CHAPTER 5

5 SOLUTION OF THE THERMAL NETWORK WITH TIME DEPENDENT PARAMETERS

In chapter 2 of this thesis we discussed the theoretical foundation of the simple building thermal analysis method of Mathews and Richards, based on the thermo–flow network of figure 2.1. It was indicated that such a simple model has a number of important advantages which makes it a very useful model on which to base a design tool. Despite it's theoretical limitations, the method has proved itself to be accurate, easy to interpret and quick. In this model, as in most other thermal models, it is assumed that the parameters of the thermo–flow network, i.e. the resistances and capacitances in figure 2.1, are time invariant. This assumption is required to enable the application of standard Fourier or Laplace techniques to obtain the solution. However, in actual buildings, some of these parameters are definitely not constant. In passive buildings, people are encouraged to open and close windows to suit themselves. This will have an effect on the ventilation resistance as well as the interior film coefficients. In active buildings, the surface convection coefficients will also vary since the air movement is controlled by the HVAC system. Furthermore, contemporary[1] trends in the design of energy efficient buildings require simulation of such exotic elements as windows with variable transmittance [1], walls with variable thermal conductance [2] and buildings with variable heat storage characteristics. These developments require the investigation of thermal models with variable parameters in order to help asses the viability of these concepts and to aid the designer.

The simple method of Mathews and Richards is ideally suited for extension to time variable systems. Its very simplicity allows the possibility of a practical and efficient implementation of a time variable solution,

[1]Actually the control of transmittance of light is not so 'contemporary'. Curtains, shutters and blinds have been used for ages to similar effect.

something which might be impossible or extremely cumbersome in a more refined model.

In this chapter we investigate the solution of the model of figure 2.1, under the assumption that the parameters are time dependent. The most important application we foresee for such a time dependent simulation, is the proper treatment of time dependent ventilation in naturally ventilated buildings. The prediction of thermal response with time dependent ventilation is sorely lacking in most programs. This serious shortcoming inhibits the practical application of many programs to passive design of buildings.

In this chapter we first examine the solution of the network of figure 2.1 with constant parameters. This is followed by a literature survey to determine possible methods for solving the time dependent system. Since very little formal techniques exist for solving time dependent systems, and those that do exist are not well known, it was felt necessary to discuss the literature in some depth in this chapter. It was discovered that most existing techniques are not applicable, since they assumed that the time dependency of the parameters can be regarded as a small perturbation on the steady value of the parameter. Nevertheless, it was still considered necessary to describe some of the more common methods here, to give some perspective. In this study, it is assumed that the parameters are subject to sudden large variations. This is required if the model is to be useful for predicting the consequences of a sudden opening of a window, or the switching on of forced ventilation. In the end it was decided that a simple numerical procedure, based on a first order difference technique is the most appropriate. But first the solution for the time independent network is discussed.

5.1 Solution of the Model with Constant Parameters: A Systems Approach

In figure 2.1 the electrical network equivalent for the heat flow problem in buildings is shown (from reference [10] of chapter 2). For convenience, the figure is repeated here as figure 5.1. To recap, in this figure T_{sa} is the combined sol–air temperature, T_o is the ventilating air temperature, T_i is the indoor air temperature, Q_r is a radiative source for penetrating radiation and interior radiative sources and Q_c is a convective source. The shell resistance and thermal capacity of the building is represented by R_o and C respectively, R_a is the internal surface heat transfer coefficient and R_v is the ventilation resistance.

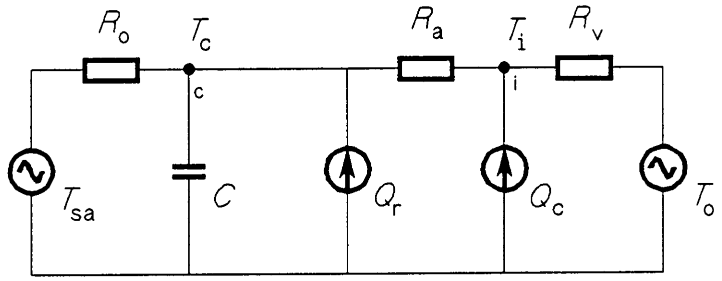

FIGURE 5.1 The thermo–flow network of Mathews and Richards. From reference [10] chapter 2.

The main purpose of this chapter is to determine the solution of this network when the R's and C become functions of time. As a point of departure we investigate the solution in the time invariant case.

5.1.1 Solution for Calculating Indoor Temperature[2]

The solution of internal temperature, T_i, for constant parameters in the Laplace domain is:

$$T_i(s) = \frac{1}{s\tau_p + 1}\Big[(s\tau_z + 1)\cdot(R_a + R_o)\cdot(T_o + R_v\cdot Q_c) + R_v\cdot(T_{sa} + R_o\cdot Q_r)\Big]\cdot\frac{1}{R_a + R_o + R_v} \tag{5.1}$$

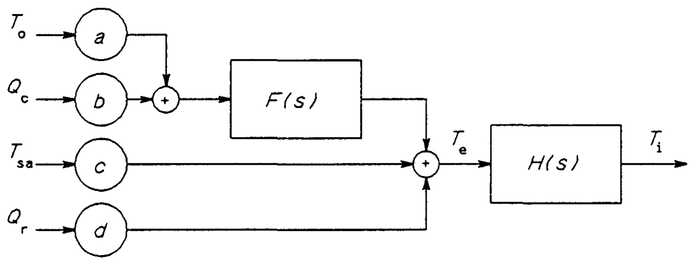

FIGURE 5.2 Block diagram of the solution of the network of figure 5.1 for interior temperature. The symbols are defined in the text.

The solution is shown schematically in figure 5.2. The definitions of the variables in this figure are:

$$a = \frac{R_a + R_v}{R_a + R_v + R_o} \quad \text{Contribution factor of outside air.} \tag{5.2}$$

$$b = \frac{R_v\,(R_o + R_a)}{R_a + R_v + R_o} \quad \text{Contribution factor of convective source.} \tag{5.3}$$

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[2]This solution, as given here, is not different in essence from the one given by Mathews and Richards. It is different though, in that Mathews and obtains the response of the system via a Fourier Transform of the response while this solution uses the frequency response of the system directly. This allows a somewhat more efficient implementation. Furthermore, the solution is given in a convenient systems format so that the influence of each individual source becomes very clear. An effective source is defined so that the response can be obtained from a single convolution between effective forcing function and response. Mathews and Richards treat the various sources individually.

$$c = \frac{R_v}{R_o+R_v+R_a} \quad \text{Contribution factor of sol–air temperature.} \tag{5.4}$$

$$d = \frac{R_v \cdot R_o}{R_o+R_v+R_a} \quad \text{Contribution factor of radiative source.} \tag{5.5}$$

$$T_e = (a \cdot T_o + b \cdot Q_c) \cdot (s\tau_z + 1) + c \cdot T_{sa} + d \cdot Q_r \tag{5.6}$$

$$F(s) = s\tau_z + 1, \qquad H(s) = 1/(s\tau_p + 1). \tag{5.7}$$

$$\tau_z = \frac{R_o \cdot R_a}{R_o+R_a} \cdot C \quad \text{Time constant of the zero.} \tag{5.8}$$

$$\tau_p = \frac{R_o\,(R_a+R_v)}{R_o+R_a+R_v} \cdot C \quad \text{Time constant of the system pole.} \tag{5.9}$$

Note the contribution of the individual sources, through their scaling factors a,b,c and d to T_e. The figure indicates how the various forcing functions contribute to the combined effective forcing function T_e. The figure indicates that the thermal network can be regarded as a single input single output system with system function given by the single pole designated $H(s)$ in the figure. The input to this system is the combined forcing function T_e and the output is T_i. Consequently, the network displays the characteristics of a single pole system, i.e. the impulse response is an exponential decay. The outside air temperature and convective source enter in the combined forcing function through a single zero network, $F(s)$, indicating that the time derivatives of these sources also effect T_e. A sudden change of the convective load Q_c will thus impart an impulsive component to T_e. Physically this implies that if e.g. the convective load suddenly increases, the interior temperature will tend to rise with it and then settle exponentially as the storage effect of the capacitor comes into play. Note also that the contributions from the sol air and radiative sources decrease as R_v decreases, implying that larger ventilation rates force the temperature to T_o. Furthermore, the position of the pole, thus the time constant, is also influenced by R_V but the position of the zero is independent of the ventilation.

From figure 5.2 we see that once the effective forcing function, T_e, has been determined, the response of the building is represented by a single pole network *H(s)* with time constant τ_P. The usual method for calculating the response of such a circuit, will obtain the indoor temperature via the convolution of the effective forcing function with the circuit's impulse response. Since the discovery of the Fast Fourier Transform this is efficiently done by multiplication in the frequency domain [3] as explained in §5.1.3.

5.1.2 Load Calculation

In active systems, the interior air–temperature is not so important, since it is a prescribed quantity. The active system attempts to control the interior temperature so that it will always be close to the set point temperature. Of more importance, in this case, is the load on the HVAC system. This energy load is the convective load, Q_c, required to keep the interior temperature at a specified level. In actual HVAC systems, the temperature control will be maintained through a closed loop system, where the convective load is adjusted to keep the interior temperature at the prescribed level. Since many different types of controllers exist, there is no unique definition of HVAC load. Normally, the system incorporates some sort of 'dead band' control where the interior temperature is allowed to swing through a specific range. The control action is only activated when the interior temperature strays outside this range.

In addition, various control laws are used. Some controllers will proportionally increase the system load as the interior temperature strays further and further from the set point. Others will switch the system on at full capacity at a certain threshold. Since a specific control law defines a specific HVAC load, a complete system simulation which incorporates the passive response of the building, the response of the plant and the controller is required to accurate determine the system load. However, the objective of a simple design tool for thermal response of buildings is not complete system simulation, it is the optimization of the passive design of the building. All that is required is for the program to predict a uniquely defined required capacity of the HVAC system. The building designer can then optimize his design by minimizing this required system capacity. To this purpose, it is convenient to define the system load as the convective load which is required to maintain the interior temperature at a prescribed setting. This definition leads to a simple procedure for determining the load. In §5.1.4 it is shown that the present treatment is easily extended to proportionally controlled systems without a deadband thermostat.

Setting T_i equal to the set–point temperature T_{ir} and solving for the system load, $Q_c = Q_{cr}$, from (5.1) gives:

$$Q_{cr}(s) = \frac{T_{ir}\cdot(s\tau_p + 1)\cdot(R_a + R_v + R_o) - R_v\cdot(T_{sa} + R_o\cdot Q_r)}{R_v\cdot(s\tau_z + 1)\cdot(R_a + R_v)} - \frac{T_o}{R_v} \tag{5.10}$$

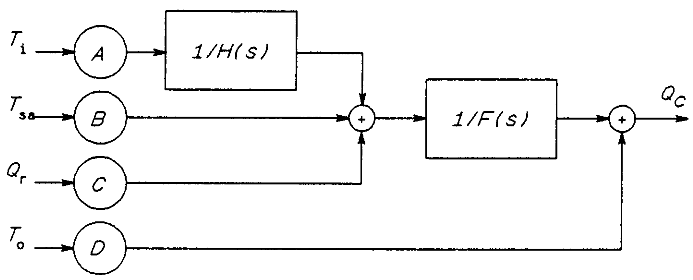

FIGURE 5.3 Block diagram for load calculation via (5.10). The symbols are defined in the text below.

Figure 5.3 shows the block diagram for this solution of the system load. The diagram shows how to find the convective load, Q_{cr}, for a specified interior air–temperature, T_{ir}. Note that in this case, the system functions, *H(s)* and *F(s)* are inverted so that the time–constant of the pole is now τ_z. Consequently, the system pole is now independent of R_v and the position of the zero τ_p varies with R_v. Note also that ventilation adds directly to the system load via the contribution factor D in figure 5.3. The definitions of the new contribution factors in figure 5.3 are:

$$A = \frac{R_a + R_v + R_o}{R_v \cdot (R_a + R_o)} \tag{5.11}$$

$$B = \frac{-1}{(R_a + R_o)} \tag{5.12}$$

$$C = \frac{-R_o}{(R_a + R_o)} \tag{5.13}$$

$$D = \frac{-1}{R_v} \tag{5.14}$$

In both systems, figure 5.2 and figure 5.1, the solution is given by the response of a single pole system to a combined forcing function. In figure 5.2, the time–constant of the system is given by τ_p and in figure 5.3 it is given by τ_z.

5.1.3 The Governing Equation

It may be remarked that, in general, when the position of a system zero is time dependent, no serious complications arise in the calculation of the response. The zero directly effects the input forcing function and not the response. This can be deduced from figure 5.2. The system

$$F = (s\tau_z + 1) = z/x$$

transforms to the differential equation

$$z(t) = \tau_Z . \frac{dx(t)}{dt} + x(t) \tag{5.15}$$

which merely adds τ_Z times the derivative of the input to the effective forcing function. However, when a system pole becomes time–dependent, the characteristics of the system itself is affected. Any change in a parameter has also another distinct effect; the contributions of the sources to the effective forcing function changes with the parameters. We may therefore conclude that variable parameters will have two separate kinds of effects on the system:

a The system input function is modulated with the parameter variations, since these variations change the contribution factors of the various sources, *viz.* a,b,c,d or A,B,C,D above, and influences the time constant of the system zero. This means that a smooth input function will become discontinuous if the parameters jump, since the contributions of the various sources to the input function will jump with the parameters.

b Secondly, the response of the system itself is affected since the time–constant of the pole also varies with the parameters. We shall see that this implies a time dependent impulse response.

Since the first type of effect essentially adds no extra complication, (the input function just becomes a stronger function of time), in the rest of this study, we concentrate on calculating the influence of the variable time–constant on the response of the single pole system.

With y identified as the response, and f the input forcing function, the differential equation of the single pole system,

$$H = 1/(s\tau_P + 1) = y/f$$

is:

$$\tau_P . \frac{dy}{dt} + y = f(t). \tag{5.16}$$

If this equation is divided by τ_P we obtain the standard first order differential equation with $\beta = 1/\tau_P$:

$$\dot{y}(t) + \beta \cdot y(t) = x(t) \tag{5.17}$$

Note that the forcing function, f, is also divided by τ_P. For constant parameters, the impulse response of the system, $h_0(t)$, is the response to $x(t) = \delta(t)$, where $\delta(t)$ is Dirac's impulse function, and is given by:

$$h_0(t) = e^{-\beta t} \tag{5.18}$$

The frequency response, $H_0(\omega)$, is given by the Fourier transform of (5.18):

$$H_0(\omega) = \frac{1}{\omega + j\beta} \tag{5.19}$$

According to the convolution theorem, the response of the system to any arbitrary forcing function $x(t)$, is given by the convolution of $h_0(t)$ with $x(t)$:

$$y(t) = x(t)^{*}h(t) = \int_{-\infty}^{t} x(u) \cdot h_0(t - u)\, du. \tag{5.20}$$

The convolution can be efficiently calculated by multiplication in the frequency domain [3,4]:

$$Y(\omega) = X(\omega)/(\omega + j\beta) \tag{5.21}$$

where $Y(\omega)$ is the Fourier transform of $y(t)$ and $X(\omega)$ is the Fourier transform of $x(t)$.

