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The Blaschke-Santaló inequality

by

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DECLARATION

I, the undersigned, hereby declare that the dissertation submitted herewith for the degree Magister Scientiae to the University of Pretoria contains my own, independent work and has not been submitted for any degree at any other university.

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Contents

Preface

In its most general form, the Blaschke-Santaló inequality is given by the following statement:

"*The volume product of any convex body is at most equal to that of an ellipsoid. Moreover, the maximum volume product is attained exclusively by ellipsoids.*"

This fundamental geometric inequality is well-known not only for its uses in a variety of fields ranging from stochastic geometry and functional analysis[13] to differential equations[9], but also for its close relation to the classical affine isoperimetric inequality [16] (see for instance Lutwak [10]). Indeed, the first proof of the Blaschke-Santaló inequality - due to Blaschke[4] for dimensions 2,3 and Santaló [16] for arbitrary dimensions - is based on that of the affine isoperimetric inequality.

The conditions for equality in the affine isoperimetric problem which gave rise to the characterization of the upper bound in the Blaschke-Santaló inequality were however only established for convex bodies with sufficiently smooth boundaries. Although it was later found (by rather technical arguments (Petty[12])) that these restrictions can be omitted, the question as to whether it is possible to prove the Blaschke-Santaló inequality without direct reference to the classical isoperimetric inequality and the therewith associated smoothness assumptions, remained open until the early 1980's. The first such proof in the case of centrally symmetric convex bodies was forwarded by Saint Raymond in 1981 in the paper entitled "*Sur le volume des corps convexes symmétriques*" [15] and is the subject of this dissertation. A direct proof of the Blaschke-Santaló inequality for general convex bodies was subsequently given by M. Meyer and A. Pajor [11] in 1990.

In the case of symmetric convex bodies, the formulation of the Blaschke-Santaló inequality is relatively simple. Let $(X, \|.\|_X)$ be a finite dimensional Banach space over $\mathbb{R}$ (also known as a Minkowski Space) with corresponding dual space $(X^*, \|.\|_{X^*})$ and let m and m^* be (associated) Haar measures defined on X and X^* respectively. A set $C \subset X$ is said to be a convex body if it is a compact convex set with non-empty interior. The volume product P of any centrally symmetric convex body $C \subset X$ is defined as the quantity

$$P(C) := m(C)m^*(C^o)$$

where C^o is the polar body associated with C. The Blaschke-Santaló inequality for centrally symmetric convex bodies can now be formulated as follows:

Theorem 0.0.1 (The Blaschke-Santaló Inequality). *Let C be any convex symmetric body contained in the n-dimensional Banach space $(X, \|.\|)$. Then*

1) *The volume product P satisfies the following inequality*

$$P(C) \leq P(E) \tag{0.0.1}$$

where E is an ellipsoid.

2) *Equality occurs in (0.0.1) if and only if C is an ellipsoid.*

Remark 0.0.2. *It will be shown in Section 2.5 that all ellipsoids have the same volume product. The existence of an upper bound for the volume product is therefore implicit in the first part of Theorem 0.0.1.*

Notation:

All vector spaces in this dissertation are assumed to be over the field $\mathbb{R}$ of real numbers. A typical n-dimensional Minkowski Space is denoted by $(X, \|.\|)$ or (X, B), where B denotes the unit ball induced by the norm $\|.\|$. For any $1 \leq p \leq \infty$, the Space l_p^n is defined as the Minkowski Space $(\mathbb{R}^n, \|.\|_p)$, where:

$$\|(x_1, ..., x_n)\|_p := \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} \quad \forall (x_1, ..., x_n) \in \mathbb{R}^n.$$

for $1 \leq p < \infty$ and

$$\|(x_1, ..., x_n)\|_\infty := \max\{|x_i| : i = 1, ..., n\} \quad \forall (x_1, ..., x_n) \in \mathbb{R}^n.$$

The n-dimensional Lebesgue measure, defined on the collection $\mathfrak{B}(\mathbb{R}^n)$ of all Borel sets in $\mathbb{R}^n$, is denoted by λ^n or λ if the underlying dimension is implicitly understood.

Outline of Dissertation:

The aim of this dissertation is to provide a relatively comprehensive exposition of the Blaschke-Santaló inequality in the case of symmetric convex bodies with particular emphasis placed on the approach outlined by Saint Raymond. The dissertation is structured as follows:

Chapter 1 provides a brief introduction to Minkowski Spaces and their associated Euclidean structure. For any n-dimensional Minkowski space X there exists a linear homeomorphism from X onto the well-known Euclidean space l_2^n and thus all Minkowski spaces are topologically equivalent. In these spaces, the collection of convex symmetric bodies is interchangeable with the collection of unit balls. Moreover, the unit ball of a Minkowski space X is an ellipsoid if and only if X is isometric to l_2^n and thus Theorem 0.0.1 can be interpreted as a measure theoretic characterization of Hilbert spaces in finite dimensions. Any linear functional acting on a Minkowski Space can be uniquely represented in terms of the Euclidean inner product (regardless of the norm-structure of X), which gives rise to the definition of the adjoint operator on the dual Minkowksi space X^*.

Chapter 2 introduces both the Haar measure, an extension of the usual volume measure on $\mathbb{R}^n$ to n-dimensional Banach spaces, and the volume product. Haar measures are closely related to to the well-known n-dimensional Lebesgue measure λ^n and can indeed be interpreted in terms of the scaled Lesbesgue measure. The rules governing the change in "volume", due to a linear change in variables, guarantee the invariance of the volume product under linear isomorphisms. The problem of finding the maximum volume product can thus, without loss of generality, be formulated in the measure space $(l_2^n, \mathfrak{B}(\mathbb{R}^n), \lambda^n)$. For unit balls of l_p^n-spaces, explicit volume formulas can be readily derived and the Blascke-Santaló inequality can thus be solved directly for this subclass.

Chapter 3 focuses on the topological properties of the collection of compact convex bodies contained in a given Minkowski Space, by defining a metric (the Hausdorff metric) on this class. It will be shown that both the Haar measure as well as the volume product are continuous functions with respect to this metric. The Blaschke selection Theorem, proved in Section 3.4, forms the basis of all existence proofs in the ensuing chapters.

Chapter 4 deals with the method of Steiner symmetrization. The most useful result in this Chapter, given by Theorem 4.3.3, asserts that for every symmetric convex body $C \subset \mathbb{R}^n$ there exists a sequence of successive Steiner symmmetrals $\{C_i\}_{i=1}^{\infty}$ of C converging to a Euclidean Ball with respect to the Hausdorff metric. This Theorem, based partly on the Blaschke selection Theorem and the continuity of the volume product, not only leads to the famous characterization of ellipsoids due to Bertrand[3] and Brunn[6] (which is invoked in Chapter 5 to prove the second part of the Blaschke-Santaló inequality), but was also used directly by Meyer and Pajor [11] in a short proof of the first part of the Blaschke-Santaló inequality.

Saint Raymond's proof of Theorem 0.0.1 is finally given in Chapter 6. This proof is essentially constructive and not only exploits the symmetry of the underlying convex body, but also various properties of convex- and concave functions. The cross-sections of any symmetric convex body C in $\mathbb{R}^n$ can be used to construct another convex body C', whose volume product is strictly greater than that of C unless C satisfies the hypothesis of a certain corollary of Brunn's theorem which in turn implies that C must be an ellipsoid. In order to maintain the fluency of Chapter 6, the necessary continuity- and differentiability properties of convex-/concave functions are discussed in the Appendix.

Chapter 1

Minkowski Spaces and their associated Euclidean Structure

1.1 Introduction

In the interest of presenting a relatively self contained text, the first chapter aims at broadly outlining the basic definitions and fundamental results concerning Minkowski Spaces that will be used throughout the dissertation.

Finite dimensional Normed spaces, also known as Minkowski Spaces, are often studied in terms of their relation to the familiar Euclidean Space l_2^n. Indeed, let $(X, \|.\|)$ be an n-dimensional Minkowski space with ordered basis $\{b_1, ..., b_n\}$. The coordinatization mapping $\zeta : X \to \mathbb{R}^n$ is defined in terms of its action on the basis vectors $\{b_1, ..., b_n\}$ as follows:

$$\zeta(b_i) := e_i \qquad \text{for } i = 1, ..., n$$

where $\{e_1, ..., e_n\}$ is the standard ordered basis for $\mathbb{R}^n$. This linear isomorphism is the key to studying both the algebraic as well as the topological structure of X and will be used frequently in sequel.

Evidently ζ is defined in terms of the ordered basis $b = \{b_1, ..., b_n\}$. To stress its dependency on this basis we use the notation ζ_b. If another ordered basis $c = \{c_1, ..., c_n\}$ of X were to be chosen then ζ_c would define a different isomorphism from X onto $\mathbb{R}^n$. The following lemma establishes a linear relation between ζ_b and ζ_c.

Lemma 1.1.1. *There exists an invertible linear mapping $T : \mathbb{R}^n \to \mathbb{R}^n$ such that*

$$\zeta_c(x) = T(\zeta_b(x)) \quad \textit{for all } x \in X \tag{1.1.1}$$

Proof. Since linear transformations preserve linear independence, it follows that the sets $\{\zeta_c(b_1), ..., \zeta_c(b_n)\}$ and $\{\zeta_c(c_1), ..., \zeta_c(c_n)\} = \{e_1, ..., e_n\}$ are both bases of $\mathbb{R}^n$. Let T be the invertible linear map defined by

$$T(e_i) := \zeta_c(b_i) \quad \text{for } i = 1, ..., n \ .$$

The linearity of T and ζ_c ensure that for any $x = \sum_{i=1}^{n} \alpha_i b_i \in X$ we have

$$\begin{aligned} \zeta_c(x) := \zeta_c\left(\sum_{i=1}^{n} \alpha_i b_i\right) &= \sum_{i=1}^{n} \alpha_i \zeta_c(b_i) = \sum_{i=1}^{n} \alpha_i T(e_i) \\ &= T\left(\sum_{i=1}^{n} \alpha_i e_i\right) =: T(\zeta_b(x)) \end{aligned}$$

□

1.2 The Norm Topology

Not only is the coordinatization mapping ζ an algebraic isomorphism, but it also defines a homeomorphism from any n-dimensional Minkowski space $(X, \|.\|)$ onto the familiar Euclidean space l_2^n *(Theorem 1.2.3 [18])*. As a consequence, all n-dimensional Minkowski Spaces are topologically equivalent to l_2^n. Particularly, any Minkowski space is locally compact, complete and has the Heine-Borel property. Moreover, every linear map between Minkowski spaces is bounded (*Theorem 1.2.4 [18]*). From this it follows that for any two norms $\|.\|$ and $\|.\|'$ defined on a given Minkowski space X there exist constants $c_1, c_2 > 0$ such that

$$c_1\|x\| \leq \|x\|' \leq c_2\|x\| \quad \text{for all } x \in X \tag{1.2.1}$$

and hence all norms on X are equivalent.

1.2.1 Convex Symmetric Bodies and the Unit Ball

Let $(X, \|.\|)$ an n-dimensional Minkowski space. The ball in $(X, \|.\|)$ with center $x_0 \in X$ and radius $r > 0$ is defined as the set

$$B(X; x_0, r) := \{x \in X : \|x - x_0\| \leq r\}.$$

Similarly, the set

$$S(X; x_0, r) := \{x \in X : \|x - x_0\| = r\}$$

is called the sphere in $(X, \|.\|)$ with center $x_0 \in X$ and radius $r > 0$. For the sake of simplicity, the unit ball $B(X; 0, 1)$ is denoted by $B(X)$ or simply by B. The following terminology will prove useful in the characterization of unit balls.

Definition 1.2.1. *Let $(X, \|.\|)$ be an n-dimensional Minkowski space with $K \subseteq X$. Then:*

(i) K is said to be convex if $\lambda K + (1-\lambda)K \subseteq K$ for every $\lambda \in [0,1]$

(ii) K is said to be (centrally) symmetric if $-K = K$

(iii) K is called a convex body if K is a compact convex set and $\text{int}K \neq \emptyset$

It follows directly from elementary properties of the norm that the unit ball $B = \{x \in X : \|x\| \leq 1\}$ of a Minkowski Space $(X, \|.\|)$ is a convex symmetric set and $0 \in \text{int}B = \{x \in X : \|x\| < 1\}$. Moreover, Minkowski spaces are characterized as those Normed spaces whose unit ball is compact (*Theorem 1.2.8[18]*) and hence B is a symmetric convex body.

Conversely, let C be an arbitrary symmetric convex body in $(X, \|.\|)$ and define the Minkowski functional $\|.\|_C : X \to [0, \infty)$ as follows:

$$\|x\|_C := \inf\{\xi > 0 : x \in \xi C\} \quad \text{for all } x \in X \tag{1.2.2}$$

It can now be shown (*Proposition 1.1.8 [18]*) that the Minkowski functional $\|.\|_C$ defines a norm on X with corresponding unit ball C. Consequently any convex symmetric body in X gives rise to a unique norm and can thus serve as the unit ball of X. The collection of symmetric convex bodies in X can subsequently be used interchangeably with the collection of unit balls in X.

1.3 The Euclidean Structure of Minkowski Spaces

Let (X, B) be a Minkowski space with unit ball B and let ζ be the coordinatization mapping defined in section 1.1. By virtue of the fact that ζ is a homeomorphism, it follows that the image $\zeta(B) \subset \mathbb{R}^n$ of B is compact and that the open set $\text{int}\zeta(B) = \zeta(\text{int}B)$ is non-empty. Moreover, the linearity of ζ ensures that the convexity and symmetry of B are preserved. $\zeta(B)$ is therefore a symmetric convex body and can hence be used to define a norm on $\mathbb{R}^n$. Now ζ defines an isometry from (X, B) onto $(\mathbb{R}^n, \zeta(B))$ and (X, B) is said to be realized as $(\mathbb{R}^n, \zeta(B))$.
It is often convenient to impose an inner product structure on an arbitrary Minkowski space (X, B). To this end, let ζ be the coordinatization mapping corresponding to ordered basis $\{b_1, ..., b_n\}$ and let

$$\langle x, y \rangle := \zeta(x) \cdot \zeta(y) \text{ for every } x, y \in X$$

where $\cdot$ denotes the usual dot product. The inner product $\langle ., . \rangle$ is known as the Euclidean structure of X in terms of $\{b_1, ..., b_n\}$ and, though useful, generally bears no relation to the norm induced by the unit ball B. It follows however from proposition 1.3.3(and the subsequent remark) that if the realization

$(\mathbb{R}^n, \zeta(B))$ of (X, B) is a Hilbert space then (X, B) itself is a Hilbert space, which is isometric to l_2^n.

Definition 1.3.1. *A subset C of a Minkowski Space $(X, \|.\|)$ is called an ellipsoid if there exists an invertible linear transformation $T : X \to \mathbb{R}^n$ such that $T(C) = B(l_2^n)$.*

Proposition 1.3.3 asserts that that a Minkowski space (X, E) is a Hilbert space if and only E is an ellipsoid. The proof is based on the well-known Jordan-von Neumann characterization of inner product spaces, which is stated here without proof *(see Amir [1])*.

Proposition 1.3.2 (Jordan-von Neumann Theorem). *A normed space $(X, \|.\|)$ is an inner product space if and only if the following relation holds:*

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2 \quad \textit{for all } x, y \in X \tag{1.3.1}$$

Proposition 1.3.3. *A Minkowski space (X, B) is a Hilbert Space if and only if B is an ellipsoid.*

Proof. Suppose (X, B) is a Hilbert space with inner product $\langle ., . \rangle$ and let $\{b_1, ..., b_n\}$ be an ordered orthonormal basis for X. The image of a typical point $x = \sum_{i=1}^n \beta_i b_i \in X$ under the coordinatization mapping ζ corresponding to this basis is given by $\zeta(x) = (\beta_1, ..., \beta_n)^T$. Let $\|.\|$ and $\|.\|_2$ denote the norms corresponding to the unit balls B and $B(l_2^n)$ respectively. Then:

$$\begin{aligned}
\|x\|^2 = \langle x, x \rangle &= \langle \sum_{i=1}^n \beta_i b_i, \sum_{i=1}^n \beta_i b_i \rangle \\
&= \sum_{i=1}^n \beta_i^2 \langle b_i, b_i \rangle + \sum_{i \neq j} \beta_i \beta_j \langle b_i, b_j \rangle \\
&= \sum_{i=1}^n \beta_i^2 = \|\zeta(x)\|_2^2
\end{aligned}$$

and thus $\zeta : X \to \mathbb{R}^n$ is the required linear bijection.
Conversely, suppose the unit ball B of the Minkowski space is an ellipsoid. By definition there then exists a linear isometric isomorphism $T : (X, B) \to l_n^2$. It can now readily be seen that the mapping $\langle ., . \rangle : X \to X$ given by:

$$\langle x, y \rangle := (Tx) \cdot (Ty) \quad \text{for } x, y \in X \tag{1.3.2}$$

defines an inner product on X whose induced norm agrees with the norm corresponding to the unit ball B. □

Remark 1.3.4. *By virtue of the linearity of the coordinatization mapping ζ and Proposition 1.3.3, a Minkowski space (X, B) is a Hilbert space if and only if there exists a realization $(\mathbb{R}^n, E)$, where E is an ellipsoid. Moreover, if there exists such a realization, then Lemma 1.1.1 guarantees that every other realization of (X, B) is also a Hilbert space.*

In light of Proposition 1.3.3, the Blaschke-Santaló inequality can be reformulated as a measure-theoretic characterization of Hilbert spaces in finite dimensions:

Theorem 1.3.5. *For any Minkowski space (X, K), the volume product P satisfies the following inequality*

$$P(K) \leq P(E) \tag{1.3.3}$$

where E is an ellipsoid. Moreover equality occurs in (1.3.3) if and only if (X, K) is a Hilbert space.

1.4 Dual Minkowski Spaces

The dual space (X^*, B^o) of an n-dimensional Minkowski space (X, B) is the collection of all linear functionals on X, equipped with the dual norm

$$\|f\| := \sup\{|f(x)| : x \in B\} \quad \forall f \in X^*$$

From the definition of the supremum and the symmetry of B, it follows readily that the unit ball B^o of X^*, also known as the polar of B, can be written in the form

$$\begin{aligned} B^o &= \{f \in X^* : \|f\| \leq 1\} = \{f \in X^* : |f(x)| \leq 1 \ \forall x \in B\} \\ &= \{f \in X^* : f(x) \leq 1 \ \forall x \in B\} \end{aligned}$$

The Natural Dual basis

Let $\{b_1, ..., b_n\}$ be an ordered basis of X. The action of any linear functional $f \in X^*$ on a vector $x \in X$ is uniquely determined by its action on $\{b_1, ..., b_n\}$. Indeed if $f, g \in X^*$ are such that $f(b_i) = g(b_i)$ for $i = 1, ..., n$ and $x = \sum_{i=1}^n \alpha_i b_i \in X$ then

$$f(x) = f(\sum_{i=1}^n \alpha_i b_i) = \sum_{i=1}^n \alpha_i f(b_i) = \sum_{i=1}^n \alpha_i g(b_i) = g(\sum_{i=1}^n \alpha_i b_i) = g(x).$$

Elementary calculations show that the collection $\{b_1^*, ..., b_n^*\}$ of functionals in X^*, defined by

$$b_i^*(b_j) := \delta_{ij} \quad \text{for } i, j = 1, ..., n$$

is linearly independent and spans X^* and is therefore called the natural dual basis of X^* corresponding to $\{b_1, ..., b_n\}$. Consequently, $\dim X^* = \dim X$ and the mapping $b_i \mapsto b_i^*$ (for $i = 1, ..., n$) defines an isomorphism from X onto X^*.

The Hilbert Adjoint Operator

Suppose the Minkowski space (X, B) is equipped with an auxiliary Euclidean structure $\langle ., . \rangle$ relative to the basis $\{b_1, ..., b_n\}$ and (X^*, B^o) is the dual with corresponding natural dual basis $\{b_1^*, ..., b_n^*\}$. By means of the isomorphism $F : b_i \mapsto b_i^*$ for $i = 1, ..., n$ and the coordinatization mapping ζ, every linear linear functional $f = \sum_{i=1}^n \gamma_i b_i^* \in X^*$ can be uniquely identified with both the vector $y_f = \sum_{i=1}^n \gamma_i b_i \in X$ and the vector $\gamma = (\gamma_1, ..., \gamma_n) \in \mathbb{R}^n$. Moreover, for any $x = \sum_{i=1}^n a_i b_i \in X$, with $a = (a_1, ..., a_n)$:

$$\langle x, y_f \rangle := \gamma \cdot a = \sum_{i=1}^n \gamma_i \sum_{j=1}^n \alpha_j b_i^*(b_j) = \sum_{i=1}^n \gamma_i b_i^*(\sum_{j=1}^n \alpha_j b_j) = f(x)$$

Remark 1.4.1. *This representation resembles the well-known Riesz representation Theorem for linear functionals on Hilbert spaces. Note that*

$$F(B^o) = \{y \in X : \langle x, y\rangle \leq 1\}$$

However, since the norm induced by the inner product $\langle ., .\rangle$ does not necessarily coincide with the norm corresponding to the unit ball B, the relation $\|y_f\| = \|f\|$ does not hold in general.

