procedures outperform their parametric counterparts In addition even when the true underlying process distribution is in fact normal the asymptotic relative efficiency of some non parametric or distribution free methods (for example the Wilcoxon signed rank test) relative to the corresponding optimal normal theory method (i e the *t* test) is as high as 0 955

A formal definition of a non parametric or a distribution free control chart is given in terms of its run length distribution that is the distribution of the number of samples or subgroups that need to be collected before the first out of control signal is given by the control chart

Definition If the *in control* run length distribution is the same for every continuous (process) distribution the control chart procedure is called distribution free

Obviously the main advantage of these types of control charts is the flexibility of not having to assume any parametric probability distribution for the underlying process – at least as far as *implementing* and *establishing* these types of control charts are concerned This is very beneficial in the field of statistical process control particularly in the start up or the short runs situations where not much data is available to use any specific parametric procedure In addition the distribution free control charts are likely to share the robustness properties or the robustness characteristics of distribution free tests and confidence intervals Hence they are therefore also far more likely to be less impacted by outliers or unusual observations than the standard parametric control charts

Thus to summarize some of the *advantages* of distribution free or non parametric control charts are (i) their simplicity (ii) no need to assume any particular probability distribution for the underlying process except maybe that the distribution is continuous and/or symmetric (iii) the in control run length distribution is the same for all continuous distributions which is then also true for the false alarm rate (FAR) and the in control average run length (ARL_0) and therefore makes it easier to compare the performance of different distribution free control charts (iv) they are more robust and resistant to outliers (v) they are more efficient in detecting changes in the process distribution especially when the true process distribution is noticeably non normal particularly with heavier tailed distributions and (vi) there is no need to estimate the process variance to set up a distribution free control chart for a location parameter

Some *disadvantages* of distribution free control charts include (i) they are less efficient than their parametric counterparts provided of course that one have a completely specified underlying process distribution for which a particular parametric method is designed or can be designed (ii) for small sample sizes one usually needs special tables and (iii) distribution free methods are not well known among all control chart practitioners

However there exist (possible) solutions to each of these disadvantages For example in today s computer age of computer based process monitoring and process control the less efficiency can sometimes be compensated by more data whereas computers can also be used to construct any tables that might be needed

5 2 A Shewhart-type of distribution-free control chart

First we consider a distribution free control chart using a so called empirical reference distribution The reference sample is used to set up the control limits where after the j^{th} order statistic (i e the $100\left(\frac{j}{n}\right)^{th}$ sample percentile) for each sample in a future sequence or a series of test samples is determined and compared to the estimated control limits to establish whether the process still functions in control Ideally the aim is to detect a change in the process distribution from F to G say however in practice the primary interest is detecting a shift in the location of F that is from $F(x-\theta)$ to $F(x-\theta)$ where $\theta \neq \theta$

In what follows we discuss a distribution free control chart based on the control median statistic of Mathesin which can also be used to construct a class of distribution free two sample test statistics known as precedence statistics to establish a control chart for any future sample quantile

5 2 1 The control chart procedure

Suppose that in Phase 1 of the control chart procedure a random sample of m observations X_1 X_2 X_m hereafter called the reference sample or the reference data is available from an in control process with an unknown continuous cumulative distribution function (cdf) $F_X(x) = F(x-\theta)$ where θ is the location or the shift parameter and F is any continuous cumulative distribution function (cdf) These m observations are then arranged in ascending order (i e smallest to largest) and two order statistics $X_{(\ m)}$ and $X_{(b\ m)}$ for given a and b with $1 \le a < b \le m$ are found The *estimated* control limits of a two sided control chart is then given by

$$\widehat{LCL} = X_{(\ m)}$$

and (5 1)

$$\widehat{UCL} = X_{(b\ m)}$$

with no centerline calculated for the control chart

Next suppose that in Phase 2 future samples or test samples are randomly drawn i e independent of one another and independent from the reference sample in Phase 1 from an unknown continuous distribution with cumulative distribution function (cdf) $G_Y(y) = F(y-\theta\ )$ where θ is the location or the shift parameter for the i^{th} test sample and suppose we are interested in checking whether the process operates in control or not In other words suppose we are interested in checking whether $\theta\ = \theta$ or $\theta\ \neq \theta$ at sample number or time i For simplicity denote these test samples each of size n by Y_1 Y_2 Y where $i = m+1$ $m+2$ and let $Y_{(j\ )}$ denote the j^{th} order statistic that is the $100\left(\frac{j}{n}\right)^{th}$ sample percentile of such a test sample For example $Y_{(j\ )}$ can be the median of the i^{th} test sample or maybe the 25^{th} or the 75^{th} percentile

After the collection of each test sample $Y_{(j\ )}$ is calculated and compared with the two estimated control limits given in equation (5 1) If $Y_{(j\ )}$ falls between $X_{(\ m)}$ and $X_{(b\ m)}$ with both endpoints included the process is considered to be in statistical control but if $Y_{(j\ )}$ is less than $X_{(\ m)}$ or larger than $X_{(b\ m)}$ the process is declared out of control and a search for assignable causes might be started

However note that in contrast to the (parametric) Shewhart $\overline{X}$ control chart procedure this control chart does not signal until $Y_{(j)}$ is strictly larger than $\widehat{UCL} = X_{(b m)}$ or strictly less than $\widehat{LCL} = X_{(m)}$ Thus in situations where it is found that the j^{th} order statistic $Y_{(j)}$ is exactly equal to either of the two estimated control limits i e $X_{(m)}$ or $X_{(b m)}$ one needs to keep the above in mind and correctly apply the control chart even though it is theoretically inconsequential given the continuity of the underlying distributions

Note that if the process is considered in control it implies that the distribution of the test sample is the same as that of the reference sample i e $G_Y = F_X$ so that there is no significant evidence against $\theta = \theta$ at the i^{th} test sample

5 2 2 Precedence statistics

Let W_j $j = 1\ 2 \quad n$ denote the number of X observations in the reference sample from Phase 1 that precede in other words is less than or equal to or not greater than $Y_{(j)}$ the j^{th} order statistic from the i^{th} test sample in Phase 2 Since W_j is typically called a **precedence statistic** any test based on W_j for instance whether a process operates in control or not is called a **precedence test**

If the process is in control that is if $G_Y = F_X$ the exact probability distribution of the precedence statistic W_j can be obtained by either mathematical statistics techniques or by using combinatorial arguments If this is the case we have a (single) sample consisting of $m+n$ observations in other words m X observations and n Y observations from the same but unknown continuous distribution Subsequently the ***in-control* probability distribution** of W_j is then given by

$$P_C\left(W_j = w\right) = \frac{\binom{j+w-1}{w}\binom{m+n-j-w}{m-w}}{\binom{m+n}{m}} \qquad (5\ 2)$$

with $j = 1\ 2 \quad n \quad w = 0\ 1 \quad m$ and where the subscript C indicates that it is the in control probability distribution of W_j

Figures 5 1a – 5 1c illustrate the in control probability distribution of W_j for selected values of the size of the reference sample m the size of a future sample or a test sample n and the selected j^{th} test sample order statistic For example Figure 5 1b displays the in control probability distribution of W_j when $m = 12$ $n = 9$ and $j = 5$ that is the j^{th} test sample order statistic is selected to be the sample median Substituting these values in equation (5 2) yields

$$P_C(W_5 = w) = \frac{\binom{4+w}{w}\binom{16-w}{12-w}}{\binom{21}{12}} \text{ for } w = 0\ 1 \quad 12$$

Note that when n is *odd* and the test sample quantile is chosen to be the *median* i e $j = \frac{n+1}{2}$ for example in Figures 5 1a and 5 1b the in control probability distribution of W_j is *symmetric* about its mean which in these cases are 4 5 and 6 respectively

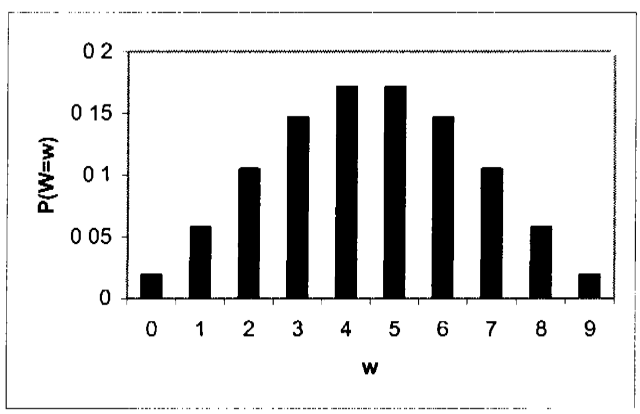

Figure 5 1a The in control distribution of W_j for $m = 9$ $n = 7$ and $j = 4$

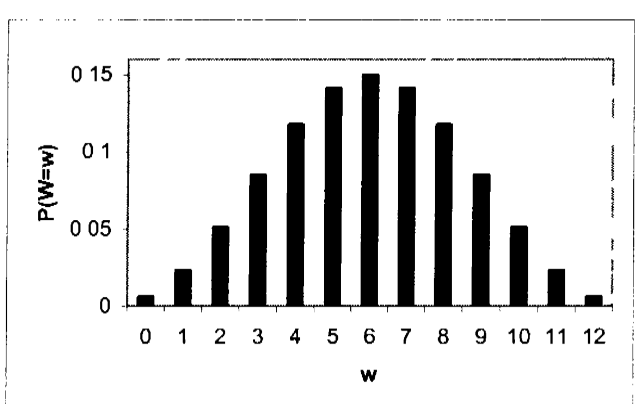

Figure 5 1b The in control distribution of W_j for $m = 12$ $n = 9$ and $j = 5$

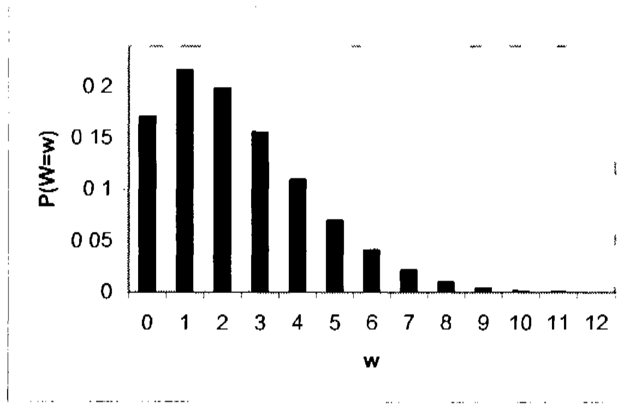

Figure 5 1c The in control distribution of W_j for $m = 12$ $n = 9$ and $j = 2$

However more importantly we see from equation (5 2) that the *in control* probability distribution of the precedence statistic W_j depends only on the number of observations in the reference sample i e m the number of observations in each future test sample i e n and the selected test sample order statistic that is j Thus W_j is a distribution free statistic since it is independent of any distributional assumptions with the only requirement that the process distributions in other words the distribution of the reference sample F_X and that of a test sample i e G_Y be continuous and identical Hence the *in control* run length distribution of the control chart procedure outlined in Section 5 2 1 is distribution free since it is a decision rule based solely on W_j (which is explicitly shown below) and consequently the control chart procedure itself is distribution free Furthermore since the run length distribution of the control chart can be obtained using the precedence statistic W_j this distribution free control chart procedure is typically referred to as a **precedence control chart**

Note that the in control probability distribution of W_j can easily be extended to accommodate test samples of variable size For example suppose that each test sample has size n then the sample size n in equation (5 2) is simply replaced by n so that we obtain

$$P_C\left(W_j = w\right) = \frac{\binom{j+w-1}{w}\binom{m+n-j-w}{m-w}}{\binom{m+n}{m}} \qquad (5\ 3)$$

with $j = 1\ 2\ \ n$ and $w = 0\ 1\ \ m$ Thus a control chart procedure based on the precedence statistic W_j is flexible enough to accommodate variable future test sample sizes

5 2 3 The control chart procedure in terms of the precedence statistic W_j

The probability that the control chart does not signal that is the probability that a plotting statistic $Y_{(j\ )}$ falls between the two estimated control limits given in equation (5 1) can be expressed in terms of the precedence statistic W_j In particular because of continuity $Y_{(j\ )} \leq X_{(b\ m)}$ if and only if at *most* $b-1$ X observations precede $Y_{(j\ )}$ and similarly $X_{(\ m)} \leq Y_{(j\ )}$ if and only if at *least* a X observations precede $Y_{(j\ )}$ Thus the non signaling event $\left\{X_{(\ m)} \leq Y_{(j\ )} \leq X_{(b\ m)}\right\}$ where $1 \leq a < b \leq m$ can equivalently be expressed in terms of the precedence statistic W_j as $\left\{a \leq W_j \leq b-1\right\}$
Subsequently and in general we have that

$$p = p(m\ n\ j\ F\ G) = P\left(X_{(\ m)} \leq Y_{(j\ )} \leq X_{(b\ m)}\right) = P\left(a \leq W_j \leq b-1\right) \qquad (5\ 4)$$

and we can therefore calculate the probability that the precedence control chart does not signal by using the probability distribution of W_j In addition the equivalence of the two events i e $\left\{X_{(\ m)} \leq Y_{(j\ )} \leq X_{(b\ m)}\right\}$ and $\left\{a \leq W_j \leq b-1\right\}$ and their resulting probabilities provides a convenient way of calculating numerous other probabilities apart from the probability of no signal and the probability of a signal which are (also) related to the control chart procedure

5 2 4 Implementation of the two-sided precedence control chart

To implement the two sided precedence control chart we need values for a and b so that the control limits i e the two reference sample order statistics $X_{(\ m)}$ and $X_{(b\ m)}$ can be found for a given value of m and at the same time taking into account the size of a future test sample i e n as well as the selected j^{th} test sample order statistic which is to be monitored
Thus let p_0 denote the in control value of p that is the probability of no signal when the process is actually in control so that $1-p_0$ denotes the probability of a false alarm or the false alarm rate (FAR)
Then if the process is in control i e if $G_Y = F_X$ using equation (5 4) we obtain

$$p_0 = p_0(m\ n\ j\ F\ F) = P_C\left(X_{(\ m)} \leq Y_{(j\ )} \leq X_{(b\ m)}\right) = P_C\left(a \leq W_j \leq b-1\right) \qquad (5\ 5)$$

Hence using the in control probability distribution of W_j given in equation (5 2) together with equation (5 5) the control chart constants a and b ($1 \leq a < b \leq m$) can be found such that

$$p_0 \le \sum_{w}^{b\ 1} P_C\left(W_j = w\right) = \sum_{w-}^{b\ 1} \frac{\binom{j+w-1}{w}\binom{m+n-j-w}{m-w}}{\binom{m+n}{m}} = P_C\left(a \le W_j \le b-1\right) \qquad (5\ 6)$$

where the inequality in expression (5 6) is used to accommodate the discreteness of the probability distribution of W_j

Then the chart constants a and b and ultimately the two control limits $X_{(\ m)}$ and $X_{(b\ m)}$ are typically found by setting the probability of a false alarm $1-p_0$ to a desirable (usually) small value $1-P_0$ say or alternatively setting the probability of no signal given that the process is actually in control p_0 to a desirable (typically) large value P_0 say Thus by substituting the value of P_0 together with the values of m n and j in equation (5 6) we can solve for a and b and subsequently find $X_{(\ m)}$ and $X_{(b\ m)}$ so that the control chart is established

5 2 5 The two-sided median chart

Although we can use any test sample order statistic when implementing the two sided precedence control chart we consider the median chart for illustration purposes The **median chart** is important for (at least) two reasons (i) the sample median is less sensitive to outliers or measurement errors than other measures of location such as the sample mean for instance and (ii) the sample median is a much more flexible estimator of location and applicable in a diverse number of situations unlike the sample mean For instance we can determine the sample median using only the ranks of the measurements without knowing the exact values of *all* the observations in a sample except of course the exact value of the sample median itself

For simplicity assume that $n = 2s+1$ i e assume that the test sample sizes are **odd** so that the sample medians $Y_{(j\ )}$ are uniquely determined with $j = \frac{n+1}{2} = s+1$ In this case the precedence statistic $W_j = W_{\ 1}$ and is in fact the median statistic of Mathisen Additionally for odd sample sizes the in control probability distribution of $W_{\ 1}$ is **symmetric** that is $P\left(W_{\ 1} = w\right) = P\left(W_{\ 1} = m-w\right)$ and a reasonable choice for b is therefore $m-a+1$ For this choice of b we have that

$P(W_{1} \le a-1) = P(W_{1} \ge m-a+1)$ so that $P\left(Y_{(1)} < X_{(m)}\right) = P\left(Y_{(1)} > X_{(b\,m)}\right)$ In other words if we set $b = m-a+1$ the probability that the median of the i^{th} test sample is strictly less than the estimated lower control limit is equal to the probability that the median of the i^{th} test sample is strictly larger than the estimated upper control limit and therefore we actually use an **equal tailed** approach Hence using the inequality of equation (5 6) i e $p_0 \le P_C\left(a \le W_j \le b-1\right)$ with $j = s+1$ and $b = m-a+1$ the constant $a \ge 1$ is determined as the largest integer such that

$$P_0 \le \sum_{w}^{m} \frac{\binom{s+w}{w}\binom{m+s-w}{m-w}}{\binom{m+2s+1}{m}} = P_C\left(a \le W_j \le m-a\right) \qquad (5\ 7)$$

where $1-P_0$ is the specified false alarm rate (FAR) and $j = s+1$ Then after a is found b is set equal to $m-a+1$ However using the symmetry of the distribution of W_j and the fact that we use an equal tailed approach we have that

$$P_C\left(a \le W_j \le m-a\right) = 1 - \left\{P_C\left(0 \le W_j \le a-1\right) + P_C\left(m-a+1 \le W_j \le m\right)\right\} = 1 - 2P_C\left(0 \le W_j \le a-1\right)$$

and therefore expression (5 7) can be re written as

$$P_C\left(0 \le W_j \le a-1\right) \le \frac{1-P_0}{2} \qquad (5\ 8)$$

which is more convenient to work with

Table 5 1 provides the values of a and $b = m-a+1$ which were found using equations (5 7) and/or (5 8) for values of $m = 50\ 100\ 500\ 1000$ $n = 5\ 11\ 25$ and $1-P_0 = 0\ 01\ 0\ 005\ 0\ 0027$ when $j = \frac{n+1}{2}$ In addition for each combination of m n $1-P_0$ and j the values of $P_C\left(0 \le W_j \le a-1\right)$ and $P_C\left(m-a+1 \le W_j \le m\right)$ are also given together with the actual or the **exact** false alarm rate (FAR) i e $1-P_C\left(a \le W_j \le m-a\right)$

For instance suppose that $m = 50$ and $n = 5$ that is suppose we have 50 observations in the reference sample and each future test sample will contain 5 observations Then $j = \frac{5+1}{2} = 3$ corresponds to the test sample median and if we desire a (maximum) false alarm rate (FAR) of 0 0027 we set $1-P_0 = 0\ 0027$ so that the (desired) probability of no signal when the process is actually in control i e P_0 is $1 - 0\ 0027 = 0\ 9973$

From Table 5 1 we obtain $a = 1$ and $b = 50$ with $P_C\left(X_{(1\ 50)} \le Y_{(3\ 5)} \le X_{(50\ 50)}\right) = 0\ 99924$ Thus the distribution free control chart is given by the $1'$ and the 50^{th} reference sample order statistics in other words $\widehat{LCL} = X_{(1\ 50)}$ and $\widehat{UCL} = X_{(50\ 50)}$ with the exact or the actual obtained false alarm rate (FAR) equal to 0 00076

			m			
$1-P$	n	j	50	100	500	1000
0 01	5	3	(3 48) (0 00359 0 00359) 0 00719	(7 94) (0 00409 0 00409) 0 00819	(40 461) (0 00477 0 00477) 0 00954	(82 919) (0 00499 0 00499) 0 00997
	11	6	(7 44) (0 00464 0 00464) 0 00928	(15 86) (0 00428 0 00428) 0 00856	(83 418) (0 00494 0 00494) 0 00989	(167 834) (0 00487 0 00487) 0 00975
	25	13	(10 41) (0 00306 0 00306) 0 00611	(23 78) (0 00402 0 00402) 0 00804	(127 374) (0 00473 0 00473) 0 00945	(258 743) (0 00497 0 00497) 0 00994
0 005	5	3	(2 49) (0 00148 0 00148) 0 00296	(5 96) (0 00176 0 00176) 0 00352	(31 470) (0 00233 0 00233) 0 00467	(64 937) (0 00246 0 00246) 0 00493
	11	6	(5 46) (0 00125 0 00125) 0 00251	(13 88) (0 00225 0 00225) 0 00449	(72 429) (0 0024 0 0024) 0 00481	(146 855) (0 00244 0 00244) 0 00488
	25	13	(9 42) (0 00157 0 00157) 0 00313	(21 80) (0 002 0 002) 0 00401	(118 383) (0 00243 0 00243) 0 00487	(239 762) (0 00245 0 00245) 0 00491
0 0027	5	3	(1 50) (0 00038 0 0038) 0 00076	(4 97) (0 00102 0 00102) 0 00204	(25 476) (0 00127 0 00127) 0 00255	(51 950) (0 00129 0 00129) 0 00257
	11	6	(5 46) (0 00125 0 00125) 0 00251	(11 90) (0 00106 0 00106) 0 00212	(64 437) (0 00131 0 00131) 0 00262	(130 871) (0 00133 0 00133) 0 00266
	25	13	(8 43) (0 0074 0 0074) 0 00149	(19 82) (0 00092 0 00092) 0 00184	(110 391) (0 00127 0 00127) 0 00254	(224 777) (0 00133 0 00133) 0 00265