In (5.21) we have written the lower integration limit at infinity to indicate that we are interested in the steady state solution; after all transients have died out.

In this study, we assume that all the sources are varying periodically and all the results apply only to periodically varying temperatures and loads. Furthermore, we also assume that variations in the parameters of the thermal network occur periodically. It may be argued that conditions in buildings are never perfectly periodic, however, since the earth has presumably been spinning round it's axis since time started, it is not possible to assign initial conditions to the climatic forcing functions. They should be regarded as at least quasi-periodic. In many thermal analysis programs arbitrary initial conditions are assumed, the solution is then integrated for a couple of days until all transients are extinct. Our objective is to find an efficient solution which do not require a long initial period of integration. Furthermore, for a design tool, the philosophy of a design day[3], or a design week is appropriate. In this case the assumption of periodic variation is fully justified.

5.1.4 Proportionally Controlled Active Systems

It is very easy to extend the model to include proportionally controlled active systems. By 'proportionally controlled' we mean systems which have no dead band, the system load is set proportional to the difference between the indoor temperature and the set point temperature. It is also assumed that the system has infinite capacity. In practice, this idealized system is realized if the deadband is small and the capacity of the system is always sufficient to maintain the indoor climate.

[3]When the designer follows a 'design day' philosphy he chooses a specific day on which he base his designs. The design day is usually statistically significant. E.g. when determining system cooling capacity, a design day is chosen so that only a small percentage of actual days will exceed the maximum temperature of the design day.

If another convective load, the system load Q_s, given by:

$$Q_s = \alpha \cdot (T_i - T_t) \tag{5.22}$$

with α the proportional feedback gain [W/K] and T_t the thermostat (set–point) temperature, is added to the convective load Q_c in (5.1), it is found that the behaviour of the system is again governed by (5.17) but with parameter $\beta(t)$ now given by:

$$\beta_S(t) = \beta_T - \frac{\alpha \cdot R_v^2}{C \cdot (R_a + R_v) \cdot (R_a + R_v - \alpha \cdot R_a R_v)}. \tag{5.23}$$

In the limit $\alpha \rightarrow \infty$, we find $\beta_S \rightarrow \beta_E$ in accordance with the root–locus theorem for closed loop systems. It is seen that by this simple redefinition of β, the method can be extended to proportionally controlled systems. Many thermostats include non–linearities such as dead bands and hysteresis. These effects can also be included in the model, but the solution of the model becomes arduous; the initial value must be found by successive iteration [5].

5.2 Variable Network

When the parameters R_o, R_v, R_a and C are functions of time, the system equations are much more complicated. In this discussion we shall concentrate on changes in the value of the ventilation resistance R_v, since in practice, variable ventilation rates are the most important operation we wish to model. Nevertheless, changes in the other parameters are also important. The ventilation rate also has an influence on the interior surface coefficients, and as explained above, many modern ideas require simulation of time dependent shell conductance etc. Since there is little distinction between changes in R_v and changes in the other resistances and also the capacitance, we derive general equations valid for all changes which conserve the amount of stored heat in the structure, as discussed in the next paragraph.

Time depend circuit parameters have a drastic effect on the thermal response. Normally buildings exhibit sluggish behaviour, the time constant of a typical office block might be 30 hours or more. Sudden interior temperature changes are therefore not expected if the parameters are invariant. A change in a circuit parameter, however, might have a very sudden effect on the interior temperature. This effect is often experienced when the door is opened on a cold windy day. The interior air–temperature immediately drops to the outdoor air–temperature but quickly recovers when the door is closed again. Changes in the value of the storage capacitance, C, may also cause drastic and enduring consequences. This is understandable in view of the fact, that, if the stored energy is assumed to be conserved when the capacitance changes, the temperature of the storage structure must necessarily change with the capacitance. It is also possible to change the capacitance without changing the temperature of the storage structure, e.g. by simultaneous with the change in value of C, extracting or adding the correct amount of heat to the reservoir. In the first case, a discontinuous change in the value of C results in a discontinuous temperature. In the second case, the time constant, hence the rate of charge or discharge of the capacitor is affected, and discontinuities in the derivative of the temperature must be expected, while the temperature itself stays continuous. In thermo–flow problems, the capacitance value is changed when a massive structure is added to or removed from the thermal system. Generally, when mass is added to a zone the new mass will be at a different temperature, and both the value of the capacitance and the amount of stored heat will be affected.

In this study we have assumed the amount of stored heat in the structure is not directly affected by the changes in the parameters. The results are therefore applicable only to changes in the value of C of the first type. Obviously, this type of change in the value of C will seldom be practicable. The solutions we derive are therefore, in practice, confined to changes in the values of the resistances, since these will have no affect on the stored charge. The effect of a change in the value of a resistance is similar to a change in the value of C of the second kind, i.e. with temperature conserved, since the time constant is affected.

Since we assume the amount of stored heat is not affected by changes in the parameters, it was found a great advantage to write the governing equations in terms of this amount and not in terms of the primary quantities of interest; namely temperature and heat flow. This indirect approach facilitates a great reduction in the formal complexity of the equations and substantially improves the accuracy of the solution.

5.2.1 Indoor temperature

The general equations, with all the variables assumed functions of time, which gives the indoor temperature resulting from the various forcing functions, are derived in appendix A: They are:

$$T_{\mathrm{i}} = \frac{T_{\mathrm{c}} \cdot R_{\mathrm{v}} + T_{\mathrm{x}} \cdot R_{\mathrm{a}}}{R_{\mathrm{v}} + R_{\mathrm{a}}} \tag{5.23}$$

where

$$T_{\mathrm{x}} = T_{\mathrm{o}} + R_{\mathrm{v}} \cdot Q_{\mathrm{c}} \quad \text{and} \quad T_{\mathrm{y}} = T_{\mathrm{sa}} + R_{\mathrm{o}} \cdot Q_{\mathrm{r}} \tag{5.24}$$

and T_{c} is the temperature at the structure node (across the capacitor in figure 5.1). T_{c} is given in terms of the stored heat q[4]

$$T_{\mathrm{c}} = q/C \tag{5.25}$$

[4]For notational consistency with Mathews and Richards ([11] chapter 2) we used Q for the loads in figure 5.1. We shall therefore use q for stored charge in this chapter as opposed to its use in chapter 2, where it indicated heat flux.

which must be obtained from the governing differential equation:

$$\dot{q} + \beta_T \cdot q = f_T. \tag{5.26}$$

The forcing function in this equation is:

$$f_T = \frac{T_x}{R_a + R_v} + \frac{T_y}{R_o} \tag{5.27}$$

and the parameter β_T is:

$$\beta_T = \frac{R_a + R_o + R_v}{C \cdot R_o \cdot (R_a + R_v)} = \frac{1}{\tau_p} \tag{5.28}$$

It is possible to substitute (5.24) to (5.28) into (5.23) to obtain an equation which directly gives T_i in terms of the sources. The equation is:

$$\begin{gathered} \frac{d}{dt}\left[\frac{(R_v + R_a) \cdot C}{R_v} \cdot T_i\right] + \beta_T \cdot \left[\frac{(R_v + R_a) \cdot C}{R_v} \cdot T_i\right] = \\ \left[\frac{R_a + R_o}{R_o \cdot R_v}\right] \cdot T_x + \frac{d}{dt}\left[\frac{C \cdot R_a}{R_v} \cdot T_x\right] + \frac{T_y}{R_o}. \end{gathered} \tag{5.29}$$

This equation, although of the same form as (5.26), is considerably more difficult to solve accurately. This fact can be demonstrated:– the denominators of the derivative terms contain R_v as a factor, this implies that the equation is not valid when R_v vanishes. Therefore, (5.29) is not applicable to the important test case where the ventilation rate is large. In contrast, the correct result immediately follows from (5.23), irrespective of the accuracy of the solution of the differential equation.

5.2.2 System Load

The convective load, Q_{cr}, required to maintain a specified indoor temperature, T_{ir}, is obtained as before, by setting $T_i = T_{ir}$ and solving for

$Q_c = Q_{cr}$ in (5.23) to (5.28). The result is:

$$Q_{cr} = \frac{R_a + R_v}{R_a \cdot R_v} \cdot T_{ir} - \frac{1}{R_a} \cdot T_{cr} - \frac{1}{R_v} \cdot T_o \tag{5.30}$$

where T_{cr} is the required temperature at the structure node given by

$$T_{cr} = q_r / C. \tag{5.31}$$

In (5.31) q_r is again the amount of heat stored in the structure, which is obtained from the differential equation:

$$\dot{q}_r + \beta_E \cdot q_r = f_E. \tag{5.32}$$

The forcing function is now:

$$f_E = \frac{T_y}{R_o} + \frac{T_{ir}}{R_a} \tag{5.33}$$

with the same definition of T_y as before. The parameter β_E is now:

$$\beta_E = \frac{R_a + R_o}{R_a \cdot R_o \cdot C} = \frac{1}{\tau_z}. \tag{5.34}$$

It is once again possible to obtain an equation linking Q_{cr} directly with T_{ir}, but the comments of §1.2.1 in this regard, is valid also here.

5.2.3 The General First Order Differential Equation

In the previous two sections, we have given the general equations for time varying systems. These equations indicate that the solutions are obtained from a linear[5] first order differential equation, with time varying coefficient β(t).

$$\dot{y}(t) + \beta(t) \cdot y(t) = x(t) \tag{5.35}$$

The forcing function, $x(t)$, of (5.35) is a strong function of the circuit parameters via the scaling factors, a,b,c,d or A,B,C,D in figures 5.2 & 5.3. In this way, the variable parameters can increase the rate of change of the effective forcing functions. It is important to realize that while the 'natural' forcing functions are probably reasonably smooth and continuous, and can be represented with a small number of Fourier components, sudden changes in the circuit parameters will cause discontinuities in the effective forcing functions, which will now require a large number of Fourier components to represent. However, the single pole circuit always tends to reject all fast changes in the forcing functions, even if the position of the pole is time dependent. It is therefore still possible to obtain fairly accurate answers with a modest number of Fourier components since the influence of the system in the input is to smooth the response.

The solution of the time dependent network is made difficult by the absence of a closed form solution for the impulse response and frequency response of the network, in the form of (5.18) and (5.19). The effect of the changes in the time–constant can be qualitatively understood by recognizing that if, in the frequency response (5.19), β is a function of time $\beta(t)$, the amplitude as well as the phase of a sinusoidally varying input signal will be modulated. In consequence, it is to be expected that the response will contain a substantial amount of superimposed signals at frequencies different from the original forcing frequency and the sinusoidal character of the forcing function will be distorted by the variable network.

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[5]Since the equation is linear, albeit time variant, it is permissible to decompose the input function in a series. The output will be given by the sum of the individual responses to the terms of the series. However, the equation is not linear in β and the parameter β can not be decomposed.

The objective of the rest of this chapter is to find a solution for the response of the time dependent one pole system of (5.35). The general solution is first examined and, in later sections, we look at specific approximate solutions described in the literature.

The general solution of the differential equation when $\beta(t)$ is a time dependent parameter is well known [6]:

$$y(t) = exp[-\Gamma(t)] \cdot \left[\int_{-\infty}^{t} exp[\Gamma(t)] \cdot x(t) \, dt\right] \tag{5.36}$$

with

$$\Gamma(t) = \int \beta(t) \, dt. \tag{5.37}$$

In the equation, the lower limit of the integral was again written as $-\infty$ to obtain the steady state response to the – now assumed periodic – forcing function $x(t)$.

Obviously the general solution can not be evaluated analytically if the function $\beta(t)$ is only known in tabulated format. Furthermore, it is clear that a numerical integration (e.g. Simpson's rule) will be problematic because of the infinite range and the oscillating nature of the integrand. The indicated range of integration $(-\infty,t]$, can be interpreted as indicating that the integration must proceed over a very long initial period, so that the transient response will be well dampened. As a rule of thumb the transient response is negligible after about 5 time–constants have elapsed. For buildings with typical time–constants of 30 hours or more, this requires an initial period of integration of more than 150 hours. It follows that direct numerical integration of the general solution will place a heavy burden on the computer. This is nevertheless the approach used by many other existing programs, although they normally integrate the original differential equation with one of the standard procedures (e.g. Runge Kutta). Obviously this method suffers from exactly the same drawback: the initial solution contains the transient response superimposed on the steady state response. Numerical methods will be discussed again later. Normally, Fourier series methods are used to find the steady state solution. These are discussed in §5.3.4.

First it should be explained why the existing methods for solution of the constant parameter model, i.e. convolution in frequency domain, can not be extended to the variable parameter case. The answer to this question is in brief: because the impulse response is not time invariant anymore, and the usual method of implementation of the convolution via Laplace or Fourier transform techniques, presupposes time invariance of the impulse response. In fact we see that it is impossible to obtain a closed form solution for the impulse response. This fact can be illustrated by calculating the impulse response of the system via the general solution above (5.35). If the input consists of an impulse $x(t) = \delta(t-t_0)$, occurring at time $t = t_0$ the general solution yields:

$$h(t,t_0) = u(t-t_0)\cdot exp[-\Gamma(t)]\cdot\left[\int_{-\infty}^{t} exp[\Gamma(t_1)]\cdot\delta(t_1-t_0)\ dt_1\right]$$

$$= u(t-t_0)\cdot exp[\Gamma(t_0) - \Gamma(t)]. \qquad (5.38)$$

where $u(t)$ is the unit step function.

