# Trading Mortality

by
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# Declaration

I, the undersigned, hereby declare that the dissertation submitted herewith for the degree Magister Scientiae of the University of Pretoria contains my own, independent work and has not been submitted for any degree at any other university.

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# Abstract

This dissertation sets out to describe a set of financial instruments whose cash flows are driven by the movements in some underlying population's mortality rates. For example, a longevity bond where the coupons are determined with reference to the proportion of the initial population that are alive at the coupon date. Other examples include mortality swaps and mortality swaptions which are analogous to interest rate swaps and interest rate swaptions.

It also aims to show there are risks associated with mortality and that these mortality driven instruments can be used to manage some of these risks.

These instruments should also enable portfolios that replicate mortality driven cash flows to be constructed. This would in turn allow the market consistent valuation of these cash flows.

To construct a pricing framework for these mortality based instruments a stochastic mortality model is needed. In this dissertation the stochastic mortality model used was the Lee-Carter model. The Lee-Carter model in essence models mortality rates per age by calendar year or cohort year using Time Series techniques.



# Key Words

Stochastic Mortality; Lee-Carter Models; Market Consistent Methods; Mortality Derivatives.

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# Terminology and notation

## Survival models

- $l_{x,t}$ the number of lives aged $x$ in year $t$
- $d_{x,t}$ the number of deaths of lives aged $x$ in year $t$
- Since $d_{x,t}$ the number of deaths of lives aged $x$ in year $t$, $l_{x+1,t+1} = l_{x,t} - d_{x,t}$
- $q_{x,t}$ = The probability of a life aged $x$ in year $t$ dies before reaching $x+1$.
- The above, leads to the following identity: $q_{x,t} = \frac{d_{x,t}}{l_{x,t}}$
- $p_{x,t}$ = The probability of a life aged $x$ in year $t$ reaches $x+1$.
- Given that a life aged $x$ in year $t$ can only be alive or dead at $x+1$ it follows: $q_{x,t} = 1 - p_{x,t}$
- ${}_np_{x,t}$ = The probability of a life aged $x$ in year $t$ reaches $x+n$.
- $s\left(x|\theta\right)$ is the survival function to age $x$ given $\theta$. It represents the proportion of lives that will live to age $x$

## Graduation

- Given $p_{x,t}$ and ${}_np_{x,t}$ it follows: ${}_np_{x,t} = \prod_{i=1}^{n} p_{x+i,t+i}$
- $D_{x,t}$ the observed number of deaths of lives aged $x$ in year $t$
- $E_{x,t}$ the observed number of lives aged $x$ in year $t$
- $\mathring{q}_{x,t} = \frac{D_{x,t}}{E_{x,t}}$ the crude initial rate of mortality of a life aged $x$ in year $t$

## Derivatives

- $T$ is the maturity date
- $k$ is the coupon rate or swap rate
- $N_t$ the notional population at $t$
- $F_t$ the benchmark population at $t$

# Nomenclature

- **Annuity:** A financial product sold by life insurance providers that pays to the annuitant a sum of money on the survival to a specified point in time. For example, a simple immediate annuity may pay the annuitant a sum of money monthly should they survive to the start of the month.

- **Assumption drag:** Where the assumptions lag rather than lead actual experience.

- **Assurance:** A life assurance policy in effect pays out on the death (or some other well defined decrement) of the life assured.

- **Basis risk:** In the context of mortality based instruments, basis risk is a residual risk as a result of the underlying population and the population the instrument is based on being different.

- **Life insurance provider:** Any institution that provides life insurance products such as life assurance policies or annuities.

- **Management actions:** The actions a life insurer may assume are taken in the process of establishing realistic values of assets and liabilities. For example, adjusting the investment strategy.

- **Mortality risk:** Mortality (Longevity) risk is the risk typically borne by life insurance providers from unanticipated increases (decreases) in mortality rates. That is, a loss is sustained if mortality increases (decreases).

- **Natural Hedging:** A method to hedge the mortality risks of a life insurance provider. The aim is to diversify the in-force population.

- **Reinsurance:** The insurance purchased by a life insurance provider. The aim is often to transfer some of the risk the life insurance provider has taken on itself.

- **Systematic mortality risk:** The mortality risk that the underlying mortality rates change fundamentally. This would cause a difference in the actual experience relative to that assumed. This is not a diversifiable source of risk. These fundamental changes in the structure of the underlying mortality rates could be driven by various things, for example, some unanticipated developments in medical science or a catastrophic event.

- **Unmanaged mortality risk:** Mortality risk that is not entirely mitigated through hedging or transfer.

- **Unsystematic mortality risk:** The mortality risk associated with the randomness of deaths in the life insurance provider's insured population. Unsystematic mortality risk is diversifiable, by the law of large numbers.

# Chapter 1

# Introduction and overview of mortality

## 1.1 Introduction

This chapter serves to introduce the major concepts that will be discussed in this dissertation. First it introduces mortality and how it is a source of risk and then presents an associated South African example. It then discusses some of the traditional methods of managing mortality risks and briefly describes mortality based instruments and how they could be used for the same. Thereafter an outline of the key objectives of this dissertation is provided.

### 1.1.1 Mortality risk

#### Overview

Using the definition from JP Morgan's LifeMetrics glossary[1]. Mortality (Longevity) risk is the risk typically borne by life insurance providers from unanticipated increases (decreases) in mortality rates. That is, a loss is sustained if mortality increases (decreases).

An example would be if a life insurance provider provides annuities where there is some payment to the annuity holder on survival. Should mortality rates turn out lighter than anticipated the annuity writer (The life insurance provider) will sustain an underwriting loss.

[1] http://www.jpmorgan.com/pages/jpmorgan/investbk/solutions/lifemetrics/glossary

Similarly, should a life insurance provider sell assurance business where there is some payment on some decrement, say death. Then should mortality rates turn out heavier than anticipated the assurance writer will sustain an underwriting loss.

Should these mortality risks not be managed or allowed for adequately, it is possible that if these losses are too large that the life insurance provider may go out of business and potentially default on their other obligations. Which could have grave consequences on the consumer.

To protect consumers against these risks the FSB sets out the capital requirements for life insurance providers. Amongst other things the larger the unmanaged mortality risk the life insurance provider carries, the larger their capital requirement. This is principally to ensure that the insurance provider can meet all obligations to policyholders as they fall due.

**How can it go wrong?**

Mortality forecasts are based on past claims and exposure data. To be able to understand the rate of mortality one needs the number of decrements (deaths) from a population and the population's size. The less accurate the claims or exposure data the less accurate the mortality forecast, so no matter what mortality data is used there will be a degree of parameter risk.

Another factor that can skew the accuracy of a mortality forecast is how well the population the forecast is being based on mirrors the population that the forecast will be applied to. An abstract, but illustrative example is that one cannot use the mortality experience of a Japanese life company to forecast the mortality experience of a South African life company, because the populations are fundamentally different. A less abstract example is that one cannot use the mortality experience of an annuity book to price an assurance book, because again the population's dynamics are fundamentally different because the different types of products meet different needs. So no matter what population is used there will be some basis risk.

Mortality forecasts might also turn out incorrect for no other reason than random error.

Also to forecast mortality, one needs to use some sort of mathematical model and no model will reflect the dynamics of reality perfectly. As a result, no matter what model is used to do a forecast of mortality, there will be a degree of model risk.

One other point to consider is that the past experience may not structurally reflect the future. For example, advancements in medical science could have a profound impact on longevity and until such an advancement takes place it can never be factored into a mortality projection - You do not know what you do not know. Take what the impact of a cure for HIV/AIDS would have on longevity. It could hypothetically double the current life expectancy of individuals with HIV/AIDS.

In addition to the above one needs to consider the potential impact of catastrophic events. Even if we make the very bold assumption that there is no model or parameter risk, an unforeseen catastrophic event could have a significant impact on the evolution of a population. If an epidemic wipes out one in every three people in some population, that population's evolutions will turn out extremely differently to any forecast based on a Gaussian distribution.

All in all, there are a host of reasons why a life insurance provider may get their mortality forecasts wrong. That said, life insurance provider's are in their game to take on mortality risks, but having the right tools to be able to manage mortality risk can dampen the effects of getting it wrong. That is the gap mortality based instruments can fill.

Essentially, mortality based instruments will enable life insurance providers to measure and manage their mortality risks more explicitly. This should in turn allow the life insurance providers to function more efficiently and better match the risks they bear to those they are comfortable bearing.

### A South African example

Dorrington, Moultrie and Timæus [9] produced some estimates of mortality rates in South Africa based on the South African Census 2001 data. They point out that South Africa's mortality rates fell for many decades prior to the mid-1980s. Then between the mid-1980s and the mid-1990's the mortality rates leveled off and have since started to rise. This is also supported by the United Nations [54] World Mortality Report 2005 from which Figure 1.1 was sourced. It shows the life expectancy at birth of South Africans by various years of birth.

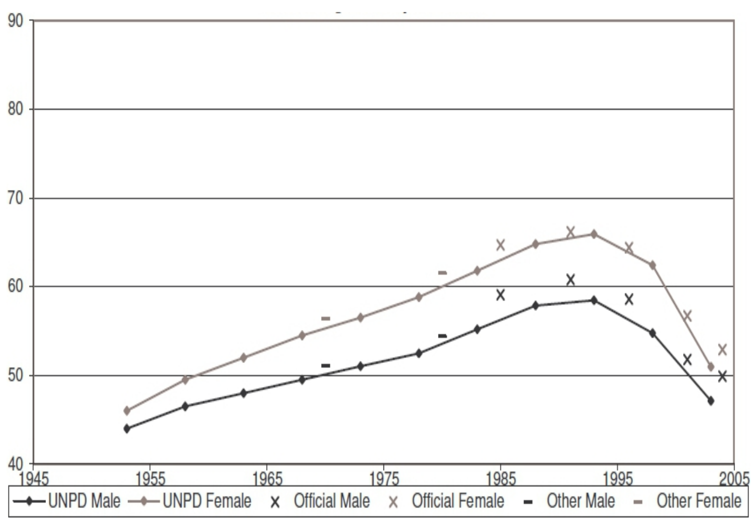


Figure 1.1: South African life expectancy at birth as determined as part of the World Mortality Report of 2005 for various cohorts.

So the question is, where will the South Africa mortality rates head to from here? Will they increase further or will they again lighten and if so how much? How much of a role will HIV/AIDS play in the evolution of an average South African's life expectancy at birth from year to year?

I am not sure we know and as a result there is a risk taking a bet on how it will turn out.

### 1.1.2 The current tools to manage mortality risk

In a life company, mortality risks are conventionally managed through some combination of natural hedging and reinsurance.

#### Natural hedging

Natural hedging can be achieved by a life insurance provider by diversifying the business they write. For example, diversifying their product offerings or region they write business in.

Assume an insurance provider writes both annuity and assurance business. Should mortality rates turn out lighter than anticipated in aggregate, then they will have a better than anticipated experience on the assurance business, but a worse than anticipated experience on the annuity business. In essence they will win on one line of business while they lose on another. That win/lose scenario is the basis of natural hedging.

It is important to note that in practice it is very difficult to manage mortality exposure by natural hedging. A life insurance provider will only have limited control over the volumes of business they write per product line or region. A further complication is that in practice even if the volumes written by product or region are as desired, it is difficult to maintain the desired mix of in-force business since the life insurance provider again only has limited control over their own lapse experience.

#### Reinsurance

Reinsurers are essentially large insurance companies that pool the risks of all their clients (insurers). These risks could range from earthquakes and hailstorms to the mortality risks outlined above. This pooling does allow for scale and for the law of large numbers to have some stabilising effect on the underlying experience. That said, these risk pools are finite, which is a natural limitation to reinsurance as a risk management tool.

All reinsurance companies exist to make profits and using their services adds an additional layer of expense that would not otherwise be incurred. Also, using only traditional reinsurance structures can be difficult and/or costly to shed some specific mortality risks. For example, the longevity risks of a book of deferred annuities.

### 1.1.3 Mortality based instruments

#### Introduced

Mortality based instruments are analogous to interest rate instruments, however their cash flows are primarily driven by the mortality experience of some underlying population. For example the cash flows of a longevity bond are driven by the number of deaths in some underlying population. That population could be the annuity book of a life insurance provider. While the price of a vanilla bond would primarily be driven by the movements in the yield curve.

Another example would be a mortality swap, which is similar to an interest rate swap, just that the mortality swap's cash flows depend on the actual number of deaths in the underlying population. One party of the mortality swap would gain if there were more death than anticipated and the other would lose.

#### As an alternative

By having a tool set that enables the tailored trade of mortality, a life insurance provider would be able to look beyond the traditional measures described above to manage their mortality risks. Those additional tools are mortality derivatives.

For example a life insurance provider could purchase a longevity bond to hedge against longevity risks, if that longevity bond had coupons structured in line with those detailed in Section 4.1[2]. Similarly if a life insurance provider

[2] Conversely by selling such a longevity bond, a life insurance provider could hedge

is concerned about large deviations in mortality experience from its best estimates, then they could enter a mortality swap (or mortality swaption) arrangement.

**Some additional benefits**

One spin-off is that as more entities start offering tools to manage mortality risk, the more fierce competition will become. This should give life insurance providers more options to choose from in terms of ways to manage their risks and to do so at a lower cost. Hopefully that will also translate to better value for money to consumers over time as the margins in the various layers needed to mange the mortality risks get tighter as a result of competition.

Another positive spin-off is that if instruments whose cash flows are driven by mortality are traded in the open market, their prices will be observable in the open market. So if those instruments can be used to replicate the cash flows of some in force business, market consistent values of those cash flows can be determined.

Basically if the cash flows of some in-force population can be replicated, a truly market consistent valuation can be done[3].

### 1.1.4 Primary objectives

The primary objectives of this dissertation are to show that:

1. There are risks associated with the mortality experience of any population.
2. That these risks can have an adverse impact on various financial institutions and the man on the street.
3. Using instruments whose cash flows are driven by the movements in some underlying population's mortality experience may be tools to help

against mortality risk.

[3]Important to note is that these additional benefits assume Credit and other risks related to such transactions can be mitigated in full. This is one of the obstacles the mortality instrument market will need to overcome if significant volumes of these instruments are to be traded.

manage some of these risks[4].

This dissertation does not aim to go into too much detail on some of the associated topics such as regulation, taxation, credit risk and mortality itself. That said most of these concepts are covered and some are elaborated on in the appendix.

There are also some simplifying assumptions that need to be made along the way. None of which detract from the concepts being discussed, but they should be borne in mind for completeness.

[4]Important to note is that these instruments could act as substitutes for traditional methods of managing mortality risks or they could complement the traditional methods.

### 1.1.5 Overview

Figure 1.2 below is a graphical representation of the major concepts presented in this dissertation, how they progress from one another and how they relate to each other.

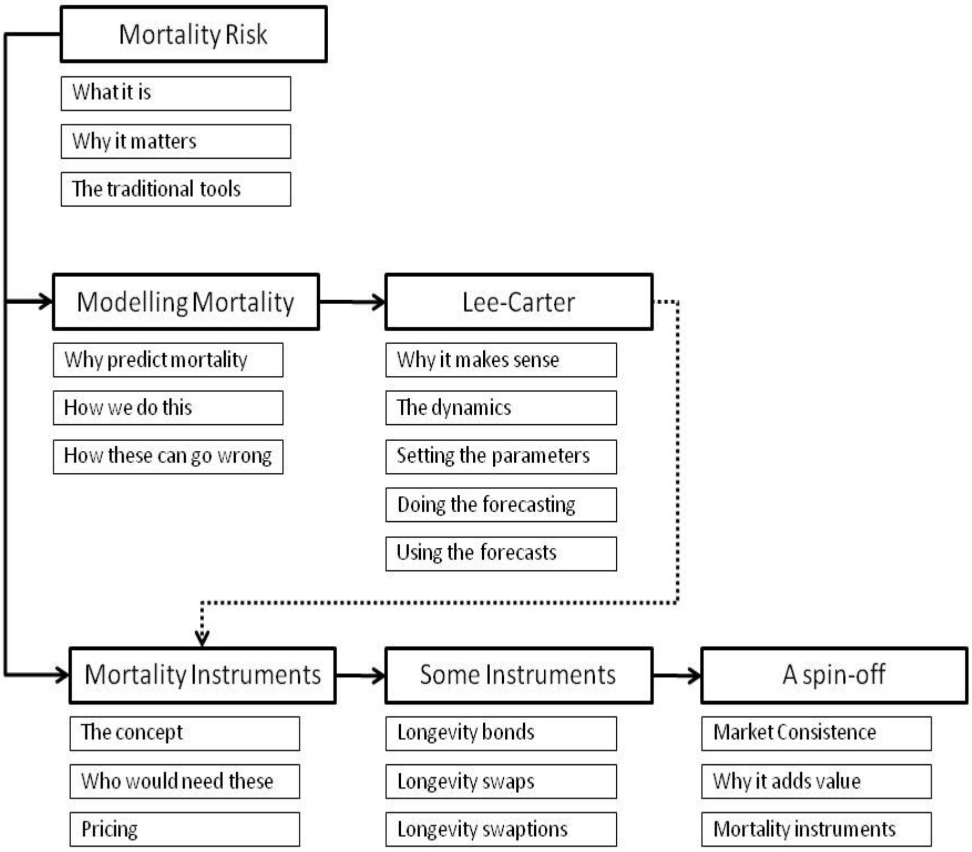


Figure 1.2: An overview of the concepts presented in this dissertation and how they progress from one another and relate to each other.

## 1.2 Mortality and its building blocks

### 1.2.1 Introduction

This chapter outlines the data that is needed to construct mortality based instruments. The two primary sources of this data for a Life Office are its own experience and their reassurer's mortality rates. It also discusses the risks associated with using either source. For example, a Life Office is exposed to basis risk if they use a reassurer's rates and parameter risk if they use their own experience.

The process of going from crude (observed) rates to graduated (smoothed) rates it also covered.

### 1.2.2 What data to use

The data used to create these instruments is the natural starting point. It goes without saying that the solutions are only as good as the data they are based on. On a practical note there are many technical difficulties to overcome when doing mortality projections, particularly the sourcing of good quality data [43].

Let us consider for example a Life Office that intends to protect against people living longer than anticipated. As discussed in the introduction, one way the Life Office could do this would be to purchase a longevity bond[5]. If this longevity bond's cash flows are based on the Life Offices own mortality experience there will be less basis risk, because the population being used as the benchmark is the same as the population being hedged. However because it takes such a long time for a credible mortality data set to emerge there is more parameter risk in using this solution.

Had the longevity bonds cash flows been based on some external data, say a reassurer's mortality experience, then there would be more basis risk since the benchmark and underlying populations would not correspond perfectly. That said, because a reassurer would have a much larger mortality data set there would be less scope for parameter error.

In South Africa there are many reinsurance companies in the life insurance space, but the major players are General Reinsurance Africa Ltd. (Gen Re),

[5]This is explained in detail in the section **Longevity Bonds**

Hannover Reinsurance Group Africa (Pty) Ltd, MRoA Munich Re Group, RGA Reinsurance Company of South Africa Limited and Swiss Re Southern Africa. All these reinsurance companies are smaller parts of very large global organisations and they feature in the majority of all reinsurance transactions in South Africa. It is this access to a broad range of clients data both locally and internationally that enables the reinsurers to be so rich in mortality data.

Knowing the above suggests that when looking to use a mortality instrument, the benchmark for that instrument should fit the need. Should the need be to protect against a general lightening of mortality then using some external data set might be fine. However if the need is to protect against mortality fluctuations of a known population, then using the experience of that population would be more appropriate.

On a practical note, the high perceived basis risk of some instruments that were developed is one of the reasons cited for those instruments not being taken up [5]. Also if the benchmark will be based on a specific population then the instrument will need to be a tailor-made instrument. However if the benchmark is based on some external population's experience, say based on population statistics, then the instrument could be standardised or exchange traded.

The Human Mortality Database [35] is a great source of data for research and this paper will be using data obtained from the HMD website www.mortality.org. That said, the data would in all likelihood not be suitable for implementing most of the ideas covered in the rest of this dissertation in practice.

### 1.2.3 The building blocks

#### Raw data to crude rates

The crude initial rate of mortality for a life aged $x$ in year $t$, call it $\mathring{q}_{x,t}$ would be the total number of observed deaths of people aged $x$ in year $t$, say $D_{x,t}$, divided by the number of people observed to be aged $x$ in year $t$, say $E_{x,t}$.

This gives rise to the equation: $\mathring{q}_{x,t} = \frac{D_{x,t}}{E_{x,t}}$.

So for example, if 597 lives aged 63 died in the year 2001, $D_{63,2001} = 597$. Then if there were $43,290$ lives aged 63 alive in 2001. Here $E_{63,2001} = 43,290$. This combined with the above gives us $\mathring{q}_{63,2001} = \frac{D_{63,2001}}{E_{63,2001}} = 0.013791$, the crude mortality rate for a life aged 63 in year 2001. These mechanics are outlined in the LifeMetrics Technical Document [18].

