CHAPTER 1

OVERVIEW

1.1 OVERVIEW

The analyses of manpower systems have become very important component of planned economic development of any organization or nation. However, manpower planning depends on the highly unpredictable human behavior and the uncertain social environment in which the system functions. Hence the study of probabilistic or stochastic models of manpower systems is very much essential. Several stochastic models of manpower systems have been proposed and studied extensively in the past (see Bartholomew (1967) and Vajda (1978)). Various stochastic models of manpower systems can be classified broadly into two types:

1. Markov Chain models
2. Renewal Models

In all these models, the manpower system is hierarchically graded into mutually exclusive and exhaustive grades so that each member of the system may be in one and only one grade at any given time. These grades are defined in terms of any relevant state variables. Individuals move between these grades due to promotions or demotions and to the outside world due to dissatisfaction, retirement or medical reasons. If the size of the grades is not fixed, then the state of the system at any time is represented by a vector $X(t) = (X_1(t), X_2(t),.........X_n(t))$ where the component $X_i(t)$ represents the number in the ith grade at any time t. Further the very nature of several manpower systems require to be observed at, say, annual intervals. Accordingly, the system behaviour is adequately described by a Markov chain, such models are called Markov chain models.

Markov chain models have been applied in examining the structure of manpower systems in terms of the proportion of staff in each grade or age profile of staff under a variety of conditions and evaluating policies for controlling manpower systems (see for example, Young and Almond (1961), Young (1971), Forbes (1971a,b), Bartholomew (1973) and Gani (1973)). In these works and in all of what followed the important question was the control of the expected numbers in the various states by recruitment control. The numbers of people in such categories change over time through wastage, promotion flows and recruitment. Some of these flows are subject to management control while others vary in a random manner. Factors such as the need to offer adequate career prospects or the requirement of the job will often dictate a desirable age or grade structure and it is the manpower planner's task to determine whether this can be achieved and , if so, how.

The limiting behavior of an expanding non-homogeneous Markov system has practical importance as shown by the literature on manpower systems (Vassiliou 1981a&b, 1982a). The limiting structure of the expected class sizes was derived under certain conditions and the relative limiting structure is shown to exist with a different set of conditions. Mehlmann (1977) and Vassiliou (1982b) studied the limiting behavior of the system with Poisson recruitment and observed that the number in the various grades are asymptotically mutually independent Poisson. Vassiliou (1984c) studied the asymptotic behavior of non-homogeneous Markov systems under the cyclical behavior assumption and provided a general theorem for the limiting structure of such systems. Vassiliou (1986) later extended the results and provided a basic theorem for the existence and determination of the limiting structure for the vector of means, variances and covariances under more general possible assumptions. He argued that the results are useful from the practical point of view since they provide valuable information about the inherent tendencies in the system.

The control of asymptotic variability of expectations, variances and covariances in a Markov chain model is a major research area in manpower systems. The earliest work on this subject was that of Pollard (1966). The results were later extended by several authors (Vassiliou and Gerontidis (1985), Vassiliou (1986), Vassiliou et al. (1990)). Attainable and maintainable structures in Markov manpower systems under recruitment control have been studied by Bartholomew (1977), Davies (1975, 1982), Vassiliou and Tsantas (1984 a&b) and later Kalamatianou (1987) analysed the same with pressure in grades. The concept of a non-homogeneous Markov system in a stochastic environment (S-NHMS) was introduced for the first time by Tsantas and

Vassiliou (1993). The problem of attaining the desired structure in an optimal way as well as maintaining relative grade sizes applying recruitment control in a stochastic environment as introduced in Bartholomew (1975, 1977) is considered. More references in this and related topics an be found in various papers by (Georgiou (1992), Tsantas (1995), Tsantas and Georgiou (1994, 1998)). A Markov model responding to promotion blockages has been proposed by Kalamatianou (1988). Raghavendra (1991) has employed a Markov chain model in obtaining the transition probabilities for promotion in a bivariate framework consisting of seniority and performance rating. Georgiou and Vassiliou (1997) have introduced phases in a Markov chain model and investigated the input policies subject to cost objective functions. Yadavalli and Natarajan (2001) studied a semi-Markov model in which a single grade system allows for wastage and recruitment. The time dependent behaviour of stochastic models of manpower system with the impact of pressure on promotion was subsequently studied by Yadavalli et al. (2002).

Although a Markov model is simple and easy to implement, it does not take into account existing knowledge of the distribution of length of service until leaving. In such cases the mathematically intractable Semi-Markov models approach is suggested (McClean 1991). The Semi-Markov processes are a generalization of Markov processes in which the probability of leaving a state at a given point in time may depend on the length of time the state has been occupied (duration of stay) and on the next state entered. However, there are several theoretical literatures on Semi–Markov Models (Pyke (1961 a & b), Ginsberg (1971), Mehlmann (1979), McClean (1978, 1980, 1986)). A stochastic model of migration, occupational and vertical mobility, based on the theory of Semi-Markov process was derived by Ginsberg (1971). McClean (1978) extended the assumption of simple Markov transitions between grades and the leaving process to semi-Markov formulation which allows for inclusion of well-authenticated leaving distributions such as the mixed exponential. Moreover, the previous assumption of Poisson recruitment is generalized to allow for a recruitment process which may vary with time, either as a mixed exponential time dependent Poisson process or by assuming that the number of recruits depends on the amount of capital owned by the firm. The previous formulation is therefore extended to take into account the fact that recruitment to a firm is a highly variable process and the assumption of Poisson recruitment to each grade is therefore restrictive. The concept of non-homogeneous semi–Markov systems found important applications in manpower system particularly in the subjects of variability, limiting distributions and maintainability of grade sizes (Vasiliou ad Papadopoulou (1992)).

On the other hand, there are several manpower systems where the grade sizes are fixed by the budget or amount of work to be done. Recruitment and promotion can occur only when vacancies arise through leaving or expansion. There may be randomness in the method by which vacancies are filled. The movements of individuals are characterised by replacements (renewals) according to some probabilistic law, and such models of manpower systems are called renewal models. The main advantage of these models over the Markov chain models is that they are closer to reality since the losses (wastages) occur continuously in time and there is always the possibility that a new recruit may also leave during the study period. White (1970) has used models of this kind to study the flows of clergy of several large American denomination. Stewmann (1975) has applied White's methods to the study of recruitment and losses in a state police force. Bartholomew (1982) has provided a detailed analysis of renewal models of manpower systems. Sirvanci (1984) has applied renewal processes to forecast the manpower losses of an organisation in order to determine whether the organisation will be able to meet its demand for manpower under present conditions. The distributions of completed length of service (CLS) in these models have been fitted to actual data from industry by several researchers (see Bartholomew, 1982). McClean (1976, 1978) has used a mixed exponential distribution for CLS and estimated the parameters using data for two companies. Agrafiotis (1983, 1984, and 1991) studied the problem of labour turnover by using renewal process type models.

A satisfactory model of manpower system should provide answers to the following questions:

1. How to provide estimates of manpower indicators of the system?
2. How to predict the future behaviour of the system under various assumptions?

3. How to find optimum solutions to various policy problems subject to various constraints given by the management?
4. How to avoid various problems by giving a warning before the situation develops?
5. How to design manpower, which is related to various problems of prediction in consultation with management?

In order to provide answers to questions raised above, the model considered should incorporate the following main factors, which predominantly determine the behaviour of a manpower system:

1. Recruitment
2. Promotion of employees
3. Wastages.

1.1.1 Recruitment

The sizes of various grades, which respond to the expansion, promotions and wastages, are maintained at the desired level at any time by a process called RECRUITMENT. The flow of recruitment can be controlled by the management authorities. The recruitment can be made in several ways. Vacancies can be filled as and when they arise or they may be allowed to accumulate and then filled up at specified periods or whenever the total number of vacancies attains a certain specific level, so as to minimize the cost. The recruitment can be made by the organization itself or by some external agencies to avoid delay and huge overhead costs. Several organizations in South Africa do not recruit employees by themselves (e.g. the preliminary process of senior level positions in Statistics South Africa) but approach recognized recruiting agencies. Usually, vacancies that arise are allowed to accumulate for a specified period of time, or to attain a specified level and then these agencies are requested to fill them up and to complete the process of recruitment in a specified period of time. However, they may not be able to fill up all the notified vacancies due to the non-availability of suitable candidates with prescribed qualifications and experience. Further additional vacancies may also arise during the period of recruitment process. Therefore there may exist some vacancies even after the process of recruitment is completed. In reality, many such manpower systems exist. However, these types of models have not been considered in the literature. Davies (1975) considered a fixed size Markov chain model that suffered losses and admits recruits to various grades in such a manner that the total grades in the system remain constant. In that paper, the recruitments take place at integral points in time and at the time of recruitment, no vacancy is left unfilled. Vassiliou et al. (1990) deal with a non-homogeneous Markov manpower system, which allows recruitment in each grade of the hierarchically graded manpower system. They have obtained the limiting expected structure of the system by control over the limit of the recruitment probabilities. Rao (1990) has considered a manpower planning model with the objective of minimizing the manpower cost with optimal recruitment policies. The recruitment size is known and fixed in this model. Hence the study of a model where vacancies are accumulated and then filled up deserves attention.

1.1.2 Promotion

Normally vacancies that arise in the lower grade are filled up by recruitments whereas those in the higher grades are filled up by promotions. Further, promotions besides giving due recognition to proficiency and credibility of the employees reduce the chance of an efficient employee leaving the organization. Some of the promotion rules are given below:

(i) The senior most in the grade is promoted.

(ii) Promotion is given at random.

(iii) Those who fill certain efficiency criterion along with some minimum completed length of service are promoted.

As per the rule (i), the length of service is the sole criterion for promotion and hence the management can control it. The rule (ii) gives full freedom for the management to promote any employee of their choice, which also is not desirable. Normally rule (iii) is preferred. Some of the reasons, which influence the promotion policies, are (a) pressure (b) efficiency and (c) length of service.

(a) Pressure

In a multi-graded hierarchical manpower system, a promotion policy that is associated with constant promotion probabilities leaves a proportion of employees qualified by completed length of service in a lower grade un-promoted. This proportion increases and pressure starts building up as time progresses. When pressure exceeds a certain level of control, a high proportion of un-promoted employees could have serious effect on the efficiency of the organization for several reasons such as productive loss and wastage. The pressure can be quantified as a function of the proportion of the people in a job grade according to Kalamatianou (1987, 1988). She has quantified pressure in three stages and suggested models to reduce the pressure by suitably changing the promotion policies well in advance.

(b) Efficiency (training)

Training of manpower has long been recognized as an important factor for improving the efficiency of the employees and for the productive improvement. Further, when it is considered as a criterion for promotion, it becomes very much effective. Mathematical models incorporating training aspects have been studied by Guardabassi et al. (1969), Grinold and Marshall (1977), Mehlmann(1980) and Vajda (1978). Goh et al. (1987) have analysed the training problem within an organisation using dynamic programming principles. These results were recently generalised using Dynamic Programming by Yadavalli et al. (2002).

(c) Length of service

Length of service in a grade should necessarily be a natural criterion for promotion in order to create a healthy atmosphere among the employees. However, for controlling the promotion, the management can include other efficiency criterion along with it for promotion. This aspect has been discussed by Bartholomew (1973, 1982), Glen (1977) and in the thesis of Kamatianou (1983).

1.1.3 Wastages

When employees move from one grade to another, they are exposed to different factors influencing them to leave the organization. Various data indicate that the reasons for leaving can be classified into the following cases:

(i) Discharge

(ii) Resignation

(iii) Redundancy

(iv) Retirement

(v) Medical retirement

Agrafotis (1984) has grouped the above cases into two main reasons, normally, (a) unnatural and (b) natural. Unnatural reasons for leaving depend on the internal structure of the company or organisation, viz, lack of promotion prospects, job satisfaction, problem of adjustment, etc., including the cases (i), (ii), and (iii) mentioned above. Natural reasons for leaving the organisation do not depend on the internal structure of the organisation, including the cases under (iv) and (v). In analysing data on a number of companies, Agrafiotis (1984) has shown that there is a significant difference in the wastage rates corresponding to reasons (a) and (b) for leaving. However, the cases (iv) and (v) relating to the natural leaving are entirely different and are to be discussed separately, for an employee leaving by way of natural retirement after having served the organisation completely cannot be grouped with an employee who leaves the organisation by way of medical reasons. As such, there are three different wastage rates:

(a) Due to internal structure

(b) Due to retirement

(c) Due to medical reasons

Unlike natural wastage the unnatural wastage can be controlled by the management by resorting to better promotional prospects, improved working conditions and training.

Some other manpower studies which investigated wastage intensities are (Vassiliou (1976, 1982), Leeson (1981, 1982), McClean et al. (1992)).

1.2 TECHNIQUES USED IN MANPOWER MODELS

In this section, we present the various techniques used in the analysis of models of manpower systems.

1.2.1 Renewal theory

Renewal theory forms an important constituent in the study of stochastic processes and is extremely used in the analysis of manpower models with recruitment. Feller (1941, 1968) made significant contributions to renewal theory giving the proper lead. Smith (1958) gave an extensive review and highlighted the applications of renewal theory to a variety of problems. A lucid account of renewal theory is given by Cox (1962).

Definition 1

Let $\{X_i : i = 1, 2, ...\}$ be a collection of random variables, which are non-negative, independent and identically distributed. Then the sequence $\{X_n\}$ is called a renewal process. We assume that each of the random variable X_i has a finite mean μ. A renewal process is completely determined by means of $f(\cdot)$, the p.d.f of X_i. Associated with the renewal process is a random variable $N(t)$, which represents the number of renewals in the time interval $(0, t]$. $N(t)$ is also known as the renewal counting process (Parzen, 1962).

Definition 2

The expected value of $N(t)$ is called the renewal function and is denoted by $H(t)$. The derivative of $H(t)$ if it exists, is denoted by $h(t)$ and is called the renewal density. The quantity $h(t)\,dt$ has the interpretation that it represents the probability that a renewal occurs in $(t, t + dt)$. We will have to identify this as what is known as the first order product density for a more general process. The renewal density satisfies the following integral equation:

$$h(t) = f(t) + \int_0^t f(u)h(t-u)\, du$$

One of the important and useful theorems in application is the key-renewal theorem (Smith, 1958).

Theorem

Let $Q(t)$ satisfy the following conditions:

(i) $Q(t) \geq 0 \quad for\ all\ t \geq 0$

(ii) $Q(t)$ is non-increasing

(iii) $\int_0^\infty Q(t)\, dt < \infty$.

Then,

$$\lim_{t\to\infty} \int_0^\infty Q(t-u)\, dH(u) = \frac{1}{\mu} \int_0^\infty Q(u)\, du.$$

Further details regarding renewal theory can be found in Smith (1958), Feller (1968), Prabhu (1965) and Srinivasan (1974). We now briefly indicate how renewal theory has been used in the study of manpower models. The stochastic element in manpower systems occur principally due to the loss mechanism arising out of staff moving out of the system. The randomness may also be due to the method by which the vacancies are filled. In the context of manpower planning, the renewal process $\{N(t), t \geq 0\}$ represents the number of recruitments required for the given position for which the first person was employed at $t = 0$. The random time X between successive replacements is called the completed length of service (CLS) and its distribution $F(x)$ is termed as the CLS distribution. Thus, during the operation period from $t = 0$ up to time t, while $N(t)$ employees leave, an equal number need to be recruited in order to keep a given position continuously staffed. To predict the value of $N(t)$ for any given time, its expected value, which is referred to as the renewal function, may be used. The relationship between the CLS distribution and the renewal density $h(t)$, the derivative of $H(t)$, is given by the renewal equation

$$h(t) = f(t) + \int_0^t f(u)h(t-u)\,du\,; \qquad t \geq 0\,.$$

Where $f(t)$ is the density of the CLS distribution $F(t)$. The renewal density $h(t)$ can be interpreted as the rate at which the losses occur. On the other hand, $F(t)$ is the distribution of the time an employee spends in the organisation before leaving. The renewal process of personnel losses has been extensively studied by Bartholomew (1962, 1982) and Bartholomew and Forbes (1979).

1.2.2 Markov renewal theory

Let E be a finite set, N the set of non-negative integers and $\Re_+ = [0, \infty)$. Suppose we have, on a probability space (Ω, B, P) random variables $X_n : \Omega \to \mathrm{E}$, $T_n : \Omega \to \Re_+$ defined for each $n \in N$ so that $0 = T_0 \leq T_1 \leq T_2 \leq ....$

Definition 1

The stochastic process $(X,T) = \{(X_{n,} T_n); \quad n \in N\}$ is said to be a Markov renewal process with the state space E provided that

$$P[X_{n+1} = j, T_{n+1} - T_n \leq t \mid X_0, X_1, ...X_n; \quad T_0, T_1, ...T_n] = P[X_{n+1} = j, T_{n+1} - T_n \leq t_n \mid X_n]$$

$$for\ all \quad n \in N, \quad j \in E\ and\ t \in \Re_+\ .$$

We assume that (X,T) is time-homogeneous, that is, for any $i, j \in \mathrm{E}$ and $t \in \Re$

$$Q(i, j, t) = P[X_{n+1} = j, T_{n+1} - T_n \leq t \mid X_n = i]$$

independent of n. The family of probabilities

$$Q = \{Q(i, j, t); \quad i, j \in \mathrm{E}, \quad t \in \Re_+\}$$

is called a semi-Markov kernel over E. We assume that

$$Q(i, j, 0) = 0 \ \ for \quad all \quad i, j \in E.$$

For each pair (i, j) the function $t \rightarrow Q(i, j, t)$ has all the properties of a distribution function except that;

$$P(i, j) = \lim_{t \to \infty} Q(i, j, t)$$

is not necessarily 1. It is easy to see that

$$P(i, j) \geq 0, \quad \sum_{j \in E} P(i, j) = 1;$$

that is, $P(i, j)$ are the transition probabilities for some Markov chain with state space E. It follows from the definition 1 and above that

$$P[X_{n+1} = j \mid X_0, X_1, \ldots X_n; T_0, T_1, \ldots T_n] = P(X_n = j)$$

$$for\ all\ n \in N, \quad j \in E.$$

This implies that $X = \{X_n; \quad n \in N\}$ is a Markov chain with state space E and the transition matrix P.

1.2.2.1 Markov Renewal Functions

We write $P_i(A)$ for the conditional probability $P[A \mid X_0 = i]$ and similarly E_i for the conditional expectations given $\{X_0 = i\}$. We also assume that $P_i[T_0 = T_1 = T_2 = \ldots\ldots = 0] = 0$.

Let us define $Q^n(i, j, t)$ as

$$Q^n(i, j, t) = P_i[X_n = j, T_n \leq t]; \quad i, j \in E, \quad t \in \Re_+ \quad for\ all \quad n \in N.$$

Then,

$$Q^0(i, j, t) = \delta_{ij} = \begin{cases} 1 & if \ \ i = j \\ 0 & if \ \ i \neq j \end{cases} \qquad for\ all\ \ t \geq 0\ \ and\ \ n \geq 0$$

where δ_{ij} is the Kronecker delta function.

We have the recursive relation

$$Q^{n+1}(i,k,t) = \sum_{j\in E}\int_0^t Q(i,j,ds)\, Q^n(j,k,t-s)$$

where the integration is over $[0,t)$. The expression $R(i,j,t)$ that gives the expected number of renewals of the position j in the interval $[0,t)$ is given by

$$R(i,j,t) = \sum_{n=0}^{\infty} Q^n(i,j,t).$$

This is finite for any $i,j\in E$ and $t<\infty$. The $R(i,j,t)$ are called Markov renewal functions and the collection $R=\{R(i,j,t);\quad i,j\in E,\ t\in\Re_+\}$ of these functions is called the Markov renewal kernel corresponding to Q. We note that for fixed $i,j\in E$, the function $t\to R(i,j,t)$ is a renewal function. We can now easily see from the various expressions above that $R_\alpha = (I-Q\alpha)^{-1}$, where I is the unit matrix.

1.2.2.2 Markov Renewal Equations

The class of functions B which we will be working with is the set of all functions

$$f\colon E \text{ X } \Re_+ \to \Re$$

such that for every $i\in E$ the function t→ f(i, t) is Borel measurable and $E\,\text{X}\,R$ bounded over finite intervals and for every fixed $j\in E$ the functions $(i,j)\to Q^n(i,j,t)\ and\ (i,j)\to R(i,j,t)$ both belong to B. For any function $f\in B$, the function Q©f defined by

$$Q©f\ (i,t) = \sum_{j\in E}\int_0^t Q(i,j,ds)\, f(j,t-s)$$

is well defined and Q©$f\in$B again. Hence the operation can be repeated, and the nth iterate is given by

$$Q©\ f(i,t) = \sum_{j\in E}\int_0^t Q^n(i,j,ds)\, f(j,t-s).$$

We can replace Q by R, which is again a well-defined function, which we will denote by R©f, that is for $f \in$B,

$$R© f = \sum_{j \in E} \int_0^t R(i, j, ds) f(j, t-s).$$

A function $f \in$B is said to satisfy a Markov renewal equation if for all $i \in E$ *and* $t \in \Re_+$,

$$f(i,t) = g(i,t) + \sum_{j \in E} \int_0^t Q(i, j, ds) f(j, t-s)$$

for some function g∈B.

Limiting ourselves to functions f, g ∈B₊ which are non-negative and denoting this by B₊ , the Markov renewal equation now becomes

$$f = g + Q © f, \quad f, g \in B_+$$

This Markov renewal equation has a solution R©g. Every solution f is of the form R©g+h, where h satisfies h=Q©h, h∈B₊. For a more detailed on Mark renewal equations see Cinclar (1975).

1.2.3 Semi-Markov processes

Let (X,T) be a Markov renewal process with state space E and semi-Markov kernel Q. Define $L = \sup_n T_n$. Then L is the lifetime of (X,T). If E is finite or if X is irreducible and recurrent, then $L = +\infty$ almost surely. By weeding out those $\omega \in \Omega$ and $t \in \Re_+$ for which $\sup_n T_n(\omega) < \infty$ we assume that $\sup_n T_n(\omega) = \infty$ for all ω. Then for any $\omega \in \Omega$ and $t \in \Re_+$ there is some integer $n \in N$ such that $T_n(\omega) \le t \le T_{n+1}(\omega)$. We can therefore define a continuous time parameter $Y = (Y_t)_{t \in \Re+}$ with state space E by putting $Y_t = X_n$ *on* $T_n \le t < T_{n+1}$. The process $Y = (Y_t)_{t \in \Re+}$ so defined is called a semi-Markov process with state space E and a semi-Markov transition kernel $Q = \{Q(i, j, t)\}$.

1.2.4 Stochastic point processes

Stochastic point processes form a class of random process more general than those considered in the previous sections. Since point processes have been studied by many researchers with varying backgrounds, there have been several definitions of them each appearing quite natural from the view point of the particular problem under study (see, for example, Bartlett (1966), Bhaba (1950), Harris (1963) and Khinchine (1955)). A stochastic process is the mathematical abstraction, which arises from considering such phenomena as a randomly located population or a sequence of events in time. Typically, there is envisaged a state space X and a set of points X_n from X representing the locations of the different members of the population or the times at which the events occur. Because a realization (or a sample path) of any of these phenomena is just a set of points in time or space, a family of such realizations has come to be called point processes (see Daley and Vere-Jones, (1971)).

A comprehensive definition of a point process is due to Moyal (1962) who deals with such process in a general space, which is not necessarily Euclidean. Consider a set of objects each of whom is described by a point x of a fixed set of points X. Such a collection of objects, which we may call a population, may be stochastic if there exists a well-defined probability distribution P on some $\sigma - field$ B of subsets of the space Φ of all states. We shall assume that the members of the population are indistinguishable from one another. The state of the population is defined as an unordered set $X^n = (x_1, x_2, \ldots\ldots, x_n)$ representing the situation where the population has n members with one of the states $x_1, x_2, \ldots\ldots, x_n$. Thus the population state space Φ is the collection of all such X^n with $n = 0, 1, 2, \ldots\ldots$ where X^0 denotes the empty population. A point process is defined to be the triplet (Ω, B, P). For a detailed treatment of stochastic point processes with special reference to its applications the reader is referred to Srinivasan (1974). A point process is called a regular point process if the probability of occurrence of more than one event in $(0, \Delta)$ is $o(\Delta)$.

1.2.5 Product densities

One of the ways of characterizing a general point process is through product densities (Ramakrishnan (1950), Srinivasan (1974)). These densities are analogous to the renewal density in the case of non-renewal processes. Let $N(t,x)$ denote the random variable representing the number of events in the interval $(t, t+x)$, $d_x N(t,x)$ the events in the interval $(t+x, t+x+dx)$ and $P(n,t,x) = P[N(t,x) = n]$.