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases} \qquad (5.39)$$

The impulse response $h(t,t_0)$ is a function of two instances of time:– the time of occurrence of the impulse t_0, and the time since the occurrence of the impulse t [7]. This is emphasized by a change of variable from t to $\tau = t-t_0$, the delay since the occurrence of the impulse, so that (5.38) becomes:

$$h(t,\tau) = u(t-t_0)\cdot exp[\Gamma(t-\tau) - \Gamma(t)]. \qquad (5.40)$$

The frequency response can now be defined as the Fourier transform of the impulse response with respect to the delay [7]:

$$H(\omega,t) = \int_{-\infty}^{\infty} h(t,\tau)\cdot exp[-j\omega(t-\tau)]\ d\tau. \tag{5.39}$$

The frequency response is dependent on time as well as frequency so that at every time instance, a different frequency response is valid or, as it is sometimes stated in the literature, the response *evolves* with time. This method will therefore require evaluation of a new impulse and frequency response, thus a Fourier transform, at every time instant. In reference [7], a full frequency domain method which utilizes the bifrequency system function, which is a double Fourier Transform from the (t,τ) domain to the (v,ω) domain, is derived. Frequency domain techniques are discussed further in §5.3.4. We will first examine some other techniques from the literature.

5.3 Methods for Solving the Variable Parameter System

Few techniques exist for obtaining solutions of equations with variable coefficients. Time variable networks are discussed in some depth by Zadeh [7,8,9], Pipes & Harvill [10] and by Gibson [11], which refers extensively to Zadeh. The earliest reference in the literature appears to be Carson [12] followed by [13]. More recent work, primarily directed at non–stationary stochastic systems, was reported by Tsao [14]. The contributions of Zadeh seems to be the most important but according to Gibson [11], the theory is still far from satisfactory: "their being as yet no general method applicable to a large class of problems". Attempts in the literature to extend the system impulse response approach to linear variable coefficient systems are not fully satisfactory [11]. The main reason seems to be the modulation effect of the variable parameters, which generate signals at different frequencies than those of the input signal, so that the Fourier components are time dependent or a double Fourier Transform is required. The mixed frequency–time domain techniques can easily lead to confusion because two independent time axes are used, t and τ in (5.40), nevertheless they are often preferred by engineers because they seem a natural extension of familiar methods. In this section some methods and results from the literature are briefly surveyed. The objective is to provide background on various problems and techniques for treating equations with variable coefficients.

We start by showing that the Laplace Transformation of the equation with variable coefficients leads to little progress [10]. Indeed, the Laplace transform of the differential equation

$$\frac{dy}{dt} + \beta(t) \cdot y = x(t) \tag{5.40}$$

is:

$$s \cdot Y(s) + y(0) + \mathscr{L}\{\beta(t) \cdot y(t)\} = X(s). \qquad (5.41$$

Using the time multiplication property of the Laplace Transform this expression is written:

$$s \cdot Y(s) + y(0) + \int_{-\infty}^{\infty} \psi(p) \cdot Y(s-p)\ dp = X(s) \tag{5.42}$$

where $\psi(s) = \mathscr{L}\{\beta(t)\}$.

We see that instead of transforming the differential equation into an algebraic equation, the Laplace transform produces an integral equation. The solution of this integral equation is discussed in §2.4.

5.3.1 Solutions for Periodic Coefficients

When the variation in the coefficient $\beta(t)$ is periodic, with the same period T, as that of the periodic forcing function (as we have previously assumed), the differential equation (5.35) is invariant under the transform $t' = t+T$ [10]. Consequently, the solution will also be periodic with period T, it need only be solved in the interval $[0, T]$.

a) Constant Segments

When the variation of the coefficient is periodic with period T and is a constant in every small segment Δt the Laplace Transform can be applied in every segment. Fodor [5] discuss the solution of an RC circuit when the resistance is periodically switched between two values. We take the case where β(t) is constant everywhere, except at two points where the value jumps discontinuously i.e. $\beta(t)$ given by:

$$\beta(t) = \begin{cases} \beta_0 & 0 \leq t < T_1 \\ \beta_1 & T_1 \leq t < T \end{cases} \tag{5.45}$$

The exact solution for a constant β and a sinusoidal input function given by $x = X\cdot(1 + m\cdot cos\omega t)$ is:

$$y(t) = y(t_0)\cdot e^{-\beta t} + A(t) \tag{5.46}$$

with

$$A(t) = \frac{X}{\beta}\cdot\left[1 - e^{-\beta t} + \frac{m\beta}{\sqrt{\beta^2+\omega^2}}\cdot\alpha(t)\right]$$

$$\alpha(t) = cos(\omega[t+t_0]-\varphi)-cos(\omega t_0-\varphi)\cdot e^{-\beta t}.$$

where $y(t_0)$ is the initial value and $tan\varphi = \omega/\beta$. Next apply this solution to the intervals in (5.45) and set $y(0) = y(T)$, $y(T_1)$ continuous. The solution for $y(0) = y_0$ is:

$$y_0 = \frac{A_1 + A_0\cdot e^{-\beta_1(T-T_1)}}{1 - e^{-\beta_0 T_1 - \beta_1(T-T_1)}} \tag{5.47}$$

with

$$A_0 = A_0(T_1) \quad \text{and} \quad A_1 = A_1(T-T_1).$$

The subscripts 0 and 1 of A and β in (5.47) refers to the first and second intervals respectively.

SOLUTION WITH TIME DEPENDENT PARAMETERS

This method can be extended to cater for the situation where the variation of β consists of more than just two constant sub–segments. In this manner, one could obtain a solution for the 24 hour period by assuming that the variation is a constant for every hour and then solving 24 equations similar to (5.46). This process will require for every harmonic component, matching of the initial and final conditions, 24 times. This procedure will be quite accurate but it is fairly intensive computationally. It requires the solution of 24 simultaneous equations for every frequency component.

b) Sinusoidal Variations of the Coefficient

When the coefficient $\beta(t)$ varies according to:

$$\beta(t) = \beta_0 \cdot (1 + m \cdot \sin\nu t), \tag{5.48}$$

where β_0, m and ν are parameters, and ν is not necessarily equal to the frequency of the forcing functions ω, we have in the general solution (5.36)

$$\Gamma(t) = \beta_0 \cdot (t + m/\nu \cdot \cos\nu t) \tag{5.49}$$

$$y(t) = \exp[-\Gamma(t)] \cdot \int_{-\infty}^{t} \Gamma(u).x(u)\, du. \tag{5.50}$$

If we assume, as before, that the input signal also varies sinusoidally and is given by $x(t) = X \cdot e^{j\omega t}$, where it is understood that only the real part of the complex phasor is physical (see §5.3.4), we have:

$$y(t) = X \cdot \exp[-\Gamma(t)] \cdot \int_{-\infty}^{t} \exp[\beta_0(u + m/\nu \cdot \cos\nu u) + j\omega u]\, du$$

$$= X \exp[-\Gamma(t)] \cdot \int_{-\infty}^{t} \exp[(\beta_0 + j\omega)u] \cdot \exp[\beta_0 m/\nu \cdot \cos\nu u]\, du \tag{5.51}$$

Now we use the well known Bessel function expansion [10]

$$e^{a\cdot \cos\theta t} = I_0(a) + 2\sum_{n=1}^{\infty} I_n(a)\cdot \cos n\theta t \tag{5.52}$$

and find

$$y(t) = X\exp[-\Gamma(t)]\cdot\int_{-\infty}^{t}\exp[(\beta_0 + j\omega)u]\cdot\Big[I_0(\beta_0 m/\nu) + 2\sum_{n=1}^{\infty} I_n(\beta_0 m/\nu)\cdot\cos n\nu u\Big]\,du$$

but

$$\int e^{bt}\cdot\cos n\theta t\,dt = K_n\,e^{bt}\cos(n\theta t-\psi_n)$$

where

$$K_n = \sqrt{b^2 + (n\theta)^2} \text{ and } \tan\psi_n = n\theta/b.$$

The integral in the solution (5.51) can therefore be written:

$$\exp[(\beta_0 + j\omega)t]\cdot\Big[I_0(\beta_0 m/\nu)/(\beta_0 + j\omega) + 2\sum_{n=1}^{\infty} I_n(\beta_0 m/\nu)\cdot K_n\cdot(\cos n\nu t - \cos\psi_n)\Big] \tag{5.51}$$

The solution contains the original frequency of the forcing function ω in the first factor of (5.53). This frequency is multiplied by a series expansion of overtones of the variation frequency ν of the coefficient. For a more general forcing function, consisting of a number of Fourier components, the solution is applicable to every component and they can be superimposed. But it is important to remember that superposition is not valid for the components of the variation of $\beta(t)$. For more complicated variations in β, consisting of a number of frequency components, the solution is useless. When $m << 1$ the number of components with significant energy rapidly diminishes and (5.53) is more practical.

5.3.2 Solution of the Volterra Integral Equation [10]

Carson [12] originally gave the solution for a circuit, in which a part of the resistance is time variant, in the from of a Volterra integral equation. This solution is significant since it shows that a successive approximation technique may be followed to obtain the solution.

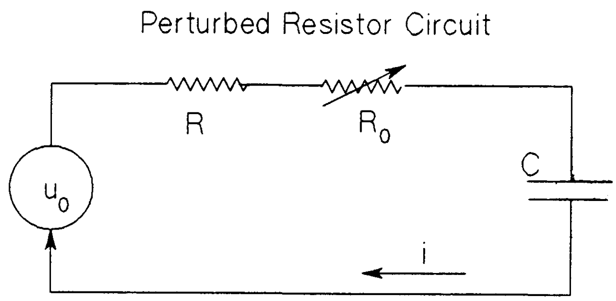

FIGURE 5.4 Circuit in which a part of the resistance, $R_0(t)$ is time variant. From [10].

In figure 5.4, the discharge resistance of the capacitor consists of a fixed resistor R, in series with a variable resistor $R_0(t)$. Ignoring the initial conditions and after application of Kirchhoff's voltage law, the circuit is found to be described by the following equation (the variables are as indicated in the figure)[6]:

$$R \cdot i + 1/C \cdot \int i(t)\, dt + R_0(t)\, i(t) = e(t)$$

After Laplace transformation (indicated by $\mathscr{L}\{\}$) this is equivalent to

$$\left[R + \frac{1}{sC}\right] I(s) + \mathscr{L}\{R_0(t) \cdot i(t)\} = E(s). \tag{5.54}$$

If we define

$$Z(s) = 1/H(s) = R + \frac{1}{sC} \text{ and } G(s) = \mathscr{L}\{R_0(t) \cdot i(t)\}$$

(5.54) becomes:

$$I(s) = H(s) \cdot E(s) - H(s) \cdot G(s). \tag{5.55}$$

the solution obtained via the inverse Laplace Transform is:

$$i(t) = \int_{-\infty}^{t} h(t-u) \cdot e(u)\, du - \int_{-\infty}^{t} h(t-u) \cdot R_0(u) \cdot i(u)\, du \tag{5.56}$$

with $h(t)$ the impulse response of the steady circuit, i.e. with $R_0 = 0$, defined by

$$h(t) = \mathscr{L}^{-1}\{H(s)\}$$

The first integral in (5.56) is the response of the steady system to the applied voltage which we shall call $i_0(t)$. The solution is then:

$$i(t) = i_0(t) - \int_{-\infty}^{t} h(t-u) \cdot R_0(u) \cdot i(u)\, du. \tag{5.57}$$

[6] In this section and elsewhere it is convenient to use electrical terminology. The reader will understand that 'current' relates to heat flux, and 'voltage' to temperature in thermo–flow problems.

This is a Volterra Integral Equation and the solution can be obtained in the following form [10,12]:

$$i(t) = i_0(t) - i_1(t) + i_2(t) - \cdots = \sum_{n=0}^{\infty} (-1)^n \cdot i_n(t)$$

with

$$i_0(t) = \int_{-\infty}^{t} h(t-u) \cdot e(u)\, du$$

$$i_1(t) = \int_{-\infty}^{t} h(t-u) \cdot R_0(u) \cdot i_0(u)\, du$$

$$\begin{matrix} \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \end{matrix}$$

$$i_n(t) = \int_{-\infty}^{t} h(t-u) \cdot R_0(u) \cdot i_{n-1}(u)\, du \tag{5.58}$$

We are interested in the voltage across the capacitor, given by:

$$v(t) = \frac{1}{C} \cdot \int_{-\infty}^{t} i(t)\, dt = \frac{1}{C} \cdot \int_{-\infty}^{t} [i_0(t) - i_1(t) + i_2(t) - \cdots]\, dt. \tag{5.59}$$

This solution involves a recursive procedure, where the zeroth approximation is the solution of the steady circuit, and higher order solutions are obtained from this solution by substituting for the forcing function, the previous solution multiplied by the time dependent resistance. The series converges quickly when R_0 is much smaller than R, in which case the circuit is often called a perturbed circuit. Unfortunately in our application we have to assume that the variation in the ventilation resistance is substantial compared to the mean value so that the Volterra series will require a large number of terms to be accurate. Nevertheless, this solution offers some insight in the physics of the time dependent circuit.

5.3.3 The Substitution Theorem [5]

In the previous section it was seen that the Volterra solution is obtained by first calculating the steady response, and then adding to this first approximation, the effects of the time variable resistor. The latter effect is obtained by calculating the voltage across the time dependent resistor, with initially the steady current and later, from more refined estimates of the current. This method is indicative of a far more powerful method based on the substitution theorem. The application of the substitution theorem to thermal networks is discussed by Athienitis [16]: "Simply stated (for thermal networks), it says that a network element across which a temperature drop is known, can be replaced by an equivalent heat source, equal to the heat flow through it." Alternatively, if the heat flux through a time dependent resistor is known, the theorem implies a time dependent temperature can be substituted for the resistor. The implementation of the method requires an initial estimate of the heat flux. From this estimate the substitution temperatures are calculated and then the solution. A recursive procedure may be used to refine the result. Since the method is very closely related to the Volterra series solution, we shall not discuss it in more detail.