Table 1.1 gives an example of some crude initial rates of mortality. These have been sourced from Bradley P. Carlin's paper: A Simple Monte Carlo Approach to Bayesian Graduation [14].

| Age($x$) | $D_x$ | $E_x$ | $\mathring{q}_x$ | Age($x$) | $D_x$ | $E_x$ | $\mathring{q}_x$ |
|---|---|---|---|---|---|---|---|
| 35 | 3 | 1771.5 | 0.0016935 | 50 | 4 | 1516.0 | 0.0026385 |
| 36 | 1 | 2126.5 | 0.0004703 | 51 | 7 | 1317.5 | 0.0051039 |
| 37 | 3 | 2743.5 | 0.0010935 | 52 | 4 | 1343.0 | 0.0029784 |
| 38 | 2 | 2766.0 | 0.0007231 | 53 | 4 | 1304.0 | 0.0030675 |
| 39 | 2 | 2463.0 | 0.0008120 | 54 | 11 | 1232.5 | 0.0089249 |
| 40 | 4 | 2368.0 | 0.0016892 | 55 | 11 | 1204.5 | 0.0091324 |
| 41 | 4 | 2310.0 | 0.0017316 | 56 | 13 | 1113.5 | 0.0116749 |
| 42 | 7 | 2306.5 | 0.0030349 | 57 | 12 | 1048.0 | 0.0114504 |
| 43 | 5 | 2059.5 | 0.0024278 | 58 | 12 | 1155.0 | 0.0103896 |
| 44 | 2 | 1917.0 | 0.0010433 | 59 | 19 | 1018.5 | 0.0186549 |
| 45 | 8 | 1931.0 | 0.0041429 | 60 | 12 | 945.0 | 0.0126984 |
| 46 | 13 | 1746.5 | 0.0074435 | 61 | 16 | 853.0 | 0.0187573 |
| 47 | 8 | 1580.0 | 0.0050633 | 62 | 12 | 750.0 | 0.0160000 |
| 48 | 2 | 1580.0 | 0.0012568 | 63 | 6 | 693.0 | 0.0086580 |
| 49 | 7 | 1467.5 | 0.0047700 | 64 | 10 | 594.0 | 0.0168350 |

Table 1.1: Sample crude initial mortality rates for various ages based on some populations observed deaths and units of exposure.

### Crude to graduated rates

The crude rates are just that - crude. As a result they will need to be smoothed out to make them more meaningful and usable in practice. The graduated rates must be both of an adequate fit and appropriately smooth.

There are many graduation methods to choose from, for example Bayesian methods as suggested by Carlin [14]. Another good example of one is the Standard Whitaker graduation process. Broffitt [8] suggests minimising the quantity $F + hS$, where $F$ and $S$ are lack-of-fit and lack-of-smoothness measures and $h$ is a constant to place weight on either fit or smoothness.

The HMD Protocol [57] also gives a detailed outline of the mechanics of how the crude rates can be calculated and how these crude rates are graduated to give smoothed rates.

### Crude and graduated rates compared

Figure 1.3 shows the difference between a set of crude and a set of graduated rates sourced from the HMD [35].

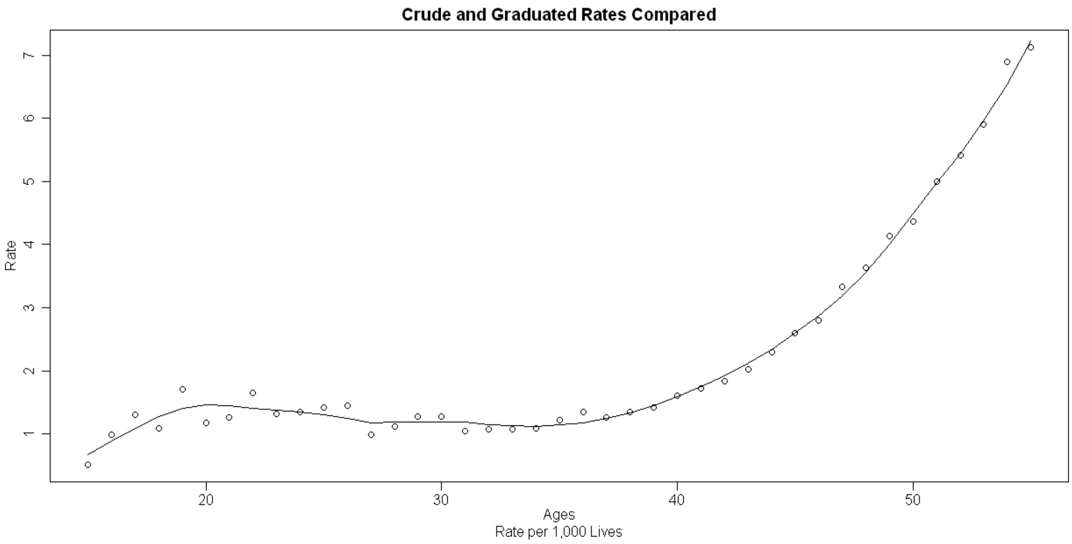


Figure 1.3: The comparison of crude and graduated initial mortality rates for various ages, expressed as a number of deaths per year per 1,000 lives.

From this, one can see that there will always be a trade-off between smoothness and fit. One can always get the fit better, but then the curve will be less smooth. Conversely one can always smooth the curve more, but at the price of lack-of-fit. Considering this, an appropriate balance between fit and smoothness needs to be struck.

## 1.2.4 Conclusions

From the preceding sections, it can be seen that the data required to construct mortality based instruments can be gathered by a life company over time. Reinsurance companies can also be good sources of mortality data for the construction of mortality based instruments.

The process of graduating crude mortality data is also important to understand when constructing mortality based instruments, as the graduation processes used play a role in the determination of the mortality rates that will ultimately be used.

There is always a balance that needs to be struck between smoothness and fit in the graduation process. This introduces a degree of subjectivity into the determination of the mortality rates and this must always be borne in mind.

The graduation processes used can have a significant impact on the eventual price of a mortality based instrument. As a result the graduation processes chosen could easily become a contentious issue in practice as it may seem to favour one party or another. Mortality based instruments could also be based purely on the crude mortality rates.

Also pricing these mortality based instruments purely on the crude rates might better serve some parties interests as it might mean a closer match between an instruments underlying cash-flows and their underlying experience.

It is also possible to decide on a graduation process that is purely mechanical and objective. Doing this upfront removes any subjectivity which might cause tension if that subjectivity seems to bias the price in any way.

All-in-all, whether crude or graduated rates are used in the construction of mortality based instruments, their respective pro's and con's must be considered in the context of that specific mortality based instrument.

## 1.3 General mortality forecasting methods

### 1.3.1 Introduction

Here the principles that underlie various methods of modelling mortality are discussed.

There may be many ways to categorise methods of forecasting mortality. Booth and Tickle [7] suggests a sensible segmentation, namely:

1. Extrapolative methods
2. Explanative methods
3. Expectation based methods

### 1.3.2 Extrapolative methods

These methods use the patterns that can be observed and then extrapolate them. For example by age or age and cohort, and the trends that can be seen emerging over time. Most of these methods also fall into one of two distinct categories - Deterministic methods and Stochastic ones.

An example of a Stochastic model would be the Lee-Carter model[6]. The Lee-Carter model in essence captures how mortality rates develop over time for each different age. So it captures how the mortality rate of say a sixty year old in 1960 would compare to that of a sixty year old in 2001 by using time-series analysis techniques.

A deterministic case would be for example, the elegant model proposed by Carriere [13]. It is an eleven parameter model to explain male and female mortality. Each of the parameters also have an intuitive appeal. Below is the definition of the model proposed by Carriere.

$$q_x = 1 - \frac{s(x+1|\theta)}{s(x|\theta)},$$

with:

[6]The Lee-Carter model is worked through in detail later in the paper

$$s(x|\theta) = \psi_1 s_1(x) + \psi_2 s_2(x) + \psi_3 s_3(x), \text{ the survival function to age } x.$$

Here we use:

$$s_1(x) = \exp\left\{-\left(\tfrac{x}{m_1}\right)^{\frac{m_1}{\sigma_1}}\right\}, \text{ is the infant mortality process.}$$

$$s_2(x) = 1 - \exp\left\{-\left(\tfrac{x}{m_2}\right)^{-\frac{m_2}{\sigma_2}}\right\}, \text{ is an accident hump process.}$$

$$s_3(x) = \exp\left\{e^{-\frac{m_3}{\sigma_3}} - e^{\frac{x-m_3}{\sigma_3}}\right\}, \text{ is a post accident hump process.}$$

$$\psi_3 = 1 - \psi_1 - \psi_2.$$

$$\theta = (\psi_1, \psi_2, \psi_3, m_1, m_2, m_3, \sigma_1, \sigma_2, \sigma_3)'.$$

The elegance of this model is that it basically says the number of lives surviving to a given age is the result of three different processes. Some infant mortality process ($s_1$), some accident hump process ($s_2$) and some post accident hump process ($s_3$). All that is required is to specify the parameters to get the combination of these processes to most accurately fit the crude rates.

The infant mortality process ($s_1$) describes the observed mortality rates of the "younger ages" in the population, typically from birth to about three or four years of age. These mortality rates start very high[7] then fall exponentially. The tail of the infant mortality process ($s_1$) generally tracks up slowly until it merges with the accident hump process.

The accident hump process ($s_2$) produces mortality rates that have a "hump shape" starting at around ages fifteen to sixteen and ending at around ages thirty to thirty five. The accident hump process should reach it's maximum in the early to middle twenties and fall from there. The accident hump effect is almost always more pronounced in males than in females of similar ages and is primarily attributed to the accidents young adult males are involved in when they start driving and drinking[8]. The accident hump process then blends into the post accident hump.

[7]The relatively high mortality rates at these younger ages are generally the result of birth complications.

[8]These two activities also tend to happen together in young adult males, sometimes with fatal results.

The post accident hump process ($s_3$) represents the mortality brought on by deteriorating health and natural aging. Mortality rates by age above the ages described by the accident hump process increase exponentially year-to-year.

Note that infant, accident hump and post accident hump mortality rates can vary significantly by populations, however the shapes remain relatively similar.

### 1.3.3 Explanative methods

Explanative methods call on Epidemiological models usually to model deaths by cause. Here the key variables are known and can be measured. An example could be how lung cancer deaths vary by the amount an individual smokes.

Structural and causal Epidemiological models often involve modelling deaths by a specific cause based on some Disease Process or known risk factors, such as smoking or drinking habits. The aforementioned Disease Processes and risk factors are generally based on medical data, knowledge and information on behavioural and environmental changes.

The main uses of Explanative methods are not to forecast population mortality rates, but rather to model the impact of changes to risk factors, behaviour or environment on the morbidity and mortality of an individual. For example, to assess what reduction in the prevalence of lung cancer can be expected based on a reduction in smoking and how this will also impact the individual's mortality and/or morbidity.

This is particularly valuable to a Life Office's rating. By better understanding the impact of changes in specific risk factors, behaviour or environment an Office could more closely align the rates they charge to an individual's expected mortality or morbidity. Doing so gives the Office a competitive advantage through sentinel and selection effects.

A great example of the results of Epidemiological studies is put forward by Bonita, Beaglehole and Kjellström [6], a publication by the World Health Organisation. *Basic Epidemiology* describes the field of Epidemiology in great detail. It describes what Epidemiology is, gives examples of practical uses, contrasts the strengths and weaknesses of various study designs and much more.

An example of a study in *Basic Epidemiology* is where the relative mortality rates of lives that never smoked, that stopped smoking at 30, that stopped smoking at 50 and those that never stopped were contrasted. It is evident that lives that never smoked have the lightest mortality rates and that the sooner a smoker stops smoking the better their recovery, no matter how old they are. Another study is where the relationships between driving speed, seat-belt use and frequency of injury are investigated. As you would imagine, people who drive faster and without seat-belts are more exposed to injury and as a result death.

Table 1.2 provides a quantitative measure of the impact of environment (exposure to asbestos) and behaviour (smoking) on lung cancer death rates for some sample of lives.

| Asbestos exposure | Tobacco use | Lung cancer death rate Per 100,000 lives |
|---|---|---|
| No | No | 11 |
| Yes | No | 58 |
| No | Yes | 123 |
| Yes | Yes | 602 |

Table 1.2: Age-standardised lung cancer death rates (per 100,000) in relation to tobacco use and occupational exposure to asbestos.

As would be expected, those individuals that used tobacco products and were exposed to asbestos exhibited the highest mortality rates as a result of lung cancer.

It is important to note that explanative models are only as good as the data the output is based on. This data is often incomplete or incorrect. For example, in order an idea of the incidence of death due to lung cancer, lung cancer would need to be stated as the cause of death. However, many individuals may have died as a result of lung cancer without it being picked up. Kruijshaar, Barendregt and Hoeymans [37] suggest that artificial data sets can be created to get around the limitations of the observed emperical data.

### 1.3.4 Expectation based methods

Expectation based methods use the (subjective) opinions of experts to forecast mortality rates. There is usually an assumed scenario given, generally with some more aggressive and conservative alternatives too.

These methods are not as popular currently as they have been in the past as there is a move to more sophisticated and robust models, as pointed out by Booth and Tickle [7]. For example, the US Social Securities Trustees use judgement to adjust trends in cause- and age-specific death rates "to reflect a reasonable path of future mortality improvement".

The advantages of these Expectation based methods is that they use demographic, epidemiological and other relevant knowledge, albeit qualitatively. The most constraining disadvantages are that the methods are subjective and that they suffer "assumption drag"[9]. Here the expectations will lag the actual experience as the experience shapes the expectations.

### 1.3.5 Conclusions

There are various approaches to modelling mortality. Each has unique advantages and disadvantages. It is important that the application of the model should be considered when deciding on the methods to use.

For example, stochastic extrapolative models are more suited to the determination of mortality based instruments' theoretical prices, while explanative methods might be more applicable if the intention is the determination of an individual's mortality rates to reach a premium for an individual life policy.

[9] The assumptions are based on what has been observed, while they should be set in a manner that suggests what will be observed.

## 1.4 Modelling mortality

### 1.4.1 Introduction

This chapter illustrates why it is important to model mortality. It also elaborates on the traditional ways in which mortality risks are dealt with. For example, the traditional approach to dealing with unsystematic mortality risk is to try and diversify it away. It also discusses the limitations of the traditional tools to manage systematic mortality risks, namely reinsurance and natural hedging.

It also outlines why stochastic mortality models are needed to establish the values of mortality based derivatives. It then introduces a few stochastic mortality models.

### 1.4.2 Why model mortality

#### Philosophically

Mortality experience can be a significant driver of economic performance at all levels. At the highest level it impacts the global economy. At a lower level it affects the performance of a nation's economy. At yet a lower level it affects the performance of the insurance industry and the players that make up that industry. Being able to understand what mortality experience to expect allows decision makers to plan more effectively for the future. Getting it wrong also has grave consequences.

Mortality is a source of risk and as a result a potential source of upside if understood (or at least understood better than by the majority). Dahl, Melchoir and Moller [26] show life insurance providers are exposed to systematic mortality risk which stems from the uncertainty related to future mortality experience.

Traditionally life insurance providers have been more concerned about unsystematic mortality risks[10].This unsystematic mortality risk is diversifiable, by the law of large numbers. That said, there will always be some residual unsystematic mortality risk as all life insurance provider's have finite in-force

[10]The mortality risk associated with the randomness of deaths in the life insurance providers populations.

population's.

As pointed out by Dahl, Melchoir and Moller (2007), life insurance providers should also be concerned about systematic mortality risks[11]. These systematic mortality risks are not diversifiable and as a result need non-traditional approaches to manage. Mortality based instruments are one such alternative.

**Practically**

To be able to run a Life Office one needs to be able to model mortality as it is a mortality risk that fundamentally drives the need for the life industry. Mortality is an integral part of any Life Insurance operation. From the setting of premiums, to the determination of the required reserves so that claims can be met with a high enough degree of confidence or to maximise the risk adjusted return on capital. Not to mention that to comply with the regulation that governs the South African life insurance industry, life insurance providers need to model their expected mortality experience.

Specifically considering that the FSB is looking to incorporate some of the principles that underpin the Solvency II framework, modelling mortality stochastically may become a necessity where it could currently be considered a luxury. The SAM Information Letter [32] signals the FSB's intention to adopt the principles of Solvency II[12] and adjust them to South Africa's specific conditions.

### 1.4.3 What models to use

A general principle is that the model chosen should be dictated by its use. One of the primary aims of this dissertation is to produce a framework to price instruments whose cash flows are driven by the mortality experience of some benchmark population.

The pricing of these mortality instruments requires a stochastic mortality model as both the time and intrinsic values of the pay-offs need to be quantified. Cairns, Blake and Dowd [10] also point out that mortality is stochastic and that there are two types of stochastic mortality - unsystematic mortality

[11] The mortality risk that the underlying mortality rates change fundamentally.

[12] More information on Solvency II is given in Section 8.3

risk[13] and systematic mortality risk[14].

### 1.4.4 Some stochastic mortality models

Some alternative methods are listed below and the dynamic's of the first four are well described in the LifeMetrics paper of JP Morgan (2007a) [18]:

1. The Lee-Carter model (1992) [40]
2. The Renshaw-Haberman model (2006) [34]
3. Currie Age-Period-Cohort model (2006) [25]
4. Cairns, Blake and Dowd model (2006) [11]
5. Extensions to Cairns, Blake and Dowd model (2007) [12]
6. Affine stochastic mortality models such as those presented by Schrager (2005) [52]

#### Notation

The models described below will use the following notation:

1. $\beta_x^{(i)}$, which reflects the age-related effects, with $x$ denoting the age
2. $\kappa_t^{(i)}$, which reflects the period-related effects, with $t$ denoting the period
3. $\gamma_c^{(i)}$, which reflects the cohort-related effects, where $c = t - x$ denoting the birth cohort
4. $q_{x,t}$, the initial mortality rate of a life aged $x$ in year $t$
5. $m_{x,t}$, the central mortality rate of a life aged $x$ in year $t$

[13]That the actual number of deaths varies from that expected because of the finite number of lives in the population. This risk is assumed to be diversifiable

[14]That the mortality rates evolve in a different way to that anticipated based on a mortality models output. This risk in not diversifiable

### Lee-Carter model

Lee and Carter [40] proposed an extrapolative model, based on historic data. It is a one-factor model of the mortality surface, by age and time. Effectively it suggests that the mortality rates of all ages are driven by a single process from one year to the next.

The dynamics are described as:

$$\log m_{x,t} = \beta_x^{(1)} + \beta_x^{(2)} \kappa_t^{(2)}. \tag{1.1}$$

The $\beta_x^{(1)}$ coefficients describe the age-specific pattern of mortality, while the $\beta_x^{(2)}$ coefficients describe the sensitivity of the mortality rates by age to changes in time. These changes in time are represented by $\kappa_t^{(2)}$.

The Lee-Carter model is one of the less complex models. So it might not be as elegant as some of the other models outlined below, but has the distinct advantage of being easier to calibrate. The parameters also have an intuitive appeal and the output is easier to interpret.

### Renshaw-Haberman model

Renshaw and Haberman [34] proposed a generalisation of the Lee-Carter model. It includes a cohort effect.

The dynamics are described as:

$$\log m_{x,t} = \beta_x^{(1)} + \beta_x^{(2)} \kappa_t^{(2)} + \beta_x^{(3)} \gamma_c^{(3)}. \tag{1.2}$$

The additional $\beta_x^{(3)} \kappa_c^{(3)}$ term is used to allow for a cohort effect. Where the cohort effect can be varied by age through $\beta_x^{(3)}$.

### Currie Age-Period-Cohort model

Currie [25] suggests a simplified version of the Renshaw-Haberman model. Here the age, period and cohort effects are assumed to impact the mortality rates independently.

The dynamics are described as:

$$\log m_{x,t} = \beta_x^{(1)} + \kappa_t^{(2)} + \gamma_c^{(3)}. \tag{1.3}$$

### Cairns, Blake and Dowd model

Cairns, Blake and Dowd [11] put forward the following dynamics:

$$\log \frac{q_{x,t}}{1 - q_{x,t}} = \kappa_t^{(1)} + \kappa_t^{(2)}(x - \bar{x}). \tag{1.4}$$

Here $\bar{x}$ is the average age over the range of ages used in the calibration. There are a number of extended versions of this model. Some with additional terms to allow for cohort effects. These models differ to those described above in that they assume a functional relationship between the mortality rates at different ages.

## 1.4.5 Conclusions

There are a host of reasons to model mortality accurately. Being able to model mortality effectively enables us to better prepare for the future and to grasp the impact of changes to mortality rates. Regulation also dictates that mortality be modelled.

Given that mortality instruments have both time and intrinsic values that need to be quantified, stochastic models are more suitable for the pricing of such mortality based instruments.

# Chapter 2

# The trade of mortality outlined

## 2.1 The concept of trading mortality

### 2.1.1 Introduction

This chapter shows how the trade of mortality is made possible through indexation. It also discusses why the trade in mortality might take place and how the trading of mortality could help Life Insurance Providers work around the limitations of some traditional approaches to dealing with mortality risk.

It then goes into detail on the parties that might get involved in the trade of mortality.

### 2.1.2 How trading mortality is possible

If a process has some degree of uncertainty and can be measured against an objective benchmark, then that process can be traded, so long as there are parties willing to take opposite views. Whether the process is one of the interest rate, the inflation rate, the gold price, the mortality rate or any other quantifiable process for that matter.

By being able to create a mortality index, analogous to say a share index, mortality can be traded. The Life & Longevity Markets Association [44] has recently been formed and published their Longevity Index Framework on the $18^{th}$ of August 2010. The stated purpose of this framework is "to provide the market for longevity risk transfer with a widely accepted set of standards for the development of longevity indices, which can be used to build public and proprietary longevity indices"[1].

There have been a few mortality indices created by various parties. These are summarised by Ménioux [45] and they include the:

1. **EIB/BNP index.**
   This was an index based on the realised mortality rates of males aged 65 in England and Wales in 2003.

2. **Credit Suisse Longevity Index.**
   This provided a standardised expected average lifetime for a general population and was based on the national statistics of US population data.

3. **JP Morgan LifeMetrics Toolkit.**
   This is an index based on the national population data of the US, England and Wales and the Netherlands.