Table 5 1 The control chart constants a and b for future test sample medians

Note that the *achieved* false alarm rate of 0 00076 is much smaller (approximately 255% less) than the nominal or specified value of 0 0027 However because of the discrete in control probability distribution of W_j it is possible that the *specified* false alarm rate (FAR) will not be *exactly* achieved Therefore in such a situation we follow a **conservative strategy** as given in expression (5 7) That is the chart constants a and b are found in such a way that the actual probability of no signal i e $P_C\left(a \le W_j \le b-1\right)$ is *larger* than or equal to the desired probability of no signal P_0 Alternatively a

and b are found in such a way that the actual probability of a false alarm is *less* than or equal to the desired or specified false alarm rate (FAR) i e $1-P_0$

In addition due to the discreteness of the statistic W_j not all desired values of P_0 might be available for all combinations of m n and j especially when m and/or n are small But this is a common occurrence when dealing with non parametric or distribution free procedures whether used for usual hypothesis testing or for setting up confidence intervals However for the larger values of m and/or n this is less likely to happen For instance if $m=1000$ and $n=25$ Table 5 1 yields $a=224$ and $b=777$ so that $\widehat{LCL}=X_{(224\ 1000)}$ and $\widehat{UCL}=X_{(777\ 1000)}$ with the actual obtained false alarm rate equal to 0 00265 which is much closer to the nominal value of 0 0027

Example 5 1
A two sided precedence control chart for the median of future test samples

Table 5 2 displays $m=100$ reference observations (arranged in ascending order) from an unknown continuous process distribution Suppose that we need to monitor the location of the process distribution using the sample medians of future test samples each of size $n=5$ In addition suppose that the maximum allowable or the specified false alarm rate (FAR) $1-P_0$ is equal to 0 01 so that $P_0=0\ 99$ in equation (5 7) Hence using Table 5 1 we find that $a=7$ and $b=100-7+1=94$ so that the distribution free precedence control chart is given by the 7^{th} and the 94^{th} reference sample order statistics Thus from Table 5 2 we find that $\widehat{LCL}=X_{(7\ 100)}=4\ 46$ and $\widehat{UCL}=X_{(94\ 100)}=9\ 60$

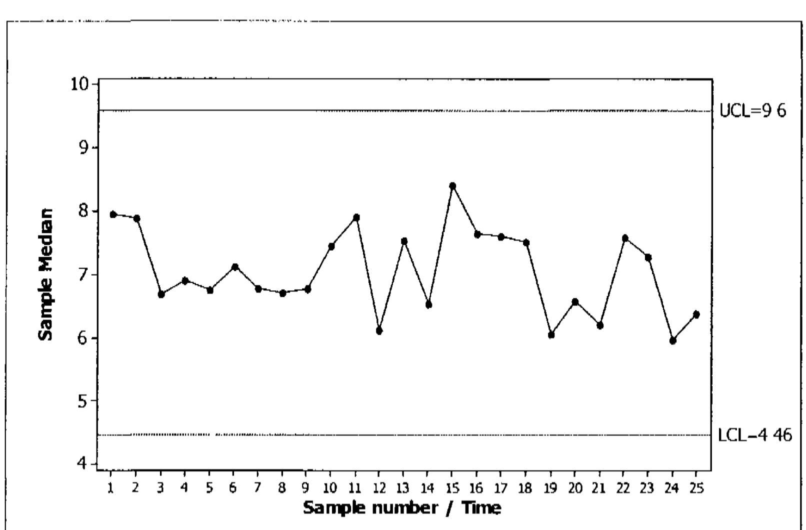

Figure 5 2 Precedence control chart for the sample median

i	$X_{(i:100)}$	i	$X_{(i:100)}$	i	$X_{(i:100)}$	i	$X_{(i:100)}$	i	$X_{(i:100)}$
1	3 13	21	5 59	41	6 45	61	7 32	81	8 30
2	3 72	22	5 62	42	6 46	62	7 37	82	8 31
3	3 82	23	5 63	43	6 51	63	7 48	83	8 35
4	4 23	24	5 73	44	6 52	64	7 67	84	8 64
5	4 34	25	5 74	45	6 53	65	7 68	85	8 67
6	4 39	26	5 77	46	6 55	66	7 70	86	8 80
7	4 46	27	5 84	47	6 64	67	7 73	87	8 86
8	4 58	28	5 90	48	6 65	68	7 81	88	8 91
9	4 66	29	5 96	49	6 72	69	7 84	89	8 96
10	4 72	30	6 02	50	6 87	70	7 89	90	9 01
11	4 83	31	6 02	51	6 87	71	7 93	91	9 17
12	4 94	32	6 05	52	6 95	72	7 96	92	9 17
13	5 08	33	6 15	53	6 96	73	8 01	93	9 49
14	5 08	34	6 21	54	7 00	74	8 04	94	9 60
15	5 20	35	6 23	55	7 04	75	8 07	95	9 83
16	5 33	36	6 32	56	7 11	76	8 14	96	9 88
17	5 37	37	6 36	57	7 17	77	8 14	97	9 96
18	5 42	38	6 39	58	7 20	78	8 24	98	10 29
19	5 46	39	6 43	59	7 20	79	8 24	99	10 31
20	5 53	40	6 44	60	7 21	80	8 29	100	10 56

Table 5 2 Phase 1 data for the precedence control chart

To monitor the process 25 test samples each of size $n = 5$ was obtained and the sample medians $Y_{(3\ 5)}$ recorded These values are displayed in panel (a) and column (b) of Table 5 3 with the distribution free two sided precedence control chart shown in Figure 5 2 Since there is no lack of statistical control no corrective action is necessary and process monitoring will continue

To summarize to construct and set up a distribution free precedence control chart the steps to follow are

Step 1 Start with a reference sample of size m when the process is thought to operate in control

Step 2 Select the size of a future test sample i e select n Then select the $100\left(\frac{j}{n}\right)^{th}$ test sample percentile to be monitored for example the sample median as was the case in Example 5 1 This determines j for a given n

Step 3 Specify the desired false alarm rate (FAR) i e specify $1-P_0$ Then using expression (5 6) find the chart constants a and b so that the *estimated* control limits are given by $\widehat{LCL} = X_{(\ m)}$ and $\widehat{UCL} = X_{(b \ m)}$ respectively

Step 4 Start collecting test samples each of size n Calculate or find the value of the j^{th} order statistic for each test sample and plot these values on the precedence control chart If any of the values plot outside the estimated control limits the process might not be in statistical control and a search for assignable causes should be started

			(a)			(b)
i	Y	Y	Y	Y	Y	$Y_{(\ )}$
1	7 97	8 08	8 62	6 82	6 56	7 97
2	8 80	7 89	2 82	8 56	7 13	7 89
3	6 70	7 26	6 51	8 00	5 61	6 70
4	3 82	5 85	7 32	7 47	6 92	6 92
5	6 77	5 13	7 13	5 96	8 25	6 77
6	7 37	8 18	4 53	7 14	6 87	7 14
7	6 01	6 78	9 14	7 85	4 10	6 78
8	6 44	5 26	6 72	7 38	7 25	6 72
9	6 87	5 19	6 78	5 69	7 37	6 78
10	6 54	7 46	8 60	7 25	8 14	7 46
11	8 66	7 92	6 54	6 18	7 97	7 92
12	4 43	4 20	6 34	6 24	6 13	6 13
13	7 35	7 54	10 28	7 69	6 30	7 54
14	4 54	9 60	5 66	6 55	6 85	6 55
15	8 43	7 32	5 65	9 72	8 77	8 43
16	8 51	8 60	7 09	7 66	6 83	7 66
17	9 11	5 06	6 09	8 54	7 61	7 61
18	7 52	9 15	7 18	5 97	8 44	7 52
19	6 55	6 05	5 41	4 68	8 00	6 05
20	7 07	8 29	6 59	4 55	3 75	6 59
21	5 81	4 89	6 57	6 53	6 21	6 21
22	6 86	8 65	6 67	7 99	7 60	7 60
23	5 47	7 29	5 92	9 25	9 32	7 29
24	8 20	7 62	5 97	3 72	5 83	5 97
25	6 26	7 98	6 38	6 09	7 45	6 38

Table 5 3 Phase 2 data for the precedence control chart

5 2 6 One-sided precedence control charts

One sided control charts are especially useful in situations where only an upward shift or only a downward shift in a particular process parameter is of interest For example we might be monitoring the breaking strength of a cable used for supporting a heavy weight or we might be monitoring the (remaining or future) lifetime of a light bulb

In the first scenario if the breaking strength of the cable decreases it might not be able to carry or hold the weight and subsequently may become a hazard However in contrast if the breaking strength of the cable increases i e shifts upward it will be a benefit since then the cable will be able to support a heavier weight than originally required

In the second scenario if the average lifetime of the light bulb is less for instance than the advertised average lifetime more customers might decide not to buy the particular brand and consequently might lead to a loss in revenue for the manufacturer Therefore it would be wise to guard against a downward shift in the average lifetime of a light bulb

In both of the above scenarios we are interested only in detecting a downward shift in a process parameter with no real interest in an upward shift Therefore a one sided lower control chart will be sufficient On the other hand a situation where only an upward shift might be of any real importance is for example the concentration of an abrasive

The required adjustment from the two sided distribution free control chart procedure to a one sided control procedure is relatively simple For example if detecting only an ***upward*** **shift** is of interest that is whether a process parameter has shifted to the *right* we only use $\widehat{UCL} = X_{(b\ m)}$ as an estimated upper control limit with no lower control limit Here the chart constant b is found such that $P_C\left(Y_{(j\ )} \geq X_{(b\ m)}\right) \leq 1 - P_0$ which is the same as requiring that $P_C\left(b \leq W_j \leq m\right) \leq 1 - P_0$

However if only detecting a ***downward*** **shift** is of interest i e whether a process parameter has shifted to the *left* we only use $\widehat{LCL} = X_{(\ m)}$ as an estimated lower control limit with no upper control limit and subsequently the chart constant a is found such that $P_C\left(Y_{(j\ )} \leq X_{(\ m)}\right) \leq 1 - P_0$ which is again the same condition as requiring that $P_C\left(0 \leq W_j \leq a-1\right) \leq 1 - P_0$

If it happens that the size of a future test sample is *odd* and if $Y_{(j\ )}$ is the sample *median* the distribution of W_j will be *symmetric* with the sample median uniquely determined If this is the case the condition for either one sided control chart i e $P_C\left(b \le W_j \le m\right) \le 1 - P_0$ for the upper one sided chart or $P_C\left(0 \le W_j \le a-1\right) \le 1 - P_0$ for the lower one sided chart with $b = m - a + 1$ is the same as the condition for the two sided control chart with the right hand side merely doubled i e

$$P_C\left(0 \le W_j \le a-1\right) \le \frac{1-P_0}{2}$$

Hence in general we can use either expression (5 7) or (5 8) which were used for the construction of Table 5 1 to obtain the control limits of a one sided control chart by setting

$$P_0^{(2)} = 1 - 2\left(1 - P_0^{(1)}\right) \quad (5\ 9)$$

where $P_0^{(2)}$ denotes the probability of no signal when the two sided control chart is actually in control and $P_0^{(1)}$ denotes the probability of no signal when either of the one sided control charts is actually in control

For example suppose that $P_0^{(1)} = 0\ 995$ that is suppose that the probability of no signal on a one sided control chart when the process is actually in control is specified to be 0 995 Then using equation (5 9) we find that $P_0^{(2)} = 1 - 2\left(1 - 0\ 995\right) = 0\ 99$ so that $1 - P_0^{(2)} = 0\ 01$ Hence if we intend to set up a lower one sided precedence control chart using $m = 50$ reference observations and future test samples of size $n = 11$ with a specified false alarm rate (FAR) of $1 - P_0^{(1)} = 0\ 005$ we find from Table 5 1 that $\widehat{LCL} = X_{(7\ 50)}$ with no upper control limit On the other hand for a upper one sided precedence control chart we find that $\widehat{UCL} = X_{(44\ 50)}$ with no lower control limit [1]

5 2 7 The run length distribution

To evaluate the performance of the two sided precedence control chart we need to derive and study its run length distribution However since we use a reference sample to estimate the control limits the control limits are subject to variability Hence we need to accommodate or account for the variability in $\widehat{LCL} = X_{(\ m)}$ and $\widehat{UCL} = X_{(b\ m)}$ To do this we proceed as in Chapters 3 and 4 that is we condition on the two observed values $x_{(\ m)}$ and $x_{(b\ m)}$ of the two reference sample order statistics $X_{(\ m)}$ and

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[1] To construct a **one sided** chart for the **test sample median** it is the value of $1 - P^{(\ )}$ and not that of $1 - P^{(\ )}$ that needs to be found in Table 5 1 in the column labeled $1 - P_0$

$X_{(b\ m)}$ to first find the *conditional* run length distribution and its associated properties Then using expectation by conditioning we find the *unconditional* run length distribution and its associated characteristics This presents the overall or the general effect of the estimation procedure on the performance of the two sided precedence control chart Finally having found the *unconditional* run length distribution and its properties we compare the performance of the two sided distribution free precedence control chart with the standard parametric CUSUM and EWMA control charts

The *conditional* run length distribution of the two sided precedence control chart

Given the observed values $x_{(\ m)}$ and $x_{(b\ m)}$ of the two reference sample order statistics $X_{(\ m)}$ and $X_{(b\ m)}$ the signaling events in Phase 2 of the control procedure are mutually independent and subsequently the run length random variable of the two sided precedence control chart follows a geometric distribution with parameter $1-p$ Here p is the *conditional* probability of no signal (derived using the probability integral transformation (PIT)) and given by

$$\begin{aligned} p &= P\left(x_{(\ m)} \le Y_{(j\ )} \le x_{(b\ m)} \mid X_{(\ m)} = x_{(\ m)}\ X_{(b\ m)} = x_{(b\ m)}\right) \\ &= P\left(F^{\ 1}\left(u_{(\ m)}\right) \le G^{\ 1}\left(U_{(j\ )}\right) \le F^{\ 1}\left(u_{(b\ m)}\right) \mid U_{(\ m)} = u_{(\ m)}\ U_{(b\ m)} = u_{(b\ m)}\right) \\ &= P\left(GF^{\ 1}\left(u_{(\ m)}\right) \le U_{(j\ )} \le GF^{\ 1}\left(u_{(b\ m)}\right) \mid U_{(\ m)} = u_{(\ m)}\ U_{(b\ m)} = u_{(b\ m)}\right) \\ &= \int_{GF\ (\ (\ m))}^{GF\ (\ (\ m))} \frac{1}{\beta(j\ n-j+1)} u^{j-1}(1-u)^{\ -j} du \end{aligned} \quad (5\ 10)$$

where $u_{(\ l)}$ is the observed value of the e^{th} order statistic $U_{(\ l)}$ in a sample of size l from a $Uniform(0\ 1)$ distribution G is the cumulative distribution function of the i^{th} test sample F is the cumulative distribution function of the reference sample from Phase 1 and

$\frac{1}{\beta(j\ n-j+1)} u^{j\ 1}(1-u)^{\ j}$ is the probability density function of $U_{(j\ )}$

In other words $\left(N \mid X_{(\ m)} = x_{(\ m)}\ X_{(b\ m)} = x_{(b\ m)}\right) \sim Geo(1-p\ )$ so that the *conditional* probability mass function (pmf) of the run length random variable is given by

$$P\left(N=k \mid X_{(\ m)} = x_{(\ m)}\ X_{(b\ m)} = x_{(b\ m)}\right) = p^{k\ 1}(1-p\ ) = p^{k\ 1} - p^k \quad \text{for} \quad k=1\ 2\ 3 \quad (5\ 11)$$

Hence the *conditional* cumulative distribution function (cdf) is

$$P\left(N \le k \mid X_{(\ m)} = x_{(\ m)}\ X_{(b\ m)} = x_{(b\ m)}\right) = 1-p^k \quad \text{for} \quad k=1\ 2\ 3 \quad (5\ 12)$$

and in addition the *conditional* average run length is

$$ARL = E\left(N \mid X_{(\ m)} = x_{(\ m)}\ X_{(b\ m)} = x_{(b\ m)}\right) = \frac{1}{1-p} \quad (5\ 13)$$

Note that in this case the subscript c indicates that p is a *conditional* probability which is different from the subscript C used in expression (5 2) to indicate the *in control* probability distribution of W_j

Example 5 2

Conditional run length distribution of the two sided precedence control chart of Example 5 1

The control chart of Example 5 1 was established using $m = 100$ reference observations and it was found that $a = 7$ and $b = 94$ so that $\widehat{LCL} = X_{(7\,100)} = 4\ 46$ and $\widehat{UCL} = X_{(94\,100)} = 9\ 60$ Hence *conditioning* on these observed values i e $x_{(7\,100)} = 4\ 46$ and $x_{(94\,100)} = 9\ 60$ expression (5 10) yields

$$p\ = P\left(4\ 46 \le Y_{(3\,5)} \le 9\ 60 \mid X_{(7\,100)} = 4\ 46\ X_{(94\,100)} = 9\ 60\right)$$

Then if we assume that the process operates **in control** in Phase 2 that is if $G_Y = F_X$ it is found from Table 5 1 that $1-p\ = 0\ 00819$

Hence the *in control conditional* probability mass function (pmf) is given by

$$P\left(N_0 = k \mid X_{(7\,100)} = 4\ 46\ X_{(94\,100)} = 9\ 60\right) = 0\ 99181^{k\ 1}\left(0\ 00819\right) \quad \text{for} \quad k = 1\ 2\ 3$$

whereas the *in control conditional* cumulative distribution function (cdf) is given by

$$P\left(N_0 \le k \mid X_{(7\,100)} = 4\ 46\ X_{(94\,100)} = 9\ 60\right) = 1 - 0\ 99181^{k} \quad \text{for} \quad k = 1\ 2\ 3$$

and the *in control conditional* average run length is

$$ARL_0 = E\left(N \mid X_{(7\,100)} = 4\ 46\ X_{(94\,100)} = 9\ 60\right) = \frac{1}{1 - 0\ 99181} = 122\ 10$$

On the other hand if we **cannot assume that the process operates in control** in Phase 2 that is if we cannot assume that $G_Y = F_X$ but instead we can assume that $F_X(x) = F(x-\theta)$ and $G_Y(y) = F(y-\theta\ )$ with $\theta\ \ i = m+1\ m+2$ which is not necessarily equal to θ we can proceed as follows

First we scrutinize the 100 reference observations $X_1\ X_2\ \ \ X_{100}$ to find a suitable parametric probability distribution that is we first find $F(\ )$

Figure 5 3 is a normal probability plot of the 100 reference observations from Phase 1 Studying Figure 5 3 along with the Anderson Darling (AD) test statistic value of 0 192 and its associated p value of 0 895 we have no significant evidence to reject the null hypothesis that the 100 reference observations are from a normal distribution Consequently we assume that in Phase 1 the process followed a normal distribution with a mean of 6 939 and a standard deviation of 1 629 where the reference sample mean i e $\bar{x} = 6\ 939$ and the reference sample standard deviation i e $s = 1\ 629$ were used to calculate the cumulative probabilities on the horizontal axis of the normal probability plot in Figure 5 3

Having found that $F(x) = \Phi\left(\frac{x-\theta}{1\ 629}\right)$ with $\theta = 6\ 939$ and where Φ is the cumulative distribution function (cdf) of the standard normal distribution we assume that in Phase 2 $G_Y(y) = \Phi\left(\frac{y-\theta}{1\ 629}\right)$ $i = m+1\ m+2$ that is we assume that the process continues to follow a normal distribution with only a possibility of a change in the process s location but no likelihood of a change in the process variation Then we estimate θ using $\bar{y} = \frac{1}{5}(y_1 + y_2 + \quad + y_5)$ and calculate the *conditional (out of control)* probability of no signal p for a future test sample