The Hilbert adjoint operator T^* of a linear isomorphism T on X relates the change of the basis $\{b_1, ..., b_n\}$ under T to the change of the corresponding dual basis $\{b_1^*, ..., b_n^*\}$.

Proposition 1.4.2. *For every isomorphism $T : X \to X$, there exists an isomorphism $T^* : X^* \to X^*$ such that $(T^*)^{-1}$ maps the natural dual basis of X onto the natural dual basis of $T(X)$.*

Proof. Define T^* at an arbitrary vector $f \in X^*$ as the mapping

$$\langle x, T^* f\rangle := \langle Tx, f\rangle \quad \text{for all } x \in X$$

Let $\{b_1, ..., b_n\}$ be a basis for X with corresponding dual basis $\{b_1^*, ..., b_n^*\}$. Evidently $(T^*)^{-1}$ exists and is defined at $f \in X^*$ as the mapping

$$\langle y, (T^*)^{-1} f\rangle := \langle T^{-1} y, f\rangle \quad \text{for all } y \in X$$

It now follows from that fact that $T^* T^{*-1}(b_i^*) = b_i^*$ for $i = 1, ..., n$ and the definition of T^* that

$$\delta_{ij} = b_i^*(b_j) = \langle b_j, T^*(T^*)^{-1} b_i^*\rangle := \langle Tb_j, (T^*)^{-1} b_i^*\rangle$$

for all $i, j = 1, ..., n$. Hence the natural dual basis of X^* corresponding to the basis $\{Tb_1, ..., Tb_n\}$ is simply $\{(T^*)^{-1} b_1^*, ..., (T^*)^{-1} b_n^*\}$ □

Remark 1.4.3. *The following two facts will prove useful in Chapter 2.*

1. *It is well known that if M_T is the standard matrix representation of T then T^* can be represented by the transpose $(M_T)^t$.*

2. *If $T : X \to X$ is an invertible linear transformation then*

$$\begin{aligned}
(T(B))^o &= \{f \in X^* : \langle Tx, f\rangle \leq 1 \forall x \in B\} \\
&= \{f \in X^* : \langle x, T^* f\rangle \leq 1 \forall x \in B\} \\
&= (T^*)^{-1}\{T^* f \in X^* : \langle x, T^* f\rangle \leq 1 \forall x \in B\} = (T^*)^{-1} B^o
\end{aligned}$$

Chapter 2

Haar Measures and the Volume Product

2.1 Introduction

Let λ^n denote the n-dimensional Lebesgue measure defined on the collection $\mathfrak{B}(\mathbb{R}^n)$ of Borel sets in $\mathbb{R}^n$. A Haar measure functions as the volume measure for arbitrary Minkowski Spaces and, as such, resembles the familiar Lebesgue measure in its most salient features. Indeed all Haar measures are are non-zero regular Borel measures which are invariant under both translations and linear transformations whose matrix representations have determinant 1. It can be shown that every locally compact topological group admits a regular Borel measure which is invariant under the action of the group *(Cohn [7])*. The proof of the existence of such measures, even within the context of abelian groups, is, however, quite tedious and doesn't contribute considerably to the development of the ideas in this chapter (The interested reader is referred to Cohn for a more comprehensive treatment of the subject). We will therefore focus on the definition and elementary properties of Haar measures on Minkowski spaces. In particular, it will be shown (in subsection 4.2.1) that all Haar measures defined on a given n-dimensional Minkowski Space are scaled versions of the well-known Lebesgue measure. As a consequence, many problems involving volumes defined on n-dimensional Minkowski Spaces, including that of finding the upper bound for the volume product, can be translated into equivalent problems in $(\mathbb{R}^n, \mathfrak{B}(\mathbb{R}^n), \lambda^n)$. Section 4.3 deals with the effect of linear transformations on the Haar measure of a given Borel set. The rules relating the change in "volume" due to a linear change of variables naturally extend the well-known rules which are valid for $(\mathbb{R}^n, \mathfrak{B}(\mathbb{R}^n), \lambda^n)$ to arbitrary Minkowski Spaces. The Volume Product, defined on symmetric convex subsets of a Minkowski space, is introduced in section 4.5. This quantity is invariant under isomorphisms and hence the general problem of finding the maximum(minimum) volume product can without loss of generality be reformulated in $(\mathbb{R}^n, \mathfrak{B}(\mathbb{R}^n), \lambda^n)$. For the subclass of unit balls $\{B(l_p^n) : 1 \leq p \leq \infty\}$, explicit volume formulas are readily available (section 4.4). In this case, the Blaschke-Santaló inequality can be established through direct computation (section 4.6).

2.2 Definition and Elementary Properties

Definition 2.2.1 (Regular Borel measures). *Let $(X, \|.\|)$ be a Minkowski space and let $\mathfrak{B}(X)$ denote the collection of Borel sets in X. A regular Borel measure $\mu : \mathfrak{B}(X) \to [0; \infty]$ is a measure with the following properties:*
(i) $\mu(U) = \sup\{\mu(K) : K \subset U$ *and* K *is compact*$\}$ $\forall$ *open sets* $U \subseteq X$.
(ii) $\mu(A) = \inf\{\mu(U) : A \subset U$ *and* U *is open*$\}$ *for every* $A \in \mathfrak{B}(X)$.

Definition 2.2.2 (Haar measures). *A non-zero, regular Borel measure μ, defined on a Minkowski space $(X, \|.\|)$, is said to be a Haar measure if it has the following properties:*
(i) $\mu(K) < \infty$ *for all compact sets* K.
(ii) $\mu(U) > 0$ *for all open sets* U
(iii) $\mu(A) = \mu(A + x)$ *for all* $A \in \mathfrak{B}(X)$ *and all* $x \in X$.

Since $\mathfrak{B}(X)$ is closed under translations, the expressions in (iii) are well-defined. It can easily be verified that the Lebesgue measure λ^n on $\mathbb{R}^n$ satisfies the criteria of the above definition and is hence an example of a Haar measure.

2.2.1 Sections of Borel sets and Fubini's Theorem

Fubini's Theorem (stated here without proof) will be frequently used in sequel, since it enables the computation of the volume of a Borel set in terms of the volumes of its cross-sections.

Definition 2.2.3. *Let $F : X \times Y \to \mathbb{R}$. Then:*
(i) The x-section F_x of F is defined as the function:

$$F_x : Y \to \mathbb{R} : y \mapsto F_x(y) := F(x, y)$$

(ii) The y-section F^y of F is defined as the function

$$F^y : X \to \mathbb{R} : x \mapsto F^y(x) := F(x, y)$$

Definition 2.2.4. .
The x-section A_x of a measurable set $A \subset X \times Y$ is defined as the x-section of the characteristic function χ_A. Hence $A_x := \{y \in Y : (x, y) \in A\}$. Similarly, the y-section A_y of A is given by $A_y := \{x \in X : (x, y) \in A\}$

Theorem 2.2.5 (Fubini's Theorem). *Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces and let $F : X \times Y \to \mathbb{R}$ be a $\mathcal{M} \times \mathcal{N}$-measurable function which is integrable with respect to the product measure $\mu \times \nu$. Then F_x is integrable μ-almost everywhere and F^y is integrable ν-almost everywhere. Moreover, the functions f and g defined by*

$$f(x) := \begin{cases} \int_Y F_x d\nu, & \text{if } F_x \text{ is integrable;} \\ 0, & \text{otherwise.} \end{cases}$$
$$g(y) := \begin{cases} \int_X F^y d\mu, & \text{if } F^y \text{ is integrable;} \\ 0, & \text{otherwise.} \end{cases}$$

are integrable and

$$\begin{aligned} & \int_X f d\mu = \int_{X \times Y} F d(\mu \times \nu) = \int_Y g d\nu \\ \Leftrightarrow \quad & \int_X \left(\int_Y F_x d\nu \right) d\mu = \int_{X \times Y} F d(\mu \times \nu) = \int_Y \left(\int_X F^y d\nu \right) d\nu \end{aligned}$$

It follows directly from Fubini's Theorem that the product measure $\mu \times \nu$ of any measurable set $A \subseteq X \times Y$ is given by

$$(\mu \times \nu)(A) = \int_X \nu(A_x) d\mu = \int_Y \mu(A_y) d\nu \tag{2.2.1}$$

2.2.2 "Uniqueness" of Haar measures

In order to prove the main assertion of this section, namely that all Haar measures, defined on a given Minkowski space are unique up to a scalar factor, the following preliminary technical lemma is necessary.

Lemma 2.2.6. *Let X be any Minkowski space and let μ be a Haar measure on X. Then the following three facts hold:*

(a) If g is a continuous, non-zero, non-negative function on X with compact support, then $0 < \int_X g d\mu < \infty$.

(b) There exists a continuous function f such that $\int_X f d\mu \neq 0$

(c) If f is a continuous function on X and μ is a Haar measure on X, then $\int_X f(x+a) d\mu(x) = \int_X f(x) d\mu(x)$ for any $a \in X$.

Proof. .
(a) μ is regular and g is continuous with compact support and hence $\int_X g d\mu < \infty$ *(see Rudin chapter 2 [14])*. Since $g \neq 0$, there exists some $x_0 \in X$

and some $\epsilon > 0$ such that $g(x_0) > \epsilon$. By the continuity of g there must be an open set U such that $g(x) > \epsilon \ \forall x \in U$. Therefore

$$\int_X g d\mu \geq \int_U g d\mu \geq \int_U \epsilon d\mu = \epsilon\mu(U) > 0 \qquad \because \text{definition 2.2.2(ii)}$$

(b) Let A be some Borel set such that $\mu(A) > 0$ and let U be any open set containing A. Since μ is regular, definition 2.2.1 (i) implies that for any such U, there must exist a compact set $K \subset U$ such that $\mu(K) > 0$. By Urysohn's lemma *(see Rudin, Lemma 2.12 [14])* we can construct a continuous function f with compact support such that $\chi_K \leq f \leq \chi_U$. For this function

$$\int_X f d\mu \geq \int_X \chi_K d\mu = \mu(K) > 0$$

(c) If χ_A is the characteristic function of measurable set $A \subset X$, then $\chi_A(x+a) = \chi_{A-a}(x)$. By the translation invariance of μ:

$$\int_X \chi_A(x) d\mu(x) = \mu(A) = \mu(A-a) = \int_X \chi_{A-a}(x) d\mu(x) = \int_X \chi_A(x+a) d\mu(x).$$

The result holds for simple functions by additivity of integrals. For any non-negative measurable function f, the result holds due to the Monotone Convergence Theorem. Finally, any measurable function f can be written as the difference of its positive- and negative parts. Since the result holds for f^+ and f^-, it must by linearity hold for f. □

We now show that all Haar measures on a given Minkowski Space are unique up to a scalar factor.

Theorem 2.2.7. *Let μ and ν be two Haar measures defined on the Minkowski space X. Then there exists a constant $c > 0$ such that $\mu = c\nu$.*

Proof. Let f be an arbitrary continuous function with compact support and g be a given continuous non-negative, non-zero function with compact support. By lemma 2.2.6 (a), $0 < \int_X g d\nu < \infty$. We use $\mu(x)$ instead of μ to stress the dependence of the integral on the underlying variable. By making repeated use of Fubini's Theorem and lemma 2.2.6 (c) we have

$$\begin{aligned}
&\int_X f(x) d\mu \int_X g(y) d\nu \\
&= \int_X \left(\int_X f(x) g(y) d\mu(x) \right) d\nu(y) = \int_X \left(\int_X f(x+y) g(y) d\mu(x) \right) d\nu(y) \\
&= \int_X \left(\int_X f(y) g(y-x) d\nu(y) \right) d\mu(x) = \int_X \left(\int_X f(y) g(-x) d\mu(x) \right) d\nu(y) \\
&= \int_X f(y) d\nu(y) \int_X g(-x) d\mu(x) \qquad (2.2.2)
\end{aligned}$$

Since g is a continuous non-negative, non-zero function with compact support, it trivially follows that the function $\hat{g}$ defined by $\hat{g}(x) = g(-x)$ has the same properties and hence $0 < \int_X \hat{g} d\mu < \infty$. Now let

$$c := \frac{\int_X \hat{g} d\mu}{\int_X g d\nu} \in (0, \infty)$$

For all measurable functions f, relation (2.2.2) can thus be rewritten as:

$$\int_X f d\mu = c \int_X f d\nu$$

Hence

$$\int_X f d(\mu - c\nu) = 0 \quad \text{for every measurable function f}$$

According to lemma 2.2.6 (c), we must therefore have

$$\mu - c\nu = 0 \Rightarrow \mu = c\nu.$$

□

All Haar measures defined on a given Minkowski space are thus scalar multiples of each other. In other words, the measurement of "volume" in Minkowski Spaces is completely determined once a fixed scale is chosen. This concept can be extended to relate Haar measures defined on isomorphic Minkowski Spaces. Indeed, the following Theorem asserts that, if m_X and m_Y are Haar measures defined on the linearly isomorphic Minkowski Spaces X and Y respectively, then m_X can be viewed as a Haar measure defined on Y and must hence be a scaled version of m_Y. This result is particularly useful when viewed in the context of the coordinatization mapping ζ. Every Haar measure on a finite dimensional Minkowski space X can then simply be expressed as a scaled version of the well-known Lebesgue measure. This makes explicit volume computations considerably easier since the Lebesgue measure is well known.

Theorem 2.2.8. *Let $T : X \to Y$ be a linear isomorphism between the n-dimensional Minkowski spaces $(X, \|.\|_X)$ and $(Y, \|.\|_Y)$ and let m_X be a Haar measure on X. Then all Haar measures m_Y on Y are of the form*

$$m_Y(C) = k m_X(T^{-1}(C)) \quad \textit{for all } C \in \mathfrak{B}(Y)$$

where $k > 0$.

Proof. Suppose T is an invertible linear mapping from X onto Y. Define the measure m on Y as follows

$$m(C) := m_X(T^{-1}(C)) \quad \text{for all } C \in \mathfrak{B}(Y) \tag{2.2.3}$$

T is continuous and hence Borel measurable which implies $T^{-1}(C) \in \mathfrak{B}(X)$ for every $C \in \mathfrak{B}(Y)$ and thus m is well-defined. Also, since T^{-1} is linear and m_X is a Haar measure, it follows that

$$\begin{aligned} m(C+x) &:= m_X(T^{-1}(C+x)) = m_X(T^{-1}(C) + T^{-1}(x)) \\ &= m_X(T^{-1}(C)) =: m(C) \end{aligned}$$

and so m is a Haar measure on Y. Moreover, Theorem 2.2.2 asserts that any Haar measure m_Y on Y must be a scalar multiple of m. So there exists some $k > 0$ such that

$$m_Y(C) = km(C) := km_X(T^{-1}(C)) \quad \text{for all } C \in \mathfrak{B}(Y)$$

□

2.3 Volume and Linear Transformations

This section discusses the effects of a bijective linear change of variables on the Haar measure of a Borel set in X. As in the case of $(\mathbb{R}^n, \mathfrak{B}(\mathbb{R}^n), \lambda^n)$, the determinant of the underlying transformation plays a central role.

2.3.1 The Determinant

We begin by defining the determinant of linear mappings on $\mathbb{R}^n$. As was mentioned in chapter 1, every such transformation T can be represented by means of an $n \times n$ matrix M_T. We can hence define the determinant of any linear map T in terms of the determinant of its corresponding standard matrix representation M_T.

Definition 2.3.1. *The determinant* $\det(T)$ *of a linear transformation* $T : \mathbb{R}^n \to \mathbb{R}^n$ *is defined as the determinant of its corresponding matrix representation* M_T.

The standard matrix representation of T is defined in terms of a specific ordered basis. It is well-known, however, that if M_T' is another standard matrix representation of T relative to a different ordered basis, then M_T and M_T' are similar matrices. In other words there exists an invertible $n \times n$ matrix U such that

$$M_T' = UM_TU^{-1}.$$

and hence $\det(M_T') = \det(UM_TU^{-1}) = \det(M_T)$.
The determinant of T is thus independent of the standard matrix representation used and is therefore well-defined.

Now consider the linear transformation T defined on an arbitrary Minkowski space X. Let $\zeta : X \to \mathbb{R}^n$ be a coordinatization mapping and define

$$T_\zeta := \zeta \circ T \circ \zeta^{-1}$$

Since $T_\zeta : \mathbb{R}^n \to \mathbb{R}^n$, it can be used to define the determinant of T.

Definition 2.3.2. *The determinant of a linear transformation $T : X \to X$ is defined as*

$$\det(T) := \det(T_\zeta)$$

Suppose ζ_1 is another coordinatization mapping. According to lemma 1.1.1 there exists an invertible linear transformation $F : \mathbb{R}^n \to \mathbb{R}^n$ such that $\zeta_1 = F\zeta$ which implies $\zeta_1^{-1} = (F\zeta)^{-1} = \zeta^{-1}F^{-1}$. Therefore

$$\begin{aligned} & T_{\zeta_1} := \zeta_1 \circ T \circ \zeta_1^{-1} = F(\zeta \circ T \circ \zeta)F^{-1} \\ \Rightarrow \quad & T_\zeta = F^{-1}T_{\zeta_1}F \end{aligned}$$

Now $\det(T_\zeta) = \det(F^{-1}\zeta_1 F) = \det(T_{\zeta_1})$. The determinant of T doesn't depend on the chosen coordinatization and is thus well-defined.

2.3.2 Linear Change of Variables

The following Theorem relates the volume of the linear image of a set in $\mathbb{R}^n$ to the volume of the set itself. The proof is based on that of Cohn [7]. The subsequent Corollary generalizes this result to arbitrary Minkowski spaces.