4. **Goldman Sachs Mortality Exchange $Q_xX$.**
   This was an index based on the mortality experience of a sample population of insured US lives aged 65 and older.

On a practical note, the formation of the Life & Longevity Markets Association was in part driven by the shortcomings of the previous mortality indices that had been created. These shortcomings include the absence of sufficient liquidity, the unacceptably high basis risk and the lack of transparency of the indices determination.

These are some of the main reasons mortality is not already actively traded.

### 2.1.3 Why trade mortality

Mortality is a source of risk as the future mortality experience of a group of lives is unknown (with certainty) today. By being able to trade mortality one could reduce risk (but give away potential upside) or search for enhanced

[1]The principles of the LLMA Framework are set out in the appendix under the section **LLMA principles**.

returns (but take on more risk).

At present there is not a vast array of tools to manage mortality risks. Ignoring mortality based instruments a life insurance provider is basically limited to natural hedging or the use of reinsurance. This is a problem pointed out by Dowd, Cairns and Blake [30]. They also point out that given the lack of tools available to manage these risks, there are even fewer tools available to quantify the mortality risks being carried.

Natural hedging would be where a life insurance provider sells both assurance and annuity business, so should mortality turn out heavier than anticipated (in aggregate), the increased claims out go on the assurance business is compensated for by the reduced annuity benefit out go. The benefits of natural hedging are pointed out by Cox and Lin [21]. However in the same paper they suggest that even if a Life Office can naturally hedge they are unlikely to be able to do so optimally and that such a strategy should be supplemented with mortality swaps. Cummins [23] also points out that natural hedging will rarely, if ever, be a complete solution.

Philosophically having the tools to shed or retain specific mortality risks gives life insurance providers the ability to match the risks to which they will be exposed, to those they are comfortable bearing more closely, all else equal. Practically the ability to trade mortality should enable life insurance providers to manage their capital requirements more effectively, which could give them a competitive advantage over market participants who cannot.

The Capital Adequacy Requirements of a South African life insurance provider requires them to keep enough capital back to ensure they can remain solvent through a set of very adverse conditions. One of which is a stress on the life insurance provider's mortality experience. So by being able to reduce the volatility around the expected mortality experience a life insurance provider could potentially lower their Capital Adequacy Requirements[2].

Effectively by reducing the Capital Adequacy Requirements less capital will be required to support a book of a given size, which means that all else equal smaller premiums would be required. Smaller premiums for a given benefit are more competitive.

---

[2]The Capital Adequacy Requirements are outlined in the appendix in the section **Capital Adequacy Requirements**

In summary, being able to trade mortality better than your competition should give you a competitive advantage. Consider the following scenario to illustrate the statement above: If the market had only two participants similar in all respects, just one being able to trade mortality and one unable. Then the one that is able to trade mortality would have a competitive advantage over the other as they would be able to manage their operation more effectively.

### 2.1.4 Who might get involved

Blake, Cairns and Dowd [5] suggest that before we try and understand the instruments, that we should first understand the potential stakeholders in a market for mortality. These stakeholders would include:

1. **Hedgers**

   Any party that is exposed to mortality risk (either systematic or unsystematic) would potentially like to be able to off load some mortality risk. As discussed earlier in the dissertation, a life insurance provider that writes annuity business is exposed to a lightening of mortality, while a life insurance provider that writes assurance business is exposed to mortality getting heavier. Both may like to shed some mortality risk.

2. **General investors**

   Given that mortality risk has a very low correlation with other market risks, mortality driven instruments may be good diversifiers.

3. **Speculators and arbitrageurs**

   Speculators who aim to profit by taking a view may want to get in on the action. Also arbitrageurs who aim to profit from irregularities in the prices of various instruments may get involved in the market.

4. **Government**

A government may issue some mortality driven instruments to facilitate a healthy life insurance industry, which is in their best interest as they effectively bear the brunt of defaults in the private sector.

5. **Regulators**

   The Financial Services Board state[3]: "The FSB is committed to promote and maintain a sound financial investment environment in South Africa". One way to help do that is to ensure the required tools to measure and manage risks (including mortality risk) are available.

6. **Other stakeholders**

   Securities managers and exchanges may also see the mortality based instrument space as a potential source of fee income.

[3] on their website www.fsb.co.za

### 2.1.5 Who has tried it before

There have been some relatively recent attempts to trade mortality. JP Morgan [18] set up the LifeMetrics framework, while BNP Paribas and the European Investment Bank (EIB) launched a longevity bond. Credit Suisse signed a longevity swap deal with a group of specialist fund managers and Deutsche Bank and Hannover Re partnered in a longevity hedging scheme.

Below is a summary of some of the mortality instruments structured between 2008 and 2010. Presented by Johan van Egmond [55].

| Date | Client | Provider | Type | Term | Size | Contract partner | Assumer of longevity risk | Comments |
|---|---|---|---|---|---|---|---|---|
| Jan/Feb 2008 | Lucida | JP Morgan | Index | 10 years | m£ 100 | Bank | Capital Markets | |
| July/Sept 2008 | Canada Life | JP Morgan | Indemnity | 40 years | m£ 500 | Bank | Capital Markets | |
| Feb 2009 | Abbey Life | Deutsche Bank | Indemnity | Run off | m£ 1500 | ? | ? | |
| March 2009 | Aviva | RBS & PartnerRe | Indemnity | 10 year | m£ 475 | Bank | Capital Markets & PartnerRe | |
| May/June 2009 | Babcock | Credit Suisse & Pacific Life Re | Indemnity | 20 year | m£ 550 + m£ 250 | Bank & InsCo | InsCo | Watson adviser |
| July 2009 | RSA | Goldman Sachs & Rothesay Life | Indemnity ? | Run off | m£ 1900 | Bank & InsCo | InsCo | DIY buy-in |
| Dec 2009 | Berkshire Pension fund | Swiss Re | Indemnity | Run off | m£ 750 | InsCo | | No adviser InsCo preferred over Investment Bank. 11.000 members |
| Feb 2010 | BMW UK | Deutsche Bank & Abbey Life | Indemnity | | m£ 3000 | InsCo | Abbey Life to 3 reinsurers | Hewitt adviser 60.0000 members |

Figure 2.1: A summary of mortality instruments developed from 2008 to 2010, including the: client, provider, term, type and size.

At the time of writing the market is still fairly small, but no longer negligible.

### 2.1.6 A real life example

JP Morgan [19] took some steps forward in the mortality based instrument space with their q-forwards. These instruments are a good proof of concept for mortality based instruments.

At a high level:

1. q-Forwards are simple capital market instruments for trading longevity and mortality risk. They are effectively instruments where the realised mortality rate of a population is exchanged for a predetermined mortality rate at some point in the future.

2. q-Forwards can form the building blocks of more complex mortality based derivatives.

3. Portfolios of q-Forwards can be used to hedge the mortality risks of assurance books or the longevity risks of annuity books.

4. Because investors require a premium to take on risk, the mortality forward rates at which the q-Forwards transact will be below the best-estimate mortality rates. This leads to a mortality spread.

#### q-Forwards introduced

JP Morgan [19] stated its intention to create a market in LifeMetrics q-Forwards in the paper q-Forwards: Derivatives for transferring longevity and mortality risk. Also the LLMA (Life and Longevity Markets Association) [44] has been established and aims to develop a liquid traded market in longevity and mortality-related risk. Citing their paper: Longevity Index Framework, A framework for building longevity indices developed by the LLMA, they aspire to achieve this through development of consistent standards, methodologies, benchmarks and best practice[4]. q-Frowards can be used to construct more complex instruments. Their pay-offs depend on some underlying indices progression.

The index could be either customised or standardised. If customised it will reflect the experience of a specific group of lives. For example, the members

[4]The LLMA's web address is www.llma.org

of some pension scheme. If standardised it will reflect the experience of some standard population which makes up the LifeMetrics Index.

### q-Forwards defined

Quoting q-Forwards: Derivatives for transferring longevity and mortality risk by JP Morgan [19]:

"A q-Forward is an agreement between two parties to exchange at a future date (the maturity of the contract) an amount proportional to the realised mortality rate of a given population (or subpopulation), in return for an amount proportional to a fixed mortality rate that has been mutually agreed at inception. In other words, a q-Forward is a zero-coupon swap that exchanges fixed mortality for realised mortality at maturity."

### q-Forwards applied

Take the following q-Forward with:

1. $T$ the maturity date
2. $q_{fixed}$ the fixed mortality rate
3. $q_{realised}$ the realised mortality rate based on the reference population
4. $N$ the notional amount
5. $S = N(q_{fixed} - q_{realised})$ the net settlement paid at $T$

If an insurer is concerned that their mortality experience may be heavier than priced for on an Assurance book they might move to a short position in the above q-Forward (i.e. one that pays fixed and receives realised mortality), since the net settlement gets smaller as the realised mortality becomes heavier.

The heavier the Assured book's mortality, the more the Office would pay in claims, but the more their income would be supplemented by smaller (or negative) net settlements. Thus by setting the fixed mortality rate at the mortality rate priced for one can hedge away the risk of mortality fluctuations. That said, should the mortality experience turn out lighter than priced for, that upside would be erroded by larger net settlements.

The graph below shows the various net settlements of a q-Forward instrument based on the data sourced from the HMD [35]. The net settlements vary by maturity and are based on:

1. $q_{fixed} = 0.016$
2. $q_{realised}$ being the observed mortality
3. $N = 100,000$

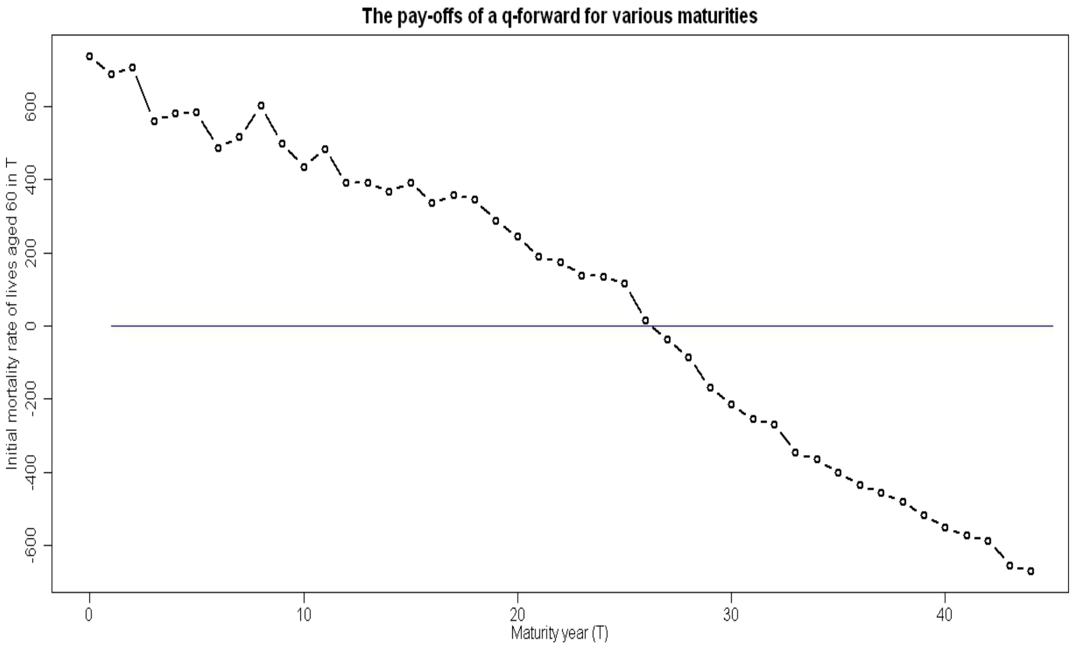


Figure 2.2: q-Forward net settlements by various maturities for a notional population of 100,000 and a fixed mortality rate of 0.016.

### Points to consider about q-Forwards

There will always be a residual basis risk when using such instruments as the reference population if unlikely to perfectly represent the underlying population.

There will also be a degree of liquidity risk associated with these instruments as they are unlikely to be traded in significant volumes (at least for the time being). Liquidity can be enhanced to a degree by using more standardised populations, but this then accentuates the basis risks. So a balance needs to be struck between Liquidity and Basis risk.

### 2.1.7 Conclusions

By using a mortality index, mortality based instruments such as q-Forwards have become traded. That mortality based instruments can then be used to manage mortality risks. For example by using q-Forwards mortality fluctuation risk can be reduced.

We have also seen from Figure 2.1 that a market for mortality based instruments is taking shape. This must confirm a need for alternative solutions for managing mortality risks.

## 2.2 The EIB/BNP Longevity Bond

### 2.2.1 Introduction

The EIB/BNP Longevity Bond was one of the first mortality based instruments to be designed. It is another example of how mortality based instruments have been constructed with the aim of providing a tool to manage mortality risks more efficiently. It also shows that various parties can get involved in the construction of these instruments. Here the European Investment Bank, BNP and Partner Re were all involved.

Blake, Cairns and Dowd [5] outline the EIB/BNP longevity bond in great detail in their paper Living with mortality: Longevity bonds and other mortality-linked securities on which the sections below are based.

### 2.2.2 The outline

The EIB/BNP longevity bond was designed primarily to appeal to pension funds and life insurance providers writing annuity business. It was issued by the European Investment Bank (EIB), designed by BNP and Partner Re was the longevity risk reinsurer.

This longevity bond had a face value of £540 million with a 25-year maturity. The bond had floating coupons linked to a survivor index based on the realised mortality rates of English and Welsh males aged 65 in 2002. The initial coupon was £50 million and there was no redemption payment.

Considering this gives rise to the following coupons, where ${}_1q_{x,t}$ is the crude central death rate for a life aged $x$ in year $t$ published by the Office for National Statistics (ONS):

$$
\begin{aligned}
N_0 &= 50^{10\times 6}, \\
N_1 &= N_0(1 -_1 q_{65,2002}), \\
N_2 &= N_1(1 -_1 q_{66,2003}), \\
&\vdots \\
N_t &= N_{t-1}(1 -_1 q_{65+t-1,2002+t-1}), \\
&\vdots \\
N_{25} &= N_{24}(1 -_1 q_{89,2026}).
\end{aligned}
$$

This in essence means the investors would have made an initial investment of £540 million and in turn received the annual mortality dependent coupons $N_1, N_2, \ldots, N_t, \ldots, N_{25}$.

The bond never made it to market, but the issue price was determined by BNP Paribs by deriving some anticipated cash flows $N_1^E, N_2^E, \ldots, N_t^E, \ldots, N_{25}^E$. These were based on the Government Actuary's Department's 2002 based projections of mortality.

Each cash flow was then discounted at LIBOR less 35 basis points. The EIB curve on average would have been around 15 basis points below the LIBOR curve, which means investors in the longevity bond were paying around 20 basis points to hedge the associated longevity risks.

### 2.2.3 The structure

The EIB/BNP longevity bond was structured as shown in Figure 2.3.

It essentially has three parts:

1. A longevity bond where the EIB pays the bond holders the mortality dependent coupons and in return receives the issue price. The coupons are paid in euros.

2. A cross-currency interest rate swap between the EIB and BNP. The EIB pays floating euros and receives fixed sterling.

3. A mortality swap between the EIB and Partner Re. This is to move the mortality exposure from the EIB to Partner Re who were the longevity reinsurer. The mortality swap is in Sterling.

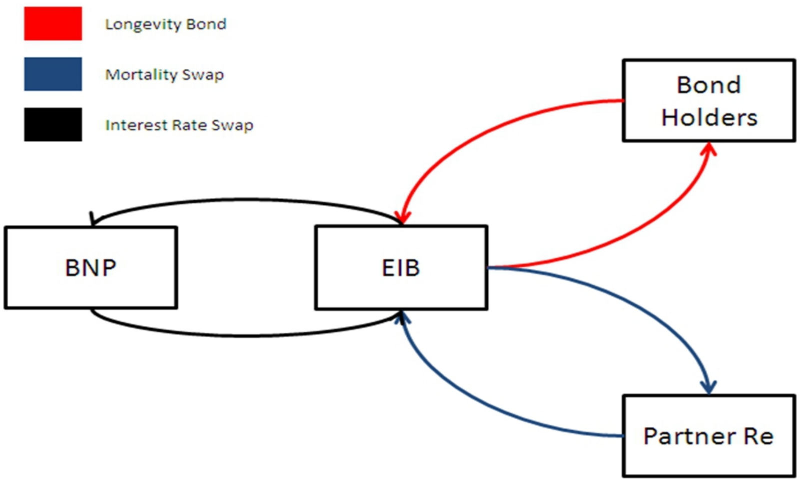


Figure 2.3: The structure of the EIB/BNP longevity bond using a Longevity Bond, a Mortality Swap and an Interest Rate Swap.

In summary, a longevity bond is issued by the EIB and as a result the EIB needs to pay the mortality dependent coupons to the bond holders in sterling. The EIB then enters an interest rate swap with BNP to alter this commitment into one to make floating euro payments. Here BNP takes on some mortality exposure which it hedges with Partner Re.

### 2.2.4 Credit risks

Important to note here, the second part of this structure means that the EIB and BNP have credit exposures to each other. Similarly the third part also means that BNP indirectly has credit exposure to Partner Re and the EIB has a direct credit exposure to Partner Re. These exposures can be managed using some of the tools available to manage credit risk. These tools include credit insurance, credit derivatives and credit enhancement schemes.

### 2.2.5 The appeal

Even though the EIB/BNP longevity bond was only ever partially subscribed and later withdrawn, it had some appeal to the market and was a major step forward for the mortality based instrument market.

The EIB/BNP longevity bond would have provided a reasonable hedge against longevity. This is because the trends in the national mortality would provide a reasonable match for the trend in annuitant and pensioner mortality[5]. This did appeal to pension funds and life insurance providers that wrote annuity business.

The survivor index was based on the crude death rates published by the ONS. This appeals to investors because this data is reliable and available to the public. This gives some reassurance to investors that the data is not being manipulated. Also the use of crude rates avoids arguments over the smoothing methodologies that would be employed.

[5]This would not be the case in South Africa where the annuitant population would exhibit significantly different mortality experience than the general South African population

## 2.3 Pricing mortality instruments

### 2.3.1 Introduction

Here a process to arrive at the theoretical price of a mortality based instrument is outlined.

### 2.3.2 How to reach a fair price

Supply and demand will ultimately dictate what the prices of mortality based instruments will be, but a first step would be to arrive at some theoretical value based on the expected present value of the net pay-offs of these instruments. This principle applies here as much as it does to any other instrument.

Chen and Hong [17] suggest using Monte Carlo methods for valuing derivatives. Primarily because of the ease of use. If a stochastic process is used to describe the various sample paths, the associated cash flows of each path can easily be established.

Effectively Chen and Hong [17] suggest to estimate the theoretical value of an instrument, one needs to:

1. Generate the various sample paths.
2. Evaluate the discounted pay-off along each sample path.
3. Average across all discounted pay-offs.

Considering the above, pricing a mortality based instrument using a Monte Carlo simulation is analogous to pricing any other vanilla derivative. For example, if one were to price a vanilla call using a Monte Carlo simulation as described by Fadina [31], one would:

1. **Define a process to represent the evolution of the underlying's price.**

   For example using the Black-Scholes model where:

$$dS(t) = rS(t)dt + \sigma S(t)dW(t), \tag{2.1}$$

which simplifies to:

$$S_T = S_0 \exp\left(\left[r - \frac{1}{2}\sigma^2\right] T + \sigma\sqrt{T}W\right), \tag{2.2}$$

where:

$$W \sim N(0,1). \tag{2.3}$$

2. **Generate the various sample paths of the underlying's price.**

   Values for $S_T$ would be calculated for a number of runs, say $10,000$, using the dynamics described above.

3. **Calculate the discounted pay-off of the call for each sample path.**

   Using the $10,000$ values for $S_T$ we could solve the pay-off of a call for each of the runs, call it $\chi_T$.

   Here:

$$\chi_T = \{S_T - K\}^+ . \tag{2.4}$$

   These pay-offs would then be discounted to arrive at the present value of the pay-off, $\chi_0$, where:

$$\chi_0 = \chi_T \exp\left(-rT\right). \tag{2.5}$$

4. **Average across all the discounted pay-offs to arrive at a price.**

   The average $\chi_0$ across the runs would then give the value of the call.

### 2.3.3 Monte Carlo methods and the the Lee-Carter model

Following a similar approach to that described in 2.3.2 by using the Lee-Carter stochastic mortality model, which will be elaborated on in Section 3.1, various sample paths of some benchmark population can be generated.

Then using the evolution of this benchmark population in conjunction with a well defined instrument[6], the discounted pay-offs along each sample path for the instrument can be established. This set of discounted cash flows can be used to value the instrument in question.

In this dissertation the discounting to arrive at the present values of the various cash flows was done using the yield curve shown in Figure 2.4[7].

[6]There exists a well defined framework to determine what cash flows the instrument will give rise to

[7]R scripts with the yield curve data used can be made available on request.