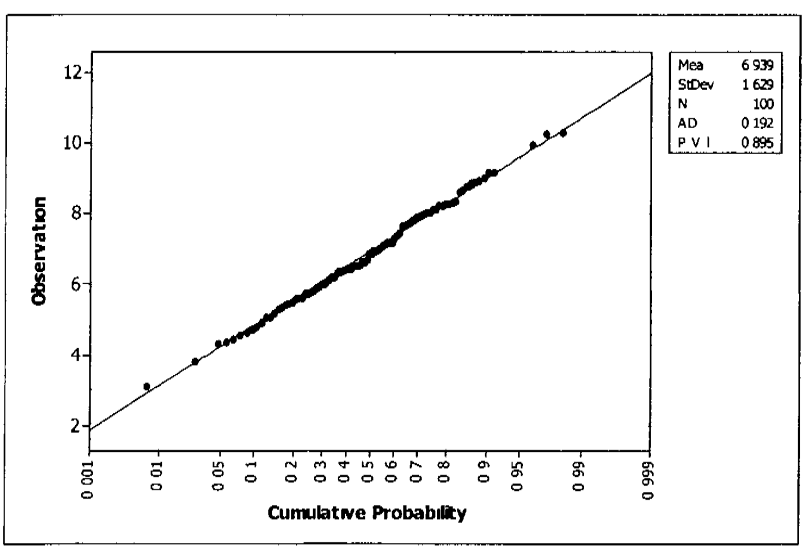

Figure 5 3 Probability plot of the 100 reference observation from Phase 1

For instance for test sample number 1 we find that $\bar{y}_1 = \frac{1}{5}(7\ 97 + 8\ 08 + 8\ 62 + 6\ 82 + 6\ 56) = 7\ 61$ and subsequently using equation (5 10) along with the fact that the cumulative distribution function (cdf) of $Y_{(3\ 5)}$ is given by

$$P\left(Y_{(3\ 5)} \le t\right) = \sum_{3}^{5} \binom{5}{i} \Phi\left(\frac{t - 7\ 61}{1\ 629}\right) \left(1 - \Phi\left(\frac{t - 7\ 61}{1\ 629}\right)\right)^{5} \quad \text{for } -\infty < t < \infty$$

where Φ denotes the cumulative distribution function (cdf) of the standard normal distribution we find that

$$\begin{aligned} p &= P\left(4\ 46 \le Y_{(3\ 5)} \le 9\ 60 \mid X_{(7\ 100)} = 4\ 46\ X_{(94\ 100)} = 9\ 60\right) \\ &= \sum_{-3}^{5} \binom{5}{i} \Phi\left(\frac{9\ 60 - 7\ 61}{1\ 629}\right) \left(1 - \Phi\left(\frac{9\ 60 - 7\ 61}{1\ 629}\right)\right)^{5} - \sum_{3}^{5} \binom{5}{i} \Phi\left(\frac{4\ 46 - 7\ 61}{1\ 629}\right) \left(1 - \Phi\left(\frac{4\ 46 - 7\ 61}{1\ 629}\right)\right)^{5} \\ &= 0\ 98852 - 0\ 00018 \\ &= 0\ 98834 \end{aligned}$$

Thus if the process mean changed from the estimated value of 6 939 in Phase 1 to 7 61 (as estimated using the first test sample) the probability of a signal increased from the in control value of 0 00819 to $1 - 0\ 98834 = 0\ 01166$ Then if we further assume that the process mean remains at 7 61 the *out of control conditional* probability mass function (pmf) is

$$P\left(N_1 = k \mid X_{(7\ 100)} = 4\ 46\ X_{(94\ 100)} = 9\ 60\right) = 0\ 98834^{k\ 1}(0\ 01166) \quad \text{for} \quad k = 1\ 2\ 3$$

and the *out of control conditional* average run length is

$$ARL_1 = E\left(N \mid X_{(7\ 100)} = 4\ 46\ X_{(94\ 100)} = 9\ 60\right) = \frac{1}{1 - 0\ 98834} = 85\ 76$$

However note that although it is unlikely that the process mean will remain (exactly) fixed at 7 61 For instance at sample number two $\bar{y}_2 = 7\ 04$ which is (already) not equal to 7 61 so that the above example and/or assumption illustrates the use and the interpretation of the *out of control conditional* run length and its associated characteristics

The *unconditional* run length distribution of the two sided precedence control chart

Having obtained the *conditional* run length distribution of the two sided precedence control chart i e $P\left(N=k \mid X_{(\ m)}=x_{(\ m)}\ X_{(b\ m)}=x_{(b\ m)}\right)$ the *unconditional* run length distribution i e $P(N=k)$ and its characteristics can easily be found using expectation by conditioning

For example the *unconditional* probability mass function (pmf) is

$$P(N=k)=E_{X_{(\ )}\ X_{(\ )}}\left(P\left(N=k \mid X_{(\ m)}=x_{(\ m)}\ X_{(b\ m)}=x_{(b\ m)}\right)\right) \quad (5\ 14)$$

which simplifies to

$$P(N=k)=E_{X_{(\ )}\ X_{(\ m)}}\left(p^{k\ 1}\right)-E_{X_{(\ m)}\ X_{(\ m)}}\left(p^{k}\right)=D\ (k-1)-D\ (k) \quad \text{for} \quad k=1\ 2\ 3 \quad (5\ 15)$$

with

$$D\ (k)=\int_0^1\int_0^t\left(\frac{1}{\beta(j\ n-j+1)}\sum_{h\ 0}^{j}\frac{(-1)^h}{j+h}\binom{n-j}{h}\left\{GF^{\ 1}(t)^{j\ h}-GF^{\ 1}(s)^{j\ h}\right\}\right)^k f(s\ t)\,dsdt \quad (5\ 16)$$

and $D\ (0)=1$ Here $f(s\ t)=\frac{m!}{(a-1)!(b-a-1)!(m-b)!}s^{\ 1}(t-s)^{b\ \ 1}(1-t)^{m-b}$ is the joint probability density function (pdf) of the two reference sample order statistics $U_{(\ m)}$ and $U_{(b\ m)}$ and as before G is the cumulative distribution function of the i^{th} test sample and F is the cumulative distribution function of the reference sample from Phase 1
(See Result 5 24 in Appendix 5 for the step by step derivation of expression (5 16))

The *in control unconditional* run length distribution is obtained by setting $G=F$ in expression (5 16) so that expression (5 15) becomes

$$P(N_0=k)=D(k-1)-D(k) \quad \text{for} \quad k=1\ 2\ 3 \quad (5\ 17)$$

with expression (5 16) changing into

$$D(k)=\int_0^1\int_0^t\left(\frac{1}{\beta(j\ n-j+1)}\sum_{h\ 0}^{-j}\frac{(-1)^h}{j+h}\binom{n-j}{h}\left\{t^{j\ h}-s^{j\ h}\right\}\right)^k f(s\ t)\,dsdt \quad (5\ 18)$$

and $D(0)=1$

To find the *out of control unconditional* average run length we start by re writing the *conditional* average run length given in equation (5 13)

By definition we have that

$$ARL = E\left(N \mid X_{(\ m)} = x_{(\ m)}\ X_{(b\,m)} = x_{(b\,m)}\right) = \sum_{k\ 1}^{\infty} kP\left(N = k \mid X_{(\ m)} = x_{(\ m)}\ X_{(b\,m)} = x_{(b\,m)}\right)$$

Then by first expanding the summation over k and then simplifying the expression once again we find that

$$ARL = \sum_{k\ 0}^{\infty} P\left(N > k \mid X_{(\ m)} = x_{(\ m)}\ X_{(b\,m)} = x_{(b\,m)}\right)$$

which in turn using the *conditional* cumulative distribution function given in equation (5 12) simply becomes

$$ARL = \sum_{k\ 0}^{\infty} p^{k} \qquad (5\ 19)$$

Finally to obtain the *out of control unconditional* average run length (of the two sided precedence control chart) we use the law of conditional expectation i e

$$ARL^{(2)} = \sum_{k\ 0}^{\infty} E_{X_{(\ m)}\, X_{(\ m)}}\left(p^{k}\right)$$

which simplifies to

$$ARL^{(2)} = \int_0^1 \int_0^1 \frac{1}{1 - F_2\left(s\ t\ j\ n\ F\ G\right)} f\left(s\ t\right) ds dt \qquad (5\ 20)$$

with

$$F_2\left(s\ t\ j\ n\ F\ G\right) = \left(\frac{1}{\beta\left(j\ n-j+1\right)} \sum_{h\ 0}^{j} \frac{(-1)^h}{j+h} \binom{n-j}{h} \left\{GF^{\ 1}(t)^{j\ h} - GF^{\ 1}(s)^{j+h}\right\}\right) \qquad (5\ 21)$$

(See Result 5 24 in Appendix 5 for the step by step derivation of expressions (5 20) and (5 21))

Then as before the *in control unconditional* average run length is obtained by setting $G = F$ in expression (5 21) so that expression (5 20) becomes

$$ARL_0^{(2)} = \int_0^1 \int_0^1 \frac{1}{1 - C_2\left(s\ t\ j\ n\right)} f\left(s\ t\right) ds dt \qquad (5\ 22)$$

with expression (5 21) becoming

$$C_2\left(s\ t\ j\ n\right) = \left(\frac{1}{\beta\left(j\ n-j+1\right)} \sum_{h\ 0}^{j} \frac{(-1)^h}{j+h} \binom{n-j}{h} \left\{t^{j\ h} - s^{j\ h}\right\}\right) \qquad (5\ 23)$$

The *conditional* run length distribution of a one sided precedence control chart

For both the one sided control charts we can proceed along the same lines to find the *conditional* as well as the *unconditional* run length distributions and their associated properties However for simplicity and ease of presentation we only consider the upper one sided control chart and refer the reader to Appendix 5 for the derivation and/or the results of the lower one sided control chart

Recall that the upper one sided control chart is given by $\widehat{UCL} = X_{(b\ m)}$ with no lower control limit Here given an observed value $x_{(b\ m)}$ of the reference sample order statistic $X_{(b\ m)}$ the run length random variable N follows a geometric distribution with parameter $1-p$ where

$$\begin{aligned} 1-p &= P\left(Y_{(j\ \ )} > x_{(b\ m)} \mid X_{(b\ m)} = x_{(\ \ m)}\right) \\ &= 1 - P\left(G^{-1}\left(U_{(j\ \ )}\right) \le F^{-1}\left(u_{(b\ m)}\right) \mid U_{(b\ m)} = u_{(\ \ m)}\right) \\ &= 1 - P\left(U_{(j\ \ )} \le GF^{\ 1}\left(u_{(b\ m)}\right) \mid U_{(b\ m)} = u_{(\ \ m)}\right) \\ &= 1 - \int_0^{GF\ \ (\ (\ \ m))} \frac{1}{\beta(j\ n-j+1)} u^{j\ 1}(1-u)^{\ \ j} du \end{aligned} \qquad (5\ 24)$$

is the *conditional* probability of a signal i e the *conditional* probability of a success (See Result 5 2 in Appendix 5)

Thus the *conditional* probability mass function (pmf) of the upper one sided precedence chart is

$$P\left(N = k \mid X_{(b\ m)} = x_{(b\ m)}\right) = p^{k\ 1}\left(1-p\ \ \right) = p^{k\ 1} - p^k \quad \text{with } k = 1\ 2\ 3 \qquad (5\ 25)$$

whereas the *conditional* cumulative distribution function (cdf) is

$$P\left(N \le k \mid X_{(b\ m)} = x_{(b\ m)}\right) = \sum_{1}^{k} p^{\ \ 1}\left(1-p\ \ \right) = 1 - p^k \qquad (5\ 26)$$

so that

$$P\left(N > k \mid X_{(b\ m)} = x_{(b\ m)}\right) = 1 - \left(1 - p^k\ \ \right) = p^k$$

In addition the expected value of N given $X_{(b\ m)} = x_{(b\ m)}$ or the *conditional* average run length is

$$ARL = E\left(N \mid X_{(b\ m)} = x_{(b\ m)}\right) = \frac{1}{1-p} \qquad (5\ 27)$$

Note that here the subscripts in p indicate that it is a conditional (c) probability of the upper (u) one sided control chart

The *unconditional* run length distribution of a one sided precedence control chart

Proceeding along the same lines that is using expectation by conditioning we find the *unconditional* run length distribution of the upper one sided control chart to be given by

$$P(N=k)=E_{X_{(\ )}}\left(P\left(N=k\mid X_{(b\ m)}=x_{(b\ m)}\right)\right) \qquad (5\ 28)$$

which implies to

$$P(N=k)=E_{X_{(\ m)}}\left(p^{k-1}\right)-E_{X_{(\ m)}}\left(p^{k}\ \right)=D\ (k-1)-D\ (k) \text{ for } k=1\ 2\ 3 \qquad (5\ 29)$$

with

$$D\ (k)=\int_0^1\left(\frac{1}{\beta(j\ n-j+1)}\sum_{h\ 0}^{-j}\frac{(-1)^h}{j+h}\binom{n-j}{h}\left\{GF^{-1}(t)\right\}^{j\ h}\right)^k\frac{m!}{(b-1)!(m-b)!}t^{b\ 1}(1-t)^{m\ b}\,dt \qquad (5\ 30)$$

and $D\ (0)=1$

(See Result 5 8 in Appendix 5 for the step by step derivation of expression (5 30))

If we set $G=F$ in expression (5 30) we obtain the *in control unconditional* run length distribution i e

$$P(N_0=k)=D\ (k-1)-D\ (k) \text{ for } k=1\ 2\ 3 \qquad (5\ 31)$$

with

$$D\ (k)=\int_0^1\left(\frac{1}{\beta(j\ n-j+1)}\sum_{h=0}^{-j}\frac{(-1)^h}{j+h}\binom{n-j}{h}\{t\}^{j\ h}\right)^k\frac{m!}{(b-1)!(m-b)!}t^{b\ 1}(1-t)^{m\ b}\,dt \qquad (5\ 32)$$

To obtain the *unconditional* average run length of the upper one sided control chart we follow the same process as for the two sided control chart That is we re write the *conditional* average run length which is (by definition) equal to

$$ARL=E\left(N\mid X_{(b\ m)}=x_{(b\ m)}\right)=\sum_{k\ 1}^{\infty}kP\left(N=k\mid X_{(b\ m)}=x_{(b\ m)}\right) \qquad (5\ 33)$$

as

$$ARL=\sum_{k\ 0}^{\infty}p^{k} \qquad (5\ 34)$$

where p is the *conditional* probability of no signal given in expression (5 24)

Then to find the *out of control unconditional* average run length of the upper one sided chart we (once again) use the law of conditional expectation i e

$$ARL^{(1)} = \sum_{k=0}^{\infty} E_{X_{( )}}\left(p^{k}\right) = \sum_{k=0}^{\infty} D\ (k) \tag{5 35}$$

which simplifies to

$$ARL^{(1)} = \int_0^1 \frac{1}{1-F_1\ (t\ j\ n\ F\ G)} f(t)\,dt \tag{5 36}$$

with

$$F_1\ (t\ j\ n\ F\ G) = \left(\frac{1}{\beta(j\ n-j+1)} \sum_{h=0}^{j} \frac{(-1)^h}{j+h}\binom{n-j}{h}\left\{GF^{\ 1}(t)\right\}^{j\ h}\right) \tag{5 37}$$

The *in control unconditional* average run length is then found by substituting $G = F$ in expression (5 37) i e

$$ARL_0^{(1)} = \int_0^1 \frac{1}{1-C_1\ (t\ j\ n)} f(t)\,dt \tag{5 38}$$

where

$$C_1\ (t\ j\ n) = \left(\frac{1}{\beta(j\ n-j+1)} \sum_{h=0}^{-j} \frac{(-1)^h}{j+h}\binom{n-j}{h}\{t\}^{j\ h}\right) \tag{5 39}$$

5 2 8 The performance of the two-sided precedence control chart

As we ve noted before the run length distribution is of interest when studying the performance of the precedence control chart For instance if the process mean has shifted from its in control value we might be interested in what the probability is that the control chart will detect the shift or signal on the first test sample or on the second test sample or more generally on the k^{th} subsequent test sample However as we have seen the *unconditional* distribution of N is not geometric Instead expressions (5 15) and (5 16) i e

$$P(N=k)=E_{X_{(\ m)}X_{(\ m)}}\left(p^{k-1}\right)-E_{X_{(\ )}X_{(\ m)}}\left(p^{k}\right)=D\ (k-1)-D\ (k) \quad \text{for} \quad k=1\ 2\ 3$$

with

$$D\ (k)=\int_0^1\int_0^t\left(\frac{1}{\beta(j\ n-j+1)}\sum_{h=0}^{j}\frac{(-1)^h}{j+h}\binom{n-j}{h}\left\{GF^{-1}(t)^{j+h}-GF^{-1}(s)^{j+h}\right\}\right)^k f(s\ t)\,dsdt$$

and $D\ (0)=1$ can be evaluated to find the entire run length distribution or any specific run length percentile of interest and answer questions as those listed above

In spite of this two (very) popular characteristics of the run length distribution is the average run length or the mean of the run length distribution and the probability of a signal or equivalently the probability of no signal Thus we rather concentrate on these characteristics as measures of the performance of the precedence control chart

The *unconditional* average run length

The *unconditional* average run length given in expression (5 20) is

$$ARL^{(2)}=\int_0^1\int_0^t\frac{1}{1-F_2(s\ t\ j\ n\ F\ G)}f(s\ t)\,dsdt$$

with

$$F_2(s\ t\ j\ n\ F\ G)=\left(\frac{1}{\beta(j\ n-j+1)}\sum_{h=0}^{j}\frac{(-1)^h}{j+h}\binom{n-j}{h}\left\{GF^{-1}(t)^{j+h}-GF^{-1}(s)^{j+h}\right\}\right)$$

From a practical point of view it is desirable that the in control average run length (ARL_0) i e when $G=F$ be high whereas the out of control average run length (ARL_1) i e when $G\neq F$ be low

Thus when it is preferred to design a two sided precedence control chart with a specified *in control* average run length (ARL_0) we can evaluate expression (5 22) i e

$$ARL_0^{(2)} = \int_0^1 \int_0^t \frac{1}{1-C_2(s\ t\ j\ n)} f(s\ t)\,dsdt$$

with

$$C_2(s\ t\ j\ n) = \left(\frac{1}{\beta(j\ n-j+1)} \sum_{h=0}^{j} \frac{(-1)^h}{j+h} \binom{n-j}{h} \left\{ t^{j+h} - s^{j+h} \right\} \right)$$

and solve for the chart constants a and b for given values of $ARL_0^{(2)}$ m n and j In principle this is not particularly difficult For instance we can use a searching algorithm (in which a and b varies) that stops when the *achieved* in control average run length is within a specified distance from the *specified* or *desired* in control average run length

For this method we need to note that the ARL is *increasing* in b and *decreasing* in a and in addition for a precedence control chart for the sample median we can find a and then set $b = m - a + 1$ which will give a unique solution However because of the complexity of this method we instead use the values of a and b presented in Table 5 1 and then calculate the in control (unconditional) average run length by evaluating expression (5 22) These values are shown in Table 5 4

			m			
$1-P$	n	j	50	100	500	1000
	5	3	635 7	214 9	114 5	104 6
0 01	11	6	642 2	245 0	113 3	108 4
	25	13	10990 0	510 8	128 3	109 8
	5	3	5671 0	678 4	242 3	215 1
0 005	11	6	9503 0	574 5	240 9	219 8
	25	13	44750 0	1488 0	261 0	227 5
	5	3	∞	1550 0	460 2	419 5
0 0027	11	6	9503 0	1630 0	456 1	409 8
	25	13	173700 0	5183 0	526 2	430 2

Table 5 4 The in control average run length (ARL) for the two sided precedence chart for the median of a future test sample[(1)]

(1) Table 5 1 and Table 5 4 should preferably be studied together

However it should be noted that yet another practical consideration for the implementation of the control chart is the fact that for both the two sided and the one sided control charts certain conditions on the chart constants a and b as well as on m n and j need to be satisfied in order for the in control average run length (ARL_0) to be **finite** These conditions are

(i) For the two sided control chart $ARL_0 < \infty \Leftrightarrow (a-j)(n-j+1)+j(m-b+1) > 0$

and

(ii) For the one sided control chart $ARL_0 < \infty \Leftrightarrow (m-b)-(n-j) > 0$

For example if we intend to construct a two sided precedence control chart for the sample median of future test samples of size $n = 5$ using $m = 50$ reference observations and with a specified false alarm rate of $1-P_0 = 0\ 0027$ Table 5 1 yields values of $a = 1$ and $b = 50$ with the exact false alarm rate equal to 0 00076 However for these values we find that $(a-j)(n-j+1)+j(m-b+1) = -3$ which is less than zero and therefore we find from Table 5 4 that $ARL_0 = \infty$

The *unconditional* probability of a signal

A second popular performance characteristic is the *unconditional* probability of a signal which is also found using the *conditional* probability of a signal and the laws of conditional expectation i e

$$P(\text{Signal}) = E_{X_{(a)}, X_{(b)}}(1-p) \qquad (5\ 40)$$

which simplifies to

$$P(\text{Signal}) = 1 - \int_0^1 \int_0^t \left(\frac{1}{\beta(j, n-j+1)} \sum_{h=0}^{j} \frac{(-1)^h}{j+h} \binom{n-j}{h} \left\{ GF^{-1}(t)^{j+h} - GF^{-1}(s)^{j+h} \right\} \right) f(s, t)\, ds\, dt \qquad (5\ 41)$$

(See Results 5 19 and 5 20 in Appendix 5 for a step by step derivation of expressions (5 40) and (5 41))

Here

$$p = \int_{GF(X_{(a)})}^{GF(X_{(b)})} \frac{1}{\beta(j, n-j+1)} u^{j-1} (1-u)^{n-j} du \qquad (5\ 42)$$

is the *conditional* probability of no signal derived as Result (5 17) in Appendix 5

The run length distribution and its properties under some specific alternatives

It is seen that the run length distribution and its associated properties depend in general on the two distribution functions F and G through the composite function $\psi = GF^{-1}$ Thus depending on the assumptions concerning F and G only then can we calculate the function ψ to evaluate the run length distribution and/or any of the performance measures For instance when the process is in control that is when $G = F$ so that $\psi(t) = t$ the in control run length distribution is obtained and given in expressions (5 17) and (5 18) However if $G \neq F$ the following results are very useful