Theorem 2.3.3. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear transformation defined on the measure space $(\mathbb{R}^n, \mathfrak{B}(\mathbb{R}^n), \lambda^n)$ then*

$$\lambda^n(T(A)) = |\det(T)|\lambda^n(A) \quad \textit{for all } A \in \mathbf{B}(\mathbb{R}^n) \tag{2.3.1}$$

Proof. Recall that a cell in $\mathbb{R}^n$ is the set

$$J = \{(x_1, ..., x_n)^t \in \mathbb{R}^n : a_i \leq x_i \leq b_i \ \text{ for } i = 1, ..., n\}$$

where $a_i \leq b_i$ for $i = 1, ..., n$. By definition, $\lambda^n(J) = \prod_{i=1}^n (b_i - a_i)$. We will prove that the relation (2.3.1) holds for every cell J. Every open set can be written as the countable union of cells in $\mathbb{R}^n$ and therefore, by the countable additivity of the Lebesgue measure, λ^n will satisfy (2.3.1) for any open set. The regularity of λ^n (particularly point (ii) in definition 2.2.1) then finally implies that (2.3.1) holds for any Borel set A. Also, the matrix representation M_T of T can be written as the product of elementary matrices and since determinants respect products, it suffices to show that (2.3.1) holds only for elementary matrices. Let Q_1 be the elementary matrix obtained from the

$n \times n$ identity matrix I by multiplying row j with a non zero constant r, so $\det(Q_1) = r$. Now

$$\begin{aligned} Q_1 J &= \{(x_1, ..., rx_j, ..., x_n)^t : a_i \leq x_i \leq b_i \quad \text{for } i = 1, ..., n\} \\ &= \{(x_1, ..., x_n)^t : a_i \leq x_i \leq b_i \quad \text{for } i \neq j, \; ra_j \leq x_j \leq rb_j\} \end{aligned}$$

and hence $\lambda^n(Q_1 J) = r \prod_{i=1}^n (b_i - a_i) = |\det(Q_1)| \lambda^n(J)$.
Let Q_2 be the elementary matrix corresponding to the interchange of row j with row k ($\det(Q_2) = -1$). Then

$$Q_2 J = \{(x_1, ..., x_n)^t : a_i \leq x_i \leq b_i \quad \text{for } i \neq j, k; \; a_j \leq x_k \leq b_j, \; a_k \leq x_j \leq b_k\}$$

which implies that $\lambda^n(Q_2 J) = \prod_{i=1}^n (b_i - a_i) = |\det(Q_2)| \lambda^n(J)$.
Finally let Q_3 be the elementary matrix obtained by replacing row j with the sum of row j and row k. The determinant of Q_3 is 1. We may assume (by multiplication with a suitable Q_2 matrix if necessary) that

$$Q_3(x_1, ..., x_{n-1}, x_n)^t = (x_1, ..., x_{n-1}, x_n + x_k)^t.$$

We can view $\mathbb{R}^n$ as the cartesian product $\mathbb{R}^{n-1} \times \mathbb{R}$. For every $x = (x_1, ..., x_{n-1})^t$ in $\mathbb{R}^{n-1}$ the x-section J_x of J is of the form $J_x = \{y \in \mathbb{R} : (x_1, ..., x_{n-1}, y)^t \in J\}$. The corresponding x-section $(Q_3 J)_x$ of $Q_3 J$ can be written as

$$\begin{aligned} (Q_3 J)_x &= \{y \in \mathbb{R} : (x_1, ..., x_{n-1}, y + x_k)^t \in Q_3 J\} \\ &= \{y \in \mathbb{R} : (x_1, ..., x_{n-1}, y)^t \in J\} + (0, .., 0, x_k)^t = J_x + (0, .., 0, x_k)^t. \end{aligned}$$

Thus, for all $x \in \mathbb{R}^n$, J_x is a translation of $(Q_3 J)_x$ and, since the Lebesgue measure λ^1 is translation-invariant, it follows that $\lambda^1((Q_3 J)_x) = \lambda^1(J_x)$. Fubini's Theorem yields:

$$\begin{aligned} \lambda^n(Q_3 J) &= \int_{\mathbb{R}^{n-1}} \lambda^1((Q_3 J)_x) d\lambda^{n-1} \\ &= \int_{\mathbb{R}^{n-1}} \lambda^1(J_x) d\lambda^{n-1} = \lambda^n(J) = |\det(Q_3)| \lambda^n(J) \end{aligned}$$

□

Corollary 2.3.4. *Let $T : X \to X$ be an invertible linear transformation defined on the Minkowski space X and let m be a Haar measure on X. Then*

$$m(T(A)) = |\det(T)| m(A) \qquad \textit{for all } A \in \mathfrak{B}(X)$$

Proof. Let T_ζ be defined as in definition 2.3.2. By making use of Theorem 2.2.8 in conjunction with Theorem 2.3.3 we obtain

$$\begin{aligned} m(T(A)) &= k\lambda^n(\zeta \circ T(A)) \qquad \text{for some } k > 0 \\ &= k\lambda^n(\zeta \circ (\zeta^{-1} \circ T_\zeta \circ \zeta)(A)) \\ &= k\lambda^n(T_\zeta(\zeta(A))) \\ &= k|\det(T_\zeta)| \lambda^n(\zeta(A)) =: |\det(T)| k\lambda^n(\zeta(A)) \\ &=: |\det(T)| m(A) \end{aligned}$$

□

An affine transformation F on a Minkowski space X is the mapping $F(x) = T(x) + a$ where T is invertible and linear and $a \in X$. It follows directly from Corollary 2.3.4 and the translation-invariance of Haar measures that

$$m(F(A)) = |det(T)|m(A) \quad \text{for all } A \in \mathfrak{B}(X)$$

If $\det(T) = 1$ then T is called a volume preserving map while F is said to be a volume preserving affine map. In this case:

$$m(F(A)) = m(A) \quad \text{for all } A \in \mathfrak{B}(X)$$

Haar measures are therefore invariant under volume preserving maps.

2.4 The Volume of the Ball $B(l_p^n, x_0, r)$ in $\mathbb{R}^n$

Let $B(l_p^n, x_0, r)$ be the set $\{x \in \mathbb{R}^n : \|x - x_0\|_p \leq r\}$. For the sake of notational simplicity, the n-dimensional Lebesgue measure on $\mathbb{R}^n$ is denoted by λ and the unit ball $B(l_p^n, 0, 1)$ is written as $B(l_p^n)$. This section not only serves as an application of the results in the previous sections, but will also be used in conjunction with section 4.6 to find both the upper- and lower bounds for the volume product of the unit balls $B(l_p^n)$, where $1 \leq p \leq \infty$.

Lemma 2.4.1. $\lambda(B(l_p^n, x_0, r)) = r^n \lambda(B(l_p^n, x_0, 1))$

Proof. Since λ is translation-invariant it can be assumed without loss of generality that $x_0 = 0$. It is readily verified that the ball $B(l_p^n, 0, r)$ can be obtained from the unit ball $B(l_p^n)$ by means of the invertible linear transformation

$$T : B(l_p^n) \to B(l_p^n, 0, r) : (y_1, ..., y_n) \mapsto T(y_1, ..., y_n) := r(y_1, ..., y_n)$$

with standard matrix representation given by $M_T = diag(r, ..., r)$. It therefore follows from Theorem 2.3.4 that

$$\lambda(B(l_p^n), 0, r) = |\det(T)|\lambda(B(l_p^n)) = r^n \lambda(B(l_p^n))$$

□

Proposition 2.4.2. *For any $n \in \mathbb{N}$ and any $p \in [1, \infty)$:*

$$\lambda(B(l_p^n)) = \frac{\left(2\Gamma(1 + \frac{1}{p})\right)^n}{\Gamma(1 + \frac{n}{p})}$$

Moreover

$$\lambda(B(l_\infty^n)) = \lim_{p \to \infty} \lambda(B(l_p^n)) = 2^n$$

Proof. Let $I_p := \int_{\mathbb{R}^n} e^{-\|x\|_p^p} dx$. By definition of $\|.\|_p$ we have:

$$I_p = \int_{\mathbb{R}^n} e^{-\|x\|_p^p} dx = \int_{\mathbb{R}^n} e^{-\sum_{i=1}^n |x_i|^p} dx = \int_{\mathbb{R}} \int_{\mathbb{R}} ... \int_{\mathbb{R}} \prod_{i=1}^n e^{-|x_i|^p} dx_1 dx_2 ... dx_n$$
$$= \prod_{i=1}^n \int_{\mathbb{R}} e^{-|x_i|^p} dx_i = \left(\int_{\mathbb{R}} e^{-|t|^p} dt \right)^n = \left(2 \int_0^\infty e^{-t^p} dt \right)^n \quad \because \text{By symmetry}$$

Now let $t = s^{\frac{1}{p}+1}$ then:

$$I_p = 2^n \left(\int_0^\infty e^{-t^p} dt \right)^n = 2^n \left(\int_0^\infty \frac{1}{p} s^{\frac{1}{p}-1} e^{-s} ds \right)^n$$
$$= 2 \frac{1}{p} \Gamma\left(\frac{1}{p} \right) = \left(2\Gamma(1 + \frac{1}{p}) \right)^n \tag{2.4.1}$$

We also have:

$$e^{-\|x\|_p^p} = \int_{\|x\|_p}^\infty p t^{p-1} e^{-t^p} dt$$

This follows easily from the Fundamental Theorem of Calculus. Indeed:

$$\int_{\|x\|_p}^\infty p t^{p-1} e^{-t^p} dt = \int_{\|x\|_p}^\infty \frac{d}{dt} \left(-e^{-t^p} \right) dt = \lim_{s \to \infty} -e^{-s^p} + e^{-\|x\|_p^p} = e^{-\|x\|_p^p}$$

We may therefore express I_p as follows:

$$I_p = \int_{\mathbb{R}^n} \int_{\|x\|_p}^\infty p t^{p-1} e^{-t^p} dt dx$$
$$= \int_{\mathbb{R}^{n+1}} \chi_{\{(x_1,\ldots,x_n,t) \in \mathbb{R}^{n+1} : \|x\|_p \leq t\}} p t^{p-1} e^{-t^p} dx_1 dx_2 ... dx_n dt$$

But

$$\{(x_1, ..., x_n, t) \in \mathbb{R}^{n+1} : \|x\|_p \leq t\} = [0, \infty) \times \{(x_1, ..., x_n) \in \mathbb{R}^n : \|x\|_p \leq t\}$$

According to Fubini's Theorem:

$$I_p = \int_0^\infty \left(\int_{\{x \in \mathbb{R}^n : \|x\|_p \leq t\}} t^{p-1} e^{-t^p} dx_1 ... dx_n \right) dt$$
$$= \int_0^\infty t^{p-1} e^{-t^p} \left(\int_{B(l_p^n, 0, t)} dx_1 ... dx_n \right) dt$$
$$= \int_0^\infty t^{p-1} e^{-t^p} \lambda(B(l_p^n, 0, t)) dt$$
$$= \int_0^\infty t^{p-1} e^{-t^p} t^n \lambda(B(l_p^n, 0, 1)) dt \quad \because \text{By lemma 2.4.1}$$
$$= \lambda(B(l_p^n, 0, 1)) \int_0^\infty t^{n+p-1} e^{-t^p} dt$$

Again let $t = s^{\frac{1}{p}+1}$ then

$$\begin{aligned}
I_p &= \lambda(B(l_p^n, 0, 1)) \int_0^\infty p\frac{1}{p} s^{\frac{1}{p}-1} (s^{\frac{1}{p}})^{p+n-1} e^{-s} ds \\
&= \lambda(B(l_p^n, 0, 1)) \int_0^\infty s^{(\frac{n}{p}+1)-1} e^{-s} ds \\
&= \lambda(B(l_p^n, 0, 1)) \Gamma(1 + \frac{n}{p}) \qquad (2.4.2)
\end{aligned}$$

Combining expressions (2.4.1) and (2.4.2) yields:

$$\lambda(B(l_p^n)) = \frac{\left(2\Gamma(1+\frac{1}{p})\right)^n}{\Gamma(1+\frac{n}{p})}$$

The unit ball $B(l_\infty^n)$ is simply the cube $[-1, 1]^n$ whose volume is given by the product of the length it's sides. Hence $\lambda(B(l_\infty^n)) = 2^n$. Since the gamma function is continuous and $\Gamma(1) = 1$, it follows that

$$\lim_{p\to\infty} \lambda(B(l_p^n)) = \lim_{p\to\infty} \frac{\left(2\Gamma(1+\frac{1}{p})\right)^n}{\Gamma(1+\frac{n}{p})} = 2^n = \lambda(B(l_\infty^n))$$

□

2.5 The Volume Product

Since the Santaló point (see [10]) of symmetric convex bodies coincides with the origin, the definition of the volume product given here can be regarded as a special case of the more commonly used definition ([10]). If m and m^* are associated Haar measures (definition 2.5.3) defined on Minkoswki spaces X and X^* respectively and C is any symmetric convex body in X, then the volume product P of C is given by $P(C) := m(C)m^*(C^o)$. Not only is this quantity independent of the choice of the associated Haar measures (Lemma 2.5.5), it is also invariant under linear isomorphisms (Proposition 2.5.6). These properties, when used in conjunction with Theorem 2.2.8, enable the reformulation of the general Blaschke-Santaló inequality (Theorem 0.0.1) in $(\mathbb{R}^n, \mathfrak{B}(\mathbb{R}^n), \lambda^n)$ without loss of generality (Theorem 2.5.6 and the subsequent Corollaries).

2.5.1 Definition and Elementary Properties

Associated Haar measures

Definition 2.5.1. *The parallelotope $\mathcal{P}(b_1, .., b_n)$ spanned by the ordered basis $\{b_1, ..., b_n\}$ of X is defined as the set:*

$$\mathcal{P}(b_1, ..., b_n) := \{x \in X : x = \sum_{i=1}^{n} \lambda_i b_i,\ 0 \le \lambda_i \le 1 \ \ for \ \ i = 1, ..., n\}$$

The parallelotope $\mathcal{P}^*(b_1, ..., b_n)$ is simply the parallelotope spanned by the dual basis vectors $\{b_1^*, ..., b_n^*\}$ of X^*.

Let m and m^* be any two Haar measures defined on X and X^* with ordered bases $\{b_1, ..., b_n\}$ and $\{b_1^*, ..., b_n^*\}$ respectively. The following Theorem makes use of properties of the Hilbert adjoint operator to show that the product $\varsigma = m(\mathcal{P}(b_1, ..., b_n))m^*(\mathcal{P}^*(b_1^*, ..., b_n^*))$ remains unchanged regardless of the chosen ordered basis. Associated measures are then simply defined as those measures for which $\varsigma = 1$.

Theorem 2.5.2. *If m and m^* are Haar measures defined on X and X^* respectively, then the product $m(\mathcal{P}(b_1, ..., b_n))m^*(\mathcal{P}^*(b_1^*, ..., b_n^*))$ is independent of the basis $\{b_1, ..., b_n\}$.*

Proof. Let $T : X \to X$ be an isomorphism mapping the basis vectors $\{b_1, ..., b_n\}$ onto the basis $\{f_1, ..., f_n\}$. According to section 1.4, the natural dual basis $\{f_1^*, ..., f_n^*\}$ is simply the set $\{(T^*)^{-1}b_1^*, ..., (T^*)^{-1}b_n^*\}$ where T^* is the Hilbert adjoint operator of T. The linearity of T implies

$$\begin{aligned}\mathcal{P}(Tb_1,...,Tb_n) &= \{x \in X : x = \sum_{i=1}^{n} \lambda_i Tb_i,\ 0 \leq \lambda_i \leq 1 \ \text{ for } \ i = 1,...,n\}\\ &= \{x \in X : T^{-1}(x) = \sum_{i=1}^{n} \lambda_i b_i,\ 0 \leq \lambda_i \leq 1 \ \text{ for } \ i = 1,...,n\}\\ &= \{Tx \in X : x = \sum_{i=1}^{n} \lambda_i b_i,\ 0 \leq \lambda_i \leq 1 \ \text{ for } \ i = 1,...,n\}\\ &= T(\mathcal{P}(b_1,...,b_n))\end{aligned}$$

Similarly it can be shown that

$$\begin{aligned}\mathcal{P}^*(Tb_1,...,Tb_n) &:= \mathcal{P}((T^*)^{-1}b_1^*,...,(T^*)^{-1}b_n^*) = (T^*)^{-1}\mathcal{P}(b_1^*,...,b_n^*)\\ &=: (T^*)^{-1}P^*(b_1,...,b_n)\end{aligned}$$

Moreover, $\det((T^*)^{-1}) = (\det(T^*))^{-1} = \det(T)$, and hence

$$\begin{aligned}&m(\mathcal{P}(f_1,...,f_n))m^*(\mathcal{P}^*(f_1,...,f_n)) = m(\mathcal{P}(Tb_1,...,Tb_n))m^*(\mathcal{P}^*(Tb_1,...,Tb_n))\\ &= m(T(\mathcal{P}(b_1,...,b_n)))m^*(T^*)^{-1}(\mathcal{P}^*(b_1,...,b_n)))\\ &= |\det(T)||\det((T^*)^{-1})|m(\mathcal{P}(b_1,...,b_n))m^*(\mathcal{P}^*(b_1,...,b_n))\\ &= m(\mathcal{P}(b_1,...,b_n))m^*(\mathcal{P}^*(b_1,...,b_n))\end{aligned}$$

□

Definition 2.5.3. *The Haar measures m and m^* are said to be associated Haar measures if:*

$$m(\mathcal{P})m^*(\mathcal{P}^*) = 1$$

The Volume Product

Definition 2.5.4. *Let (X, B) and (X^*, B^o) be a pair of n-dimensional dual Minkowski spaces and let m and m^* be the associated Haar measures on X and X^* respectively. The volume product $P(B)$ of B is defined as:*

$$P(B) := m(B)m^*(B^o)$$

while the reduced volume product γ of B is given by

$$\gamma(B) := (n!P(B))^{\frac{1}{n}}$$

Lemma 2.5.5. *The volume product is independent of the choice of associated Haar measures m and m^*.*

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Proof. Suppose (m_1, m_1^*) and (m_2, m_2^*) are two pairs of associated Haar measures. Since m_1 and m_2 are both Haar measures on X, it follows that there exists a scalar $\alpha \in \mathbb{R}$ such that $m_1 = \alpha m_2$. Similarly there exists a scalar $\beta \in \mathbb{R}$ such that $m_1^* = \beta m_2^*$. Now for any parallelotope $\mathcal{P}(e_1, ..., e_n)$ we have:

$$\begin{aligned} & m_1(\mathcal{P}(e_1, ..., e_n))m_1^*(\mathcal{P}(e_1^*, ..., e_n^*)) = 1 \\ \Rightarrow\ & \alpha m_2(\mathcal{P}(e_1, ..., e_n))\beta m_2^*(\mathcal{P}(e_1^*, ..., e_n^*)) = 1 \end{aligned}$$

But

$$m_2(\mathcal{P}(e_1, ..., e_n))m_2^*(\mathcal{P}(e_1^*, ..., e_n^*)) = 1$$

Since m_2 and m_2^* are associated Haar measures. From this it follows that $\alpha\beta = 1$. So for the unit sphere B we have

$$m_1(B)m_1^*(B^o) = \alpha m_2(B)\beta m_2^*(B^o) = m_2(B)m_2^*(B^o)$$

Therefore $P(X)$ is independent of the choice of associated Haar measures. $\square$

Proposition 2.5.6. *Let X and Y be n-dimensional Minkowski Spaces and let $T : X \to Y$ be an linear isomorphism. Then $P(T(B)) = P(B)$ for any symmetric convex body $B \subset X$.*

Proof. This result is a direct consequence of Theorem 2.2.8. Indeed let T^* be the Hilbert adjoint operator of T and let (m_Y, m_Y^*) and (m_X, m_X^*) be two pairs of associated Haar measures on the spaces X, X^*, Y and Y^* respectively. According to Theorem 2.2.8 there exist non-negative scalars $\alpha, \beta \in \mathbb{R}$ such that

$$\begin{aligned} 1 &= m_Y(\mathcal{P}(b_1, ..., b_n))m_Y^*(\mathcal{P}(b_1^*, ..., b_n^*)) \\ &= \alpha\beta m_X(\mathcal{P}(T^{-1}b_1, ..., T^{-1}b_n))m_X^*(\mathcal{P}(T^*(b_1^*), ..., T^*(b_n^*))) \end{aligned}$$

for any basis $\{b_1, ..., b_n\}$ of Y. Since m_X and m_X^* are associated, $\alpha\beta = 1$. Recall from remark 1.4.3 that $(T(B))^o = (T^*)^{-1}(B^o)$ and hence

$$\begin{aligned} P(T(B)) &= m_Y(T(B))m_Y^*((T^*)^{-1}(B^o)) \\ &= \alpha\beta m_X(B)m_X^*(B^o) = m_X(B)m_X^*(B^o) \\ &= P(B) \end{aligned}$$

$\square$

Corollary 2.5.7. *For any Minkowski Space (X, B) with dual space (X^*, B^o) we have $P(B) = P(B^o)$ and hence $\gamma(B) = \gamma(B^o)$.*

Proof. The mapping $T : X \to X^*|b_i \mapsto Tb_i := b_i^*$ for $i = 1, ..., n$ defines a linear isomorphism from X onto X^* such that $T(B) = B^o$. Proposition 2.5.6 now implies that $P(B) = P(T(B)) =: P(B^o)$ and therefore $\gamma(B) = \gamma(B^o)$. $\square$

Corollary 2.5.8. *The Blaschke-Santaló inequality can, without loss of generality, be reformulated in* $(\mathbb{R}^n, \mathfrak{B}(\mathbb{R}^n), \lambda^n)$.