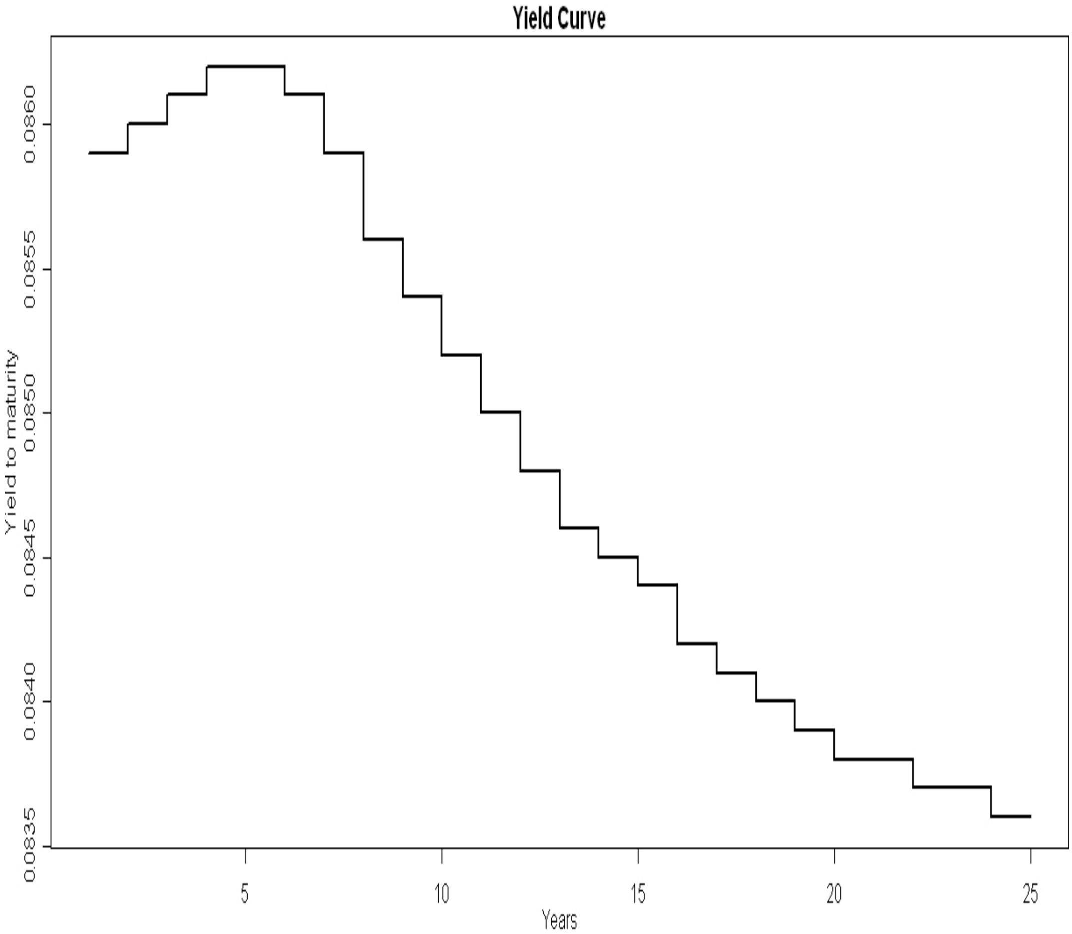


Figure 2.4: A sample yield curve as observed on the South African Bond exchange on 21/09/2006.

# Chapter 3

# The Lee-Carter model

## 3.1 The Lee-Carter model outlined

### 3.1.1 Introduction

This chapter gives an outline of the Lee-Carter model and why it's a natural choice for simulating mortality rates to use in the valuation of mortality based instruments. The Lee-Carter model essentially revolves around the facts that there are trends in mortality rates by age for a given cohort and by cohort for a given age. Why these trends exist is also discussed.

Subsequent to that, the dynamics, parameterisation and implementation of the Lee-Carter model is detailed.

### 3.1.2 A look at some mortality rates

Figure 3.1 is a plot representing a sample set of mortality rates. These mortality rates were sourced from the HMD [35] and are the graduated initial mortality rates of males of various ages born in England and Wales in the years 1961 until 2005.

There are two general trends observable in the above surface.

The first that the likelihood of dying gets larger as one gets older. This is intuitively obvious and also empirically observable.

The second that the mortality rates for a given age fall by year. This could be attributable to many things, for example improvements in medical sci-

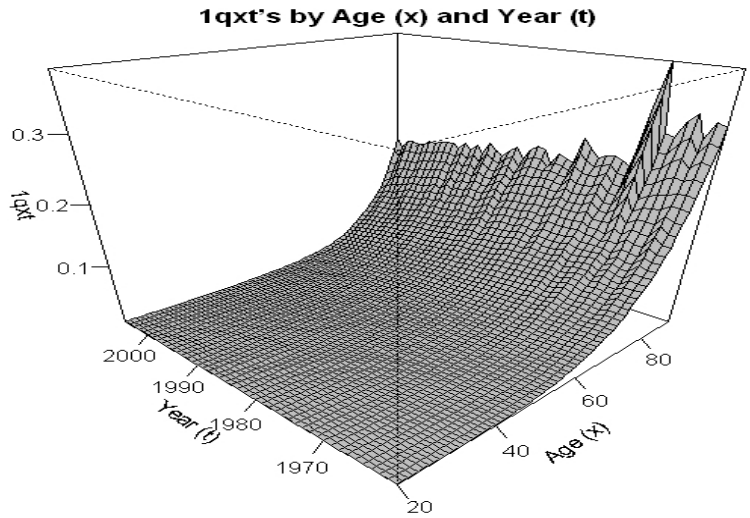


Figure 3.1: A plot of a set of observed initial mortality rates for males of various ages born in England and Wales in the years 1961 to 2005.

ence and the average quality of life. This lightening of mortality over time is discussed at length by Vaupel [56].

Costa [20] cites a few reasons for the lightening of mortality rates for a given age by year, including:

1. **Medical care**

   In essence the better the medical care available the longer people are expected to live.

2. **Reduced infectious disease rates**

   Many infectious diseases have an impact on cardiac function, for example: late stage syphilis, measles, typhoid fever and malaria. So as these infectious diseases become less prevalent, mortality rates will fall.

3. **Reduced occupational stresses**

   The move in the work force from manual and hazardous occupations to white-collar work will also help bring down mortality rates. Fewer occupational hazards and better working conditions[1] are the main drivers

[1] For example less exposure to dust, fumes, gases and chemicals.

of this effect.

4. **Improved nutritional intake**

   There is evidence to suggest that some degenerative conditions can be linked to nutritional status. These conditions include non-insulin-dependent diabetes and autoimmune thyroiditis. So as our nutritional intake improves with time, mortality rates should fall.

5. **Life style improvements**

   People have become significantly more aware of the impact of their lifestyles on their health. As a result of this increased awareness more people look to maintain more healthy and balanced lifestyles than in the past. A smaller proportion of people smoke and more people are aware of their diets and exercise habits. The resulting health benefits mean lighter mortality.

6. **Rising incomes**

   As incomes rise people will have improved access to health care, eat better food and be able to manage their health more proactively. For example people who can afford to are more likely to go for regular health screening tests to detect illness earlier. This in turn improves the likelihood of the illness being successfully treated. Also people who can afford to are more likely to prevent the onset of some conditions and should they contract an illness they are more likely to be able to afford the cures.

7. **Rising education**

   There is evidence of a strong positive correlation between education and adult health. Basically, the more educated an individual the longer they will live.

### 3.1.3 The dynamics defined

The Lee-Carter model as described by Girosi and King [33] forecasts a set of log mortality rates $m_{x,t} = \ln(q_{x,t}) = \ln(1 - p_{x,t})$ using the dynamics:

$$m_{x,t} = \alpha_x + \beta_x \gamma_t + \epsilon_{x,t}. \tag{3.1}$$

1. $\alpha_x$ reflects the mean mortality rates by age, over time.
2. $\beta_x$ reflects the relationship between the general mortality rates for a specific age and the general trend in mortality rates.
3. $\gamma_t$ reflects the general mortality rates over time.
4. $\epsilon_{x,t}$ captures the randomness of the underlying process. It is this factor that essentially makes this a stochastic model, since $\epsilon_{x,t}$ is itself a random variable. The other factors $\alpha_x$, $\beta_x$ and $\gamma_t$ are solved entirely from the observed mortality experience.

### 3.1.4 The dynamics explained

Lee [40] suggests that the log mortality rates can be linearly represented by $\alpha_x$, $\beta_x$ and $\gamma_t$ over time per age. This makes intuitive sense if one looks at Figure 3.2, which shows the log mortality rates of the sample population as shown in Figure 3.1.

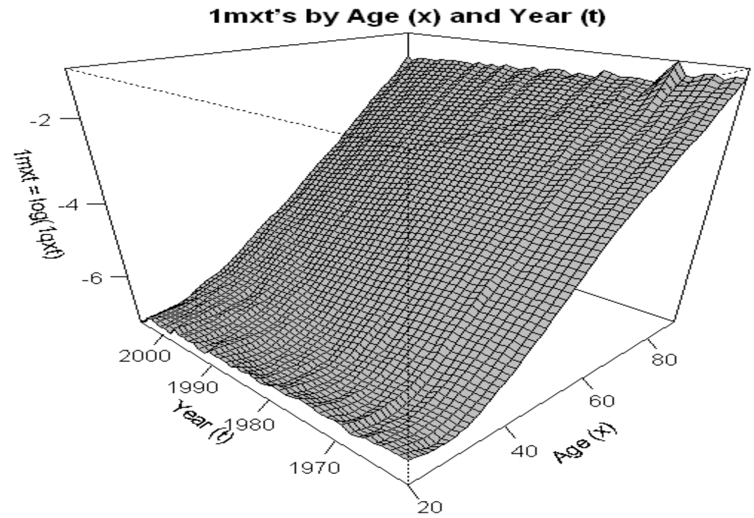


Figure 3.2: A plot of the log of the observed initial mortality rates for males of various ages born in England and Wales in the years 1961 until 2005.

These dynamics suggest the log mortality rate for a specific age $x$ over a specific year will have a component that varies by year only ($\gamma_t$) and a component that varies by age only ($\alpha_x$).

Assuming the data set spreads across ages $x_{min}$ to $x_{max}$ and calendar years $t_{min}$ to $t_{max}$, imposing two constraints on the dynamics described we can ensure a unique solution set. This is because $m_{x,t} = \alpha_x + \beta_x\gamma_t + \epsilon_{x,t}$ is over-parameterised according to Li and Chan [41].

These constraints are:

$$\sum_{t=t_{min}}^{t_{max}} \gamma_t = 0 \tag{3.2}$$

and

$$\sum_{x=x_{min}}^{x_{max}} \beta_x^2 = 1. \tag{3.3}$$

$\sum_{t=1}^{T} \gamma_t = 0$ implies $\alpha_x = \bar{m}_x$ where $\bar{m}_x$ is the mean of $\{m_{x,t_{min}}, \ldots, m_{x,t_{max}}\}$.

The dynamics of (3.1) can be rewritten in terms of the mean centered log-mortality rate giving:

$$\tilde{m}_{x,t} = m_{x,t} - \alpha_x. \tag{3.4}$$

If we assume $\epsilon_{x,t}$ is normally distributed then it follows that:

$$\tilde{m}_{x,t} \sim N(\mu_{x,t}, \sigma^2),$$

where:

$$\mu_{x,t} = \beta_x \gamma_t$$

and

$$\tilde{m} = \begin{bmatrix} \tilde{m}_{x_{min},t_{min}} & \ldots & \tilde{m}_{x_{min},t_{max}} \\ \vdots & \ddots & \vdots \\ \tilde{m}_{x_{max},t_{min}} & \ldots & \tilde{m}_{x_{max},t_{max}} \end{bmatrix}.$$

### 3.1.5 Determining the parameters

Lazar [38] suggested finding the parameter vectors $\bar{\beta}$ and $\bar{\gamma}$ using the singular value decomposition[2] of $\tilde{m} = UDV'$. The estimate for $\bar{\beta}$ is the first column of $U$ and the estimate for $\bar{\gamma}$ is $\bar{\beta}'\tilde{m}_t$. With $\tilde{m}_t$ the t$^{th}$ column of $\tilde{m}$ and $\bar{\gamma}' = [\gamma_{t_{min}}, \ldots, \gamma_{t_{max}}]$.

[2]Singular value decomposition is described in 8.5

Using Time Series analysis long-term forecasts can be done where the model's parameters are solved with reference to the observed experience of the population being modeled.

Using this framework has a few implicit assumptions. Firstly, it assumes that $\bar{\beta}$ remains constant over the forecast period and secondly, that forecasts for $\{\gamma_{f_{min}}, \ldots, \gamma_{f_{max}}\}$ are from a standard univariate time series model - such as a random walk with drift. Here the forecast horizon is set to span $f_{min}$ to $f_{max}$.

This suggests that:

$$\gamma_f = \gamma_{f-1} + \theta + \xi_f, \tag{3.5}$$

where, typically:

$$\xi_f \sim N(0, \sigma_{rw}^2). \tag{3.6}$$

Here $\theta$ is the drift parameter and it has the $MLE$:

$$\hat{\theta} = \frac{\gamma_{t_{max}} - \gamma_{t_{min}}}{t_{max} - t_{min}}, \tag{3.7}$$

furthermore:

$$\sigma_{rw}^2 = \frac{1}{(t_{max} - t_{min}) - 1} \sum_{t=t_{min}}^{t_{max}-1} \left\{ (\gamma_{t+1} - \gamma_t) - \hat{\theta} \right\}^2. \tag{3.8}$$

### 3.1.6 Solving the mortality rates

Once $\theta$ and $\sigma_{rw}^2$ are known we can use the recursive relationship $\gamma_f = \gamma_{f-1} + \theta + \xi_f$ to solve $\bar{\gamma}'_f = [\gamma_{f_{min}}, \ldots, \gamma_{f_{max}}]$, which is the parameter set that reflects the expected general mortality rates over time.

Using (3.5) and (3.7) gives rise to the following set of equations:

$$\gamma_{f_{min}} = \gamma_{t_{max}} + \theta + \xi_{f_{min}}, \tag{3.9}$$

$$\gamma_{f_{min}+1} = \gamma_{f_{min}} + \theta + \xi_{f_{min}+1}, \tag{3.10}$$

$$\vdots$$

$$\gamma_{f_{max}} = \gamma_{f_{max}-1} + \theta + \xi_{f_{max}}, \tag{3.11}$$

with:

$$\xi_f \sim N(0, \sigma^2_{rw}), \forall f \in \{f_{min}, \ldots, f_{max}\}.$$

Then using the relationship $m_{x,t} = \alpha_x + \beta_x \gamma_t + \epsilon_{x,t}$ and the vector $\bar{\gamma}'_f$, we can determine a set of simulated log mortality rates for all the ages $\{x_{min}, \ldots, x_{max}\}$ and all years $\{f_{min}, \ldots, f_{max}\}$.

Call this, $\tilde{m}_f$, where:

$$\tilde{m}_f = \begin{bmatrix} \tilde{m}_{x_{min},f_{min}} & \ldots & \tilde{m}_{x_{min},f_{max}} \\ \vdots & \ddots & \vdots \\ \tilde{m}_{x_{max},f_{min}} & \ldots & \tilde{m}_{x_{max},f_{max}} \end{bmatrix}. \tag{3.12}$$

From (3.12) and the relationship $q_{x,f} = \exp\{\tilde{m}_{x,f}\}$ we can solve a set of mortality rates, call this $q_f$, for all the ages $\{x_{min}, \ldots, x_{max}\}$ and all years $\{f_{min}, \ldots, f_{max}\}$.

$$q_f = \begin{bmatrix} q_{x_{min},f_{min}} & \ldots & q_{x_{min},f_{max}} \\ \vdots & \ddots & \vdots \\ q_{x_{max},f_{min}} & \ldots & q_{x_{max},f_{max}} \end{bmatrix}. \tag{3.13}$$

### 3.1.7 Using these mortality rates

It is this set of simulated mortality rates, $q_f$ in (3.13) that can be used to determine a simulated notional population, against which the pay-offs of the various mortality instruments can be determined.

Defining $N_{x,f}$ as the in-force notional population of lives aged $x$ in year $f$ leads to the relationship:

$$N_{x+1,f+1} = N_{x,f}(1 - q_{x,f}), \forall x \in \{x_{min}, \ldots, x_{max}\}, f \in \{f_{min}, \ldots, f_{max}\}. \tag{3.14}$$

Effectively, the anticipated population of lives aged $x+1$ in year $f+1$ (the LHS of (3.14)) is the proportion of population aged $x$ in year $f$ that are not expected to die between $f$ and $f+1$ (the RHS of (3.14)).

This in turn results in the following iterative scheme:

$$\begin{aligned} N_{x+1,f_{min}+1} &= N_{x,f_{min}}(1 - q_{x,f_{min}}), \\ &= N_{x,f_{min}}(p_{x,f_{min}}). \end{aligned}$$

$$\begin{aligned} N_{x+2,f_{min}+2} &= N_{x+1,f_{min}+1}(1 - q_{x+1,f_{min}+1}), \\ &= N_{x+1,f_{min}+1}(p_{x+1,f_{min}+1}), \\ &= N_{x,f_{min}}(p_{x,f_{min}} p_{x+1,f_{min}+1}), \\ &= N_{x,f_{min}} \prod_{i=0}^{1} p_{x+i,f_{min}+i}. \end{aligned}$$

$$\begin{aligned} N_{x+3,f_{min}+3} &= N_{x+2,f_{min}+2}(1 - q_{x+2,f_{min}+2}), \\ &= N_{x+2,f_{min}+2}(p_{x+2,f_{min}+2}), \\ &= N_{x,f_{min}}(p_{x,f_{min}} p_{x+1,f_{min}+1} p_{x+2,f_{min}+2}), \\ &= N_{x,f_{min}} \prod_{i=0}^{2} p_{x+i,f_{min}+i}. \end{aligned}$$

$$\vdots$$

$$\begin{aligned}
N_{x+F,f_{min}+F} &= N_{x+F-1,f_{min}+F-1}(1 - q_{x+F-1,f_{min}+F-1}), \\
&= N_{x+F-1,f_{min}+F-1}(p_{x+F-1,f_{min}+F-1}), \\
&= N_{x,f_{min}}(p_{x,f_{min}} \cdots p_{x+F-1,f_{min}+F-1}), \\
&= N_{x,f_{min}} \prod_{i=0}^{F-1} p_{x+i,f_{min}+i}.
\end{aligned}$$

Using the scheme of equations above allows $N$ to be determined recursively, where:

$$N = \begin{bmatrix} N_{x_{min},f_{min}} & \ldots & N_{x_{min}+f_{max}-f_{min},f_{max}} \\ \vdots & \ddots & \vdots \\ N_{x_{max},f_{min}} & \ldots & N_{x_{min}+f_{max}-f_{min},f_{max}} \end{bmatrix}. \tag{3.15}$$

$N$ represents the population per age and year of forecast, which can be used to value various mortality based instruments.

### 3.1.8 Conclusions

The Lee-Carter model is a suitable choice of stochastic mortality model since it has dynamics that adequately mirror reality.

We have also seen that from a set of graduated mortality rates the various parameters of the Lee-Carter model can be solved using a singular value decomposition, since (3.1) defines a linear framework. Then using those parameters a set of mortality rates can be determined. From those mortality rates the evolution of the underlying population can be simulated.

This simulated population's evolution forms the reference against which the cash flows of some mortality based instruments can be determined.

## 3.2 Step-by-step implementation in R

### 3.2.1 Introduction

This section shows a step-by-step implementation of the Lee-Carter model outlined in Chapter 3.1 in R [51]. Here the mortality data used is sourced from the HMD [35] and spans the ages 20 to 90 and the years 1961 to 2005.

### 3.2.2 The global parameters

A few global parameters need to be defined and set:

1. Some notional initial population size. This is set to $100,000$ in total and will be used to determine $N_{x,f_{min}}, \forall x \in \{x_{min}, \ldots, x_{max}\}, \forall f \in \{f_{min}, \ldots, f_{max}\}$.

2. The number of simulations. This needs to be large enough for the results to be significant and $3,250$ should be sufficient.

3. The forecast horizon. Here 20 years is used.

```
# Set up Global Parameters
#-------------------------

Starting_Notional_Population = 100000
Number_Of_Simulations        = 3250
Years_Of_Forecast            = 20
```

### 3.2.3 The data-set specific parameters

The minimum and maximum, ages and years are set based on the raw mortality data being used[3].

In effect:

1. $x_{min} = 20$

2. $x_{max} = 90$

[3] R scripts with the raw mortality data can be made available on request.

3. $t_{min} = 1961$
4. $t_{max} = 2005$

```
# Stochastic Mortality Model (Lee Carter)
#----------------------------------------

# Data Input
#-----------

MIN_AGE    = 20
MAX_AGE    = 90
AGE_RANGE  = MIN_AGE:MAX_AGE
MIN_YEAR   = 1961
MAX_YEAR   = 2005
YEAR_RANGE = MIN_YEAR:MAX_YEAR
AGES       = MAX_AGE-MIN_AGE+1
YEARS      = MAX_YEAR-MIN_YEAR+1
ENTRIES    = AGES*YEARS
```

### 3.2.4 Parameterising the model

```
# Model Parameterisation
#-----------------------
```

First one needs to determine $M$ from $Q$ using the relationship $m_{x,t} = \ln(q_{x,t})$, where:

$$m = \begin{bmatrix} m_{20,1961} & \cdots & m_{20,2005} \\ \vdots & \ddots & \vdots \\ m_{90,1961} & \cdots & m_{90,2005} \end{bmatrix} = \ln q$$

and

$$q = \begin{bmatrix} q_{20,1961} & \cdots & q_{20,2005} \\ \vdots & \ddots & \vdots \\ q_{90,1961} & \cdots & q_{90,2005} \end{bmatrix}.$$

This can be accomplished by using the code below.

```
M = log(Q)
```

also, having determined $m$, we can now determine $\bar{\alpha}$, since $\bar{\alpha}' = [\alpha_{20}, \ldots, \alpha_{90}]$. Where $\alpha_x = \bar{m_x}$ and $\bar{m_x}$ is the mean of $\{m_{x,t_{min}}, \ldots, m_{x,t_{max}}\}$. So where $x = 20$, $\alpha_{20} = \bar{m}_{20}$ which is the mean of $\{m_{20,1961}, \ldots, m_{20,2005}\}$.

```
# Finding m_x_bar = alpha_x
#--------------------------

alpha_x = array(1:AGES, dim = c(1,AGES))
for(x in 1:AGES)
{
  alpha_x[x] = mean(M[x, ])
}
```

Using the relationship $m_{x,t} = \ln(q_{x,t})$ we can back-solve the mean mortality rate per age, as can be seen in the graph below.