(i) For **location** alternatives that is when the process follows the same parametric distribution in Phase 1 and Phase 2 with merely a shift in the location or the scale parameter let $F(x) = H(x - \theta_1)$ and $G(x) = H(x - \theta_2)$ where H is a continuous cumulative distribution function $x \in \Re$ and $\theta_1\ \theta_2 \in \Re$ so that $\psi(t) = H(\theta_1 - \theta_2 + H^{-1}(t))$

(ii) For **scale** alternatives that is when the process follows the same parametric distribution in Phase 2 as in Phase 1 but in this case with only a possibility of a change in the process variation or spread let $F(x) = H(\frac{x}{\gamma_1})$ and $G(x) = H(\frac{x}{\gamma_2})$ where H is a continuous cumulative distribution function $x \in \Re$ and $\gamma_1\ \gamma_2 \in \Re^+$ so that $\psi(t) = H(\frac{\gamma_1}{\gamma_2} H^{-1}(t))$

(iii) For **location scale** alternatives let $F(x) = H(\frac{x - \theta_1}{\gamma_1})$ and $G(x) = H(\frac{x - \theta_2}{\gamma_2})$ where H is a continuous cumulative distribution function $x \in \Re$ $\theta_1\ \theta_2 \in \Re$ $\gamma_1\ \gamma_2 \in \Re$ so that

$$\psi(t) = H(\frac{\theta_1 - \theta_2}{\gamma_2} + \frac{\gamma_1}{\gamma_2} H^{-1}(t))$$

Thus it is clear that the run length distribution can only be calculated and/or evaluated when the change from the in control to the out of control process distribution is known or in general if we know the relationship between G and F through the composite function ψ

For instance if the underlying process distribution is normal and we are only interested in a change in location result (i) gives $\psi(t) = \Phi(\theta_1 - \theta_2 + \Phi^{-1}(t))$ where θ_1 is the in control value of the location or the shift parameter θ in Phase 1 and θ_2 is the out of control value of the location or shift parameter in Phase 2

The robustness of the two sided precedence control chart

(i) The in control average run length

As pointed out earlier one of the nicest properties of non parametric or distribution free control charts is the fact that the *in control* average run length is independent of any parametric process distribution In contrast the *in control* average run length of the standard normal theory control charts such as the standard parametric Shewhart $\overline{X}$ control chart the CUSUM and the EWMA control charts are sometimes greatly affected by changes in the underlying process distribution

For example Table 5 5 displays the *in control* average run lengths of the two sided distribution free precedence control chart along with the two sided (parametric) CUSUM and the two sided (parametric) EWMA control charts – all based on the normal distribution with mean μ and variance σ^2 The CUSUM chart constants h and k as well as the EWMA control chart constants L and λ were chosen such that they are either good at detecting a small shift i e $\frac{1}{2}\sigma$ in this case or a large process shift i e 2σ – given that the assumption of normality is of course satisfied

Distribution	Precedence	CUSUM		EWMA	
	$m = 1000$	*Large shift*	*Small shift*	*Large shift*	*Small shift*
	$n = 5$ $j = 3$	$h = 4\ 61$	$h = 0\ 86$	$L = 2\ 81$	$L = 3\ 05$
	$P_0 = 0\ 0078$	$k = 0\ 56$	$k = 2\ 24$	$\lambda = 0\ 1$	$\lambda = 0\ 4$
	$a = 48$ $b = 953$	$n = 5$	$n = 5$	$n = 5$	$n = 5$
$N(0\ 1)$	501 89	500 00	500 00	500 00	500 00
$Laplace(0\ 1)$	501 89	390 64	196 18	418 77	289 82
$Unif(0\ 1)$	501 89	549 16	1820 13	494 73	703 72
$N(0\ 1\ 1)$	501 89	206 38	196 69	214 49	199 79
$Gamma(1\ 1)$	501 89	310 74	121 53	404 27	207 62
$Gamma(4\ 1)$	501 89	462 43	345 68	472 33	370 20
$t(4)$	501 89	266 86	131 90	380 98	190 39
$t(40)$	501 89	456 76	413 44	470 87	380 63

Table 5 5 The in control average run length values for several distributions

For the non normal distributions Student s t distribution and the Laplace distribution (also known as the double exponential distribution) represent the effect of **heavy** tails on the in control average run length (ARL_0) whereas the uniform distribution can be used to study the effect of **light** tails In addition the two different gamma distributions display the possible effect that **skewness** might have on the ARL_0 Lastly for a **change in the process variance** we can compare the average run lengths of the $N(0\ 1)$ distribution with that of the $N(0\ 1\ 1)$ distribution

For quick and easy (visual) reference Figure 5 4 displays some of the distributions that are considered

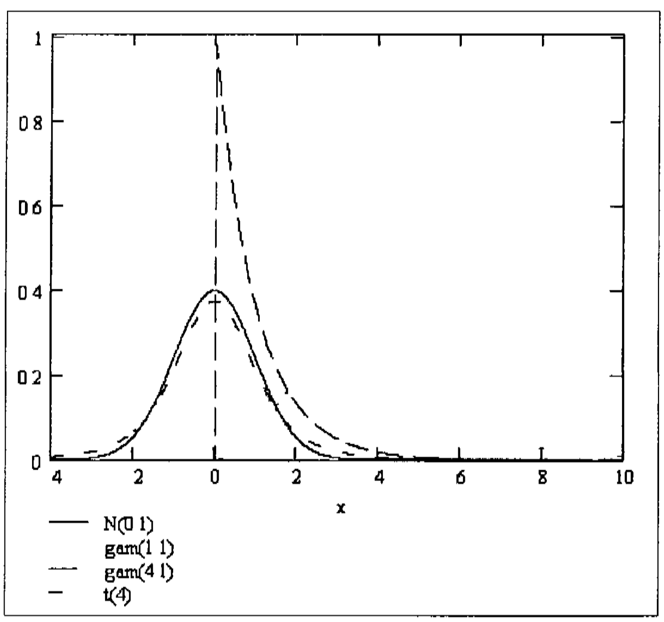

Figure 5 4 Probability distributions for the evaluation of the in control average run lengths in Table 5 5

Studying Table 5 5 we observe the following The in control average run length (ARL_0) of the CUSUM chart is particularly sensitive to departures from the assumption of normality and for a change in the variance of the process especially when the CUSUM chart is designed specifically to detect a large shift in the location of the process

On the other hand the EWMA control chart is relatively robust against departures from normality especially when the weighting constant λ is small However the EWMA chart is not particularly robust for a change in the variance of the process distribution

(ii) The out of control average run length

For comparison of the out of control average run length of the two sided precedence control chart with that of the standard parametric Shewhart $\overline{X}$ chart the CUSUM chart and the EWMA chart Tables 5 6a – 5 6c present the average run lengths for the $N(0\ 1)$ the $\Gamma(1\ 1)$ as well as the $t(4)$ distributions

From Tables 5 6a and 5 6b we see that the CUSUM and the EWMA charts are better than the precedence control chart in detecting (especially) small shifts in a normal distribution and a $\Gamma(1\ 1)$ distribution However for the normal distribution the differences between the precedence control chart and the parametric CUSUM and EWMA charts are not very large for shift larger than 1 5σ say But this shouldn t be surprising considering that the precedence chart is a Shewhart type of control chart

	Precedence	CUSUM		EWMA	
Shift / σ	$m = 1000$ $n = 5\ j = 3$ $P_0 = 0\ 0078$ $a = 48\ b = 953$	*Large* $h = 4\ 61$ $k = 0\ 56$ $n = 5$	*Small* $h = 0\ 86$ $k = 2\ 24$ $n = 5$	*Large* $L = 2\ 81$ $\lambda = 0\ 1$ $n = 5$	*Small* $L = 3\ 05$ $\lambda = 0\ 4$ $n = 5$
0 00	501 89	500 00	500 00	500 00	500 00
0 25	240 93	33 35	162 72	25 75	56 05
0 50	71 70	8 78	36 09	8 86	11 00
0 75	24 22	4 86	10 60	5 33	4 76
1 00	9 79	3 40	4 21	3 87	2 97
1 50	2 70	2 22	1 56	2 57	1 77
2 00	1 37	1 77	1 09	2 04	1 25
2 50	1 07	1 34	1 01	1 81	1 04
3 00	1 01	1 06	1 00	1 40	1 00
4 00	1 00	1 00	1 00	1 01	1 00

Table 5 6a The average run length values for the $N(0\ 1)$ distribution

Shift / σ	Precedence	CUSUM		EWMA	
	$m = 1000$, $n = 5$ $j = 3$, $P_0 = 0\ 0078$, $a = 48$ $b = 953$	*Large* $h = 5\ 18$, $k = 0\ 56$, $n = 5$	*Small* $h = 1\ 75$, $k = 2\ 24$, $n = 5$	*Large* $L = 2\ 81$, $\lambda = 0\ 1$, $n = 5$	*Small* $L = 3\ 05$, $\lambda = 0\ 4$, $n = 5$
0 00	501 89	496 43	498 22	509 76	505 68
0 25	439 01	41 40	182 85	25 18	80 83
0 50	256 45	10 28	96 94	7 66	18 73
0 75	125 31	5 44	24 50	3 91	9 98
1 00	61 87	3 75	9 05	2 47	3 70
1 50	15 81	2 43	2 32	1 38	1 76
2 00	4 83	1 90	1 36	1 03	1 20
2 50	1 63	1 61	1 02	1 00	1 00
3 00	1 02	1 15	1 00	1 00	1 00
4 00	1 00	1 00	1 00	1 00	1 00

Table 5 6b The average run length values for the $Gamma(1\ 1)$ distribution

From Table 5 6c we see that for the $t(4)$ distribution that is for a symmetric distribution with heavier tails than the standard normal distribution small shifts are better detected by the CUSUM chart (with a small value for the chart constant k) as well as the EWMA chart (with a small value for the weighting constant λ) but larger shifts are once again detected sooner by the precedence control chart

Shift / σ	Precedence	CUSUM		EWMA	
	$m = 1000$, $n = 5$ $j = 3$, $P_0 = 0\ 0078$, $a = 48$ $b = 953$	*Large* $h = 5\ 38$, $k = 0\ 56$, $n = 5$	*Small* $h = 1\ 81$, $k = 2\ 24$, $n = 5$	*Large* $L = 3\ 04$, $\lambda = 0\ 1$, $n = 5$	*Small* $L = 3\ 65$, $\lambda = 0\ 4$, $n = 5$
0 00	501 89	495 23	499 47	507 91	504 44
0 25	300 09	41 88	340 58	28 28	128 87
0 50	32 70	10 24	136 21	7 86	22 45
0 75	11 16	5 54	35 50	3 98	7 11
1 00	2 26	3 86	9 84	2 47	3 77
1 50	1 16	2 49	2 31	1 38	1 20
2 00	1 02	1 98	1 34	1 06	1 03
2 50	1 00	1 63	1 06	1 01	1 00
3 00	1 00	1 20	1 01	1 00	1 00
4 00	1 00	1 01	1 00	1 00	1 00

Table 5 6c The average run length values for the $t(4)$ distribution

5 3 Dıstrıbutıon-free control charts based on the Sıgn Test statıstıc

5 3 1 The Shewhart Sıgn chart

Consıder the sıtuatıon ın whıch samples or subgroups of n observatıons are taken randomly at regular tıme ıntervals from an *unknown* contınuous process dıstrıbutıon wıth a *known* target value μ_0 Generally μ_0 can be any value but typıcally taken to be the medıan Thus ıf ıt happens that the dıstrıbutıon ıs symmetrıc the target value ıs equal to the process mean

Let X_{1} X_{2} X $i = 1\ 2\ 3$ be the sample or the subgroup at the i^{th} poınt ın tıme and suppose that each observatıon X_j $j = 1\ 2$ n ıs compared to μ_0 and the number of observatıon above and below μ_0 ıs recorded for each sample

Then let

$$SN = \sum_{j\ 1} \varphi\left(X_j - \mu_0\right) \qquad (5\ 43)$$

where

$$\varphi(t) = \begin{cases} 1 & if \quad t > 0 \\ 0 & if \quad t = 0 \\ -1 & if \quad t < 0 \end{cases} \qquad (5\ 44)$$

so that SN ıs the *dıfference* between the number of observatıons *above* μ_0 and the number of observatıons *below* μ_0 ın the i^{th} sample Sınce the process dıstrıbutıon ıs assumed to be contınuous $P\left(X_j = \mu_0\right) = 0$ and the scenarıo where $\varphi\left(X_j - \mu_0\right) = 0$ ı e zeros or zero dıfferences wıth μ_0 should at least ın theory never occur However where observatıons are rounded off the occasıonal zero mıght be observed but as long as the number of zeros does not occur too often we can safely compute SN as defined ın expressıon (5 43)

The control lımıts and the centerlıne of the **Shewhart sıgn chart** are

$$\begin{aligned} UCL &= a \\ CL &= 0 \\ LCL &= -a \end{aligned} \qquad (5\ 45)$$

where $a > 0$ is a positive integer Therefore the Shewhart sign chart signals that a shift occurred if $|SN| \geq a$ that is if $SN \geq a$ or if $SN \leq -a$ On the other hand as long as $-a < SN < a$ the process is considered in control at the target value μ_0

Example 5 3
A Shewhart sign chart

Panel (a) of Table 5 7 displays the individual observations of 15 independent samples or subgroups from an *unknown* continuous process distribution each of size $n = 10$ which should be used to monitor the *known* target value $\mu_0 = 0$ In addition column (b) of Table 5 7 contains the values of the plotting statistic SN for $i = 1\ 2 \quad 15$

For example in sample number $i = 4$ six of the values are larger than zero four of the values are less than zero with none of the values equal to zero Thus $SN_4 = \sum_{j\ 1}^{10}\left(X_{4j} - 0\right) = 2$

	(a)										(b)
Sample i	X	X	X	X	X	X	X	X	X	X	SN
1	0 30	1 28	0 24	1 28	1 20	1 73	2 18	0 23	1 10	1 09	0
2	0 69	1 69	1 85	0 98	0 77	2 12	0 57	0 40	0 13	0 37	8
3	0 33	0 37	1 34	0 09	0 19	0 51	1 97	0 87	2 38	0 65	2
4	1 66	1 61	0 54	0 90	1 92	0 08	0 52	0 68	0 38	0 76	2
5	1 44	0 85	1 52	0 36	0 03	0 03	0 32	2 19	1 74	0 74	6
6	2 58	1 45	1 28	0 65	0 76	0 47	0 87	0 60	1 37	1 12	0
7	0 69	0 32	0 94	0 24	0 13	0 56	0 14	0 91	1 88	0 49	4
8	0 07	0 83	0 86	0 64	0 92	1 11	1 20	1 56	0 71	0 64	2
9	2 21	1 44	1 30	0 11	0 00	0 45	0 03	1 05	1 77	0 83	3
10	0 44	0 62	0 21	1 03	1 24	0 31	0 84	0 82	0 43	0 45	2
11	0 52	0 85	0 51	0 61	1 30	1 76	0 55	0 12	0 04	0 65	0
12	0 55	0 85	0 80	0 46	0 69	1 63	0 30	0 59	1 85	0 34	6
13	1 04	0 14	1 14	0 15	0 78	1 08	0 58	0 53	0 55	0 32	2
14	0 44	1 37	1 99	0 57	0 09	0 23	2 84	1 25	0 88	1 33	2
15	0 19	0 54	0 25	1 22	1 27	0 29	1 31	0 76	0 78	0 43	2

Table 5 7 Data for the Shewhart sign chart

If we select $a = 10$ we have that the control limits and centerline are given by

$$UCL = 10$$
$$CL = 0$$
$$LCL = -10$$

with the resulting Shewhart sign chart shown in Figure 5 5

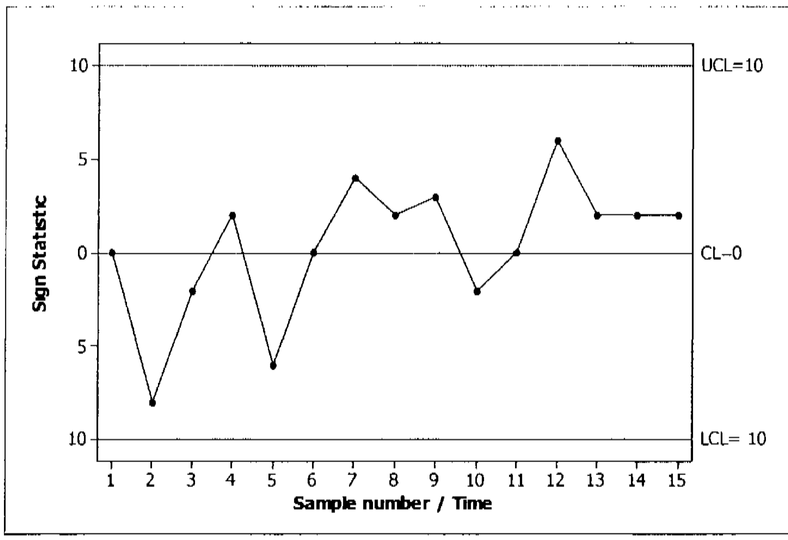

Figure 5 5 A Shewhart sign chart

Since none of the plotting statistics fall on or outside the two control limits the process is considered in control at the target value $\mu_0 = 0$ and no corrective action or search for assignable causes is necessary

5 3 2 The tabular CUSUM Sign chart

A non parametric or distribution free CUSUM type of control chart based on the sign test statistic SN can be obtained by replacing Y_t in expressions (4 8a) and (4 8b) with SN_t

In other words for a **one sided upper** CUSUM type of control chart (based on the sign test statistic) for detecting positive deviations from the in control or known target value μ_0 we use

$$SN_t = \max\left(0\ SN_{t-1} + SN_t - k\right) \qquad (5\ 45)$$

with the starting value $SN_0 = 0$ and which signals at the first t for which $SN_t^+ \geq h$

On the other hand for a **one sided lower** CUSUM type of control chart (based on the sign test statistic) for detecting negative deviations from the in control or known target value μ_0 we use

$$SN_t = \max\left(0\ SN_{t-1} - SN_t - k\right) \qquad (5\ 46a)$$

or

$$SN_t = \min\left(0\ SN_{t-1} + SN_t + k\right) \qquad (5\ 46b)$$

with the starting value $SN_0^- = 0$ and which signals at the first t for which $SN_t \geq h$ or $SN_t \leq -h$

A corresponding **two sided symmetric** CUSUM type of control chart signals as the first t for which either of the one sided chart signals i e whenever $SN_t \geq h$ or $SN_t \geq h$ Note that here the decision interval $h > 0$ and the reference value $k > 0$ are the parameters of the control chart procedure

Example 5 4
A two sided tabular CUSUM sign chart

For the data of Table 5 7 the two one sided CUSUM plotting statistics SN_t^+ and SN_t with $SN_0^+ = SN_0 = 0$ and $k = 2$ were calculated and added to columns (b) and (d) of Table 5 8 The decision interval h was chosen to be 8

To illustrate the calculations consider sample number 1 The equations for the plotting statistics SN_1 and SN_1^- are

$$SN_1 = \max\left(0 \; SN_0 + SN_1 - k\right) = \max\left(0 \; 0 + 0 - 2\right) = 0$$

and

$$SN_1 = \min\left(0 \; SN_0 + SN_0 + k\right) = \min\left(0 \; 0 + 0 + 2\right) = 0$$

	(a)	(b)	(c)	(d)	(e)
Sample t	SN	SN	N	SN	N
1	0	0	0	0	0
2	8	0	0	6	1
3	2	0	0	6	2
4	2	0	0	2	3
5	6	0	0	6	4
6	0	0	0	4	5
7	4	2	1	0	0
8	2	2	2	0	0
9	3	3	3	0	0
10	2	0	0	0	0
11	0	0	0	0	0
12	6	4	1	0	0
13	2	4	2	0	0
14	2	4	3	0	0
15	2	4	4	0	0

Table 5 8 A tabular two sided CUSUM sign chart

The remaining calculations are summarized in columns (b) and (d) of Table 5 8 Along with SN_t and SN_t the quantity N which counts the number of consecutive periods since SN_t became non zero and N which counts the number of consecutive periods since SN_t became non zero were calculated The graphical display of the tabular CUSUM control chart the so called CUSUM status chart is shown in Figure 5 6 and does not signal Thus neither a search for assignable causes nor any corrective action is necessary