5.3.4 A Recursive Solution[7]

Another recursive solution is obtained directly from the differential equation (5.35) written in the form:

$$\dot{y}(t) + \gamma_0(t)\cdot y = \xi_0(t) \tag{5.60}$$

which is obtained by taking $\beta(t) = \gamma_0(t)$ and $x(t) = \xi_0(t)$. We attempt to find a solution in the form of an initial approximation y_0, and an error

[7]This recursive procedure was devised by the author himself. No serious attempt was made to investigate whether it was previously given in the literature. It probably has a direct connection with the Volterra solution of §5.3.2 and the transfer function approximations of Tsao, discussed later.

term y_1:

$$y(t) = y_0 + y_1 = \xi_0(t)/\gamma_0(t) + y_1(t). \tag{5.61}$$

Taking the derivative of (5.61)

$$\dot{y}(t) = \frac{\gamma_0\cdot\dot{\xi}_0 - \dot{\gamma}_0\cdot\xi_0}{\gamma_0{}^2} + \dot{y}_1$$

and substituting it in the differential equation (5.60) produces:

$$\dot{y}_1 + \gamma_0\cdot y_1 = -\frac{\gamma_0\cdot\dot{\xi}_0 - \dot{\gamma}_0\cdot\xi_0}{\gamma_0{}^2} \tag{5.62}$$

which is of exactly the same form as (5.60) if

$$\xi_1 = -\frac{\gamma_0\cdot\dot{\xi}_0 - \dot{\gamma}_0\cdot\xi_0}{\gamma_0{}^2}. \tag{5.63}$$

The whole procedure is now repeated for y_1 to obtain y_2 etc. The final solution is:

$$y(t) = 1/\gamma_0\cdot[\xi_0 + \xi_1 + \xi_2 + \cdots] \tag{5.64}$$

where we have

$$\xi_0(t) = x(t)$$

$$\xi_n(t) = \frac{\xi_{n-1}\dot{\gamma}_0 - \gamma_0\dot{\xi}_{n-1}}{\gamma_0{}^2} \qquad n = 1,2,3,\cdots$$

The convergence of this solution has not been determined. The solution is determined in higher orders of derivatives of $\gamma_0\cdot\xi_0/\gamma_0{}^2$, and should converge fast if the derivatives of γ_0 and ξ_0 vanish, or if $\gamma_0{}^2$ is large. For sinusoidal forcing functions, the magnitudes of the derivatives are proportional to the frequency and consequently the series will diverge for large frequencies. In general, it appears that it will diverge unless all the functions are reasonably smooth.

5.3.5 Fourier Methods and the Modulation Function Equation[8]

The standard approach to obtaining the steady state solution to a time invariant system, is via the phasor (Fourier series) representation as applied in §5.1.4. The application of this method to the differential equation (5.35) leads to a first order differential equation in complex quantities, similar in form to the original equation, which we shall call the modulation function equation (MFE), after Tsao [14]. This equation is well known in the literature where procedures are described to derive it, by inspection, directly from the original equation [7,14]. The advantage of this method is that it is applicable to systems of higher order, and also to non–stationary stochastic systems. We shall derive it by assuming the input and output functions can be represented by a complex Fourier series expansion. [4]

a) Phasor Representation of Periodic Forcing Functions

If the forcing function $x(t)$, is a periodic function with period T, it can be expanded in a complex Fourier series:

$$x(t) = \sum_{n=-\infty}^{\infty} C_n \cdot e^{jn\omega_0 t} \tag{5.65}$$

with $\omega_0 = \frac{2\pi}{T}$ and

$$C_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\cdot e^{-jn\omega_0 t}\, dt. \tag{5.66}$$

Because the system is linear, the contributions from each phasor, $C_n \cdot e^{jn\omega_0 t}$, can be calculated independently and the results can be summed

[8]This frequency domain method is described in full detail since it is a viable solution method. Although it is the conclusion of this thesis that the numerical procedure of §5.4 is more suitable, future developments may require the use of this alternative method. It is in any case illuminating.

to find the total solution. At this stage it is important to remember, that in the circuit of figure 5.1, we require the forcing function T_e to be periodic with period T, and therefore, since T_e is dependent on the circuit parameters, they must also be periodic with the same period and hence also β. This condition is satisfied if all signals are specified over a given period (e.g. 24 hours) and it is assumed that all the signals are periodic with this period. This is exactly the assumption we made in §5.1.3.

We propose to use the phasor $x(t) = X \cdot e^{-j\omega t}$ with X a constant complex amplitude, as forcing function in equation (5.35) and assume that the response is given by $y(t) = Y(t,\omega) \cdot e^{-j\omega t}$. $Y(t,\omega)$ is the complex function describing the amplitude and phase modulation introduced by the time dependent system. It is explicitly written a function of t, to emphasize that we expect the amplitude and phase of the response to be functions of time. Substituting these definitions of x and y in (5.35) yields:

$$\frac{\partial Y}{\partial t} + [\beta(t) + j\omega] \cdot Y = X. \tag{5.67}$$

This is the MFE. The frequency response of the system is now defined by

$$H(t,\omega) = \frac{Y(t,\omega)}{X}, \tag{5.68}$$

corresponding with the definition for a time invariant system. The frequency response must satisfy:

$$\frac{\partial H(t,\omega)}{\partial t} + [\beta(t) + j\omega] \cdot H(t,\omega) = 1. \tag{5.69}$$

The impulse response is defined as the inverse Fourier Transform of $H(t,\omega)$ with respect to ω. These definitions correspond with the definitions in §5.1.4. In [7,14] a general method for obtaining the MFE is derived and solutions via expansions are given. These solutions will be discussed later.

Equation (5.69) is of exactly the same form as (5.35) but involves complex quantities. However, the forcing function is a constant. To solve this equation, we need initial conditions for either $y(0,\omega)$ or $H(0,\omega)$. Zadeh [7] indicates that a method for deriving initial conditions is to assume the system was constant and equal to $H(0,\omega)$ prior to $t = 0$. However, we are looking for periodic solutions with $H(T,\omega) = H(0,\omega)$. It is therefore not possible to assume that the system was initially invariant, and the initial condition remains unspecified, as with the time domain representation. In the next section, we indicate how the problem of the unknown initial condition can be circumvented by using a suitable approximate transfer function. With the initial condition known, the MFE can be solved by any of the methods discussed so far.

b) Expansions for the Transfer Function

Zadeh [7] gives a differential equation which must be satisfied by the system function in the following form:

$$\left[\frac{1}{j\omega + \beta}\right] \cdot \frac{\partial H}{\partial t} + H = \frac{1}{j\omega + \beta}. \tag{5.70}$$

This equation obviously corresponds with equation (5.67). But Zadeh gives the following interesting interpretation of (5.70): "The system function of a variable network may be formally regarded as the response of an initially unexcited system, of which [(5.70)] is the fundamental equation, to the frozen system function of the network." And also: "The frozen system function may be regarded as a first approximation to the actual system function of a variable network whenever the coefficients of the fundamental equation do not vary appreciably over the width of the impulse response of the system." Zadeh defines the frozen system function as the function determined by freezing the variable network at the instant of consideration. In the case of the single pole network the frozen system function is, compare with (5.19):

$$H_f(t,\omega) = \frac{1}{j\omega + \beta(t)} \tag{5.71}$$

We shall encounter (5.71) again later. In reference [7] two approximate methods for solving (5.70) with series expansions are given. These methods are not applicable to our problem where the variations in the parameter β are sudden and large. The problems which arise under these conditions are illustrated in the next paragraph for a similar expansion due to Tsao.

According to Tsao [14] the solution of the MFE equation can be obtained iteratively, where the zeroth order approximation is the frozen system function H_f. The m th order approximation is given by[9]:

$$_{m}Y(t,\omega) = H_0(t,\omega)\cdot\left\{X - \frac{\partial\{_{m-1}Y\}}{\partial t}\right\}. \tag{5.72}$$

The zeroth order approximation is thus the frozen response as defined above:

$$_{0}H(t,\omega) = H_f = 1/(j\omega + \beta(t))$$

The first order approximation is:

$$_{1}Y(t,\omega) = H_f(t,\omega)\cdot\left\{X(\omega) - \frac{\partial\ _{0}Y}{\partial t}\right\} \tag{5.73}$$

but $$\frac{\partial\ _{0}Y}{\partial t} = H_f(t,\omega)^2\cdot X(\omega)\ [-\dot{\beta}(t)]$$

$$_{1}H(t,\omega) = H_f(t,\omega)\cdot[1 + \dot{\beta}(t)\cdot H_f^2]. \tag{5.74}$$

[9]A small subscript before a variable is used to indicate the approximation order.

This expression agrees with that obtained in §5.3.3 when $\xi_0 = X$ (a complex constant) in (5.60). The error, ϵ, in these approximations are given approximately by the next higher order term, which is for the zeroth order approximation:

$$ {}_0\epsilon(t,\omega) = \dot{\beta}(t)/(\beta + j\omega)^2. \tag{5.75}$$

The error depends on the speed of variation of β and will vanish if $\dot{\beta}$ vanishes. For the first order approximation one finds:

$$\begin{aligned} {}_1\epsilon(t,\omega) &= -\, H_f{}^3\cdot\ddot{\beta} + 3\cdot H_f{}^4\cdot\dot{\beta}^2 \\ &= 3[{}_0\epsilon(t,\omega)]^2 - H_f{}^3\cdot\ddot{\beta}. \end{aligned} \tag{5.76}$$

If the zeroth order error ${}_0\epsilon$ is small the errors will decrease with higher order approximations. For sudden changes in β, i.e. $\dot{\beta}$ large one can expect that the expansion (5.72) will diverge, and it appears that greater accuracy is obtained from the frozen system, than from higher order approximations. The zeroth and first order approximations were implemented in a program to test these conclusions and it was indeed found that the expansion diverges.

However, the solution obtained by using just the frozen response, H_f, appeared to be quite useful. The frozen response is a very convenient approximate solution. H_f is uniquely specified at each time instant so that the solution is straightforward; the frequency components of the forcing function are simply multiplied with H_f and added at each time instant, as in (5.77) below. However, at first glance it appears to be a very crude approximation. It amounts to ignoring the derivative term, $\partial H/\partial t$, in (5.69). Since we are assuming that $\beta(t)$ is subject to sudden large changes, $\partial H/\partial t$ could not possibly be small if it is approximately given by H_f. However, it was subsequently discovered that the true system function, obtained from (5.69), responds very sluggishly to changes in β, since it was governed by the same long time–constant as the original equation (5.35). Therefore the action of the system (5.69) is to dampen the quick changes in β, so that the derivative $\partial H/\partial t$ may indeed be small. From this point of view, it is seen that the approximation $H \approx H_f$ will tend to accentuate the variability of the system and the approximate solution will, in a certain sense, overreact to the variation in β. However, investigation showed that if β remained sufficiently small, that is, if the thermal time constant remained sufficiently large, this over–reaction is not visible in the response, and the frozen system gives useful results. The approximate solution of (5.35) using the frozen transfer function is given by:

$$y(t) = \sum_k X_k \cdot exp(j \cdot k \cdot \omega_0)/(\beta(t) + j \cdot k \cdot \omega_0) \qquad (5.77)$$

where k is an index running over the frequency components, $\omega = k \cdot \omega_0$, of the forcing function:

$$x(t) = \sum_k X_k \cdot exp(j \cdot k \cdot \omega_0). \qquad (5.78).$$

The method was implemented in the thermal analysis program. A listing of the relevant routines are supplied on the accompanying floppy diskette.

Intensive investigation of the conduct of this approximate solution was carried out. It indicated some disturbing trends which required further corrective procedures. The thermal forcing functions normally possess a large mean component plus a diurnal swing component. The effective forcing function will therefore exhibit the same characteristic. In the solution of the thermal network it is essential that the mean component be calculated accurately. However, in those cases where the mean component of the input was substantial, the computed mean as well as the swing of the output were inaccurate. The reason seems to be that the too strong modulation effect of the frozen system approximation on the mean component of the input. It is crucial to understand that the time variable circuit generates an additional swing component, by modulation of the input, even if the input is held constant.

c) <u>Separate Treatment of Mean– and Swing Component</u>

It appeared reasonable to expect that better accuracy could be obtained by treating the mean and swing components separately. The original differential equation (5.35) is:

$$\dot{y} + \beta\cdot y = x. \tag{5.79}$$

if x,y and β consist of a mean part indicated by an over–bar plus a swing component indicated by a tilde, i.e.

$$x = \bar{x} + \tilde{x}, \qquad y = \bar{y} + \tilde{y}, \qquad \beta = \bar{\beta} + \tilde{\beta} \tag{5.80}$$

(5.79) becomes

$$\dot{\tilde{y}} + \bar{\beta}\cdot\bar{y} + \tilde{\beta}\cdot\bar{y} + \bar{\beta}\cdot\tilde{y} + \tilde{\beta}\cdot\tilde{y} = \bar{x} + \tilde{x}. \tag{5.81}$$

In this equation the terms $\bar{\beta}\cdot\bar{y}$ and $\bar{x}$ are constants while the terms $\dot{\tilde{y}}$, $\tilde{\beta}\cdot\bar{y}$, $\bar{\beta}\cdot\tilde{y}$ and $\tilde{x}$ have zero mean components. The only remaining term is $\tilde{\beta}\cdot\tilde{y}$ which contains again a mean and a swing indicated by:

$$\tilde{\beta}\cdot\tilde{y} = \overline{\tilde{\beta}\cdot\tilde{y}} + \widetilde{\tilde{\beta}\cdot\tilde{y}} \tag{5.82}$$

Equation (5.81) is now separated into two equations; one for the constant terms and another for the swing terms:

$$\bar{\beta}\cdot\bar{y} + \overline{\tilde{\beta}\cdot\tilde{y}} = \bar{x} \tag{5.83}$$

$$\dot{\tilde{y}} + \tilde{\beta}\cdot\bar{y} + \bar{\beta}\cdot\tilde{y} + \widetilde{\tilde{\beta}\cdot\tilde{y}} = \tilde{x}. \tag{5.84}$$

These equations must be solved simultaneously for $\bar{y}$ and $\tilde{y}$. The following approximate method may be used, in which the solution of the differential equations are obtained from the frozen system function:

- Assume $\overline{\tilde{\beta}\cdot\tilde{y}} << \widetilde{\tilde{\beta}\cdot\tilde{y}}$ so that $\tilde{\beta}\cdot\tilde{y} \approx \widetilde{\tilde{\beta}\cdot\tilde{y}}$.

- From (5.83) follows that a first approximation for the mean of the output is:

$$\bar{y}_0 = \bar{x}/\bar{\beta} \tag{5.85}$$

- Next apply equation (5.84) in the following form:

$$\dot{\tilde{y}}_0 + \beta\cdot\tilde{y}_0 = \tilde{x} - \tilde{\beta}\cdot\bar{y}_0 = x\cdot(1 - \beta/\bar{\beta}) \tag{5.86}$$

  to find $\tilde{y}_0$, the first estimate of the swing.

- Next calculate the mean of $\tilde{y}_0\cdot\tilde{\beta} = \overline{\tilde{y}_0\cdot\tilde{\beta}}$ and set the next higher approximation:

$$\bar{y}_1 = \frac{\bar{x} - \overline{\tilde{y}_0\cdot\tilde{\beta}}}{\bar{\beta}} = \bar{y}_0 - \frac{\overline{\tilde{y}_0\cdot\tilde{\beta}}}{\bar{\beta}}. \tag{5.87}$$

- If necessary the iteration can continue by solving again

$$\dot{\tilde{y}}_1 + \beta\cdot\tilde{y}_1 = \tilde{x} - \tilde{\beta}\cdot\bar{y}_1 = x\cdot(1 - \beta/\bar{\beta}) \text{ etc.} \tag{5.88}$$

This procedure was programmed (see the listing on the accompanying floppy diskette) and the answers were compared to accurate answers, obtained via the exact solution of the special case, given in §5.3.1a. It was found that a single iteration through (5.85) to (5.87) was sufficiently accurate. Note that the differential equation which is actually approximately solved with the frozen transfer function, is (5.86), which contains no mean output component.