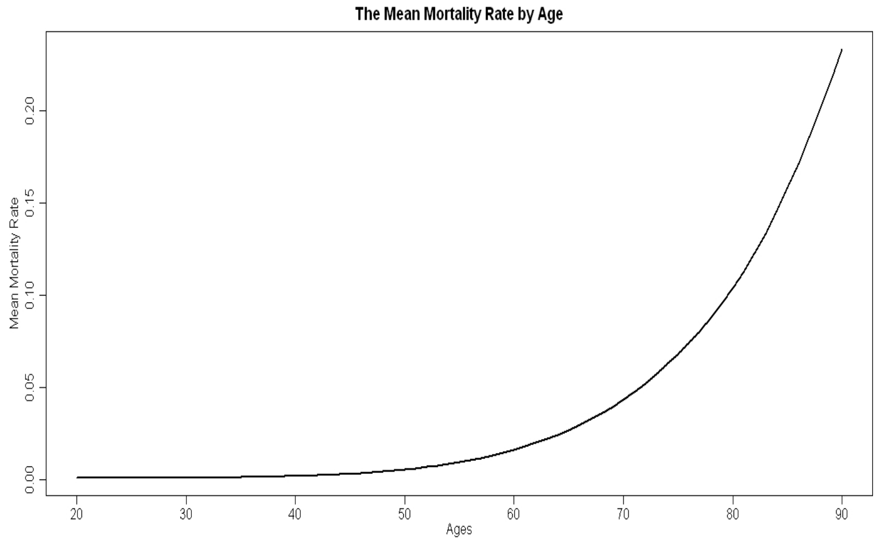


Figure 3.3: The mean initial mortality rate for ages 20 to 90, of males born in England and Wales between 1961 and 2005.

From $m$ we can also determine $\tilde{m}$, the mean centered log-mortality rates using the relationship: $\tilde{m}_{x,t} = m_{x,t} - \alpha_x$.

So:

$$\tilde{m} = \begin{bmatrix} \tilde{m}_{20,1961} & \cdots & \tilde{m}_{20,2005} \\ \vdots & \ddots & \vdots \\ \tilde{m}_{90,1961} & \cdots & \tilde{m}_{90,2005} \end{bmatrix}.$$

```
# Finding M_tilda
#----------------

M_tilda = array(1:ENTRIES, dim = c(AGES,YEARS))
for(x in 1:AGES)
{
  for(t in 1:YEARS)
  {
    M_tilda[x, t] = M[x, t] - alpha_x[x]
  }
}
```

The singular value decomposition of $\tilde{m}$ gives the parameter vectors $\bar{\beta}$ and $\bar{\gamma}$ where $\tilde{m} = UDV'$.

```
# Singular Value Decomposition of M_tilda = U D t(V)
#----------------------------------------------------

SVD_M_tilda   = svd(M_tilda)
D             = diag(SVD_M_tilda$d)
U             = SVD_M_tilda$u
V             = SVD_M_tilda$v
M_tilda_check = U %*% D %*% t(V)
```

$\bar{\beta}$ is the first column of $U$.

```
# Set beta_x
#-----------

beta_x = U[ ,1]
```

$\bar{\gamma} = \bar{\beta}'\bar{m}_t$. With $\bar{m}_t$ the t$^{th}$ column of $\tilde{m}$, so $\bar{\gamma}' = [\gamma_{1961}, \ldots, \gamma_{2005}]$.

```
# Create gamma_t
#---------------

gamma_t = array(1:YEARS, dim = c(YEARS))

# Populate gamma_t
#-----------------

for(t in 1: YEARS)
{
  gamma_t[t] = t(beta_x) %*% M_tilda[ ,t]
}
```

Now having completed the parameterisation of the model, we can check that the constraints assumed are maintained, namely that: $\sum_{t=1961}^{2005} \gamma_t = 0$ and $\sum_{x=20}^{90} \beta_x^2 = 1$.

```
# Check constraints
#------------------

beta_x_sqrd = array(1:AGES, dim = c(AGES))
for(x in 1:AGES)
{
  beta_x_sqrd[x] = (beta_x[x])^2
}

Check_beta_x  = sum(beta_x_sqrd)  #Should be 1
Check_gamma_t = sum(gamma_t)      #Should be 0
```

### 3.2.5 The simulations

Having determined all the required parameters of the model, these can now be used to forecast a set of mortality rates. These forecasted rates can then in turn be used to determine the notional in-force population. This notional population will then be used to price some sample mortality instruments.

#### Setting the parameters for forecasting

Setting the forecast horizon to 20 years from $t_{max} = 2005$ gives us the following:

1. $f_{min} = 2006$
2. $f_{max} = 2025$

```
# Forecasting
#------------

RUNS           = Number_Of_Simulations
F_YEARS        = Years_Of_Forecast
F_START_YEAR   = MAX_YEAR+1
F_END_YEAR     = F_START_YEAR + F_YEARS - 1
F_YEAR_RANGE   = F_START_YEAR:F_END_YEAR
```

```
F_ENTRIES       = AGES*F_YEARS
RUNS_RANGE      = 1:RUNS

gamma_MAX_YEAR = array(1:RUNS, dim = c(RUNS,1))
gamma_MIN_YEAR = array(1:RUNS, dim = c(RUNS,1))
gamma_t_TEMP   = array(1:RUNS, dim = c(RUNS, YEARS))
theta          = array(1:RUNS, dim = c(RUNS,1))
MAX_YEAR_F     = array(1:RUNS, dim = c(RUNS,1))
MIN_YEAR_F     = array(1:RUNS, dim = c(RUNS,1))
```

Considering that $\hat{\theta} = \frac{\gamma_{2005} - \gamma_{1961}}{2005-1961}$, we can determine $\theta$ using the code below.

```
for(N in 1:RUNS)
{
  for(t in 1: YEARS)
  {
    gamma_t_TEMP[N,t] = gamma_t[t]
  }
}

for(N in 1:RUNS)
{
  gamma_MAX_YEAR[N,1] = gamma_t[YEARS]
  gamma_MIN_YEAR[N,1] = gamma_t[1]
  MAX_YEAR_F[N,1]     = MAX_YEAR
  MIN_YEAR_F[N,1]     = MIN_YEAR
  TOP                 = (gamma_MAX_YEAR[N,1]-gamma_MIN_YEAR[N,1])
  BOTTOM              = (MAX_YEAR_F[N,1]-MIN_YEAR_F[N,1])
  theta[N,1]          = TOP/BOTTOM
}
```

Given the definition of $\sigma^2_{rw}$ above, we can solve $\sigma^2_{rw}$, as below.

$$\sigma^2_{rw} = \frac{1}{(2005-1961)-1} \sum_{i=1961}^{2005-1} \left\{ (\gamma_{i+1} - \gamma_i) - \hat{\theta} \right\}^2 .$$

```
dummy_1 = array(1:(RUNS*(YEARS-1)), dim = c(RUNS, YEARS-1))
```

```
one = array(1:1, dim = c(RUNS,1))
rw_var = array(1:1, dim = c(RUNS,1))

for(N in 1:RUNS)
{
  for(i in 1:( YEARS-1))
  {
    dummy_1[N,i] = (gamma_t_TEMP[N,i+1]-gamma_t_TEMP[N,i]-theta[N,1])^2
  }
}

for(N in 1:RUNS)
{
  rw_var[N,1] = 1/(MAX_YEAR_F[N,1]-MIN_YEAR_F[N,1]-one[N,1])*sum(dummy_1[N, ])
}

# Drop dummy_1
#------------

rm(dummy_1)
```

With the variance of the Random Walk process calibrated, a set of random normal variables can be generated using: $\xi_t \sim N(0, \sigma^2_{rw})$.

```
gamma_F_t = array(1:(F_YEARS*RUNS), dim = c(RUNS,F_YEARS))
xi = array(1:(F_YEARS*RUNS), dim = c(RUNS,F_YEARS))

for(N in 1:RUNS)
{
  for(t in 1:F_YEARS)
  {
    xi[N,t] = rnorm(1, mean=0, sd=(rw_var[N,1])^0.5)
  }
}
```

Given the set of random normal variables generated above and the $MLE$ for $\theta$, we can solve $\bar{\gamma}_f$, where $\bar{\gamma}'_f = [\gamma_{2006}, \ldots, \gamma_{2025}]$. This can be done using: $\gamma_f = \gamma_{f-1} + \theta + \xi_f$ as set out below.

```
for(N in 1:RUNS)
{
  gamma_F_t[N,1] = gamma_MAX_YEAR[N,1] + theta[N,1] + xi[N,1]
```

```
  for(t in 2:F_YEARS)
  {
    gamma_F_t[N,t] = gamma_F_t[N,t-1] + theta[N,1] + xi[N,t]
  }
}
```

The following section of code is to ensure that the dimensions of the various objects are as required for further computations.

```
beta_x_TEMP = array(1:(RUNS*AGES), dim = c(RUNS,AGES))
alpha_x_TEMP = array(1:(RUNS*AGES), dim = c(RUNS,AGES))
for(N in 1:RUNS)
{
  for(x in 1:AGES)
  {
    beta_x_TEMP[N,x] = beta_x[x]
    alpha_x_TEMP[N,x] = alpha_x[x]
  }
}
```

### A forecast of the log-mortality rates

Having determined $\bar{\alpha}$, $\bar{\gamma_f}$ and $\bar{\beta}$ we can solve $\tilde{m}_f$ using the relationship: $\tilde{m}_{x,f} = \alpha_x + \beta_x \gamma_f, \forall x \in \{20, \ldots, 90\}, f \in \{2006, \ldots, 2025\}$.

```
# Estimates of M_tilda = M_F_tilda
#--------------------------------

M_F_tilda = array(1:(RUNS*F_ENTRIES), dim = c(AGES,F_YEARS,RUNS))
for(x in 1:AGES)
{
  for(t in 1:F_YEARS)
  {
    for(N in 1:RUNS)
    {
      M_F_tilda[x, t, N] = beta_x_TEMP[N,x]*(gamma_F_t[N,t])+alpha_x_TEMP[N,x]
    }
  }
}
```

Figure 3.4 is an example of a set of simulated log mortality rates.

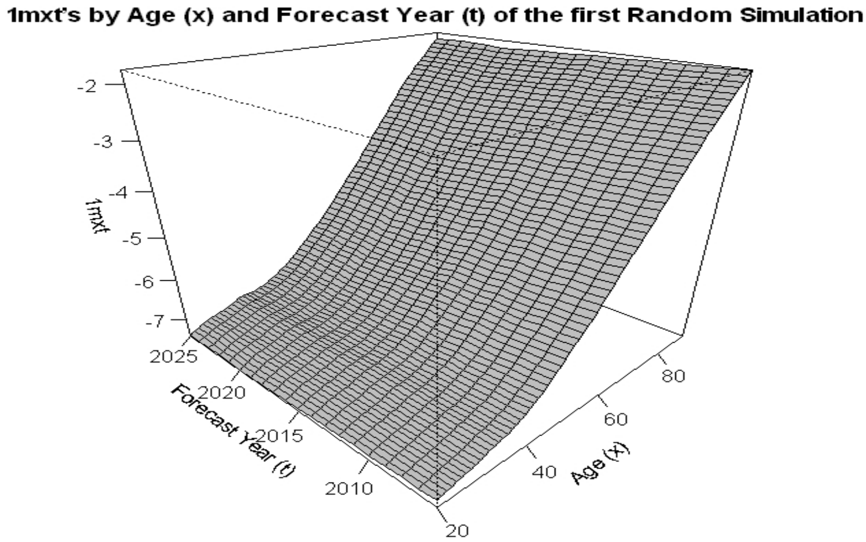


Figure 3.4: Simulated log mortality for the years 2006 until 2025 for ages 20 to 90, based on 3,250 runs.

Using the relationship $m_{x,f} = \ln(q_{x,f})$ we can now forecast a set of mortality rates per age for each year of the forecast. This mortality forecast can then be used to establish a fair value for the various mortality instruments as outlined in section 2.3.2.

Having the forecast mortality rates will allow various sample paths of pay-offs to be be projected. Those sample pay-offs can then each be discounted and the average across all runs taken to establish the expected discounted pay-off. The expected discounted pay-off of an instrument would be the fair value of that instrument.

```
Q_F = array(1:F_ENTRIES, dim = c(AGES,F_YEARS,RUNS))

# Populate Q for all AGES and YEARS
for (x in 1:AGES)
{
  for (t in 1:F_YEARS)
  {
    for(N in 1:RUNS)
    {
      Q_F[x,t,N] = exp(M_F_tilda[x,t,N])
    }
  }
}
```

Figure 3.5 is the set of simulated mortality rates solved from the simulated log mortality rates.

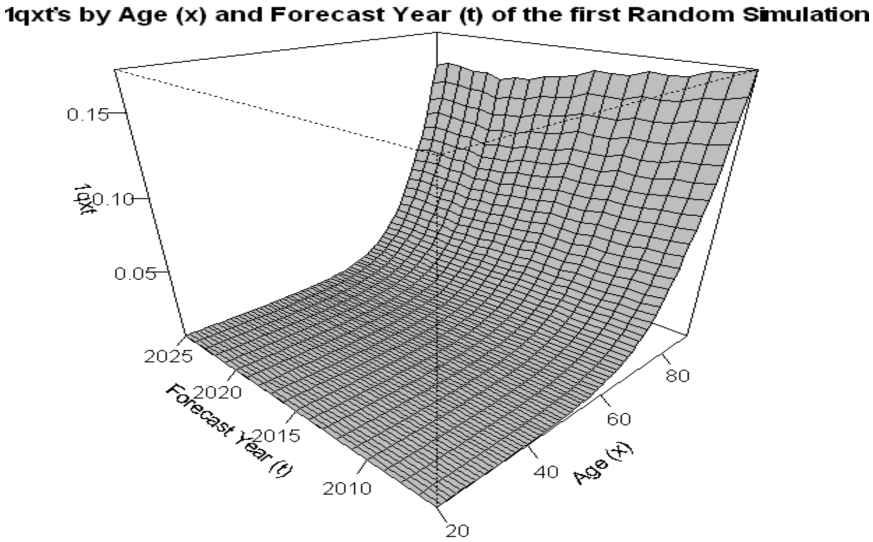


Figure 3.5: Simulated mortality for the years 2006 until 2025 for ages 20 to 90, based on 3,250 runs.

### 3.2.6 A simulated population

Using the simulated mortality rates, a notional population can be modelled over time. The notional population will generally represent the actual in-force population that would be hedged/traded by a Life Office in practice. It could for example be the in-force population of assured lives or a population of annuitants. This population's evolution relative to some benchmark will determine the pay-offs of the various mortality instruments.

Some mix of lives in the notional population by age will need to be assumed. In practice this mixture would be known as the population in-force is known. Here there is some assumed population mixture by age[4].

By using this assumed mix of lives by age and the assumed initial size of the notional population, the starting notional population can be determined as set out below.

Define:

1. $N_{x,t_{max}}$ as the number of lives aged $x$ in the starting notional population.

[4]The R script with the assumed mix of lives by age can be seen in 8.6

2. $M_x$ as the proportion of lives aged $x$ in the starting notional population.

This suggests that:

$$N_{x,t_{max}} = 100,000M_x, \forall x \in \{x_{min}, \ldots, x_{max}\}.$$

The resulting starting notional population is depicted in Figure 3.6:

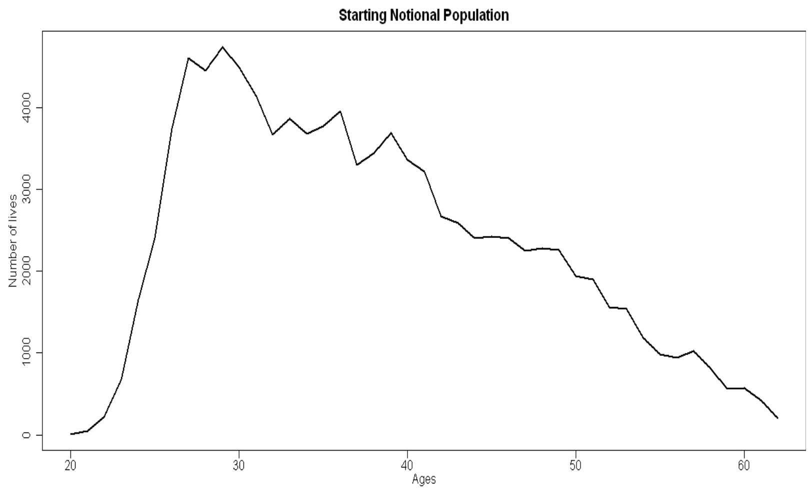


Figure 3.6: A plot of the assumed starting notional population based on a starting population of 100,000 and an assumed mixture of lives by age.

Using the recursive relationship $N_{x+F,f_{min}+F} = N_{x+F-1,f_{min}+F-1}(1 - q_{x+F-1,f_{min}+F-1})$ and the assumed starting notional population, we can then project the notional population for each age over the forecast horizon.

```
# Simulated Population Overtime Per Simulation
#----------------------------------------------

SIM_INFORCE = array(0, dim = c(AGES,F_YEARS+1,RUNS))

for(x in 1:AGES)
{
```

```
  SIM_INFORCE[x, 1, ] = Starting_Notional_Population*MIX[x, 2]
}

AGES_TO_USE = 43  #Ages used based on the assumed mix


for(t in 1:F_YEARS)
{
  for(x in 1:(AGES_TO_USE+F_YEARS))
  {
    SIM_INFORCE[x+1, t+1, ]=SIM_INFORCE[x, t, ]*(1-Q_F[x,t, ])
  }
}
```

Using this assumed mixture of lives by age and the simulated mortality rates, gives a sample simulated in-force population as can be seen in the terrain plot of Figure 3.7.


Figure 3.7: A terrain plot of the simulated evolution of the notional population over the years 2005 to 2025 for ages 20 to 90.

# Chapter 4

# Mortality based instruments

## 4.1 Longevity Bonds

### 4.1.1 Introduction

This chapter describes longevity bonds, their cash flows and some of their uses. It also introduces a framework to solve their theoretical values.

It also illustrates how by using the output of the Lee-Carter model a longevity bond can be valued. Here a longevity bond is defined and priced from first principles.

### 4.1.2 Definition

Longevity bonds are structures similar in many ways to standard corporate bonds. They pay regular coupons and a redemption payment, with the exception that the coupon payments are linked to the experience of some notional population. One example would be the Swiss Re mortality bond of December 2003 as discussed by Alistar and Harrison [1]. It had a three-year maturity and was priced at LIBOR + 1.35%.

### 4.1.3 Cash flows outlined

At time $t$ there is a coupon of $kN_t$ payable for $t \in \{1, 2, \ldots, T\}$.

At time $T$ there is a redemption payment of $N_0$.

We define:

1. $T$ as the maturity date.
2. $k$ as the coupon rate.
3. $N_t$ as the notional population at $t$, with $N_t = \sum_{x=x_{min}}^{x=x_{max}} N_{x,t}$.
4. $N_0$ as the notional starting population.

$N_0$, the notional starting population, is generally based on the population that is being used as the underlying.

So essentially, the coupon at $t$ is the coupon rate multiplied by the proportion of lives still in the population at $t$ and the redemption payment is equal to the number of lives that started in the population.

This means the larger the population in-force when the coupon is determined the larger the coupon, which can be seen as a hedge against longevity. To illustrate this point, take a population of annuitants, the longer they live, the more annuity payments will be made. So if people are living longer than originally anticipated more annuity payments than anticipated would be paid, but being long in a longevity bond would then also mean that larger coupons than originally anticipated would be received. This is also a point made by Blake, Boardman and Cairns [4]. Ngai [46] explores how effective different hedging strategies using various mortality based instruments are.

### 4.1.4 Pricing framework

Below is a pricing framework to arrive at the price of a longevity bond using the output of the stochastic simulation of the in-force population. It boils down to that the price of the longevity bond is the sum of the expected values of the coupons and the redemption payment, discounted to arrive at a present value[1].

Define:

[1]This approach is almost equivalent to that used by Bauer and Ruß[3], they however suggest the survival probabilities should be those implied by the market from annuity market quotes.

1. $P$ as the price of the longevity bond.
2. $V^t$ as the discounting factor to determine the present value of R1.00 received at time $t$.

This leads to the following:

$$\begin{aligned} P &= kE[N_1]V^1 + kE[N_2]V^2+, \ldots, +kE[N_{T-1}]V^{T-1} + kE[N_T]V^T + N_0V^T \\ &= k\sum_{t=1}^{t=T} E[N_t]V^t + N_0V^T. \end{aligned}$$

Here:

$$E[N_i] = \sum_{x=x_{min}}^{x=x_{max}} E[N_{x,i}], \forall i \in \{1, \ldots, T\}.$$

Note that $E[N_{x,t}]$ is the mean of $N_{x,t}$ across all the runs in the stochastic simulation of the in-force population.

### 4.1.5 Cash-flows illustrated

**Longevity bond coupons**

Averaging across the various runs gives the expected coupon per year. These coupons can be seen in the figure below. As one would expect to see, they fall over time as the starting notional population runs off [2].