Proof. Let X be an n-dimensional Minkowski space and suppose the Blascke-Santaló inequality holds in $\mathbb{R}^n$. Then $P(B) \leq P(E)$ for any symmetric convex body $B \subset \mathbb{R}^n$, where $E \subset \mathbb{R}^n$ is an ellipsoid. Since the coordinatization mapping ζ defines an isomorphism from the Minkowski Space X onto $\mathbb{R}^n$, it follows from Proposition 2.5.6 and the definition of ellipsoids that for all convex symmetric bodies $C \subset X$,

$$P(C) = P(\zeta(C)) \leq P(E) = P(\zeta^{-1}(E)),$$

where $\zeta^{-1}(E)$ is an ellipsoid in X. Moreover, if $K \subset X$ is a symmetric convex body such that $P(C) \leq P(K)$ for all other symmetric convex bodies $C \in X$, then $P(\zeta(C)) = P(C) \leq P(K) = P(\zeta(K))$ for all sets $\zeta(C) \subset \mathbb{R}^n$. By the second part of the Blascke-Santaló inequality for $\mathbb{R}^n$ and by Proposition 1.3.3, it follows that $\zeta(K)$ and hence K is an ellipsoid. □

Remark 2.5.9. *Let* $\{e_1, ..., e_n\}$ *be the standard ordered basis for* $\mathbb{R}^n$ *with corresponding dual basis* $\{e_1^*, ..., e_n^*\}$ *and let* m^* *denote the Haar measure on* $(\mathbb{R}^n)^*$, *associated with the n-dimensional Lebesgue measure* λ.
The mapping $T : (\mathbb{R}^n)^* \to \mathbb{R}^n | e_i^* \mapsto T(e_i^*) = e_i$ *defines a linear isomorphism and hence, according to Theorem 2.2.8, there exists a scalar* $k > 0$ *such that* $m^*(B) = k\lambda(T(B))$ *for all* $B \in \mathfrak{B}((\mathbb{R}^n)^*)$. *Since the volume of the cube* $\mathcal{P}(e_1, ..., e_n)$ *is equal to 1, it follows from the association of* λ *and* m^* *that*

$$k\left(\lambda(\mathcal{P}(e_1, ..., e_n)\right)^2 = m^*(\mathcal{P}(e_1^*, ..., e_n^*))\lambda(\mathcal{P}(e_1, ..., e_n) = 1 \Rightarrow k = 1$$

and hence

$$P(C) = \lambda(C)\lambda(T(C^o)) \quad \textit{for all convex symmetric bodies } C \subset \mathbb{R}^n$$

Recall from Section 1.4 that $T(C^o)$ *can be written in the form*

$$T(C^o) = \{y \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \quad \textit{for all } x \in C\}.$$

In order to simplify notation in subsequent chapters, C^o *will from now on implicitly be understood to denote the set* $T(C^o)$.

2.6 The Blaschke-Santaló Inequality for the Collection $\{B(l_p^n) : 1 \leq p \leq n\}$

The following proposition makes use of the explicit formulae derived in section 4.4 to find both the lower- and upper bound of the volume product for the unit balls $B(l_p^n)$, where $1 \leq p \leq \infty$. For this restricted class of convex bodies, the maximal volume product is attained exclusively for the Euclidean ball $B(l_2^n)$, which complies with the more general version of the The Blaschke-Santaló inequality. The use of the reduced volume product γ instead of P is simply for notational convenience. The proof of Proposition 2.6.1 is based largely on properties of the gamma function Γ which are used here without proof. An exhaustive discussion of these properties can be found in Artin [2].

Proposition 2.6.1 (St. Raymond). *For all* $p \in [1, \infty]$

$$4 = \gamma(B(l_1^n)) = \gamma(B(l_\infty^n)) \leq \gamma(B(l_p^n)) \leq \gamma(B(l_2^n)) \tag{2.6.1}$$

Proof. Let $1 \leq q \leq \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Recall from Proposition 2.4.2 that the volume of $B(l_p^n)$ is given by:

$$\lambda(B(l_p^n)) = \frac{\left(2\Gamma(1 + \frac{1}{p})\right)^n}{\Gamma(1 + \frac{n}{p})}$$

Proposition 2.6.1 is proved by considering the equivalent problem of maximizing and minimizing the function

$$F\left(\frac{1}{p}\right) := \ln P(B(l_p^n)) = \ln \lambda(B(l_p^n)) + \ln \lambda(B(l_q^n))$$

over all possible values of $p \geq 1$. The following arguments show that $F : (0, 1] \to \mathbb{R}$ is a concave function.
The gamma function is infinitely differentiable and can be expanded as follows:

$$\Gamma(x) = \lim_{n\to\infty} \frac{n!n^x}{(n + x)(n - 1 + x)...(1 + x)x} < \infty \quad \text{for all } x > 0 \tag{2.6.2}$$

Consider the function $x :\mapsto \ln(\Gamma(x + 1))$. By the above expansion and the continuity of the natural logarithmic function, $\ln(\Gamma(x + 1))$ can be written as:

$$\begin{aligned}\ln(\Gamma(x + 1)) &= \lim_{n\to\infty} \ln\left[\frac{n!n^x}{(n + x)(n - 1 + x)...(2 + x)(1 + x)}\right] \\ &= \lim_{n\to\infty} [\ln(n!) + (x + 1)\ln(n) - \sum_{j=1}^{n+1} \ln(x + j)]\end{aligned}$$

Since this sequence is absolutely convergent we may interchange differentiation and limits:

$$\frac{d}{dx}\left[\ln(\Gamma(x+1))\right] = \lim_{n\to\infty}\left[\ln(n) - \sum_{j=0}^{n}\frac{1}{x+1+j}\right]$$

$$\Rightarrow \frac{d^2}{dx^2}\left[\ln(\Gamma(x+1))\right] = \lim_{n\to\infty}\left[\sum_{j=0}^{n}\frac{1}{(x+1+j)^2}\right] = \sum_{j=0}^{\infty}\frac{1}{(x+1+j)^2}$$

Now

$$\frac{d^2}{dx^2}\left[\ln\left(\frac{\Gamma(x+1)^n}{\Gamma(1+nx)}\right)\right] = \frac{d^2}{dx^2}\left[n\ln(\Gamma(x+1))\right] - \frac{d^2}{dx^2}\left[\ln(\Gamma(1+nx))\right]$$

where

$$\frac{d^2}{dx^2}\left[n\ln(\Gamma(x+1))\right] = \sum_{j=0}^{\infty}\frac{n}{(x+1+j)^2} = \sum_{j=0}^{\infty}\sum_{k=1}^{n}\frac{1}{(x+1+j)^2} \tag{2.6.3}$$

and

$$\frac{d^2}{dx^2}\left[\ln(\Gamma(1+nx))\right] = \lim_{m\to\infty}\sum_{k=1}^{m+1}\frac{n^2}{(nx+k)^2} = \lim_{m\to\infty}\sum_{k=1}^{m+1}\frac{1}{(x+\frac{k}{n})^2}$$

$$= \sum_{m=1}^{\infty}\frac{1}{(x+\frac{m}{n})^2} = \sum_{k=1}^{n}\frac{1}{(x+0+\frac{k}{n})^2} + \sum_{k=1}^{n}\frac{1}{(x+1+\frac{k}{n})^2} + \ldots$$

$$= \sum_{j=0}^{\infty}\sum_{k=1}^{n}\frac{1}{(x+j+\frac{k}{n})^2} \tag{2.6.4}$$

Combining expressions (2.6.3) and (2.6.4) yields:

$$\frac{d^2}{dx^2}\left[\ln\left(\frac{\Gamma(x+1)^n}{\Gamma(1+nx)}\right)\right] = \sum_{j=0}^{\infty}\sum_{k=1}^{n}\frac{1}{(x+j+1)^2} - \frac{1}{(x+j+\frac{k}{n})^2} < 0$$

for $x \geq 0$. The function $F:(0,1]\to[0,\infty)$ is rewritten in terms of x as follows:

$$F(x) := \ln\left(P(B(l^n_{\frac{1}{x}}))\right) = \ln\left(\lambda(B(l^n_{\frac{1}{x}}))\right) + \ln\left(\lambda(B(l^n_{1-\frac{1}{x}}))\right)$$

From the definition of $B(l^n_{\frac{1}{x}})$ and the above inequality it follows directly that F is a concave function. In addition for any $x \in (0,\frac{1}{2})$:

$$\begin{aligned}
F(\frac{1}{2}+x) &:= \ln\left(\lambda(B(l^n_{\frac{1}{x+\frac{1}{2}}}))\right) + \ln\left(\lambda(B(l^n_{\frac{1}{1-(x+\frac{1}{2})}}))\right) \\
&= \ln\left(\lambda(B(l^n_{\frac{1}{1-(\frac{1}{2}-x)}}))\right) + \ln\left(\lambda(B(l^n_{\frac{1}{\frac{1}{2}-x}}))\right) \\
&= F(\frac{1}{2}-x)
\end{aligned}$$

From this it follows that F is symmetric about $\frac{1}{2}$ and therefore it attains its maximum at the point $x = \frac{1}{2}$. In other words $P(B(l_p^n)) \leq P(B(l_2^n))$ for all $1 \leq p \leq \infty$. Moreover, since F is a concave function, it decreases monotonically as x tends towards 1 or 0. This implies that F, and hence $P(B(l_p^n))$, attains its minimum at the point $x = 1 = p$. Since $(l_1^n)^* = l_\infty^n$, it follows from Corollary 2.5.7 that

$$P(B(l_\infty^n)) = P(B(l_1^n)) \leq P(B(l_p^n)) \quad \forall 1 \leq p \leq \infty.$$

□

Chapter 3

The Hausdorff Metric

3.1 Introduction

In the ensuing chapters it is often more convenient to consider the distance between sets rather than between points. For any Minkowski space $(X, \|.\|)$ with unit ball B it is possible to define a metric δ, called the Hausdorff metric on the collection $\mathcal{K}$ of non-empty compact subsets of X. Throughout this Chapter a variety of different subclasses of $\mathcal{K}$ play a role. In aid of distinguishing between these subclasses, the following notation will prove useful:

$\mathcal{K}$: The collection of all non-empty compact subsets of X.
$\mathcal{C}$: The collection of all non-empty convex compact subsets of X.
$\mathcal{C}_b$: The collection of all convex bodies in X.
$\mathcal{C}_{b_0}$: The collection of all convex bodies whose interior contains the origin.
$\mathcal{C}_s$: The collection of all symmetric convex bodies in X.

Remark 3.1.1. *It can readily be seen that* $\mathcal{C}_s \subset \mathcal{C}_{b_0} \subset \mathcal{C}_b \subset \mathcal{C} \subset \mathcal{K}$.

Section 6.2 serves as an introduction to the Hausdorff metric. Even though this metric depends on the underlying norm $\|.\|$, it will be shown that the equivalence of norms on Minkowski Spaces ensures the equivalence of the corresponding Hausdorff metrics. Under the Hausdorff metric, both the Haar measure and the volume product can be regarded as continuous functions defined on the collections C_b and C_s respectively (Section 6.3). Section 6.4 sets out to prove the Blaschke selection theorem, namely:

"*In the Minkowski space (X, B), the collection of all convex bodies contained within the set aB (where $a > 0$) is sequentially compact with respect to the Hausdorff metric*".

This theorem, especially when used in conjunction with the continuity of the Haar measure and the volume product, is pivotal in proving Theorem 4.3.3, which will in turn be used not only to establish the first part of the Blaschke-Santaló inequality directly, but also to prove Brunn's theorem (Theorem 4.4.2) in Chapter 4.

3.2 Definition and General Properties

Let $(X, \|.\|)$ be an n-dimensional Minkowski Space with fixed unit ball B. The Hausdorff metric δ on $\mathcal{K}$ is expressed in terms of the quantity δ', defined as follows:

Definition 3.2.1. *Let $C, D \in \mathcal{K}$. Define:*

$$\delta'(C, D) := \inf\{\epsilon > 0 : C \subset D + \epsilon B\} \tag{3.2.1}$$

Remark 3.2.2. *Since D is non-empty, it follows that $D + \epsilon B$ is non-empty for all $\epsilon > 0$. Moreover, C is compact and hence bounded, which implies $C \subset D + \epsilon B$ for a large enough ϵ. Hence δ' is well defined.*

Proposition 3.2.3 (Properties of δ'). *For any $C, D \in \mathcal{K}$ the following hold:*

1. *$\delta' \geq 0$ and $\delta'(C, D) = 0 \Leftrightarrow C \subset D$*
2. *If $A, C, D \in \mathcal{K}$ then $\delta'(A, D) \leq \delta'(A, C) + \delta'(C, D)$*

Proof. .
1. It is clear from the definition that $\delta'(C, D) \geq 0 \ \forall C, D \in \mathcal{K}$.
Suppose $\delta'(C, D) = 0$. By definition of the infimum, there exists a sequence $\{\epsilon_i\}$ of positive scalars such that $\lim_{i\to\infty} \epsilon_i = 0$ and $C \subset D + \epsilon_i B$ for all $i \in \mathbb{N}$. This means that for any $c \in C$ we have $c = d_i + \epsilon_i b_i$ for some $d_i \in D$, $b_i \in B$. Since $\|b_i\| \leq 1$ for all $i \in \mathbb{N}$, it follows that

$$0 \leq \|c - d_i\| = \|\epsilon_i b_i\| = \epsilon_i \|b_i\| \leq \epsilon_i \to 0 \quad \text{as } n \to \infty.$$

Hence the sequence $\{d_i\} \subset D$ converges to $c \in C$ and since D is compact (and therefore closed), it follows that $c \in D$. But $c \in C$ was arbitrary, and thus $C \subset D$.
Conversely, suppose $C \subset D$. But $D \subset D + \epsilon B$ for any $\epsilon > 0$ (since $0 \in B$). This implies that $C \subset D + \epsilon B$ for all $\epsilon > 0$ and hence
$0 \leq \delta'(C, D) = \inf\{\epsilon > 0 : C \subset D + \epsilon B\} = 0$.

2. Suppose $A \subset C + \eta_1 B$ and $C \subset D + \eta_2 B$. It then follows by properties of Minkowski addition that $A \subset D + \eta_1 B + \eta_2 B = D + (\eta_1 + \eta_2)B$.

Therefore

$$
\begin{aligned}
&\{\epsilon : A \subset C + \epsilon B\} + \{\epsilon : C \subset D + \epsilon B\} \subset \{\epsilon : A \subset D + \epsilon B\} \\
&\Rightarrow \inf\{\epsilon : A \subset D + \epsilon B\} \leq \inf\left[\{\epsilon : A \subset C + \epsilon B\} + \{\epsilon : C \subset D + \epsilon B\}\right] \\
&\qquad\qquad\qquad (\because \text{The infimum over a larger set is smaller}) \\
&\Rightarrow \inf\{\epsilon : A \subset D + \epsilon B\} \leq \inf\{\epsilon : A \subset C + \epsilon B\} + \inf\{\epsilon : C \subset D + \epsilon B\} \\
&\Rightarrow \delta'(A, D) \leq \delta'(A, C) + \delta'(C, D)
\end{aligned}
$$

□

Although Proposition 3.2.3(1) indicates that δ' itself can clearly not be used as a metric on $\mathcal{K}$, the Hausdorff metric δ is defined in terms of δ' as follows:

Definition 3.2.4. *The Hausdorff metric δ defined on $\mathcal{K}$ is defined for all $C, D \in \mathcal{K}$ as:*

$$\delta(C, D) := \max\{\delta'(C, D), \delta'(D, C)\} \tag{3.2.2}$$

Proposition 3.2.5. *The Hausdorff metric is indeed a metric on $\mathcal{K}$.*

Proof. .

1. Since $\delta(C, D) := \max\{\delta'(C, D), \delta'(D, C)\}$, it is clear that $\delta(C, D) \geq 0$ for all $C, D \in \mathcal{K}$ since $\delta' \geq 0$. From this it also follows that $\delta(C, D) = 0 \Leftrightarrow$ both $\delta'(C, D) = 0$ and $\delta'(D, C) = 0$.From properties of δ' we know that this is only possible when $C \subset D$ and $D \subset C \Rightarrow C = D$.

2. *Triangle inequality*: From the previous proposition, we know that for any sets $A, C, D \in \mathcal{K}$ we have $\delta'(A, D) \leq \delta'(A, C) + \delta'(C, D)$ and $\delta'(D, A) \leq \delta'(D, C) + \delta'(C, A)$. It therefore follows that:

$$
\begin{aligned}
\delta(A, D) &= \max\{\delta'(A, D), \delta'(D, A)\} \\
&\leq \max\{\delta'(A, C) + \delta'(C, D), \delta'(D, C) + \delta'(C, A)\} \\
&\leq \max\{\delta'(A, C), \delta'(C, A)\} + \max\{\delta'(C, D), \delta'(D, C)\} \\
&= \delta(A, C) + \delta(C, D)
\end{aligned}
$$

3. *Symmetry*:
$\delta(A, C) = \max\{\delta'(A, C), \delta'(C, A)\} = \max\{\delta'(C, A), \delta'(A, C)\} =: \delta(C, A)$

□

Generally, different norms give rise to different Hausdorff metrics. It will shown in the next theorem, however, that for Minkowski spaces these Hausdorff metrics are equivalent.

Proposition 3.2.6. *If $\|.\|_1$ and $\|.\|_2$ are equivalent norms on X with associated Hausdorff metrics δ_1 and δ_2 respectively, then there exist scalars $c_1, c_2 > 0$ such that for all $C, D \in \mathcal{K}$*

$$c_1\delta_2(C, D) \leq \delta_1(C, D) \leq c_2\delta_2(C, D).$$

Proof. Since $\|.\|_1$ and $\|.\|_2$ are equivalent, there exist positive scalars $c_1, c_2 > 0$ such that

$$c_1\|x\|_2 \leq \|x\|_1 \leq c_2\|x\|_2 \quad \text{for all } x \in X \tag{3.2.3}$$

Let B_1 and B_2 be the unit balls corresponding to norms $\|.\|_1$ and $\|.\|_2$ respectively. For any $y \in c_1B_1$, we have $\|y\|_2 \leq \frac{1}{c_1}\|y\|_1 \leq 1 \Rightarrow y \in B_2$. Similarly for $x \in B_2$, we have $\frac{1}{c_2}\|x\|_1 \leq \|x\|_2 \leq 1 \Rightarrow x \in c_2B_1$. The inequalities in (3.2.3) thus imply:

$$c_1B_1 \subset B_2 \subset c_2B_1 \tag{3.2.4}$$

Now consider any $C, D \in \mathcal{K}$:

$$\begin{aligned}\{\epsilon > 0 : C \subset D + \epsilon B_1\} &= \{c_1\epsilon > 0 : C \subset D + c_1\epsilon B_1\} \\ &= c_1\{\epsilon > 0 : C \subset D + \epsilon c_1 B_1\} \\ &\subset c_1\{\epsilon > 0 : C \subset D + \epsilon B_2\} \qquad \because c_1B_1 \subset B_2\end{aligned}$$

Therefore

$$\begin{aligned}c_1\delta_2'(C, D) &= \inf c_1\{\epsilon > 0 : C \subset D + \epsilon B_2\} \\ &\leq \inf\{\epsilon > 0 : C \subset D + \epsilon B_1\} = \delta_1'(C, D)\end{aligned}$$

Similarly $c_1\delta_2'(D, C) \leq \delta_1'(D, C)$ and hence

$$c_1\delta_2(C, D) \leq \delta_1(C, D) \tag{3.2.5}$$

Also:

$$\begin{aligned}c_2\{\epsilon > 0 : C \subset D + \epsilon B_2\} &= \{c_2\epsilon > 0 : C \subset D + c_2\epsilon\frac{1}{c_2}B_2\} \\ &= \{\eta > 0 : C \subset D + \eta\frac{1}{c_2}B_2\} \\ &\subset \{\eta > 0 : C \subset D + \eta B_1\} \qquad \because \tfrac{1}{c_2}B_2 \subset B_1\end{aligned}$$

So

$$\begin{aligned}\delta_1'(C, D) &= \inf\{\epsilon > 0 : C \subset D + \epsilon B_1\} \\ &\leq \inf c_2\{\epsilon > 0 : C \subset D + \epsilon B_2\} = c_2\delta_2'(C, D)\end{aligned}$$

Similarly $\delta_1'(C, D) \leq c_2\delta_2'(C, D)$ and therefore

$$\delta_1(C, D) \leq c_2\delta_2(C, D) \tag{3.2.6}$$

Together, relations (3.2.5) and (3.2.6) imply

$$c_1\delta_2(C, D) \leq \delta_1(C, D) \leq c_2\delta_2(C, D)$$

□

3.3 Continuity of the Haar measure and the Volume Product

This section aims to show that any Haar measure $m : \mathcal{C}_b \longrightarrow [0, \infty)$ is a continuous mapping with respect to the Hausdorff metric δ. The continuity of the volume product $P : \mathcal{C}_s \longrightarrow [0, \infty)$ (Theorem 3.3.6) follows directly from this result. The proof of the continuity of m is outlined as follows:
On the collection $\mathcal{C}_{b_0} \subset \mathcal{C}_b$, a relatively simple metric Δ_2 can be defined, which is equivalent to δ on $\mathcal{C}_{b_0}$ (Lemma 3.3.3). It can readily be shown (Theorem 3.3.4) that m is continuous on $\mathcal{C}_{b_0}$ with respect to the metric Δ_2 and hence with respect to δ. The continuity of m in the case of the more general class $\mathcal{C}_b$ can then be established by means of the translation-invariance of Haar measures (Theorem 3.3.5).