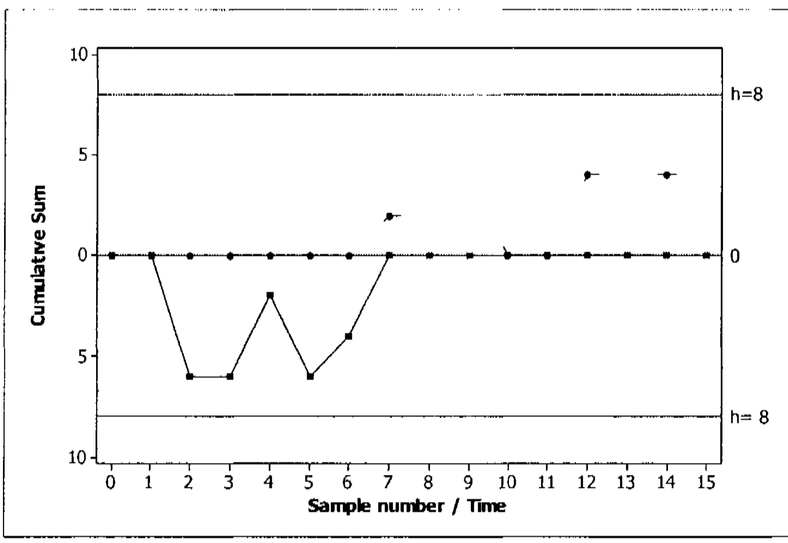

Figure 5 6 A CUSUM status chart of the two sided tabular CUSUM sign chart

5 4 Distribution-free control charts based on the Wilcoxon Signed-Rank statistic

Let $X_1, X_2, \ldots, X_i$ $i = 1\ 2\ 3$ be samples or subgroups of independent observations taken at random (and sequentially) from an *unknown* but continuous process distribution symmetric about a *known* value μ_0 typically called the target value or the control value

If R_j is the rank of $|X_j|$ *within* the group $(|X_1|\ |X_2| \ldots |X_n|)$ for $i = 1\ 2\ 3$ and $j = 1\ 2 \ldots n$ then

$$U_j = \varphi(X_j - \mu_0) R_j \tag{5 47}$$

where $\varphi(t)$ is defined in expression (5 44) are the usual Wilcoxon signed ranks of the observations *within* the i^{th} sample or subgroup and

$$SR_i = \sum_{j=1} U_j \qquad i = 1\ 2\ 3 \tag{5 48}$$

is the *difference* between signed ranks associated with the *positive differences* i e those observations for which $X_j - \mu_0 > 0$ and the signed ranks associated with the *negative differences* i e those observations for which $X_j - \mu_0 < 0$ for the i^{th} sample or subgroup If this is the case $SR_1\ SR_2\ SR_3$ is a sequence of independent Wilcoxon signed rank statistics each based on n independent and identically distributed observations which can then be used to detect departures from a distribution symmetric about the control value μ_0

5 4 1 The tabular CUSUM Wilcoxon Signed-Rank chart

A **one sided upper** CUSUM type of control chart based on the Wilcoxon signed rank test statistic SR_t for detecting positive deviations from the in control or known target value μ_0 is defined as

$$SR_t^+ = \max\left(0\ SR_{t-1}^+ + SR_t - k\right) \tag{5 49}$$

with the starting value $SR_0^+ = 0$ and which signals at the first t for which $SR_t^+ \geq h$

One the other hand for a **one sided lower** CUSUM type of control chart based on the Wilcoxon signed rank test statistic for detecting negative deviations from the in control or known target value μ_0 we define

$$SR_t^{} = \max\left(0\ SR_{t\ 1} - SR_t - k\right) \tag{5 50a}$$

or

$$SR_t^{} = \min\left(0\ SR_{t\ 1} + SR_t + k\right) \tag{5 50b}$$

with the starting value $SR_0 = 0$ and which signals at the first t for which $SR_t \geq h$ or $SR_t \leq -h$ A corresponding **(symmetric) two sided** CUSUM type of control chart signals as the first t for which either of the one sided chart signals i e whenever $SR_t \geq h$ or $SR_t^- \geq h$ Note that here the decision interval $h > 0$ and the reference value $k > 0$ are (as in the past) the parameters of the control chart procedure

Example 5 5
A two sided tabular CUSUM Wilcoxon Signed Rank chart

Panel (a) of Table 5 9a contains the individual observations of 20 independent samples each of size $n = 4$ from an *unknown* continuous distribution symmetric about the *known* value $\mu_0 = 0$ The corresponding absolute ranks R_j and the Wilcoxon signed ranks U_j are shown in panel (c) of Table 5 9a and panel (b) Table 5 9b respectively The sum of the Wilcoxon signed ranks i e SR are shown in panel (c) of Table 5 9b

	(a)				(b)				(c)			
Sample i	X	X	X	X	$\lvert X \rvert$	$\lvert X \rvert$	$\lvert X \rvert$	$\lvert X \rvert$	R	R	R	R
1	0 39	0 03	0 25	0 30	0 39	0 03	0 25	0 3	4	1	2	3
2	0 61	1 32	2 84	0 07	0 61	1 32	2 84	0 07	2	3	4	1
3	2 04	0 40	0 54	1 12	2 04	0 4	0 54	1 12	4	1	2	3
4	0 51	1 63	0 28	0 96	0 51	1 63	0 28	0 96	2	4	1	3
5	0 20	0 01	0 21	0 82	0 2	0 01	0 21	0 82	2	1	3	4
6	2 03	0 70	1 21	0 26	2 03	0 7	1 21	0 26	4	2	3	1
7	0 64	1 48	0 91	1 23	0 64	1 48	0 91	1 23	1	4	2	3
8	0 19	2 74	0 85	1 25	0 19	2 74	0 85	1 25	1	4	2	3
9	0 51	1 77	1 69	0 24	0 51	1 77	1 69	0 24	2	4	3	1
10	0 84	0 71	0 82	0 19	0 84	0 71	0 82	0 19	4	2	3	1
11	0 29	0 47	0 04	0 39	0 29	0 47	0 04	0 39	2	4	1	3
12	0 38	2 18	1 24	0 80	0 38	2 18	1 24	0 8	1	4	3	2
13	0 86	0 61	0 70	1 13	0 86	0 61	0 7	1 13	3	1	2	4
14	0 31	1 09	0 43	0 69	0 31	1 09	0 43	0 69	1	4	2	3
15	0 42	1 35	0 51	1 80	0 42	1 35	0 51	1 8	1	3	2	4
16	0 39	0 65	1 57	0 60	0 39	0 65	1 57	0 6	1	3	4	2
17	1 31	0 59	0 21	0 22	1 31	0 59	0 21	0 22	4	3	1	2
18	0 64	0 27	0 26	0 08	0 64	0 27	0 26	0 08	4	3	2	1
19	0 33	0 49	0 38	1 54	0 33	0 49	0 38	1 54	1	3	2	4
20	0 48	0 54	1 73	1 00	0 48	0 54	1 73	1	1	2	4	3

Table 5 9a Data for the Wilcoxon signed rank charts

	(a)				(b)				(c)
Sample i	$\varphi(X\ )$	$\varphi(X\ )$	$\varphi(X\ )$	$\varphi(X\ )$	U	U	U	U	SR
1	1	1	1	1	4	1	2	3	2
2	1	1	1	1	2	3	4	1	10
3	1	1	1	1	4	1	2	3	10
4	1	1	1	1	2	4	1	3	8
5	1	1	1	1	2	1	3	4	2
6	1	1	1	1	4	2	3	1	2
7	1	1	1	1	1	4	2	3	10
8	1	1	1	1	1	4	2	3	2
9	1	1	1	1	2	4	3	1	8
10	1	1	1	1	4	2	3	1	8
11	1	1	1	1	2	4	1	3	10
12	1	1	1	1	1	4	3	2	6
13	1	1	1	1	3	1	2	4	8
14	1	1	1	1	1	4	2	3	4
15	1	1	1	1	1	3	2	4	2
16	1	1	1	1	1	3	4	2	0
17	1	1	1	1	4	3	1	2	2
18	1	1	1	1	4	3	2	1	10
19	1	1	1	1	1	3	2	4	6
20	1	1	1	1	1	2	4	3	2

Table 5 9b Calculations for the Wilcoxon signed rank charts

The two one sided CUSUM plotting statistics SR_t and SR_t^- with $SR_0^+ = SR_0 = 0$ and $k = 3$ were calculated and added together with two counters N^+ and N to Table 5 9c The decision interval h was chosen to be equal to 8

To illustrate the calculations consider sample number 1 The expressions for the CUSUM plotting statistics SR_1 and SR_1 are

$$\begin{aligned} SR_1^+ &= \max\left(0\ SR_0 + SR_1 - k\right) \\ &= \max\left(0\ 0+2-3\right) \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} SR_1 &= \min\left(0\ SR_0 + SR_1 + k\right) \\ &= \min\left(0\ 0+2+3\right) \\ &= 0 \end{aligned}$$

The remaining calculations are summarized in panels (a) and (b) of Table 5 9c The CUSUM status chart of the two sided tabular CUSUM Wilcoxon Signed Rank chart displayed in Figure 5 7 signals twice The first signal is given on sample number 10 when $S_{10} = 10$ plots below the lower decision interval i e $h = -8$ whereas the second signal is given shortly afterwards on sample number 12 when $S_{12} = 10$ falls above the upper decision interval i e $h = 8$

If investigation revealed any assignable cause(s) for the first signal the lower one sided control CUSUM plotting statistic would have been restarted from zero while the upper one sided CUSUM plotting statistics would have continued as usual whereas the opposite would have been done if any assignable cause(s) were found on the second sample However considering the current situation that is a signal on the lower side (indicating a possible downward shift) on sample number 10 followed shortly afterwards with a signal on the upper side (indicating a possible upward shift) on sample number 12 these two signals were considered false alarms and neither of the CUSUM plotting statistics were reset to zero After all although not impossible it is unlikely that a downward shift in a process will be followed so soon by an upward shift unless there was over correction in the positive direction from one of the control chart personnel

	(a)		(b)	
Sample i	SR	N	SR	N
1	0	0	0	0
2	0	0	7	1
3	7	1	0	0
4	0	0	5	1
5	0	0	4	2
6	0	0	0	0
7	7	1	0	0
8	2	2	0	0
9	0	0	5	1
10	0	0	10	2
11	7	1	0	0
12	10	2	0	0
13	0	0	5	1
14	1	1	0	0
15	0	0	0	0
16	0	0	0	0
17	0	0	0	0
18	7	1	0	0
19	0	0	3	1
20	0	0	2	2

Table 5 9 c A tabular two sided CUSUM Wilcoxon signed rank chart

The two sided tabular CUSUM Wilcoxon signed rank chart signals

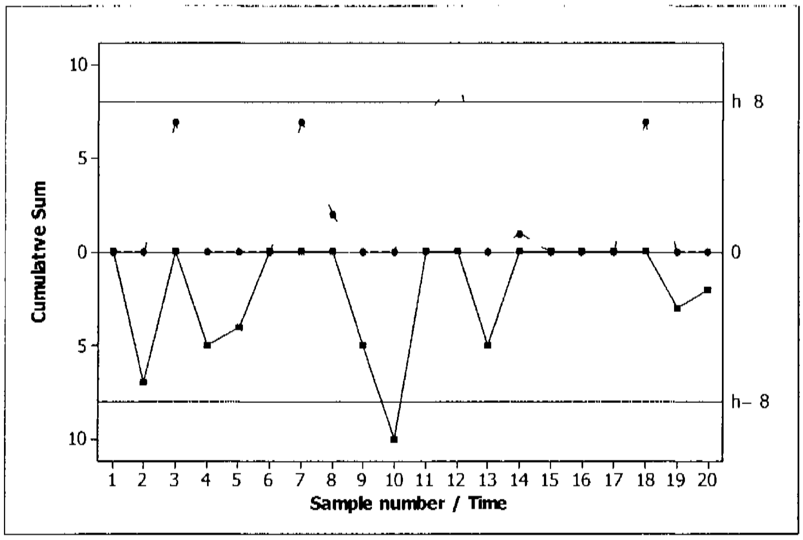

Figure 5 7 A CUSUM status chart of the two sided tabular CUSUM Wilcoxon signed rank chart

5 4 2 The EWMA Wilcoxon Signed-Rank chart

An EWMA control chart procedure based on the Wilcoxon signed rank statistic SR defined in expression (5 48) accumulates the statistics SR_1 SR_2 SR_3 with the EWMA plotting statistics defined as

$$Z_t = \lambda SR_t + (1-\lambda)Z_{t-1} \qquad (5\ 51)$$

with the starting value $Z_0 = 0$ and the weighting constant $0 < \lambda \le 1$

The centerline and the **exact** control limits are given by

$$\begin{aligned} UCL &= L\sigma_{SR}\sqrt{\frac{\lambda}{2-\lambda}\left(1-(1-\lambda)^{2t}\right)} \\ CL &= 0 \\ LCL &= -L\sigma_{SR}\sqrt{\frac{\lambda}{2-\lambda}\left(1-(1-\lambda)^{2t}\right)} \end{aligned} \qquad (5\ 52)$$

whereas the **asymptotic** or the **steady state** control limits that is when $t \to \infty$ are given by

$$\begin{aligned} UCL &= L\sigma_{SR}\sqrt{\frac{\lambda}{2-\lambda}} \\ LCL &= -L\sigma_{SR}\sqrt{\frac{\lambda}{2-\lambda}} \end{aligned} \qquad (5\ 53)$$

where $\sigma_{SR} = \sqrt{\frac{n(n+1)(2n+1)}{24}}$ is the in control standard deviation of the Wilcoxon signed rank statistic if there are no ties *within* a sample or subgroup

The process is considered to be in control while all the plotting statistics Z_t $t = 1\ 2\ 3$ fall between the two control limits but as soon as a plotting statistic falls on or outside the upper or the lower control limit the process is declared out of control and typically a search for assignable causes would be started

Example 5 6
An EWMA Wilcoxon Signed Rank chart

A two sided tabular CUSUM Wilcoxon Signed Rank chart for the data in column (a) of Table 5 9a was created in Example 5 5 As alternative an EWMA control chart (also based on the Wilcoxon Signed Rank statistic calculated in column (c) of Table 5 9c) is now created

For this purpose we will use $\lambda = 0\ 15$ to calculate the plotting statistics and $L = 2\ 25$ to find the control limits To illustrate the calculations consider sample number 1 with $SR_1 = 2$ and suppose that the starting value $Z_0 = 0$ Then the first plotting statistic on the EWMA Wilcoxon Signed Rank chart in Figure 5 8 is

$$\begin{aligned} Z_1 &= \lambda SR_1 + (1-\lambda)Z_0 \\ &= 0\ 15(2) + (1-0\ 15)0 \\ &= 0\ 30 \end{aligned}$$

The value of the second plotting statistic is

$$\begin{aligned} Z_2 &= \lambda SR_2 + (1-\lambda)Z_1 \\ &= 0\ 15(-10) + (1-0\ 15)0\ 30 \\ &= -1\ 245 \end{aligned}$$

with the remaining calculations summarized in column (b) Table 5 10

	(a)	(b)
Sample i	SR	$Z = \lambda SR + (1-\lambda)Z$
1	2	0 300
2	10	1 245
3	10	0 442
4	8	0 825
5	2	1 001
6	2	0 551
7	10	1 032
8	2	0 577
9	8	0 709
10	8	1 803
11	10	0 033
12	6	0 872
13	8	0 459
14	4	0 210
15	2	0 121
16	0	0 103
17	2	0 212
18	10	1 681
19	6	0 528
20	2	0 149

Table 5 10 Calculations for the EWMA Wilcoxon Signed Rank chart

For simplicity we use the steady state control limits on the EWMA Wilcoxon Signed Rank chart which is found from expression (5 53) i e

$$
\begin{aligned}
LCL &= -L\sigma_{SR}\sqrt{\frac{\lambda}{2-\lambda}} \\
&= -2\ 25\sqrt{\frac{4(4+1)(2\times 5+1)}{24}}\sqrt{\frac{0\ 15}{2-0\ 15}} \\
&= -1\ 94
\end{aligned}
$$

and

$$
\begin{aligned}
UCL &= L\sigma_{SR}\sqrt{\frac{\lambda}{2-\lambda}} \\
&= 2\ 25\sqrt{\frac{4(4+1)(2\times 5+1)}{24}}\sqrt{\frac{0\ 15}{2-0\ 15}} \\
&= 1\ 94
\end{aligned}
$$

respectively

The resulting EWMA Wilcoxon signed rank chart is shown in Figure 5 8 Since none of the plotting statistics fall on or outside the lower and/or the upper control limits the process is considered in control with no remedial action needed

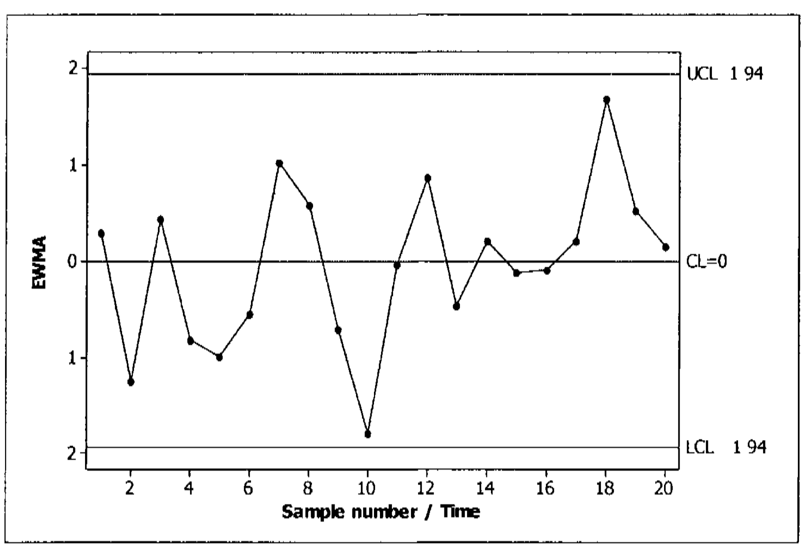

Figure 5 8 An EWMA Wilcoxon signed rank chart

5 5 Appendıx 5

5 5 1 Upper one-sıded control chart

Result 5 1 Probabılıty of no sıgnal Condıtıonal

$$p \quad = \int_0^{GF\ \left(\ (\ m)\right)} \frac{1}{\beta\left(j\ \ n-j+1\right)} u^{j\ \ 1}\left(1-u\right)^{\ \ j} du$$

Proof

$$\begin{aligned}
P\left(\text{No Sıgnal} \mid X_{(b\ m)} = x_{(\ \ )}\right) &= P\left(Y_{(j\ \ )} \le x_{(b\ m)} \mid X_{(b\ m)} = x_{(\ \ )}\right) \\
&= P\left(G^{\ 1}\left(U_{(j\ \ )}\right) \le F^{\ 1}\left(u_{(b\ m)}\right) \mid U_{(b\ m)} = u_{(\ \ m)}\right) \\
&= P\left(U_{(j\ \ )} \le GF^{\ 1}\left(u_{(b\ m)}\right) \mid U_{(b\ m)} = u_{(\ \ m)}\right) \\
&= \int_0^{GF\ \left(\ (\ m)\right)} f_{U_{(\ \ )}}\left(u\right) du \\
&= \int_0^{GF\ \left(\ (\ m)\right)} \frac{1}{\beta\left(j\ \ n-j+1\right)} u^{j\ \ 1}\left(1-u\right)^{\ \ j} du \\
&= p
\end{aligned}$$

Result 5 2 Probability of a signal Conditional

$$1-p \quad =1-\int_{0}^{GF\left(x_{(b\,m)}\right)}\frac{1}{\beta\left(j\;n-j+1\right)}u^{j-1}\left(1-u\right)^{n-j}du$$

Proof

$$\begin{aligned}P\left(\text{Signal}\mid X_{(b\,m)}=x_{(b\,m)}\right)&=P\left(Y_{(j\,)}>x_{(b\,m)}\mid X_{(b\,m)}=x_{(b\,m)}\right)\\&=1-P\left(Y_{(j\,)}\le x_{(b\,m)}\mid X_{(b\,m)}=x_{(b\,m)}\right)\\&=1-P\left(\text{No Signal}\mid X_{(b\,m)}=x_{(b\,m)}\right)\\&=1-\int_{0}^{GF\left(x_{(b\,m)}\right)}\frac{1}{\beta\left(j\;n-j+1\right)}u^{j-1}\left(1-u\right)^{n-j}du\\&=1-p\end{aligned}$$

Result 5 3 Probabılıty of no sıgnal Uncondıtıonal

$$p = \int_0^1 \left(\frac{1}{\beta(j \; n-j+1)} \sum_{h \; 0}^{j} \frac{(-1)^h}{j+h} \binom{n-j}{h} \left\{ GF^{\;1}(t) \right\}^{j \; h} \right) \frac{m!}{(b-1)!(m-b)!} t^{b \; 1} (1-t)^{m \; b} \, dt$$