The product density of order n is defined as

$$h_n(x_1, x_2 ......, x_n) = \lim_{\Delta_1, \Delta_2, .., \Delta_n \to 0} \frac{P[N(x_i, \Delta_i) \geq 1; \quad i = 1, 2, ... , n]}{\Delta_1 \Delta_2 ......\Delta_n}$$

where $x_1 \neq x_2 \neq .... \neq x_n$, or equivalently for a regular process

$$h_n(x_1, x_2 ......, x_n) = \lim_{\Delta_1, \Delta_2, .., \Delta_n \to 0} \frac{\left[\prod_{i=1}^{n} N(x_i, \Delta_i) \geq 1; \quad i = 1, 2, ..., n \right]}{\Delta_1 \Delta_2 ......\Delta_n}$$

where $x_1 \neq x_2 \neq .... \neq x_n$.

These densities represent the probability of an event in each of the intervals $(x_1, x_1 + \Delta x_1)$, $(x_2, x_2 + \Delta x_2), \ldots, (x_n, x_n + \Delta x_n)$. Even though the functions $h_n(x_1, x_2 ......, x_n)$ are called densities it is important to note that their integration will not give probabilities but will yield the factorial moments. The ordinary moments can be obtained by relaxing the condition that all the x_i's are different.

1.3 HETEROGENEITY

The validity of the models described under section 1.2 depends highly on the assumption that the manpower study be based on homogeneous groups of individuals. This is a huge task, which can never be attained in practice because human behaviour is highly unpredictable and the environment on which the system operates is uncertain. However, it is paramount that the researcher ensures that there is no major

source of heterogeneity. Individuals' differences depend on many factors such as their motivation, performance and commitment to the employer.
The subject of homogeneity of individuals is fundamental in virtually all fields of study. However, in the biomedical literature, it is a well known fact that individuals differ substantially in their endowment for longevity (see Manton (1981); Keyfitz (1978); Shepard and Zeckhauser (1977). Hence it is important to try and understand the impact of heterogeneity on the study results. In demography and public policy analysis studies, it has been found that ignoring heterogeneity in frailty results in biased results (Vaupel et al. (1979, 1985)).

According to Bartholomew et al. (1991) the analysis of individual differences is of fundamental importance in the study of manpower system, in particular, wastages (losses from the system). Any attempt to describe wastage pattern must reckon with the fact that an individual's propensity to leave a job depends on a great many factors, both personal and environmental. Failure to recognise the effects of heterogeneity may not only result in erroneous results when applying manpower models but also complicate both the theoretical and empirical research due to the composition of the population and the differential impact of economic, environmental and social forces.
The flow of people in manpower systems, moving employees in various states can be subdivided into recruitment stream, the transition between the state and the outflow from the system. Considering a discreet time t =0, 1,.. we assume that the individuals' transitions between the states take place either according to a homogeneous Markov chain. Most of the work was based on homogeneous Markov chain model introduced by Young and Almond (1961), Gani (1963), Young (1971), and Sales (1971).
Later on Young and Vassiliou (1974), Vassiliou (1976, 1978) introduced the non-homogeneous Markov chain model, which was reported by many researchers to provide a good prediction in practice. Vassiliou (1982a) introduced the more general framework of non-homogeneous Markov model, which incorporates a great variety of applied probability models. As the literature shows, the theory of non-homogeneous Markov systems (NHMS) has flourished since then (Vassiliou, et al. (1990); Tsantas and Vassiliou (1993); Georgiou (1992); Tsantas (1995)).

A number of authors suggested tackling the problem of heterogeneity by dividing the personnel system into more homogeneous subsystems. The pioneering work on mover-stayer models of labor mobility by Blumen et al. (1955), Goodman (1961) and later Bartholomew (1982) was one form of subdividing the population into categories- the 'stayers' who hardly change their jobs and the 'movers' who tend to change jobs frequently. Ugwuowo and McClean (2000) proposed some techniques to deal with heterogeneity for modeling wastage, though the problem exits in other flows within the personnel system. To incorporate population heterogeneity into manpower modeling, two strategies have been suggested: the use of observable sources of heterogeneity as it affects wastage and the latent source of heterogeneity that are impossible to observe but are known to affects the key parameters of the model. Although the division of individuals in homogeneous subcategories is a fundamental and important step in application of the manpower planning techniques, there is still lack of attention towards the way homogeneous groups can be attained in practice. De Feyer (2006) presented a general framework to get more homogeneous subgroups for using Markov Chain theory in manpower planning. A general splitting-up approach is suggested as well as the use of some statistical multivariate techniques is proposed to support the splitting-up process. The main sources of heterogeneity within an organization are summarized in Figure 1.1. An example of a splitting up process is depicted in Figure 1.2.

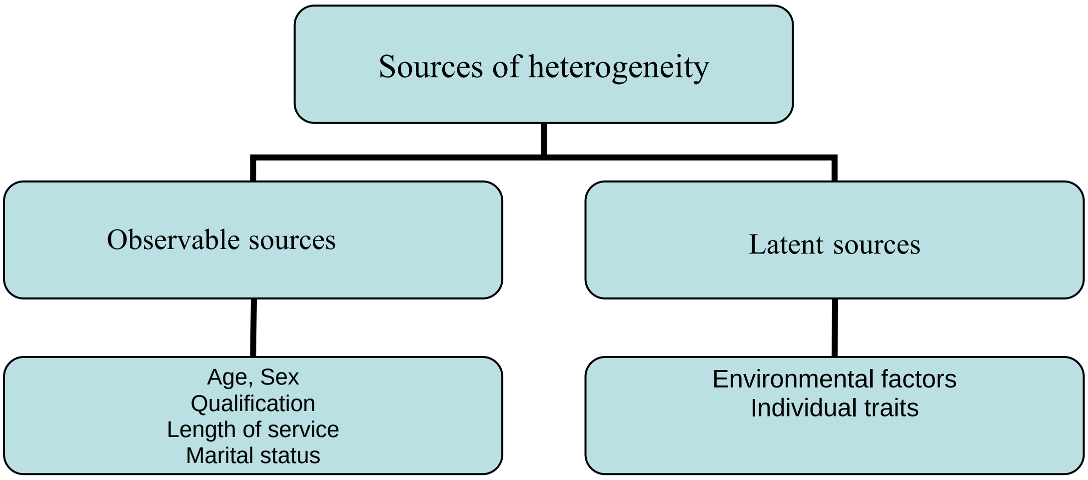

Figure 1.1: Summary of Heterogeneity

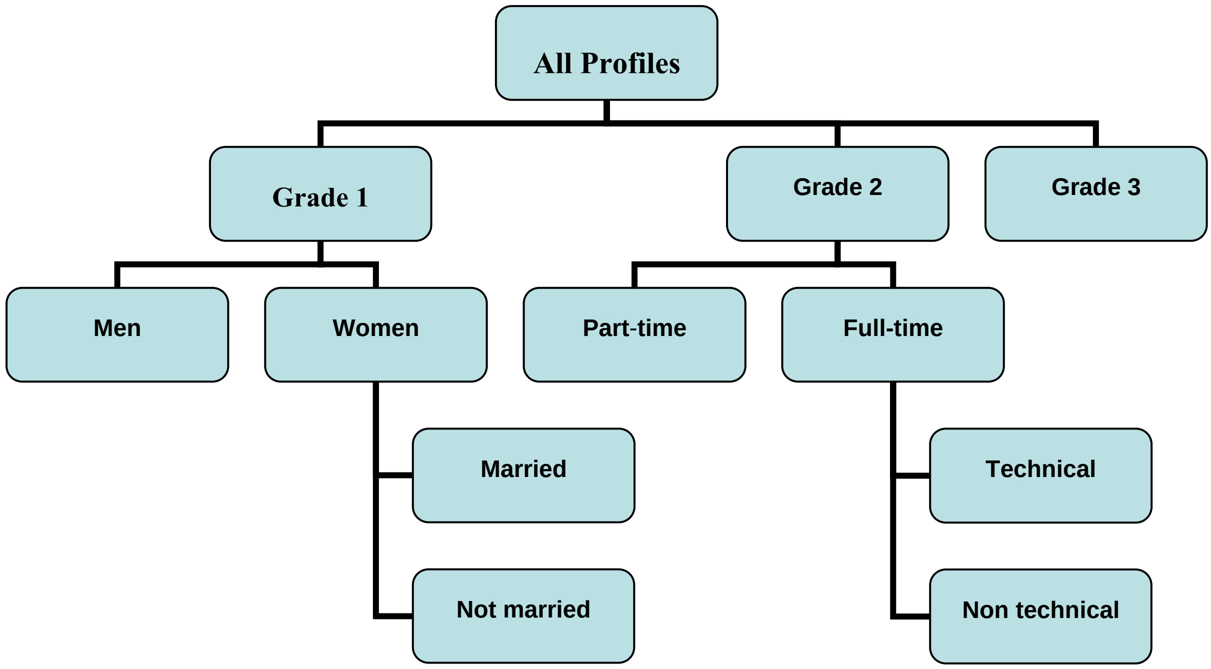

Figure 1.2: Illustration of splitting up process

1.4 SCOPE OF THE WORK

An attempt is made in this thesis to study stochastic models of manpower systems with reference to the following aspects: (i) recruitment (ii) promotion (iii) training and (iv) wastage.

For the various models considered, expressions for the relevant measures of system performance of the system are derived. Appropriate cost models are developed to obtain the optimal policies. Numerical illustrations are also shown to highlight the results obtained.

CHAPTER 2

APPLICATION OF MARKOV CHAINS IN A MANPOWER SYSTEM WITH EFFICIENCY AND SENIORITY

2.1 INTRODUCTION

Vacancies in any grade of an organization are filled either with promotions from next lower grades or by new recruitments. In general, promotions can be classified under the dichotomous policy namely, promotion based on efficiency and promotion based on seniority. Here, seniority means the length of service an employee acquired in each grade and efficiency means the measure of specialized skills or performance in the jobs which could be rated on a scale amenable to quantitative analysis and ranked in ascending order depending on their performance. If the efficiency is not rewarded by means of promotion the so called brilliant people termed as High fliers who would discharge the duties more effectively may leave the organization (this is presently happening in South Africa). So to retain them every organization should follow promotions based on efficiency.

Raghavendra (1991) obtained promotional probabilities and recruitment vectors embedding Markovian theory with certain assumptions on the promotional policies of the organization such as promotions allowed to the next grade and no demotion, without maintaining the grade structure over a period of time. Model 1 is the extension of Raghavendra (1991), where maintainability of grades is considered. In model 2 we give importance to efficiency and skills of the employees by allowing multiple promotions. That is, an employee is promoted to the next higher grade due to seniority and efficiency, whereas he is prompted to other higher grades due to efficiency only. Here two cases are discussed as (i) maintainability (ii) non-maintainability of grade structures. The promotional probabilities and recruitment vectors and cut-off levels of seniority and efficiency for promotions are found. The models developed require the following assumptions and notation.

2.2 ASSUMPTIONS AND NOTATION

2.2.1 Notation

Let $t = 1, 2, ..., T$; t being the horizon, usually t represents a year.

$i,\ j = 1, 2, ..., k$ states of the system representing the various grades, with total number of grades being k.

$N_j(t)$: Number of staff in grade j at the beginning of period t.

$P_{ij}(t)$: P [a member of staff in grade i at the beginning of period t is in grade j at the beginning of the next time period $(t+1)$].

$R_j(t)$: Number of new recruits to grade j during period t.

$w_j(t)$: Wastage factor expressed as a proportion of members of staff of grade j.

e_{ij}^p : Proportion of staff promoted from grade i to j; $(i < j)$.

e_j^r: Proportion of newly recruited staff to grade j.

$e_j^p = \sum_{i=1}^{j-1} e_{ij}^p$ = proportion of staff promoted to grade j.

$$e_j^r = \begin{cases} 1 - e_{ij}^p & \textit{if there is promotion only to the next grade} \\ 1 - e_j^p & \textit{if there are multiple promotions} \end{cases}$$

2.2.2 Assumptions

1. The system sates are mutually exclusive.
2. $N(1) = (N_1(1), N_2(1), ..., N_k(1))$, the vector of existing staff structure is known and $N(t) = (N_1(t), N_2(t), ..., N_k(t))$, the vector of staff requirements for the future periods are assumed to be known over a finite period of time T, $(t = 1, 2, 3, ..., T)$.
3. The expected strength of staff at any grade j at time point $t = 1, 2, 3, ..., T$ is known.
4. w (t), the wastage vectors are known, $t = 1, 2, 3, ..., T$.
5. Promotion to a grade from the next lower grade is allowed under both aspects of seniority and efficiency.
6. Promotions from other lower grades to an upper grade are allowed based only on their performance ratings (efficiency levels).
7. The bivariate distribution of employees under seniority and performance rating (efficiency) is known for all grades at various times.

The following section explains how Markovian theory is applied in Manpower models.

2.3 APPLICATION OF MARKOV CHAINS IN MANPOWER MODELS

Consider an organization, which satisfies all the above assumptions under Markovian assumptions (Bartholomew, 1982).

We have

$$N_j(t+1) = \sum_{i=1}^{k} p_{ij}(t) N_i(t) + R_j(t); \quad \forall j = 1, 2, ..., k. \tag{2.1}$$

Which implies that the staff in the grade j at time $t+1$ is the sum of employees staying in the same grade j during the time interval $(t, t+1)$ and the employees coming from various grades to grade j either by promotion or by demotion during $(t, t+1)$ and the new recruits into grades j during $(t, t+1)$.

Since at any point of time a member of the staff would either stay in the same grade, move to another grade either by promotion or by demotion or leave the system as wastage, we have

$$\sum_{j=1}^{k} p_{ij}(t) + w_i(t) = 1; \qquad \forall i = 1, 2, ..., k \tag{2.2}$$

Under Model-1 we determine the promotion probabilities and recruitment vector of various grades of an organization under maintainability of grade structure.

2.4 ANALYSIS OF MODEL-1: ONE STEP TRANSITION UNDER MAINTAINABLE GRADE STRUCTURE

Here we assume that the strength of staff at any grade is the same at various time points over a finite interval $(0, T)$.

That is

$$N_j(1) = N_j(2) = ... = N_j(T); \qquad \forall j = 1, 2, ..., k$$

As there are no double promotions and demotions, and promotion only to the next higher grade is allowed, equations (2.1) and (2.2.) take the form

$$N_j(t+1) = N_j(t) = P_{jj}(t)N_j(t) + P_{(j-1)j}(t)N_{j-1}(t) + R_j(t) \tag{2.3}$$

$$P_{jj}(t) + P_{(j-1)j}(t) + w_j(t) = 1\,; \qquad \forall j = 1, 2, ..., k \tag{2.4}$$

With the above assumptions, the number of staff to be promoted and the number to be recruited for various grades can be estimated as follows. For $t = 1$ *and* $j = k$ (the highest grade), equations (2.3) and (2.4) become

$$N_k(2) = N_k(1) = P_{kk}(1)N_k(1) + P_{(k-1)k}(1)N_{k-1}(1) + R_k(1) \tag{2.5}$$

$$P_{kk}(1) = 1 - w_k(1) \tag{2.6}$$

(As there is no promotion from the highest grade, $P_{k(k+1)}(1) = 0$).

Therefore the total number of promotions and recruitment is obtained from equations (2.5) and (2.6) as

$$\begin{aligned} P_{((k-1)k}(1)N_{k-1}(1) + R_k(1) &= N_k(1) - N_k(1)[1 - w_k(1)] \\ &= N_k(1)w_k(1) \\ &= N'_k(2), \text{ (say)} \end{aligned} \tag{2.7}$$

Since the number of promotions and recruitments are in the ratio $e_k : (1 - e_k)$, we have

$$P_{(k-1)k}(1)N_{k-1}(1) = e_k N'_k(2) \tag{2.8}$$

$$R_k(1) = (1 - e_k)N'_k(2) \tag{2.9}$$

Equations (2.8) and (2.9) give the number of promotions from grade (k-1) to grade k and the number of new recruits to grade k respectively. From equation (2.8), we have

$$P_{(k-1)k}(t) = \frac{e_k N'_k(2)}{N_{k-1}(1)} \tag{2.10}$$

In equation (2.3), $t = 1$ *and* $j = k - 1$ yields

$$P_{(k-1)(k-1)}(1) = 1 - w_{k-1}(1) - P_{(k-1)k}(t) \tag{2.11}$$

Proceeding in a similar manner for variations in j, the number of promotions and recruitment and the transition probabilities can be estimated for all other states of the system at various time points.

While in model-1 promotion only to the next higher grade is considered, multiple promotions are allowed in model 2 and are discussed under two cases of maintainable and non-maintainable grade structures.

2.5 ANALYSIS OF MODEL-2: MULTIPLE PROMOTIONS

Here we assume that the strength of the staff in any grade is the same at various time points. That is

$$N_j(1) = N_j(2) = ... = N_j(T)\,; \qquad \forall j = 1, 2, ..., k$$

Along with the maintainability of grade structure over a period of time T, equation (2.1) and (2.2) take the form

$$N_j(t+1) = N_j(t) = \sum_{i=1}^{j} p_{ij}(t) N_i(t) + R_j(t); \quad \forall j = 1, 2, ..., k \tag{2.12}$$

$$\sum_{i=j}^{k} p_{ji}(t) + w_j(t) = 1\,; \qquad \forall\ j = 1, 2, ..., k \tag{2.13}$$

With the above assumptions, the number of employees to be promoted and the number of employees to be recruited for various grades at time t are obtained as follows:

For $t = 1$ *and* $j = k$ (the highest grade) equations (2.12) and (2.13) reduced to

$$N_k(2) = N_k(1) = \sum_{i=1}^{k} P_{ik}(1)N_i(1) + R_k(1) \tag{2.14}$$

$$P_{kk}(1) = 1 - w_k(1)\,. \tag{2.15}$$

Therefore the total number of promotions and recruitment to the k^th grade, at time t=2 are obtained from equations (2.14) and (2.15) as

$$\begin{aligned}\sum_{i=1}^{k-1} P_{ik}(1)N_i(1) + R_k(1) &= N_k(1) - P_{kk}(1)N_k(1)\\ &= N_k(1)w_k(1)\\ &= N'_k(2)\,, \quad (say)\,. \end{aligned}\tag{2.16}$$

Since the number of promotions and recruitment to the k^th grade are in the ratio $e_k^p : (1-e_k^p)$, where $e_k^p = \sum_{i=1}^{k-1} e_{ik}^p$, we have the number of promotions as

$$\begin{aligned}\sum_{i=1}^{k-1} P_{ik}(1)N_i(1) &= e_k^p N'_k(2)\\ &= \sum_{i=1}^{k-1} e_{ik}^p N'_k(2)\,. \end{aligned}\tag{2.17}$$

And the number of recruitments to the grade k as

$$\begin{aligned}R_k(1) &= (1-e_k^p)N'_k(2)\\ &= e_k^r N'_k(2)\,. \end{aligned}\tag{2.18}$$

From equation (2.17), we have

$$P_{ik}(1) = \frac{e_{ik}^p N'_k(2)}{N_i(1)}\,; \qquad \forall\ i = 1, 2, ..., k-1. \tag{2.19}$$

Putting $t = 1 \quad and \quad j = k-1$ in (2.13) we have

$$P_{(k-1)(k-1)}(1) = 1 - w_{k-1}(1) - P_{(k-1)k}(1) \tag{2.20}$$

By proceeding in a similar manner, the numbers of promotions and recruitments and the transitional probabilities can be obtained for all other states of the system at various time points.

2.5.1 Case-2: Non-maintainable grade structures

Here we assume that the strength of staff and any grade is not necessarily the same at various time points. That is, $N_j(t_1) \neq N_j(t_2)$ for at least one pair of $t_1, t_2 \quad (t_1 \neq t_2)$ for all $j = 1, 2, ..., k$.

With the above assumptions, proceeding in a similar manner as in the case-1, equations (2.12) takes the form

$$N_j(t+1) = \sum_{i=1}^{j} p_{ij}(t)N_i(t) + R_j(t); \quad \forall j = 1, 2, ..., k . \tag{2.21}$$

Whereas as the equation (2.13) remains the same, equation (2.21) reduces to

$$N_k(2) = \sum_{i=1}^{k} P_{ik}(1)N_i(1) + R_k(1) \tag{2.22}$$

along with equation (2.15). Therefore the total number of promotions and recruitments at grade k at time t=2 are obtained from equation (2.22) and is given by

$$\begin{aligned}\sum_{i=1}^{k-1} P_{ik}(1)N_i(1) + R_k(1) &= N_k(2) - P_{kk}(1)N_k(1) \\ &= N_k''(2), \qquad (say) .\end{aligned} \tag{2.23}$$

Since the number of promotions and recruitment at grade k are in the ratio $e_k^p : (1 - e_k^p)$ where $e_k^p = \sum_{i=1}^{k-1} e_{ik}^p$, we have the number of promotions given by

$$\begin{aligned}\sum_{i=1}^{k-1} P_{ik}(1)N_i(1) &= e_k^p N_k''(2) \\ &= \sum_{i=1}^{k-1} e_{ik}^p N_k''(2) .\end{aligned} \tag{2.24}$$

And the number of recruitments to the grade k is given by

$$R_k(1) = (1 - e_k^p)\, N_k''(2)\,. \tag{2.25}$$

From equation (2.26) we have

$$P_{ik}(1) = \frac{e_{ik}^p N_k''(2)}{N_i(1)}\,; \qquad \forall\; i = 1, 2, \ldots k-1\,. \tag{2.26}$$

Using equation (2.20) and proceeding in a similar manner as in case-1, the numbers of promotions and recruitments and the transitional probabilities can be obtained for all other states of the system at various time points.

2.6 BIVARIATE FRAMEWORK TO DETERMINE THE CUT-OFF LEVELS FOR PROMOTION UNDER SENIORITY AND EFFICIENCY

Let X and Y be discrete random variables representing seniority and efficiency respectively. Let $P_j(x, y)$ be the joint probability mass function of these two variables for members of staff in grade j in the organization and $F_j(x, y)$ be the cumulative joint probability that $X \le x$ and $Y \le y$. Let $g_j(x) = \sum_y P(x, y)$ and $h_j(y) = \sum_x P_j(x, y)$ be the respective marginal probabilities. Let the corresponding cumulative distribution functions be $G_j(x)$ *and* $H_j(y)$.

Suppose an organization's policy requires the proportion of promotions based on seniority and on efficiency as $s_{(j-1)j}$ and $(1 - s_{(j-1)j})$ respectively from grade (j-1) to j for all $j = 2, 3, \ldots, k$, and multiple promotions (promotions with jumps) are to be based only on efficiency, then the minimum levels of X and Y required for promotion can be evaluated.

The minimum cut-off level x for seniority required for promotion from grade (j-1) to grade j, can be obtained from the following equation

$$s_{(j-1)j}\, e^{p}_{(j-1)j} N'_j(t+1) = N_{j-1}(t)\,\left[1 - G_{j-1}(x)\right]. \tag{2.27}$$

Similarly the minimum cut-off level y for efficiency required for promotion from grade (j-1) to j is obtained from the equation

$$(1 - s_{(j-1)j})\, e^{p}_{(j-1)j} N'_{j-1}(t+1) = N_{j-1}(t)\,\left[1 - H_{j-1}(y)\right]. \tag{2.28}$$

For $i < j-1$, promotions from grade i to grade j are based only on efficiency. Hence in these cases the minimum levels of efficiency for promotions are given by

$$e^{p}_{ij} N'_j(t+1) = N_i(t)\left[1 - H_i(y)\right]. \tag{2.29}$$

The order in which promotions are made is based on the two factors; i.e. seniority and efficiency may also influence the chance of a specific member of staff getting promoted. It does not affect the person with high values X and Y, it is likely to affect those around the cut-off values of X and Y (see (2.27) and (2.28)). These cut-off values are influenced by the degree of correlation between X and Y.

2.7 CONCLUSION

In this chapter the Markovian model is embedded in a bivariate framework to generate promotion probabilities and recruitments. The bivariate aspect of seniority and efficiency associated with promotion is also studied. It clearly establishes the bounds for promotion under seniority and efficiency so that unambiguity is created. Our approach well suits the present day requirements of most of the organization as they follow the dual criteria of seniority and efficiency.

CHAPTER 3

MODELING OF AN INTERMITTENTLY BUSY MANPOWER SYSTEM[1]

[1] A modified version of this chapter was presented at the IASTED conference Sept 11-13, 2006 in Gaborone Botswana. (The paper has been refereed and published in the proceedings).

3.1 INTRODUCTION

While many authors directly discuss the economics of minimizing a manpower system, this chapter deals with the aspect of the image of Goodwill that an organization aspires to achieve economy directly. In any organization employees look forward for better opportunities and hop to other organizations in search for them. This behavior affects the normal routine work of the organization. The adverse effects are felt more where a person leaves the organization during a busy period of the organization.