3.3.1 The Metric Δ_2

Definition 3.3.1. *Let K_1 and K_2 be any two convex bodies in X whose interior contains the origin. Let*

$$\Delta'(K_1, K_2) := \inf\{\alpha \geq 0 : K_1 \subset \alpha K_2\}.$$

The metric Δ_2 is defined as:

$$\Delta_2(K_1, K_2) := \ln\left(\max\{\Delta'(K_1, K_2), \Delta'(K_2, K_1)\}\right)$$

Remark 3.3.2. *Since the interior of K_2 contains the origin, there exists a ball $B_2 \subset K_2$ with center 0. Now K_1 is compact and therefore bounded, so there exists an $\alpha > 0$ such that $K_1 \subset \alpha B_2 \subset \alpha K_2$. The set $\{\alpha > 0 : K_1 \subset \alpha K_2\}$ is thus non-empty and Δ' is well-defined.*

It can be directly verified that Δ_2 is indeed a metric.

Lemma 3.3.3. *Let (X, B) be an n-dimensional Minkowski Space. Then the metrics Δ_2 and δ defined on $\mathcal{C}_{b_0}$ are equivalent.*

Proof. We will show that any sequence $\{K_n\}_{n=1}^{\infty}$ in $\mathcal{C}_{b_0}$ converges to some $K \in \mathcal{C}_{b_0}$ with respect to Δ_2 if and only if it converges to K with respect to δ:
Since $K \in \mathcal{C}_{b_0}$, the origin is an interior point of K. Hence there exists a positive scalar β_1 such that the ball $\beta_1 B \subset K$. Furthermore K is bounded and therefore there exists a $\beta_2 > 0$ such that $K \subset \beta_2 B$. So

$$\beta_1 B \subset K \subset \beta_2 B \tag{3.3.1}$$

(i) Suppose $K_n \longrightarrow_{\Delta_2} K$ as $n \to \infty$:
For any given $\gamma > 0$, let $\epsilon = \frac{1+\sqrt{1+\frac{4\gamma}{\beta_2}}}{2} > 1$. Then $\gamma = \epsilon(\epsilon - 1)\beta_2$.

Since $K_n \longrightarrow_{\Delta_2} K$, there exists an $N_\gamma \in \mathbb{N}$ such that $\Delta_2(K, K_n) < \ln(\epsilon)$ for all $n \geq N_\gamma$. This implies

$$K \subset \epsilon K_n \qquad \text{and} \qquad K_n \subset \epsilon K \subset \beta_2 \epsilon B. \tag{3.3.2}$$

Since $K_n \subset K$, any element $k_n \in K_n$ can be written in the form $k_n = \epsilon k = k + (\epsilon - 1)k$ for some $k \in K$. Therefore $K_n \subset K + (\epsilon - 1)K \subset K + (\epsilon - 1)\beta_2 B$. Similarly $K \subset K_n + (\epsilon - 1)K_n \subset K_n + \epsilon(\epsilon - 1)\beta_2 B$. Now $\epsilon(\epsilon - 1)\beta_2 > (\epsilon - 1)\beta_2$ and hence the definition of δ implies that $\delta(K, K_n) \leq \epsilon(\epsilon - 1)\beta_2 = \gamma$ for all $n \geq N_\gamma$. Hence $K_n \longrightarrow_\delta K$.

(ii) Conversely, suppose that $K_n \longrightarrow_\delta K$ as $n \to \infty$:
Again, let $\gamma > 0$ be given and choose $\epsilon > 0$ such that $\gamma = \ln(1 + \frac{2}{\beta_1}\epsilon)$. Assume, without any loss of generality, that $\gamma < \ln(2) \Rightarrow \epsilon < \frac{\beta_1}{2}$.
There exists an $N_\gamma \in \mathbb{N}$ such that $\delta(K, K_n) < \epsilon$ for all $n \geq N_\gamma$. From the inclusions in (3.3.1) it follows that:

$$K_n \subset K + \epsilon B \subset (1 + \epsilon \beta_1^{-1})K. \tag{3.3.3}$$

Moreover, by combining the inclusion in (3.3.1) with the definition of δ and the fact that $\epsilon < \frac{\beta_1}{2}$ we obtain:

$$\beta_1 B \subset K \subset K_n + \epsilon B \subset K_n + \frac{\beta_1}{2} B$$

and hence

$$\begin{aligned} \frac{\beta_1}{2} B &\subset \frac{1}{2} K_n + \frac{\beta_1}{4} B \\ &\subset \frac{1}{2} K_n + \frac{1}{4} K_n + \frac{\beta_1}{8} B = (\frac{1}{2} + \frac{1}{4}) K_n + \frac{\beta_1}{8} B \qquad \because \text{convexity of } K_n \\ &\subset (\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots) K_n \end{aligned}$$

So any element $b \in \frac{\beta_1}{2} B$ can be written as $b = (\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots)k = k$ for some $k \in K_n$ and hence $\frac{\beta_1}{2} B \subset K_n$. So

$$K \subset K_n \epsilon B \subset (1 + 2\epsilon \beta_1^{-1}) K_n. \tag{3.3.4}$$

From (3.3.3) and (3.3.4) it now follows that $\Delta_2 \leq \ln((1 + 2\epsilon\beta_1^{-1})) = \gamma$ for all $n \geq N_\gamma$ and so $K_n \longrightarrow_{\Delta_2} K$. □

3.3.2 The Continuity of $m : \mathcal{C}_b \longrightarrow [0, \infty)$

Theorem 3.3.4. *The mapping $m : \mathcal{C}_{b_0} \to [0, \infty) : K \mapsto m(K)$ is a continuous mapping on $(\mathcal{C}_{b_0}, \Delta_2)$ and hence on $(\mathcal{C}_{b_0}, \delta)$.*

Proof. Let $K \in \mathcal{C}_{b_0}$ and $\epsilon > 0$ be given. Choose $\eta > 1$ such that

$$\eta^n(\eta^n - 1)m(K) < \epsilon.$$

Then for any $K' \in \mathcal{C}_{b_0}$ such that $\Delta_2(K, K') < \ln(\eta)$ we have:

$$\begin{aligned} & K' \subset \eta K \\ \Rightarrow \quad & m(K') \leq m(\eta K) = \eta^n m(K) \\ \Rightarrow \quad & m(K') - m(K) \leq (\eta^n - 1)m(K) < \eta^n(\eta^n - 1)m(K) < \epsilon \end{aligned} \tag{3.3.5}$$

Also

$$\begin{aligned} & K \subset \eta K' \\ \Rightarrow \quad & m(K) \leq m(\eta K') = \eta^n m(K') \\ \Rightarrow \quad & m(K) - m(K') \leq (\eta^n - 1)m(K') \leq \eta^n(\eta^n - 1)m(K) < \epsilon \end{aligned} \tag{3.3.6}$$

From inequalities (3.3.5) and (3.3.6) it follows that $|m(K) - m(K')| \leq \epsilon$ for all K' in the $\ln(\eta)$-neighborhood of K. Therefore m is continuous at $K \in \mathcal{C}_{b_0}$. Since K was arbitrary, m is continuous on $(\mathcal{C}_{b_0}, \Delta_2)$ and hence on $(\mathcal{C}_{b_0}, \delta)$ (by Lemma 3.3.3). □

Theorem 3.3.5. *The mapping* $m : \mathcal{C}_b \longrightarrow [0, \infty)$ *is continuous with respect to* δ*.*

Proof. Let K be an arbitrary set in $\mathcal{C}_b$. Since K is a convex body, there exists some point $x \in \text{int}(K)$. The interior of the translated set $K - x$ thus contains the origin and hence $K - x \subset \mathcal{C}_{b_0}$. Theorem 3.3.4 now implies that m is continuous at $K - x$ with respect to δ and hence there exists for every $\epsilon > 0$ an $\eta > 0$ such that

$$|m(K - x) - m(K' - x)| < \epsilon \quad \text{whenever} \quad \delta(K - x, K' - x) < \eta.$$

Now if $\delta(K, K') < \eta$ then

$$\begin{aligned} K \subset K' + \eta B \quad & \text{and} \quad K' \subset K + \eta B \\ \Rightarrow K - x \subset K' - x + \eta B \quad & \text{and} \quad K' - x \subset K - x + \eta B \\ \Rightarrow \ \delta(K - x, K' - x) < \eta \quad & \end{aligned}$$

Hence, by the translation-invariance of m:

$$|m(K) - m(K')| = |m(K - x) - m(K' - x)| < \epsilon \quad \text{whenever} \quad \delta(K, K') < \eta.$$

□

3.3.3 The Continuity of $P : \mathcal{C}_s \longrightarrow [0, \infty)$

Suppose (X, B) is an n-dimensional Minkowski space with Dual space (X^*, B^o). Let $(\mathcal{C}_s^*, \delta_{B^o})$ denote the collection of all symmetric convex bodies in X^*, equipped with the Hausdorff metric defined in terms of B^o. Similarly, let δ_B denote the Hausdorff metric defined in terms of B.

Theorem 3.3.6. *The volume product $P : \mathcal{C}_s \to [0, \infty)$ is continuous with respect to the Hausdorff metric δ_B.*

Proof. Let m and m^* be two associated Haar measures on X and X^* respectively. Define the mapping

$$\varphi : (\mathcal{C}_s, \delta_B) \to (\mathcal{C}_s^*, \delta_{B^o}) | C \mapsto \varphi(C) = C^o.$$

The volume product P evaluated at an arbitrary set $C \in \mathcal{C}_s$ can thus be written as the product

$$P(C) = m(C) \cdot (m^* \circ \varphi)(C).$$

It is clear from Theorem 3.3.5 that m is continuous on $\mathcal{C}_s \subset \mathcal{C}_{b_0}$. It thus remains to show that the composite function

$$m^* \circ \varphi : \mathcal{C}_s \longrightarrow [0, \infty)$$

is continuous. Since all polar bodies are contained in $\mathcal{C}_s^*$, φ can be regarded as a mapping from $\mathcal{C}_s$ to $\mathcal{C}_s^*$. According to Lemma 3.3.3, there exist positive real numbers $c_1, c_2, d_1, d_2 > 0$ such that

$$c_1 \delta_B \leq \Delta_2 \leq c_2 \delta_B \quad \text{and} \quad d_1 \delta_{B^o} \leq \Delta_2^* \leq d_2 \delta_{B^o}.$$

Given $C \in \mathcal{C}_s$ and $\epsilon > 0$, let $\eta = \frac{d_1 \epsilon}{c_2}$. If $D \in \mathcal{C}_s$ is such that $\delta_B(C, D) < \eta$, then

$$\begin{aligned}
& \Delta_2(C, D) < d_1 \epsilon \\
\Rightarrow \quad & C \subset e^{d_1 \epsilon} D \quad \text{and} \quad D \subset e^{d_1 \epsilon} C \\
\Rightarrow \quad & D^o \subset e^{d_1 \epsilon} C^o \quad \text{and} \quad C^o \subset e^{d_1 \epsilon} D^o \qquad \because \text{Properties of Polar Bodies} \\
\Rightarrow \quad & \Delta_2^*(C^o, D^o) < d_1 \epsilon \Rightarrow \delta_{B^o}(C^o, D^o) < \epsilon
\end{aligned}$$

This implies that φ is continuous. In addition, Theorem 3.3.5 guarantees the continuity of the mapping $m^* : \mathcal{C}_s^* \longrightarrow [0, \infty)$ and, since the composition of two continuous functions is continuous, the result follows. $\square$

3.4 The Blaschke selection theorem

Consider the n-dimensional Minkowski space (X, B). A set $K \in X$ is said to be uniformly bounded by the scalar $a > 0$ if $K \subset aB$. For the sake of notational convenience, let $\mathcal{K}_a := \{K \in \mathcal{K} : K \subset aB\}$ and $\mathcal{C}_a := \{C \in \mathcal{C} : C \subset aB\}$. The Blaschke selection theorem can therefore be formulated as follows: "*For any $a > 0$, the collection $\mathcal{C}_a$ is sequentially compact in $(\mathcal{K}, \delta)$.*"

A set $K \subset X$ is called totally bounded if for every $\epsilon > 0$ there exist points $x_1, x_2, ..., x_n \in K$ such that $K \subset \bigcup_{i=1}^{n} B(X, x_i, \epsilon)$. The strategy used in this section to prove the Blaschke selection theorem resembles the approach used in *Thompson, Section 2.5 [18]* and relies mainly on the following well-known characterization of compactness in metric spaces, which is stated without proof (*see for instance Dunford and Schwartz [8]*).

Theorem 3.4.1. *For any set K in a metric space, the following statements are equivalent:*
(a) K is compact.
(b) K is sequentially compact.
(c) K is complete and totally bounded.

More specifically, the Blaschke selection theorem will be proved according to the method outlined by the following steps:

1. $(\mathcal{K}, \delta)$ is a complete metric space (Theorem 3.4.2).
2. $\mathcal{K}_a$ is a totally bounded (Theorem 3.4.3) and closed subset of the complete space $\mathcal{K}$ and is therefore compact (Corollary 3.4.4).
3. $\mathcal{C}$ is a closed subset of $\mathcal{K}$ and is therefore complete (Lemma 3.4.6).
4. $\mathcal{C}_a$ is a closed subset of the compact set $\mathcal{K}_a$ and must hence be compact (Theorem 3.4.7).

Theorem 3.4.2. *If $\{K_n\}$ is a Cauchy sequence in $(\mathcal{K}, \delta)$ then K_n converges to*

$$K_0 = \bigcap_{i=1}^{\infty} c\ell \left(\bigcup_{j \geq i}^{\infty} K_j \right) \in \mathcal{K}$$

Hence $(\mathcal{K}, \delta)$ is a complete metric space.

Proof. Since the arbitrary intersection of closed sets is also closed, it follows trivially that K_0 is a closed set. Let $\epsilon > 0$ be given. Then there exists an $n_0(\epsilon) > 0$ such that $\delta(K_i, K_j) < \epsilon$ for all $i, j \geq n(\epsilon)$. This implies that $K_j \subset K_i + \epsilon B$ for all $i, j \geq n(\epsilon)$.
In particular, for all $j \geq i \geq n_0(\epsilon)$ we have $K_j \subset K_i + \epsilon B$ and hence $\bigcup_{j \geq i} K_j \subset$

$K_i + \epsilon B$. Since both K_i and B are compact sets, $K_i + \epsilon B$ is closed, which implies

$$c\ell\left(\bigcup_{j\geq i} K_j\right) \subset K_i + \epsilon B \subset K_i + 2\epsilon B$$

This inclusion holds for all $i \geq n_0(\epsilon)$ and therefore it follows that

$$K_0 = \bigcap_{i=1}^{\infty} c\ell\left(\bigcup_{j\geq i} K_j\right) \subset \bigcap_{i\geq n_0(\epsilon)} c\ell\left(\bigcup_{j\geq i} K_j\right) \subset K_i + 2\epsilon B \tag{3.4.1}$$

for all $i \geq n_0(\epsilon)$.

Conversely, for any $k \in \mathbb{N}$ there exists an $n_k(\epsilon) \in \mathbb{N}$ such that $\delta(K_i, K_j) \leq 2^{-k}\epsilon$ for all $j, i > n_k(\epsilon)$. Choose an arbitrary positive integer $m_0 \geq n_0(\epsilon)$ and let $x_0 \in K_{m_0}$ (this is possible since $K_{m_0} \in \mathcal{K}$ and is therefore non-empty). Now let $m_1 > \max\{m_0, n_1(\epsilon)\}$. Then

$$\delta(K_{m_1}, K_{m_0}) < 2^{-1}\epsilon < \epsilon \Rightarrow K_{m_0} \subset K_{m_1} + 2^{-1}\epsilon B.$$

Therefore, x_0 can be written in the form $x_0 = x_1 + y$, where $x_1 \in K_{m_1}$ and $y \in 2^{-1}\epsilon B$, from which it follows that $\|x_1 - x_0\| = \|y\| \leq 2^{-1}\epsilon < \epsilon$. Now choose an $m_2 > \max\{m_1, n_2(\epsilon)\}$. Similarly $K_{m_1} \subset K_{m_2} + 2^{-2}\epsilon B$, which implies that there exist $x_2 \in K_{m_2}$ and $y \in 2^{-2}\epsilon B$ such that $x_1 = x_2 + y$ and hence $\|x_1 - x_2\| = \|y\| \leq 2^{-2}\epsilon < 2^{-1}\epsilon$. Generally if $m_1, ..., m_{k-1}$ and $x_1, ..., x_{k-1}$ have been chosen, take $m_k > \max\{m_{k-1}, n_k(\epsilon)\}$. By the same argument there is an $x_k \in K_{m_k}$ such that $\|x_k - x_{k-1}\| < 2^{k-1}\epsilon$.
Now $\{x_i\}_{i=1}^{\infty}$ is a Cauchy sequence in $(X, \|.\|)$ and must hence converge (by the completeness of Minkowski spaces) to some $\bar{x}$. Since the sequence $\{m_k\}_{k=1}^{\infty}$ is strictly increasing and unbounded, it follows that for any $n \in \mathbb{N}$ there exists an $m_{n'} > n$ and hence $x_{n'} \in K_{m_{n'}} \subset \bigcup_{j\geq n} K_j$. In fact for any $k \geq n'$ we have $m_k > m_{n'} \Rightarrow x_k \in \bigcup_{j\geq m_k} K_j \subset \bigcup_{j\geq n} K_j$. Since the sequence $\{x_k\} \subset \bigcup_{j\geq n} K_j$ converges to $\bar{x}$ it thus follows that $\bar{x} \in cl\left(\bigcup_{j\geq n} K_j\right)$. This holds for any n. Therefore $\bar{x} \in \bigcap_{n=1}^{\infty} c\ell\left(\bigcup_{j\geq n} K_j\right) = K_0$. Hence $K_0 \neq \emptyset$. By continuity of the norm we have:

$$\begin{aligned}\|\bar{x} - x_0\| &= \lim_{n\to\infty} \|x_n - x_0\| = \lim_{n\to\infty} \|x_n - x_{n-1} + x_{n-1} - x_{n-2} + ... + x_1 - x_0\| \\ &\leq \lim_{n\to\infty} \sum_{k=1}^{n} \|x_k - x_{k-1}\| \leq \lim_{n\to\infty} \sum_{k=1}^{n} 2^{-(k-1)}\epsilon = 2\epsilon\end{aligned}$$

Hence for any $m_0 \geq n_0(\epsilon)$ and any $x_0 \in K_{m_0}$ we can find a $\bar{x} \in K_0$ such that $\|\bar{x} - x_0\| \leq 2\epsilon$, which implies $x_0 - \bar{x} \in 2\epsilon B$ and hence $x_0 \in K_0 + 2\epsilon B$. But $m_0 \geq n_0(\epsilon)$ is an arbitrary integer. Thus

$$K_i \subset K_0 + 2\epsilon B \quad \text{for all } i \geq n_0(\epsilon) \tag{3.4.2}$$

From inclusions (3.4.1) and (3.4.2) it finally follows that $\delta(K_0, Ki) \leq 2\epsilon$ for all $i \geq n_0(\epsilon)$ which implies that $\delta(K_0, K_i) \to 0$ as $i \to \infty$ □

Theorem 3.4.3. *$\mathcal{K}_a$ is totally bounded.*

Proof. Let $\epsilon > 0$ be given. The ball aB is compact and hence there exists a finite collection of balls $\{B(x_1, \epsilon), ...B(x_k, \epsilon)\}$ with centers $x_1, ..., x_k$ that cover aB. Let J be the collection of all non-empty subsets of $\{x_1, ..., x_k\}$. J contains $2^k - 1$ sets that are all bounded by aB, and (being finite unions of compact point sets) are compact. Thus $J \subset (\mathcal{K}_a, \delta)$. For any $K \in \mathcal{K}_a$ define $F_K := \{x_i \in F : \text{there exists an } x \in K \text{ such that } \|x_i - x\| < \epsilon\}$. By the definition of F_k we have $F_k \subset K + \epsilon B$. Conversely, since $\bigcup_{i=1}^{k} B(x_i, \epsilon)$ is a finite covering of aB and hence of K, we can find for every $x \in K$ an $x_i \in F_K$ such that $\|x - x_i\| < \epsilon$. It follows that $x \in F_K + \epsilon B$ and thus $K \subset F_K + \epsilon B$. Therefore $\delta(K, F_K) < \epsilon$. In other words, every $K \in \mathcal{K}_a$ is contained in some ball in $(\mathcal{K}_a, \delta)$ with radius ϵ whose center is an element of the finite set $J \subset mathcal K_a$. Therefore $\mathcal{K}_a$ can be covered by a finite number of ϵ-balls which implies that $(\mathcal{K}_a, \delta)$ is totally bounded. □

Corollary 3.4.4. *$\mathcal{K}_a$ is compact with respect to δ.*

Proof. $\mathcal{K}_a$ is a closed subset of $\mathcal{K}$: Let $\{K_i\}$ be a sequence in $\mathcal{K}_a$ converging to some K. Then $\{K_i\}$ is a Cauchy sequence which (by completeness of $\mathcal{K}$) converges to $K_0 \Rightarrow K = K_0$. Now $K_i \subset aB$ for all $i \in \mathbb{N} \Rightarrow K_0 = \bigcap_{i=1}^{\infty} c\ell\left(\bigcup_{j\geq i} K_j\right) \subset aB \Rightarrow K_0 \in \mathcal{K}_a$. This implies that $\mathcal{K}_a$ is complete (since $\mathcal{K}$ is complete). Also, $\mathcal{K}_a$ is totally bounded, according to Theorem 3.4.3. It now follows from Theorem 3.4.1 that $(\mathcal{K}_a, \delta)$ is compact. □

The following technical Lemma establishes a closed-form expression for the limit of any convergent sequence in $\mathcal{K}$, which is used in Lemma 3.4.6 to show that $\mathcal{C}$ is a closed (and hence complete) subset of $(\mathcal{K}, \delta)$.