The implementation obtains the frequency components (5.78) of the effective forcing function, via the prime factor FFT algorithm described in [3]. Note that, even though the forcing function contains many harmonics when the parameters change discontinuously – due to the changes of the contribution factors of the various sources in figures 5.2 and 5.3 – not many of these harmonics are required in the solution. The basic response of the differential equation, although time dependent, still remains sluggish, since we have written the governing equation in terms of the stored heat in the structure. From physical considerations it follows that the stored heat must remain a smooth function of time, even when the temperatures are discontinuous. It was found that – for almost all buildings – 5 frequency components are sufficient to obtain an accuracy of 0.5 ^{o}C for the predicted interior temperature.

The phasor components Y are obtained from the components of the forcing function X, multiplied by the system response H as given by (5.77). When only a small number of Fourier components is used the method is quick. For j time points and i frequency components, the approximation (5.77) requires i^*j complex multiplications. For 24 time points, and 5 frequency components, the total is 120 complex multiplications, which is the same as the number required by frequency domain convolution of time invariant systems. In addition, the method requires one 24 point FFT to obtain the spectrum of the forcing function.

This Fourier series method, which uses the frozen system transfer function as an approximation to the actual transfer function, was found to be well behaved and quite accurate. However, in §5.4 a numerical method is described which is far simpler, more efficient and potentially more accurate. Since this latter numerical method subsequently replaced the Fourier series method, the above detailed discussion of the Fourier series method, is strictly speaking, redundant to this thesis. The author nevertheless feels that the method merits the considerable space given to it here. It is a natural extension of the method used for solving the time invariant case, and as such, provides much insight in the behaviour of the time dependent system.

5.4 A New Efficient Numerical Algorithm

In §5.3.1a a method was indicated which is based on the assumption that the input functions are constant in small time intervals. It was shown that this method will require the solution of a large number of simultaneous equations. However, since we have written the governing equation (5.35) in terms of the stored heat in the structure, it is reasonable to expect that also the output, will be a fairly smooth function. In this section we obtain an approximate method, based on the assumption that all variables, input output and also β, are constant between sampling points.

The big advantage of writing the governing equation in terms of the amount of stored energy is; it reduces the sensitivity of the solution of T_i to errors in the solution of the differential equation (5.35). A large part of the variation in T_i is accurately included in the final calculation of T_i and Q_{cr} from q, in (5.23) and (5.30) respectively. This is easily demonstrated by noting that, in the limit, when the ventilation rate is very large ($R_v \sim 0$), the sole contributor to T_i in equation (5.23) is T_x. On the other hand, when R_v approaches infinity, the sole contributor is T_c. This is in accordance with the network of figure 5.1.

a) A First Difference Equation

To solve (5.35) we rewrite the equation in the form of an integral equation[10]:

$$\begin{aligned} y(t) &= \int_{-\infty}^{t} x(t_1) - \beta(t_1)\cdot y(t_1)\, dt_1 \\ &= \int_{-\infty}^{0} xt_1) - \beta(t_1)\cdot y(t_1)\, dt_1 + \int_{0}^{t} x(t_1) - \beta(t_1)\cdot y(t_1)\, dt_1 \\ &= y(0) + \int_{0}^{t} x(t_1) - \beta(t_1)\cdot y(t_1)\, dt_1 \end{aligned} \tag{5.89}$$

For periodic, steady state solutions with period T, it is required that the initial value of every period equals the final value of the previous period:

$$y(0) = y(T) = y(0) + \int_{0}^{T} x(t_1) - \beta(t_1)\cdot y(t_1)\, dt_1$$

and therefore

$$\int_{0}^{T} x(t_1) - \beta(t_1)\cdot y(t_1)\, dt_1 = 0. \tag{5.90}$$

The steady, periodic solution of (5.35) is given by (5.89) with initial condition stipulated by (5.90). For discrete data at $t = t_i = i\cdot\Delta T$, $i = 0,1,2,3....$, $T = \mathrm{N}\cdot\Delta T$, these equations take the form:

$$y_k = y_0 + \sum_{i=0}^{k-1} \Delta T\cdot(x_i - \beta_i\cdot y_i) \tag{5.91}$$

and

$$\sum_{i=0}^{N-1} \Delta T\cdot(x_i - \beta_i\cdot y_i) = 0. \tag{5.92}$$

[10]A similar approach using the general solution (5.36) fails, apparently because (5.36) does not contain $y(0)$ under the integral.

It is assumed all variables are constant between sampling points and $x_i = x(t_i)$ as well as $\beta_i = \beta(t_i)$ are known tabulated functions. Equation (5.91) can be written in the following corresponding, iterative form:

$$y_k = \Delta T \cdot x_{k-1} + (1 - \Delta T \cdot \beta_{k-1}) \cdot y_{k-1} \qquad k = 1,2,3...N-1 \qquad (5.93)$$

Equation (5.93) is a simple first difference equation, and the technique is known in the numerical analysis literature as Euler's method. If one value of y_k is known, the other values are easily found from (5.93), provided the iteration is stable. The essence of an efficient method is an effective methodology for obtaining one value, so that the others can be obtained from the iteration. In the next section an explicit equation for the initial value is derived.

b) A Closed Form Solution for the Initial Value[11]

A closed form solution for the initial value y_0 is obtained by starting with

$$y_N = \Delta T \cdot x_{N-1} + (1 - \Delta T \cdot \beta_{N-1}) \cdot y_{N-1} = y_0 \qquad (5.94)$$

and substituting previous values of y_k.

$$y_T = \Delta T \cdot x_{N-1} + (1 - \Delta T \cdot \beta_{N-1}) \cdot (\Delta T \cdot x_{N-2} +$$

$$(1 - \Delta T \cdot \beta_{N-2}) \cdot (\Delta T \cdot x_{N-3} + (\;...\; \Delta T \cdot x_0)...))) +$$

$$(1 - \Delta T \cdot \beta_{N-1}) \cdot (1 - \Delta T \cdot \beta_{N-2}) \cdot ... \cdot (1 - \Delta T \cdot \beta_0) \cdot y_0 = y_0.$$

[11]Although this closed from solution for the initial value is very natural and its derivation straightforward, it appears to be unknown in the literature. However, no exhaustive search in the numerical analysis literature was undertaken and it is probable that this result is common knowledge in some circles.

Solving for y_0, this is rewritten compactly:

$$y_0 = \frac{\left[\sum_{k=0}^{N-2} \Delta T \cdot x_k \cdot \prod_{j=k+1}^{N-1} (1 - \Delta T \cdot \beta_k) \right] + \Delta T \cdot x_{N-1}}{1 - \prod_{k=0}^{N-1} (1 - \Delta T \cdot \beta_k)}. \tag{5.95}$$

When programming equation (5.95), advantage can be taken of the fact that the product expression appears both in the numerator and the denominator, by starting with the highest value of k and counting down instead of up. By storing at each step k the partial sum and the partial product, the product expression can be evaluated successively for every term in the sum. The product expression in the denominator is found by multiplying the final product of the numerator with $(1 - \Delta T \cdot \beta_0)$.

The complete approximate solution is given by (5.93) with initial value given explicitly by (5.95). We maintain that the method is efficient; no initial period of integration or iterations are required to get rid of the transients and the numerical integration step size is fixed at the sampling period of the data. The accuracy of the solution is discussed later.

c) Stability

First we need to establish the stability of (5.93). This question is important despite the physical constraints which ensure the absolute stability of the original differential equation. It is well known that the sampling interval has an influence on the stability of a sampled data system [17]. To investigate the stability of (5.93) the $\mathcal{Z}$ transform method [17] will be used under the assumption that β is independent of time. This assumption is required since it is difficult to establish the stability of the time dependent form. The transfer function of (5.93) in the z domain is:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1} \cdot \Delta T \cdot \beta}{1 - z^{-1} \cdot (1 - \Delta T \cdot \beta)}. \tag{5.96}$$

The time invariant system is stable if the pole of the transfer function lies inside the unit circle on the complex z–plane, i.e.

$$|1 - \Delta T \cdot \beta| \leq 1. \tag{5.97}$$

For the time variable circuit, the position of the pole is time dependent but stability may be assumed if, at all times, the pole never strays outside the unit circle. This is not a stringent condition. Gibson [11] discusses the stability of time variable systems and he gives an example where condition (5.97) is satisfied and yet the system is unstable. However, we have found that (5.97) will indicate stability of (5.93), so that if the sampling interval is chosen so that $\Delta T \cdot \beta < 2$, the recursion is stable. For most practical buildings a sampling interval of 1 hour suffices. For some very light constructions, it was found necessary to decrease ΔT to 15 minutes.

d) Accuracy of the Method

The method is essentially Euler's method for the numerical integration of a differential equation. An upper bound for the total propagated error is [18]:

$$|e| \leq \frac{\Delta T \cdot |\ddot{y}|}{2 \cdot \beta} \cdot [exp(\beta \cdot T) - 1]. \tag{5.98}$$

The error grows rapidly when $\beta \cdot \Delta T = \Delta T / \tau > 0.1$. Since we have written the governing equation in terms of the stored heat, which is a slowly varying quantity, one would normally expect $|\ddot{y}|$ to be quite small. In fact, the derivative of (5.35) gives:

$$\begin{aligned} \ddot{y} &= \dot{x} - \beta \cdot \dot{y} - \dot{\beta} \cdot y \\ &= \dot{x} - \beta \cdot (x - \beta \cdot y) - \dot{\beta} \cdot y \\ &= \dot{x} - \beta \cdot x + (\beta^2 - \dot{\beta}) \cdot y. \end{aligned} \tag{5.99}$$

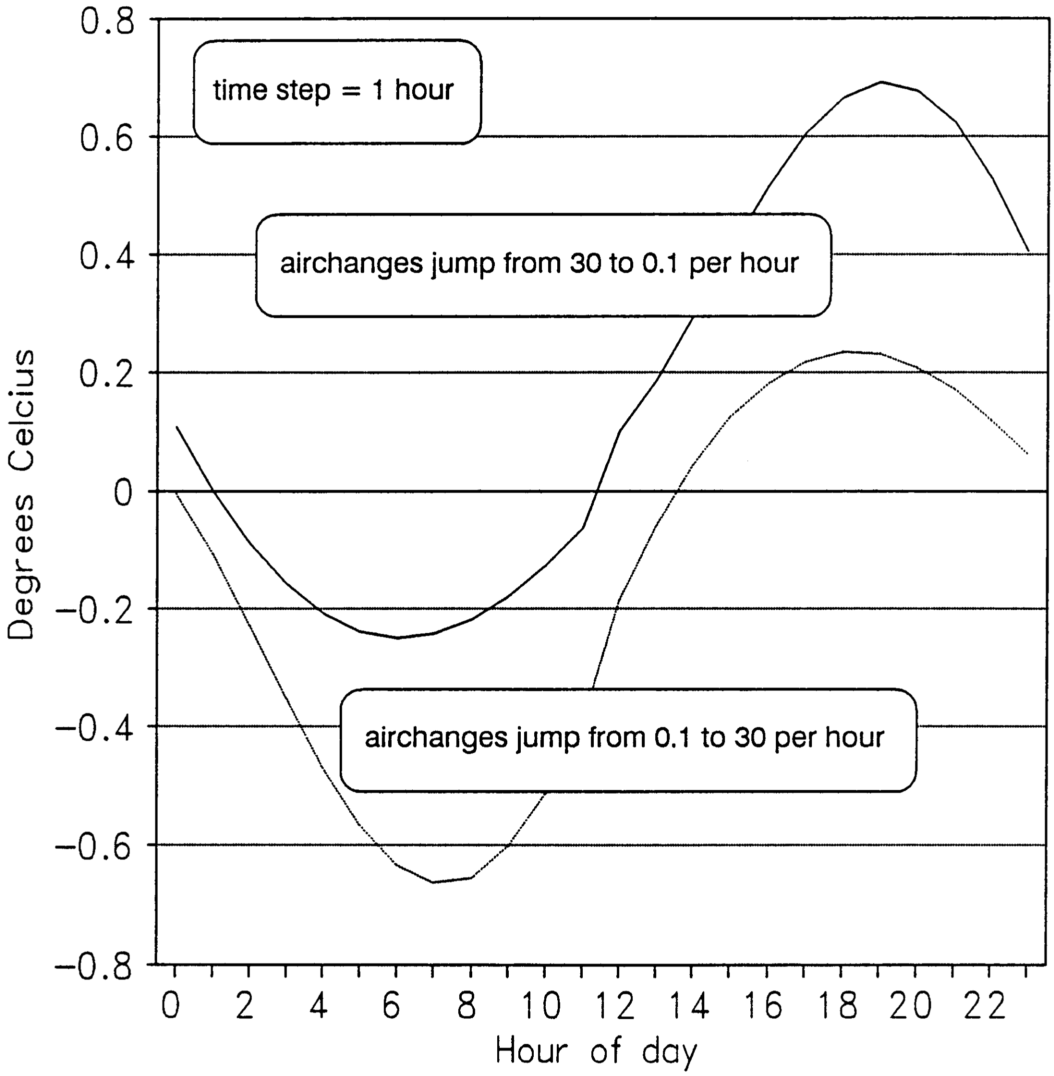

FIGURE 5.5 Difference between analytically derived exact prediction of interior temperature, for sinusoidal forcing functions, and numerical algorithm. The sampling period is 1 h. Building: low–mass agricultural shed, $\tau = 5$ h. The upper and lower traces show the error when the air–change–rate jumps from 0.1 to 30 /h, and from 30 to 0.1 /h respectively. The forcing functions are given in the text.

$|\ddot{y}|$ contains a term proportional to $\dot{\beta}$ which might be large for large variations of $\beta(t)$. Actually, the numerical integration technique is rigorously exact if all variables assume constant values between sampling points, even with discontinuous derivatives at the sampling points. Equation (5.98) must not be taken too seriously, it is not a very tight upper bound and, furthermore, it is derived under the assumption that the functions are all continuous.