Here we use:

1. $k = 5\%$
2. $T = 20$
3. $N_0 = 100,000$

[2]The R code to determine all the cash flows associated with the sample longevity bond can be seen in the Appendix in the section **Longevity Bond**

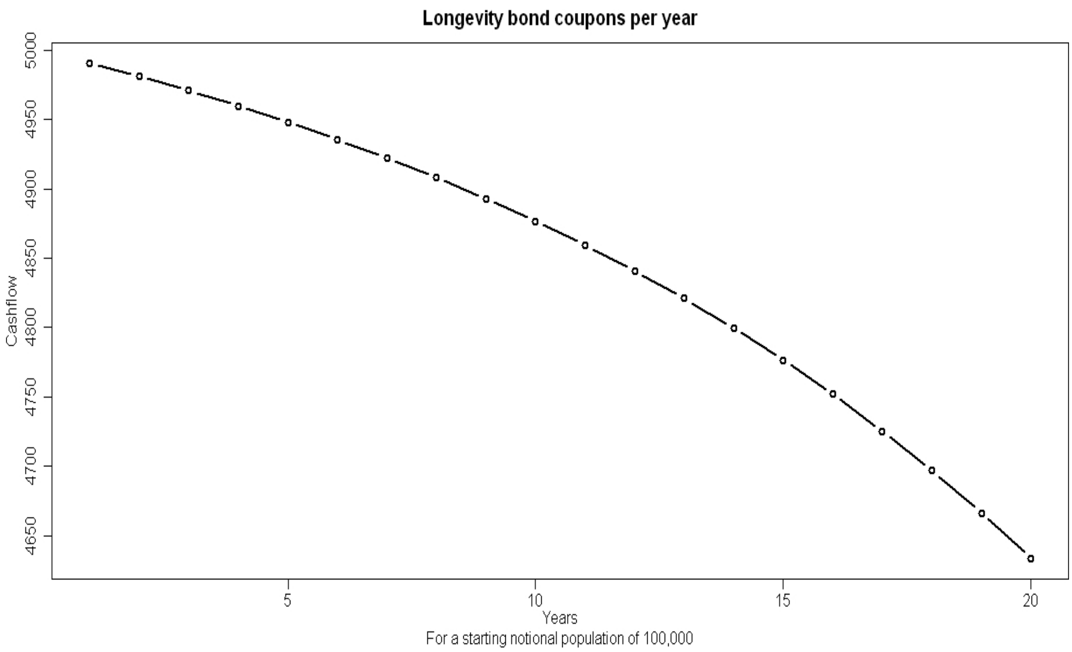


Figure 4.1: The coupons of a longevity bond based on the evolution of the sample notional population with a coupon rate of 5%.

### 4.1.6 Cash flow component summary

Below is a table summarising the cash flow components of the longevity bond. The price is equal to the sum of the discounted cash flows (R 66,319.93) per hundred thousand nominal. Each cash flow is the sum of the coupon and the redemption payments, if applicable.

| Year | Cash Flow | Discounted Cash Flow |
|---|---|---|
| 1 | R 4,990.82 | R 4,596.02 |
| 2 | R 4,981.08 | R 4,223.42 |
| 3 | R 4,970.75 | R 3,879.83 |
| 4 | R 4,959.76 | R 3,563.05 |
| 5 | R 4,948.07 | R 3,272.55 |
| 6 | R 4,935.62 | R 3,006.93 |
| 7 | R 4,922.34 | R 2,764.67 |
| 8 | R 4,908.17 | R 2,544.26 |
| 9 | R 4,893.03 | R 2,340.29 |
| 10 | R 4,876.83 | R 2,152.98 |
| 11 | R 4,859.49 | R 1,980.91 |
| 12 | R 4,840.91 | R 1,822.77 |
| 13 | R 4,820.99 | R 1,677.38 |
| 14 | R 4,799.62 | R 1,541.68 |
| 15 | R 4,776.67 | R 1,416.72 |
| 16 | R 4,752.01 | R 1,303.55 |
| 17 | R 4,725.48 | R 1,197.48 |
| 18 | R 4,696.97 | R 1,099.74 |
| 19 | R 4,666.33 | R 1,009.67 |
| 20 | R 104,633.39 | R 20,926.05 |
| Total | R 196,958.35 | R 66,319.93 |

Table 4.1: A summary of the components of the longevity bond's price based on a 5% coupon rate.

### 4.1.7 Hedging with longevity bonds

By taking a long position in longevity bonds the coupons received would increase as longevity increases. This could prove to be a useful tool to hedge against longevity risk. This is a point made in various papers, for example, Antolin and Blommestein [2][3]. Another example would be where a life office that has written a large book of annuity business is exposed to longevity risks, since the lighter mortality turns out relative to that priced for the smaller the underwriting profit or possibly the larger the underwriting loss.

Similarly for defined benefit pension schemes, the longer the pensioners live the more benefits the fund needs to provide for. If they live significantly longer than originally anticipated the fund might become unable to provide

[3]They also suggest that government should get involved and structuring mortality-indexed government bonds or to facilitate the creation of a mortality index

the benefits it is obliged to.

Partner Re [49] discuss longevity risks and how longevity bonds may play a role in helping life offices and pension schemes manage these mortality risks. They put forward three different sets of tools, namely:

1. Traditional Reinsurance.
2. Mortality Swaps and their variants[4].
3. Market Solutions, which include Longevity Bonds.

### 4.1.8 Away from just the theory

#### A real life example

Again citing Life Reinsurance Pricing, Longevity Risk, The Longevity Bond [49], the first longevity bond was designed in 2004 for the UK pension scheme market based on the mortality data of Partner Re and in collaboration with BNP Paribas/EIB. It was a 25 year term instrument. It provided coupons linked to the mortality experience (based on National Statistics) of males in England and Wales aged 65 in 2003.

Sadly because there was not a liquid market for such instruments and because the concept was very new, this specific bond never made it to market. That said, since then mortality instruments have become far more common and there is more and more incentive for these instruments to be created. The Cass Business School [15] suggests it would be in a Government's best interests to get involved in issuing longevity bonds for various reasons, including[5]:

1. **The expected cost of Government funding could be reduced by issuing longevity bonds.**

   Longevity bonds would mean the Government would gain access to another source of long-term funding. At the same time these longevity bonds could be issued with a longevity risk premium. This is similar in concept to why the Government issues CPI linked bonds.

[4] These will be touched on later in this paper
[5] The paper is written in a U.K. context, but it is as relevant in a South African context

2. **Government has an interest in ensuring there is an orderly transfer of liabilities to the capital markets.**

   At some point the insurance industry may want to start transferring longevity risk to the capital market. Particularly if it becomes a cost effective way to manage mortality risk. If the Government gets involved in the establishment of the mortality instrument space they can ensure it develops into a healthy, robust industry.

3. **Government has an interest in ensuring there is an efficient annuity market.**

   Government would want to promote efficient and cost effective savings vehicles for the nation. Also, establishing a term structure of mortality rates should help Life Offices manage their capital requirements more efficiently under SAM[6], which will require an efficient annuity market.

4. **Government is possibly best placed to equitably manage the impact of longevity across generations.**

   It would be inequitable if the burden of lightening mortality needed to born out by only a few generations.

[6]The South African equivalent of Solvency II that is expected to come into effect in 2014.

## 4.2 Mortality Swaps (fixed for floating)

### 4.2.1 Introduction

This chapter describes mortality swaps and builds on the concepts introduced in Section 4.1. The cash flows of mortality swaps are described and some of their uses are discussed. As for longevity bonds, a framework to arrive at their theoretical values is also introduced.

Again the output of the Lee-Carter model is used to value a sample mortality swap which is priced from first principles.

### 4.2.2 Definition

Mortality swaps are comparable to normal interest swaps, where a set of fixed and floating cash flows are exchanged based on the evolution of the actual experience relative to some benchmark experience.

### 4.2.3 Cash-flows outlined

At time $t$ there is a payment of $kF_t$ payable and a payment of $kN_t$ receivable for $t \in \{1, 2, \ldots, T\}$. $F_t$ is known at the outset so is $Fixed$ and since $N_t$ will only become known over time it is $Floating$.

Let us define:

1. $T$ as the maturity date.
2. $k$ as the swap rate.
3. $N_t$ as the notional population at $t$, with $N_t = \sum_{x=x_{min}}^{x=x_{max}} N_{x,t}$.
4. $F_t$ as the pre-defined population at $t$, set at $t = 0$ with $F_t = \sum_{x=x_{min}}^{x=x_{max}} F_{x,t}$.
5. $\chi_t$ as the net pay-off at $t$, with $\chi_t = k(N_t - F_t)$.

Essentially a fixed for floating mortality swap boils down to: where the swap rate is $k$, $k$ of the deviation between the actual and benchmark experience is swapped. Here $F_t$ is the benchmark experience and this is decided at the outset. $N_t$ represents the actual experience of some notional population which in practice will only become known as it's progression is observed.

This in turn means some part $(k)$ of the deviation between the benchmark and actual experience can be stripped away. This is handy for a life office as a tool to manage mortality fluctuation risk. Take for example a life office that sells assurance business that makes payouts when insured individuals die. If they die at a rate significantly higher than that assumed in the pricing of these products, the life office will make smaller underwriting profits and possibly underwriting losses.

The benefits of being able to use mortality swaps are reinforced in the paper: Pricing Survivor Swaps with Mortality Jumps and Default Risk, by Chang, Chen and Tsay [16]. This paper also puts forward a far more comprehensive pricing framework than the one described below. That said it is based on the same fundamentals.

### 4.2.4 Pricing framework

The following is a pricing framework to arrive at the price of a mortality swap, again using the output of the stochastic simulation of the in-force population[7]. Fundamentally the price of the swap is the expected present value of the net cash flows that the swap will result in[8].

Define:

1. $S$ the price of the mortality swap.
2. $V^t$ the discounting factor to determine the present value of R1.00 received at time $t$.

This leads to the following:

[7]Dawson, Dowd, Cairns and Blake [27] set out a comprehensive framework to price an array of Survivor Derivatives

[8]The R code to determine all the cash flows associated with the sample mortality swap can be seen in the Appendix in the section **Mortality swap**

$$\begin{aligned}
S &= kE[N_1 - F_1]V^1 + kE[N_2 - F_2]V^2 +, \ldots \\
&\quad \ldots, +kE[N_{T-1} - F_{T-1}]V^{T-1} + kE[N_T - F_T]V^T, \\
&= kE[\chi_1]V^1 + kE[\chi_2]V^2 +, \ldots, +kE[\chi_{T-1}]V^{T-1} + kE[\chi_T]V^T, \\
&= k\sum_{t=1}^{t=T} E[\chi_t]V^t.
\end{aligned}$$

Note that:

$$\begin{aligned}
E[N_t] &= \sum_{x=x_{min}}^{x=x_{max}} E[N_{x,t}], \forall t \in \{1, \ldots, T\}. \\
E[F_t] &= \sum_{x=x_{min}}^{x=x_{max}} E[F_{x,t}], \forall t \in \{1, \ldots, T\}.
\end{aligned}$$

Similarly here, $E[N_{x,t}]$ is the mean of $N_{x,t}$ across all the runs in the stochastic simulation of the in-force population. While $E[F_{x,t}]$ is predefined at the outset of the contract for all $x \in \{x_{min}, \ldots, x_{max}\}$ and $t \in \{1, \ldots, T\}$.

### 4.2.5 Cash-flows illustrated

Taking an annuity book where each life is entitled to receive R1.00 annually while they are alive. This would give rise to some annual annuity benefit out go, say $A_t$, in year $t$. This leads to the following set of equations:

$$\begin{aligned}
A_t &= E[N_t], \\
&= \sum_{x=x_{min}}^{x=x_{max}} E[N_{x,t}].
\end{aligned}$$

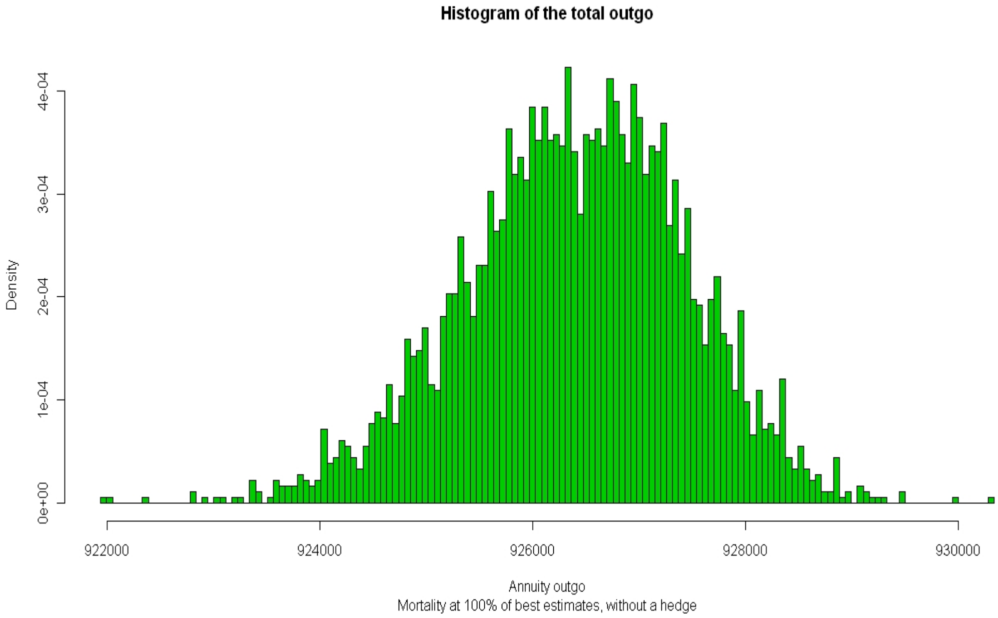


Figure 4.2: A histogram of the annuity outgo at 100% of best-estimates of the mortality of the notional population.

Now considering this outgo over the horizon until $T$ leads to the relationships:

$$\begin{aligned} A &= A_1V^1 + A_2V^2+, \ldots, +A_{T-1}V^{T-1} + A_TV^T \\ &= E[N_1]V^1 + E[N_2]V^2+, \ldots, +E[N_{T-1}]V^{T-1} + E[N_T]V^T \\ &= \sum_{t=1}^{t=T} E[N_t]V^t \end{aligned}$$

Setting the notional population to 100,000 and using the simulated population from earlier in the paper results in the following distribution of $A$ over the 3,250 simulations[9].

[9]The R code to determine all the cash flows associated with this annuity outgo can be seen in the Appendix in the section **Annuity outgo**

### 4.2.6 Hedging using a mortality swap

#### A practical example

By using a mortality swap, part of the difference between actual and expected experience can be hedged away. A point made by Sweeting [53]. An easy way to visualise this is to consider the net cash flows where there is a mortality swap as a hedge on the annuity book introduced above[10]:

$$\begin{aligned}
A - S &= \left\{E[N_1]V^1+,\ldots,+E[N_T]V^T\right\} - \left\{kE[N_1 - F_1]V^1+,\ldots,+kE[N_T - F_T]V^T\right\}, \\
&= \left\{E[N_1]V^1+,\ldots,+E[N_T]V^T\right\} - \ldots \\
&\quad \ldots - \left\{kE[N_1]V^1 - kF_1V^1+,\ldots,+kE[N_T]V^T - kF_TV^T\right\}, \\
&= \left\{(1-k)E[N_1]V^1+,\ldots,+(1-k)E[N_T]V^T\right\} - \left\{kF_1V^1+,\ldots,+kF_TV^T\right\}, \\
&= (1-k)\left\{E[N_1]V^1+,\ldots,+E[N_T]V^T\right\} - k\left\{F_1V^1+,\ldots,+F_TV^T\right\}, \\
&= (1-k)\sum_{t=1}^{t=T} E[N_t]V^t - k\sum_{t=1}^{t=T} F_tV^t.
\end{aligned}$$

Here $k, A, S$ and $T$ are known at the outset. As are $V_t$ and $F_t, \forall t \in \{1,\ldots,T\}$. So the cash flows that are unknown at the outset are: $(1-k)E[N_t], \forall t \in \{1,\ldots,T\}$. Had the mortality swap not been in place the cash flows unknown at the outset would be: $E[N_t], \forall t \in \{1,\ldots,T\}$.

As can be seen, by introducing a mortality swap less cash flow in unknown from the outset. The larger $k$ is, the less uncertainty there is in the cash flows.

The figure below shows the distribution of the total outgo where the annuity outgo is hedged with a mortality swap, with a swap rate of 20%. When comparing this to the distribution of the annuity outgo only it can be seen that the mean outgo is very similar, but there is less weight in the tails of the distribution. In essence, by having a mortality swap in place a degree of the volatility in the annuity benefit outgo has been removed.

[10] Also, the R code to determine all the net cash flows associated with the sample mortality swap as a hedging instrument can be seen in the Appendix in the section **Mortality swap as a hedge**

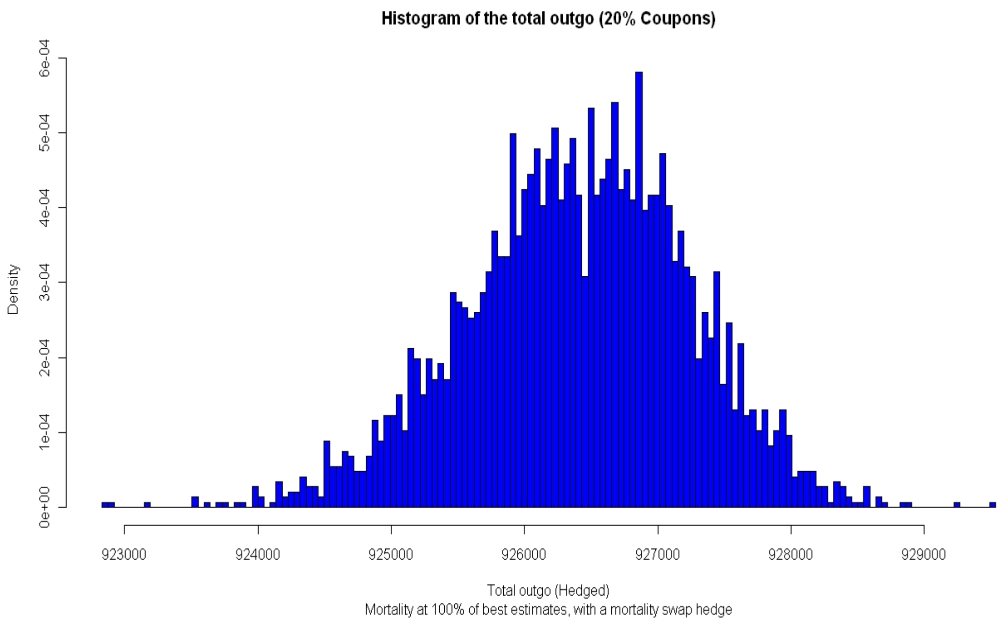


Figure 4.3: A histogram of the total outgo at 100% of best-estimates of the mortality of the notional population, with a mortality swap hedge.

The above can also be confirmed by comparing the CDF of total outgo with and without the mortality swap hedge. The CDF of the total outgo with and without the hedge in place can be seen to have very similar means. However the CDF of the total outgo with a mortality swap in place has less weight in the tails.

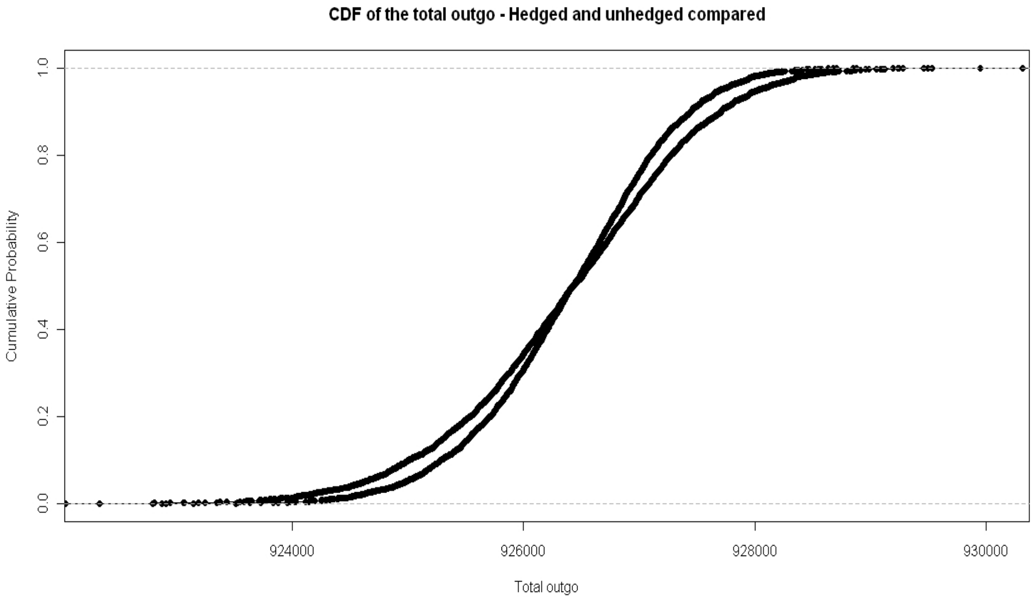


Figure 4.4: A CDF plot of the total outgo at 100% of best-estimates of the mortality of the notional population, with and without a mortality swap hedge.