Lemma 3.4.5. *If $\{K_n\}$ is a sequence in $(\mathcal{K}, \delta)$ such that $K_n \longrightarrow K_0$, then*

$$K_0 = \bigcap_{\epsilon>0} \bigcup_{i=1}^{\infty} \bigcap_{j\geq i} (K_j + \epsilon B)$$

Proof. The convergence $K_i \to K_0$ implies that for every $\epsilon > 0$ there exists an $i(\epsilon) \in \mathbb{N}$ such that $K_0 \subset K_j + \epsilon B$ for all $j \geq i(\epsilon)$. Hence

$$K_0 \subset \bigcap_{\epsilon>0} \bigcap_{j\geq i(\epsilon)} (K_j + \epsilon B) \subset \bigcap_{\epsilon>0} \bigcup_{i=1}^{\infty} \bigcap_{j\geq i} (K_j + \epsilon B) \tag{3.4.3}$$

For the converse inclusion, note that since all convergent sequences are Cauchy sequences, it follows from Theorem 3.4.2, that K_0 is non-empty and can be written in the form

$$K_0 = \bigcap_{i=1}^{\infty} c\ell \left(\bigcup_{j \geq i}^{\infty} K_j \right).$$

and hence it suffices to show that

$$\bigcap_{\epsilon>0} \bigcup_{i=1}^{\infty} \bigcap_{j \geq i} (K_j + \epsilon B) \subset \bigcap_{i=1}^{\infty} c\ell \left(\bigcup_{j \geq i}^{\infty} K_j \right).$$

Suppose

$$x \in \bigcap_{\epsilon>0} \bigcup_{i=1}^{\infty} \bigcap_{j \geq i} (K_j + \epsilon B).$$

Then for any $\epsilon > 0$ there exists an $i(\epsilon)$ such that $x \in K_j + \epsilon B$ for all $j \geq i(\epsilon)$. Let $m \in \mathbb{N}$ be arbitrary and choose $n \in \mathbb{N}$ such that $n \geq \max\{m, i(\epsilon)\}$. Then

$$x \in K_n + \epsilon B \subset \bigcup_{n \geq m} K_n + \epsilon B$$

So for every $\epsilon > 0$, there exist $x_{i(\epsilon)} \in \bigcup_{n \geq m} K_n$ and $y \in \epsilon B$ such that $x = x_{i(\epsilon)} + y$. If $x_{i(\epsilon)} = x$ then $x = x_{i(\epsilon)} \in \bigcup_{n \geq m} K_n \subset c\ell \left(\bigcup_{n \geq m} K_n \right)$.
Even if $x_{i(\epsilon)} \neq x$, it still holds that $\|x - x_{i(\epsilon)}\| = \|y\| < \epsilon$. This implies that for any ϵ-neighborhood $B(x, \epsilon)$ of x, we can find $x_{i(\epsilon)} \in \bigcup_{n \geq m} K_n \in$ such that $x_{i(\epsilon)} \in B(x, \epsilon)$ and is distinct from x. Thus x is an accumulation point of $\bigcup_{n \geq m} K_n$ and hence $x \in c\ell \left(\bigcup_{n \geq m} K_n \right)$. Since m was arbitrary,

$$x \in \bigcap_{m=1}^{\infty} c\ell \left(\bigcup_{n \geq m} K_n \right).$$

□

Lemma 3.4.6. *The set $\mathcal{C}$ is a closed subset of $(\mathcal{K}, \delta)$ and is therefore complete.*

Proof. Let $\{C_n\}$ be any sequence of sets in $\mathcal{C}$ converging to $K_0 \in \mathcal{K}$. By the previous lemma,

$$K_0 = \bigcap_{\epsilon>0} \bigcup_{i=1}^{\infty} \bigcap_{j \geq i} (C_j + \epsilon B)$$

Now since the sum of two convex sets is always convex, we have $C_j + \epsilon B$ is convex. further the arbitrary intersection of convex sets is convex which implies that $\bigcap_{j \geq i} (C_j + \epsilon B)$ is convex. Now the sequence $\{\bigcap_{j \geq i} (C_j + \epsilon B)\}_{i=1}^{\infty}$ is a nested sequence of convex sets therefore

$$\bigcup_{i=1}^{\infty} \bigcap_{j \geq i} (C_j + \epsilon B) = \bigcap_{j \geq 1} (C_j + \epsilon B)$$

which is convex. Finally, the set

$$K_0 = \bigcap_{\epsilon>0} \bigcup_{i=1}^{\infty} \bigcap_{j\geq i} (C_j + \epsilon B)$$

is again the intersection of convex sets and hence is convex. So $K_0 \in \mathcal{C}$ which implies that $\mathcal{C}$ is closed. □

Theorem 3.4.7 (The Blaschke Selection Theorem). *The collection $\mathcal{C}_a$ is sequentially compact with respect to δ.*

Proof. $\mathcal{C}_a = \mathcal{K}_a \bigcap \mathcal{C}$, where both $\mathcal{C}$ and $\mathcal{K}_a$ are closed (according to Theorem 3.4.6 and Corollary 3.4.4 respectively). Therefore $\mathcal{C}_a$ is a closed subset of $\mathcal{K}_a$. Since $\mathcal{K}_a$ is compact (Corollary 3.4.4), it follows that $\mathcal{C}_a$ must also be compact. □

Chapter 4

Steiner Symmetrization and the First Part of the Blaschke-Santaló Inequality

Let A be a non-empty compact convex set in $\mathbb{R}^n$ and let $\mathcal{H}$ be a hyperplane in $\mathbb{R}^n$ with unit normal vector u. The Steiner symmetral $A_{\mathcal{H}}$ of A about $\mathcal{H}$ is computed by translating all the chords of A which are perpendicular to $\mathcal{H}$, in the direction u until their midpoints lie on $\mathcal{H}$. The union of all these chords is symmetric about $\mathcal{H}$ and is called the Steiner symmetral $A_{\mathcal{H}}$ of A about $\mathcal{H}$. Section 4.1 develops the formal method of Steiner symmetrization while section 4.2 establishes some elementary properties of the Steiner symmetral. In particular, it will be shown that the Steiner symmetral preserves convexity and compactness (Propositions 4.2.1 and 4.2.2) as well as the volume (Theorem 4.2.3) of the original set. These two sections, as well as Section 4.3 are largely based on Section 6.6 of the book "Convexity" *(Webster[19])*. Section 4.3 sets out to show that for every convex body in $A \subset \mathbb{R}^n$ there exists a sequence of Steiner symmetrals of A converging in the Hausdorff metric to a Euclidean ball B_0. This result not only leads to the famous characterization of ellipsoids due to H. Brunn [6] (Theorem 4.4.2), which is invoked in Chapter 5 to prove the second part of the Blaschke-Santaló inequality, but was also used directly by Meyer and Pajor [11] in a short proof of the first part of the Blaschke-Santaló inequality (Lemma 4.4.6 and Theorem 4.4.7).

4.1 Definitions

Let $A \subset \mathbb{R}^n$ be a non-empty compact convex set and $\mathcal{H}$ be a hyperplane with unit normal vector u. The definition of the Steiner symmetral $A_{\mathcal{H}}$ of A about $\mathcal{H}$ relies on the orthogonal decomposition of $\mathbb{R}^n = \mathcal{H} \oplus \mathcal{H}^{\perp} = \mathcal{H} \oplus \mathrm{span}\{u\}$. Indeed, any vector $a \in A$ can be written uniquely as $a = p + \theta u$, where $p \in \mathcal{H}$

and $\theta \in \mathbb{R}$. The following definitions will prove useful in this context.

Definition 4.1.1. *The projection $\mathcal{H}(A)$ of A onto the hyperplane $\mathcal{H}$ is defined as the set $\mathcal{H}(A) := \{p \in \mathcal{H} : p + \theta u \in A$ for some $\theta \in \mathbb{R}\}$. Also, for any $p \in \mathcal{H}(A)$, let $I_A(p) := \{\theta \in \mathbb{R} : p + \theta u \in A\}$.*

It follows directly from the convexity of A that both $\mathcal{H}(A)$ and $I_A(p)$ are convex sets. Moreover, the compactness of A ensures that $I_A(p)$ is a non-empty compact interval for any $p \in \mathcal{H}(A)$. The following functions are therefore well-defined.

Definition 4.1.2. *Define the functions $\alpha_A, \beta_A, \gamma_A : \mathcal{H}(A) \to \mathbb{R}$ by:*

$$\begin{aligned}
\alpha_A(p) &:= \min\{\theta : \theta \in I_A(p)\} \\
\beta_A(p) &:= \max\{\theta : \theta \in I_A(p)\} \\
\gamma_A(p) &:= \beta_A(p) - \alpha_A(p)
\end{aligned}$$

When no ambiguities are apparent, the subscript A will be omitted. In terms of the above notation, $A = \{p + \theta u : p \in \mathcal{H}(A),\ \theta \in I_A(p)\}$. In other words, A is the union of chords of the form $\{p + \theta u : \theta \in I_A(p)\}$ where $p \in \mathcal{H}(A)$. The Steiner symmetral $A_{\mathcal{H}}$ of A about $\mathcal{H}$ is obtained by translating every such chord along the line $\ell = \text{span}\{u\}$ so that its midpoint $\frac{1}{2}(\alpha(p) + \beta(p))$ lies on $\mathcal{H}(A)$. More explicitly:

$$\begin{aligned}
A_{\mathcal{H}} :&= \{p + \theta u : p \in \mathcal{H}(A),\ \theta \in I_A(p) - \frac{1}{2}(\alpha(p) + \beta(p))\} \\
&= \{p + \theta u : p \in \mathcal{H}(A),\ |\theta| \leq \gamma(p)\} \qquad (4.1.1)
\end{aligned}$$

Another, equivalent definition, which will prove useful in Lemma 4.4.6 can be understood in terms of the reflection of A about $\mathcal{H}$. The Steiner symmetral is constructed by mapping each chord of the form $\{p + \theta u : \theta \in I_A(p)\}$ with $p \in \mathcal{H}(A)$, onto the chord $\frac{1}{2}\{p + \theta u : \theta \in I_A(p)\} + \frac{1}{2}\{p - \theta u : \theta \in I_A(p)\}$. More concisely:

$$A_{\mathcal{H}} = \{p + \frac{1}{2}(\theta_1 - \theta_2)u : p \in \mathcal{H}(A),\ \theta_i \in I_A(p) \text{ for } i = 1, 2\} \qquad (4.1.2)$$

In order to prove that the sets defined by (4.1.1) and (4.1.2) are equal, it suffices to show that for a given $p \in \mathcal{H}(A)$, $\theta + \frac{1}{2}(\alpha(p) + \beta(p)) \in I_A(p)$ if and only if θ can be written as $\theta = \frac{1}{2}(\theta_1 - \theta_2)$, where $\theta_i \in I_A(p)$ for $i = 1, 2$.
For any $\theta \in I_A(p) - \frac{1}{2}(\alpha(p) + \beta(p))$, it can easily be seen that

$$\theta_1 := \theta + \frac{1}{2}(\alpha(p) + \beta(p)) \quad \text{and} \quad \theta_2 := -\theta + \frac{1}{2}(\alpha(p) + \beta(p))$$

are both contained in $I_A(p)$ and $\theta = \frac{1}{2}(\theta_1 - \theta_2)$. Conversely, if $\theta = \frac{1}{2}(\theta_1 - \theta_2)$, where $\theta_i = \lambda_i\alpha(p) + (1 - \lambda_i)\beta(p)$ with $\lambda_i \in [0, 1]$ for $i = 1, 2$, it follows that $\theta + \frac{1}{2}(\alpha(p) + \beta(p))$ can be written as

$$\theta + \frac{1}{2}(\alpha(p) + \beta(p)) = \mu\alpha(p) + (1 - \mu)\beta(p)$$

where $\mu = \frac{1+\lambda_1-\lambda_2}{2} \in [0,1]$. Hence $\theta + \frac{1}{2}(\alpha(p) + \beta(p)) \in I_A(p)$.

4.2 Elementary Properties

Proposition 4.2.1. .

(i) *$A_{\mathcal{H}}$ is symmetric about $\mathcal{H}$. Moreover, if A is symmetric about $\mathcal{H}$ then $A_{\mathcal{H}} = A$.*

(ii) *$A_{\mathcal{H}}$ is convex.*

(iii) *If $p \in \mathcal{H}$ then the Euclidean ball $B := \{x \in \mathbb{R}^n : .\langle x-p, x-p\rangle \leq r^2\}$ is symmetric about $\mathcal{H}$ and hence $B_{\mathcal{H}} = B$.*

(iv) *For any $A, C \subset \mathcal{C}$ with $C \subset A$, it follows $C_{\mathcal{H}} \subset A_{\mathcal{H}}$.*

Proof. .

(i) If $p + \theta u \in A_{\mathcal{H}}$ then $p \in \mathcal{H}(A)$ and $|-\theta| = |\theta| \leq \frac{1}{2}\gamma(p)$. Therefore $p+\theta(-u) = p-\theta u \in A_{\mathcal{H}}$ and hence $A_{\mathcal{H}}$ is symmetric about $\mathcal{H}$. Moreover, suppose that for all $p \in \mathcal{H}(A)$, $p - \theta u \in A$ whenever $p + \theta u \in A$. In this case $\alpha(p) = -\beta(p)$ and hence $I_A(p) = [-\beta, \beta]$. Thus

$$A_{\mathcal{H}} = \{p + \theta u : p \in \mathcal{H},\ |\theta| \leq \beta\} = A$$

(ii) Suppose $a, a' \in A_{\mathcal{H}}$. According to definition (4.1.2),

$$a = p + \frac{1}{2}(\theta_1 - \theta_2)u \quad \text{and} \quad a' = p' + \frac{1}{2}(\theta_1' - \theta_2')u$$

where $p, p' \in \mathcal{H}(A)$, $\theta_i \in I_A(p)$ and $\theta_i' \in I_A(p')$ for $i = 1, 2$. For any $\mu \in [0,1]$,

$$\mu a+(1-\mu)a' = \mu p+(1-\mu)p'+\frac{1}{2}\left[(\mu\theta_1 + (1-\mu)\theta_1') - (\mu\theta_2 + (1-\mu)\theta_2')\right]u$$

The convexity of $\mathcal{H}(A)$ and of $I_A(p)$ now imply that $\mu a+(1-\mu)a' \in A_{\mathcal{H}}$.

(iii) Consider a typical point $q + \theta u \in B$, where $q \in \mathcal{H}$, $\theta \in \mathbb{R}$. Since the vector $q - p$ is parallel to $\mathcal{H}$ and therefore $\langle u, q-p\rangle = 0$, it follows that

$$\begin{aligned}\|q + \theta u - p\|^2 &= \|q-p\|^2 + 2\theta\langle q-p, u\rangle + \theta^2\|u\|^2 \\ &= \|q-p\|^2 + \theta^2\|u\|^2 = \|q - \theta u - p\|^2\end{aligned}$$

It therefore follows that $q - \theta u \in B$ and hence $B = B_{\mathcal{H}}$.

(iv) For any $p \in \mathcal{H}(C)$ there exists a $\theta \in \mathbb{R}$ such that $p + \theta u \in C \subset A$. Therefore $p \in \mathcal{H}(A)$ which implies $\mathcal{H}(C) \subset \mathcal{H}(A)$.
By making use of definition (4.1.2),

$$\begin{aligned} C_{\mathcal{H}} :&= \{p + \frac{1}{2}(\theta_1 - \theta_2)u : p \in \mathcal{H}(C), p + \theta_i u \in C \subset A \text{ for } i = 1, 2\} \\ &\subset \{p + \frac{1}{2}(\theta_1 - \theta_2)u : p \in \mathcal{H}(A), p + \theta_i u \in A \text{ for } i = 1, 2\} \\ &=: A_{\mathcal{H}} \end{aligned}$$

□

The following Proposition asserts that the Steiner symmetral $A_{\mathcal{H}}$ about $\mathcal{H}$ preserves certain topological properties of the original set A. This leads to the useful result that if $A \in \mathcal{C}_b$ then $A_{\mathcal{H}} \in \mathcal{C}_b$.

Proposition 4.2.2. *If A is a non-empty compact convex set in $\mathbb{R}^n$ then then $A_{\mathcal{H}}$ is also in $\mathcal{C}$. Moreover, if A is a convex body, then $A_{\mathcal{H}}$ is also a convex body.*

Proof. Let A be a non-empty compact convex set in $\mathbb{R}^n$. It was seen in Proposition 4.2.1 (ii) that the Steiner symmetral preserves convexity. Since all Minkowski Spaces have the Heine-Borel property, it suffices to show in addition that $A_{\mathcal{H}}$ is closed and bounded and has a non-empty interior whenever A does.

$A_{\mathcal{H}}$ is closed:
Let $\{x_i\}_{i=1}^{\infty}$ be a sequence in $A_{\mathcal{H}}$ converging to some $x \in \mathbb{R}^n$ with respect to the Euclidean norm. By definition, each $x_i \in A_{\mathcal{H}}$ is of the form $x_i = p_i + \theta_i u$, where $p_i \in \mathcal{H}(A)$ and $|\theta_i| \leq \frac{1}{2}\gamma(p_i)$ for $i \in \mathbb{N}$. Moreover, the orthogonal decomposition of $\mathbb{R}^n$ into $\mathcal{H} \oplus \text{span}\{u\}$ ensures that x can be written as $x = p + \theta u$, where $p \in \mathcal{H}$ and $\theta \in \mathbb{R}$. Since $p - p_i \perp u$, it follows that for all $i \in \mathbb{N}$

$$\begin{aligned} \|x - x_i\|^2 &= \langle x - x_i, x - x_i \rangle = \langle p - p_i + (\theta - \theta_i)u, p - p_i + (\theta - \theta_i)u \rangle \\ &= \langle p - p_i, p - p_i \rangle + 2(\theta - \theta_i)\langle p - p_i, u \rangle + \langle p - p_i, p - p_i \rangle \\ &= \|p - p_i\|^2 + (\theta - \theta_i)^2 \\ &\geq \|p - p_i\|^2 \end{aligned}$$

So $0 \leq \|p - p_i\| \leq \|x - x_i\| \to 0$ as $i \to \infty$ and hence p_i converges to p. This in turn implies that the sequence $\{\theta_i u\} = \{x_i - p_i\}$ is convergent. Now span$\{u\}$ is closed and hence $\theta_i u \to \theta u \Rightarrow \theta_i \to \theta$ for some $\theta \in \mathbb{R}$.
Consider the points $y_i = p_i + \alpha(p_i)u$ and $z_i = p_i + \beta(p_i)u$ in A. Since A is compact, there exist subsequences $\{y_{i_k}\}$ and $\{z_{i_k}\}$ that converge to $y, z \in A$ respectively. But by the above argument, $p_{i_k} \to p$ as $k \to \infty$. The fact that span$\{u\}$ is closed hence implies that $\alpha(p_{i_k}) \to a$ and $\beta(p_{i_k}) \to b$ for some

$a, b \in \mathbb{R}$ as $k \to \infty$. Moreover, since $\alpha(p_{i_k}) \leq \beta(p_{i_k})$ for all $i \in \mathbb{N}$, it follows that $a \leq b$. The vectors z and y can thus be written as:

$$y = p + au \quad \text{and} \quad z = p + bu$$

where $\alpha(p) \leq a \leq b \leq \beta(p)$ ($\because y, z \in A$). This also implies that $p \in \mathcal{H}(A)$. Finally:

$$|\theta| = \lim_{k\to\infty} |\theta_{i_k}| \leq \lim_{k\to\infty} \frac{1}{2}(\alpha(p_{i_k}) - \beta(p_{i_k})) = \frac{1}{2}(b-a) \leq \frac{1}{2}\gamma(p)$$

Therefore $x = p + \theta u \in A_{\mathcal{H}} \Rightarrow A_{\mathcal{H}}$ is closed.