However, it is not necessary that the staff strength be always full for satisfactory performance of the functions. Thus, there are 'lean' periods when full staff strength is not needed. The 'busy and lean' periods, whose duration is random, occur alternately in an organization. Such a manpower system may be called an intermittently busy manpower system.

In the context of reliability of an intermittently used system, Gaver (1964) who has studied the system performance defines the point event called 'disappointment'. Still in Gaver (1964) it is pointed out that it is pessimistic to evaluate the performance on an intermittently used system solely on the basis of the distribution of the time to system failure. The point event, called a disappointment, is characterized as follows:

- The system fails during a need period or
- A need for the system arises, but it is in the failed state.

It is well known that the steady state availability is a satisfactory measure for systems, which are operated continuously, such as for manpower planning system. Confidence limits for the steady state availability of a two-unit standby system was investigated by Chandrasekhar and Natarajan (1997) while Yadavalli, et al. (2002) examined the same for a two unit system with the introduction of preparation time for the service facility. Recently the confidence limits for the stationary rate of disappointment of an intermittently used system have been studied by Yadavalli and Botha (2002). In this chapter, an attempt is made to obtain the expression for the stationary rate of crisis in an intermittently busy manpower system and derive the 100(1- α) % confidence limits for the same, when both the busy and lean times have an exponential distribution.

Definition 1

The organization is said to face a crisis if a vacancy is caused by the departure of a person during the 'busy' period or alternately if a busy period arises when there exists at least one vacancy. In both the situations the recruitment process is immediately initiated.

Definition 2

Stationary rate of crisis of an organization is the annual frequency (i.e. the number of times the crisis occurs in a unit of time, usually taken as a year) in the long run (as $t \to \infty$) with which crisis occur in the organization.

3.2 ASSUMPTIONS

1. The 'busy' and 'lean' periods occur alternately.
2. The time T for which the staff strength remains 'full' is exponentially distributed with parameter λ and the time R required to complete recruitment for filling up vacancies is exponentially distributed with parameter μ.
3. T and R are independently distributed random variables.
4. The 'busy' period is exponentially distributed with parameter α and the 'lean' period is also exponentially distributed with parameter β.
5. There is a recruitment board of the organization, which starts its functions as soon as a vacancy arises.
6. The wastages (resignations, retirement, dismissals, and deaths) of employees are immediately taken as 'alert signal' by the recruitment board.
7. If an employee leaves the organization during lean/busy period, the recruitment process is immediately initiated and the recruitment is done regardless of whether the busy/lean period arises or not.

3.3 SYSTEM ANALYSIS

Let $\{Z(t), t > 0\}$ be the stochastic process depicting the state of the manpower system with state space $\{0, 1, 2, 3\}$ corresponding to various situations that arise in the organization described in Table 3.1.

Table 3.1 System states

State	Staff strength	Busy/lean period
0	Full	Busy
1	Full	Lean
2	Understaffed	Busy
3	Understaffed	Lean

In this problem, state 2 represents the crisis state in the organization. Let $p_i(t) = P[Z(t) = i] \qquad i = 0, 1, 2, 3$

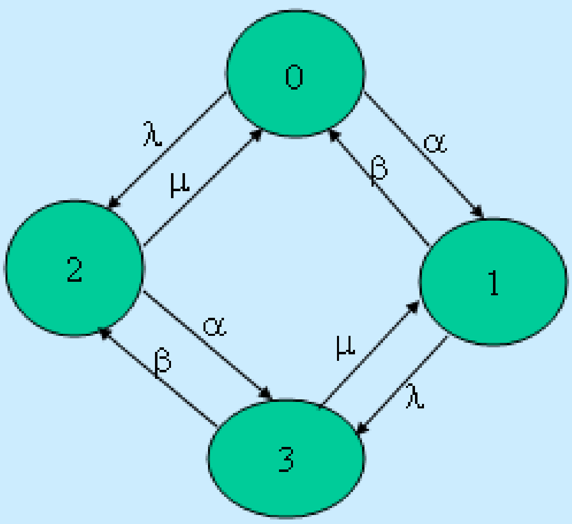

Figure 3.1:Transition diagram

Since the interest is in the stationary behavior of the process, we need $\lim_{t\to\infty} p_i(t) = p_i$

Using the transition diagram, Beichelt and Fatti (2002), the following differential-difference equations can be obtained.

$$P_0(t+\Delta) = P\left[Z(t+\Delta) = 0 / Z(t) = 1\right]P_1(t) + P\left[Z(t+\Delta) = 0 / Z(t) = 2\right]P_2(t) + P\left[Z(t+\Delta) = 0 / Z(t) = 0\right]P_0(t) + o(\Delta)$$

$$= \beta\Delta P_1(t) + \mu\Delta P_2(t) + \left[1 - (\lambda + \alpha)\Delta\right]P_0(t) + o(\Delta)\ .$$

Hence

$$\lim_{\Delta\to\infty} \frac{P_0(t+\Delta) - P_0(t)}{\Delta} = \beta P_1(t) + \mu P_2(t) - (\lambda + \alpha)P_0(t)$$

so that

$$P_o^{'}(t) = -(\lambda+\alpha)P_o(t) + \beta P_1(t) + \mu P_2(t)\,. \qquad (3.3.1)$$

Similarly

$$P_1^{'}(t) = -(\lambda+\beta)P_1(t) + \alpha P_0(t) + \mu P_3(t) \qquad (3.3.2)$$

$$P_2^{'}(t) = -(\alpha+\mu)P_2(t) + \lambda P_0(t) + \beta P_3(t) \qquad (3.3.3)$$

and $$P_3^{'}(t) = -(\mu+\beta)P_3(t) + \alpha P_2(t) + \lambda P_1(t)\,. \qquad (3.3.4)$$

The following steady state equations can be easily obtained using (3.3.1)-(3.3.4)

$$(\alpha+\lambda)P_0 = \beta P_1 + \mu P_2 \qquad (3.3.5)$$

$$(\lambda+\beta)P_1 = \alpha P_0 + \mu P_3 \tag{3.3.6}$$

$$(\alpha+\mu)P_2 = \lambda P_0 + \beta P_3 \tag{3.3.7}$$

$$(\mu+\beta)P_3 = \alpha P_2 + \lambda P_1 \tag{3.3.8}$$

These equations are linearly dependent and can be solved by using the fact that $\sum_{i=0}^{3} P_i = 1$.

Therefore

$$P_0 = \frac{\beta\mu}{(\alpha+\beta)(\lambda+\mu)} \tag{3.3.9}$$

$$P_1 = \frac{\alpha\mu}{(\alpha+\beta)(\lambda+\mu)} \tag{3.3.10}$$

$$P_2 = \frac{\beta\lambda}{(\alpha+\beta)(\lambda+\mu)} \tag{3.3.11}$$

$$P_3 = \frac{\alpha\lambda}{(\alpha+\beta)(\lambda+\mu)} \quad . \tag{3.3.12}$$

The main interest is to find the ‘rate of crisis in a steady state’ (C_∞)

$$P[crisis \quad in \ (t, t+\Delta)] = P[crisis \quad in \ (t, t+\Delta)/Z(t)=0]\,P[Z(t)=0]$$

$$+ P[crisis \quad in \ (t, t+\Delta)/Z(t)=3]\,P[Z(t)=3] + o(\Delta)$$

$$= P[Z(t,t+\Delta)=2/Z(t)=0]\,P[Z(t)=0]$$

$$+ P[Z(t,t+\Delta)=2/Z(t)=3]\,P[Z(t)=3] + o(\Delta)$$

$$= (\lambda\Delta + 0\Delta)P_0(t) + (\beta\Delta + 0\Delta)P_3(t) + o(\Delta)$$

$$= \lambda P_o(t) + \beta P_3(t) + o(\Delta) .$$

The rate of crisis in the organization at time t, is C_t

$$C_t = \lambda P_o(t) + \beta P_3(t) .$$

Hence, the stationary rate of crisis is

$$C_\infty = \lim_{t\to\infty} C_t = \lambda P_0 + \beta P_3$$

namely

$$C_\infty = \frac{\beta\lambda(\alpha + \mu)}{(\alpha + \beta)(\lambda + \mu)} .$$

3.4 SPECIAL CASE

It should be noted that for an organization with some busy time and full-staff strength, that is, $\alpha = \lambda$ whatever be the recruitment time, the stationary rate of crisis is $C_\infty = \frac{\beta\lambda}{(\lambda + \beta)}$ is dependent only on β, the full staff strength. When β is a fixed constant, C_∞ becomes a constant.

3.5 ASYMPTOTIC CONFIDENCE LIMITS FOR THE STATIONARY RATE OF CRISIS

In this section we obtain 100(1- α) % confidence limits for the stationary rate of crisis in the organization.

Let $X_1, X_2, ...X_n$ be a sample of leaving times with p.d.f. given by

$$f_1(x) = \lambda e^{-\lambda x}, \qquad 0 < x < \infty, \quad \lambda > 0 .$$

Let $Y_1, Y_2, ...Y_n$ be a sample of recruitment times with p.d.f. given by

$$f_2(y) = \mu e^{-\mu y}, \qquad 0 < y < \infty, \quad \mu > 0 .$$

Let $Z_1, Z_2, ...Z_n$ be a sample of busy periods with p.d.f. given by

$$g_1(z) = \alpha e^{-\alpha z}, \qquad 0 < z < \infty, \quad \alpha > 0.$$

Let $V_1, V_2, ...V_n$ be a sample of lean periods with p.d.f. given by

$$g_2(v) = \beta e^{-\beta v}, \qquad 0 < v < \infty, \ \beta > 0.$$

Let $\overline{X}$, $\overline{Y}$, $\overline{Z}$, *and* $\overline{V}$ be the sample means of the time to leaving the system, the time to recruitment of staff into the system, the time to busy service periods and lean service periods of the system, respectively. Then

$$E(\overline{X}) = \frac{1}{\lambda}, \qquad E(\overline{Y}) = \frac{1}{\mu}, \qquad E(\overline{Z}) = \frac{1}{\alpha} \ \text{ and } \ E(\overline{V}) = \frac{1}{\beta}.$$

It can be shown that $\overline{X}, \ \overline{Y}, \ \overline{Z}, \ \textit{and} \ \overline{V}$ are the MLE's of

$$\theta_1 = \frac{1}{\lambda}, \qquad \theta_2 = \frac{1}{\mu}, \qquad \theta_3 = \frac{1}{\alpha} \ \text{ and } \ \theta_4 = \frac{1}{\beta} \ \text{ respectively.}$$

The stationary rate of crisis is

$$C_\infty = \frac{(\theta_3 + \theta_4)(\theta_1 + \theta_2)}{\theta_4 \theta_1 (\theta_3 + \theta_2)}$$

and hence, the estimator of C_∞ is given by

$$\hat{C}_\infty = \frac{(\overline{Z} + \overline{V})(\overline{X} + \overline{Y})}{\overline{V}\,\overline{X}\,(\overline{Z} + \overline{Y})}.$$

Using the application of the Multivariate Central Theorem (see Rao, 1973), it follows that

$$\sqrt{n}\left[\left(\overline{X}, \ \overline{Y}, \ \overline{Z}, \ \overline{V}\right) - \left(\theta_1, \theta_2, \theta_3, \theta_4\right)\right] \xrightarrow{d} N(0, \Sigma) \quad as \quad n \to \infty$$

where

$$(\theta_1, \theta_2, \theta_3, \theta_4) = \left(\frac{1}{\lambda}, \frac{1}{\mu}, \frac{1}{\alpha}, \frac{1}{\beta}\right)$$

and the dispersion matrix $\sum = [\sigma_{i,j}^2]_{4X4}$ is given by

$$\sum = diag\left(\theta_1^2, \theta_2^2, \theta_3^2, \theta_4^2\right).$$

From Rao (1973), we have $\sqrt{n}\left(\hat{C}_\infty - C_\infty\right) \xrightarrow{d} N(0, \sigma^2(\theta))$

with

$$\sigma^2(\theta) = \sum_{i=1}^{4}\left(\frac{\partial C_\infty}{\partial \theta_i}\right)^2 \sigma_{ii}$$

$$= \sum_{i=1}^{4}\left(\frac{\partial C_\infty}{\partial \theta_i}\right)^2 \theta_i^{\,2}$$

Let $\sigma^2(\hat{\theta})$ be the estimator of $\sigma^2(\theta)$ which is obtained by replacing θ by a consistent estimator $\hat{\theta} = \left(\overline{X},\ \overline{Y},\ \overline{Z},\ \overline{V}\right)$. Since $\sigma^2(\theta)$ is a continuous function of θ, we know that $\sigma^2(\hat{\theta})$ is a consistent estimator of $\sigma^2(\theta)$.

Thus

$$\sigma^2(\hat{\theta}) \to \sigma^2(\theta) \quad as \quad n \to \infty.$$

Using the Slutsky's theorem, we have

$$\frac{\sqrt{n}\left(\hat{C}_\infty - C_\infty\right)}{\hat{\sigma}} \xrightarrow{d} N(0, 1) \quad as \quad n \to \infty .$$

This implies that $\Pr\left[-z_{\alpha/2} \le \frac{\sqrt{n}\left(\hat{C}_\infty - C_\infty\right)}{\hat{\sigma}} \le z_{\alpha/2}\right] = 1 - \alpha$

Where $z_{\alpha/2}$ is obtained from the normal tables. Hence, the asymptotic $100(1-\alpha)\%$ confidence limits for C_∞ are given by $C_\infty \pm z_{\alpha/2}\frac{\hat{\sigma}}{\sqrt{n}}$.

3.6 NUMERICAL ILLUSTRATION

Table 3.2 gives the 95% confidence limits for the stationary rate of crisis for different values of θ_1 and θ_2 for the values of θ_3 and θ_4 fixed at $\theta_3 = 100$ and $\theta_4 = 50$. Figure 3.2 gives a graphical display of the stationary rate of crisis against leaving rate for the model in figure 3.1. The parametric values used in the equations for C_∞ were $\mu = \frac{1}{60}$, $\alpha = \frac{1}{100}$, $\beta = \frac{1}{50}$. For these values the leaving rate ranged from $\frac{1}{100}$ *to* $\frac{1}{10}$. The graph shows that an increase in leaving rate increased the rate of crisis.

Table 3.2: 95% confidence limits

n=100	θ1	Confidence Limits	θ2	Confidence Limits
	20	(0.05924, 0.09076)	20	(0.0089, 0.0577)
	40	(0.03839, 0.05540)	40	(0.0262, 0.0452)
	60	(0.03129, 0.04370)	60	(0.0313, 0.0437)
	80	(0.02768, 0.03791)	80	(0.0339, 0.0437)
	100	(0.02549, 0.03450)	100	(0.0359, 0.0441)
n=500	20	(0.06794, 0.08206)	20	(0.0224, 0.0442)
	40	(0.04318, 0.05062)	40	(0.0314, 0.0399)
	60	(0.03476, 0.04024)	60	(0.0313, 0.0437)
	80	(0.03045, 0.03515)	80	(0.0366, 0.0409)
	100	(0.02798, 0.03202)	100	(0.0382, 0.0418)
n=1000	20	(0.07002, 0.07998)	20	(0.0256, 0.0410)
	40	(0.04421, 0.04958)	40	(0.0327, 0.0387)
	60	(0.03554, 0.03946)	60	(0.0355, 0.0395)
	80	(0.03119, 0.03441)	80	(0.0373, 0.0403)
	100	(0.02859, 0.03141)	100	(0.0387, 0.0413)

An increase in leaving rate λ will increase the rate of crisis. Conversely, reduction in leaving rate increases the average time to leave and consequently reduces the rate of crisis. While an increase in recruitment rate μ, reduces the rate of crisis, decreasing the recruitment rate will increase the average time to leave and the rate of crisis (see Figure 3.2).

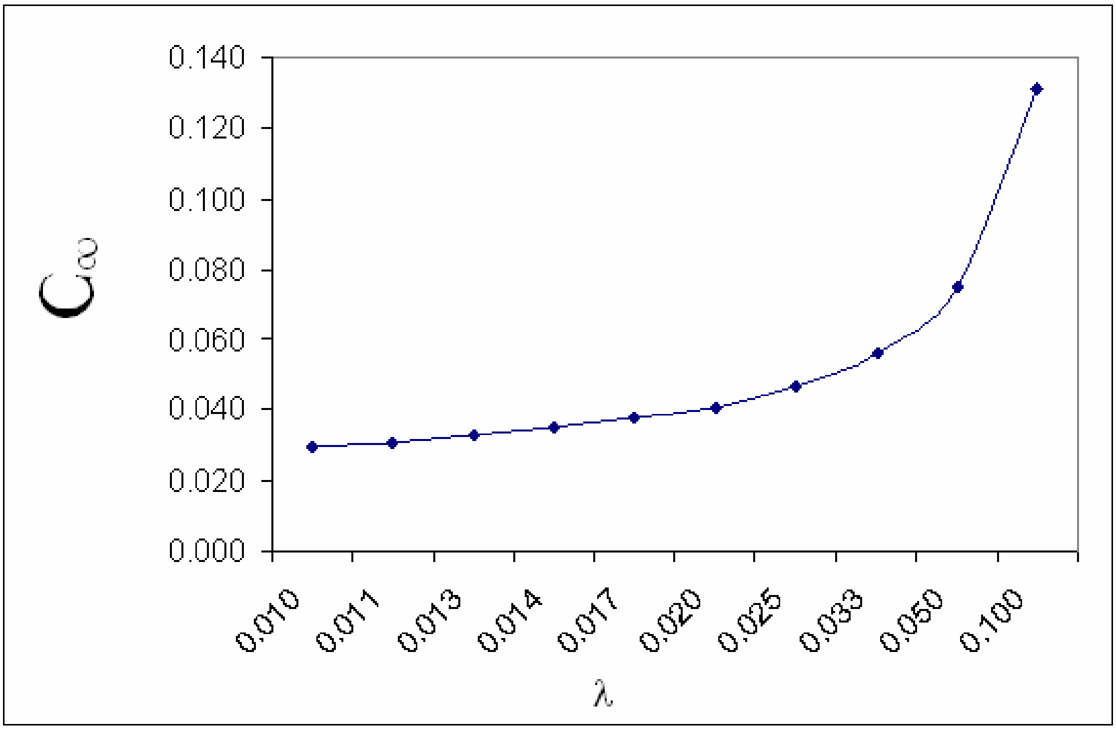

Figure 3.2 Stationary rate of crisis for the model

3.7 NON-MARKOVIAN MODEL OF INTERMITTENTLY BUSY MANPOWER SYSTEM

For reliability systems Baxter (1981) obtained some general measures for the reliability of a repairable one-unit system, by identifying the sequence of periods of operation and repair as an alternating renewal process (Cox, 1962). This type of modeling was possible because the uptime and down time in a reliability system are independent random variables. Two-unit standby systems in which the lifetime and repair time of a unit are generally distributed random variables are also considered by Subramanian et al. (1983). Yadavalli and Hines (1991) subsequently studied the joint distribution of the up time and disappointment time of an intermittently used two unit system. At the epoch of failure of a unit) operating online), if the other unit is in a state of failure undergoing repair, the system enters the down state and the duration of the down state depends on the elapsed repair time of the unit under repair. Thus in this example the uptime and the down time are correlated random variables. The entire process can be thought of as a sequence of cycles where a cycle consists of an uptime plus the subsequent down time. The evolution of the system can be modeled by an alternating renewal; process in which the random variables representing the uptime and the random variable representing the subsequent down time are correlated. We call such a process 'correlated alternating renewal process' if it satisfies some more additional conditions. Earlier, the joint distribution of the up time and down time has been obtained by Nakagawa and Osaki (1976). In this sub-section of the chapter we apply the correlated alternating renewal process to a manpower system. This is achieved with the help of the joint forward recurrence time to a system busy period and system lean period. The alternating renewal process discussed by Baxter (1981) is shown to be a particular case of the correlated alternating renewal process studied here and the results are deduced as a special case.

Assumptions

All the assumptions in section 3.2 are the same in this model except 2 and 4.

2'. The busy period is an exponentially distributed random variable with parameter α .

4'. Lean period is a random variable having p.d.f $g(\cdot)$.

8. Periods of full strength of staff and the period of under staffed are distributed random variables with parameters λ and μ respectively.

3.7.1 Notation

An event E is characterized by system recovery that is the system enters the state 0 from state 2. Event E_1 is characterized by the event that 'the staff strength is full and the staff is busy'. Event C is characterized by the event 'crisis of the system' when the system enters state 2 from state 3. The two-valued stochastic process Z(t) describes the state of full staff strength and state of understaffed for the system at time t, that is

$$Z(t) = \begin{cases} 0 & \textit{if the system is in state of full staffed} \\ 1 & \textit{if the system is in state of undestaffed} \end{cases} .$$

Associated with the process $\{Z(t)\,; t \geq 0\}$, we define the following auxiliary functions $\pi_{ij}(t)$, useful to our analysis:

$$\pi_{ij}(t) = pr\{Z(t) = j \,/\, Z(0) = i\}, \qquad i,\ j = 0, 1, \qquad t \geq 0 .$$

These can be obtained by renewal theoretic arguments

$$\pi_{00}(t) = \frac{\{\beta + \alpha \exp[-(\alpha + \beta)\, t]\}}{(\alpha + \beta)}$$

$$\pi_{01}(t) = \frac{\alpha\{1 - \exp[-(\alpha + \beta)\, t]\}}{(\alpha + \beta)}$$

$$\pi_{11}(t) = \frac{(\alpha + \beta)\exp\{-(\alpha + \beta)\, t\}}{(\alpha + \beta)}$$

$$\pi_{10}(t) = \frac{\beta\{1 - \exp[-(\alpha + \beta)\, t]\}}{(\alpha + \beta)}$$

3.7.2 Joint distribution of the uptime and disappointment time

If X is the time interval between an E event and the next C event and Y is the time interval between the C event and the following E event, then the joint density of X and Y is given by

$$f_{X,Y}(x,y) = \pi_{00}(x)\lambda\exp(-\lambda x)g(x+y)$$

$$+\int_0^x \pi_{01}(u)\lambda\exp(-\lambda u)\beta\exp[-\beta(x-u)]g(x+y)\,du$$

$$+g(y)\int_0^x\int_0^{x-u} \pi_{01}(u)\lambda\exp(-\lambda u)g(u+v)$$

$$x\,(1-\exp\{-\lambda[x-(\alpha+v)]\})\beta\exp[-\beta(x-u)]\,dv\,du$$

$$+2g(y)\int_0^x\int_0^{x-u} g(u)\exp(-\lambda u)\pi_{01}(u+v)\lambda\exp(-\lambda v)$$

$$x\,\{\exp[-\lambda(x-u)]\}\beta\exp\{-\beta[x-(u+v)]\}\,dv\,du$$

$$+\int_0^x h_{e_1}(u)\pi_{00}(x-u)\lambda\exp[-\lambda(x-u)]g(x+y-u)\,du$$

$$+\int_0^x\int_0^{x-u} h_{E_1}(u)\pi_{01}(v)\lambda\exp(-\lambda v)\beta\exp\{-\beta[x-(v+u)]\}$$

$$x\,g(x+y-u)\,dv\,du+g(y)$$

$$x\int_0^x\int_0^{x-w}\int_0^{x-(u+w)} h_{E_1}(w)\pi_{01}(u)\lambda\exp(-\lambda u)g(v+u)$$

$$x\,(1-\exp\{-\lambda[x-(u+v+w)]\})\beta\;x\;\exp\{-\beta[x-(u+w)]\}\,dv\,du\,dw$$

Where $h_{E_1}(t)$ the renewal density of E_1 events is given by

$$h_{E_1}(t) = \sum_{n=1}^{\infty} \begin{bmatrix} 2\int_0^t g(u)\,\exp\,(-\lambda u)\pi_{00}(t)\;\lambda\exp[-\lambda(t-u)]\exp\,[-\lambda(t-u)]\;du \\ +\int_0^t\int_o^{t-u} g(u)\exp(-\lambda u)\pi_{01}(u+v)\;\lambda \\ x\,\exp\,(-\lambda v)\exp[-\lambda(t-u)]\;\beta\exp\{-\beta[t-(u+v)]\,\}\;dv\;du \\ +\int_0^t\int_0^{t-u} \pi_{01}(u)\lambda\exp(-\lambda u)g(u+v)\;x\,\exp\{-\lambda[t-(u+v)]\}\,\beta\exp[-\beta(t-u)]\;dv\;du \end{bmatrix}^{(n)}$$