Comparing the total outgo using the table below also illustrates the point that a mortality swap can be used to strip out volatility in mortality experience.

| | Total Outgo (Unhedged) | Total Outgo (Hedged) |
|---|---|---|
| Minimum | R 921,432 | R 922,425 |
| 1st Quartile | R 923,794 | R 924,315 |
| Median | R 926,155 | R 926,204 |
| Mean | R 926,155 | R 926,204 |
| 3rd Quartile | R 928,517 | R 928,093 |
| Maximum | R 930,878 | R 929,982 |

Table 4.2: Summary statistics of total outgo, with and without a mortality swap hedge

### 4.2.7 A practical consideration

Assuming the fixed leg of the fixed for floating mortality swap has a shorter duration. In that case the mortality swap will be cash flow positive for purchaser for a number of years before turning cash flow positive for the seller.

This in turn means the purchaser won't know if the purchase is favourable or not until well into the contracts life. It also means that the seller builds up their credit exposure to the purchaser over time.

Again the importance of managing the credit risk around these instruments is highlighted. This is of particular practical importance because of the long life spans of these instruments.

## 4.3 Mortality Swaptions (fixed for floating)

### 4.3.1 Introduction

This chapter describes mortality swaptions (which are essentially mortality swaps with caps or floors). Similarly the cash flows of mortality swaptions are described and some of their uses outlined.

Here too the output of the Lee-Carter model is used to value a sample mortality swaption.

### 4.3.2 Definition

Mortality swaps are comparable to normal interest swaptions. Here the floating cash-flows can be traded for fixed cash-flows, subject to a non-negative difference.

### 4.3.3 Cash-flows outlined

At time $t$ there is a payment of $k \times max\{N_t - F_t, 0\}$ receivable for $t \in \{1, 2, \ldots, T\}$. $F_t$ is known at the outset so is $Fixed$ and since $N_t$ will only become known over time it is $Floating$. This outline is consistent with that set out by Dawson, Dowd, Cairns and Blake [28].

Define:

1. $T$ as the maturity date.
2. $k$ as the swap rate.
3. $N_t$ as the notional population at $t$, with $N_t = \sum_{x=x_{min}}^{x=x_{max}} N_{x,t}$.
4. $F_t$ as the pre-defined population at $t$, set at $t = 0$ with $F_t = \sum_{x=x_{min}}^{x=x_{max}} F_{x,t}$.
5. $\psi_t$ as the net pay-off at $t$, with $\psi_t = k \times max\{N_t - F_t, 0\}$.

Considering the above, a fixed for floating mortality swaption is a fixed for floating mortality swap just that the long party's repayments are capped at zero. Equivalently, $k$ of the downside between the actual and benchmark experience is swapped away. As for the mortality swap, $F_t$ is the benchmark experience and this is known from the outset. $N_t$ represents the actual experience of some notional population which emerges over time.

This structure means that $k$ of the deviation in experience in the 'wrong' direction can be stripped away. Here this fixed for floating structure would allow the swaption purchaser to protect against the downside of mortality being lighter than anticipated. The lighter mortality experience turns out to be, the larger $N_t$ will be and so the larger the net pay-off $k \times max\{N_t - F_t, 0\}$ will be. This would allow a life office to protect against longevity risk, but not give away as much of the upside if mortality turns out higher than anticipated since $(N_t - F_t)$ is capped at zero.

### 4.3.4 Pricing framework

The following is a pricing framework to arrive at the price of a mortality swaption. As for the mortality swap it'll use the stochastic simulation of the in-force population. Fundamentally the price of the swaption is the expected present value of the net cash flows that the swaption will result in[11]. Dawson, Dowd, Cairns and Blake [29] also show this using a straight forward application of the Black-Scholes-Merton model.

We define:

1. $O$ the price of the mortality swaption.
2. $V^t$ the discounting factor to determine the present value of R1.00 received at time $t$.

This leads to the following:

[11]The R code to determine all the cash flows associated with the sample mortality swaption can be seen in the Appendix in the section **Mortality swaption**

$$\begin{aligned}
O &= kE[max\{N_1 - F_1, 0\}]V^1 + kE[max\{N_2 - F_2, 0\}]V^2 +, \ldots \\
&\quad \ldots, +kE[max\{N_{T-1} - F_{T-1}, 0\}]V^{T-1} + kE[max\{N_T - F_T, 0\}]V^T, \\
&= kE[\psi_1]V^1 + kE[\psi_2]V^2 +, \ldots, +kE[\psi_{T-1}]V^{T-1} + kE[\psi_T]V^T, \\
&= k\sum_{t=1}^{t=T} E[\psi_t]V^t.
\end{aligned}$$

Here:

$$\begin{aligned}
E[N_t] &= \sum_{x=x_{min}}^{x=x_{max}} E[N_{x,t}], \forall t \in \{1, \ldots, T\}. \\
E[F_t] &= \sum_{x=x_{min}}^{x=x_{max}} E[F_{x,t}], \forall t \in \{1, \ldots, T\}.
\end{aligned}$$

Similarly here, $E[N_{x,t}]$ is the mean of $N_{x,t}$ across all the runs in the stochastic simulation of the in-force population. While $E[F_{x,t}]$ is pre-defined at the outset of the contract for all $x \in \{x_{min}, \ldots, x_{max}\}$ and $t \in \{1, \ldots, T\}$.

### 4.3.5 Hedging using a mortality swaption

By using a mortality swaption, some downside mortality risk can be hedged away[12] but potentially less of the upside of mortality turning out more favourable than anticipated is given away[13].

Consider the net cash flows where there is a mortality swap as a hedge on the annuity book introduced above:

[12] For example the longevity risk of an annuity book

[13] Also, the R code to determine all the net cash flows associated with the sample mortality swaption as a hedging instrument can be seen in the Appendix in the section **Mortality swaption as a hedge**

$$\begin{aligned}
A - O &= \left\{E[N_1]V^1+, \ldots, +E[N_T]V^T\right\} -, \ldots \\
&\quad \ldots - \left\{kE[max\left\{N_1 - F_1, 0\right\}]V^1+, \ldots, +kE[max\left\{N_T - F_T, 0\right\}]V^T\right\}, \\
&= \left\{E[N_1]V^1+, \ldots, +E[N_T]V^T\right\} -, \ldots \\
&\quad \ldots - k\left\{max\left\{E[N_1] - F_1, 0\right\}V^1+, \ldots, +max\left\{E[N_T] - F_T, 0\right\}V^T\right\}, \\
&= \sum_{t=1}^{t=T} E[N_t]V^t - k\sum_{t=1}^{t=T} max\left\{E[N_t] - F_t, 0\right\}V^t.
\end{aligned}$$

Again, here $k, A, O$ and $T$ are known at the outset. As are $V_t$ and $F_t, \forall t \in \{1, \ldots, T\}$. So the cash flows that are unknown at the outset related to $E[N_t], \forall t \in \{1, \ldots, T\}$.

As can be seen, by introducing a mortality swaption less cash flow in unknown from the outset. The larger $k$ is, the less uncertainty there is in the cash flows.

# Chapter 5

# Market consistent methods

## 5.1 Market Consistence outlined

### 5.1.1 Introduction

This chapter details what Market Consistent methods are and how they can be used. It also explains why there needs to be a complete and liquid market for market consistent methods to be practical. There is also detail on why market consistent methods should be adopted.

### 5.1.2 Market Consistent methods defined

The Market Consistent value of a cash-flow is determined by the cost of a perfectly replicating portfolio, constructed of traded financial instruments. So if the valuation method determines the value of the cash-flow by constructing a perfectly replicating portfolio, then the method is said to be market consistent.

This implies the market needs to be Complete and Liquid. Complete in the sense that any pay-off would need to be replicable given the instruments traded in the market. If this condition is not met it could mean that the required replicating portfolio can't be constructed to value a specific set of cash-flows. If the market is not sufficiently liquid, then it means that the instruments required to construct replicating portfolios are not traded/tradeable at their fair price in sufficient volume. This in turn would mean that it may not be possible to construct the required replicating portfolio.

### 5.1.3 The South African mortality instrument market

At present the South African mortality instrument market is virtually non-existent. This means it is not Complete or Liquid, making it practically impossible to construct such perfectly replicating portfolios as required above to arrive at Market Consistent values using mortality instruments.

Using the more developed economies as a guide, where mortality instruments are being seen more frequently, it can only be a matter of time before such a market develops on South Africa[1]. The rate at which this happens will also depend on the relative advantage Life Offices using the instruments find themselves in compared to those who don't. The more competitive the market, the quicker the various (major) participants will need to adjust should they find themselves at a relative disadvantage. As shown above if a Life Office can efficiently trade mortality they will, all else equal, be at a relative advantage.

### 5.1.4 Why should market consistent models be adopted

There is great value in a frame work that can ensure objectivity, comparability and consistency as pointed out by Lebel [39]. This is also the driving force behind Market Consistent reporting and supplementary reporting as detailed in the paper by Pollard and Whitlock [50].

Embedded values have long been seen to give a more insightful view of Life Office. Embedded values immediately reflect the impact of various actions and decisions of the current management rather than showing the unwinding of the legacy of the past. This is in contrast to any primary accounts.

One shortfall of the traditional embedded value methods is that there is a significant amount of discretion in how the embedded value is determined. This makes it extremely difficult to do an apples-for-apples comparison across different market participants. This shortfall could be addressed by using Market Consistent Embedded Values.

By being able to use traded assets to establish the values of the underlying cash flows, much of the subjectivity will be removed, since the value is ob-

[1] The need for mortality based derivatives will also be brought to the fore by SAM, which is the proposed risk based capital requirement framework being proposed by the FSB.

servable in the market and the market[2] is efficient and driven by supply and demand.

At present the market consistent values of Life Office liabilities are actually only pseudo market consistent, as the values are established using an Economic Scenario Generator (ESG) and not on physically traded instruments. O'Brien [47] explains in great detail how the United Kingdoms Financial Services Authority currently requires assets and liabilities to be valued on a market consistent basis. This is also outlined in section 8.4.

Such an approach removes fall less of the subjectivity out of the calculation than a truly market consistent method would. If a market of traded mortality instruments existed, then truly market consistent values of Life Office cash flows could be established.

[2] In theory

## 5.2 Market consistent valuations using longevity bonds

### 5.2.1 Introduction

The following chapter shows that there are uses for mortality based instruments beyond protecting against adverse mortality movements. One such alternative is the determination of market consistent values of mortality driven cash flows.

### 5.2.2 Mechanics

So far hedging has been the sole use put forward for these instruments. There is also great scope for them to be used to determine market consistent values of liabilities[3].

To establish a market consistent value of the liabilities of a population in-force, one needs to find a set of instruments that will exactly replicate the outgo that the population will give rise to. The value of those instruments is then equivalent to the liability of the population, assuming there is no arbitrage.

Considering that the cash-flows of a longevity bond are defined by $P$ as below, which results in:

$$\begin{aligned}
P &= kE[N_1]V^1 + kE[N_2]V^2 +, \ldots \\
&\quad \ldots, +kE[N_{T-1}]V^{T-1} + kE[N_T]V^T + N_0V^T, \\
P - N_0V^T &= kE[N_1]V^1 + kE[N_2]V^2 +, \ldots, +kE[N_{T-1}]V^{T-1} + kE[N_T]V^T, \\
\frac{P - N_0V^T}{k} &= E[N_1]V^1 + E[N_2]V^2 +, \ldots, +E[N_{T-1}]V^{T-1} + E[N_T]V^T.
\end{aligned}$$

Also, the annuity outgo of a book of annuitants each receiving R 1.00 each year on survival is defined by $A$, giving:

[3] In contrast to using mortality based instruments, Olivieri and Pitacco [48] aim to outline a setting for the valuation of a Life Annuity portfolio using Risk-Neutral arguments involving reinsurance arrangements and bonds

$$\begin{aligned} A &= A_1V^1 + A_2V^2+, \ldots, +A_{T-1}V^{T-1} + A_TV^T, \\ &= E[N_1]V^1 + E[N_2]V^2+, \ldots, +E[N_{T-1}]V^{T-1} + E[N_T]V^T. \end{aligned}$$

Using the above two sets of equations gives:

$$A = \frac{P - N_0V^T}{k}.$$

Thus, since $A, P, N_0, V^T$ and $k$ are known and observable in a market[4], a market consistent value of $A$ can be determined, where $A$ is the annuity outgo of an annuity population.

[4]assuming one exists

# Chapter 6

# Summaries and conclusions

This chapter summarises the conclusions of the preceding chapters and shows how the primary objectives set out in Section 1.1.4 have been achieved and show that:

1. There are risks associated with the mortality experience of any population.
2. These risks can have an adverse impact on various financial institutions and the man on the street.
3. Using instruments whose cash flows are driven by the movements in some underlying populations mortality experience may be tools to manage some of these risks[1].

## 6.1 Summaries

### 6.1.1 Mortality and the impact thereof

Mortality experience at the end of the day impacts everyone of us in some way or another. Consider that if mortality rates are heavier than expected, say due to higher prevalence of HIV and AIDS, the employable work force may shrink. Which has repercussions on economic growth.

Conversely, if mortality is lighter than anticipated, because of some unexpected medical advancements, we might find ourselves with an aging population that needs to be supported. This could result in an economic strain if

[1] Important to note is that these instruments could act as substitutes for traditional methods of managing mortality risks or they could compliment the traditional methods.

the employed population is not large enough relative to the retired population.

To a Life Insurance Provider if mortality experience turns out worse than expected, then underwriting losses may be the end result.

We can't *manage* mortality rates easily, but we can create tools to manage the impact of mortality turning out different to what we anticipate.

### 6.1.2 How can it go wrong

Mortality projections could be wrong for various reasons including, but not limited to the fact:

1. Models require exposure and claims data, which in reality is often far from perfect.
2. There is always some residual basis risk.
3. Projections can turn out wrong for no reason other than random error.
4. Models won't ever perfectly describe reality.
5. Models can not allow for structural changes in mortality that haven't been observed in past data.
6. Tail/catastrophic events do occur.

### 6.1.3 Tools to manage mortality

There are traditional tools for managing mortality risks. For example natural hedging and reinsurance, but these tools have some drawbacks.

It is difficult for a life insurance provider to manage the mix of their in force business because there are various factors outside their direct control, such as lapses and new business volumes. This reduces the efficiency of natural hedging.

Reinsurance on the other hand always comes at a cost as reinsurers are in business to make profits. They also incur costs in the process of making profits.

A new set of tools to supplement the traditional methods is emerging. These tools include mortality based instruments. Not only do the mortality based derivatives address some of the short-falls of the traditional methods, but there are also some additional spin-off's. One such spin-off is that once a market for mortality based derivatives emerges, it should be easier to establish the market consistent values of life insurance liabilities.

### 6.1.4 Modelling mortality

Accurate modelling of mortality starts with good quality data. The best data is that gathered over time by a life insurance provider. Reinsurer's are also a reliable source of good quality mortality data.

Once good quality data is available, a model needs to be decided on. The choice of model should depend on what the model's output is going to be used for. For example, if the model's output is going to be used for the pricing of mortality driven instruments, then a stochastic mortality model is generally the most suiteable.

An example of a stochastic mortality model is the Lee-Carter model, which models mortality rates per age, per year. This "per age, per year" structure lends itself well to the pricing of instruments that protect against adverse mortality experience overtime. Some other suitable alternatives would include the Renshaw-Haberman model, the Currie Age-Period-Cohort model, the Cairns, Blake and Dowd model or some extension thereof or an affine mortality model.

### 6.1.5 Trading mortality

By having a mortality index against which the level of mortality can be measured, it can be traded. This is analogous to say a share price index against which instruments are traded.

Some frameworks for such indices have been put forward. The Life & Longevity Markets Association [44] suggests a framework. JP Morgan [18] has also set out the LifeMetrics framework. These indices can be either standard (and as a result observable in the open market) or customised for a specific instrument for a specific use.

A customised index might be created for a mortality based instrument to hedge the mortality risks of a specific population. However such instruments will be less liquid. On the other hand a standardised index could be used in a similar manner, just that instruments traded against these standardised indices will be more liquid, but they do introduce some basis risk.

### 6.1.6 Pricing mortality instruments

The principles behind pricing mortality based instruments are the same as those underpinning the pricing of other derivatives. In this dissertation the instruments are priced using Monte Carlo methods.

The basic process is to:

1. Specify a process to represent the evolution of the underlying's[2] value.
2. Generate various sample paths of the underlying's price.
3. Calculate the resulting cash flows the instrument would give rise to.
4. Determine the present value of these cash flows.
5. Solve the value of the instrument, which is the sum of the expected present values of all the cash flows.

### 6.1.7 The Lee-Carter model

The Lee-Carter model is used to forecast the future levels of mortality per age, by year using the dynamics as set out in (3.1):

$$m_{x,t} = \alpha_x + \beta_x \gamma_t + \epsilon_{x,t}.$$

[2]Here the underlying is mortality

The dynamics of the Lee-Carter model capture two trends. The first, that the likelihood of an individual dying gets larger with each passing year of age. The second, that the mortality rates for a given age change from year to year[3].

The lightening of mortality from one year to the next for a given age can be attributed to many things including:

1. Improvements in medical care.
2. Reductions in infectious disease rates.
3. Reduced occupational stresses and risks.
4. Improved diet.
5. Lifestyle improvements.
6. Rising incomes.
7. Rising levels of education.

From the mortality rates forecast using the Lee-Carter model, the evolution of a notional population can be simulated. Having this notional population per age, by year gives the information needed to determine the cash flows of various mortality based instruments such as: Longevity bonds, mortality swaps and mortality swaptions.

### 6.1.8 Mortality instruments

Various instruments can be designed to meet an array of different needs. A theme echoed throughout this paper and re-iterated by various authors, including Cummins [24]. Whether the need is to protect against longevity risk, as pointed out by Cui [22] or to reduce the exposure to volatility in mortality experience to smooth operating profits. One just needs to have a well defined framework to dictate who gets what and under what conditions.

[3] An individual aged 50 in the year 2001 is more likely to reach 51 than an individual aged 50 in the year 2000. This is attributable to the lightening of mortality from one year to the next.

Having such a framework and a stochastic mortality model allows for both the time and intrinsic values of these instruments to be calculated. Hopefully once the need for these instruments is established a market for these instruments might develop. This would in turn bring down the costs of these instruments as the market place becomes more competitive and efficient.

Another positive spin-off would be that market consistent values of mortality driven cash flows will be easier to establish.

## 6.2 Conclusion

Considering Sections 6.1.1 and 6.1.2 it is clear that mortality impacts everyone of us daily and is a source of risk. This is in line with the first and second of the three primary objectives.

Sections 6.1.3, 6.1.5 and 6.1.8 again show that mortality based instruments can help us manage the risks associated with mortality. This is in line with the third of the three primary objectives.

### 6.2.1 Closing thoughts

Considering all of the above, one might ask: "So what does this all mean?".

Essentially, mortality rates and their unpredictable nature are a source of risk. We might think we know how to deal with these risks and that we have sufficient tools to do so, but it has been shown that we must not be too sure of ourselves. Our prediction's about mortality will most probably turn out wrong and when they do we'll need tools to manage the impact's of these error's.

It has been shown how mortality based instruments can compliment traditional ways of managing mortality risk's. The more tool's we have to deal with the uncertainty of tomorrow, the more likely we will be able to weather the storm's we are already on course for today - whether we know we are heading for them or not.

To be able to construct these mortality based instruments we need to be able to model mortality and to model it stochastically. This presents a few unique challenges.

Considering all the above I would like to ask: "If these instruments are so great, why are they not already extensively traded?".

There are a host of reason's why these instrument's are not extensively traded (yet). For example the indices that have been constructed to date do not adequately address the market's concern's about basis risk, transparency and liquidity. For a complete and liquid market to emerge these issue's need to be addressed.

There are also major practical tax and regulatory hurdles to overcome.

That said, a few of these instruments are making it to market and there has been an increase in demand for such instruments.

Figure 2.1 (A summary of mortality instruments developed from 2008 to 2010) Suggests these instruments are finding their way into the market despite all the initial teething problems. I personally see that more and more of these instruments will move off drawing boards and onto balance sheets. Possibly not in the same volumes as other more traditional instruments, but the market for mortality based instruments will grow.

You might also ask: "If you can replicate a set of mortality driven cash-flows (say those of a life assurance portfolio) with these mortality based instruments, then surely you can arrive at a market consistent value for those cash-flows?".

The answer is yes, in theory and this has been shown. However this is not the case in practice. If we assume a complete and liquid market for mortality based derivatives does evolve then we may be able to use them in the process of determining market consistent values, but most probably not until that point in time.

Reaching that point does however require one to stretch one's imagination, but I trust after reading this dissertation it is now less of a stretch.

Given that all the objectives the dissertation set out to achieve have been achieved, where to from here?

Now that a need for mortality based instruments has been established and that the theory has been detailed, the next step would be to delve into the technical nitty-gritty of such instruments with the intention of finding remedies for the practical short-falls.

Once all the practical obstacles have been cleared, the use of the instruments to determine market consistent values of mortality driven liabilities can could be studied and detailed further.

Another path to follow could be to see how these instruments fit into the Solvency II framework. Solvency II aims to ensure companies adopt good risk management practices. By adopting mortality based instruments, companies enhance their ability to manage mortality risks which is in line with the objectives of Solvency II.

# Chapter 7

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# Chapter 8

# Appendix

## 8.1 LLMA principles

The following sets out what the LLMA believes need to be met in order for a longevity index to be successful. These are outlined by the LLMA [44] in the LLMA's Longevity Index Framework.

### 8.1.1 Tradeability

The index needs to be tradeable. This in turn means it needs to be credible, robust and have broad buy-in from the market.

### 8.1.2 Transparency

The index needs to enjoy full disclosure in terms of data sources, methodology, algorithms, rules, degree of discretion and governance procedures. All historic data needs to be freely available and easily accessible. There must be clear rules on how data anomalies are handled, including missing and late data. Minimal scope for discretion and discretion should be based on objective analysis and communicated to the market in a full and timely manner.