To obtain a practical evaluation of the accuracy of the method, the approximate solution can be compared with the exact solution in a special case. In §5.3.1a the analytical solution for a system with constant β in subintervals was given. Figure 5.5 shows the error between analytic solution (5.46), (5.47), and numerical solution (5.93), (5.95), for a low–mass building. The building (an agricultural shed) has a time–constant of about 5 hours (see Table 5.1) with closed windows. This is a very short thermal time–constant for a building, and a practical sampling rate would be 15 min, but to show the robustness of the method, a sampling period of 1 h is used in the calculation. The air change–rate varies between 0.1 and 30 /h resulting in a time–constant jump from 5.7 to 3.8 h, the jump occurring at $T_1 = 11$ h. The forcing functions used for the calculation are:

$$T_{sa} = 20+10\cdot \cos(2\pi/24\cdot t)\ ^{o}\mathrm{C},$$
$$T_o = 20+5\cdot \cos(2\pi/24\cdot t)\ ^{o}\mathrm{C} \quad \text{and}$$
$$Q_c = Q_r = 0\ \mathrm{kW}\ .$$

The calculation is given in appendix 4A.

Figure 5.5 shows the error obtained by a sudden increase in the number of air changes as well as a sudden decrease of similar strength. In this worst case, $\beta\cdot\Delta T = 0.25$, the temperature error is less than 1^{o}C. The error is decreased to insignificant levels by decreasing the sampling period to 15 min, with linear interpolation between sampling points. Figure 5.6 shows

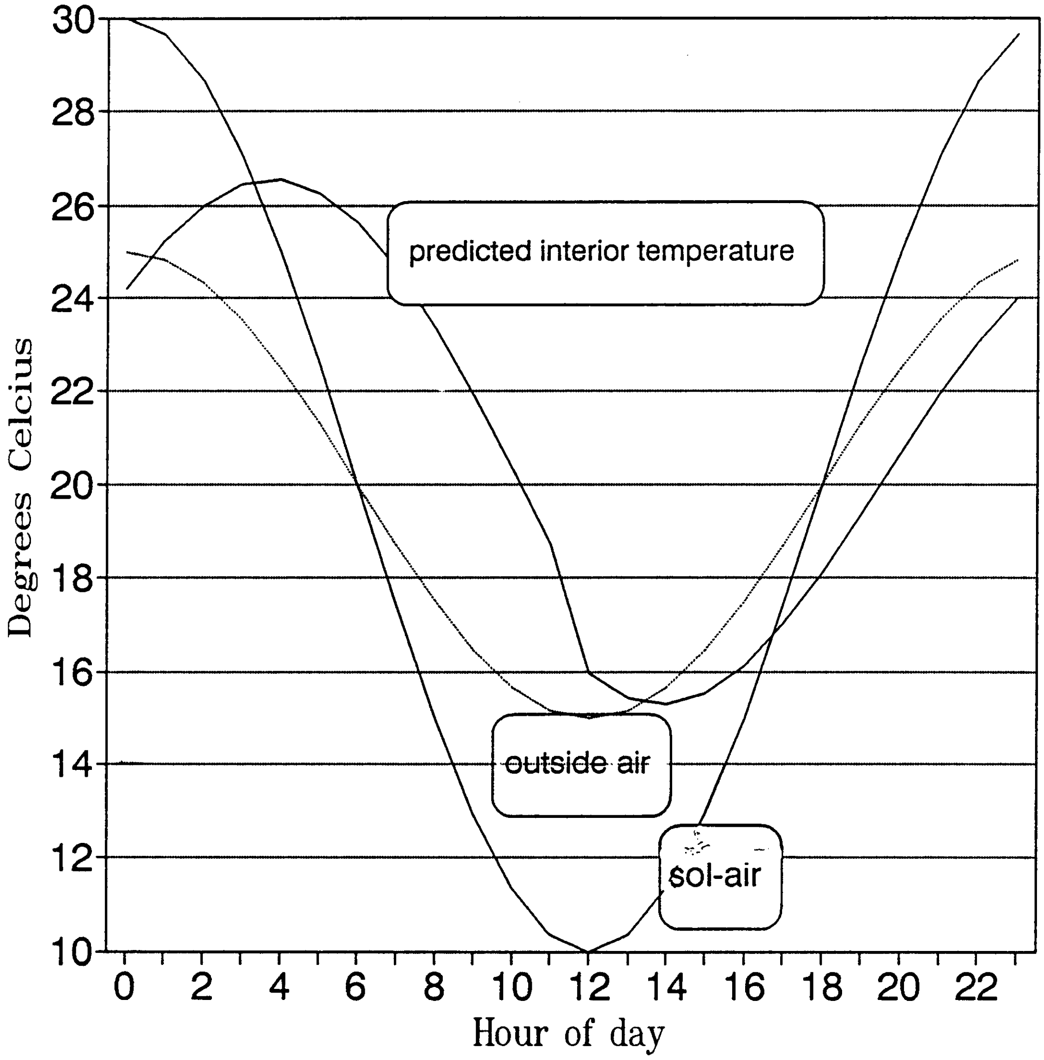

FIGURE 5.6 Predicted interior temperature for the low–mass building when the air–change rate jumps between 0.1 and 30 /h. The thick–solid line is the assumed sol–air temperature and the thick–broken line the assumed outside air temperature.

the resulting interior temperatures. Note the sharp discontinuity. In practice, the heat capacitance and the finite mixing time of the interior air (both neglected in the model) will tend to smooth the discontinuity so that a smooth transition will be measured. (It was also found that, with sudden changes in the ventilation rate, the time constant of the thermograph can often not be neglected.)

The calculations were repeated for a massive building (office block) where the time constant jumped from 144 to 38 h when the ventilation rate was increased from 0.1 to 30 /h. The error between the analytic and approximate solutions in this case, $T = 1$ h, $\beta \cdot \Delta T = 0.025$, were less than 0.1 °C.

When evaluating the numerical error it must be borne in mind that the accuracy of the thermal modelling is definitely limited by the extreme simplicity of the model. It makes no sense to strive for infinite accuracy in the calculation procedure when both the assumptions inherent in the model, and uncertainties in the detail of the construction and ventilation rates limit the practical, attainable accuracy. The objective of creating a simplified and easy to use tool, which will give results quickly, must remain in the forefront.

e) <u>More Accurate Algorithms</u>

To decrease error propagation for continuous functions one can use a higher order numerical approximation technique. Beginning with the trapezoid rule, for instance, an exactly similar scheme with local error theoretically proportional to the third derivative and square step size results. These and a host of other higher order approximation techniques [5,18] are not as advantageous for discontinuous input functions. Unless the integration is done over continuous subintervals, they tend to smooth the discontinuities and to make the solution appear non–causal, since they pre–empt the sudden change. The effect is easily explained by noting that the higher order techniques in effect interpolate between the sampling points so that values in the immediate future will influence the present result. In practice, we have found the Euler algorithm sufficiently accurate and quick. Further reduction in computation time is possible from implicit and higher order methods [e.g. 19], but the simple method is already so fast that the matter is of academic importance only. It is possible to find an expression similar to (5.95) for an implicit discretization of (5.35)[12] The implicit methods have the advantage that they are generally stable, however, they were not investigated in depth.

f) Extension to Higher Order Systems

The simple numerical method is readily extended to higher order systems. We have, in chapter 2, suggested a refined thermo–flow model, figure 2.22, in which the interior masses are treated separately. By using the state space approach [17], it is possible to represent this system as two first order differential equations, where the 2 state variables must be chosen to respectively correspond with: i) the stored heat in the shell of the building, and ii) the stored heat in the interior masses. By redefining y to be the state vector, and x the input vector, the numerical recursion is directly applicable. The initial value takes a somewhat different form because of the matrix operators. In similar manner the method can be extended to systems of any order.

g) Implementation

A listing of the implementation of the numerical procedure is given on the accompanying floppy diskette. It is quite straightforward. In the implementation provision was made for dynamic selection of the step size by linking the step size to the thermal time–constant of the building through (5.97). In this way, numerical accuracy may be assured while efficiency is maximized. However, it was found that a step size of 1 h is adequate for all buildings, except for some extremely light constructions, where the numerical accuracy is not sufficient. Since the calculation proved to be very efficient, it was decided to use a fixed time step of 15 minutes for all buildings.

[12]The implicit method uses a backward difference to discretize the derivative in (5.35) and not the forward difference of (5.95).

The implementation caters for the prediction of internal temperatures as well as the convective loads, to maintain a specified interior temperature, and, since the two simulations are completely distinct although based on similar equations, an appropriate method for verifying the accuracy of the computer implementation is:

- first determine the load required to maintain a specified interior temperature for an arbitrary building and climate
- add this load to the convective load just specified
- calculate the interior temperature with this load
- verify that the predicted temperature is equal to the initially specified interior temperature.

This method applies the numerical solution twice in a back to back manner. It can also be used to determine the accuracy of the numerical procedure. It was carried out on a number of different buildings with different climatic data. The difference between the specified temperature and the final temperature is insignificant for massive buildings and below 1 °C for very lightweight structures, so that the numerical implementation is well verified.

h) <u>Verification Measurements</u>[13]

Although this thesis is aimed at the entrenchment and extension of the method of Mathews and Richards, and not so much with the merit and verification of the method, it was deemed necessary to perform at least one set of measurements in order to verify that the predictions of the model with variable parameters are plausible. For this reason two similar experimental test huts at the Division for Building Technology, Council for Scientific and Industrial Research, Pretoria were used. For details of the buildings see reference [5] of chapter 3. In one of the buildings a mechanical ventilation system was installed which delivered 17 air changes per hour. The interior air–temperatures and outdoor air temperature were recorded with Thies Clima thermographs. In addition the temperature of

[13]The experiment was suggested by Prof. E.H. Mathews.

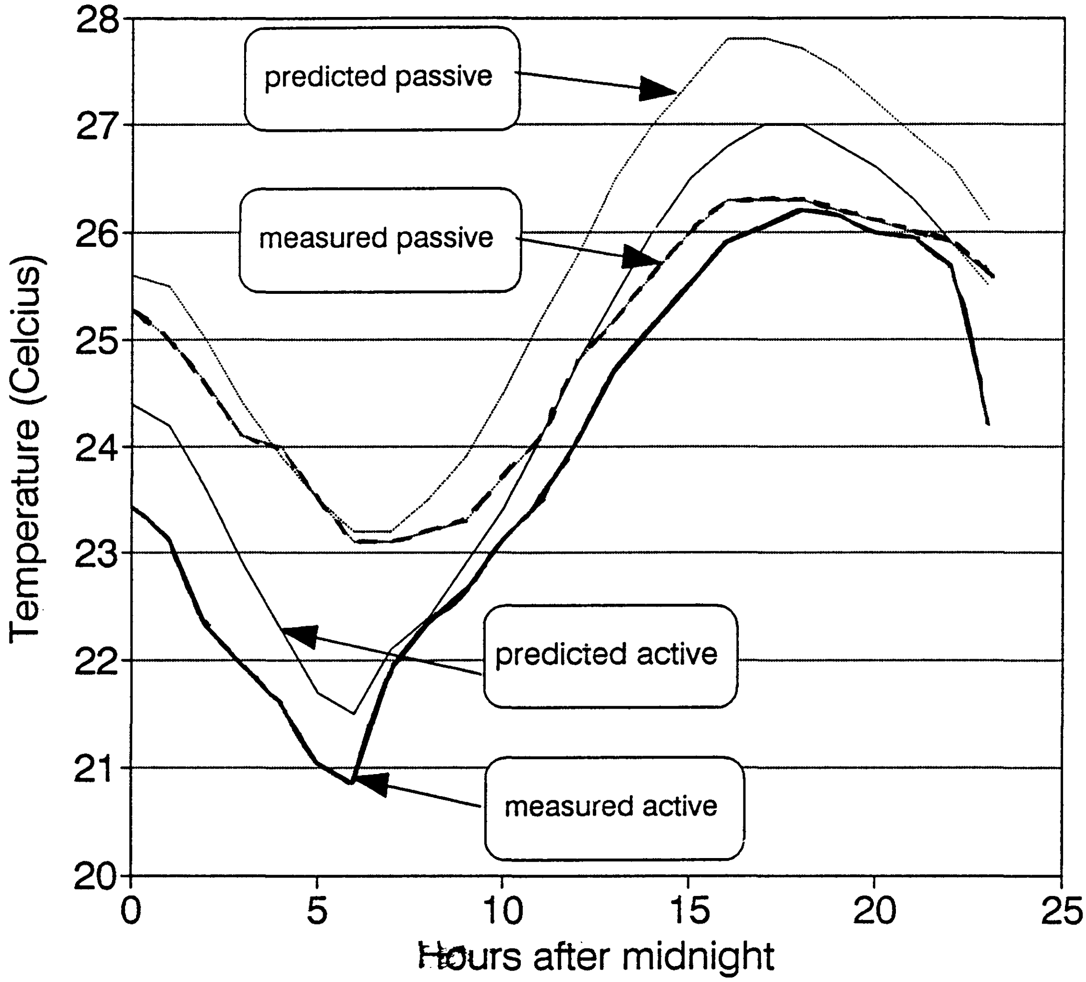

FIGURE 5.7 Measured and predicted interior air temperatures in two huts. The dotted lines indicate the measured– (thick) and predicted (thin) air temperature in the passive control hut. The solid lines indicate the measured– (thick) and predicted (thin) air temperature in the hut subjected to a forced ventilation of 17 ach from 22h00 – 06h00.

the air delivered to the zone by the ventilation system was recorded with thermocouples. Initially the ventilation system was not switched on and it was verified that the air temperatures in both huts are indeed identical. Afterwards the ventilation system was switched on from 22h00 to 06h00 for a few days. The computer program was also used to predict the interior air temperature. The ventilation rate was estimated at 0.5 air changes per hour in the passive hut and the same value was assumed valid in the period 06h00 – 22h00 in the active hut. The measured ventilating air temperature was used in the calculation as well as the measured outdoor air temperature. Results for a typical day are presented in figure 5.7.

Unfortunately, it was not possible to obtain sun radiation data for the period of duration of the measurements so that data of a previous year for the same month and location had to be used. For this reason, the predictions are not as accurate as others which were made at a previous occasion for the same buildings with constant ventilation rates. From the figure it appears that the effect of the night ventilation is somewhat underestimated by the procedure, since the measured drop in temperature between the huts exceeds the predicted value. However, this is probably not significant, given the relative minor effect of the night ventilation. Clearly, these experiments need to be repeated a substantial number of times, with a larger difference between the passive and the active hut (higher ventilation rates and cooler ventilating air[14]), to come to a statistically valid conclusion. However, despite the shortcomings of the experiment it does appear that the influence of the sudden change in the ventilation rate is predicted within reasonable limits.