$A_{\mathcal{H}}$ is bounded:
Since A is bounded there exists a Euclidean ball B such that $A \subset B$. Assume without loss of generality that the midpoint of B lies on $\mathcal{H}$. Parts (iii) and (iv) of Proposition 4.2.1 now imply that $A_{\mathcal{H}} \subset B_{\mathcal{H}} = B$. Therefore $A_{\mathcal{H}}$ is bounded.

If $\text{int}A \neq \emptyset$ then $\text{int}A_{\mathcal{H}} \neq \emptyset$:
If A has a non-empty interior there exists a Euclidean ball $B \subset A$. It follows readily from part (iii) of Proposition 4.2.1 that $B_{\mathcal{H}}$ is also a Euclidean Ball. Moreover, part (iv) of proposition 4.2.1 implies that $B_{\mathcal{H}} \subset A_{\mathcal{H}}$ and hence $\text{int}A_{\mathcal{H}} \neq \emptyset$. $\square$

Theorem 4.2.3. *Let $A \subset \mathbb{R}^n$ be any non-empty compact convex set and $\mathcal{H}$ be a hyperplane. Then $\lambda^n(A_{\mathcal{H}}) = \lambda^n(A)$.*

Proof. This Theorem is proved by induction. Let λ^n denote the n-dimensional Lebesgue measure.
For n=1:
A is a non-empty closed interval $[a, b]$ and $\mathcal{H}$ is a point in $\mathbb{R}$. Any point $x \in A_{\mathcal{H}}$ is of the form $x = \mathcal{H}+\theta$, where $|\theta| \leq \frac{1}{2}(b-a)$. The length of the closed interval $A_{\mathcal{H}}$ is given by:

$$\begin{aligned}\lambda(A_{\mathcal{H}}) &= \max\{x : x \in A_{\mathcal{H}}\} - \min\{x : x \in A_{\mathcal{H}}\}\\ &= \mathcal{H} + \frac{1}{2}(b-a) - (\mathcal{H} - \frac{1}{2}(b-a)) = b - a = \lambda(A)\end{aligned}$$

Suppose the Theorem holds for dimension n-1:
Let v be a vector parallel to $\mathcal{H}$ and let $\mathcal{H}_\nu = \{x \in \mathbb{R}^n : x \cdot v = \nu\}$ for $\nu \in \mathbb{R}$. The ν-section A_ν of A is defined as the intersection $A \cap H_\nu$. If it can be shown that $(A_{\mathcal{H}})_\nu = (A_\nu)_{\mathcal{H}}$ for all $\nu \in \mathbb{R}$ then Fubini's Theorem, used in conjunction with the induction hypothesis, implies:

$$\lambda^n(A) = \int_{\mathbb{R}} \lambda^{n-1}(A_\nu)d\nu = \int_{\mathbb{R}} \lambda^{n-1}((A_\nu)_{\mathcal{H}})d\nu = \int_{\mathbb{R}} \lambda^{n-1}((A_{\mathcal{H}})_\nu)d\nu = \lambda^n(A_{\mathcal{H}})$$

It thus suffices to show $(A_{\mathcal{H}})_\nu = (A_\nu)_{\mathcal{H}}$ for all $\nu \in \mathbb{R}$. Note that since $v \perp u$

$$\begin{aligned}&p \in \mathcal{H}(A_\nu)\\ \Leftrightarrow\ &p \in \mathcal{H},\ p + \theta u \in A \ \text{ and } \ p + \theta u \in H_\nu \text{ for some } \theta \in \mathbb{R}\\ \Leftrightarrow\ &p \in \mathcal{H}(A) \ \text{ and } \ p \cdot v = (p + \theta u) \cdot v = \nu\\ \Leftrightarrow\ &p \in \mathcal{H}(A) \ \text{ and } \ p \in \mathcal{H}_\nu\\ \Leftrightarrow\ &p \in (\mathcal{H}(A))_\nu\end{aligned}$$

Therefore

$$\begin{aligned}&p + \frac{1}{2}(\theta_1 - \theta_2)u \in (A_\nu)_{\mathcal{H}}\\ \Leftrightarrow\ &p \in \mathcal{H}(A_\nu) = \mathcal{H}(A) \cap \mathcal{H}_\nu \text{ and } \ p + \theta_i u \in A \cap H_\nu \text{ for } i = 1, 2\\ \Leftrightarrow\ &p \in \mathcal{H}(A), p + \theta_i u \in A \text{ for } i = 1, 2 \text{ and } p + \frac{1}{2}(\theta_1 - \theta_2)u \in \mathcal{H}_\nu\\ \Leftrightarrow\ &p + \frac{1}{2}(\theta_1 - \theta_2)u \in (A_{\mathcal{H}}) \cap H_\nu = (A_{\mathcal{H}})_\nu\end{aligned}$$

□

4.3 Sequences of Steiner Symmetrals

Theorem 4.3.1. *Let $\{A_k\}$ be a sequence of convex bodies that converge to a convex body $A \subset \mathbb{R}^n$. Then the sequence $\{(A_k)_{\mathcal{H}}\}$ of Steiner symmetrals of A_k will converge in the Hausdorff metric to the Steiner symmetral $A_{\mathcal{H}}$ of A.*

Proof. Let B denote the Euclidean unit ball and assume, without loss of generality that the origin is an interior point of A lying on $\mathcal{H}$. Hence $A \in C_{b_0}$ and $\mathcal{H}$ is a subspace. Since A is bounded and has a non-empty interior, there exist positive numbers $s, r > 0$ such that

$$rB \subset A \subset sB \Rightarrow rB = (rB)_{\mathcal{H}} \subset A_{\mathcal{H}} \subset (sB)_{\mathcal{H}} = sB \tag{4.3.1}$$

Also, by Lemma 3.3.3 it follows that $\{A_k\}$ converges to A with respect to the metric $\triangle_2$. Thus there exists an $N_1 \in \mathbb{N}$ such that

$$rB \subset A_k \subset sB \quad \text{for all } k \geq N_1$$

and hence

$$rB = (rB)_{\mathcal{H}} \subset (A_k)_{\mathcal{H}} \subset (sB)_{\mathcal{H}} = sB \quad \text{for all } k \geq N_1 \tag{4.3.2}$$

Let $\epsilon > 0$ be given. Since A_k converges to A, there exists an $N_2 \in \mathbb{N}$ such that:

$$A_k \subset A + \frac{r\epsilon}{s}B \;\text{ and }\; A \subset A_k + \frac{r\epsilon}{s}B \;\text{ for all } k \geq N_2 \tag{4.3.3}$$

Let $k \geq \max\{N_1, N_2\}$. Hence, according to (4.3.1), (4.3.2) and (4.3.3)

$$\begin{aligned} A_k &\subset A + \frac{r\epsilon}{s}B \subset A + \frac{\epsilon}{s}A = (1 + \frac{\epsilon}{s})A \\ \Rightarrow (A_k)_{\mathcal{H}} &\subset (1 + \frac{\epsilon}{s})A_{\mathcal{H}} = A_{\mathcal{H}} + \frac{\epsilon}{s}A_{\mathcal{H}} \subset A_{\mathcal{H}} + \epsilon B \end{aligned}$$

Similarly

$$\begin{aligned} A &\subset A_k + \frac{r\epsilon}{s}B \subset A_k + \frac{\epsilon}{s}A_k = (1 + \frac{\epsilon}{s})A_k \\ \Rightarrow A_{\mathcal{H}} &\subset (1 + \frac{\epsilon}{s})(A_k)_{\mathcal{H}} = (A_k)_{\mathcal{H}} + \frac{\epsilon}{s}(A_k)_{\mathcal{H}} \subset (A_k)_{\mathcal{H}} + \epsilon B \end{aligned}$$

From these two inclusions it follows that $\delta((A_k)_{\mathcal{H}}, A_{\mathcal{H}}) < \epsilon$ which implies convergence. □

Definition 4.3.2. *Let $\mathcal{S}(A)$ be the family of sets in $\mathbb{R}^n$ which can be obtained by applying a finite number of Steiner symmetrizations to A . In other words*

$$\mathcal{S}(A) = \{A_N : N \in \mathbb{N}, A_k = (A_{k-1})_{\mathcal{H}} \textit{ for } k = 2, ..., N \textit{ and } A_1 = A_{\mathcal{H}}\}$$

Theorem 4.3.3. *For any convex body $A \subset \mathbb{R}^n$ there exists a sequence of sets in $\mathcal{S}(A)$ that converges to the closed ball B_0 of volume $\lambda^n(A)$ whose center is the origin.*

Proof. Let B denote the Euclidean unit ball and let

$$r_0 := \inf\{r > 0 : \text{ there is a } C \in \mathcal{S}(A) \text{ such that } C \subset rB\}$$

By the definition of the infimum there exists a sequence of sets $\{A_k\}$ in $\mathcal{S}(A)$ such that $A_k \subset (r_0 + \frac{1}{k})B \subset (r_0 + 1)B$ for all $k \in \mathbb{N}$. This sequence is thus contained $\mathcal{C}_a$ with $a = r_0 + 1$ must hence, according to the Blaschke selection theorem, have a convergent subsequence $\{A_{k_i}\}$, which converges to a convex body B_0. Since $B_0 \subset (r_0 + \frac{1}{k})B$ for all $k \in \mathbb{N}$, it follows that $B_0 \subset r_0 B$. Also, by Theorem 4.2.3, all sets $\{A_{k_i}\}$ have the same volume as A. The continuity of the volume measure (Theorem 3.3.5) now implies that $\lambda^n(B_0) = \lambda^n(A)$.
If it can be shown that B_0 is not a proper subset of $r_0 B$, then the result follows. Suppose that there is an $x_0 \in r_0 B$ such that $x_0 \notin B_0$. Since $r_0 B = \text{conv}(\text{bd}(r_0 B))$, it can be assumeed, without loss of generality, that $x_0 \in \text{bd}(r_0 B)$. B_0 is closed and hence there exists an $s > 0$ such that $B_0 \cap B(x_0, s) = \emptyset$. Moreover, $\text{bd}(r_0 B)$ is a compact set, from which it follows that there exist elements $x_1, ..., x_m \in \text{bd}(r_0 B)$ such that $\text{bd}(r_0 B) \subset B(x_0, s) \cup B(x_1, s) \cup ... \cup B(x_m, s)$.

Construction

Define $C_0, C_1, C_2, ..., C_m$ to be the sets $C_i := \text{bd}(r_0 B) \cap B(x_i, s)$. It immediately follows that $\text{bd}(r_0 B) = \bigcup_{i=0}^m C_i$. Also let $\mathcal{H}_i$ be the hyperplane which orthogonal to the vector $x_i - x_0$ and passes through the origin. It can readily be seen that if $\mathcal{H}_i$ is defined as above, then C_0 is the reflection of C_i about $\mathcal{H}_i$. Consider $(B_0)_{\mathcal{H}_1}$. Since $B_0 \subset r_0 B$ it follows that $(B_0)_{\mathcal{H}_1} \subset (r_0 B)_{\mathcal{H}_1} = r_0 B$. Moreover, $B_0 \cap B(x_0, s) = \emptyset$ and therefore $B_{\mathcal{H}_1}$ must be disjoint from $C_0 \cup C_1$. Similarly $(B_{\mathcal{H}_1})_{\mathcal{H}_2}$ is disjoint from $C_0 \cup C_1 \cup C_2$. Applying successive symmetrizations to B_0 about the hyperplanes $\mathcal{H}_1, ..., \mathcal{H}_m$ thus yields a convex body B_0' which is disjoint from $C_0 \cup C_1 \cup ... \cup C_m$ and hence from $\text{bd}(r_0 B)$. This implies $B_0' \subset \text{int}\{r_0 B\}$ and therefore there exists an $0 < \epsilon < r_0$ such that $B_0' \subset (r_0 - \epsilon)B$. For every $k \in \mathbb{N}$, let A_k' be the set obtained by applying the same sequence of Steiner symmetrizations to A_k about the hyperplanes $\mathcal{H}_1, ..., \mathcal{H}_m$. Since A_k converges to B_0 it follows from Theorem 4.3.1 that A_k' converges to B_0'. But $B_0' \subset (r_0 - \epsilon)B$ and hence there must exist an $N \in \mathbb{N}$ such that $A_k' \subset (r_0 - \frac{1}{2}\epsilon)B$ for all $k \geq N$. This implies that

$$A_{N+1}' \subset (r_0 - \frac{1}{2}\epsilon)B. \tag{4.3.4}$$

Since A_{N+1} (and therefore A_{N+1}') is contained in $\mathcal{S}(A)$, inclusion (4.3.4) contradicts the definition of r_0. Hence $B_0 = r_0 B$. □

4.4 Brunn's Theorem and the First Part of the Blaschke-Santaló Inequality

Theorem 4.3.3 is used in this section to prove the following two fundamental results. Theorem 4.4.2, due to Bertrand[3] for dimension 2 and Brunn[6] higher dimensions, establishes a characterization of ellipsoids which is used in chapter 5 to prove the second part of the Blaschke-Santaló inequality, whereas Theorem 4.4.7, due to M. Meyer and A. Pajor [11], makes direct use of Theorem 4.3.3 to prove the first part of the Blaschke-Santaló inequality.

Lemma 4.4.1. *Let K be a fixed convex body contained in the Minkowski space (X, B) and let $\{T_i\}_{i=1}^{\infty}$ be a sequence of linear transformations such that the sequence $\{T(K_i)\}_{i=1}^{\infty} \subset \mathcal{C}_s$ converges with respect to the Hausdorff metric δ to some $\hat{K} \in \mathcal{C}$. Then there exists a linear transformation T such that $\hat{K} = T(K)$.*

Proof. Since $K \in C_s$, there exists a scalar $r_1 > 0$ such that $r_1 B \subset K$. Also, since $\hat{K}$ is in $\mathcal{C}$, it is compact and hence bounded, therefore there exists a scalar $r_2 > 0$ such that $\hat{K} \subset r_2 E$.
Let $K_i := T_i(K)$ for all $i \in \mathbb{N}$ and let $\epsilon > 0$ be given. Then there exists an $N \in \mathbb{N}$ such that $K_i \subset \hat{K} + \epsilon B \subset (r_2 + \epsilon)B$ for all $i \geq N$ and thus

$$r_1 T(B) = T_i(r_1 B) \subset T_i K = K_i \subset (r_2 + \epsilon)B.$$

From this it follows that $\|T_i\| \leq \frac{1}{r_1}(r_2 + \epsilon)$ for all $i \geq N$. Let $k = \max\{T_1(E), T_2(E), ..., \frac{1}{r_1}(r_2 + \epsilon)\}$. Then $\|T_i\| \leq k$ for all $i \in \mathbb{N}$. In other words the sequence $\{T_i\}$ is bounded and, since the set of all linear transformations on $\mathbb{R}^n$ is a finite dimensional space, it follows that this sequence has a subsequence $\{T_{i_k}\}$ which converges in norm to some linear transformation T. □

Theorem 4.4.2 (Bertrand/Brunn). *Let C be a centrally symmetric convex body in $\mathbb{R}^n$ with following property:*

P1: For every vector $v \neq 0$, the centers of all the cross-sections of C by lines parallel to v lie in a hyperplane $\mathcal{M}$.

Then C is an ellipsoid.

Proof. Let $v \neq 0$ be an arbitrary vector in $\mathbb{R}^n$ and let $\mathcal{M}$ be the associated hyperplane containing the midpoints of all cross-sections of C parallel to v. Since $v \notin \mathcal{M}$, the set span$\{\mathcal{M}, v\}$ spans $\mathbb{R}^n$ and hence every $x \in C$ can be expressed as $x = m + \gamma v$ where $m \in \mathcal{M}$ and $\gamma \in \mathbb{R}$. In this case, property P1 is equivalent to the statement '$m + \gamma v \in C \Leftrightarrow m - \gamma v \in C$'.

If P1 holds for some symmetric convex body C, then P1 also holds for the image $T(C)$ of C under any invertible linear transformation T. Indeed, suppose P1 holds for $C \in \mathcal{C}_s$. Let $v \neq 0$ be an arbitrary vector and let $\mathcal{M}$ be the hyperplane containing the midpoints of all chords through C which are parallel to $T^{-1}(v) \neq 0$. The symmetry of C readily implies that $0 \in \mathcal{M}$ and hence $T(\mathcal{M})$ is also a hyperplane. Furthermore,

$$m + \gamma T^{-1}(v) \in C \Leftrightarrow m - \gamma T^{-1}(v) \in C.$$

and therefore

$$T(m) + \gamma v \in T(C) \Leftrightarrow T(m) - \gamma v \in T(C).$$

$T(\mathcal{M})$ is thus the desired hyperplane corresponding to v.

It will now be shown that for any symmetric convex body C satisfying P1, the computation of the Steiner symmetral of C with respect to any hyperplane $\mathcal{H}$ can be represented by means of an invertible linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$. Let $\mathcal{H}$ be an arbitrary hyperplane with unit normal v and let $\mathcal{M}$ be the corresponding hyperplane such that $m + \gamma v \in C \Leftrightarrow m - \gamma v \in C$ for all $m \in \mathcal{M}$. Any two points $x_1, x_2 \in C$ can be written as $x_i = m_i + \gamma_i v$, where $m_1, m_2 \in \mathcal{M}$ and $\gamma_i \in \mathbb{R}$ for $i = 1, 2$. Furthermore, m_1, m_2 can be written as $m_i = h_i + \theta_i v$, where $\theta_i \in \mathbb{R}$, $h_i \in \mathcal{H}$ and hence $x_i = h_i + (\theta_i + \gamma_i)v$. Since $v \perp \mathcal{H}$, the Steiner symmetral of C about $\mathcal{H}$ can now be computed by means of the mapping

$$x_i = h_i + (\theta_i + \gamma_i)v \mapsto T(x_i) := h_i + \gamma_i v \quad \text{for } i = 1, 2.$$

It readily follows that $T(x_1 + x_2) = T(x_1) + T(x_2)$ and $T(\alpha x_1) = \alpha T(x_1)$ for all $\alpha \in \mathbb{R}$. Moreover, the inverse of T is given by $T^{-1}(h_1 + \gamma_1 v) := h_1 + (\xi_1 + \theta_1)v = x_1$ for every $x_1 \in C$.

According to Theorem 4.3.3, it is possible to obtain a sequence $\{C_i\}$ of Steiner symmetrals converging to the Euclidean ball B_0, by making successive symmetrizations about the appropriate hyperplanes $\mathcal{H}_1, \mathcal{H}_2, ...$ corresponding to direction vectors $v_1, v_2, ...$ Since these Steiner symmetrizations can be represented by invertible linear transformations, the sequence $\{C_i\}_{i=1}^{\infty}$ can thus be written in the form $\{T_i(C)\}_{i=1}^{\infty}$, where $T_i : \mathbb{R}^n \to \mathbb{R}^n$ is a linear isomorphism for $i \in \mathbb{N}$. Lemma 4.4.1 now implies that there must be some linear mapping T such that $B_0 = T(C)$. In addition, since Steiner symmetrizations preserve volume,

$$\lambda(B_0) = \lambda(T(C)) = |\det(T)|\lambda(C) = \lambda(C) \Rightarrow \det(T) \neq 0$$

and hence T is invertible. Therefore, $C = T^{-1}(B_0)$ from which it follows that C is an ellipsoid. $\square$

A set $A \subset \mathbb{R}^n$ is said to have a center of symmetry $y \in A$ if for every $x \in A$, $2y - x \in A$. The proof of the second part of the Blaschke-Santaló inequality, outlined in Chapter 5, relies mainly on showing that any symmetric convex body C at which the upper bound of the volume product is attained, satisfies the following property:

For any hyperplane $\mathcal{H} \in \mathbb{R}^n$ containing the origin, every cross-section of C by hyperplanes parallel to $\mathcal{H}$ has a center of symmetry and these centers lie in a line

Although this property does not correspond directly to the hypothesis used in Brunn's Theorem, it will be shown in Theorem 4.4.3 and the subsequent corollary that it nevertheless suffices to guarantee that C is an ellipsoid.