### 8.1.3 Robustness

The index needs robust procedures and processes for:

1. Data collection.
2. Data validation.
3. Calculation rules and algorithms.
4. Production.
5. Publication.

### 8.1.4 Objectivity

The index must be perceived to be objective, so that governance, ownership and production must be:

1. Independent.
2. Impartial.
3. Free from conflict.

### 8.1.5 Simplicity

Simplicity should be a key focus as it will aid transparency and robustness. It should also aid liquidity.

### 8.1.6 Clear governance

Ownership of the index should be distinguished from its oversight and production. Also the oversight of the index should involve an appropriately staffed 'Index Oversight Committee'. This committee must include external experts and have a mandate to maintain an index that encourages trading and liquidity.

Production of the index should be conducted by an independent 'Index Calculation Agent'. It also goes without saying that access to index data prior to publication needs to be controlled, to prevent access being gained to proprietary information unfairly.

### 8.1.7 Timeliness

The index must be updated regularly, with a frequency appropriate to the market.

### 8.1.8 Continuity

Ongoing production of the index must be ensured and sub-indices should be stable and reliable.

### 8.1.9 Consistency

There should be consistency in:

1. The methodology of the construction of the index, and
2. in the provision of data.

### 8.1.10 Universality

To ensure the longevity market grows as broadly and consistently as possible, the index methodology should be widely applicable to creating indices for:

1. Different countries.
2. Other broad-based populations of lives, such as industry or occupational groupings.
3. Customised pools of lives[1].

The index should be relevant for a broad range of uses. For example to manage the mortality risks of:

1. Pension plans.

[1] For example for large pension plans or groups of pension plans

2. Annuity portfolios.
3. Life portfolios.
4. Equity release portfolios.
5. General investors.

## 8.2 Capital Adequacy Requirements

The following section sets out in some detail how the Capital Adequacy Requirements (CAR) of South African life insurance providers are calculated and why there is a Capital Adequacy Requirement. This is based on the professional guidance issued by the Actuarial Society of South Africa [42] in PGN 104.

Compliance with PGN 104 is mandatory for Statutory Actuaries performing valuations of long-term insurance registered in South Africa for the purpose of published financial reporting, statutory reporting and tax liability calculation.

### 8.2.1 The background to CAR

In determining a life insurance provider's liabilities a degree of financial resilience is introduced by adding compulsory margins to the best-estimate assumptions used in the valuation. The compulsory margins to be added to the best-estimates of mortality are 7.5%. Additional discretionary margins may also be applied if it is felt these are required.

The compulsory (and possibly discretionary margins) aim to ensure the life insurance provider has adequate assets to meet all future outgo if actual experience deviates from the valuation assumptions. These margins are not however sufficient to ensure that the life insurance provider remains solvent where there are significant deviations in actual experience from that assumed.

To protect against these more adverse scenarios the excess of assets over liabilities needs to be large enough to see it through these scenarios. As a result the minimum assets to be held in excess of liabilities to be able to survive these shocks is referred to as the Capital Adequacy Requirement (CAR) and CAR is pitched at a level that the life insurance provider should be 95% certain it will meet all liabilities as they fall due.

### 8.2.2 The principles behind CAR

There needs to be a balance between having a large enough CAR to protect the various stakeholders, but it must not be so large as to endanger the viability of the long-term insurance industry. Considering this it would be too conservative to assume all shocks happened simultaneously, which leads to one of the following methods below being adopted.

The size of a number of cushions to cover specific events is assessed assuming only that event occurs, where the target confidence is 95%.

The overall cushion is not the sum of these individual cushions, but some lower amount to allow for the fact that not all the shocks will happen simultaneously. Some simplifying assumptions are made and the correlation of the various shocks do achieve this.

For example:

1. If events A and B can be assumed to be strongly negatively correlated, then the larger of A and B's cushion is used.
2. If events A and B can be assumed to be strongly positively correlated, then the sum of A and B's cushions is used.
3. If events A and B can be assumed to be uncorrelated, then the root of the sum of the squared cushions is used.

### 8.2.3 The CAR formula

$CAR$ is the maximum of:

1. Ordinary CAR, $OCAR$.
2. Termination CAR, $TCAR$.
3. Minimum CAR, $MCAR$.

Effectively: $CAR = \max(TCAR, OCAR, MCAR)$.

The dynamics of $MCAR$ and $TCAR$ are not impacted by the concepts of this dissertation, so will not be given much more attention.

### 8.2.4 OCAR

The $OCAR$ formula uses a factor based approach which isolates each major risk category and establishes the capital needed to cushion a blow from that source of risk. The results are summed with an adjustment to the sum to allow for the correlations of the different sources of risk, as outlined above.

The $OCAR$ figure then also needs to be grossed up to allow for the effects of a fall in the fair value of the assets backing it as a result of credit risks.

This is done by first calculating the Intermediate Ordinary CAR, the $IOCAR$, then grossing this $IOCAR$ figure up for the credit risks in the backing assets. This gives the relationships:

$$OCAR = \frac{IOCAR}{(1-k)},$$

where:

$$k = \sqrt{(g + \frac{h}{2})^2 + \frac{3}{4}h^2}.$$

and $g$ is the investment risk item and $h$ is the credit risk item in the $IOCAR$ calculation as outlined below.

### 8.2.5 The IOCAR formula

The $IOCAR$ figure is calculated using the formula below:

$$IOCAR = \sqrt{a^2 + b^2 + ci^2 + cii^2 + ciii^2 + d^2 + e^2 + f^2 + (g + \frac{1}{2}h)^2 + \frac{3}{4}h^2 + i}.$$

Where the various components are:

1. $a$ for lapse risk.
2. $b$ for surrender risk.
3. $ci$ for mortality risk.
4. $cii$ for morbidity risk.
5. $ciii$ for medical risk.
6. $d$ for annuitant mortality risk.
7. $e$ for mortality, morbidity and medical assumption risk.
8. $f$ for expense risk.
9. $g$ for investment risk.
10. $h$ for credit risk.
11. $i$ for operational and other risk.

It is the components $ci, cii, ciii, d$ and $e$ that can be reduced by using mortality based instruments. By having the tools to manage the mortality, morbidity and medical fluctuations, less capital will be required to reach the 95% degree of confidence.

## 8.3 Solvency II

As described by Zugic, et al. [58], Solvency II is a new regulatory framework that life insurance providers within the EU will need to comply with. It is an extremely comprehensive framework that has three primary functions:

1. Defining a framework to determine the required solvency capital levels.
2. Ensuring life insurance companies adopt good corporate governance and risk management practices.
3. Encouraging transparent disclosure.

The aim of the solvency capital levels are to ensure that a life insurer has sufficient capital to weather a 1 in 200 year storm. The aggregate solvency capital level is made up of several parts, for example one of which is a longevity risk solvency capital component.

The current prescribed stress as set out in QIS 5[2] (The fifth Quantitative Impact Study of Solvency II) to determine the longevity solvency capital would be an instant reduction in mortality rates of 20%. As you can imagine, this could result in a significant amount of capital needing to be kept at hand to weather such a storm.

One way to reduce the the longevity solvency requirement would be to put in place some hedges against falling mortality rates. Such hedges could be in part comprised of mortality based instruments such as mortality swaps or longevity bonds.

[2]This is guided the CEIOPS (Committee of European Insurance and Occupational Pensions Supervisors)

## 8.4 Market consistent valuations

O'Brien [47] discusses the principle of market consistent valuations in great detail in the paper: Market-Consistent Valuations of Life Insurance Business: The U.K. Experience.

The most realistic value for an asset or liability is the value that it would be bought or sold at in a free market. This is the principle of a market consistent method as there is nothing subjective about that value that is arrived at in this manner.

Considering that in the United Kingdom (and in South Africa) most the assets of life insurance providers are traded it is not very difficult to determine the market consistent values of the assets. It is however not as simple to determine the market consistent values of life insurance liabilities.

Take for example an annuity that pays an amount each year while the annuitant is alive. Here the market consistent value of the liability is driven by the annuitants mortality - something that is not (yet) observable in the open market.

### 8.4.1 The ESG

As a consequence of having to place a market consistent value on various cash flows not observable in the open market, Economic Scenario Generators (ESG's) are built. These ESG's are used to do stochastic projections of various scenarios of interest rates, inflation rates, equity values, fixed interest security values and other asset prices.

The ESG's are calibrated to traded instruments such as puts and calls. Then thousands of runs are done to value sets of instruments that can be used as replicating portfolios of the assets and liabilities. Then in turn the value of a liability is determined from the value of the portfolio that replicates that liability as determined using the ESG.

### 8.4.2 The model office and management actions

How all the cash flows of a life insurance provider are assumed to interact is dictated by the way the **model office** is constructed. A model office is essentially a mathematical representation of an entire life insurance provider. One way to visualise what a model office is, is to see it as a large model of the income statement and balance sheet, that takes account every cash flow item.

Given the model office and the fact that a life insurance provider has some ability to adjust to an ever changing landscape some **management actions** can be allowed for in the simulations. For example it could be assumed that the life insurance provider adjusts their investment strategy as a result of some economic scenario that develops - say out of the corporate bonds backing their annuity liabilities into government bonds instead.

### 8.4.3 The practical implementation

In practice the construction and calibration of an ESG, the design of the model office and the management actions that will introduce subjectivity into the way these market consistent values are calculated. This introduction of subjectivity is fundamentally out of line with the objectives of market consistent valuations.

## 8.5 SVD

Kalman [36] shows that for any arbitrary real $m \times n$ matrix $A$ there is an SVD. There are orthogonal matrices $U$ and $V$ and some diagonal matrix $\Sigma$, such that $A = U\Sigma V'$. Here, $U$ will be $m \times m$ and $V$ will be $n \times n$ so that $\Sigma$ is $m \times n$.

Some applications of SVD include:

1. The computation of the eigenvalue decomposition of a matrix product $A'A$, which is typically a problem encountered while doing principle component analysis or solving correlation matrices.

2. The estimation of the rank of a matrix.

3. Solving the generalised inverse of a matrix.

## 8.6 The assumed mix of lives by age

```
# The assumed mixture by age
#---------------------------

MIX = array(0, dim = c(AGES,2))

for(x in 1:AGES)
{
  MIX[x,1] = x+19
}

MIX[1,2] = 0.0000352238290675
MIX[2,2] = 0.000493130328228254
MIX[3,2] = 0.0021838412008394
MIX[4,2] = 0.00676288210287704
MIX[5,2] = 0.0164845406641771
MIX[6,2] = 0.0240223107140262
MIX[7,2] = 0.0374071400057074
MIX[8,2] = 0.0460016021857938
MIX[9,2] = 0.0445221057819547
MIX[10,2] = 0.0474102522090604
MIX[11,2] = 0.0449445195871259
MIX[12,2] = 0.0414218224872554
MIX[13,2] = 0.0366316460833674
MIX[14,2] = 0.0386743604402439
MIX[15,2] = 0.0367723173225275
MIX[16,2] = 0.0376876589680577
MIX[17,2] = 0.0395193120783812
MIX[18,2] = 0.0329679107302305
MIX[19,2] = 0.0344471856697987
MIX[20,2] = 0.0368423314055725
MIX[21,2] = 0.0336016966419752
MIX[22,2] = 0.032122520653337
MIX[23,2] = 0.026698220561101
MIX[24,2] = 0.0258530116225733
MIX[25,2] = 0.0240213504766319
MIX[26,2] = 0.0242325174486194
MIX[27,2] = 0.0240212732777772
MIX[28,2] = 0.0224715918255491
MIX[29,2] = 0.0227534189368132
MIX[30,2] = 0.022542195606118
MIX[31,2] = 0.0194424770056663
MIX[32,2] = 0.0189494527571228
MIX[33,2] = 0.0155680465986564
MIX[34,2] = 0.0154272038769879
MIX[35,2] = 0.0119050437942424
MIX[36,2] = 0.00986226185906554
```

```
MIX[37,2] = 0.00943954695474512
MIX[38,2] = 0.0102848126103678
MIX[39,2] = 0.00810113242731273
MIX[40,2] = 0.00556504519116783
MIX[41,2] = 0.00570596133683165
MIX[42,2] = 0.0041562426767603
MIX[43,2] = 0.00204288206628567
```

## 8.7 R code

### 8.7.1 Longevity bond

Below the coupon rate is set to 5%, the starting notional population to 100,000 and the term to 20 years.

```
# Longevity Bond
#---------------

# Coupon at t of k*INFORCE(t)
# Redemption payment of the Notional Amount = N
# For t in 1:T
# k the Coupon Rate

k = 0.05
N = Starting_Notional_Population
T = 20
```

As can be seen below the coupons are the product of the in-force notional population and the coupon rate $k$.

```
COUPONS = array(0, dim = c(AGES,T,RUNS))

for(x in 1:AGES)
{
  for(t in 1:T)
    {
      for(r in 1:RUNS)
      {
        COUPONS[x, t, r] = SIM_INFORCE[x, t+1, r]*k
      }
    }
}
```

```
MEAN_COUPONS = array(0, dim = c(AGES,T))

for(x in 1:AGES)
{
```

```
  for(t in 1:T)
  {
    MEAN_COUPONS[x, t] = mean(COUPONS[x, t, ])
  }
}

A_COUPONS = array(0, dim = c(T,2))

for(t in 1:T)
{
  A_COUPONS[t,1] = t
  A_COUPONS[t,2] = sum(MEAN_COUPONS[ ,t])
}
```

The redemption payment is equal to the starting notional population.

```
A_REDEMPTION = array(1:T,dim = c(T,2))

for(t in 1:T)
{
  A_REDEMPTION[t,2] = 0
}

A_REDEMPTION[20,2] = N
```

By discounting the coupons and the redemption payments using the yield curve described above, we arrive at the expected present value of the pay-off's. This is essentially the price of the longevity bond.

```
PV_A_COUPONS = A_COUPONS
PV_A_REDEMPTION = A_REDEMPTION

for(t in 1:T)
{
  PV_A_COUPONS[t,2]    = A_COUPONS[t,2]*YIELD_DATA[t,3]
  PV_A_REDEMPTION[t,2] = A_REDEMPTION[t,2]*YIELD_DATA[t,3]
}

PRICE_LONGEVITY_BOND = sum(PV_A_COUPONS)+sum(PV_A_REDEMPTION)
```

### 8.7.2 Annuity outgo

```
# Annuity Outgo (1) - Mortality 100% of best-est
#-----------------------------------------------

OUT = array(0, dim = c(AGES,T,RUNS))

for(t in 1:T)
{
  for(r in 1:RUNS)
  {
    for(x in 1:AGES)
    {
      OUT[x,t,r] = SIM_INFORCE[x,t+1,r]
    }
  }
}

PV_OUT = OUT

for(t in 1:T)
{
  PV_OUT[ ,t, ] = OUT[ ,t, ]*YIELD_DATA[t,3]
}
```

### 8.7.3 Annuity outgo - Histogram

```
# Histogram for outgo (No hedge)- 100% of best-est
#-------------------------------------------------

HIST_PV_OUT = array(0, dim = c(RUNS))

for(r in 1:RUNS)
{
  HIST_PV_OUT[r] = sum(PV_OUT[ , ,r])
}

MIN_HIST_PV_OUT = min(HIST_PV_OUT)
MAX_HIST_PV_OUT = max(HIST_PV_OUT)

BUNCHES   = 150
INTERVALS = trunc((MAX_HIST_PV_OUT-MIN_HIST_PV_OUT)/BUNCHES)+1

BREAKS = seq(MIN_HIST_PV_OUT-INTERVALS,...
             ...MAX_HIST_PV_OUT+INTERVALS,by = INTERVALS)
```

```
hist(HIST_PV_OUT,breaks = BREAKS,xlab = 'Annuity outgo',...
     ...ylab = 'Density', main = 'Histogram of the total outgo',...
     ...sub = 'Mortality at 100% of best estimates, without a hedge',...
     ...freq = FALSE,col = 3)
```

### 8.7.4 Mortality swap

Below the swap rate is set to 20%, the notional amount to 100,000 and the term to 20 years.

```
# Mortality Swaps (Fixed For Floating)
#-------------------------------------

# Fixed for Floating
# Paying FIXED       = k*MEAN_SIM_INFORCE(t)
# Recieving FLOATING = k*SIM_INFORCE(t)
# Notional = N
# T = 20
# k = 20

T = 20
k = 0.05
N = Starting_Notional_Population

SWAP_OUT = array(0, dim = c(AGES, F_YEARS, RUNS))

for(x in 1:AGES)
{
  for(t in 1:F_YEARS)
  {
    for(r in 1:RUNS)
    {
      SWAP_OUT[x, t, r] = k*(SIM_INFORCE[x, t+1, r]-...
                          ...MEAN_SIM_INFORCE[x,t+1])
    }
  }
}

PV_SWAP_OUT = array(0, dim = c(AGES, F_YEARS, RUNS))

for(x in 1:AGES)
{
  for(t in 1:F_YEARS)
```

```
  {
    for(r in 1:RUNS)
    {
      PV_SWAP_OUT[x, t, r] = SWAP_OUT[x, t, r]*YIELD_DATA[t,3]
    }
  }
}

EPV_SWAP_OUT = array(0, dim = c(AGES,F_YEARS))

for(x in 1:AGES)
{
  for(t in 1:F_YEARS)
  {
    EPV_SWAP_OUT[x, t] = mean(PV_SWAP_OUT[x, t, ])
  }
}

EPV_SWAP = sum(EPV_SWAP_OUT)
```

### 8.7.5 Mortality swap as a hedge

```
# Annuity book hedged with a Mortality Swap
#-----------------------------------------

NET_OUT = array(0, dim = c(AGES, F_YEARS, RUNS))

for(x in 1:AGES)
{
  for(t in 1:F_YEARS)
  {
    for(r in 1:RUNS)
    {
      NET_OUT[x, t, r] = SIM_INFORCE[x,t+1,r]-...
      ...k*(SIM_INFORCE[x, t+1, r]-MEAN_SIM_INFORCE[x,t+1])
    }
  }
}

PV_NET_OUT = array(0, dim = c(AGES, F_YEARS, RUNS))

for(x in 1:AGES)
{
  for(t in 1:F_YEARS)
  {
```

```
    for(r in 1:RUNS)
    {
      PV_NET_OUT[x, t, r] = NET_OUT[x, t, r]*YIELD_DATA[t,3]
    }
  }
}

EPV_NET_OUT = array(0, dim = c(AGES,F_YEARS))

for(x in 1:AGES)
{
  for(t in 1:F_YEARS)
  {
    EPV_NET_OUT[x, t] = mean(PV_NET_OUT[x, t, ])
  }
}
```

### 8.7.6 Mortality swaption

Below the swap rate is set to 20%, the notional amount to 100, 000 and the term to 20 years.

```
# Mortality Swaption (Fixed For Floating)
#----------------------------------------

# Fixed for Floating Swaption
# Paying FIXED        = 0
# Recieving FLOATING = k*max(SIM_INFORCE(t)-MEAN_SIM_INFORCE(t))
# Notional = N
# T = 20
# k = 20

T = 20
k = 0.20
N = Starting_Notional_Population

SWAPTION_OUT = array(0, dim = c(AGES, F_YEARS, RUNS))

for(x in 1:AGES)
{
  for(t in 1:F_YEARS)
  {
    for(r in 1:RUNS)
    {
```

```
      SWAPTION_OUT[x, t, r] = k*max(SIM_INFORCE[x, t+1, r]-...
      ...MEAN_SIM_INFORCE[x,t+1],0)
    }
  }
}

PV_SWAPTION_OUT = array(0, dim = c(AGES, F_YEARS, RUNS))

for(x in 1:AGES)
{
  for(t in 1:F_YEARS)
  {
    for(r in 1:RUNS)
    {
      PV_SWAPTION_OUT[x, t, r] = SWAPTION_OUT[x, t, r]*YIELD_DATA[t,3]
    }
  }
}

EPV_SWAPTION_OUT = array(0, dim = c(AGES,F_YEARS))

for(x in 1:AGES)
{
  for(t in 1:F_YEARS)
  {
    EPV_SWAPTION_OUT[x, t] = mean(PV_SWAPTION_OUT[x, t, ])
  }
}

EPV_SWAPTION = sum(EPV_SWAPTION_OUT)
```

### 8.7.7 Mortality swap as a hedge

```
# Annuity book hedged with a Mortality Swaption
#-----------------------------------------------

SWAPTION_NET_OUT = array(0, dim = c(AGES, F_YEARS, RUNS))

for(x in 1:AGES)
{
  for(t in 1:F_YEARS)
  {
    for(r in 1:RUNS)
    {
      SWAPTION_NET_OUT[x, t, r] = SIM_INFORCE[x,t+1,r]-...
```

```
      ...k*max(SIM_INFORCE[x, t+1, r]-MEAN_SIM_INFORCE[x,t+1],0)
    }
  }
}



PV_SWAPTION_NET_OUT = array(0, dim = c(AGES, F_YEARS, RUNS))

for(x in 1:AGES)
{
  for(t in 1:F_YEARS)
  {
    for(r in 1:RUNS)
    {
      PV_SWAPTION_NET_OUT[x, t, r] = NET_OUT[x, t, r]*YIELD_DATA[t,3]
    }
  }
}

EPV_SWAPTION_NET_OUT = array(0, dim = c(AGES,F_YEARS))

for(x in 1:AGES)
{
  for(t in 1:F_YEARS)
  {
    EPV_SWAPTION_NET_OUT[x, t] = mean(PV_SWAPTION_NET_OUT[x, t, ])
  }
}
```