3.7.3 Marginal densities

If the marginal densities of the random variables X and Y are $f_X(x)$ and $f_Y(y)$, respectively, then

$$f_X(x) = \int_0^{\infty} f_{X,Y}(x,y)\;dy$$

$$= \int_0^x \pi_{00}(x)\lambda\exp(-\lambda x)G(x) + \int_0^x \pi_{01}(u)\lambda\exp(-\lambda u)\beta\exp[-\beta(x-u)]G(x)\;du$$

$$+\int_0^x\int_0^{x-u} \pi_{01}(u)\lambda\exp(-\lambda u)g(u+v)$$

$$x\,(1-\exp\{-\lambda[x-(u+v)]\})\beta\exp[-\beta(x-u)]\;dv\;du$$

$$+\,2\int_0^x\int_0^{x-u} g(u)\exp(-\lambda u)\pi_{01}(u+v)\lambda\exp(-\lambda v)$$

$$x\,\{\exp(-\lambda v)-\exp[-\lambda(x-u)]\}\beta\exp\{-\beta[x-(u+v)]\}\;dv\;du$$

$$+\int_0^x h_{E_1}(u)\pi_{00}(x-u)\lambda\exp[-\lambda(x-u)]\,\overline{G}(x-u)\;du$$

$$+\int_0^x\int_0^{x-u} h_{E_1}(u)\pi_{01}(v)\lambda\exp(-\lambda v)\beta\exp\{-\beta[x-(v+u)]\}\,\overline{G}(x-u)\;dv\;du$$

$$+\int_0^x\int_0^{x-w}\int_0^{x-(u+w)} h_{E_1}(w)\lambda_{01}(u)\lambda\exp(-\lambda u)g(u+v)$$

$$x\,(1-\exp\{-\lambda[x-(u+v+w)]\})\beta\exp\{-\beta[x-(u+w)]\}\,dv\,du\,dw$$

$$+\,2\int_0^x\int_0^{x-w}\int_0^{x-(u+w)} h_{E_1}(w)g(u)\exp(-\lambda u)\pi_{01}(u+v)\lambda\exp(-\lambda v)$$

$$x(\exp(-\lambda v)\exp\{-\lambda[x-(u+w)]\}\;\beta\;x\;\;\exp\{-\beta[x-(u+v+w)]\}\,dv\,du\,dw$$

and

$$f_Y(Y)=\int_0^\infty\;f_{X,Y}(x,y)\,dx$$

$$=\int_0^\infty\pi_{00}(x)\lambda\exp(-\lambda x)G(x+y)+\int_0^\infty\,dx\int_0^x\pi_{01}(u)\lambda\exp(-\lambda u)\beta\exp[-\beta(x-u)]G(x+y)\,du$$

$$+\,g(y)\int_0^\infty\,dx\int_0^x\int_0^{x-u}\;\pi_{01}(u)\lambda\exp(-\lambda u)g(u+v)$$

$$x\,(1-\exp\{-\lambda[x-(u+v)]\})\beta\exp[-\beta(x-u)]\,dv\,du$$

$$+\,2\,g(y)\int_0^\infty\,dx\int_0^x\int_0^{x-u}\;g(u)\exp(-\lambda u)\pi_{01}(u+v)\lambda\exp(-\lambda v)$$

$$x\,\{\exp(-\lambda v)-\exp[-\lambda(x-u)]\}\beta\exp\{-\beta[x-(u+v)]\}\,dv\,du$$

$$+\int_0^\infty\,dx\int_0^x h_{E_1}(u)\pi_{00}(x-u)\lambda\exp[-\lambda(x-u)]g(x+y-u)\;\;du$$

$$+\int_0^\infty\,dx\int_0^x\int_0^{x-u}\;h_{E_1}(u)\pi_{01}(v)\lambda\exp(-\lambda v)\beta\exp\{-\beta[x-(v+u)]\}\;g(x+y-u)dv\,du$$

$$+\,g(y)\int_0^\infty\,dx\int_0^x\int_0^{x-w}\int_0^{x-(u+w)} h_{E_1}(w)\pi_{01}(u)\lambda\exp(-\lambda u)g(u+v)$$

$$x\,(1-\exp\{-\lambda[x-(u+v+w)]\})\beta\exp\{-\beta[x-(u+w)]\}\,dv\,du\,dw$$

$$+\,2\,g(y)\int_0^\infty\,dx\int_0^x\int_0^{x-w}\int_0^{x-(u+w)} h_{E_1}(w)g(u)\exp(-\lambda u)\pi_{01}(u+v)\lambda\exp(-\lambda v)$$

$$x\,(\exp(-\lambda v)\exp\{-\lambda[x-(u+w)]\})\beta\;x\exp\{-\beta[x-(u+v+w)]\}\,dv\,du\,dw$$

The density of the random variable $X+Y$ representing the cycle length is

$$f_{X+Y}(t)=\int_0^t f_{X+Y}(u,t-u)\,du\,.$$

3.7.4 Joint forward recurrence time

Let t be a time instant when the system is up. We say that the joint forward recurrence time ψ, is the bivariate random variable (U_t, W_t) where U_t corresponds to the time interval from t to the next C event and W_t the time interval from t to the subsequent E event.

$$\psi_{C,E}(t,x,y)=\begin{cases} f_{X,Y}(t+x,\quad y-x)+\int_0^t h_E(u)f_{X,Y}(t-u+x,\quad y-x)\,du, & for\quad y>x\geq 0 \\ 0, & otherwise \end{cases}$$

where

$$h_E(t)=\sum_{n=1}^{\infty} f_{X+Y}^{(n)}(t)$$

is the renewal density of E events.

3.7.5 Marginal forward recurrence times

The marginal forward recurrence times are given by

$$\psi_C(t,x)=\int_0^{\infty}\psi_{C,E}(t,x,y)\,dy = f_X(t+x)+\int_0^{\infty} h_E(u)\,f_X(t-u+x)\,du$$

and

$$\begin{aligned}\psi_E(t,y)&=\int_0^{\infty}\psi_{C,E}(t,x,y)\,dx\\ &=\int_0^{\infty} f_{X,Y}(t+u,\,y-u)\,du+\int_0^{\infty} du\int_0^t h_E(u)\,f_{X,Y}(t-w+u,\,y-u)\,dw\,.\end{aligned}$$

3.7.6 Stationary values of the forward recurrence times

As defined earlier,

$$\psi_{C,E}(x,y) = \lim_{t\to\infty} \psi_{C,E}(t,x,y) = \frac{1}{\mu_1+\mu_2}\int_x^\infty f_{X,Y}(t, y-x)\, dt$$

$$\psi_C(x) = \lim_{t\to\infty} \psi_C(t,x) = \frac{1}{\mu_1+\mu_2}\int_x^\infty f_X(t)\, dt$$

$$\psi_E(y) = \lim_{t\to\infty} \psi_E(t,y) = \frac{1}{\mu_1+\mu_2}\int_y^\infty f_{X+Y}(t)\, dt$$

where

$$\mu_1 = E(X) = \int_0^\infty x\ f_X(x)\, dx$$

and

$$\mu_2 = E(y) = \int_0^\infty y\ f_y(y)\, dy$$

3.7.7 Operating characteristics of the system

3.7.7.1 Time to first C event

Let C be the random variable denoting the time to the first C event, then, T_C has p.d.f. given by

$$f_{T_C} = \psi_C(0,t) = f_x(t).$$

Thus

$$pr\{T_C > t\} = \int_t^\infty f_x(u)\, du.$$

The mean value of T_C is given by

$$\text{Mean time to crisis} = \int_0^\infty x\ f_x(x)\, dx.$$

3.7.7.2 Number of C events in the interval (0, t)

The first order product density for C events is given by

$$h_1(x) = \lim_{\Delta\to 0} \frac{1}{\Delta} E[N(x,\Delta)]$$

$$= \lim_{\Delta \to 0} \frac{1}{\Delta} \; pr\{N(x,\Delta)=1\} .$$

Where $N(x,\Delta)$ denotes the number of C events in the time interval $(x, x+\Delta)$. Hence

$$h_1(x) = \psi_C(0,x) = \int_0^x \; h_E(u)\, \psi_C(u, x-u)\, du$$

Therefore the expected number of C events in an arbitrary time interval $(0,t]$ is given by

$$E[N(0,t)] = \int_0^t h_1(x)\, dx$$

$$= \int_0^t \psi_C(0,x)\, dx + \int_0^t dx \int_0^x h_E(u) \psi_C(u, x-u)\, du$$

$$= \int_0^t f_X(x)\, dx$$

$$= \int_0^t dx \int_0^x h_E(u)\, f_X(x-u)\, du .$$

The expected duration of crisis is given by the expected value of the random variable Y, and

$$E(Y) = \int_0^\infty y \; f_Y(y)\, dy .$$

3.8 SPECIAL CASE

When $\alpha = \infty$, i.e when the busy period is large,

then

$$\pi_{00}(t) = 1$$

$$\pi_{01}(t) = 0 \qquad for \quad all \quad t \geq 0$$

and

$$f_{X,Y}(x,y) = \lambda \exp(-\lambda x) g(x+y) + \int_0^x h_{E_1}(u) \lambda \exp[-\lambda(x-u)] g(x+y-u)\, du .$$

where

$$h_{E_1}(t) = \sum_{n=1}^{\infty} \left[2 \int_0^t g(u) \exp(-\lambda u) \lambda \exp[-\lambda(t-u)] \exp[-\lambda(t-u)]\, du \right]^{(n)} .$$

Furthermore, the p.d.f of the random variable $X + Y$ is given by

$$f_{X+Y}(t) = g(t)[1-\exp(-\lambda t)] + \int_0^t du \int_0^u h_{E_1}(w)\lambda \exp[-\lambda(u-w)]g(t-w)\, dw$$

The marginal densities of X and Y are given by

$$f_x(x) = \lambda \exp(-\lambda x)\, \overline{G}(x) + \int_0^x h_{E_1}(u)\lambda \exp[-\lambda(x-u)]\overline{G}(x+y)\, du$$

and

$$f_y(y) = \int_0^\infty \lambda \exp(-\lambda x)\, g(x+y)\, dx + \int_0^\infty \int_0^x h_{E_1}(u)\lambda \exp[-\lambda(x-u)]g(x+y-u)\, du..$$

3.9 CONCLUSION

In this chapter, we derive the stationary rate of crisis for a manpower planning system. Confidence limits for a system steady state crisis are developed for the system. We also provide the numerical example to examine the effects of varying the system parameters that govern rates of attrition (λ), recruitment (μ), busy period (α), and lean periods (β), which gain some insight on the system performance measures. A non-Markovian model is studied for the above model in the last section. Important measures such as the amount of crisis, time taken to observe the first crisis and the expected number of crisis events observed within a specified period of time are calculated. These are all important tools for management to use to manage their organizations effectively and timely.

CHAPTER 4

STOCHASTIC STRUCRURES OF GRADED SIZE IN MANPOWER PLANNING SYSTEMS[2]

[2] A modified version of this chapter was presented at the IASTED conference Sept 11-13, 2006 in Gaborone Botswana. (The paper has been refereed and published in the proceedings)

4.1 INTRODUCTION

Graded manpower systems have been studied from different points of view by several researchers (see Bartholomew, 1973, 1982), Young and Vassiliou (1974), Vassiliou (1978), Bartholomew and Forbes (1979), Vassiliou and Gerontidis (1985), McClean (1991) and Vassiliou et al. (1990).

A particular aspect which has received much attention is the examination of moment structure of the state of these systems in terms of the proportion of staff in each grade; and the evaluation of recruitment and promotion policies for controlling them. In these works, the graded structure is analyzed with grade dependent promotion probabilities and the length of service is considered as an important criterion in determining the staff flows (see Morgan (1979), Vassiliou (1981), Leeson (1979, 1980, and 1982)). In a large number of manpower organizations such as a civil service, each grade is further subdivided into several categories for administration reasons.

These categories may be several departments or sections within grades or divisions consisting of persons who have completed zero years of service, one year of service, two years of service, etc. and promotions are considered at the end of each year for all the employees of a lower grade to higher grades. The proportion of promotion will be different for each category and hence dependent not only on the grade size but also on the category size. By varying the family of promotion probabilities, the structure of the system can be steered to a desired level. Further, for a given set of promotion probabilities, it is worthwhile to find the probability distribution of the state of the system.

In this chapter, an attempt is made to analyse the impact of category and grade dependent promotion probabilities on the grade structure of hierarchical manpower systems. To be specific, we consider multi-grade manpower systems in which each grade is subdivided into several categories according to length of service in that grade. The last category of each lower grade consists of persons who have completed a specified period of service in that grade and do not get promotion. An employee in a lower grade is eligible for promotion to the most junior category of the next higher grade and the probability of promotion is dependent on the grade and category of the employee. Un-promoted employee of the category of a lower grade will move to the next higher category of the grade in the next unit of time until he reaches the last category of the grade from where he is either promoted or leaves the system. The unit of time may be taken as a year. The movement to the system are allowed in the lowest category of the lowest grade. Wastages are allowed from any category of any grade and no demotions take place.

The organisation of this chapter is as follows: in section 4.2, the basic model is described and the assumptions and notation are provided. The probability distribution of the state of the system is defined in section 4.3. The expected time to reach the top most grade by a new entrant in the lowest grade are found in section 4.4. The recurrence relation for the moments of the grade sizes is derived in section 4.5. A numerical example is provided in section 4.6 to highlight the impact of category and grade dependency on the grade structure of a particular organisation.

4.2 ASSUMPTIONS AND NOTATION

1. There are L grades arranged in descending order of seniority, grade 1 representing the senior most and grade L, the junior most.
2. Each grade i is further subdivided into k_i+1 categories $C_j^i \quad j=1,\ldots k_i$.
3. The category consists of those employees who have completed j years of service in grade i .
4. The category $C_{k_i}^i$ consists of employees with k_i and more years of completed service in grade i.
5. Any employees of the ith grade can be promoted to the (i-1)st grade and they are put in the lowest category of the (i-1)st grade.
6. Each employee of the category $C_j^i \quad j=1,\ldots k_i$ has equal probability p_{ij} of promotion to the category C_0^{i-1}.
7. Promotions take place at the end of each year.

8. Recruitment is made only at the beginning of each year and is of fixed size R.
9. Wastages can occur from any category of any grade.
10. q_{ij}: the probability that an employee of the category C_j^i leaves the system.
11. $N(i, j, t)$: random variable denoting the number of employees in C_j^i at time t.
12. $\overline{N}(i, j, t)$: mean number of employees in C_j^i.
13. n_{ij}: the mean number of employees promoted from C_j^i.
14. l_{ij}: the mean number of employees who have left the system from C_j^i.
15. T: number of years required for an employee to reach the top most grade from his last time of entry into the system.
16. T_i: number of years required for an employee to reach the grade (i-1)st since the time of his entry into grade i.

4.3 THE PROBABILITY DISTRIBUTION OF THE STATE OF THE SYSTEM

Given the promotion and wastage probabilities, we proceed to determine the probability distribution of the state of the Markov system at any time t. For the sake of simplicity we assume that there are 4 grades arranged in descending order of seniority of which grade 1 is the senior-most and grade 4 is the junior most. The grade 1 consists of 2 categories, the grade 2 consists of 3 categories, the grade 3 consists of 4 categories and the grade 4 consists of 3 categories. We also assume that no promotions occur from the first category of each lower grade and no wastages occur from all the categories except the last category of each grade, that is,

$$p_{21} = 0.0, \quad p_{31} = 0.0, \quad p_{41} = 0.0$$
$$q_{11} = 0.0, \quad q_{21} = 0.0, \quad q_{22} = 0.0$$
$$q_{31} = 0.0, \quad q_{32} = 0.0, \quad q_{33} = 0.0$$
$$q_{41} = 0.0, \quad q_{42} = 0.0\,.$$

The system configuration and the promotion probabilities are given in Table 4.1.

Table 4.1 System description

Grade 1	0.0	0.0		
Grade 2	0.0	p_{22}	p_{23}	
Grade 3	0.0	p_{32}	p_{33}	p_{34}
Grade 4	0.0	p_{42}	p_{43}	

First, we note that, since a fixed size R of recruitment is made at the beginning of each year and that it is made only into category C_1^4, the probability distribution of N(4,1,t) is known for all time t. In fact, we have

$$P[N(4,1,t)=n]=\delta(n-R), \quad n=0,\ 1,\ 2,....;\ t=0,1,2,....$$

where δ_{ij} is the Kronecker delta function.

As initial condition, we have

$$P[N(i,\ j,0)=n]=0, \quad i\neq 4,\ j\neq 1; \quad N(4,1,0)=R.$$

Now, observing all the possible flows of staff starting from time t=0, we can obtain the state probabilities at any time t. For the purpose of illustrations, we do this for times t=1, t=2, t=3.

At time t=1, only the categories C_1^4, C_2^4 are occupied so that the others are empty. Hence, we have:

$$P[N(4,1,1)=n_1, N(4,2,1)=n_2\,|\,N(4,1,0)=i_1]=P[N(4,\ 1,\ 1)=n_1]\,\delta(n_2-i_1).$$

Next, at time t=2, only the categories $C_1^4, C_2^4, C_3^4, C_1^3$ are occupied and the others unoccupied. Hence, we have,

$$P\left[N(4,1,2)=n_1, N(4,2,2)=n_2, N(4,3,2)=n_3, N(3,1,2)=n_4 \mid N(4,1,1)=i_1, N(4,2,1)=i_2\right]$$

$$=P\left[N(4,1,2)=n_1\right]\delta(i_2-n_2)\sum_{n_{12}}^{i_2}\delta(n_3-i_2-n_{12})\,\delta(n_4-n_{12})\binom{i_2}{n_{12}}\,p_{42}^{n_{12}}(1-p_{42})^{i_2-n_{12}}.$$

In the same way, observing that at time $t=3$, the categories $C_1^4, C_2^4, C_3^4, C_1^3, C_2^3$ are occupied and the other categories are unoccupied, we have,

$$P\left[\begin{array}{c} N(4,1,3)=n_1,\ N(4,2,3)=n_2,\ N(4,3,3)=n_3,\ N(3,1,3)=n_4,\ N(3,2,3)=n_5 \mid \\ N(4,1,2)=i_1,\ N(4,2,2)=i_2,\ N(4,2,3)=i_3,\ N(3,1,2)=i_4 \end{array}\right]$$

$$=P\left[N(4,1,3)=n_1\right]\left\{\delta(n_2-i_1)\,\delta\left[n_3-(i_2-n_{42})-(n_3-n_{43}-l_{43})\right]\delta\left[n_4-(n_{42}-n_{43})\right]\delta(i_4-n_5)\right\}$$

$$*\left\{\sum_{n_{42}=0}^{i_2}\sum_{n_{43}=0}^{i_3}\sum_{l_{43}=0}^{i_3-n_{43}}\binom{i_2}{n_{42}}\binom{i_3}{n_{43}+l_{43}}\binom{n_{43}+l_{43}}{l_{43}}p_{42}^{n_{42}}\,p_{43}^{n_{43}}\,q_{43}^{l_{43}}(1-p_{42})^{i_2-n_{42}}\,(1-p_{43}-q_{43})^{i_3-n_{43}-l_{43}}\right\}.$$

Proceeding in this way, we can find all the conditional probabilities for all time t. Since P[N(4,1,0) $=i_1$] is known, all the state probabilities can be computed forward in time and till the probabilistic structure of the state of the manpower systems is completely determined.

4.4 EXPECTED TIME TO REACH THE TOP-MOST GRADE

Since we want to find the mean time to reach the top-most grade, we assume that the probability that an employee leaves a grade is zero, that is $q_{ij}=0$, $\forall\ i, j$. Also assume $p_{ij}=0$. Since C_j^i consists of those employees who have completed j years of service in grade i and the probability that he is promoted to C_0^{i-1} is p_{ij}, the probability that an employee is promoted to grade (i-1) after he has put in a service of j years in grade i is p_{ij}. Accordingly, the probability distribution of, the time spent in grade i is given by

$$P\left[T_i = 1\right] = 0$$

$$P[T_i = 2] = p_{i2}$$

$$P\left[T_i = m\right] = (1-p_{12})(1-p_{13}).....(1-p_{i(m-1)})p_{im} \qquad 2 \le m \le k_i;$$

$$P\left[T_i = k_i + m\right] = \prod_{l=2}^{k_i}(1-p_{il})(1-p_{ik_i})^{m-1}p_{ik_i} \qquad m = 1, 2, ... \quad .$$

Hence, we have

$$E(T_i) = \sum_{j=2}^{k_i}\prod_{m=2}^{j-1}(1-p_{im})\, p_{ij} + \left\{\prod_{m=2}^{k_i}(1`-p_{im})\right\}(1-p_{ik_i})\left(k + \frac{1}{p_{ik_i}}\right) \quad .$$

The mean time to reach the top-most grade is given by

$$E(T) = \sum_{i=2}^{L} E(T_i) \; .$$

We find the mean number of years an employee has to remain in a grade before being promoted to the next grade for two different sets of promotion probabilities and present the results in Tables 4.4 and 4.6 .

4.5 MOMENTS OF THE GRADED SIZES

The stochastic process describing the behavior of the system is a Markov chain on the state space

$$E = \{(i,\, j), \quad i = 1, 2,......L; \quad j = 1, 2,....k_i,.....L+1\}$$

where L+1 represents the state to which employees are leaving the system. Let the transition probability matrix P be defined by

$$P\,[(\,l\,, m\,)\,/(\,i\,,\; j\,)] \quad \text{where} \quad P\,[(\,l\,, m\,)\,/(\,i\,,\; j\,)]$$

represents the probability that an individual found in state (i, j) at time t moves on to the state (l, m) in time t+1, for all t. Then we have;

$$P[(i-1,0)\,|\,(i,\ j)] = p_{ij}$$

$$P[i,\ j+1)\,|\,(i,\ j)] = 1-\ p_{ij}$$

$$P\left[(i,k_i)\,|\,(i,k_i)\right] = 1 - p_{ik_i} - q_{ik_i}$$

$$P[L+1)\,|\,(i,k_i)] = q_{ik_i}$$

and

$$P\,[(l,m)\,|\,(i,\ j)] = 0\,,\ \text{for all other values.}$$

Let R (t) denotes the vector corresponding to the recruitment. Since recruitment are allowed only in the category C_0^L and is a constant R for all t, we have all the elements of R(t) as zero except the term corresponding to C_0^L. Then the expected number in the system at time t is given by the recursive equation (Bartholomew, 1967).

$$\overline{N}(t+1) = \overline{N}(t)P + R(t+1)r$$

where

$\overline{N}(t+1)$ is the expected number of employees in the system in the i[th] state at time t+1,

P is the transition matrix whose element p_{ij} is the probability of a move from state i to j in any time interval, $R(T+1)$ is the number of recruits at time $T+1$ and $r=(r_1, r_2,...r_k)$ is the recruitment vector.

4.6 NUMERICAL EXAMPLE

Some numerical examples have been carried out of this model. Tables 4.2 to 4.5 give different scenarios for promotion probabilities to each category within grades, for instance an employee would move from grade 4 categories 6 to grade 3 categories 0 with probability 0.3. Tables 4.4 and 4.6 give the average time it takes for an individual to move from one grade to another. Tables 4.7 to 4.10 indicate the number of employees leaving the system within each category of the grades given that 40 people were recruited each year while Tables 4.11 to 4.14 give the corresponding scenario while organization started with 80 recruits. It is observed that if promotions are time dependent it will take an employee about 36 years to reach the topmost grade, whereas if promotion is based on efficiency it will take only 14.7 years to reach the top most grade.

Table 4.2a Transition probabilities within and between grades

Grade 2	0.0 0.2 0.4 0.5 0.6 0.8 0.9 0.8
Grade 3	0.0 0.2 0.4 0.6 0.7 0.8 0.9 0.8
Grade 4	0.0 0.2 0.3 0.4 0.8 0.3
Grade 5	0.0 0.2 0.4 0.5

The non-zero leaving probabilities are given below:

Table 4.2b: probability of leaving wastage

Grade 1	**Grade 2**	**Grade 3**	**Grade 4**	**Grade 5**
1.00	0.80	0.70	0.60	0.50

With the above probabilities and R=40, we have obtained the expected numbers of employees who will leave the system in the various categories of the grades at different times $t = 6$, $t = 11$, $t = 16$, $t = 21$ and present them respectively in Tables 4.7 to 4.10 without changing the promotion and wastage probabilities, if we change only the recruitment size as R=80, we observe that for the same time points all the mean numbers are almost doubled and this fact is exhibited in Tables 4.11 to 4.14.

Table 4.3a: Promotion probabilities

Grade 2	0.0 0.2 0.4 0.5 0.6 0.8 0.9 0.8
Grade 3	0.0 0.2 0.4 0.6 0.7 0.8 0.9 0.8
Grade 4	0.0 0.2 0.3 0.4 0.8 0.3
Grade 5	0.0 0.2 0.4 0.5

Non-zero leaving probabilities

Table 4.3 b Probability of leaving through wastage

Grade 1	Grade 2	Grade 3	Grade 4	Grade 5
1.00	0.80	0.70	0.60	0.50

Table 4.4: Mean time to reach grades

From grade	To Grade	Mean time
5	4	3.7
4	3	3.9
3	2	3.5
2	1	3.6

The mean time to reach the top-grade is 14.7 years.