5.5 Conclusion, Chapter 5

Of the many methods in the literature for solving time variant systems two seems practical. They are the Fourier series technique of §5.3.5 which uses the frozen system response as an approximation. Another more efficient method is the numerical method of §5.4. Both these methods were investigated in some detail, implemented in computer programs and evaluated. Of the two methods the numerical procedure seems more fitting because of its efficiency and also because the approximation inherent in using the frozen system response, is difficult to quantify. The numerical method is also more flexible and can be easily adapted to higher order

[14]The fan blew air into the room so that the fan load to some extend cancelled the cooling effect of the ventilation. This can be avoided by changing the arrangement so that the fan actually sucks air from the room.

systems, different discretization methods etc. The Fourier series method has the advantage that it is a straightforward extension of the usual frequency domain methods used for time–invariant systems.

We conclude that it proved possible to find a numerical technique for solving the time invariant system which does not require a long initial period of integration. The method is efficient and sufficiently accurate. The following points and assumptions are of crucial importance for the solution procedure:

- the influence of the changes of the system parameters on the response of the system must be carefully analyzed to determine which types of changes are allowed and what quantities are conserved during the changes,
- the system differential equations must be written in terms of conserved quantities, in this case the stored heat,
- both the forcing functions and the variations must be periodic with the same period.

These assumptions are not overly restrictive and it is possible that the method of §5.4 will find application in diverse fields. It is applicable to any discrete system where periodic solutions are required.

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SYMBOLS Chapter 5

A	Load contribution factor of interior air temperature [kW/K], term of (5.45).
a	Interior temperature contribution factor of outside air temperature.
B	Load contribution factor of sol–air temperature [kW/K].
b	Interior temperature contribution factor of convective source [K/kW].
C	Heat storage capacitance of massive structures [kJ/K], load contribution factor of radiative source.
C_{n}	Fouries series expansion coefficient.
D	Load contribution factor of outside air [kW/K].
d	Interior temperature contribution factor of radiative source [K/kW].
f	System forcing function, sometimes with subscripts T for temperature prediction or E for load calculation.
H	System transfer function.
h_{o}	System impulse response.
I_{n}	Bessel function.
j	Imaginary number.
K_{n}	Bessel function.
Q_{cr}	Convective load which will maintain a specified interior temperature [kW].
Q_{c}	Convective load [kW].
Q_{r}	Radiative load [kW].
Q_{s}	Active system load [kW].
q	Heat energy stored in the massive structure [kW].
R_{a}	Mean film resistance from interior surface of shell to interior air [K/kW].
R_{o}	Conductive shell resistance [K/kW].
R_{v}	Equivalent ventilation resistance [K/kW].
s	Independent variable in Laplace domain [1/h].

T	Diurnal period of 24 hours [h], often temperature [^{o}C].
T_c	Mean structure temperature [^{o}C].
T_{cr}	Required structure temperature for comfort [^{o}C].
T_{ir}	Required interior comfort temperature [^{o}C].
T_i	Zone interior air temperature [^{o}C].
T_{sa}	Effective sol–air external temperature [^{o}C].
T_o	Temperature of ventilating air [^{o}C].
T_x, T_y	Effective forcing temperatures [^{o}C].
$\mathrm{T_t}$	Thermostat temperature [^{o}C].
$\Delta\mathrm{T}$	Time interval between sampling points [h].
t	Independent variable – time [h].
t_o	Initial value of time axis [h].
u	Unit step function.
X	System input in frequency domain.
x	System input.
Y	System output in frequency domain.
y	System output.

α	Proportional feedback constant [W/K], factor in (5.45).
β	Coefficient of differential equation, equal to inverse of time–constant [/h]. Subscript T refers to interior temperature, E to energy loads and S to active systems.
$\delta(t)$	Dirac's impulkse function.
ϵ	Error term
Γ	Anti–derivative of β(t).
τ	Thermal time–constant of building [h], elapsed time since stimulation of the system [h].
τ_p	Time–constant of system pole [h].
τ_z	Time–constant of system zero [h].
ν	Frequency [rad/h]
ω	Frequency [rad/h].

APPENDIX 5A

Evaluation of numerical technique for variable RC against an exact solution for β constant in intervals.

Units: length $m \equiv 1L$ time $s \equiv 1T$ mass $kg \equiv 1M$ temperature $K \equiv 1Q$

$J \equiv kg\cdot m^{2}\cdot s^{-2}$ $W \equiv J\cdot s^{-1}$

$kJ \equiv J\cdot 10^{3}$ $kW \equiv W\cdot 10^{3}$

$h \equiv 60\cdot 60\cdot s$ $rad \equiv 1$

$deg \equiv \frac{\pi}{180}\cdot rad$ $\Omega \equiv K\cdot W^{-1}$

$F \equiv kJ\cdot K^{-1}$ $qh \equiv \frac{h}{1}$

$f \equiv 1\cdot 24^{-1}\cdot h^{-1}$ $\omega \equiv 2\cdot\pi\cdot f$

$T := 24\cdot h$ $T1 := 11\cdot h$

Air: $\rho := 1\cdot kg\cdot m^{-3}$ $cp := 1\cdot kJ\cdot kg^{-1}\cdot K^{-1}$

$N := \frac{T}{qh}$

Variable ventilation. Analytic solution for square wave modulation of β.

Data vir 'STORE.ZNE'.

$Ao := 1290.8\cdot m^{2}$ $Ra := .000046\cdot\Omega$

$C := 410.61\cdot Ao\cdot F\cdot m^{-2}$ $Ro := \frac{0.050075}{Ao}\cdot\Omega\cdot m^{2}$

$ach := \begin{bmatrix}30\\.1\end{bmatrix}\cdot h^{-1}$ $vol := 3624\cdot m^{3}$

$Rv := \overrightarrow{\left[\frac{1}{vol\cdot\rho\cdot ach\cdot cp}\right]}$ $Rv = \begin{bmatrix}0.000033\\0.009934\end{bmatrix}\cdot\Omega$

$\beta := \overrightarrow{\left[\frac{Ro + Ra + Rv}{Ro\cdot(Ra + Rv)\cdot C}\right]}$ $\beta = \begin{bmatrix}0.261\\0.176\end{bmatrix}\cdot h^{-1}$

Time constants:

$$\tau := \overrightarrow{\left[\beta^{-1}\right]} \qquad \tau = \begin{bmatrix} 3.832 \\ 5.689 \end{bmatrix} \cdot h$$

$$t := 0 \,..N - 1$$

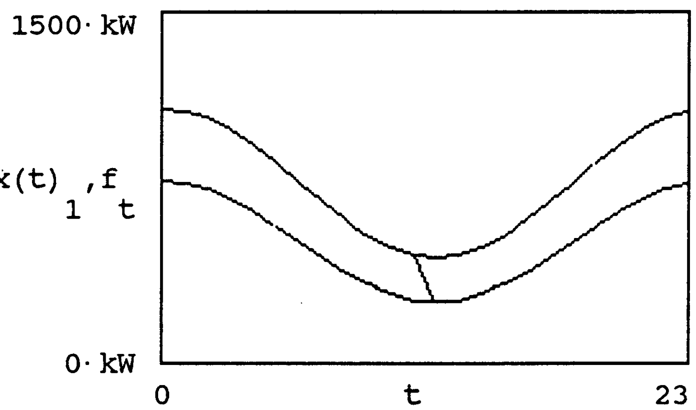
Forcing functions:

$$Tsa(t) := (20 + 10\cdot\cos(\omega\cdot t\cdot qh))\cdot K$$

$$To(t) := (20 + 5\cdot\cos(\omega\cdot t\cdot qh))\cdot K$$

$$fx(t) := \overrightarrow{\left[\frac{To(t)}{Ra + Rv} + \frac{Tsa(t)}{Ro}\right]}$$

$$f_t := if\left[t\cdot qh \le T1, fx(t)_0, fx(t)_1\right]$$



Solve first order differential equation.

(I) Analytic solution.

$$X := \overrightarrow{\left[20\cdot K\cdot\frac{Ro + Ra + Rv}{Ro\cdot(Ra + Rv)}\right]} \qquad X = \begin{bmatrix} 768.351 \\ 517.551 \end{bmatrix}\cdot kW$$

$$m := \overrightarrow{\left[\frac{Ro\cdot 0.25 + (Ra + Rv)\cdot 0.5}{Ro + Ra + Rv}\right]} \qquad m = \begin{bmatrix} 0.418 \\ 0.499 \end{bmatrix}$$

$$Ti := \begin{bmatrix} T1 \\ T - T1 \end{bmatrix} \qquad t0 := \begin{bmatrix} 0\cdot h \\ T1 \end{bmatrix}$$

$$\phi := \left[\overrightarrow{\left[\operatorname{atan}\left[\frac{\omega}{\beta}\right]\right]}\right] \qquad \phi = \begin{bmatrix}45.094\\56.123\end{bmatrix}\cdot \mathrm{deg}$$

$$\alpha(t) := \overrightarrow{(\cos(\omega\cdot(t+t0)-\phi)-\cos(\omega\cdot t0-\phi)\cdot \exp(-\beta\cdot t))}$$

$$A(t) := \overrightarrow{\left[\frac{X}{\beta}\cdot\left[1+\frac{m\cdot\beta}{\sqrt{\beta^{2}+\omega^{2}}}\cdot\alpha(t)-e^{-\beta\cdot t}\right]\right]}$$

$$\alpha(T1) = \begin{bmatrix}-0.539\\0.114\end{bmatrix}$$

$$A0 := A(T1)_{0} \qquad A0 = 2.31\cdot 10^{3}\cdot \mathrm{kW}\cdot \mathrm{h}$$

$$A1 := A(T-T1)_{1} \qquad A1 = 3.128\cdot 10^{3}\cdot \mathrm{kW}\cdot \mathrm{h}$$

$$y00 := \frac{A0\cdot \exp\left[-\beta_{1}\cdot(T-T1)\right]+A1}{1-\exp\left[-\beta_{0}\cdot T1-\beta_{1}\cdot(T-T1)\right]}$$

$$y00 = 3.383\cdot 10^{3}\cdot \mathrm{kW}\cdot \mathrm{h}$$

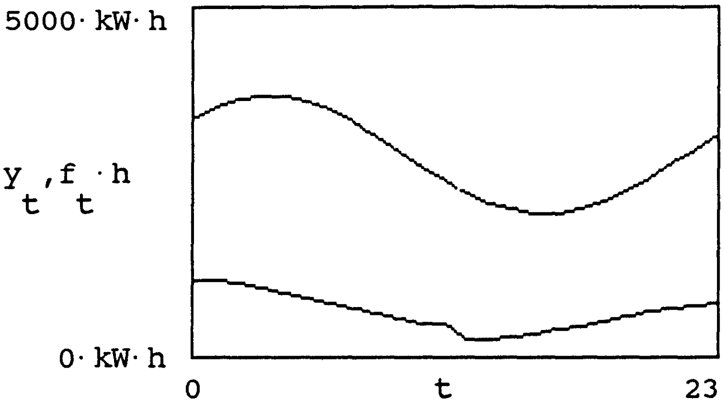
$$y0(t) := y00\cdot \exp\left[-\beta_{0}\cdot t\right]+A(t)_{0}$$

$$y10 := y0(T1) \qquad y10 = 2.502\cdot 10^{3}\cdot \mathrm{kW}\cdot \mathrm{h}$$

$$y1(t) := y10\cdot \exp\left[-\beta_{1}\cdot(t-T1)\right]+A(t-T1)_{1}$$

$$y1(T) = 3.383\cdot 10^{3}\cdot \mathrm{kW}\cdot \mathrm{h}$$

$$y_{t} := \mathrm{if}(t\cdot qh \le T1, y0(t\cdot qh), y1(t\cdot qh))$$

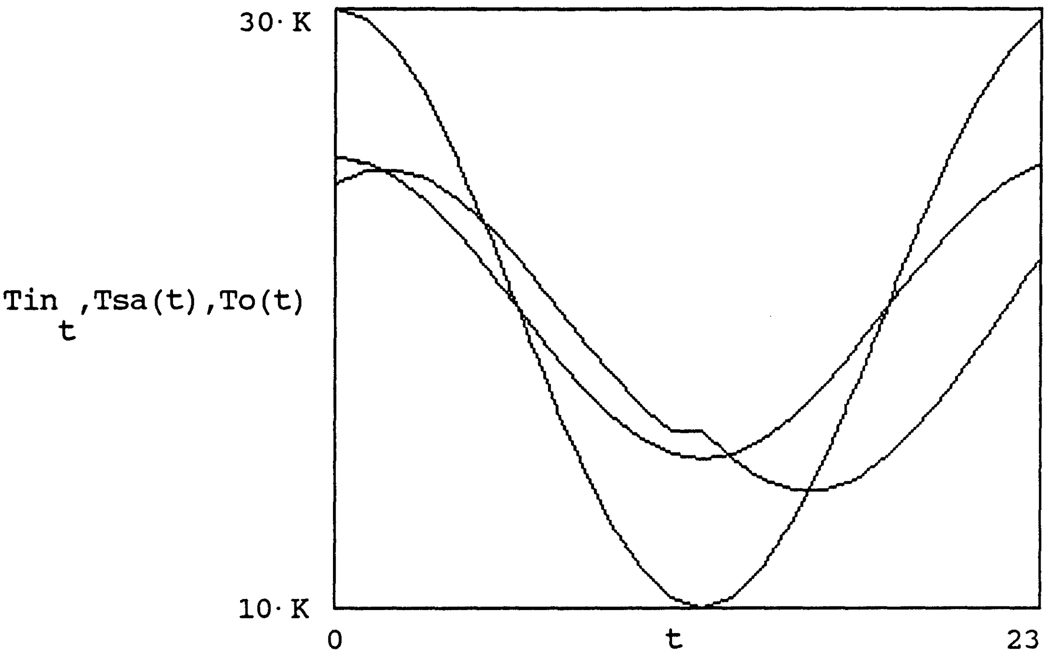


Interior temperature:

$$\mathrm{Tin}(t) := \overrightarrow{\left[\frac{\frac{y_t}{C}\cdot \mathrm{Rv} + \mathrm{To}(t)\cdot \mathrm{Ra}}{\mathrm{Rv} + \mathrm{Ra}}\right]}$$

$$\mathrm{Tii}(t) := \mathrm{if}\left[t\cdot \mathrm{qh} \le \mathrm{T1}, \mathrm{Tin}(t)_0, \mathrm{Tin}(t)_1\right]$$

$$\mathrm{Tin}_t := \mathrm{Tii}(t)$$



(II) Numerical Approximation

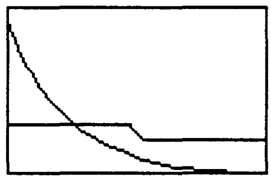
$k := 1 .. N - 1$

$Beta_t := if\left[t \cdot qh \le T1, \beta_0, \beta_1\right]$

$prod_0 := \left[1 - Beta_{N-1} \cdot 1 \cdot qh\right]$

$prod_k := prod_{k-1} \cdot \left[1 - Beta_{N-k-1} \cdot 1 \cdot qh\right]$

$l := 0 .. N - 2$



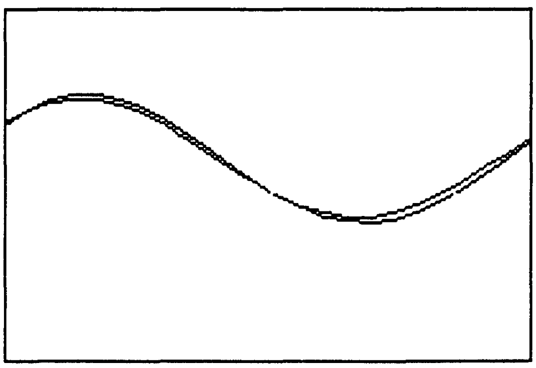
$$ya_0 := \frac{\left[\sum_{l} f_l \cdot qh \cdot prod_{N-l-2}\right] + f_{N-1} \cdot qh}{1 - prod_{N-1}}$$

