PART II: Investigation into the possible implementation of non-linear response reconstruction

This second part of the thesis presents an investigation into the possible implementation of polynomial non-linear system identification routines in response reconstruction. Non-linear system identification has been well researched from a mathematical formulation point of view – see Billings *et al.* [5][6][7] Chen [18], Fasol [26], Korenberg [36], etc. To the author's knowledge, non-linear dynamic system identification has not previously been applied to the field of response reconstruction.

The development of a non-linear response reconstruction technique is presented in a concise, almost chronological manner. Only the most relevant theory is included in the body of the thesis, with more detail included in the appendices. Linear system identification is introduced in Chapter 4, at the hand of the ARX [40][41] time domain model formulation. The ARX fundamentals are extended to accommodate polynomial non-linear modelling capabilities with the NARX [9][46] model structure in Chapter 5. Application of the developed NARX modelling and response reconstruction techniques is presented in Chapter 6. Finally some conclusions and recommendations for future research are made in Chapter 7.

Chapter 4

Linear time domain system identification: ARX

Dynamic response reconstruction for fatigue tests has the global aim to reproduce operational measured response stresses in the test structure as accurately as possible using servo-hydraulic actuators. To calculate actuator drive signals from knowledge of operational measured responses a dynamic model that describes the complete system is required. Such a dynamic model is found by using some system identification formulation. In choosing from the multitude of different system identification model types and structures, two factors are of prime importance: accuracy and ease of operation. System identification for use in response reconstruction is predominantly frequency based, and linear. Accuracy, ease of use, low calculation time, and minimal computing requirements prompted investigation into a time domain approach. More specifically the ARX model format, as presented by Ljung [40][41] combined with a time domain state space description was indicated by Raath [51] to be ideal for use in response reconstruction. The characteristics of the ARX model structure and the modifications thereof are briefly discussed.

4.1. ARX model structure

A basic single input, single output (SISO) dynamic system is presented in Figure 4.1, with a system input *u(t)*, output *y(t)* and a disturbance signal *e(t)*; *t* = 1,2,....., *N*. The development of the ARX model from the generalise time-invariant model structure is adapted from the work presented by Ljung [40].

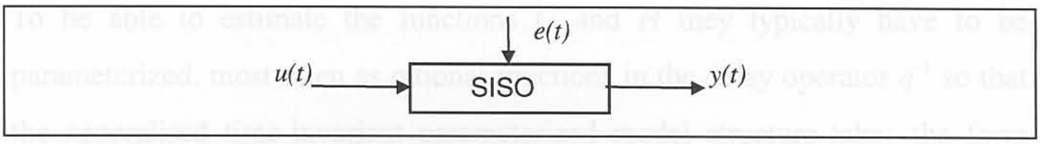

Figure 4.1: Basic SISO configuration

Assuming the signals are related by a linear system, we can write:

$$y(t) = G(q)u(t) + v(t) \qquad (4\text{-}1)$$

where q is the shift operator and $G(q)u(t)$ is short for

$$G(q)u(t) = \sum_{k=1}^{\infty} g(k)u(t-k) \qquad (4\text{-}2)$$

In (4-1) $v(t)$ is an additional, unmeasurable disturbance (noise). Its properties can be expressed in terms of its spectrum $\Phi_v(\omega)$, which is defined as:

$$\Phi_v(\omega) = \sum_{\tau=-\infty}^{\infty} R_v(\tau)e^{-i\omega\tau} \qquad (4\text{-}3)$$

where $R_v(t)$ is the covariance function of $v(t)$ with E the mathematical expectation.

$$R_v(\tau) = Ev(t)v(t-\tau) \qquad (4\text{-}4)$$

Alternatively, the disturbance $v(t)$ can be described as filtered white noise:

$$v(t) = H(t)\,e(t) \qquad (4\text{-}5)$$

Substitution into (4-1) gives the complete time domain description for the system of Figure 4.1.

$$y(t) = G(q)u(t) + H(t)\, e(t) \qquad (4\text{-}6)$$

To be able to estimate the functions *G* and *H* they typically have to be parameterized, most often as rational functions in the delay operator q^{-1} so that the generalised time-invariant parameterized model structure takes the form (Ljung [40]):

$$A(q)\,y(t) = \frac{B(q)}{F(q)}u(t) + \frac{C(q)}{D(q)}e(t) \qquad (4\text{-}7)$$

where *A(q)*, *B(q)*, *C(q)*, *D(q)* and *F(q)* are polynomials in the delay operator q^{-1}. Various simplifications can be applied, one of which leads to the ARX-model (Auto Regressive with eXogenous input).