Table 4.5a: Promotion probabilities

Grade 2	0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.2
Grade 3	0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.4
Grade 4	0.0 0.0 0.0 0.0 0.0 0.3
Grade 5	0.0 0.0 0.0 0.0 0.3

In Table 4.5a promotions are allowed only when an employee reaches the top category of each grade and Table 4.5b gives the non-zero leaving probabilities.

Table 4.5b: Probability of leaving through wastage

Grade 1	Grade 2	Grade 3	Grade 4	Grade 5
1.00	0.80	0.70	0.60	0.50

Table 4.6: Mean Time to reach grades

From grade	To Grade	Mean time
5	4	6.3
4	3	8.3
3	2	9.5
2	1	12.0

The mean-time to reach the top-grade is 36.1 years.

Table 4.7: Time =6years

Grade 1	00 00 00 00
Grade 2	00 00 00 00 00 00 00 00
Grade 3	02 00 00 00 00 00 00 00
Grade 4	30 21 06 00 00 00 00
Grade 5	40 40 32 19

The expected number of employees leaving the system is 10.

Table 4.8: Time = 11years

Grade 1	18 13 08 00
Grade 2	29 28 21 11 00 04 01 00
Grade 3	29 29 23 14 06 02 00 0 0
Grade 4	30 30 24 17 10 02
Grade 5	40 40 32 19

The expected number of employees leaving the system=11

Table 4.9: Time = 16years

Grade 1	29 29 29 00
Grade 2	29 29 23 14 07 03 01 00
Grade 3	29 29 23 14 06 02 00 00
Grade 4	30 30 24 17 10 02
Grade 5	40 40 32 19

The expected number of employees leaving the system =11

Table 4.10: Time = 21years

Grade 1	29 29 29 00
Grade 2	29 29 23 14 07 03 01 00
Grade 3	29 29 23 14 06 02 00 00
Grade 4	30 30 24 17 10 02
Grade 5	40 40 32 19

The expected number of employees leaving the system=11.

Table 4.11: Time = 6years

Grade 1	00 00 00 00
Grade 2	00 00 00 00 00 00 00 00
Grade 3	03 00 00 00 00 00 00 00
Grade 4	61 42 13 00 00 00
Grade 5	80 80 64 38

The expected number of employees leaving the system=19

Table 4.12: Time = 11years

Grade 1	36 26 16 00
Grade 2	57 56 41 21 08 02 00 00
Grade 3	58 58 46 28 11 03 00 00
Grade 4	61 61 49 34 20 04
Grade 5	80 80 34 38

The expected number of employees leaving the system=22

Table 4.13: Time = 16years

Grade 1	58 58 58 00
Grade 2	58 58 46 28 14 06 01 00
Grade 3	58 58 46 28 11 03 01 00
Grade 4	61 61 49 34 20 05
Grade 5	80 80 64 38

The expected number of employees leaving the system=22

Table 4.14: Time = 21years

Grade 1	58 58 58 00
Grade 2	58 58 46 28 14 06 01 00
Grade 3	58 58 46 28 11 03 01 00
Grade 4	61 61 49 34 20 05
Grade 5	80 80 64 38

The expected number of employees leaving the system=22

4.7 CONCLUSION

This chapter has presented a method for analyzing the impact of category and grade dependent probabilities on grade structure of a hierarchical manpower system, under certain assumptions. The probability distribution of the expected time spent in a grade is derived. Numerical examples indicate that doubling the recruitment size from 40 to 80 employees leads to the mean numbers leaving to be almost double in each category and grade. Restricting promotions within categories also lead to long waiting times to reach the top.

CHAPTER 5

ANALYSIS OF OPTIMAL PROMOTION POLICY FOR A MANPOWER SYSTEM BY A QUEUEING APPROACH[3]

[3] A modified version of this chapter is published in Management Dynamics, Vol . 15, No. 2 (2006).

5.1 INTRODUCTION

In the competitive world of today which is characterized by a large number of qualified persons, manpower planning draws the serious attention of researchers engaged in this field, since each organization requires employees with specialized skills in various fields to accomplish its business objectives, both now and in the future. Through manpower planning the management of any organization not only optimizes the expertise and skills of its human resources, but may also select the optimal number and correct type of employees available at the right place at the right time.

Determining manpower planning policies is one of the most critical and difficult aspects of an organization. In particular, after the recruitment, determining promotion policies from one grade to another becomes more difficult as the organization requires more expertise since it is linked to the productivity enhancement of the organization.

Various models applicable to manpower planning have been developed and studied in the past by many well-known researchers such as Marshall and Olkin (1967), Smith (1970), Bartholomew (1971), and Forbes (1971). Moreover, there are special features associated with the methods and models relevant to manpower systems, which arise in various fields.

Considering recruitment and promotion as some of the main activities of the organization, Vajda (1975) discussed the mathematical aspect of manpower planning. The concepts of linear programming are used to develop a graded population structure where both the recruitment rates and transfer rates between the various grades are controlled by management. Davies (1975) discussed the maintainability structures in Markov chain models under recruitment control. Leeson (1984) considered the recruitment policies and their effects on internal structures. Recruitment control refers to an effective control of recruitment policies to obtain an optimal supply of recruits for a system at any time. Generally recruitment levels are connected with wastage and promotions in a system as well as the desired growth of the system, hence controlling recruitment policies may help attain the desired structure, which could be maintained over time.

Kalamatianou (1987) obtained an attainable and maintainable grade structure in Markov manpower system with pressure in grades. Furthermore, the work of Vassiliou (1976) and Leeson (1982) determines the wastage and promotion rates required to bring about any desired future personnel structure. Grinold (1976) placed emphasis on uncertain requirements. The main purpose was to provide a framework to regulate the supply of adequately qualified employees for naval aviation. Sathiyamoorty (1980) discussed a cumulative damage model of manpower planning with correlated inter-arrival times of shocks. Rao (1990) proposed a dynamic programming approach to determine optimal recruitment policies. A bivariate model under efficiency and seniority embedded with stochastic theory was studied by Raghvendra (1991).

Young and Vassiliou (1974) have considered a non-linear model for the promotion of staff. In particular, a stochastic model of promotion based on an ecological principle, which states that promotions should be proportional to the number of skilled employees available for promotion and the number of vacancies for promotion was proposed. Subramanian (1996a, 1996b) developed an optimal policy for time bound promotion in a hierarchical manpower system and a model on optimum promotion rate. Sathiyamoorty and Elangovan (1997, 1998, 1999) studied an optimal recruitment policy for training prior to placement. A semi-Markov model of a manpower system was studied by Yadavalli and Natarajan (2001) with the interest focused on the total number of vacancies available in the entire organization. Recently a study on training dependent promotions and wastage was also carried out by Yadavalli et al. (2002b).

Gross and Harris (1974) and Takacs (1960) have presented basic concepts of various queuing models. Further, queuing and inventory concepts are grouped as interdisciplinary subjects by Morse (1958) and applied to manpower planning problems by Yadavalli et al. (2005). Mishra and Pal (2003) have discussed the computational approach to the M/M/1/N interdependent queuing model. Further Mishra and Mishra (2004) evaluated the total optimal cost of the machine interference model as an important performance measure of the system. Very recently, Rajalakshmi and Jeeva (2003), Jeeva, Rajalakshmi, Charles and Yadavalli (2004) discussed stochastic programming in cluster based optimum allocation of recruitment.

Thus a close review of the aforesaid publications on manpower planning reveals that so far many aspects and approaches have been discussed in various literature sources pertaining to manpower planning. However, these models are of no use, as long as they cannot be converted into effective tools usable within organizations.

In this chapter a fresh attempt has been made to analyze the promotion policy component of manpower planning by mapping the system to a queuing model, where we describe employees eligibility for promotion by a Poisson arrival and lengths of waiting for promotion are modeled using an Erlang distribution. The optimal promotion policy and total optimal cost of the system for promotion have been computed. To highlight the importance of the model, a hypothetical example is used for illustration.

5.2 THE DESCRIPTION OF THE MODEL

We consider an $M/E_K/1:\infty/FIFO$ queuing model with Markovian input and Erlangian service having k phases. In this model, it is assumed that the employees in grade i become eligible at a rate, which is randomly distributed according to a Poisson distribution and employees proceed to be serviced on a first come, first out basis (FIFO). Let the mean value of the rate be λ_i. It is further assumed that the interval between two consecutive instances of a vacancy arising in grade $(i+1)$ is exponentially distributed such that the expected number of vacancies arising during unit time is μ_i with the traffic intensity $\frac{\lambda_i}{\mu_i} < 1$. This is a very restrictive assumption since $\lambda_i < \mu_i$ it is meant to control the queue size otherwise the queue built up could be infinite. The promotion time (service time) distribution is assumed to be an Erlangian distribution with mean $\frac{1}{k\mu}$ where μ is the parameter of the exponential distribution. A single service channel is operated and there is no limit placed on the number of employees applying for promotion.

The employees applying for promotion are kept on the waiting list and considered for promotion as and when vacancies arise. Thus the manpower system is mapped onto a queuing system and studied.

Let c_0 be the fixed cost of promotion, which is incurred as the establishment cost per unit of time for any organization, c_1 be the promotion cost (service cost) per unit per unit time and c_2 be the holding/waiting cost per unit per unit time for the model. Since eligibility (arrivals) follows a random distribution, fluctuations will occur in the expected queue length for the promotion in the manpower planning system. On the part of the management (policy makers of the organization), since the exact number of persons applying for the promotion are not known, this state of indecision hampers and further delays the promotion policy of the organization. Consequently, the productivity of the organization is affected. Let c_3 be per unit cost per phase associated with a hamper- situation and be known as the hamper cost per unit of the fluctuations in the expected queue length of the system.

The total expected queue length of the system, average number of phases and per phase fluctuations in the system are obtained as follows.

Expected queue length in the system (Ls) $= \frac{(k+1)\lambda}{2k\mu(\mu-\lambda)}$

Average number of phases $= \frac{k(k+1)\rho}{2(1-k\rho)}$, $\rho = \lambda / k\mu$

Per phase fluctuations in the queue length of the system by

$$\sum_{n=0}^{\infty}(n-L)^2 P_n$$

$$=(1-\rho)\sum_{n=0}^{\infty} n^2 \rho^n - L^2$$

where L is the expected number of employees in the system, P_n is the probability of finding n employees in the system and $n = 0, 1, \ldots \infty$.

For the model developed here, three phases $(k = 3)$ are considered. The first phase is used for basic screening such as minimum time (minimum number of years of service put in), minimum qualification and required training for promotion, the second phase for evaluation of the performance towards target and quality achievement and the third and the final phase is considered for interviewing the staff. Therefore, for purposes of evaluating the model $k = 3$ will be assumed in the next section of this chapter.

5.3 THE ANALYSIS OF THE MODEL

The total cost incurred by the organization for implementing the promotion policy consists of the sum of the fixed cost of promotion, the promotion cost, the cost of waiting for a vacancy to be created multiplied by the average number of phases and the hamper cost per unit multiplied by per phase variability in the queue length of the system.

The cost function as total optimal cost (TOC) is constructed as follows:

$$TOC = c_0 + c_1\mu + c_2 \frac{(k+1)\lambda}{2k\mu(\mu-\lambda)} + c_3 \frac{k(k+1)\rho}{2(1-k\rho)}\left((1-\rho)\sum_{n=0}^{\infty} n^2 \rho^n - L^2\right)$$

After simplification, (see Gross and Harris, 1974) the above expression reduces to

$$TOC = c_0 + c_1\mu + c_2 \frac{(k+1)\lambda}{2k\mu(\mu-\lambda)} + c_3 \frac{\mu k(k+1)\rho^2}{2(\mu-\lambda)(1-\rho)^2}$$

Let

$$TOC = c_0 + A_1 + A_2 + A_3 \tag{5.1}$$

where,

$A_1 = c_1\mu$,

$A_2 = c_2 \frac{(k+1)\lambda}{2k\mu(\mu-\lambda)}$,

and $A_3 = c_3 \frac{\mu k(k+1)\lambda^2}{2(\mu-\lambda)(k\mu-\lambda)^2}$

For the optimum promotion policy (μ), equation (5.1) yields a non-linear equation in μ after taking the first derivative of the same, which is solved by making use of the fast converging Newton-Raphson method and developing a program in C language.

5.4 NUMERICAL ILLUSTRATION AND DISCUSION OF THE RESULTS

In the numerical illustration, since the model under consideration is studied for the steady state, the costs of the model are considered to vary in such a way that at least one cost must be contradictory to other costs. This is a basic requirement for the formation of the queue. Moreover, the selection of the arrival rate is also considered as per the steady state condition, that is $\lambda < 3\mu$. If the aforesaid conditions are violated, then the model shows erroneous output by giving a negative total optimal cost of the system, which is never possible. In Table 5.1, it is assumed that c_0 is fixed and is taken as a constant value. The table illustrates the optimal promotion policy (μ*) and the total optimal cost of the manpower system for the promotion. Starred values of parameters in the row 9 of Table 5.1 show the optimal promotion and total optimal cost of the system corresponding to various parameters.

Table 5.1: Relationship between TOC and optimal promotion policy, μ when c_0 is fixed

λ	c_0 (Dollars)	c_1 (Dollars)	c_2 (Dollars)	c_3 (Dollars)	k	μ*	TOC (Dollars)
1	700	50	25	15	3	8.95	1147.89
2	700	53	24	14	3	8.91	1173.46
3	700	67	23	13	3	8.89	1298.40
4	700	69	22	12	3	8.89	1318.83
5	700	74	21	11	3	8.85	1365.12
6	700	77	20	10	3	8.8	1397.16
7	700	81	19	9	3	8.75	1444.76
8	700	88	18	8	3	8.78	1589.46
9*	700*	90*	17*	7*	3*	8.81*	912.73*
10	700	92	16	6	3	8.81	1401.61
11	700	100	15	5	3	8.83	1515.70
12	700	103	14	4	3	8.87	1563.69
13	700	107	13	3	3	8.9	1614.03
14	700	112	12	2	3	8.9	1668.88
15	700	121	11	1	3	8.92	1762.97

Further, assuming that the promotion cost c_1 to be constant, which sometimes happens to the organization when it has budgetary constraints, then the resultant trend between the different costs and total optimal cost are shown in Table 5.2 below.

Table 5.2: Relationship between TOC and optimal promotion policy μ, when both c_0 and c_1 are fixed

λ	c_0 (Dollars)	c_1 (Dollars)	c_2 (Dollars)	c_3 (Dollars)	k	μ*	TOC (Dollars)
1	700	177	19	5	3	8.95	2285.04
2	700	177	42	8	3	8.91	2290.29
3	700	177	76	10	3	8.89	1018.46
4	700	177	80	15	3	8.89	2106.30
5	700	177	91	23	3	8.85	1940.41
6	700	177	111	28	3	8.8	1537.73

In Table 5.3, it is assumed that waiting and hamper costs are constant while assessing the change in the total optimal cost with the change in the promotion cost.

Table 5.3: Relationship between TOC and optimal promotion policy μ, when only c_1 is allowed to vary

λ	c_0 (Dollars)	c_1 (Dollars)	c_2 (Dollars)	c_3 (Dollars)	k	μ*	TOC (Dollars)
1	700	207	47	104	3	8.95	2554.14
2	700	194	47	104	3	8.91	2434.82
3	700	189	47	104	3	8.89	2397.13
4	700	175	47	104	3	8.89	2293.95
5	700	142	47	104	3	8.85	2038.51
6	700	129	47	104	3	8.8	2012.48

In Table 5.4 we looked at the special case when $\lambda = \mu$. In this the case employee's eligibility for the job and the expected number of vacancies that arise occur at the same rate. An analytic expression for the case is given in the appendix A. We notice that the optimal policy is achieved when $\lambda = \mu = 1$.

Table 5.4: Relationship between TOC and optimal promotion policy when $\lambda=\mu$

λ	c_0 (Dollars)	c_1 (Dollars)	c_2 (Dollars)	c_3 (Dollars)	k	μ^*	TOC (Dollars)
1	700	50	25	15	3	1	772.50
2	700	53	24	14	3	2	827.00
3	700	67	23	13	3	3	920.50
4	700	69	22	12	3	4	994.00
5	700	74	21	11	3	5	1086.50
6	700	77	20	10	3	6	1177.00
7	700	81	19	9	3	7	1280.50
8	700	88	18	8	3	8	1416.00
9	700	90	17	7	3	9	1520.50
10	700	92	16	6	3	10	1629.00
11	700	100	15	5	3	11	1807.50
12	700	103	14	4	3	12	1942.00
13	700	107	13	3	3	13	2095.50
14	700	112	12	2	3	14	2271.00
15	700	121	11	1	3	15	2516.50

5.5 CONCLUSION

While analyzing the variation over different parameters in Table 5.1, it is interesting to note that when c_0 is fixed and the other two costs which are in contravention to each other are varying, the values of the optimal promotion policy and total optimal cost of the promotion are obtained and this trend of variation in various parameters is worth noticing in an organization.

In Table 5.2 where c_0 and c_1 are fixed and other costs are varying, it is noticeable that the variation in the total optimal cost is significant. Table 5.3 shows significant variation in TOC when c_0, c_2 and c_3 are fixed.

Manpower planning is about ensuring that the right types of employees are available at the right place at the right time. The success of the manpower planning is paramount to the survival of the organization and the complexities associated with the planning process and environment. Quantitative techniques such as queuing theory applied in this study can enhance problem-solving abilities and hence improve decision-making effectiveness of an organization.

The most practical implication is that of controlling the internal structure through hiring, promotions, internal transfers, redundancies and retirement planning. The problem is to precisely plan and control these interrelated organizational activities in order to achieve a stable organization capable of meeting its objectives.

Application of manpower planning techniques means organizational effectives, i.e. it may maximize the overall effectiveness of promotion policies to retain the best skilled employees. As a result of using this model and trying alternative manpower policies one can discover and explore the cost performance that exists. The following studies give application of manpower planning techniques in different organizational problems (Meehan and Ahmed (1990); Gass et al. (1988); Andrew and Abodunde (1977); Leeson (1982); Gorunescu, McClean and Millard (2002)).

Lastly, management may implement the human resource planning models in their functional areas of business to develop policies on recruitment and selections, training and development, hiring, promotion and retention benefits to foster the spirit of organizational citizenship.

CHAPTER 6

LIFE TABLE TECHNIQUES IN THE ANALYSIS OF ATTRITION IN A MANPOWER SYSTEM WITH REFERENCE TO HIGHER EDUCATIONAL INSTITUTIONS[4]

[4] A modified version of this chapter was presented at the 'SAIMS' conference Sept 13-15, 2006 in Stellenbosch, South Africa. (The paper has been refereed and published in the proceedings)

6.1 INTRODUCTION

Various stochastic models of manpower systems have been studied in the past (Yadavalli & Natarajan (2001); Yadavalli et al. (2002); Yadavalli et al. (2005)). Several studies have shown that socially-valued and demographic factors such as income, length of service, age, sex, marital status and the general conditions of service have a significant contribution on an individuals attrition (see Lane and Andrew (1955), Bartholomew (1959; 1971), Young (1971)). An earlier study by Wolfbein was not only to show the relevance of demographic factors but adopted the technique of life table to a measurement of working life span.

In this chapter we focus on educational qualification as a primary contributor to attrition and employ the life table technique to analyse the wastage and attrition rates of staff of an Educational Institution. In particular we analyse the length of service expectation and survival rates of staff using the terminology of demography as a matter of convenience.

A life table gives mortality rate and expectation of life of the population with different ages. It is mostly employed by life insurance companies to determine premiums to be set for life insurance and for determining rate of disability and retirement benefits, etc. It is also used in other fields such as demography and public health to study population growth, patient survivorship after diagnosis, and length of widowhood as well as married life. Life table is a convenient method for summarizing the mortality experience of any population. It particularly it provides a comprehensive and concise measure of longevity of that population. A life table is quite useful to a business organization attempting to assess its health benefits liabilities for both current workers and future retirees (Pol and Thomas, 1997).

In this particular problem we use a life table to calculate the survivorship of a cohort of employees in an educational institution before they could leave the job. Institutions of higher education are experiencing major problems of recruiting and retaining expertise and knowledge base due to competitiveness. This coupled with high costs of recruitment and time taken to search for people with skill has great effects on the institution budgets and development. It is, therefore, of considerable importance to institution planners to determine the likelihood of leaving and the distribution of the staff length of service in order to better understand the complex phenomena of institution staff movement and wastage rates. Wastage or attrition are used in the place of death and completed length of service [CLS] in the place of age.

This life table gives a summary of wastage or attrition of manpower of a cohort during an interval of their service. It will provide extensive information about the impact of wastage on service life expectancy and show any trend in wastage.

6.2 NOTATION AND TERMINOLOGY

This section defines the basic life table functions, shows how life tables can be calculated and the relationships between them.

i : Exact number of years of service [i – integer]

n :length of interval

l_i : Number of persons with i completed years of service

${}_nd_i$: Number of wastages while passing from i and [$i+n$] years of service

n : Width of classes defined by length of service, $n = 1, 2......, k$

${}_nq_i$: Probability of leaving the job between i and [$i+n$] years of service following the attainment of length i

${}_np_i$: Probability of continuing in the service between i and [$i+n$] years of service

nL_i : Persons years serviced by the cohort between i to [$i+n$] years

T_i : Total persons-years serviced by the cohort from i years of service

e_i^0 : Expected length of service in years left from the year of service

CLS : Completed length (in years) of service

${}_nm_i$: The attrition rate for the cohort between i to [$i+n$] years

$G(i)$: is the probability of one not facing attrition until he reaches the ith year of service.

Relation between life table functions:

$$ {}_nq_i = \frac{d_i}{l_i} $$

$$ {}_nd_i = l_i - l_{i+n} \quad \text{for all } n=1, 2, 3, \ldots k; $$

i=1, 2, 3…k_1: k, k_1 finite and $k_1 \geq k$

$$ p_i = 1 - q_i $$

$$ {}_nm_i = \frac{{}_nd_i}{{}_nL_i} $$ is the central rate of attrition

Where $$ {}_nL_i = \int_0^n l_{i+t}\, dt \approx [n/2][l_i + l_{i+n}] $$

$$ T_i = L_i + L_{i+1} + \ldots\ldots\ldots + L_{i+n} = \int_0^\infty l_{i+t}\, dt $$

$$ e_i^0 = \frac{T_i}{l_i} $$

$$ G(i) = \frac{l_{i+n}}{l_i} $$

6.3 SYSTEM DESCRIPTION

1. We consider a cohort of persons who joined the service from the inception of the Educational institution and study only their wastage rates.
2. Minimum qualification required to work in the institution is post graduate.
3. Maximum length of service a person can put in the institution is 30 years.
4. In this approach the rates are calculated for classes defined by length of service (see Tables 6.1, 6.2, 6.3 and 6.4).
5. We assume that there are no significant differences in attrition between males and females.

6.4 STRUCTURE OF THE TABLES OF LENGTH OF SERVICE

Perhaps the most natural way of collecting data to investigate the pattern of wastage is to observe homogeneous groups of entrants and note how long each remains in the organization before leaving. Such a group, joining at about the same time is known as a cohort. We employ the cohort life table as it presents a historical record of what actually happened to the recruits. By recording their service lengths, one would know how many survived attrition to attain a certain length of service, the probability of not leaving until the i th year of service, the wastage/attrition rates and expected length of service. As leaving is a process which can occur virtually at any time in a person's career, it is reasonable to treat completed length of service as a continuous variable.

A conventional life table starts with an initial group of 100 000 at birth and follows it through life, subject to a pattern of mortality (Shryock, Siegel and. Associates, 1954). Since the focus here is on the span of service duration, the life table starts with the completed length of service since the inception of the institution or since year zero and follows it through life, subject to a pattern of attrition determined by a specified set of mortality rate. In this note we give importance to an Educational Institutions where people working have different qualifications including Postgraduate [P.G.], Master of Philosophy [M.Phil.], Master of Science [M.Sc], Doctorate of Philosophy [Ph.D.]. We consider persons who leave the institution as wastage or attrition at various stages of completed length of service with different qualifications and present the results in Tables 6.1-6.6.