Proof

$$\begin{aligned}
P(\text{No Sıgnal}) &= E_{X_{(\;)}}\left(p \quad\right) \\
&= E_{U_{(\;)}}\left(\int_0^{GF\;(\,(\;)\,)} \frac{1}{\beta(j \; n-j+1)} u^{j \; 1} (1-u)^{\;\; j} \, du \right) \\
&= \int_0^1 \left(\int_0^{GF\;(\;)} \frac{1}{\beta(j \; n-j+1)} u^{j \; 1} (1-u)^{\;\; j} \, du \right) \frac{m!}{(b-1)!(m-b)!} t^{b-1} (1-t)^{m \; b} \, dt \\
&= \int_0^1 \left(\int_0^{GF\;(t)} \frac{1}{\beta(j \; n-j+1)} u^{j \; 1} \left\{ \sum_{h \; 0}^{j} (-1)^h \binom{n-j}{h} u^h \right\} du \right) \frac{m!}{(b-1)!(m-b)!} t^{b \; 1} (1-t)^{m \; b} \, dt \\
&= \int_0^1 \left(\frac{1}{\beta(j \; n-j+1)} \sum_{h \; 0}^{-j} (-1)^h \binom{n-j}{h} \int_0^{GF\;(t)} u^{j \; h \; 1} du \right) \frac{m!}{(b-1)!(m-b)!} t^{b \; 1} (1-t)^{m \; b} \, dt \\
&= \int_0^1 \left(\frac{1}{\beta(j \; n-j+1)} \sum_{h \; 0}^{j} (-1)^h \binom{n-j}{h} \frac{u^{j \; h}}{j+h} \Big|_0^{GF\;(t)} \right) \frac{m!}{(b-1)!(m-b)!} t^{b \; 1} (1-t)^{m \; b} \, dt \\
&= \int_0^1 \left(\frac{1}{\beta(j \; n-j+1)} \sum_{h \; 0}^{j} \frac{(-1)^h}{j+h} \binom{n-j}{h} \left\{ GF^{\;1}(t) \right\}^{j \; h} \right) \frac{m!}{(b-1)!(m-b)!} t^{b \; 1} (1-t)^{m \; b} \, dt \\
&= p
\end{aligned}$$

Result 5 4 Probability of a signal Unconditional

$$1-p \; = 1-\int_0^1\left(\frac{1}{\beta(j,\, n-j+1)}\sum_{h=0}^{j}\frac{(-1)^h}{j+h}\binom{n-j}{h}\left\{GF^{-1}(t)\right\}^{j+h}\right)\frac{m!}{(b-1)!(m-b)!}t^{b-1}(1-t)^{m-b}\,dt$$

Proof

$$\begin{aligned}
P(\text{Signal}) &= E_{X_{( )}}\left(1-p \;\right)\\
&= 1-E_{X_{( )}}\left(p \;\right)\\
&= 1-P(\text{No Signal})\\
&= 1-\int_0^1\left(\frac{1}{\beta(j,\, n-j+1)}\sum_{h=0}^{n-j}\frac{(-1)^h}{j+h}\binom{n-j}{h}\left\{GF^{-1}(t)\right\}^{j+h}\right)\frac{m!}{(b-1)!(m-b)!}t^{b-1}(1-t)^{m-b}\,dt\\
&= 1-p
\end{aligned}$$

Result 5 5 Probabilıty of a false alarm Condıtional

$$CFAR = 1 - \int_0^{u_{(\ )}} \frac{1}{\beta(j \ n-j+1)} u^{j \ 1}(1-u)^{\ j} du$$

Proof

$$P\left(\text{False Alarm} \mid G = F \ X_{(b\,m)} = x_{(\ m)}\right) = P\left(Y_{(j \ )} > x_{(b\,m)} \mid G = F \ X_{(b\,m)} = x_{(\ m)}\right)$$
$$= 1 - P\left(Y_{(j \ )} \le x_{(b\,m)} \mid G = F \ X_{(b\,m)} = x_{(\ )}\right)$$
$$= 1 - P\left(F^{\ 1}\left(U_{(j \ )}\right) \le F^{\ 1}\left(u_{(b\,m)}\right) \mid U_{(b\,m)} = u_{(\ m)}\right)$$
$$= 1 - P\left(U_{(j \ )} \le u_{(b\,m)} \mid U_{(b\,m)} = u_{(\ m)}\right)$$
$$= 1 - \int_0^{u_{(\ m)}} f_{U_{(\ )}}(u)\,du$$
$$= 1 - \int_0^{u_{(\ m)}} \frac{1}{\beta(j \ n-j+1)} u^{j \ 1}(1-u)^{\ j} du$$
$$= CFAR$$

Thıs expressıon could also have been obtaıned by sımply substıtutıng $G = F$ so that $GF^{-1}(u) = u$ ın the condıtıonal probabılıty of a sıgnal ın Result 5 2 Thus Result 5 5 ıs sımılar to Result 5 2 ı e $1-p \ = 1 - \int_0^{GF\left((\ )\right)} \frac{1}{\beta(j \ n-j+1)} u^{j \ 1}(1-u)^{\ -j} du$ with $G = F$

Result 5 6 Probabılıty of a false alarm Unconditional

$$FAR = 1 - \int_0^{u_{(\ \ )}} \frac{1}{\beta(j\ n-j+1)} u^{j\ 1}(1-u)^{\ j}\, du$$

Proof

$$\begin{aligned}
P(\text{False Alarm}) &= E_{X_{(\ m)}}\left(P\left(\text{False Alarm} \mid G=F\ \ X_{(b\,m)} = x_{(\ m)}\right)\right) \\
&= E_{U_{(\ m)}}\left(1 - \int_0^{u_{(\ \ )}} \frac{1}{\beta(j\ n-j+1)} u^{j\ 1}(1-u)^{\ j}\, du\right) \\
&= 1 - E_{U_{(\ m)}}\left(\int_0^{u_{(\ \ )}} \frac{1}{\beta(j\ n-j+1)} u^{j\ 1}(1-u)^{-j}\, du\right) \\
&= 1 - \int_0^1\left(\int_0^t \frac{1}{\beta(j\ n-j+1)} u^{j\ 1}(1-u)^{\ j}\, du\right)\frac{m!}{(b-1)!(m-b)!} t^{b\ 1}(1-t)^{m\ b}\, dt \\
&= 1 - \int_0^1\left(\int_0^t \frac{1}{\beta(j\ n-j+1)} u^{j-1}\left\{\sum_{h\ 0}^{-j}(-1)^h\binom{n-j}{h}u^h\right\}du\right)\frac{m!}{(b-1)!(m-b)!} t^{b\ 1}(1-t)^{m\ b}\, dt \\
&= 1 - \int_0^1\left(\frac{1}{\beta(j\ n-j+1)}\sum_{h\ 0}^{j}(-1)^h\binom{n-j}{h}\int_0^t u^{j\ h\ 1}du\right)\frac{m!}{(b-1)!(m-b)!} t^{b\ 1}(1-t)^{m\ b}\, dt \\
&= 1 - \int_0^1\left(\frac{1}{\beta(j\ n-j+1)}\sum_{h\ 0}^{j}(-1)^h\binom{n-j}{h}\frac{u^{j+h}}{j+h}\Big|_0^t\right)\frac{m!}{(b-1)!(m-b)!} t^{b\ 1}(1-t)^{m\ b}\, dt \\
&= 1 - \int_0^1\left(\frac{1}{\beta(j\ n-j+1)}\sum_{h\ 0}^{j}\frac{(-1)^h}{j+h}\binom{n-j}{h}\{t\}^{j\ h}\right)\frac{m!}{(b-1)!(m-b)!} t^{b\ 1}(1-t)^{m-b}\, dt \\
&= FAR
\end{aligned}$$

This expression could also have been obtained by simply substituting $G = F$ so that $GF^{\ 1}(u) = u$ in the unconditional probability of a signal in Result 5 4 Thus Result 5 6 is similar to Result 5 4 i e

$$1-p\ = 1 - \int_0^1\left(\frac{1}{\beta(j\ n-j+1)}\sum_{h\ 0}^{j}\frac{(-1)^h}{j+h}\binom{n-j}{h}\{GF^{\ 1}(t)\}^{j\ h}\right)\frac{m!}{(b-1)!(m-b)!} t^{b\ 1}(1-t)^{m\ b}\, dt$$

with $G = F$

Result 5 7 Run length distribution Conditional

$$P\left(N=k \mid X_{(b\ m)}=x_{(b\ m)}\right)=p^{k\ 1}\left(1-p\ \ \right)=p^{k\ 1}-p^{k} \quad \text{for } k=1\ 2\ 3$$

Proof

Given that $X_{(b\ m)}=x_{(b\ m)}$ the run length distribution is geometric with parameter or probability of success (signal) $1-p\ \ $ where $p\ \ =P\left(Y_{(j\ \ )}\leq x_{(b\ m)} \mid X_{(b\ m)}=x_{(b\ m)}\right)$ In other words $\left(N \mid X_{(b\ m)}=x_{(b\ m)}\right)\sim GEO\left(1-p\ \ \right)$

Thus the conditional probability mass function (pmf) is

$$P\left(N=k \mid X_{(b\ m)}=x_{(b\ m)}\right)=p^{k\ 1}\left(1-p\ \ \right)=p^{k-1}-p^{k} \quad \text{with } k=1\ 2\ 3$$

whereas the conditional cumulative distribution function (cdf) is

$$P\left(N\leq k \mid X_{(b\ m)}=x_{(b\ m)}\right)=\sum_{1}^{k} p\ \ ^{1}\left(1-p\ \ \right)=1-p^{k}$$

so that

$$P\left(N>k \mid X_{(b\ m)}=x_{(b\ m)}\right)=1-\left(1-p^{k}\ \ \right)=p^{k}$$

In addition the expected value of N given $X_{(b\ m)}=x_{(b\ m)}$ or the conditional average run length is

$$ARL=E\left(N \mid X_{(b\ m)}=x_{(b\ m)}\right)=\frac{1}{1-p}$$

These results follow directly and conveniently from the properties of the geometric distribution

To obtain a more convenient expression for the conditional average run length we proceed as follows

$$ARL = E\left(N \mid X_{(b\,m)} = x_{(b\,m)}\right)$$

$$= \sum_{k=1}^{\infty} kP\left(N = k \mid X_{(b\,m)} = x_{(b\,m)}\right)$$

$$= P\left(N = 1 \mid X_{(b\,m)} = x_{(b\,m)}\right) \quad + \quad 2P\left(N = 2 \mid X_{(b\,m)} = x_{(b\,m)}\right) \quad + \quad 3P\left(N = 3 \mid X_{(b\,m)} = x_{(b\,m)}\right) \quad + \cdots$$

$$\begin{array}{lllll}
 & P\left(N = 1 \mid X_{(b\,m)} = x_{(b\,m)}\right) & + \; P\left(N = 2 \mid X_{(b\,m)} = x_{(b\,m)} = x_{(b\,m)}\right) & + \; P\left(N = 3 \mid X_{(b\,m)}\right) & + \\
= & & + \; P\left(N = 2 \mid X_{(b\,m)} = x_{(b\,m)}\right) & + \; P\left(N = 3 \mid X_{(b\,m)} = x_{(b\,m)}\right) & + \\
 & & & + \; P\left(N = 3 \mid X_{(b\,m)} = x_{(b\,m)}\right) & +
\end{array}$$

$$= P\left(N > 0 \mid X_{(b\,m)} = x_{(b\,m)}\right) + P\left(N > 1 \mid X_{(b\,m)} = x_{(b\,m)}\right) + P\left(N > 2 \mid X_{(b\,m)} = x_{(b\,m)}\right) + \cdots$$

$$= \sum_{k=0}^{\infty} P\left(N > k \mid X_{(b\,m)} = x_{(b\,m)}\right)$$

$$= \sum_{k=0}^{\infty} p^{k}$$

Result 5 8 Run length dıstrıbutıon Uncondıtıonal

$$P(N=k)=E_{X_{(m)}}\left(p^{k\ 1}\right)-E_{X_{(\ )}}\left(p^{k}\ \right)=D\ (k-1)-D\ (k)\ \text{for } k=1\ 2\ 3$$

Proof

$$P(N=k)=E_{X_{(\ )}}\left(P\left(N=k\mid X_{(b\ m)}=x_{(b\ m)}\right)\right)$$
$$=E_{X_{(\ )}}\left(p^{k\ 1}-p^{k}\ \right)$$
$$=E_{X_{(\ )}}\left(p^{k\ 1}\right)-E_{X_{(m)}}\left(p^{k}\ \right)$$

However

$$E_{X_{(m)}}\left(p^{k}\ \right)=E_{U_{(m)}}\left(\left(\int_0^{GF\ \ (\ (\ m)}\frac{1}{\beta(j\ n-j+1)}u^{j\ 1}(1-u)^{\ -j}du\right)^{k}\right)$$
$$=\int_0^1\left(\int_0^{GF\ \ (t)}\frac{1}{\beta(j\ n-j+1)}u^{j\ 1}(1-u)^{\ \ j}du\right)^{k}f(t)dt$$
$$=\int_0^1\left(\int_0^{GF\ \ (t)}\frac{1}{\beta(j\ n-j+1)}u^{j\ 1}(1-u)^{\ \ j}du\right)^{k}\frac{m!}{(b-1)!(m-b)!}t^{b\ 1}(1-t)^{m\ b}dt$$
$$=\int_0^1\left(\int_0^{GF\ \ (t)}\frac{1}{\beta(j\ n-j+1)}u^{j\ 1}\left\{\sum_{h\ 0}^{j}(-1)^{h}\binom{n-j}{h}u^{h}\right\}du\right)^{k}\frac{m!}{(b-1)!(m-b)!}t^{b\ 1}(1-t)^{m-b}dt$$
$$=\int_0^1\left(\frac{1}{\beta(j\ n-j+1)}\sum_{h\ 0}^{j}(-1)^{h}\binom{n-j}{h}\int_0^{GF\ \ (t)}u^{j\ \ h-1}du\right)^{k}\frac{m!}{(b-1)!(m-b)!}t^{b\ 1}(1-t)^{m\ b}dt$$
$$=\int_0^1\left(\frac{1}{\beta(j\ n-j+1)}\sum_{h\ 0}^{j}(-1)^{h}\binom{n-j}{h}\frac{u^{j\ \ h}}{j+h}\Big|_0^{GF\ \ (t)}\right)^{k}\frac{m!}{(b-1)!(m-b)!}t^{b\ 1}(1-t)^{m\ b}dt$$
$$=\int_0^1\left(\frac{1}{\beta(j\ n-j+1)}\sum_{h\ 0}^{j}\frac{(-1)^{h}}{j+h}\binom{n-j}{h}\left\{GF^{\ 1}(t)\right\}^{j+h}\right)^{k}\frac{m!}{(b-1)!(m-b)!}t^{b\ 1}(1-t)^{m\ b}dt$$
$$=D\ (k)$$

Thus

$$P(N=k)=E_{X_{(\ )}}\left(p^{k\ 1}\right)-E_{X_{(\ )}}\left(p^{k}\ \right)=D\ (k-1)-D\ (k)\ \text{for } k=1\ 2\ 3$$

with

$$D(0) = \int_0^1 \left(\frac{1}{\beta(j, n-j+1)} \sum_{h=0}^{j} \frac{(-1)^h}{j+h} \binom{n-j}{h} \left\{ GF^{-1}(t) \right\}^{j+h} \right)^0 \frac{m!}{(b-1)!(m-b)!} t^{b-1} (1-t)^{m-b} dt$$

$$= \int_0^1 \frac{m!}{(b-1)!(m-b)!} t^{b-1} (1-t)^{m-b} dt$$

$$= 1$$

If $G(\,) = F(\,)$ the **in control** run length distribution is obtained, i.e.

$$P(N = k) = D(k-1) - D(k) \text{ for } k = 1, 2, 3, \ldots$$

with

$$D(k) = \int_0^1 \left(\frac{1}{\beta(j, n-j+1)} \sum_{h=0}^{j} \frac{(-1)^h}{j+h} \binom{n-j}{h} \{t\}^{j+h} \right)^k \frac{m!}{(b-1)!(m-b)!} t^{b-1} (1-t)^{m-b} dt$$

Thus the **unconditional** (*out of control*) average run length can be found as follows

$$ARL^{(1)} = \sum_{k=0}^{\infty} E_{X_{(b,m)}}\left(p^k\right)$$

$$= \sum_{k=0}^{\infty} D(k)$$

$$= \sum_{k=0}^{\infty} \int_0^1 \left(\frac{1}{\beta(j, n-j+1)} \sum_{h=0}^{j} \frac{(-1)^h}{j+h} \binom{n-j}{h} GF^{-1}(t)^{j+h} \right)^k f(t) dt$$

Let $F_1(t, j, n, F, G) = \left(\frac{1}{\beta(j, n-j+1)} \sum_{h=0}^{j} \frac{(-1)^h}{j+h} \binom{n-j}{h} \left\{ GF^{-1}(t) \right\}^{j+h} \right)$ so that

$$ARL^{(1)} = \sum_{k=0}^{\infty} \int_0^1 \left(F_1(t, j, n, F, G) \right)^k f(t) dt$$

$$= \int_0^1 \sum_{k=0}^{\infty} \left(F_1(t, j, n, F, G) \right)^k f(t) dt$$

$$= \int_0^1 \frac{1}{1 - F_1(t, j, n, F, G)} f(t) dt$$

The **unconditional** (*in control*) average run length can then be found by substituting $GF^{-1}(t)$ with t in $D(k)$ i.e.

$$ARL_0^{(1)} = \sum_{k=0}^{\infty} D(k)$$

$$= \sum_{k=0}^{\infty} \int_0^1 \left(\frac{1}{\beta(j, n-j+1)} \sum_{h=0}^{j} \frac{(-1)^h}{j+h} \binom{n-j}{h} \{t\}^{j+h} \right)^k f(t)\,dt$$

Let $C_1(t, j, n) = \left(\frac{1}{\beta(j, n-j+1)} \sum_{h=0}^{j} \frac{(-1)^h}{j+h} \binom{n-j}{h} \{t\}^{j+h} \right)$ so that

$$ARL_0^{(1)} = \sum_{k=0}^{\infty} \int_0^1 \left(C_1(t, j, n)\right)^k f(t)\,dt$$

$$= \int_0^1 \sum_{k=0}^{\infty} \left(C_1(t, j, n)\right)^k f(t)\,dt$$

$$= \int_0^1 \frac{1}{1 - C_1(t, j, n)} f(t)\,dt$$

5 5 2 Lower one-sided control chart

Result 5 9 Probability of no signal Conditional

$$p_l = \int_{GF\ (\ (\ \ ))}^{1} \frac{1}{\beta(j\ n-j+1)} u^{j\ 1}(1-u)^{\ j}\, du$$

Proof

$$\begin{aligned}
P\left(\text{No Signal} \mid X_{(\ m)} = x_{(\ m)}\right) &= P\left(Y_{(j\ )} \geq x_{(\ m)} \mid X_{(\ m)} = x_{(\ m)}\right) \\
&= P\left(G^{\ 1}\left(U_{(j\ )}\right) \geq F^{\ 1}\left(u_{(\ m)}\right) \mid U_{(\ m)} = u_{(\ m)}\right) \\
&= P\left(U_{(j\ )} \geq GF^{\ 1}\left(u_{(\ m)}\right) \mid U_{(\ m)} = u_{(\ m)}\right) \\
&= \int_{GF\ (\ (\ m))}^{1} f_{U_{(\ )}}(u)\, du \\
&= \int_{GF\ (\ (\ m))}^{1} \frac{1}{\beta(j\ n-j+1)} u^{j-1}(1-u)^{\ j}\, du \\
&= p_l
\end{aligned}$$

Result 5 10 Probability of a signal Conditional

$$1-p_l = 1-\int_{GF^{-1}(u_{(m)})}^{1}\frac{1}{\beta(j,n-j+1)}u^{j-1}(1-u)^{n-j}du$$

Proof

$$\begin{aligned}
P\left(\text{Signal} \mid X_{(m)}=x_{(m)}\right) &= P\left(Y_{(j)}<x_{(m)} \mid X_{(m)}=x_{(m)}\right)\\
&=1-P\left(Y_{(j)}\geq x_{(m)} \mid X_{(m)}=x_{(m)}\right)\\
&=1-\left(\text{No Signal} \mid X_{(m)}=x_{(m)}\right)\\
&=P\left(U_{(j)}\geq GF^{-1}\left(u_{(m)}\right)\right)\\
&=1-\int_{GF^{-1}(u_{(m)})}^{1}\frac{1}{\beta(j,n-j+1)}u^{j-1}(1-u)^{n-j}du\\
&=1-p_l
\end{aligned}$$

Result 5 11 Probabılıty of no sıgnal Unconditıonal

$$p_l = \int_0^1 \left(\frac{1}{\beta(j\ n-j+1)} \sum_{h\ 0}^{-j} \frac{(-1)^h}{j+h} \binom{n-j}{h} \left\{1-GF^{\ 1}(t)\right\}^{j\ h} \right) \frac{m!}{(a-1)!(m-a)!} t^{\ 1} (1-t)^{m}\ dt$$

Proof

$$\begin{aligned}
P(\text{No Sıgnal}) &= E_{X_{(\ m)}}\left(p_l\ \right) \\
&= E_{U_{(\ )}}\left(\int_{GF\ (\ (\ m))}^{1} \frac{1}{\beta(j\ n-j+1)} u^{j\ 1}(1-u)^{\ j}\, du \right) \\
&= \int_0^1 \left(\int_{GF\ (\ (\ m))}^{1} \frac{1}{\beta(j\ n-j+1)} u^{j\ 1}(1-u)^{\ j}\, du \right) \frac{m!}{(a-1)!(m-a)!} t^{\ 1}(1-t)^{m}\ dt \\
&= \int_0^1 \left(\int_{GF\ (\ (\ m))}^{1} \frac{1}{\beta(j\ n-j+1)} u^{j\ 1} \left\{ \sum_{h\ 0}^{j} (-1)^h \binom{n-j}{h} u^h \right\} du \right) \frac{m!}{(a-1)!(m-a)!} t^{\ 1}(1-t)^{m}\ dt \\
&= \int_0^1 \left(\frac{1}{\beta(j\ n-j+1)} \sum_{h\ 0}^{j} (-1)^h \binom{n-j}{h} \int_{GF\ (t)}^{1} u^{j\ h\ 1} du \right) \frac{m!}{(a-1)!(m-a)!} t^{\ 1}(1-t)^{m}\ dt \\
&= \int_0^1 \left(\frac{1}{\beta(j\ n-j+1)} \sum_{h\ 0}^{j} (-1)^h \binom{n-j}{h} \frac{u^{j\ h}}{j+h}\Big|_{GF\ (t)}^{1} \right) \frac{m!}{(a-1)!(m-a)!} t^{\ 1}(1-t)^{m}\ dt \\
&= \int_0^1 \left(\frac{1}{\beta(j\ n-j+1)} \sum_{h\ 0}^{j} \frac{(-1)^h}{j+h} \binom{n-j}{h} \left\{1-GF^{\ 1}(t)\right\}^{j\ h} \right) \frac{m!}{(a-1)!(m-a)!} t^{\ 1}(1-t)^{m}\ dt \\
&= p_l
\end{aligned}$$