$$A(q)\,y(t) = B(q)u(t) + e(t) \qquad (4\text{-}8)$$

with:

$$A(q) = 1 + a_1 q^{-1} + a_2 q^{-2} + \cdots + a_{na} q^{-na} \qquad (4\text{-}9)$$

$$B(q) = b_0 + b_1 q^{-1} + b_2 q^{-2} + \cdots + b_{nb} q^{-nb} \qquad (4\text{-}10)$$

If *nk* is the number of delays from input to output the model is usually written as:

$$A(q)\,y(t) = B(q)u(t - nk) + e(t) \qquad (4\text{-}11)$$

If expanded, the ARX's polynomial structure may be written as a linear difference equation of the form:

$$\begin{aligned} y(t) = a_1 \cdot y(t-1) + \cdots + a_{na} \cdot y(t-na) + \cdots \\ \cdots + b_0 \cdot u(t-nk) + b_1 \cdot u(t-nk-1) + \cdots + b_{nb} \cdot u(t-nk-nb) \end{aligned} \quad (4\text{-}12)$$

This is the one-step-ahead predictor for a SISO system. A novel method of expanding the ARX formulation for MIMO systems was proposed by Raath [52]. For a system with *ny* outputs, the ARX difference equation of (4-12) is expanded to *ny* multiple input, single output (MISO) one step ahead predictors, so that for each output channel *k:*

$$\begin{aligned} y_k(t) = y_1(t-1),\ldots,y_1(t-na_1), y_2(t-1),\ldots,y_2(t-na_2),\ldots y_{ny}(t-1),\ldots,y_{ny}(t-na_{ny}),\ldots \\ \ldots u_1(t-nk_1),\ldots,u_1(t-nk_1-nb_1), u_2(t-nk_2),\ldots,u_2(t-nk_2-nb_2),\ldots \\ \ldots,u_{nu}(t-nk_{nu}),\ldots,u_{nu}(t-nk_{nu}-nb_{nu}) \end{aligned} \quad (4\text{-}13)$$

These MISO models are combined into a MIMO discrete state space model description where:

$$\begin{aligned} x_{k+1} &= \Phi x_k + \Gamma u_k \\ y_k &= \mathbf{C} x_k + \mathbf{D} u_k \end{aligned} \quad (4\text{-}14)$$

and: $x_k =$ state vector
$u_k =$ input vector
$\mathrm{y_k} =$ output vector
$\Phi =$ state matrix
$\Gamma =$ input matrix
$\mathbf{C} =$ output matrix
$\mathbf{D} =$ direct transmission matrix

For brevity the combination of MISO models into a single MIMO model description is not included in this thesis, it is however presented in detail by Raath [51][52]. To further simplify the model description the "full order" approach to structure detection is presented in the next Section.

4.2. ARX Structure selection

For application in response reconstruction a black box model, which requires minimal information about the system prior to modelling is needed. The ARX formulation in (4-13) is a general black-box type model capable of accurately describing the dynamic behaviour of most linear engineering structures. It is however required to estimate the model structure prior to modelling, which for each output channel k involves:

- Selection of the number of a parameters, na_k
- Selection of the number of b parameters, nb_k
- Selection of the number of delays, nk_k

For MIMO systems the number of model structure parameters to be estimated explodes for increasing numbers of channels. This problem is solved by utilisation of the "full order" approach, presented by Raath [51], which defines that for each output channel $y_k(t)$ only the dynamic model order, n, must be selected. Equation (4-15) presents the relation between the parameters na, nb, and nk and the dynamic model order n for each output channel k:

$$: \quad \left.\begin{aligned} na_i &= n_k \\ nb_j &= n_k + 1 \\ nk_j &= 0 \end{aligned}\right\} \begin{aligned} i &= 1,2,\cdots,ny \\ j &= 1,2,\cdots,nu \end{aligned} \qquad (4\text{-}15)$$

Each MISO model is identified for increasing model orders to detect the optimal model structure, a relatively simple procedure, even for MIMO systems with large numbers of channels.

This full order MIMO-ARX model is the basis of the linear QanTiM package, and showed potential to be extended to include non-linear modelling terms. This expansion of the linear ARX model to the non-linear NARX model is presented in the next chapter.