Table 6.1: Structure of a Life Table for staff with PhD qualification

Exact number of Years of Service	No. of persons with exact no. of completed years of service	No. of persons leaving the job between i to i+n years of service	Probability of leaving the job between i to i+n years of service	Average person years service by the cohort between i to i+n years	Total person years serviced by the cohort from i years of service	Expected length of service in years left from the year of service	Probability of a person will not face any attrition till ith year of service	Hazard rate of leaving the job after a given CLS
i-(i+n)	l_i	${}_nd_i$	${}_nq_i$	${}_nL_i$	T_i	e_i^o	G(i)	h(i)
0-1	33	9	0.2727	28.50	92.00	2.79	0.7273	---
1-2	24	8	0.3333	20.00	63.50	2.65	0.4849	0.0088
2-3	16	8	0.5000	12.00	43.50	2.72	0.2424	0.0198
3-4	8	5	0.6250	4.50	31.50	3.94	0.0909	0.0496
4-8	1	0	0.0000	22.50	27.00	27.00	0.0909	0.0000
8-10	1	1	1.0000	2.00	4.50	4.50	0.0000	0.0304
10-15	0	0	0.0000	0.00	0.00	0.00	0.0000	0.0000
15-20	0	0	0.0000	0.00	0.00	0.00	0.0000	0.0000
20-30	0	0	0.0000	0.00	0.00	0.00	0.0000	0.0000

Table 6.2: Structure of a Life Table for staff with M. Phil./M.Sc qualification

Exact number of Years of Service	No. of persons with exact no. of completed years of service	No. of persons leaving the job between i to i+n years of service	Probability of leaving the job between i to i+n years of service	Average person years service by the cohort between i to i+n years	Total person years serviced by the cohort from i years of service	Expected length of service in years left from the year of service	Probability of a person will not face any attrition till ith year of service	Hazard rate of leaving the job after a given CLS
i-(i+n)	l_i	${}_nd_i$	${}_nq_i$	${}_nL_i$	T_i	e_i^o	G(i)	h(i)
0-1	64	10	0.1563	59.00	916.50	14.00	0.8437	---
1-2	54	9	0.1667	49.50	857.50	15.88	0.7030	0.0038
2-3	45	9	0.2000	40.50	808.00	17.96	0.5624	0.0055
3-4	36	3	0.0833	33.50	767.50	21.32	0.5156	0.0028
4-8	31	3	0.0968	167.50	734.00	23.68	0.4657	0.0036
8-10	28	2	0.0714	59.00	566.50	20.23	0.4324	0.0029
10-15	26	1	0.0385	135.00	507.50	19.52	0.4158	0.0017
15-20	25	1	0.0400	127.50	372.50	14.90	0.3990	0.0018
20-30	24	24	1.00	245.00	245.00	10.21	0.0000	0.0480

Table 6.3: Structure of a Life Table for staff with P.G. (Honors) qualification

Exact number. of Years of Service	No. of persons with exact no. of completed years of service	No. of persons leaving the job between i to i+n years of service	Probability of leaving the job between i to i+n years of service	Average person years service by the cohort between i to i+n years	Total person years serviced by the cohort from i years of service	Expected length of service in years left from the year of service	Probability of a person will not face any attrition till ith year of service	Hazard rate of leaving the job after a given CLS
i-(i+n)	l_i	${}_nd_i$	${}_nq_i$	${}_nL_i$	T_i	e_i^o	G(i)	h(i)
0-1	143	8	0.0559	139.00	3685.5	25.78	0.9941	---
1-2	135	7	0.0519	131.50	3546.5	26.27	0.8951	0.0011
2-3	128	5	0.0391	125.50	3415.0	26.68	0.8547	0.0008
3-4	123	2	0.0163	121.00	3289.5	26.74	0.8407	0.0004
4-8	119	2	0.0168	605.00	3168.5	26.63	0.8266	0.0004
8-10	117	0	0.0000	236.00	2563.5	21.91	0.8266	0.0000
10-15	117	1	0.0085	585.00	2327.5	19.89	0.8195	0.0019
15-20	116	0	0.0000	582.50	1742.5	15.02	0.8195	0.0000
20-30	116	116	1.0000	1160.00	1160.0	10.00	0.0000	0.0235

Table 6.4: Structure of a Life Table for all staff of the institution

Exact No. of Years of Service	No. of persons with exact no. of completed years of service	No. of persons leaving the job between i to i+n years of service	Probability of leaving the job between i to i+n years of service	Average person years service by the cohort between i to i+n years	Total person years serviced by the cohort from i years of service	Expected length of service in years left from the year of service	Probability of a person will not face any attrition till ith year of service	Hazard rate of leaving the job after a given CLS
i-(i+n)	l_i	${}_nd_i$	${}_nq_i$	${}_nL_i$	T_i	e_i^o	G(i)	h(i)
0-1	240	27	0.1125	226.50	4694.0	19.56	0.8875	---
1-2	213	24	0.1127	201.00	4467.5	20.97	0.7875	0.0024
2-3	189	22	0.1164	178.00	4266.5	22.57	0.6958	0.0028
3-4	167	10	0.0599	159.00	4088.5	24.48	0.6541	0.0018
4-8	151	5	0.0331	795.00	3929.5	26.02	0.6325	0.0010
8-10	146	3	0.0205	297.00	3134.5	21.47	0.6195	0.0006
10-15	143	2	0.0140	722.50	2837.5	19.84	0.6108	0.0004
15-20	141	1	0.0071	710.00	2115.0	15.00	0.6065	0.0002
20-30	140	140	1.0000	1405.00	1405.0	10.04	0.0000	0.0317

6.5 SURVIVAL AND HAZARD RATES

We consider the survival rates of employees in the system as well as the hazard rates of leaving employment after completing a certain length of service in the job. Completed lengths of service are best described by duration models. Defining a duration model precisely requires a time origin, a time scale and a precision definition of the event ending the duration. In a manpower system, different individuals will often have different time origins for the duration of their employment. In practice one would like individuals in the study to be as homogeneous as possible, after controlling for observable differences.

Survival rates and hazard rates are useful for completed lengths of service analysis. Survival Rate $G(i)$ is defined as the probability that a person will not face any attrition till "i" years of service. For instance, the probability that a person with a Ph.D qualification will not leave the service till 5 years of service is

$$[1-{}_nq_0][1-{}_nq_1][1-{}_nq_2][1-{}_nq_3][1-{}_nq_4];$$

therefore from Table 6.1, G [4] = 0.0909. This measure shows that survival rate of highly qualified person within the institution is least as compared to those with Masters Degree and Honours.

Survival ratios use the life table to calculate the proportion of persons surviving attrition between i and $i+n$ years of service. These ratios can be used to determine the percentage of persons in the systems at a particular point in time who can be expected to still be in the system at some point in the future. The survival ratio from Table 6.4, for the service length 4-8 years surviving attrition to the service length 8-10 years is

$$\text{Survival ratio} = \frac{{}_{10}L_8}{{}_8L_4} = \frac{297}{795} = 0.374\,.$$

Thus approximately 37.4 percent of the persons who were in the system after serving 4 years will still be with the institution after 8 years. This institution clearly undergoes significant attrition as only a few recruits will be in the system after rendering their 8 years of service.

Similarly the Hazard Rate $h(i)$ of leaving the job after 'i' years of service is the conditional probability of leaving the job in a unit time given that the person has not left the job till then. The hazard rate $h[10]$ in the case of staff of the institution overall is

$$h[10] = \left\{ \frac{[1-{}_n q_0]}{52} \right\} / \{[1-{}_n q_0][1-{}_n q_1][1-{}_n q_2][1-{}_n q_3][1-{}_n q_4][1-{}_n q_8]\} = 0.0004 .$$

This result shows that the wastage rate of people having put some considerable years of service [10 years] is [4 / 10000] per unit time (e.g. a week), which is almost, zero. These results confirm the results found in Silcock (1954).

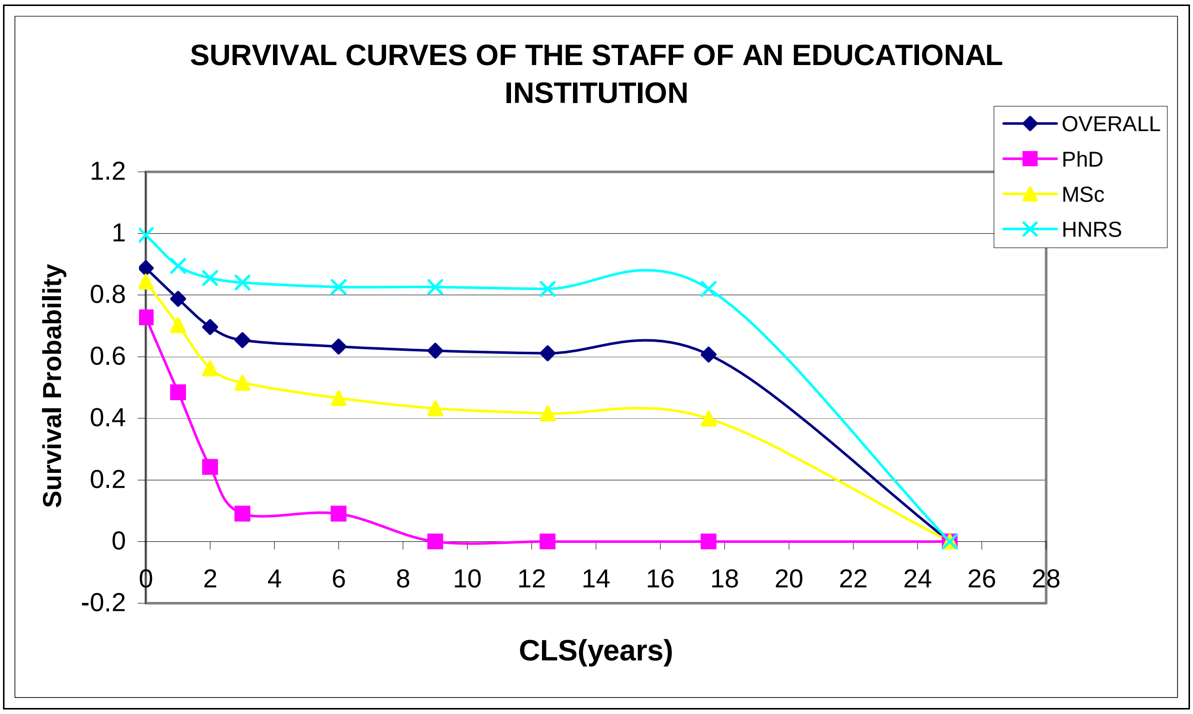

Figure 6.1: Survival rates of persons with Ph.D, MPhil/MSc, P.G.(Hons) and overall staff

6.6 RESULTS

Tables 6.1-6.4 give the length of service distribution of staff with PhD, Masters Degree, Honours and the overall staff of the Educational Institution respectively. Table 6.4 starts with the number of employees who completed i years of service, l_i out of a given number employed. It is observed that 240 employees were recruited at the beginning of year 0 or at the inception of the institution. Out of the 240 staff, 27 left within the first year of service, leaving 213 who survived attrition. The ${}_n d_i$ column shows the number of employees who left work between i and $i+n$ years of service. Since everyone must eventually leave the job, $d_{20-30} = l_{20-30}$.

The ${}_n q_i$ column shows the probability of leaving between i and $i+n$. Since everyone leaves, $q_{20-30} = 1$. The value of q_0 indicates that just over 11% of all recruits left before completing their first year of service. Out of the original cohort of 240 recruits only 213 persons completed one year of service and hence 0.1127x213=24 persons will leave before completing two years of service. The T_i column shows the sum of ${}_n L_i$ values at and above i years of service. Accordingly, the value of ${}_{30}T_{20}$ is the same as ${}_{30}L_{20}$. T_0 is the sum of duration of service in years of all recruits at retirement. Thus, according to Table 6.4 the 240 recruits would have served 4088.5 person-years after their fourth year of service. Over their length of service time, the 240 cohort of recruits serviced a total of 4694 total person years.

It is common for the expected length of service remaining after attaining one year of service, e_i^0 to increase at the earliest period after assuming duty followed by a steady decline. This gives a *hump* kind of survival curve showing that recruits are at high risk of leaving the institution during their early years of service and later settle down when they feel their job is secure enough. This is confirmed by Column (3) ${}_n d_i$, the number of people leaving the job in an institution between i and $i+n$ years declines rapidly, but then starts to drop gradually for those who served between 4 and 20 years before reaching a peak probably due to the effect of retirement.

Table 6.4 shows that at recruitment, employees are expected to work for 19.56 years. After one year of service, a person is expected to work for 20.97 years, because that person has already survived the risk of attrition during the first year of service. These figures are seen to differ from qualification to qualification. For instance, for staff with a PhD qualification they are 2.79 and 2.65 years respectively while for Masters Degree holders the figures are 14.4 and 15.88 years respectively. This is a clear indication that persons with higher qualification tend to easily find jobs elsewhere and are likely to be more mobile than other persons with lower qualifications. e_0^0 which can be interpreted as the mean length of service at work is an important measure of the remaining years of service for an employee. Its usefulness lies in helping the management to plan for future staffing situations.

From the tables we observe that q_i, the probability of wastage is a decreasing function indicating that the propensity to leave falls away with increasing service and this is what is usually found (Silcock, 1954). On comparison we observe that e_i^0, the expected length of service left and G_i, the survival rate are high while $h[i]$, the conditional probability of leaving after a given CLS is low. This shows that persons with high qualifications pursue for better jobs as shown in figure 6.1. This graph of a survivorship function G(x) is continually decreasing. It is fairly rapid at the first few years of service when recruits are indecisive, and the rate of fall slows down over the middle of the lifespan where leaving is gradual. The curve then falls steeply at higher years where wastage for employees is again comparatively lower.

We know that the annual rate of wastage is $q_i = \dfrac{d_i}{l_i}$ and $m_i = \dfrac{d_i}{L_i}$ is the central rate of wastage can be expressed as functions of l_i, the number of employees surviving attrition to age i out of a given number recruited. These equations show that q_i can be expressed in terms of m_i as

$$q_i = \frac{2nm_i}{2 + nm_i}$$

where n is the width of class interval.

For example, from Table 6.4, $q_3 = 0.0599$, we can calculate m_3 to be 0.0618 which shows that the two rates are more or less the same in this case. Tables 6.5 and 6.6 show the cumulative wastage rate and cumulative hazard rate of persons with various qualifications against their CLS respectively.

Since the risk of leaving increases with duration of service, figures 6.1-6.4 show that the wastage rate and hazard rate of persons with Ph.D. degrees are very rapid and steep in their increase whereas the other two categories are almost similar. This is partly due to the smaller numbers in PhD category in comparison. We note that as the number of years of service increase, the curves become almost straight lines and the overall graph always lies between the graphs of M.Phil/M.Sc. and P.G./Honours showing that the wastage rate of cohort is the average of the above two categories. The hazard and wastage rates increase steadily until after 7 or 8 years and then rises rapidly to a high of 0.07 in probability and to almost 2 persons for hazard and wastage rates respectively.

Table 6.5: Cumulative wastage rate of persons with different qualifications

CLS	Ph. D.	M.Phil. /MSc	P.G./Hons.	OVER ALL
0	0.2727	0.1563	0.0559	0.1150
1	0.6060	0.3230	0.0780	0.2252
2	1.1060	0.5230	0.1469	0.3416
3	1.7310	0.6063	0.1632	0.4015
6	1.7310	0.7031	0.1800	0.4346
9	2.7310	0.7745	0.1800	0.4551
12.5	----	0.8130	0.1885	0.4691
17.5	----	0.8530	0.1885	0.4762
25	----	1.8530	1.1885	1.4762

Table 6.6: Cumulative hazard rate of persons with different qualifications

CLS	Ph. D.	M.Phil. /M.Sc	P.G./Hons.	OVER ALL
0	-----	----	----	----
1	0.0088	0.0038	0.0011	0.0024
2	0.0286	0.0093	0.0019	0.0052
3	0.0782	0.0121	0.0023	0.0070
6	0.0782	0.0157	0.0027	0.0080
9	0.1086	0.0186	0.0027	0.0086
12.5	----	0.0203	0.0046	0.0090
17.5	----	0.0221	0.0046	0.0092
25	----	0.0701	0.0281	0.0409

Note: In the case of CLS $[i, i+n]$ the mid values of the intervals are taken.

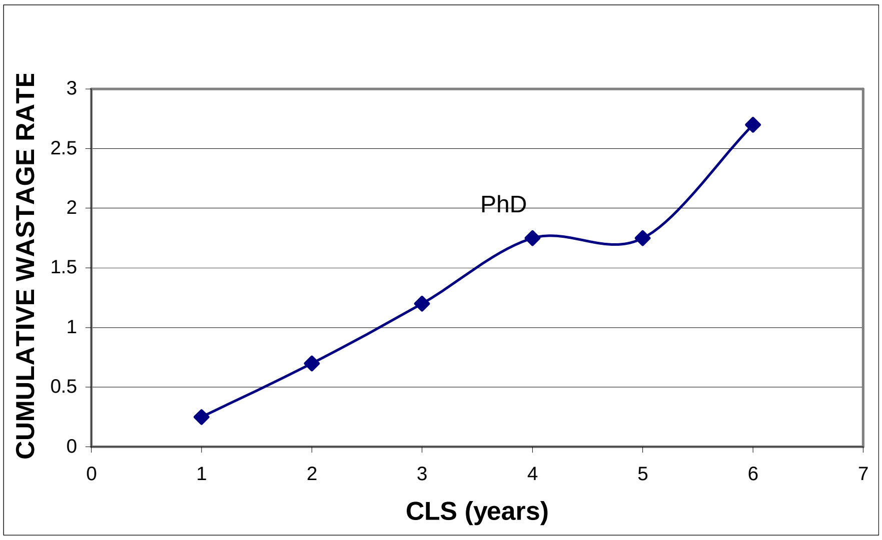

Figure 6.2: Cumulative Wastage Rate of persons with PhD

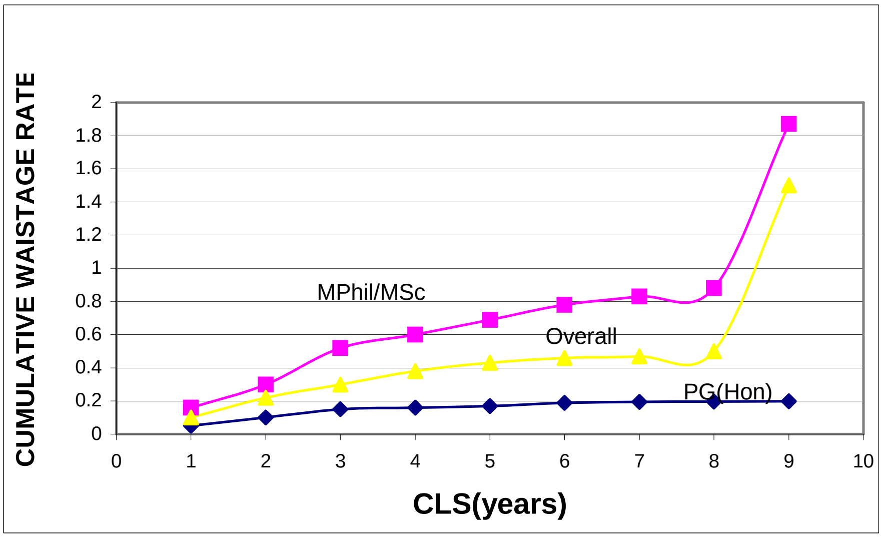

Figure 6.3: Cumulative Wastage Rate of persons with MPhil/MSc, P.G. (Hons) and overall staff.

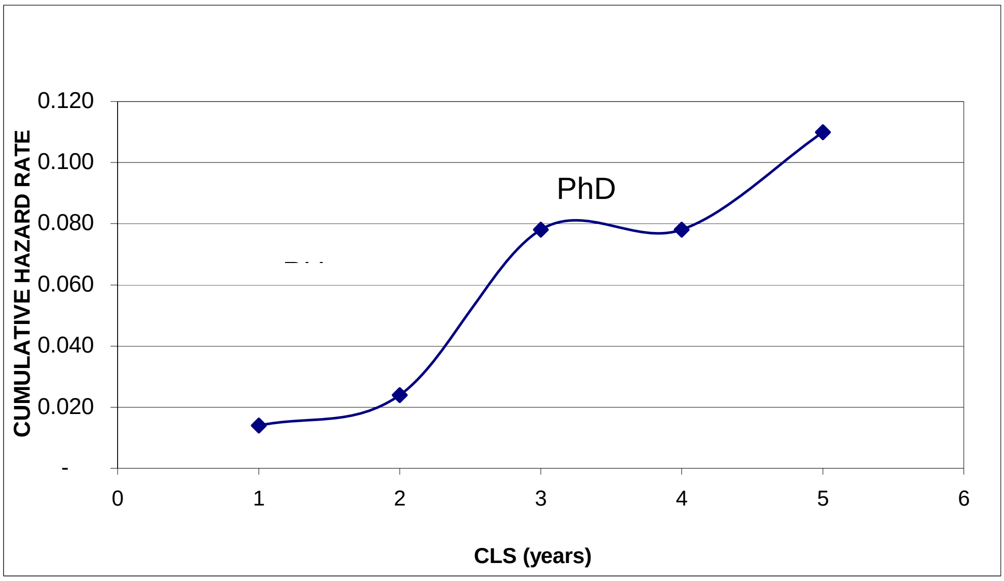

Figure 6.4: Cumulative Hazard Rate for persons with Ph.D

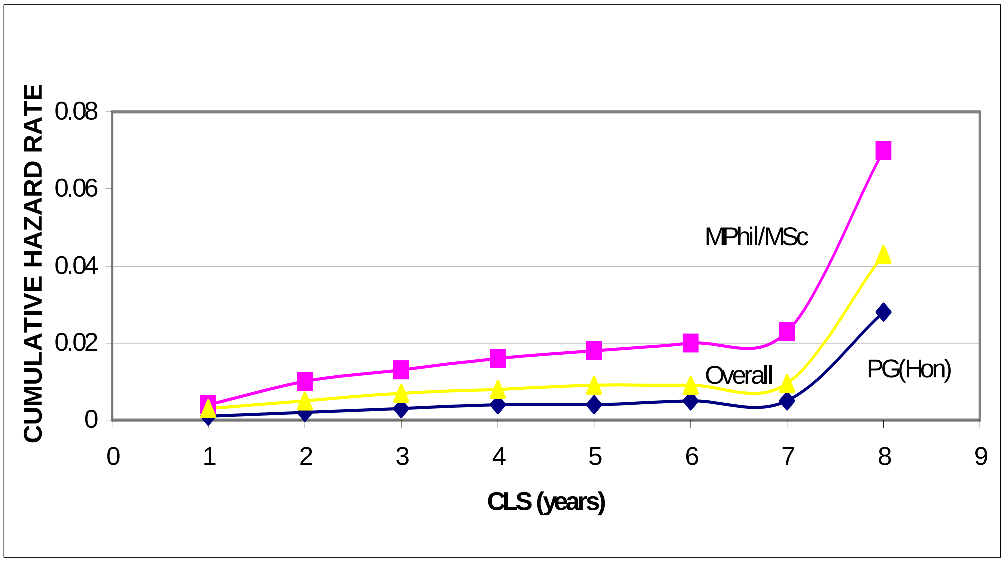

Figure 6.5: Cumulative Hazard Rate for persons with MPhil/MSc, P.G. (Hons) and overall staff.

6.7 CONCLUSION

The outcome of the discussion shows that the wastage of people with higher qualification is more than for the people with minimum qualifications, which is not negligible. Wastage has a direct implication on the organisational/institutional environment. According to Geerlings and Verbraeck (2000), the influence of the environment, through the rise of technology, changing needs of persons, political and economic situations, legislation and any others are factors that further complicate the problem of wastage. Hence there is a need for management to transform their manpower needs on a continuous basis. The work has provided a frame work for management decisions. Perhaps the management could look into the contributory factors to wastage such as :

- policy and benefits planning;
- academic programme planning;
- deteriorating condition of service;
- strength and clarity of the institutions mission as well as
- the effectiveness of the recruitment and retention programs in order to shape their organisational/institutional environment.

Management should not only be mindful of the outcome of the performance reward systems but also the process of how to implement those systems.

A life table technique was used to analyse the length of service of an educational institution. It has been observed that academic staff with higher qualification tends to leave employment more easily than their counterparts. This is attributable to the fact that staff with PhD competes more easily for jobs perhaps due to their marketability or having the right skills required by the organizations /institutions.

CHAPTER 7

STUDY OF THREE MODELS ON OPTIMAL PROMOTION IN A MANPOWER PLANNING SYSTEM

7.1 INTRODUCTION

Any organizational structure is generally built on a graded manpower system in which a member of the organization can belong to only one of the several mutually exclusive grades. One of the main aspects of manpower planning is to decide on policies related to the promotion of staff members as promotion is one of the critical factors that can be controlled by the management.

Having done fairly extensive research on managerial aspects, Young (1965) has given models of planning recruitment and providing promotion avenues for the members of the staff. Forbes (1970) studied promotion and recruitment policies for the control of quasi –stationary hierarchical system. Young and Vassiliou (1974) considered a non-linear model on the promotion of staff while Vassiliou (1978) has discussed another non-linear Markovian model for promotion in a manpower system. Later Leeson (1982) came out with yet another model which introduces grade profiles and are in-built mechanism pertaining to promotions that results in a significant reduction in wastage of human resources.