Result 5 12 Probability of a signal Unconditional

$$1-p_l=1-\int_0^1\left(\frac{1}{\beta(j\ n-j+1)}\sum_{h\ 0}^{j}\frac{(-1)^h}{j+h}\binom{n-j}{h}\left\{1-GF^{\ 1}(t)\right\}^{j\ h}\right)\frac{m!}{(a-1)!(m-a)!}t^{\ \ 1}(1-t)^{m}\ dt$$

Proof

$$\begin{aligned}P(\text{Signal})&=E_{X_{(\ )}}\left(1-p_l\ \right)\\&=1-E_{X_{(\ )}}\left(p_l\ \right)\\&=1-P(\text{No Signal})\\&=1-\int_0^1\left(\frac{1}{\beta(j\ n-j+1)}\sum_{h\ 0}^{j}\frac{(-1)^h}{j+h}\binom{n-j}{h}\left\{1-GF^{\ 1}(t)\right\}^{j+h}\right)\frac{m!}{(a-1)!(m-a)!}t^{\ \ 1}(1-t)^{m-}\ dt\\&=1-p_l\end{aligned}$$

Result 5 13 Probabılıty of a false alarm Condıtıonal

$$CFAR = 1 - \int_{(\ m)}^{1} \frac{1}{\beta(j\ n-j+1)} u^{j\ 1}(1-u)^{\ j}\, du$$

Proof

$$\begin{aligned}
P\left(\text{False Alarm} \mid G=F\ X_{(\ m)}=x_{(\ m)}\right) &= P\left(Y_{(j\ )} < x_{(\ m)} \mid G=F\ X_{(\ m)}=x_{(\ m)}\right)\\
&= 1-P\left(Y_{(j\ )} \geq x_{(\ m)} \mid G=F\ X_{(\ m)}=x_{(\ m)}\right)\\
&= 1-P\left(F^{\ 1}\left(U_{(j\ )}\right) \geq F^{\ 1}\left(u_{(\ m)}\right) \mid U_{(\ m)}=u_{(\ m)}\right)\\
&= 1-P\left(U_{(j\ )} \geq u_{(\ m)} \mid U_{(\ m)}=u_{(\ )}\right)\\
&= 1-\int_{u_{(\ m)}}^{1} f_{U_{(\ )}}(u)\, du\\
&= 1-\int_{u_{(\ m)}}^{1} \frac{1}{\beta(j\ n-j+1)} u^{j\ 1}(1-u)^{-j}\, du\\
&= CFAR
\end{aligned}$$

Thıs expressıon could also have been obtaıned by sımply substıtutıng $G=F$ so that $GF^{\ 1}(u)=u$ ın the condıtıonal probabılıty of a sıgnal ın Result 5 ? Thus Result 5 ? ıs sımılar to Result 5 ? ı e $1-p_{l}\ =1-\int_{GF\ (\ (\ m))}^{1} \frac{1}{\beta(j\ n-j+1)} u^{j\ 1}(1-u)^{\ j} du$ wıth $G=F$

Result 5 14 Probabılıty of a false alarm Unconditıonal

$$FAR = 1 - \int_0^1 \left(\frac{1}{\beta(j\ n-j+1)} \sum_{h\ 0}^{j} \frac{(-1)^h}{j+h} \binom{n-j}{h} \{1-t\}^{j\ h} \right) \frac{m!}{(a-1)!(m-a)!} t^{\ 1} (1-t)^{m}\ \ dt$$

Proof

$$\begin{aligned}
P(\text{False Alarm}) &= E_{X_{(\ m)}}\left(P\left(\text{False Alarm} \mid G = F\ \ X_{(\ m)} = x_{(\ m)}\right)\right) \\
&= E_{U_{(\ m)}}\left(1 - \int_{u_{(\ m)}}^1 \frac{1}{\beta(j\ n-j+1)} u^{j\ 1}(1-u)^{\ \ j}\, du \right) \\
&= 1 - E_{U_{(\ m)}}\left(\int_{u_{(\ m)}}^1 \frac{1}{\beta(j\ n-j+1)} u^{j\ 1}(1-u)^{\ \ j}\, du \right) \\
&= 1 - \int_0^1 \left(\int_t^1 \frac{1}{\beta(j\ n-j+1)} u^{j\ 1}(1-u)^{\ \ j}\, du \right) \frac{m!}{(a-1)!(m-a)!} t^{\ 1}(1-t)^{m}\ \ dt \\
&= 1 - \int_0^1 \left(\int_t^1 \frac{1}{\beta(j\ n-j+1)} u^{j\ 1} \left\{ \sum_{h=0}^{-j} (-1)^h \binom{n-j}{h} u^h \right\} du \right) \frac{m!}{(a-1)!(m-a)!} t^{\ 1}(1-t)^{m}\ \ dt \\
&= 1 - \int_0^1 \left(\frac{1}{\beta(j\ n-j+1)} \sum_{h\ 0}^{j} (-1)^h \binom{n-j}{h} \int_t^1 u^{j\ h\ 1} du \right) \frac{m!}{(a-1)!(m-a)!} t^{\ 1}(1-t)^{m}\ \ dt \\
&= 1 - \int_0^1 \left(\frac{1}{\beta(j\ n-j+1)} \sum_{h\ 0}^{j} (-1)^h \binom{n-j}{h} \frac{u^{j+h}}{j+h}\Big|_t^1 \right) \frac{m!}{(a-1)!(m-a)!} t^{\ 1}(1-t)^{m}\ \ dt \\
&= 1 - \int_0^1 \left(\frac{1}{\beta(j\ n-j+1)} \sum_{h\ 0}^{j} \frac{(-1)^h}{j+h} \binom{n-j}{h} \{1-t\}^{j\ h} \right) \frac{m!}{(a-1)!(m-a)!} t^{\ 1}(1-t)^{m}\ \ dt \\
&= FAR
\end{aligned}$$

Thıs expressıon could also have been obtaıned by sımply substıtutıng $G = F$ so that $GF^{\ 1}(u) = u$ ın the uncondıtıonal probabılıty of a sıgnal ın Result 5 ? Thus Result 5 ? ıs sımılar to Result 5 ? ı e

$$1 - p_l = 1 - \int_0^1 \left(\frac{1}{\beta(j\ n-j+1)} \sum_{h\ 0}^{j} \frac{(-1)^h}{j+h} \binom{n-j}{h} \{1 - GF^{\ 1}(t)\}^{j\ h} \right) \frac{m!}{(a-1)!(m-a)!} t^{\ 1}(1-t)^{m}\ \ dt$$

wıth $G = F$

Result 5 15 Run length dıstrıbutıon Condıtıonal

$$P\left(N=k \mid X_{(\ m)}=x_{(\ m)}\right)=p_l^{k\ 1}\left(1-p_l\ \right)=p_l^{k\ 1}-p_l^{k} \quad \text{for } k=1\ 2\ 3$$

Proof

Gıven $X_{(\ m)}=x_{(\ m)}$ the run length dıstrıbutıon ıs geometrıc wıth parameter or probabılıty of success (sıgnal) $1-p_l$ where $p_l\ =P\left(Y_{(j\ )}\geq x_{(\ m)} \mid X_{(\ m)}=x_{(\ m)}\right)$ In other words

$$\left(N \mid X_{(\ m)}=x_{(\ m)}\right)\sim Geo\left(1-p_l\ \right)$$

Thus the condıtıonal probabılıty mass functıon (pmf) ıs

$$P\left(N=k \mid X_{(\ m)}=x_{(\ m)}\right)=p_l^{k\ 1}\left(1-p_l\ \right)=p_l^{k\ 1}-p_l^{k} \quad \text{with } k=1\ 2\ 3$$

whereas the condıtıonal cumulatıve dıstrıbutıon functıon (cdf) ıs

$$P\left(N\leq k \mid X_{(\ m)}=x_{(\ m)}\right)=\sum_{1}^{k}p_l^{\ 1}\left(1-p_l\ \right)=1-p_l^{k}$$

so that

$$P\left(N>k \mid X_{(\ m)}=x_{(\ m)}\right)=1-\left(1-p_l^{k}\ \right)=p_l^{k}$$

In addıtıon the expected value of N gıven $X_{(\ m)}=x_{(\ m)}$ or the condıtıonal average run length ıs

$$ARL=E\left(N \mid X_{(\ m)}=x_{(\ m)}\right)=\frac{1}{1-p_l}$$

These results follow dırectly and convenıently from the propertıes of the geometrıc dıstrıbutıon

To obtaın a more convenıent expressıon for the condıtıonal average run length we proceed as follows

$$
\begin{aligned}
ARL &= E\left(N \mid X_{(\ m)} = x_{(\ m)}\right) \\
&= \sum_{k\ 1}^{\infty} kP\left(N = k \mid X_{(\ m)} = x_{(\ m)}\right) \\
&= P\left(N=1 \mid X_{(\ m)} = x_{(\ m)}\right) \;+\; 2P\left(N=2 \mid X_{(\ m)} = x_{(\ m)}\right) \;+\; 3P\left(N=3 \mid X_{(\ m)} = x_{(\ m)}\right) \;+ \\
&\quad\; P\left(N=1 \mid X_{(\ m)} = x_{(\ m)}\right) \;+\; P\left(N=2 \mid X_{(\ m)} = x_{(\ m)}\right) \;+\; P\left(N=3 \mid X_{(\ m)} = x_{(\ m)}\right) \;+ \\
&= \qquad\qquad\qquad\qquad\qquad\quad\; +\; P\left(N=2 \mid X_{(\ m)} = x_{(\ m)}\right) \;+\; P\left(N=3 \mid X_{(\ m)} = x_{(\ m)}\right) \;+ \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\; +\; P\left(N=3 \mid X_{(\ m)} = x_{(\ m)}\right) \;+ \\
&= P\left(N>0 \mid X_{(\ m)} = x_{(\ m)}\right) + P\left(N>1 \mid X_{(\ m)} = x_{(\ m)}\right) + P\left(N>2 \mid X_{(\ m)} = x_{(\ m)}\right) + \\
&= \sum_{k\ 0}^{\infty} P\left(N > k \mid X_{(\ m)} = x_{(\ m)}\right) \\
&= \sum_{k\ 0}^{\infty} p_l^k
\end{aligned}
$$

Result 5 16 Run length distribution Unconditional

$$P(N=k)=E_{X_{(\ )}}\left(p_l^{k\ 1}\right)-E_{X_{(\ m)}}\left(p_l^{k}\ \right)=D_l\ (k-1)-D_l\ (k)\ \text{ for } k=1\ 2\ 3$$

Proof

$$\begin{aligned}P(N=k)&=E_{X_{(\ m)}}\left(P\left(N=k\mid X_{(\ m)}=x_{(\ m)}\right)\right)\\&=E_{X_{(\ m)}}\left(p_l^{k\ 1}-p_l^{k}\ \right)\\&=E_{X_{(\ )}}\left(p_l^{k\ 1}\right)-E_{X_{(\ )}}\left(p_l^{k}\ \right)\end{aligned}$$

However

$$\begin{aligned}E_{X_{(\ )}}\left(p_l^{k}\ \right)&=E_{U_{(\ m)}}\left(\left(\int_{GF\ \ (\ (\ m)}^{1}\frac{1}{\beta(j\ n-j+1)}u^{j\ 1}(1-u)^{\ \ j}du\right)^{k}\right)\\&=\int_0^1\left(\int_{GF\ \ (t)}^{1}\frac{1}{\beta(j\ n-j+1)}u^{j\ 1}(1-u)^{\ \ j}du\right)^{k}f(t)dt\\&=\int_0^1\left(\int_{GF\ \ (t)}^{1}\frac{1}{\beta(j\ n-j+1)}u^{j\ 1}(1-u)^{\ \ j}du\right)^{k}\frac{m!}{(a-1)!(m-a)!}t^{\ \ 1}(1-t)^{m\ \ }dt\\&=\int_0^1\left(\int_{GF\ \ (t)}^{1}\frac{1}{\beta(j\ n-j+1)}u^{j\ 1}\left\{\sum_{h\ 0}^{j}(-1)^{h}\binom{n-j}{h}u^{h}\right\}du\right)^{k}\frac{m!}{(a-1)!(m-a)!}t^{\ \ 1}(1-t)^{m\ \ }dt\\&=\int_0^1\left(\frac{1}{\beta(j\ n-j+1)}\sum_{h\ 0}^{\ -j}(-1)^{h}\binom{n-j}{h}\int_{GF\ \ (t)}^{1}u^{j\ h\ 1}du\right)^{k}\frac{m!}{(a-1)!(m-a)!}t^{\ \ 1}(1-t)^{m\ \ }dt\\&=\int_0^1\left(\frac{1}{\beta(j\ n-j+1)}\sum_{h\ 0}^{\ -j}(-1)^{h}\binom{n-j}{h}\frac{u^{j+h}}{j+h}\Big|_{GF\ \ (t)}^{1}\right)^{k}\frac{m!}{(a-1)!(m-a)!}t^{\ \ 1}(1-t)^{m\ \ }dt\\&=\int_0^1\left(\frac{1}{\beta(j\ n-j+1)}\sum_{h\ 0}^{j}\frac{(-1)^{h}}{j+h}\binom{n-j}{h}\left\{1-GF^{\ 1}(t)\right\}^{j\ h}\right)^{k}\frac{m!}{(a-1)!(m-a)!}t^{\ \ 1}(1-t)^{m\ \ }dt\\&=D_l\ (k)\end{aligned}$$

Thus

$$P(N=k)=E_{X_{(\ m)}}\left(p_l^{k\ 1}\right)-E_{X_{(\ )}}\left(p_l^{k}\ \right)=D_l\ (k-1)-D_l\ (k)\ \text{ for } k=1\ 2\ 3$$

with

$$D_l(0) = \int_0^1 \left(\frac{1}{\beta(j\ n-j+1)} \sum_{h\ 0}^{j} \frac{(-1)^h}{j+h} \binom{n-j}{h} \{1-GF^{\ 1}(t)\}^{j\ h} \right)^0 \frac{m!}{(a-1)!(m-a)!} t^{\ 1} (1-t)^{m} \ dt$$
$$= \int_0^1 \frac{m!}{(a-1)!(m-a)!} t^{\ 1} (1-t)^{m} \ dt$$
$$= 1$$

If $G(\ ) = F(\ )$ the **in control** run length distribution is obtained i e

$$P(N = k) = D_l(k-1) - D_l(k) \text{ for } k = 1\ 2\ 3$$

with

$$D_l(k) = \int_0^1 \left(\frac{1}{\beta(j\ n-j+1)} \sum_{h\ 0}^{j} \frac{(-1)^h}{j+h} \binom{n-j}{h} \{1-t^{j\ h}\} \right)^k \frac{m!}{(b-1)!(m-b)!} t^{b\ 1} (1-t)^{m\ b} dt$$

Thus the **unconditional** (*out of control*) average run length can be found as follows

$$ARL^{(1)} = \sum_{k\ 0}^{\infty} E_{X_{(\ )}}(p^k)$$
$$= \sum_{k\ 0}^{\infty} D\ (k)$$
$$= \sum_{k\ 0}^{\infty} \int_0^1 \left(\frac{1}{\beta(j\ n-j+1)} \sum_{h\ 0}^{-j} \frac{(-1)^h}{j+h} \binom{n-j}{h} GF^{\ 1}(t)^{j\ h} \right)^k f(t) dt$$

Let $F_1\ (t\ j\ n\ F\ G) = \left(\frac{1}{\beta(j\ n-j+1)} \sum_{h\ 0}^{j} \frac{(-1)^h}{j+h} \binom{n-j}{h} \{GF^{-1}(t)\}^{j+h} \right)$ so that

$$ARL^{(1)} = \sum_{k\ 0}^{\infty} \int_0^1 (F_1\ (t\ j\ n\ F\ G))^k f(t) dt$$
$$= \int_0^1 \sum_{k\ 0}^{\infty} (F_1\ (t\ j\ n\ F\ G))^k f(t) dt$$
$$= \int_0^1 \frac{1}{1 - F_1\ (t\ j\ n\ F\ G)} f(t) dt$$

The **unconditional** (*in control*) average run length can then be found by substituting $GF^{\;1}(t)$ with t in $D\;(k)$ i e

$$ARL_0^{(1)} = \sum_{k\;0}^{\infty} D\;(k)$$

$$= \sum_{k\;0}^{\infty} \int_0^1 \left(\frac{1}{\beta(j\;\;n-j+1)} \sum_{h\;0}^{j} \frac{(-1)^h}{j+h} \binom{n-j}{h} \{t\}^{j\;\;h} \right)^k f(t)\,dt$$

Let $C_1\;u(t\;\;j\;\;n) = \left(\frac{1}{\beta(j\;\;n-j+1)} \sum_{h\;0}^{j} \frac{(-1)^h}{j+h} \binom{n-j}{h} \{t\}^{j\;\;h} \right)$ so that

$$ARL_0^{(1)} = \sum_{k\;0}^{\infty} \int_0^1 \left(C_1\;\;(t\;\;j\;\;n)\right)^k f(t)\,dt$$

$$= \int_0^1 \sum_{k\;0}^{\infty} \left(C_1\;\;(t\;\;j\;\;n)\right)^k f(t)\,dt$$

$$= \int_0^1 \frac{1}{1-C_1\;\;(t\;\;j\;\;n)} f(t)\,dt$$

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5 5 3 Two-sided control chart

Result 5 17 Probability of no signal Conditional

$$p = \int_{GF\left(\ (\ )\right)}^{GF\left(\ (\ )\right)} \frac{1}{\beta(j\ n-j+1)} u^{j\ 1}(1-u)^{\ j} du$$

Proof

$$P\left(\text{No Signal} \mid X_{(\ m)} = x_{(\ m)}\ X_{(b\ m)} = x_{(b\ m)}\right)$$
$$= P\left(x_{(\ m)} \le Y_{(j\ )} \le x_{(b\ m)} \mid X_{(\ m)} = x_{(\ m)}\ X_{(b\ m)} = x_{(b\ m)}\right)$$
$$= P\left(F^{-1}\left(u_{(\ m)}\right) \le G^{\ 1}\left(U_{(j\ )}\right) \le F^{\ 1}\left(u_{(b\ m)}\right) \mid U_{(\ m)} = u_{(\ m)}\ U_{(b\ m)} = u_{(b\ m)}\right)$$
$$= P\left(GF^{\ 1}\left(u_{(\ m)}\right) \le U_{(j\ )} \le GF^{\ 1}\left(u_{(b\ m)}\right) \mid U_{(\ m)} = u_{(\ m)}\ U_{(b\ m)} = u_{(b\ m)}\right)$$
$$= \int_{GF\left(\ (\ m)\right)}^{GF\left(\ (\ m)\right)} \frac{1}{\beta(j\ n-j+1)} u^{j\ 1}(1-u)^{\ j} du$$
$$= p$$

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Result 5 18 Probability of a signal Conditional

$$1-p \;=1-\int_{GF\left(\;(\;)\right)}^{GF\left(\;(\;)\right)} \frac{1}{\beta(j\;\,n-j+1)} u^{j\;\,1}(1-u)^{\;\,j} du$$

Proof

$$P\left(\text{Signal} \mid X_{(\;\,m)}=x_{(\;\,m)}\;\, X_{(b\,m)}=x_{(b\,m)}\right)$$

$$=P\left(Y_{(j\;\,)}<x_{(\;\,m)} \mid X_{(\;\,m)}=x_{(\;\,m)}\;\, X_{(b\,m)}=x_{(b\,m)}\right)+P\left(Y_{(j\;\,)}>x_{(b\,m)} \mid X_{(\;\,m)}=x_{(\;\,m)}\;\, X_{(b\,m)}=x_{(b\,m)}\right)$$

$$=1-P\left(x_{(\;\,m)} \leq Y_{(j\;\,)} \leq x_{(b\,m)} \mid X_{(\;\,m)}=x_{(\;\,m)}\;\, X_{(b\,m)}=x_{(b\,m)}\right)$$

$$=1-P\left(\text{No Signal} \mid X_{(\;\,m)}=x_{(\;\,m)}\;\, X_{(b\,m)}=x_{(b\,m)}\right)$$

$$=1-\int_{GF\left(\;(\;\,m)\right)}^{GF\left(\;(\;\,m)\right)} \frac{1}{\beta(j\;\,n-j+1)} u^{j\;\,1}(1-u)^{\;\,j} du$$

$$=1-p$$

Result 5 19 Probability of no signal Unconditional

$$p=\int_0^1\int_0^t\left(\frac{1}{\beta(j,n-j+1)}\sum_{h=0}^{n-j}\frac{(-1)^h}{j+h}\binom{n-j}{h}\left\{GF^{-1}(t)^{j+h}-GF^{-1}(s)^{j+h}\right\}\right)f(s,t)\,ds\,dt$$