$ya_0 = 3.345 \cdot 10^3 \cdot kW \cdot h$ $\quad y_0 = 3.383 \cdot 10^3 \cdot kW \cdot h$

$ya_k := f_{k-1} \cdot qh + \left[1 - Beta_{k-1} \cdot qh\right] \cdot ya_{k-1}$

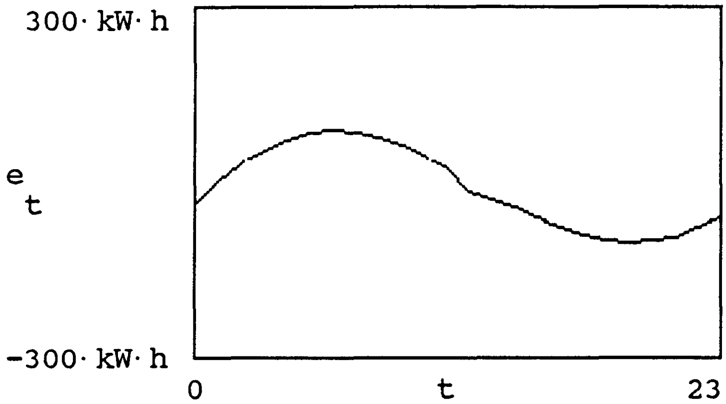
% error:

$$\frac{ya_0 - y_0}{y_0} \cdot 100 = -1.117$$



Stored Energy Error:

$e := \overrightarrow{(ya - y)}$

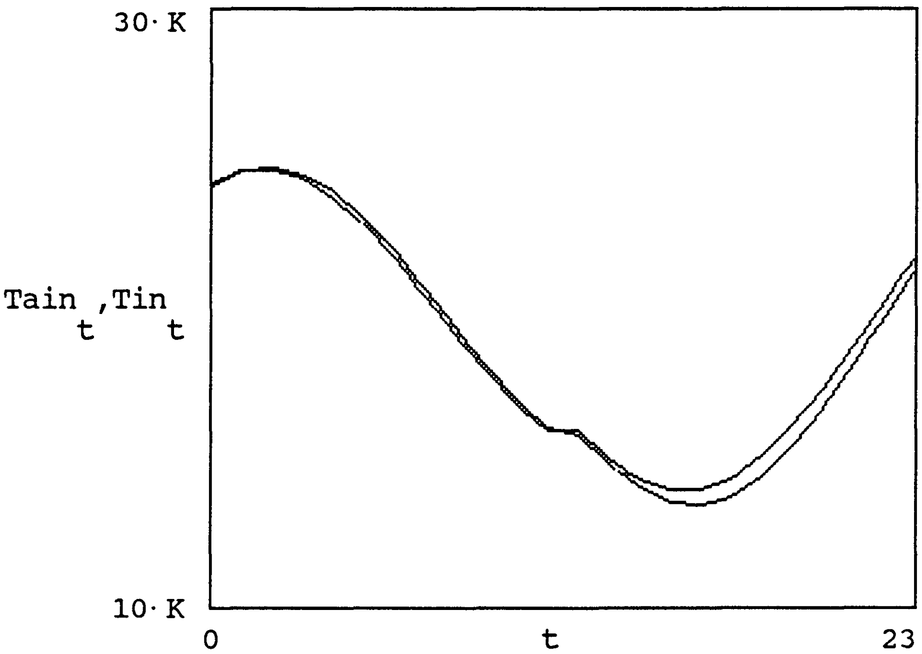


Temperatures:

$$Tain(t) := \overrightarrow{\left[\frac{\frac{ya_t}{C}\cdot Rv + To(t)\cdot Ra}{Rv + Ra}\right]}$$

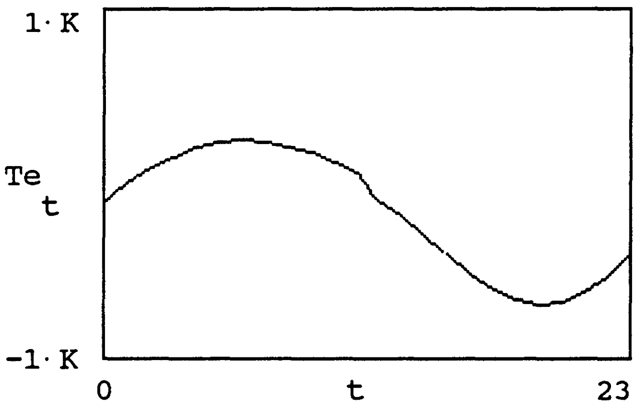
$$Taii(t) := if\left[t\cdot qh \le T1, Tain(t)_0, Tain(t)_1\right]$$

$$Tain_t := Taii(t)$$



$Te := \overrightarrow{(Tain - Tin)}$

Temperature error



CHAPTER 6

6 CLOSURE

In §1.3 the objectives of this study were spelled out. In this final chapter, we examine these objectives again in the light of the results and conclusions of chapters 2 to 5. It is our aim to determine the extend to which we have succeeded in meeting these objectives, and to distill from the conclusions, given at the end of each chapter, some final remarks and suggestions.

6.1 The Theoretical Underpinnings of the Method of Mathews and Richards

We set out in chapter 2 to examine the method of Mathews and Richards from a theoretical point of view, and to determine the extend to which it is possible to derive their novel thermo–flow network, rigorously from a comprehensive model. It must be stressed that this objective was identified consequential to the successful verification programme of the method of Mathews and Richards, in order to elucidate certain aspects of the method, and not in an attempt to be critical. In fact the success of the method, as testified by measurements, coupled with the simplicity and clear physical interpretation, precludes criticism. However, a rigorous derivation of the simplified model from a comprehensive model, shows that it can still be improved in the following aspects:

a) The definition of the mean sol–air temperature can be improved by including the time constants of the individual structures in the definition. This will to some extend correct the phase shift discrepancy observed in some measurements [4], and relieve the method of the *ad hoc* empirical phase correction.

b) A more satisfactory lumped model would be obtained by representing the massive elements with an RCR section, as in figure 2.14, instead of an RC section. The advantage of this representation is that a clearer physical interpretation is possible since the interior surface temperatures are uniquely identified and the capacitance is more remote from the interior surface node.

c) The treatment of interior elements described in [5] are expedient, but may lead to errors in passive zones with massive internal elements. If the internal elements are kept separate from the shell elements, a second order network with two time constants results. This second order network is closer to reality than the single time constant network and may improve the accuracy of the model in the case of zones with massive internal elements.

d) An analysis of temperature differences between interior surfaces indicates that, in general, heat exchange between interior surfaces can not be ignored and must be included in a model which, in addition to the bulk temperature of the interior air, attempts also to predict the mean interior surface temperature. Since comfort criteria include contributions from both the air– and surface temperatures, the utility of the method will be enhanced by a more refined model of interior heat exchange. This need not complicate the method excessively since various simplifying assumptions are available. One could, for instance, empirically discriminate between northern and southern parts of the shell, and internal walls. These enhancements would obviously require extensive experimentation and verification.

The implementation, solution and ease of use of the method will not be compromised by these enhancement to any significant extend. Note however, one can not realistically expect a large improvement in the accuracy of the method with the incorporation of these enhancements, since the accuracy is already high. However, in the author's opinion these improvements are justifiable on the grounds that the main advantage of these very simplified methods is their clear physical interpretation, and it is precisely the physical interpretation which will most benefit from the improvements.

6.2 Structural Storage

In chapter 3 we successfully extended the model of Mathews and Richards to cater for buildings with structures which are directly subjected to cooling, by air being forced over their surfaces. To incorporate this addition, it was required to define the convective heat transfer coefficient in terms of the bulk temperature of the wall, since the model of Mathews and Richards does not clearly identify the surface temperatures; which are usually used in the definition. This difficulty will disappear if suggestion b) above is implemented.

6.3 Multi–Zone Thermo–Flow

We have shown in chapter 4 that a natural extension of the method to coupled zones is potable. This extended model can be fairly efficiently solved if the parameters are time invariant. However, in the time variant case iterative techniques are required. It was found that the implementation of suggestion c) above will be beneficial for such a multi–zone model.

The implementation of the multi–zone model will increase the accuracy of the temperature predictions in passive buildings with large temperature differences between adjacent zones. It will make possible accurate predictions in buildings which include sun–spaces, solar heated ceiling spaces etc. However, it will be of little benefit to designers of conditioned buildings. Since it will seriously compromise the speed of the solution and also the ease of use of the program, it seems advisable to include it as an appendage to the program, in preference to integrating it completely with the other facilities. In this case, if this extra capability is not required, it can be ignored and will not compromise the rest of the program.

6.4 Numerical Solution of a System with Time Dependent Parameters

In chapter 5 we present a new efficient solution of the thermo–flow network of Mathews and Richards. This solution is based on the assumption that both the time dependent parameters of the circuit, as well as the forcing functions, are cyclic with the same period. It proved possible to find a closed form expression for the initial value of the discrete solution, so that the initial value can be found after effectively integrating through just one cycle. This method is very efficient compared to the usual numerical technique, which starts with arbitrary initial conditions, and then integrates until at least 5 time–constants have elapsed, to get rid of transients.

The method is very general and can be applied to any discrete system of cyclic equations to find cyclic solutions. It does not depend on the method, or order, of discretization of the underlying continuous system. The only requirement is a fixed time–step size. It therefore appears to have applications also in celestial mechanics, cyclic combustion etc.

In addition it was shown that a mixed time–frequency domain method can be used which is a logical extension of the Fourier series method to systems with time dependent parameters. It was shown that the method can be used to obtain sufficiently accurate approximate solutions, if the mean part of the solution is treated separately. However, this method requires the solution of an evolution equation for the system and it has no computational benefit compared to the efficient numerical solution of the previous paragraphs.

6.5 Suggestions for the Future

This thesis must be seen as part of an ongoing effort, by the Centre for Experimental and Numerical Thermo–flow, to develop and promote a viable tool for building thermal analysis. The results of this study suggests that: (in order of priority)

a) Methodology of Thermal Modelling. It proved possible to deduce simplified thermal networks from more detailed networks by suitable simplifying assumptions. These assumptions are to a large extend arbitrary and depend on the degree of simplification required. Future simplified models can be constructed logically, by reduction of comprehensive models to the required degree of simplicity. The accuracy of the simplified model can be theoretically deduced from the degree to which actual buildings comply with the simplifying assumptions. However, it must be remembered that the attainable accuracy of a thermal model is often not determined by the assumptions, but rather by the accuracy of the input data. Therefore, even though buildings may violate some of the assumptions of the model, it does not necessarily follow that the model is useless.

b) Mean Sol–Air. In their definition of the mean sol–air forcing function, Mathews and Richards ignores the heat storage capability of the walls. This contributes to the phase discrepancy they experience. A better definition of the mean sol–air is given in §2.5.4, in terms of the fundamental diurnal frequency of the temperature variation.

c) Structural Storage. Structural storage systems are easily included in the thermal analysis method. However, a somewhat unusual definition of the heat transfer coefficient is required.

d) Second Order Network. The second order thermal network of figure 2.22 has some distinct advantages compared to the model of figure 2.1. It provides surface temperatures which are required for comfort criteria, is easier to interpret physically and treats the interior massive elements more correctly. It is also more suitable for extension to multi–zone thermal analysis as well as structural storage since surface temperatures are clearly defined. In the author's opinion it warrants implementation and testing.

e) Numerical Solution. The numerical solution presented in chapter 5 seems very general and may have applications in diverse fields. Its extension to higher order systems and application to other discretization methods should be investigated. However, it does not seem possible to extend the method to non–linear systems, such as deadband controllers etc., a fact which requires further investigation.

f) Multi–Zone Thermal Analysis. The multi–zone method can be implemented but it is limited to time invariant systems. For the analysis of tome dependent systems, full numerical methods should also be investigated.

6.6 The Utility of Highly Simplified Models

In conclusion it is perhaps necessary to repeat our justification for the emphasis we placed, in this study, on highly simplified lumped thermo–flow networks, when, with modern computers, it is feasible to solve the comprehensive models.

It is our objective to create a modern building designer's tool. The comprehensive models may be marginally more accurate than the simplified models, but, they are not suitable for a total systems design approach where the interaction between the passive response of the building, and the system, is of primary importance. Eventually, a design tool must provide more than just a simulation for a given set of conditions, it must aid the designer in his grasp of the problem. The simple network, whether derived from theoretical considerations and simplifying assumptions, or empirical correlations, can have a much clearer physical interpretation since many unessential details are discarded. It is easier to comprehend, to solve, and to extend. In conditioned buildings, the trend is to emphasize the passive response of the building and the system controller. The control engineer needs a simple, sufficiently accurate model of the passive response, to optimize his design.

The success of the original empirical model of Wentzel, Page–Shipp and Venter [1], as well as the later enhancements by Mathews [2], Joubert and Mathews [3], Mathews and Richards [4] and Mathews, Rousseau, Richards and Lombard [5], testifies to the utility of these simplified models. In this study, a firm theoretical foundation is given to these models and their utility is enhanced.

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