In a subsequent investigation Leeson (1982) had shown that from computed wastage and promotion proportions it is possible to return to original principles of stationary probabilities and thereby compute the wastage and promotion intensities which produce the proportions corresponding to some desired planning proposals. Agrafiotis (1984) suggested a grade specific stochastic model which accounts for the effect on wastage of the internal structure and the promotion experience of its employees.

Feuer and Schinnar (1984) carried out sensitivity analysis of promotion opportunities in graded organization, highlighting the links between personnel flows and vacancy flows. Leeson (1994) employed projection and promotion models for graded manpower system to consider recruitment policies and their effects on internal structures. Earlier Kalamatianou (1988) proposed a model in which promotion probabilities are functions of the seniority structure within the grades. The model suggests a method of overcoming the problem of promotion blockages. However, despite the fact that the various methods discussed above are highly comprehensive, certain aspects of an optimum promotion policy have been left out.

Time bound promotions are very common in organizations with employees in different grades. In order to avoid stagnation of personnel in a single grade such promotions are given to those who could not get elevated under competitive conditions. In this chapter, three models have been studied. In model 1, a continuous time manpower model is proposed in which an optimum promotion policy is discussed when the cost of promoting a person from grade i $(i = 1, 2, ..., n)$ at time t is a function of the number of persons in that grade. The solution is obtained with the help of Euler-Langrage equation. A deduction is also made considering the cost to be a constant, independent of the grade size.

In the other two models, a manpower system with M-grades $(i = 1, 2, ..., M)$ is considered over a time interval $(0, T_i)$ during which two types of promotions are contemplated from i^{th} grade to $(i+1)^{th}$ grade. The first type of promotion is to promote an individual as and when the vacancies arise. The second type is called an automatic promotion which takes place at the end of $(0, T_i)$ and all those who remain stagnant in grade i throughout the interval $(0, T_i)$ are automatically promoted to the next $(i+1)^{th}$ grade. Vacancies which arise in the $(i+1)^{th}$ grade give rise to promotion from the i^{th} grade. In model 2 the vacancy in the next higher grade is only one at any point in time, otherwise promotion is given only to a single person at every demand epoch. In model 3 it is assumed that at every instant a random number of persons can have promotions. The optimal value of T_i is arrived at for the general case and the results are derived assuming specific distributions for the number of vacancies that arise. Numerical results justify the results obtained in the models.

7.2 MODEL-1

The following notation is used in the analysis of this model.

7.2.1 Notation

Let

$S_i(t)$: Number of persons in grade i at time t

$F(S_i)$: Rate of promotion of employees from grade $(i-1)$ to grade i.

$P_i(t)$: Rate of promotion of employees from grade i to grade $(i+1)$ at time t.

$C(S_i)$: Cost of promoting a person from grade i to grade $(i+1)$ when the size of the grade i at time t is $S_i(t)$

7.2.2 Mathematical model

From the relation between $S_i(t)$, $F(S_i)$ and $P_i(t)$ we get the following equation

$$\frac{dS_i}{dt} = F(S_i) - P_i(t) \qquad (7.2.1)$$

$S_i(t)P_i(t)\Delta t$ denotes the number of persons promoted from grade i to grade (i+1) during the interval $(t, t+\Delta t)$. Since $C(S_i)$ is the cost of promoting a person from grade i to grade (i+1), the cost of promoting $S_i(t)P_i(t)\Delta t$ persons in the interval $(t, t+\Delta t)$ is $C(S_i)S_i(t)P_i(t)\Delta t$. Therefore, the total cost involved in this case is given by

$$C = \int_0^\infty C(S_i)S_i(t)P_i(t)\, dt$$

$$C = \int_0^\infty C(S_i)S_i(t)[F(S_i) - S_i'(t)\,]\, dt \qquad (7.2.2)$$

where $S_i'(t) = \dfrac{dS_i(t)}{dt}$

We know that if $I = \int_a^b f(x, y, \frac{dy}{dx})\, dx$, the problem of calculus of variations is to find that function $y(x)$ for which I is maximum or minimum. The answer is given by the solution of the Euler-Lagrange equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0 . \tag{7.2.3}$$

Here

$$f(t, S_i, S_i') = C(S_i)S_i(t)\left[F(S_i) - S_i'(t)\right]$$

and

$$\frac{\partial f}{\partial S_i} - \frac{d}{dt}\left(\frac{\partial f}{\partial S_i}\right) = 0$$

gives

$$[(C'(S_i)S_i(t) + C(S_i)][F(S_i) - S_i'(t)] + C(S_i)S_i(t)F'(S_i) + \frac{d}{dt}[C(S_i)S_i(t)] = 0 . \tag{7.2.4}$$

That is,

$$[C'(S_i)S_i(t) + C(S_i)]\,[F(S_i) - S_i'(t)] + [C(S_i)S_i(t)F'(S_i)] + C'(S_i)S_i'(t)S_i(t) + C(S_i)S_i't) = 0 ,$$

$$S_i(t)[C'(S_i)F(S_i) + C(S_i)F'(S_i)] = -C(S_i)F(S_i) \tag{7.2.5}$$

and so

$$S_i(t)\frac{d}{dS_i}[C(S_i)F(S_i)] = -C(S_i)F(S_i)$$

giving $$\frac{d[C(S_i)F(S_i)]}{C(S_i)F(S_i)} = -\frac{dS_i}{S_i}$$

Hence we get

$$\ln\,[C(S_i)F(S_i)] = -\ln S_i + \ln k$$

or

$$S_i(t)C(S_i)F(S_i) = k \qquad \text{(a constant)}$$

which can be determined from the initial conditions.

Thus

$$F(S_i) = \frac{k}{S_i(t)C(S_i)} \quad . \tag{7.2.6}$$

Further if S_i^* is the value of S_i when the cost is minimum then,

$$F(S_i^*) = \frac{k}{S_i^*(t)C(S_i^*)} \, .$$

Therefore (7.2.1) gives

$$P_i(t) = F(S_i^*) = \frac{k}{S_i^*(t)C(S_i^*)}$$

which gives the promotion rate from grade i at time t. Thus the promotions rate from grade i at time t depends on the optimum grade size at time t. Since the cost function $C(S_i^*)$ is always an increasing function of $S_i(t)$, we see that $P_i(t)$ is a decreasing function of $S_i(t)$.

If the initial grade size in any grade i is less than the optimum size $S_i^*(t)$, then the management may decide that it is better not to promote the employees from grade i till the grade size increases to S_i^*. On the other hand if promotion is essential the recruitment to grade i can be made to make the grade size to be $S_i^*(t)$ and then promotion can be effected at a constant rate of $\frac{k}{S_i^*(t)C(S_i^*)}$. Hence the promotion on seniority basis is preferred. If the initial grade size is already greater than S_i^* then promotion can be given at a faster rate which is permissible under the promotion policies of the organization or voluntary retirement scheme can be made attractive so that more persons opt for it, till the grade size decreases to $S_i^*(t)$. After that the constant promotion rate $\dfrac{k}{S_i^*(t)C(S_i^*)}$ can be practiced.

7.2.3 Special case

When the cost of promotion at time t is taken to be independent of the time t and grade size $S_i^*(t)$, we have

$$C(S_i) = c$$

Then the equation (7.2.5) becomes

$$cS_i(t)F'(S_i) = -cF(S_i)$$

and so

$$S_i(t)F'(S_i) + F(S_i) = 0$$

$$\frac{d}{dS_i}[S_i(t)F(S_i)] = 0 .$$

This means

$$S_i(t)F(S_i) = k_1 \qquad \text{(a constant).}$$

Therefore

$$F(S_i) = \frac{k_1}{S_i(t)} = \frac{ck_1}{cS_i(t)} = \frac{k_2}{cS_i(t)}$$

where $k_2 = ck_1$.

Thus the optimum promotion rate is given by

$$F(S_i^*) = \frac{k_2}{cS_i^*(t)}$$

7.3 MODEL-2

7.3.1 Assumptions and notation

(i) Each vacancy arising in the next higher grade, say (i+1) gives rise to a demand for a regular promotion from the ith grade at any instant.

(ii) The demand for each instant is only one.

(iii) All those who remain stagnant in grade i at the end of the interval $(0, T_i)$ are automatically promoted to the grade (i+1).

N_i : The size of the ith grade

K : Number of regular promotions during (0, Ti) which is a discrete random variable (each regular promotion is for one unit only).

C_{1i} : Cost of one regular promotion in the ith grade during $(0, T_i)$.

C_{2i} : Cost of one automatic promotion in the ith grade at the end of $(0, T_i)$.

$F(.)$: Distribution function of the inter-arrival times between two regular promotions.

$F_n(t) : [F(t)]^{(n)}$; n-fold convolution of $F(t)$

Now, the expected cost of regular promotions and automatic promotions for the ith grade in the interval $(0, T_i)$ is given by

$$E(C_{T_i}) = C_{1i} \sum_{k=0}^{N_i} k.P\left[\, exactly \quad k \quad regular \quad promotions \quad during \quad (0, T_i)\right]$$
$$+ C_{2i} \sum_{k=0}^{N_i} (N_i - k).P\left[\, exactly \quad k \quad regular \quad promotions \quad during \quad (0, T_i)\right]$$

Using renewal theory

$$E(C_{T_i}) = C_{1i} \sum_{k=0}^{N_i} k.\left[F_k(T_i) - F_{k+1}(T_i)\right] + C_{2i} \sum_{k=0}^{N_i} (N_i - k)\left[F_k(T_i) - F_{k+1}(T_i)\right] \qquad (7.3.1)$$

To find the optimum T_i , we have

$$\frac{d}{dT_i}\left[E(C_{T_I})\right]=0 \tag{7.3.2}$$

$$\Rightarrow C_{1i}\sum_{k=0}^{N_i} k\left[f_k(T_i)-f_{k+1}(T_i)\right]+C_{2i}\sum_{k=0}^{N_i}(N_i-k)\left[f_k(T_i)-f_{k+1}(T_i)\right]=0$$

where $f_k(T_i)=\left[f(T_i)\right]^{(k)}$ is the k-fold convolution of the density $f(T_i)$

$$\Rightarrow \quad \frac{N_i\sum_{k=0}^{N_i}\left[f_k(T_i)-f_{k+1}(T_i)\right]}{\sum_{k=0}^{N_I}k\left[f_k(T_i)-f_{k+1}(T_i)\right]} = \frac{C_{2i}-C_{1i}}{C_{2i}}$$

$$\Rightarrow \quad \frac{-N_i f_{N_{I+1}}(T_i)}{\sum_{k=1}^{N_I} f_k(T_i)-N_i f_{N_i+1}(T_i)} = \frac{C_{2i}-C_{1i}}{C_{2i}} \tag{7.3.3}$$

This is the general result for obtaining T_i. For a set of given values of N_i, C_{1i} *and* C_{2i} and also the distribution of inter-arrival times, the optimal value of T_i can be obtained by solving the equation (7.3.1).

7.3.2 Special case (model -2)

Inter-arrival times between regular promotions are assumed to be identically exponentially distributed with parameter λ.

Hence

$$f_{N_i}(T_i)=\frac{\lambda(\lambda T_i)^{N_i-1}}{(N_i-1)!}e^{-\lambda T_i}.$$

Then the equation (7.3.3) becomes

$$\frac{\dfrac{-(\lambda T_i)^{N_i} N_i}{N_i}}{\sum_{k=1}^{N_i} \dfrac{(\lambda T_i)^{k-1}}{(k-1)\,!} - \dfrac{N_i (\lambda T_i)^{N_i}}{N_i\,!}} = \frac{C_{2i} - C_{1i}}{C_{1i}}$$

when $C_{1i} = C_{2i}$, we have from (7.3.3), $f_{N_i+1}(T_i) = 0$

i.e.

$$\frac{\lambda e^{-\lambda T_i} (\lambda T_i)^{N_i}}{N_i!} = 0 \quad \Rightarrow \quad \lambda e^{-\lambda T_I} (\lambda T_i)^{N_i} = 0$$

$$\lambda \neq 0, \;\; T_i \neq 0 \quad \Rightarrow \quad (\lambda T_i)^{N_i} \neq 0$$

$$e^{-\lambda T_i} = 0 \quad \Rightarrow \quad \lambda T_i = \infty$$

In such a case we have the following:

Case (i): λ is large and T_i is small so that $\lambda T_i = \infty$. But it is impossible since $(0, T_i)$ contains several intervals with parameter λ.

Case (ii): λ is small and T_i is very large. This is possible.

Case (iii): λ and T_i are very large. This is also possible.

We consider case (ii) namely λ is finite and $T_i = \infty$; in this case nobody will be there for automatic promotions.

If $C_{2i} > C_{1i}$, no solution exists for T_i. Numerical illustration is obtained when $C_{1i} > C_{2i}$, assuming inter-arrival times between regular promotions as exponential, and for specific values of N_i.

Let us suppose that $C_{1i} = \$300$, $C_{2i} = \$100$.

Then

$$\frac{C_{2i} - C_{1i}}{C_{2i}} = -2$$

For different values of N_i, the equations are obtained and they are such that each has only one positive root. The positive roots have been obtained by using Horner's method (See Table (7.3.1))

Table 7.3.1: Positive roots for different grade sizes

N_i	EQUATIONS	$\hat{T}_i$
2	$3\lambda T_i - 2 = 0$	$\frac{2}{3\lambda}$
3	$3(\lambda T_i)^2 - 2(\lambda T_i) - 4 = 0$	$\frac{4.6}{3\lambda}$
4	$3(\lambda T_i)^3 - 2(\lambda T_i)^2 - 6(\lambda T_i) - 12 = 0$	$\frac{2.2}{\lambda}$
5	$3(\lambda T_i)^4 - 2(\lambda T_i)^3 - 8(\lambda T_i)^2 - 24(\lambda T_i) - 48 = 0$	$\frac{3.1}{\lambda}$

For specific value of λ, $\hat{T}_i$ corresponding to N_i can be obtained as above. The optimum T_i will be decided depending upon the value of N_i for any given λ (see Table 7.3.1).

7.4 MODEL-3

In this model it is assumed that a random number of persons can be given regular promotions, at each instant. So, at any epoch in which regular promotions are made, a random number of persons k can be promoted during $(0, T_i)$. In this case the expected cost of regular promotions in $(0, T_i)$ and automatic promotions at the end of $(0, T_i)$ is given by

$$E(C_{T_i}) = C_{1i}\sum_{k=1}^{N_I} k \sum_{j=1}^{k}\left[F_j(T_i) - F_{j+1}(T_i)\right] p_j(k)$$
$$+ C_{2i}\sum_{k=0}^{N_i}(N_i - k)\sum_{j=0}^{k}\left[F_j(T_i) - F_{j+1}(T_i)\right] p_j(k) \quad for \quad j \le k$$

(7.4.1)

where

$P_j(k)$=P[exactly k regular promotions in j instants].

This is given by the coefficient of s^k in the expansion of $\phi^j(s)$, where

$$\phi(s) = \sum_{r=1}^{\infty} p_r s^r \text{, with } p_r = P[X = r].$$

X is in the random number of persons given regular promotions at each instant. Here $\phi^{(j)}(s)$ stands for the j-fold convolution of $\phi(s)$.

Therefore

$$\phi^{(j)}(s) = [\phi(s)]^j .$$

If the $X's$ are independent and identically distributed random variables to obtain the optimum value of T_i we have

$$\frac{d}{dT_i}\left[E(C_{T_i})\right] = 0$$

$$\Rightarrow \quad C_{1i}\sum_{k=1}^{N_I} k \sum_{j=1}^{k}\left[f_j(T_i) - f_{j+1}(T_i)\right] p_j(k)$$
$$+ C_{2i}\sum_{k=0}^{N_i}(N_i - k)\sum_{j=1}^{k}\left[f_j(T_i) - f_{j+1}(T_i)\right] p_j(k) = 0$$

$$\Rightarrow \quad \frac{\sum_{k=1}^{N_i} k \sum_{j=1}^{k}\left[f_j(T_i) - f_{j+1}(T_i)\right] p_j(k)}{\sum_{i=1}^{N_I} N_i \sum_{j=1}^{k}\left[f_j(T_i) - f_{j+1}(T_i)\right] p_j(k)} = \frac{C_{2i}}{C_{2i} - C_{1i}}. \tag{7.4.2}$$

The solution for T_i can be obtained from the above equation for general distributions. Solutions for assumptions of specific distributions may be obtained with tedious computational work.

7.4.1 Special case

When $N_i = 3, \quad C_{2i} = \$100 \quad and \quad C_{1i} = \$300$, the numerator of the LHS of equation (7.4.2) becomes

$$\begin{aligned} & f_1(T_i)\left[P_1(1) + 2P_1(2) + 3P_1(3)\right] \\ & - f_2(T_i)\left[P_1(1) + 2P_1(2) - 2P_2(2) + 3P_1(3) - 3P_2(3)\right] \\ & - f_3(T_i)\left[2P_2(2) + 3P_2(3) - 3P_3(3)\right] - 3f_4(T_i)P_3(3). \end{aligned} \tag{7.4.3}$$

The denominator of LHS of (7.4.2) becomes

$$\begin{aligned} & \{f_1(T_i)\left[P_1(1) + P_1(2) + P_1(3)\right] \\ & - f_2(T_i)\left[P_1(1) + P_1(2) - P_2(2) + P_1(3) - P_2(3)\right] \\ & - f_3(T_i)\left[P_2(2) + P_2(3) - P_3(3)\right] - f_4(T_i)\, P_3(3)\}. \end{aligned} \tag{7.4.4}$$

Let the inter-arrival times between two regular promotions be independently and identically distributed exponential with parameter α. Let $\alpha = 0.05$.

We have

$$f_k(T_i) = \frac{\alpha(\alpha T_i)^{k-1}}{(k-1)\,!} e^{-\alpha T_i},$$

so that

$$f_1(T_i) = 0.5e^{-0.5T_i}$$

$$f_2(T_i) = 0.25T_i e^{-0.5T_i}$$

$$f_3(T_i) = 0.0625T_i^2 e^{-0.5T_i}$$

and

$$f_4(T_i) = 0.0104T_i^3 e^{-0.5T_i}.$$

Let us suppose that X follows a Poisson distribution with parameter λ and it is evidently truncated at $X = 0$. T he probability density function of truncated Poisson distribution is

$$\psi(s) = \sum_{k=1}^{\infty} \frac{e^{-\lambda}\lambda^k s^k}{k!(1-e^{-\lambda})} = \frac{e^{-\lambda}(e^{\lambda s}-1)}{1-e^{-\lambda}}.$$

Let $\lambda = 1.5$

$P_1(1) = P[X_1 = 1] =$ Coefficient of s^1 in $\psi(s)$

$$= \frac{\lambda e^{-\lambda}}{1-e^{-\lambda}} = \frac{1.5e^{-1.5}}{1-e^{-1.5}} = 0.4307$$

$P_1(2) = P[X_1 = 2] =$ Coefficient of s^2 in $\psi(s)$

$= 0.3230$

$P_1(3) = 0.1615$

$P_2(2) = P[X_1 + X_2 = 2] =$ Coefficient of s^2 in $\psi^2(s)$

$$= \left[\frac{\lambda e^{-\lambda}}{1-e^{-\lambda}}\right]^2$$

$= 0.1855$

$P_2(3) = P[X_1 + X_2 = 3]$

$= 0.2782$

$$P_3(3) = P[X_1 + X_2 + X_3 = 3]$$

$$= 0.0799.$$

Now, equation (7.4.3) gives

$$e^{-0.5T_i}[0.7806 - 0.889T_i - 0.0604T_i^2 - 0.0025T_i^3]$$

and that of (7.4.4) is

$$e^{-0.5T_i}[1.3728 - 0.3387T_i - 0.0702T_i^2 - 0.0024T_i^3].$$

Hence we get

$$0.0074T_i^3 + 0.1910T_i^2 + 5.165T_i^2 - 2.934 = 0.$$

This equation has only one positive root and it lies between 2 and 3 and it is $\hat{T}_i = 2.7$.

So, the optimal period of the cycle for the ith grade is found to be $\hat{T}_i = 2.7$ years.

7.5 CONCLUSION

In this chapter it is shown that the optimum promotion rate for any grade depends on the grade size though the cost of promotion may or may not be dependent on it.

A number of extensions of this model are possible. A simulation model can be developed to study the effect of various optimum promotion policies on the system for different cost structures. The optimal cycle for giving the time bound promotion can be obtained for any specific grade, under given values of the parameter, costs and distributions. It is also possible to obtain a common optimal policy for all the grades put together.

CHAPTER 8

OPTIMAL TIME FOR THE WITHDRAWAL OF THE VOLUNTARY RETIREMENT SCHEME, AND OPTIMAL TIME INTERVAL BETWEEN SCREENING TESTS FOR PROMOTIONS

8.1 INTRODUCTION

In any organisation the required staff strength is maintained through new recruitments. The exit of personnel from an organisation is a common phenomenon, which is known as wastage. Many stochastic models dealing with wastage are found in Bartholomew and Forbes (1979). In production-oriented organisations wherever there is surplus staff strength a reduction becomes a necessity. The staff strength in the organisation depends on the market demand for the products. If the staff strength is more than the requested level, attempts are made for the exit of personnel on a voluntary basis tempting then with suitable financial packages.

During a period of T years the voluntary retirement scheme is operated on k epochs. At each of these epochs a random number of employees opt to retire under the scheme and this in turn reduces the staff strength. If the total number of persons who retire crosses a level called the threshold level, the scheme is withdrawn. A salient feature of the investigation is to determine the optimal length of time (0,T) and this cycle length is obtained under some specific assumptions using the concept of cumulative damages process of the reliability theory. For a detailed description and analysis of shock models one can refer to Ramanarayan (1977) who analysed the system exposed to a cumulative damage process of shock. Sathiyamurthy (1980) discussed cumulative damage shock models correlating the inter-arrival times between shocks. Similarly, recruitment of persons based on their satisfactory performance in screening tests is a common procedure in vogue in many organisations. The use of compartmental models in manpower planning is quite common. For a detailed study of the compartmental models in manpower systems, one can refer to Agrafiotis (1991).

Consider a system which has two compartments c_1 and c_2 . The size of c_1 is fixed as n. Transition of persons from c_1 to c_2 is allowed and in between there is a screening test to evaluate the competence of individuals to get into c_2. The compartment c_2 may be thought of as one consisting of persons with greater skills, efficiency and administrative capabilities. The qualities are evaluated by the screening test. The persons in c_1 are first recruited and kept in the reserve list. Assuming that they are given some training to improve their capabilities, keeping these persons in c_1 and training them involves a maintenance cost or reserve cost. Conducting the test but with no persons getting entry to c_2 involves some cost namely screening test cost which is a total loss. In case no persons get selected and enter into c_2, the vacancies in c_2 remain unfilled and each such unfilled vacancy gives rise to some shortage cost in terms of loss productivity. To make good this loss, recruitment of persons from outside to compartment c_2 is made on an emergency basis. The longer the time interval between the screening tests the greater will be the cost of maintenance of persons in c_1 which in turn increases the cost of shortages in c_2. Frequent screening tests results in higher test costs. With a view to minimize the above said costs, the optimal time interval namely T between successive screening tests is attempted here. The results have been applied on some special cases of distributions.

The organisation of this chapter is as follows: In section 8.2, model 1 is described. System description and notation is discussed in section 8.2.1. In section 8.2.2, the cost analysis of the model for which the optimal time for the withdrawal of the voluntary retirement is studied. Model 2 is a study of optimal time interval between screening tests for promotion in manpower planning. In section 8.3.1, the model assumptions and notation have been described. The cost analysis for this model is studied in section 8.3.2. Some special cases are studied in section 8.3.3. Numerical examples illustrated results in the last section.

8.2 MODEL-1

8.2.1 Notation

k **:** Number of epochs in (0,T) at which voluntary retirement is permitted.

X_i: A discreet random variable representing the number of persons retiring at the ith epoch.

$V_k(T)$: P [there are k epochs during (0,T)]

L: A discreet random variable denoting the number of persons in total who opt for retirement in k epochs.

$P_L(k)$: P [L persons opt for retirement in k such epochs]

Y : Threshold level

C_V : Cost of voluntary retirement per person at each of these epochs

C_F : Cost of failure of the scheme

$f(\cdot)$: pdf of inter-arrival times between epochs

$f^{(k)} f(\cdot)$: k-fold convolution

$F(\cdot)$: Distribution function corresponding to $f(\cdot)$.