Proof

$$\begin{aligned}
P(\text{No Signal}) &= E_{X_{(a)},X_{(b)}}(p) \\
&= E_{U_{(a,m)},U_{(b,m)}}\left(\int_{GF^{-1}(U_{(a,m)})}^{GF^{-1}(U_{(b,m)})}\frac{1}{\beta(j,n-j+1)}u^{j-1}(1-u)^{n-j}du\right) \\
&= \int_0^1\int_0^t\left(\int_{GF^{-1}(s)}^{GF^{-1}(t)}\frac{1}{\beta(j,n-j+1)}u^{j-1}(1-u)^{n-j}du\right)f(s,t)\,ds\,dt \\
&= \int_0^1\int_0^t\left(\int_{GF^{-1}(s)}^{GF^{-1}(t)}\frac{1}{\beta(j,n-j+1)}u^{j-1}\left\{\sum_{h=0}^{n-j}(-1)^h\binom{n-j}{h}u^h\right\}du\right)f(s,t)\,ds\,dt \\
&= \int_0^1\int_0^t\left(\frac{1}{\beta(j,n-j+1)}\sum_{h=0}^{n-j}(-1)^h\binom{n-j}{h}\int_{GF^{-1}(s)}^{GF^{-1}(t)}u^{j+h-1}du\right)f(s,t)\,ds\,dt \\
&= \int_0^1\int_0^t\left(\frac{1}{\beta(j,n-j+1)}\sum_{h=0}^{n-j}(-1)^h\binom{n-j}{h}\frac{u^{j+h}}{j+h}\Bigg|_{GF^{-1}(s)}^{GF^{-1}(t)}\right)f(s,t)\,ds\,dt \\
&= \int_0^1\int_0^t\left(\frac{1}{\beta(j,n-j+1)}\sum_{h=0}^{n-j}\frac{(-1)^h}{j+h}\binom{n-j}{h}\left\{GF^{-1}(t)^{j+h}-GF^{-1}(s)^{j+h}\right\}\right)f(s,t)\,ds\,dt \\
&= p
\end{aligned}$$

where $f(s,t)=\frac{m!}{(a-1)!(b-a-1)!(m-b)!}s^{a-1}(t-s)^{b-a-1}(1-t)^{m-b}$ is the joint probability density function (pdf) of the two order statistics $U_{(a,m)}$ and $U_{(b,m)}$

Result 5 20 Probability of a signal Unconditional

$$1-p = 1-\int_0^1\int_0^t\left(\frac{1}{\beta(j\ n-j+1)}\sum_{h\ 0}^{j}\frac{(-1)^h}{j+h}\binom{n-j}{h}\left\{GF^{\ 1}(t)^{j\ h} - GF^{\ 1}(s)^{j\ h}\right\}\right)f(s\ t)\,ds\,dt$$

Proof

$$\begin{aligned}
P(\text{Signal}) &= E_{X_{(\ )},X_{(\ )}}(1-p\ ) \\
&= 1-E_{X_{(\ )},X_{(\ )}}(p\ ) \\
&= 1-P(\text{No Signal}) \\
&= 1-\int_0^1\int_0^t\left(\frac{1}{\beta(j\ n-j+1)}\sum_{h=0}^{j}\frac{(-1)^h}{j+h}\binom{n-j}{h}\left\{GF^{\ 1}(t)^{j\ h} - GF^{\ 1}(s)^{j\ h}\right\}\right)f(s\ t)\,ds\,dt \\
&= 1-p
\end{aligned}$$

where $f(s\ t) = \frac{m!}{(a-1)!(b-a-1)!(m-b)!}s^{\ 1}(t-s)^{b-\ 1}(1-t)^{m\ b}$ is the joint probability density function (pdf) of the two order statistics $U_{(\ m)}$ and $U_{(b\ m)}$

Result 5 21 Probabılıty of a false alarm Condıtıonal

$$CFAR = 1 - \int_{u_{(\ )}}^{u_{(\ )}} \frac{1}{\beta(\jmath\ \ n-\jmath+1)} u^{\jmath\ \ 1}(1-u)^{\ \ \jmath} du$$

Proof

$$P\left(\text{False Alarm} \mid G = F\ \ X_{(\ \ m)} = x_{(\ \ m)}\ \ X_{(b\ m)} = x_{(b\ m)}\right)$$
$$= P\left(Y_{(\jmath\ \ )} < x_{(\ \ m)} \mid G = F\ \ X_{(\ \ m)} = x_{(\ \ m)}\ \ X_{(b\ m)} = x_{(b\ m)}\right) + P\left(Y_{(\jmath\ \ )} > x_{(b\ m)} \mid G = F\ \ X_{(\ \ m)} = x_{(\ \ m)}\ \ X_{(b\ m)} = x_{(b\ m)}\right)$$
$$= 1 - P\left(F^{\ \ 1}\left(u_{(\ \ m)}\right) \le F^{\ \ 1}\left(U_{(\jmath\ \ )}\right) \le F^{\ \ 1}\left(u_{(b\ m)}\right) \mid U_{(\ \ m)} = u_{(\ \ m)}\ \ U_{(b\ m)} = u_{(b\ m)}\right)$$
$$= 1 - P\left(u_{(\ \ m)} \le U_{(\jmath\ \ )} \le u_{(b\ m)} \mid U_{(\ \ m)} = u_{(\ \ m)}\ \ U_{(b\ m)} = u_{(b\ m)}\right)$$
$$= 1 - \int_{u_{(\ \ m)}}^{u_{(\ \ m)}} f_{U_{(\ \ )}} du$$
$$= 1 - \int_{u_{(\ \ m)}}^{u_{(\ \ m)}} \frac{1}{\beta(\jmath\ \ n-\jmath+1)} u^{\jmath\ \ 1}(1-u)^{\ \ \jmath} du$$
$$= CFAR$$

Thıs expressıon could also have been obtaıned by sımply substıtutıng $G = F$ so that $GF^{\ \ 1}(u) = u$ ın the condıtıonal probabılıty of a sıgnal ın Result 5 18 Thus Result 5 21 ıs sımılar to Result 5 18 ı e $1 - p\ \ = 1 - \int_{GF\ (\ (\ \ ))}^{GF\ (\ (\ \ m))} \frac{1}{\beta(\jmath\ \ n-\jmath+1)} u^{\jmath\ \ 1}(1-u)^{\ \ \jmath} du$ wıth $G = F$

Result 5 22 Probability of a false alarm Unconditional

$$FAR = 1 - \int_0^1 \int_0^t \left(\frac{1}{\beta(j, n-j+1)} \sum_{h=0}^{j} \frac{(-1)^h}{j+h} \binom{n-j}{h} \left\{ t^{j+h} - s^{j+h} \right\} \right) f(s,t)\, ds\, dt$$

Proof

$$P(\text{False Alarm}) = E_{X_{(a,m)}, X_{(b,m)}} \left(P\left(\text{False Alarm} \mid X_{(a,m)} = x_{(a,m)}, X_{(b,m)} = x_{(b,m)} \right) \right)$$

$$= E_{U_{(a,m)}, U_{(b,m)}} \left(1 - \int_{u_{(a,m)}}^{u_{(b,m)}} \frac{1}{\beta(j, n-j+1)} u^{j-1} (1-u)^{n-j} du \right)$$

$$= 1 - \int_0^1 \int_0^t \left(\int_s^t \frac{1}{\beta(j, n-j+1)} u^{j-1} (1-u)^{n-j} du \right) f(s,t)\, ds\, dt$$

$$= 1 - \int_0^1 \int_0^t \left(\int_s^t \frac{1}{\beta(j, n-j+1)} u^{j-1} \left\{ \sum_{h=0}^{j} (-1)^h \binom{n-j}{h} u^h \right\} du \right) f(s,t)\, ds\, dt$$

$$= 1 - \int_0^1 \int_0^t \left(\frac{1}{\beta(j, n-j+1)} \sum_{h=0}^{j} (-1)^h \binom{n-j}{h} \int_s^t u^{j+h-1} du \right) f(s,t)\, ds\, dt$$

$$= 1 - \int_0^1 \int_0^t \left(\frac{1}{\beta(j, n-j+1)} \sum_{h=0}^{j} (-1)^h \binom{n-j}{h} \frac{u^{j+h}}{j+h} \Big|_s^t \right) f(s,t)\, ds\, dt$$

$$= 1 - \int_0^1 \int_0^t \left(\frac{1}{\beta(j, n-j+1)} \sum_{h=0}^{j} \frac{(-1)^h}{j+h} \binom{n-j}{h} \left\{ t^{j+h} - s^{j+h} \right\} \right) f(s,t)\, ds\, dt$$

$$= FAR$$

where $f(s,t) = \frac{m!}{(a-1)!(b-a-1)!(m-b)!} s^{a-1} (t-s)^{b-a-1} (1-t)^{m-b}$ is the joint probability density function (pdf) of the two order statistics $U_{(a,m)}$ and $U_{(b,m)}$

This expression could also have been obtained by simply substituting $G = F$ so that $GF^{-1}(u) = u$ in the unconditional probability of a signal in Result 5 ? Thus Result 5 ? is similar to Result 5 ? i e

$$1 - p = 1 - \int_0^1 \int_0^t \left(\frac{1}{\beta(j, n-j+1)} \sum_{h=0}^{j} \frac{(-1)^h}{j+h} \binom{n-j}{h} \left\{ GF^{-1}(t)^{j+h} - GF^{-1}(s)^{j+h} \right\} \right) f(s,t)\, ds\, dt$$ with

$G = F$

Result 5 23 Run length dıstrıbutıon Condıtıonal

$$P\left(N = k \mid X_{(\ m)} = x_{(\ m)}\ X_{(b\,m)} = x_{(b\,m)}\right) = p^{k\ 1}\left(1 - p\ \right) = p^{k\ 1} - p^{k} \text{ for } k = 1\ 2\ 3$$

Proof

Gıven $X_{(\ m)} = x_{(\ m)}$ and $X_{(b\,m)} = x_{(b\,m)}$ the run length dıstrıbutıon ıs geometrıc wıth parameter or probabılıty of success (sıgnal)

$1 - p\ = 1 - P\left(x_{(\ m)} \le Y_{(\jmath\ )} \le x_{(b\,m)} \mid X_{(\ m)} = x_{(\ m)}\ X_{(b\,m)} = x_{(b\,m)}\right)$ In other words

$\left(N \mid X_{(\ m)} = x_{(\ m)}\ X_{(b\,m)} = x_{(b\,m)}\right) \sim Geo\left(1 - p\ \right)$

Thus the condıtıonal probabılıty mass functıon (pmf) ıs

$$P\left(N = k \mid X_{(\ m)} = x_{(\ m)}\ X_{(b\,m)} = x_{(b\,m)}\right) = p^{k\ 1}\left(1 - p\ \right) = p^{k\ 1} - p^{k} \text{ for } k = 1\ 2\ 3$$

whereas the condıtıonal cumulatıve dıstrıbutıon functıon (cdf) ıs

$$P\left(N \le k \mid X_{(\ m)} = x_{(\ m)}\ X_{(b\,m)} = x_{(b\,m)}\right) = \sum_{-1}^{k} p\ ^{1}\left(1 - p\ \right) = 1 - p^{k}$$

so that

$$P\left(N > k \mid X_{(\ m)} = x_{(\ m)}\ X_{(b\,m)} = x_{(b\,m)}\right) = 1 - \left(1 - p^{k}\right) = p^{k}$$

In addıtıon the expected value of N gıven $X_{(\ m)} = x_{(\ m)}$ and $X_{(b\,m)} = x_{(b\,m)}$ or the condıtıonal average run length ıs

$$ARL = E\left(N \mid X_{(\ m)} = x_{(\ m)}\ X_{(b\,m)} = x_{(b\,m)}\right) = \frac{1}{1 - p}$$

To obtaın a more convenıent expressıon for the condıtıonal average run length we proceed as follows

$$
\begin{aligned}
ARL &= E\left(N \mid X_{(\ m)}\ X_{(b\,m)}\right) \\
&= \sum_{k\ 1}^{\infty} kP\left(N = k \mid X_{(\ m)}\ X_{(b\,m)}\right) \\
&= P\left(N = 1 \mid X_{(\ m)}\ X_{(b\,m)}\right) \quad + \quad 2P\left(N = 2 \mid X_{(\ m)}\ X_{(b\,m)}\right) \quad + \quad 3P\left(N = 3 \mid X_{(\ m)}\ X_{(b\,m)}\right) \quad + \\
&\quad\ P\left(N = 1 \mid X_{(\ m)}\ X_{(b\,m)}\right) \quad + \quad P\left(N = 2 \mid X_{(\ m)}\ X_{(b\,m)}\right) \quad + \quad P\left(N = 3 \mid X_{(\ m)}\ X_{(b\,m)}\right) \quad + \\
&= \qquad\qquad\qquad\qquad\qquad\quad + \quad P\left(N = 2 \mid X_{(\ m)}\ X_{(b\,m)}\right) \quad + \quad P\left(N = 3 \mid X_{(\ m)}\ X_{(b\,m)}\right) \quad + \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad + \quad P\left(N = 3 \mid X_{(\ m)}\ X_{(b\,m)}\right) \quad + \\
&= P\left(N > 0 \mid X_{(\ m)}\ X_{(b\,m)}\right) + P\left(N > 1 \mid X_{(\ m)}\ X_{(b\,m)}\right) + P\left(N > 2 \mid X_{(\ m)}\ X_{(b\,m)}\right) + \\
&= \sum_{k\ 0}^{\infty} P\left(N > k \mid X_{(\ m)}\ X_{(b\,m)}\right) \\
&= \sum_{k\ 0}^{\infty} p^{k}
\end{aligned}
$$

Result 5 24 Run length distribution Unconditional

Proof

$$
\begin{aligned}
P(N=k) &= E_{X_{(\ m)}\ X_{(\ m)}}\left(P\left(N=k \mid X_{(\ m)}=x_{(\ m)}\ X_{(b\,m)}=x_{(b\,m)}\right)\right) \\
&= E_{X_{(\ m)}\ X_{(\ m)}}\left(p^{k\ 1}-p^{k}\right) \\
&= E_{X_{(\ m)}\ X_{(\ )}}\left(p^{k\ 1}\right)-E_{X_{(\ m)}\ X_{(\ )}}\left(p^{k}\right)
\end{aligned}
$$

However

$$
\begin{aligned}
E_{X_{(\ m)}\ X_{(\ m)}}\left(p^{k}\right) &= E_{U_{(\ m)}\ U_{(\ )}}\left(\left(\int_{GF\ (\ (\ m))}^{GF\ (\ (\ m))}\frac{1}{\beta(j\ n-j+1)}u^{j\ 1}(1-u)\ ^{j}du\right)^{k}\right) \\
&= \int_0\int_0\left(\int_{GF\ (\ )}^{GF\ (t)}\frac{1}{\beta(j\ n-j+1)}u^{j\ 1}(1-u)\ ^{j}du\right)^{k}f(s\ t)\,dsdt \\
&= \int_0\int_0\left(\int_{GF\ (\ )}^{GF\ (t)}\frac{1}{\beta(j\ n-j+1)}u^{j\ 1}\left\{\sum_{h=0}^{j}(-1)^{h}\binom{n-j}{h}u^{h}\right\}du\right)^{k}f(s\ t)\,dsdt \\
&= \int_0\int_0\left(\frac{1}{\beta(j\ n-j+1)}\sum_{h=0}^{-j}(-1)^{h}\binom{n-j}{h}\int_{GF\ (\ )}^{GF\ (t)}u^{j\ h\ 1}du\right)^{k}f(s\ t)\,dsdt \\
&= \int_0\int_0\left(\frac{1}{\beta(j\ n-j+1)}\sum_{h=0}^{-j}(-1)^{h}\binom{n-j}{h}\frac{u^{j\ h}}{j+h}\Big|_{GF\ (\ )}^{GF\ (t)}\right)^{k}f(s\ t)\,dsdt \\
&= \int_0\int_0\left(\frac{1}{\beta(j\ n-j+1)}\sum_{h=0}^{-j}\frac{(-1)^{h}}{j+h}\binom{n-j}{h}\left\{GF^{\ 1}(t)^{j\ h}-GF^{\ 1}(s)^{j\ h}\right\}\right)^{k}f(s\ t)\,dsdt \\
&= D\ (k)
\end{aligned}
$$

Thus

$$
P(N=k)=E_{X_{(\ )}\ X_{(\ )}}\left(p^{k\ 1}\right)-E_{X_{(\ )}\ X_{(\ m)}}\left(p^{k}\right)=D\ (k-1)-D\ (k)\ \text{ for } k=1\ 2\ 3
$$

with

$$D\ (0)=\int_0^1\int_0^t\left(\frac{1}{\beta(j\ n-j+1)}\sum_{h=0}^{j}\frac{(-1)^h}{j+h}\binom{n-j}{h}\left\{GF^{-1}(t)^{j+h}-GF^{-1}(s)^{j+h}\right\}\right)^0 f(s\ t)\,dsdt$$

$$=\int_0^1\int_0^t f(s\ t)\,dsdt$$

$$=\int_0^1\int_0^t\frac{m!}{(a-1)!(b-a-1)!(m-b)!}s^{a-1}(t-s)^{b-a-1}(1-t)^{m-b}\,dsdt$$

$$=1$$

If $G(\ )=F(\ )$ the **in control** run length distribution is obtained i e

$$P(N=k)=D(k-1)-D(k)\ \text{for}\ k=1\ 2\ 3$$

with

$$D(k)=\int_0^1\int_0^t\left(\frac{1}{\beta(j\ n-j+1)}\sum_{h=0}^{j}\frac{(-1)^h}{j+h}\binom{n-j}{h}\left\{t^{j+h}-s^{j+h}\right\}\right)^k f(s\ t)\,dsdt$$

Thus the **unconditional** (*out of control*) average run length can be found as follows

$$ARL^{(2)}=\sum_{k=0}^{\infty}E_{X_{(\ )}X_{(\ )}}\left(p^k\right)$$

$$=\sum_{k=0}^{\infty}D\ (k)$$

$$=\sum_{k=0}^{\infty}\int_0^1\int_0^t\left(\frac{1}{\beta(j\ n-j+1)}\sum_{h=0}^{j}\frac{(-1)^h}{j+h}\binom{n-j}{h}\left\{GF^{-1}(t)^{j+h}-GF^{-1}(s)^{j+h}\right\}\right)^k f(s\ t)\,dsdt$$

$$F_2(s\ t\ j\ n\ F\ G)=\left(\frac{1}{\beta(j\ n-j+1)}\sum_{h=0}^{j}\frac{(-1)^h}{j+h}\binom{n-j}{h}\left\{GF^{-1}(t)^{j+h}-GF^{-1}(s)^{j+h}\right\}\right)\ \text{so that}$$

$$ARL^{(2)}=\sum_{k=0}^{\infty}\int_0^1\int_0^t\left(F_2(s\ t\ j\ n\ F\ G)\right)^k f(s\ t)\,dsdt$$

$$=\int_0^1\int_0^t\sum_{k=0}^{\infty}\left(F_2(s\ t\ j\ n\ F\ G)\right)^k f(s\ t)\,dsdt$$

$$=\int_0^1\int_0^t\frac{1}{1-F_2(s\ t\ j\ n\ F\ G)}f(s\ t)\,dsdt$$

The **unconditional** (*in control*) average run length can then be found by substituting $GF^{-1}(t)$ with t in $D\ (k)$ i e

$$ARL_0^{(2)} = \sum_{k\ 0}^{\infty} D(k)$$

$$= \sum_{k\ 0}^{\infty} \int_0^1 \int_0^t \left(\frac{1}{\beta(j\ \ n-j+1)} \sum_{h\ 0}^{j} \frac{(-1)^h}{j+h} \binom{n-j}{h} \left\{ t^{j\ h} - s^{j\ h} \right\} \right)^k f(s\ \ t)\,dsdt$$

Let $C_2(s\ \ t\ \ j\ \ n) = \left(\frac{1}{\beta(j\ \ n-j+1)} \sum_{h\ 0}^{j} \frac{(-1)^h}{j+h} \binom{n-j}{h} \left\{ t^{j\ h} - s^{j\ h} \right\} \right)$ so that

$$ARL_0^{(2)} = \sum_{k\ 0}^{\infty} \int_0^1 \int_0^t \left(C_2(s\ \ t\ \ j\ \ n) \right)^k f(s\ \ t)\,dsdt$$

$$= \int_0^1 \int_0^t \sum_{k\ 0}^{\infty} \left(C_2(s\ \ t\ \ j\ \ n) \right)^k f(s\ \ t)\,dsdt$$

$$= \int_0^1 \int_0^t \frac{1}{1 - C_2(s\ \ t\ \ j\ \ n)} f(s\ \ t)\,dsdt$$

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