$C(T)$: Total cost

8.2.2 Cost analysis

The total cost arising due to the (i) the failure of the scheme with no persons retiring (ii) a random number of persons retiring but below the threshold level which renders the scheme a failure are put together as follows:

$$C(T) = [1 - F_1(T)]\, C_F + [\sum_{k=1}^{\infty} V_L(k)][\sum_{L \ge k} P_L(k)]\, P(Y > L)\,(LC_V + C_F)$$

$$+ \sum_{k=1}^{\infty} \int_0^T f^{(k)}(t)\, dt\, [\sum_{L \ge k} P_L(k)]\, P[Y > L)\; x\; LC_V$$

$$= [1 - F_1(T)]\, C_F + \sum_{k=1}^{\infty} \left[F^{(k)}(T) - F^{(k+1)}(T)\right] \left[\sum_{L \ge k} P_L(k)\right] P(Y > L)(LC_V + C_F)$$

$$+ \sum_{k=1}^{\infty} \int_0^T f^{(k)}(t)\, dt \left[\sum_{L \ge k} P_L(k)\right] P[Y > L]\; x\; LC_V .$$

The main purpose of this chapter is to find the optimal value of T, which minimises the total cost $C(T)$. For a continuous variable t, we have,

$$\frac{dC(T)}{dT} = f_1(T)\,C_F + \sum_{k=1}^{\infty}\left[f^{(k)}(T) - f^{(k+1)}(T)\right]\left[\sum_{L\geq k} P_L(k)\right] P(Y > L)(LC_V + C_F)$$

$$+\sum_{k+1}^{\infty} f^{(k)}(T) \sum_{L\geq k} P_L(k)\; P\left[Y > L\right] x\; LC_v = 0.$$

This gives

$$-\,f_1(T)\,C_F + C_V \left\{ \begin{array}{l} \sum_{k=1}^{\infty} f^{(k)}(T)\left[\sum_{L\leq k} LP_L(k)\right] P(Y > L) \\ +\sum_{k=1}^{\infty} f^{(k)}(T)\left[\sum_{L\geq k} LP_L(k)\right] P(L \geq Y) \\ -\sum_{k=1}^{\infty} f^{(k+1)}(T)\left[\sum_{L\geq k} LP_L(k)\right] P(L \geq Y) \end{array} \right\}$$

$$+\,C_F \sum_{k=1}^{\infty}\left[f^{(k)}(T) - f^{(k+1)}(T)\right]\left[\sum_{L\geq k} P_L(k)\right] P(Y > L) = 0$$

$$= -\,f_1(T)\,C_F + C_V \left\{ \begin{array}{l} \sum_{k=1}^{\infty} f^{(k)}(T)\left[\sum_{L\geq k} LP_L(k)\right] P(Y > L) + P(Y \leq L) \\ -\sum_{k=1}^{\infty} f^{(k+1)}(T)\left[\sum_{L\geq k} LP_L(k)\right] P(Y > L) \end{array} \right\}$$

$$+\,C_F \sum_{k=1}^{\infty}\left[f^{(k)}(T) - f^{(k+1)}(T)\right]\left[\sum_{L\geq k} P_L(k)\right] P(Y > L) = 0$$

Therefore

$$\frac{\sum_{k=1}^{\infty} f^{(k)}(T)\left[\sum_{L\geq k} LP_L(k)\right] - \sum_{k=1}^{\infty} f^{(k+1)}(T)\left[\sum_{L\geq k} LP_L(k)\right] P(Y>L)}{f_1(T) - \sum_{k=1}^{\infty}\left[f^{(k)}(T) - f^{(k+1)}(T)\right]\left[\sum_{L\geq k} LP_L(k)\right](P>L)} = \frac{C_F}{C}. \qquad (8.1)$$

Any value of T which satisfies the equation (8.1) for a given set of values of the cost and other parameters like k and Y is the optimal value of T and T is unique since it gives the local minimum. The only criterion to choose optimum is based in the total cost.

8.2.3 Special case

When the threshold level of Y is taken to be random variable that follows geometric distribution with parameter θ, we have

$$P(Y=k) = (1-\theta)\theta^{k-1} \qquad k = 1, \quad 2, \ldots .$$

For given L we have $P(L \geq Y) = 1 - \theta^L \quad or \quad P(Y > L) = \theta^L$.

Also

$$P_L(k) = P\left(X_1 + X_2 + \ldots X_K = L\right)$$

and so

$$\sum_{L\geq k} P_L(k)\, P(Y>L) = \sum_{L\geq k} P_L(k)\, \theta^L = \psi^k(\theta) \qquad (say).$$

Hence

$$\sum_{L\geq k} L\, P_L(k)\, P(Y>L) = \sum_{L\geq k} L\, P_L(k)\, \theta^L = \psi^k(\theta) \qquad (say)$$

$$= \theta \sum_{L\geq k} L\, P_L(k)\, \theta^{L-1}$$

$$= \theta\left[\psi^k(\theta)\right]'$$

Let us define

$$P(s) = \sum_{k=0}^{\infty} P_k s^k$$

so that

$$p'(s) = \sum_{k=1}^{\infty} k \; P_k s^{k-1} .$$

In view of the fact that L is a random variable we have

$$\sum_{k=1}^{\infty} f^{(k)}(T) \sum_{L \geq k} L \; P_L(k) = \sum_{k=1}^{\infty} f^{(k)}(T) \; k \, E(L) .$$

From (8.1)

$$\frac{\sum_{k=1}^{\infty} f^{(k)}(T) \; k \; E(L) - \sum_{k=1}^{\infty} f^{(k+1)}(T) \; \theta \left[\psi^k(\theta)\right]'}{f_1(T) - \sum_{k=1}^{\infty} f^{(k)}(T) \; \psi^k(\theta) + \sum f^{(k+1)}(T) \; \psi^k(\theta)} = \frac{C_F}{C_V} . \tag{8.2}$$

Let X follow a Poisson distribution with parameter λ

$$P\left[X = r\right] = \frac{e^{-\lambda} \lambda^r}{r!} \qquad r = 0, 1, 2, \ldots\ldots..$$

The probability generating function of a Poisson distribution is

$$\psi(\theta) = \sum P_n \theta^r = e^{-\lambda(1-\theta)} .$$

Now

$$\sum_{k=1}^{\infty} f^{(k)}(T) \psi^k(\theta) = \sum_{k=1}^{\theta} f^{(k)}(T) \; e^{-k\lambda(1-\theta)} .$$

Let

$$f(t) = \alpha \, e^{-\alpha t}$$

then,

$$f^{k}(T)=\frac{\alpha(\alpha T)^{k-1}e^{-\alpha T}}{(k-1)\,!}.$$

Therefore

$$\sum_{k=1}^{\infty}f^{(k)}(T)\psi^{k}(\theta)=\sum_{k=1}^{\infty}\alpha e^{-\alpha T}e^{-k\lambda(1-\theta)}\frac{(\alpha T)^{k-1}}{(k-1)\,!}$$

$$=\alpha e^{-\alpha T}e^{-\lambda(1-\theta)}\sum_{k=1}^{\infty}\frac{[\alpha Te^{-\lambda(1-\theta)}]^{k-1}}{(k-1)\,!}$$

$$=\alpha e^{-\alpha T}e^{-\lambda(1-\theta)}[1-e^{-\lambda(1-\theta)}]$$

and $\qquad E(L)=\sum nP_n=\lambda.$

Using these results we get from (8.2)

$$\frac{\alpha\lambda+\alpha^{2}\lambda T[1-\theta e^{-\lambda(1-\theta)}e^{-\alpha T[1-e^{-\lambda(1-\theta)}]}]}{\alpha e^{-\alpha T}[1-e^{-\lambda(1-\theta)}][1-e^{-\lambda(1-\theta)}]}=\frac{C_F}{C_V}. \qquad (8.3)$$

8.3 MODEL 2

8.3.1 Assumptions

(i) There is a fixed size or strength of persons in compartment c_1

(ii) Transition from c_1 to c_2 is permitted on the basis of screening test

(iii) Shortages are permitted in c_2

(iv) In every screening test a person has a constant probability p of getting selected and permitted to join c_2

(v) If k vacancies exist in c_2, r out of k are selected from c_1 with constant probability p and (k-r) are selected outside c_1 with probability q and $p+q=1$

Notation

n : Size of the compartment c_1.

C_L: Cost of retention of each person c_1 to c_2. In other words the screening test results in the selection of nobody from c_1.

C_s: Cost of each unfilled vacancy in c_2 per unit time.

$f(\cdot)$: pdf of inter-arrival times of the screening test.

$F^{(k)}(t)$: k-fold convolution of $F(t)$.

$F(\cdot)$: Cumulative Distribution function of inter-arrival times of screening test.

8.3.2 Cost analysis

The total expected cost of retention in c_1, cost of wastages in futile screening tests and cost of shortages in c_2 is given by

$$E(C(T)) = [1 - F(T)]\, C_n + TC_h + \sum_{k=1}^{\infty} \left[F^{(k)}(T) - F^{(k+1)}(T)\right] \sum_{r=0}^{k} {}^{k}c_r\, p^r q^{k-r} (k-r) c_s . \tag{8.4}$$

Differentiating (8.4) w.r.t. t and equating to zero, we get

$$- f(T) C_n + C_h + \sum_{k=1}^{\infty} \left[f^{(k)}(T) - f^{(k+1)}(T) \right] kqc_s = 0 .$$

Since

$$\sum_{k=1}^{\infty} k \left[f^{(k)}(T) - f^{(k+1)}(T) \right] = \frac{f(T) C_n - C_L}{C_S q} \tag{8.5}$$

Special case (model 2)

(i) Let

$$f(T) = \lambda e^{-\lambda t}$$

$$E(C(T)) = e^{-\lambda T} C_n + TC_h + \sum_{k=1}^{\infty} e^{-\lambda T} \frac{(\lambda T)^k}{k!} . \sum_{r=0}^{k} {}^k c_r \, p^r q^{k-r} (k-r) \, c_s$$

$$= e^{-\lambda T} C_n + TC_h + qc_s \lambda T \tag{8.6}$$

Differentiating (8.) w.r.t. T and equating to zero, we get

$$\frac{C_2 + qC_S \lambda}{\lambda C_n} = e^{-\lambda T}$$

T satisfying the above equation is optimal.

(ii) Let $f(T)$ be a two-stage Erlangian with parameter λ, then we get

$$2\lambda TC_n + C_s q e^{-\lambda T} = \frac{e^{-\lambda T}\left[2C_h + C_S q \lambda\right]}{\lambda}$$

T satisfying the above equation is optimal.

8.4 NUMERICAL ILLUSTRATION (MODEL 1)

Let

$$\alpha = 0.5, \qquad \lambda = 1, \qquad \theta = 0.5, \qquad C_F = \$5000, \qquad C_V = \$500.$$

Then

$$e^{-\lambda(1-\theta)} = e^{-0.5} = 0.6065$$

$$1 - e^{-\lambda(1-\theta)} = 0.3935$$

$$\theta \quad e^{-\lambda(1-\theta)} = 0.5 x 0.3935 = 0.3033$$

$$e^{-\alpha T}[1 - e^{-\lambda(1-\theta)}] = e^{-0.5T[0.3935]}$$

$$= e^{-0.1968T}.$$

Therefore

$$\frac{0.5 + 0.25T\left[1 - 0.3033e^{=0.1968T}\right]}{0.5x0.3935e^{-0.1968T}} = 10.$$

Taking first approximation to $e^{-0.1968\,T}$

$$\frac{0.5 + 0.25T\left[1 - 0.3033\,(1 - 0.1968T)\right]}{0.1968\,[1 - 0.1968T]} = 10$$

$$\Rightarrow 0.0149T^2 + 0.5612T - 1.468 = 0$$

Let

$$f(T) = 0.0149T^2 + 0.5612T - 1.468.$$

Then

$$f(2) < 0, \qquad f(3) > 0$$

This implies that the optimum value of T lies between 2 and 3.

By Newton's method of approximation $\hat{T} = 2.45$ years. Such similar results can be obtained for a given set of values λ, α, θ, C_F and C_V. It would be interesting to investigate the variation in $\hat{T}$ when one of the above parameters is allowed to vary keeping the other parameters and costs fixed. The variations in $\hat{T}$ as suggested above are dealt with by representing them by graphs.

8.5 CASE (1)

Let

$$\lambda = 1, \qquad \theta = 0.5, \qquad C_F = \$5000, \qquad C_V = \$500.$$

Fix all these parameters and allow α to vary; α is the exponential parameter and hence $\alpha > 0$. For various values of α we have the following:

Table 8.1: Increasing Inter-arrival times (Model I)

α	equation	$\hat{T}$
0.5	$0.0149T^2+0.5612T-1.468=0$	2.5
0.6	$0.0258T^2+0.8082T-1.761=0$	2.0
0.7	$0.0409T^2+1.10004T-2.055=0$	1.7
0.8	$0.0955T^2+1.4368T-2.348=0$	1.5
0.9	$0.11074T^2+1.8193T-2.642=0$	1.3
1.0	$0.11937T^2+2.2447T-2.935=0$	1.2
1.1	$0.2061T^2+2.7163T-3.229=0$	1.1
1.2	$0.2062T^2+3.2932T-3.522=0$	1.0
1.3	$0.2623T^2+3.7944T-3.816=0$	0.9
1.5	$0.4028T^2+5.0506T-4402=0$	0.8
2.0	$0.9548T^2+9.5808T-5.87=0$	0.6

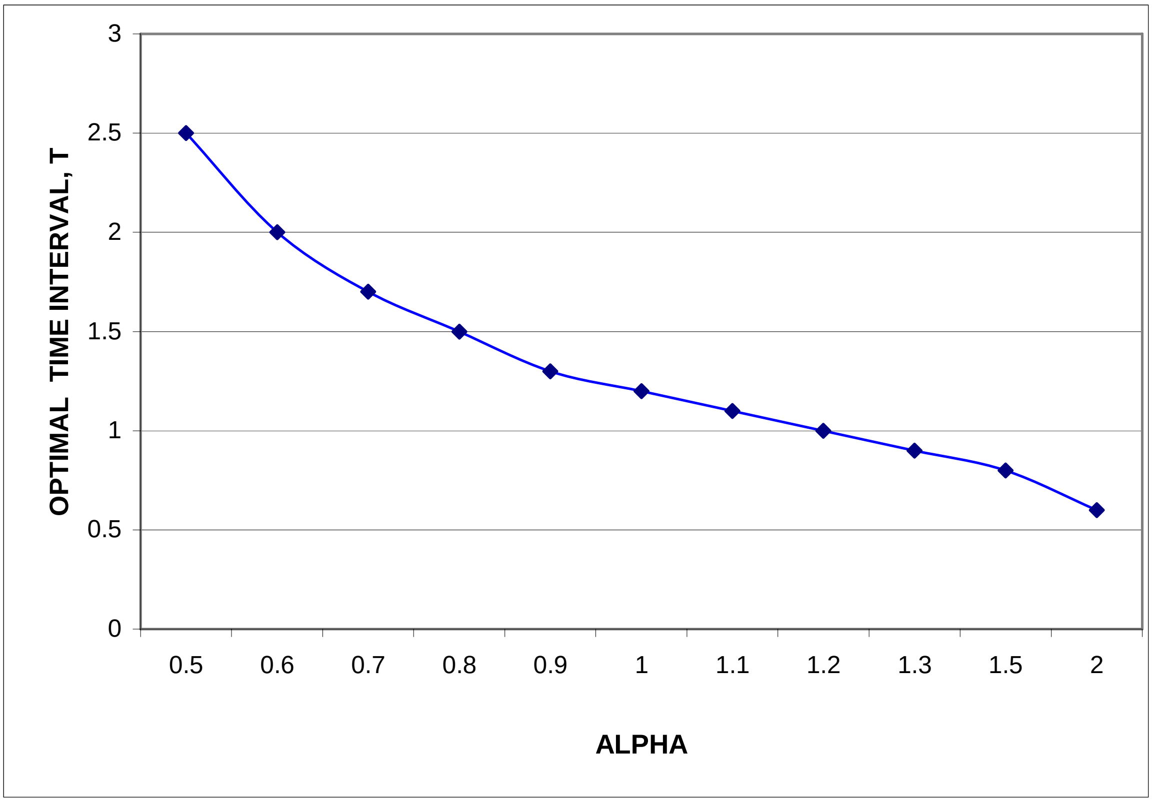

Figure 8.1: Model I

It may be observed that $\hat{T}$ values decrease when the value of α increases keeping other parameters and costs fixed. It shows that if the inter-arrival times between decision-making epochs are made shorter, it results in the optimal period becoming shorter because many decisions are made at shorter intervals thereby creating more vacancies.

Case (ii)

Let

$$\alpha = 0.5, \quad \theta = 0.5, \quad C_F = \$5000, \quad C_V = \$500.$$

Fix these values and allow λ to vary since λ is the Poisson parameter $\lambda > 0$, for various values of λ ,we have Table 8.2.

Table 8.2: increasing rate of leaving (Model I)

λ	equation	$\hat{T}$
1.0	$0.0149T^2+0.5612T - 1.468=0$	2.5
1.1	$0.01687T^2+0.643T -1.566=0$	2.3
1.2	$0.0185T^2+0.0.7266T -1.656=0$	2.1
1.5	$0.0234T^2+0.9824T -1.889=0$	1.8
2.0	$0.0291T^2+1.3998T-2.161=0$	1.5
2.5	$0.0319T^2+1.8064T-2.318=0$	1.3
3	$0.0325T^2+2.1743T-2.384=0$	1.1

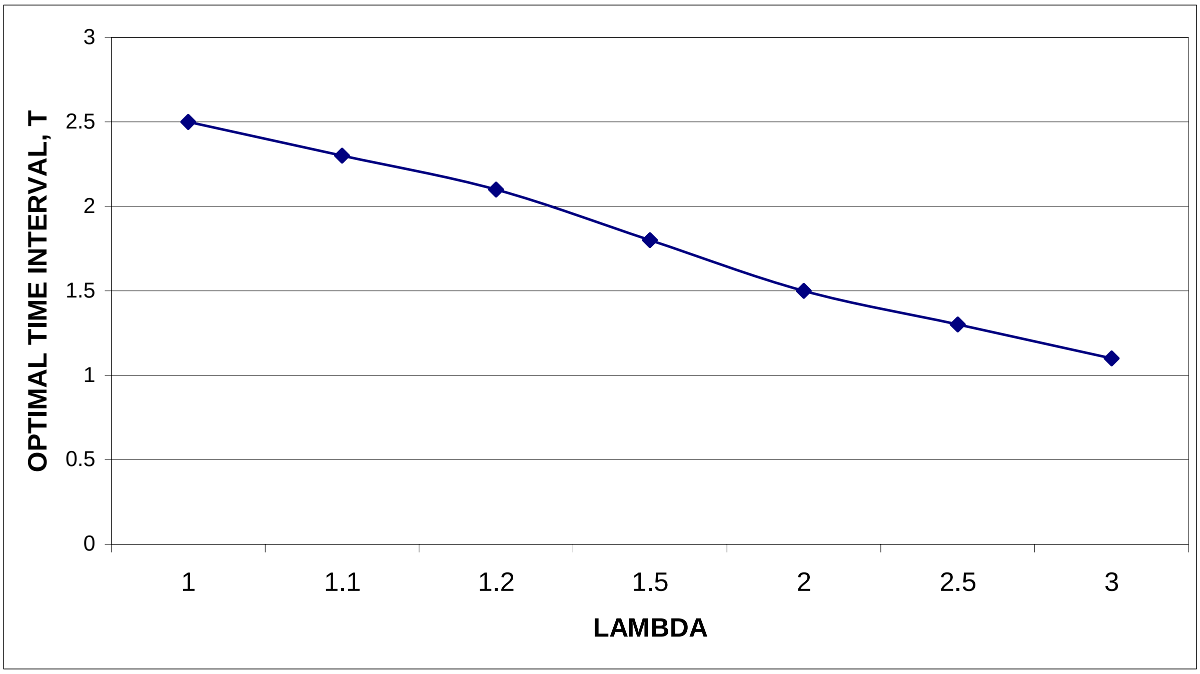

Figure 8.2: Model I

From Table 8.2, we infer that if λ increases, the number of persons leaving on average at each decision epoch increases, which in turn compels the withdrawal of the scheme or closure of the policy at an earlier date. Hence $\hat{T}$ decreases.

Case (iii)

Let

$$\alpha = 0.5, \quad \lambda = 1, \qquad C_F = \$5000, \qquad C_V = \$500.$$

Allow to θ vary. Since θ is the parameter of the geometric distribution, $(0 < \theta < 1)$ for various values of θ, we have:

Table 8.3: Model I

θ	**equation**	$\hat{T}$
0.1	$0.005T^2$+1.1198t-2.467=0	2.2
0.2	$0.0061T^2$+0.9845t-2.253=0	2.3
0.3	$0.0093T^2$+0.8458t-2.017=0	2.3
0.4	$0.0124T^2$+0.7021t-1.756=0	2.4
0.5	$0.0149T^2$+0.5612t-1.468=0	2.5
0.	$0.0166T^2$+0.4215t-1.149=0	2.5
0.7	$0.0168T^2$+0.2833t-0.796=0	2.4
0.8	$0.0148T^2$+0.168t-0.406=0	2.1

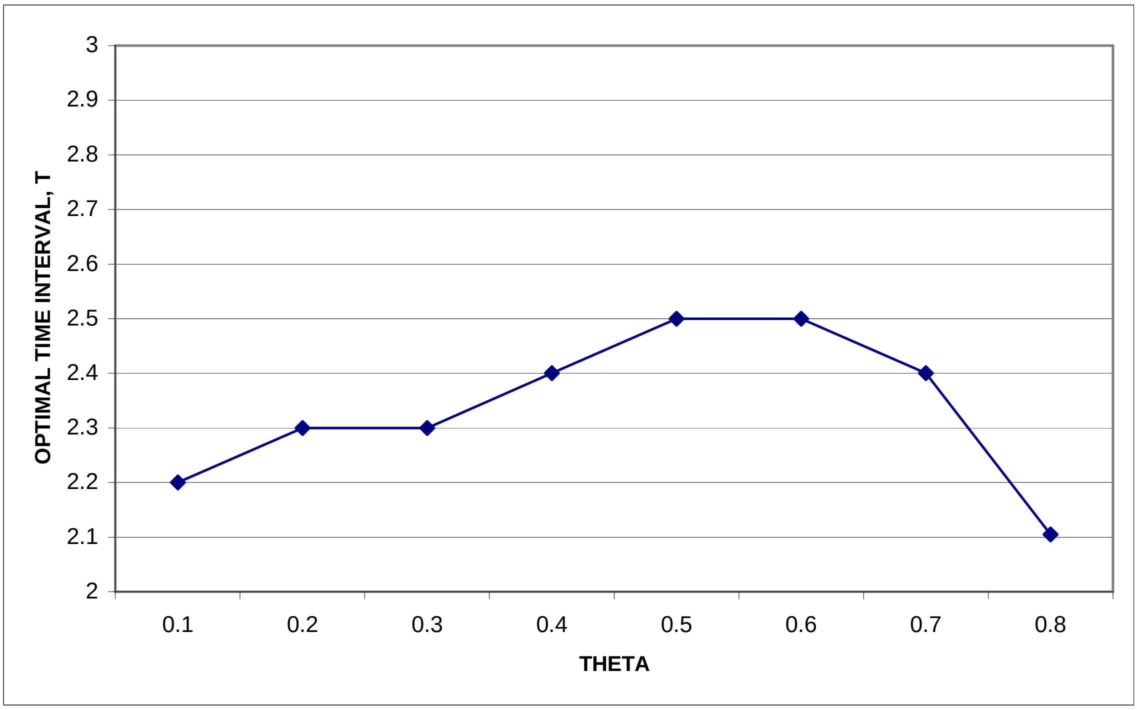

Figure 8.3: Model I

From Table 8.3 the value of $\hat{T}$ increases initially with respect to θ and then starts decreasing. For θ=0.5 and θ=0.6 , the value of $\hat{T}$ is the maximum. Figure 8.3 depicts the same.

8.6 NUMERICAL ILLUSTRATION (MODEL2)

The value of T which satisfies equation (8.6) is the optimal T and it can be obtained for specific values of λ, q, C_n, C_s *and* C_h.

For example if we take

$$\lambda = 3, \quad q = 0.5 \quad C_s = \$5000, \quad C_n = \$20\,000, \quad C_h = \$500$$

the optimal T=1.3483 units.

8.7 CONCLUSION (MODEL 1)

It may be observed that the very essence of this result lies in the fact that the absence of voluntary retirement introduced will be withdrawn and will not be re-introduced again till the end of $\hat{T}$. In practical applications the estimates of the parameters λ, α, and θ may be obtained by using appropriate methods of estimation on the basis of the past data available in the organisation.

8.8 CONCLUSION (MODEL 2)

It is inferred that the optimal value of T depends upon the parameters like λ and q and the costs involved such as C_n, C_s and C_h. For every combination of these quantities, the optimal T can be obtained by solving the corresponding non-linear equation. It would be interesting to investigate the behaviour of T consequent to the changes in λ, keeping all other values fixed. It can also be seen by calculation that as q increases the optimal value of T increases. While considering the inter-arrival times between screening tests for different distributions it has been noted that the equation that provides the optimal T changes with every change.