Theorem 4.4.3 (Meyer and Pajor). *Let $C \subset \mathbb{R}^n$ be a convex symmetric body and $\mathcal{H}$ be a hyperplane containing the origin. Then the following two statements are equivalent*

a) The centers of all the cross-sections of C^o by lines orthogonal to $\mathcal{H}$ lie on a hyperplane $\mathcal{M}$.

b) Every cross-section of C by a hyperplane parallel to $\mathcal{H}$ has a center of symmetry and these centers if symmetry are in line.

Proof. Let v and u be the unit normal vectors of $\mathcal{M}$ and $\mathcal{H}$ respectively. According to Remark 2.5.9 and a corollary to the Hahn-Banach Theorem (*see for instance Theorem 4.3, Rudin [14]*),

$$C^o = \{y \in \mathbb{R}^n : \langle x, y\rangle \leq 1 \text{ for all } x \in C\}$$

and

$$C = \{x \in \mathbb{R}^n : \langle x, y\rangle \leq 1 \text{ for all } y \in C^o\}.$$

Suppose (a) holds. Evidently u cannot lie in $\mathcal{M}$ and hence every $y \in \mathbb{R}^n$, particularly every $y \in C^o$, can be written in the form $y = m + \gamma u$, where $m \in \mathcal{M}$ and $\gamma \in \mathbb{R}$. Moreover, hypothesis (a) together with the symmetry of C^o imply that $m + \gamma u \in C^o \Leftrightarrow -m + \gamma u \in C^o$. Since $v \notin \mathcal{H}$, any hyperplane $\mathcal{H}_\lambda$ parallel to $\mathcal{H}$ is of the form $\mathcal{H}_\lambda = \{x \in \mathbb{R}^n : \langle x - \lambda v, u\rangle = 0\}$, from which it follows that $\langle x, u\rangle = \lambda\langle v, u\rangle$ for all $x \in \mathcal{H}_\lambda$. For any $x \in C \cap \mathcal{H}_\lambda$ and any $y = m + \gamma u \in C^o$ it thus follows that,

$$\langle(2\lambda v - x) - \lambda v, u\rangle = \langle\lambda v - x, u\rangle = -\langle x - \lambda v, u\rangle = 0 \quad \because x \in \mathcal{H}_\lambda$$

and

$$\begin{aligned}\langle 2\lambda v - x, m + \gamma u\rangle &= 2\lambda\langle v, m\rangle + 2\lambda\gamma\langle v, u\rangle - \langle x, m + \gamma u\rangle \\ &= 0 + \langle x, 2\gamma u\rangle + \langle x, -m - \gamma u\rangle && \because v \perp m \text{ and } \langle x, u\rangle = \lambda\langle v, u\rangle \\ &= \langle x, -m + \gamma u\rangle \leq 1 && \because x \in C\end{aligned}$$

Hence $2\lambda v - x \in C \cap \mathcal{H}_\lambda$ whenever $x \in C \cap \mathcal{H}_\lambda$, from which it follows that λv is the center of symmetry for each cross-section $C \cap \mathcal{H}_\lambda$.
Since $C = \{x \in \mathbb{R}^n : \langle x, y\rangle \leq 1 \text{ for all } x \in C^o\}$, the above arguments can easily be reversed to show that the converse holds. $\square$

Corollary 4.4.4. *Suppose $C \subset \mathbb{R}^n$ is a symmetric convex body such that for any hyperplane $\mathcal{H} \in \mathbb{R}^n$ containing the origin, every cross-section of C by hyperplanes parallel to $\mathcal{H}$ has a center of symmetry and these centers lie in a line. Then C is an ellipsoid.*

Proof. It follows directly from Theorem 4.4.3 that, under these conditions, C^o satisfies the hypothesis of Brunn's Theorem, which in turn implies that C^o is an ellipsoid. By the Riesz representation Theorem for linear functionals on Hilbert spaces, the bidual space $(\mathbb{R}^n, (C^o)^o)$ is isometrically isomorphic to $(\mathbb{R}^n, C^o)$ and must therefore also be a Hilbert space. The reflexivity of Minkowski Spaces now guarantees that $(\mathbb{R}^n, C)$ is a Hilbert space or equivalently that C is an ellipsoid. $\square$

Theorem 4.4.7, due Meyer and Pajor [11], provides a short proof of the first part of the Blaschke-Santaló inequality. Lemma 4.4.6 is partly based on the Brunn-Minkowski inequality (*see [5],[6] or Theorem 6.1.1 Schneider [17]*), which is given below without proof, and establishes the fact that the volume product is non-decreasing under Steiner symmetrizations. Theorem 4.4.7 then invokes Theorem 4.3.3 and the continuity of the volume product to show that the upper bound of the set $\{P(C) : C \in \mathcal{C}_s\}$ is attained by ellipsoids.

Theorem 4.4.5 (The Brunn-Minkowski Theorem). *Let A and B be convex bodies in $\mathbb{R}^n$, λ the n-dimensional Lesbesgue measure and $\mu \in [0,1]$. Then*

$$\lambda(\mu A + (1-\mu)B)^{\frac{1}{n}} \geq \mu\lambda(A)^{\frac{1}{n}} + (1-\mu)\lambda(B)^{\frac{1}{n}}$$

and this inequality is strict unless $A = kB + t$, where $k \in \mathbb{R}$ and $t \in \mathbb{R}^n$.

Lemma 4.4.6 (M. Meyer, A. Pajor). *Let $C_{\mathcal{H}}$ be the Steiner symmetral of a centrally symmetric convex body $C \subset \mathbb{R}^n$ about a hyperplane $\mathcal{H}$ containing the origin. Then*

$$\lambda^n((C_{\mathcal{H}})^o) \geq \lambda^n(C^o) \quad \textit{and hence } P(C) \geq P(C_{\mathcal{H}}) \tag{4.4.1}$$

Proof. By applying a suitable volume preserving map , we may assume without loss of generality that $\mathcal{H} = \{(x_1, ..., x_n) \in \mathbb{R}^n : x_n = 0\}$. Moreover, we make the identification $\mathbb{R}^n \equiv \mathcal{H} \times \mathbb{R}$. Then

$$\begin{aligned}
C^o :=& \{(Y,y) \in \mathcal{H} \times \mathbb{R} : \langle Y, X\rangle + yx \leq 1 \text{ for all } X \in \mathcal{H}(C) \\
& \text{and } x \text{ such that } (X,x) \in C\} \\
C_{\mathcal{H}} :=& \{(X,x) \in \mathcal{H} \times \mathbb{R} : X \in \mathcal{H}(C), x = \frac{1}{2}(x_1 - x_2) \\
& \text{where } (X, x_i) \in C \text{ for } i = 1,2\} \\
(C_{\mathcal{H}})^o :=& \{(Y,y) \in \mathcal{H} \times \mathbb{R} : \langle Y, X\rangle + \frac{1}{2}y(x_1 - x_2) \leq 1 \ \forall X \in \mathcal{H}(C) \\
& \text{and } x|\ (X, x_i) \in C, i = 1,2\}
\end{aligned}$$

For any $y \in \mathbb{R}$, let $(C^o)_y := \{Y \in \mathcal{H} : (Y,y) \in C\}$ denote the y-section of C^o. It can directly be seen that $\frac{1}{2}((C^o)_y + (C^o)_{-y}) \subset ((C_{\mathcal{H}})^o)_y$ for all $y \in \mathbb{R}$. Therefore

$$\lambda^{n-1}[((C_{\mathcal{H}})^o)_y] \geq \left(\lambda^{n-1}[\frac{1}{2}(C^o)_y + \frac{1}{2}(C^o)_{-y}]\right) \tag{4.4.2}$$

Also, since C is centrally symmetric, $(C^o)_y = -(C^o)_{-y}$. Hence, according to the Brunn Minkowski inequality,

$$\begin{aligned}
\left(\lambda^{n-1}[\frac{1}{2}(C^o)_y + \frac{1}{2}(C^o)_{-y}]\right)^{\frac{1}{n-1}} &\geq \frac{1}{2}\left(\lambda^{n-1}[(C^o)_y]\right)^{\frac{1}{n-1}} + \frac{1}{2}\left(\lambda^{n-1}[(C^o)_{-y}]\right)^{\frac{1}{n-1}} \\
&= \frac{1}{2}\left(\lambda^{n-1}[(C^o)_y]\right)^{\frac{1}{n-1}} + \frac{1}{2}\left(\lambda^{n-1}[-(C^o)_y]\right)^{\frac{1}{n-1}} \\
&= \left(\lambda^{n-1}[(C^o)_y]\right)^{\frac{1}{n-1}}
\end{aligned}$$

Thus

$$\lambda^{n-1}[(C^o)_y] \leq \lambda^{n-1}[\frac{1}{2}(C^o)_y + \frac{1}{2}(C^o)_{-y}] \leq \lambda^{n-1}[((C_{\mathcal{H}})^o)_y] \quad \text{for all } y \in \mathbb{R}$$

Hence, by Fubini's Theorem,

$$\lambda^n[C^o] = \int_{\mathbb{R}} \lambda^{n-1}[(C^o)_y]dy \leq \int_{\mathbb{R}} \lambda^{n-1}[((C_{\mathcal{H}})^o)_y]dy = \lambda^n[(C_{\mathcal{H}})^o] \tag{4.4.3}$$

Theorem 4.2.3 asserts that $\lambda^n(C) = \lambda^n(C_{\mathcal{H}})$, from which it follows that $P(C) \leq P(C_{\mathcal{H}})$. $\square$

Theorem 4.4.7 (M. Meyer, A. Pajor). *For any $C \in \mathcal{C}_s$,*

$$P(C) \leq P(E) \quad \textit{where } E \textit{ is an ellipsoid}$$

Proof. According to Theorem 4.3.3, there exists a sequence $\{C_n\}$ of successive Steiner symmetrals of C (i.e. $C_n = (C_{n-1})_{\mathcal{H}}$ for all $n \in \mathbb{N}$) converging to the Euclidean ball B_0 with respect to the Hausdorff metric. The previous Lemma now implies that the corresponding sequence $\{P(C_n)\}$ of Volume Products is increasing and, by the continuity of the volume product (Theorem 3.3.6), it follows that

$$P(C) \leq \lim_{n\to\infty} P(C_n) = P(B_0).$$

Furthermore, since ellipsoids are simply defined as the images of the Euclidean ball under linear isomorphisms, it follows from Proposition 2.5.6 that $P(E) = P(B_0)$ for any ellipsoid $E \subset \mathbb{R}^n$ and hence the upper bound of $\{P(C) : C \in \mathcal{C}_s\}$ is attained by all ellipsoids. $\square$

Chapter 5

Saint Raymond's Proof of the Blaschke-Santaló Inequality

5.1 Introduction

Let $\mathcal{H} \subset \mathbb{R}^{n+1}$ be an n-dimensional hyperplane containing the origin and identify $\mathcal{H} \times \mathbb{R}$ with $\mathbb{R}^{n+1}$. For any $C \subset \mathbb{R}^{n+1}$ in $\mathcal{C}_s$, let the t-section C_t of C be given by:

$$C_t = C \cap \mathcal{H} = \{x \in \mathbb{R}^n : (x, t) \in C\} \text{ for any } t \in \mathbb{R}.$$

Furthermore, identify the dual space $(\mathbb{R}^{n+1})^* = (\mathcal{H} \times \mathbb{R})^*$ with $\mathcal{H} \times \mathbb{R}$ by means of the linear isomorphism $e_i^* \mapsto e_i$. The action of a given vector $(\xi, \gamma) \in (\mathcal{H} \times \mathbb{R})^*$ on an arbitrary point $(x, t) \in \mathcal{H} \times \mathbb{R}$ is thus given by

$$(\xi, \gamma)(x, t) = \langle (\xi, \gamma), (x, t) \rangle = \xi(x) + \gamma t.$$

where $\langle ., . \rangle$ denotes the Euclidean inner product. Chapter 5 discusses Saint Raymond's proof of the Blaschke-Santaló inequality for centrally symmetric convex bodies [15]. This proof relies mainly on the construction described in Lemma 5.3.1. Indeed, it will be shown that for any $C \in \mathcal{C}_s$ there exists a convex symmetric body C' whose volume product is strictly greater than that of C, unless C satisfies the hypothesis of Corollary 4.4.4, from which it follows that C must be an ellipsoid. This not only proves that the upper bound of the set $\{P(C) : C \in \mathcal{C}_s\}$ is attained by ellipsoids, but also that ellipsoids are the only sets in $\mathcal{C}_s$ with this property. The set C' is defined in terms of the t-sections C_t of C. Accordingly, section 6.2 introduces the preliminary properties of the t-sections C_t of C and their associated polar bodies $(C_t)^o$, which are necessary to prove that C' is a symmetric convex body and to find an expression for the volume product of C' in terms of that of C. The proof the Blaschke-Santaló inequality is finally presented in Section 5.3.

5.2 Sections of Centrally Symmetric Convex Bodies

5.2.1 Preliminary Results

The following elementary properties relating to t-sections C_t of the convex body C result directly from the corresponding properties of C.

Lemma 5.2.1. *Let $C \subset \mathbb{R}^{n+1} = \mathcal{H} \times \mathbb{R}$ be a convex set. Then $C_t \subset \mathcal{H}$ is also convex.*

Proof. Suppose $x_1, x_2 \in C_t$ for some $t \in \mathbb{R}$. Then $(x_1, t) \in C$ and $(x_2, t) \in C$. For any $\lambda \in [0, 1]$:

$$\begin{aligned} &(\lambda x_1 + (1-\lambda)x_2, t) \\ =&(\lambda x_1 + (1-\lambda)x_2, \lambda t + (1-\lambda)t) \\ =&\lambda(x_1, t) + (1-\lambda)(x_2, t) \in C \qquad \because \text{C convex} \end{aligned}$$

Therefore $\lambda x_1 + (1-\lambda)x_2 \in C_t$ and hence C_t is convex. □

Lemma 5.2.2. *Let $C \subset \mathcal{H} \times \mathbb{R}$ be a symmetric set. Then $-C_t = C_{-t} \quad \forall t \in \mathbb{R}$.*

Proof.

$$\begin{aligned} & x \in -C_t \Leftrightarrow (-x, t) \in C \\ \Leftrightarrow\ & (x, -t) = -(-x, t) \in C \quad \because \text{symmetry} \\ \Leftrightarrow\ & x \in C_{-t} \end{aligned}$$

□

Lemma 5.2.3. *Let $C \subset \mathcal{H} \times \mathbb{R}$ be a convex body. Then $I := \{t \in \mathbb{R} : C_t \neq \emptyset\}$ is a closed bounded interval in $\mathbb{R}$*

Proof. Let $t_1, t_2 \in I$ and suppose x_1 and x_2 are elements of C_{t_1} and C_{t_2} respectively. This implies $(x_1, t_1), (x_2, t_2) \in C$. The convexity of C now implies that for all $\lambda \in [0, 1]$:

$$(\lambda x_1 + (1-\lambda)x_2, \lambda t_1 + (1-\lambda)t_2) = \lambda(x_1, t_1) + (1-\lambda)(x_2, t_2) \in C$$

and hence $C_{\lambda t_1 + (1-\lambda)t_2} \neq \emptyset \quad \forall \lambda \in [0, 1]$. Therefore I is an interval.
In order to show that I is a closed subset of $\mathbb{R}$, let $\{t_n\}$ be any sequence in I converging to some $t \in \mathbb{R}$. Since $C_{t_n} \neq \emptyset \ \forall n \in \mathbb{N}$, there exists an $x_n \in \mathcal{H}$ such that $(x_n, t_n) \in C \ \forall n \in \mathbb{N}$. Moreover, the compactness of C guarantees that the sequence $\{(x_n, t_n)\}$ has a subsequence which converges to some point $(x_0, t_0) \in C$. Since $t_n \to t$, it follows (by uniqueness of limits) that $t = t_0 \Rightarrow (x_0, t) \in C \Rightarrow C_t \neq \emptyset \Rightarrow t \in I$ and hence I is closed. The fact that I is bounded follows directly from the bondedness of C. □

Lemma 5.2.4. *Any set $C \in \mathcal{H} \times \mathbb{R} = \mathbb{R}^{n+1}$ is convex if and only if*

$$C_{\lambda t_1+(1-\lambda)t_2} \supseteq \lambda C_{t_1} + (1-\lambda)C_{t_2} \tag{5.2.1}$$

for any $t_1, t_2 \in I = \{t \in \mathbb{R} : C_t \neq \emptyset\}, \lambda \in [0,1]$.

Proof. Suppose the set C is convex and

$$x \in \lambda C_{t_1} + (1-\lambda)C_{t_2}$$

for some $\lambda \in [0,1]$. Then $x = \lambda x_1 + (1-\lambda)x_2$ where $(x_1,t_1),(x_2,t_2) \in C$. Since C is convex:

$$\begin{aligned}(x, \lambda t_1 + (1-\lambda)t_2) &= (\lambda x_1 + (1-\lambda)x_2, \lambda t_1 + (1-\lambda)t_2)\\ &= \lambda(x_1,t_1) + (1-\lambda)(x_2,t_2) \in C\end{aligned}$$

Therefore

$$x \in C_{\lambda t_1+(1-\lambda)t_2} \Rightarrow C_{\lambda t_1+(1-\lambda)t_2} \supseteq \lambda C_{t_1} + (1-\lambda)C_{t_2}$$

Conversely, suppose the inclusion (5.2.1) holds and let $(x_1,t_1),(x_1,t_2) \in C$. Then $x_1 \in C_{t_1}$ and $x_2 \in C_{t_2}$. For any $\lambda \in [0,1]$:

$$\lambda x_1 + (1-\lambda)x_2 \in \lambda C_{t_1} + (1-\lambda)C_{t_2} \subseteq C_{\lambda t_1+(1-\lambda)t_2}$$

Therefore

$$\lambda(x_1,t_1) + (1-\lambda)(x_2,t_2) = (\lambda x_1 + (1-\lambda)x_2, \lambda t_1 + (1-\lambda)t_2,) \in C$$

and hence C is convex. □

5.2.2 Sections of Polar Bodies

Sections of a polar body are conveniently described in terms of the body's Minkowski functional.

Definition 5.2.5. *For every $t \in \mathbb{R}$, let p_t denote the Minkoswki functional of the polar body $(C_t)^o$.In other words*

$$p_t(\xi) := \begin{cases} \sup\{\xi(x) : x \in C_t\}, & \text{if } C_t \neq \emptyset; \\ -\infty, & \text{if } C_t = \emptyset. \end{cases} \qquad \forall \xi \in (\mathbb{R}^n)^* = \mathbb{R}^n$$

Lemma 5.2.6. .

(i) For a given $\xi \in (\mathbb{R}^n)^*$, the mapping $t \mapsto p_t(\xi)$ is concave and continuous on I.

(ii) For a given $t \in I$, the mapping $\xi \mapsto p_t(\xi)$ is convex and continuous on $(\mathbb{R}^n)^*$.

Proof. .

The mapping $t \mapsto p_t(\xi)$:

Let $\xi \in (\mathbb{R}^n)^*$, $\lambda \in [0,1]$ and $t_1, t_2 \in I$ be given. Now

$$\begin{aligned} p_{\lambda t_1+(1-\lambda)t_2} &= \sup\{\xi(x) : x \in C_{\lambda t_1+(1-\lambda)t_2}\} \\ &\geq \sup\{\xi(x) : x \in \lambda C_{t_1} + (1-\lambda)C_{t_2}\} \quad \because \text{Section 5.2.1} \\ &= \lambda \sup\{\xi(x) : x \in C_{t_1}\} + (1-\lambda)\sup\{\xi(x) : x \in C_{t_2}\} \\ &= \lambda p_{t_1}(\xi) + (1-\lambda)p_{t_2}(\xi) \end{aligned}$$

and hence p_t is concave. Since p_t is concave and finite on the interval I it follows from Corollary A.1.4 that p_t is continuous on int$I$. It only remains to be shown that p_t is continuous at the endpoints of I. Let t_0 be the right endpoint of I and let $\{t_i\}_{i=1}^{\infty}$ be a sequence in I converging to $t_0 \in I$. Since I is convex and contains 0, any t_i can be expressed as $t_i = \lambda_i t_0$ where $\lambda_i \in [0,1]$. The set C_{t_i} is compact for every $t_i \in I$ and since ξ is a continuous functional, there exists an $x_i \in C_{t_i}$ such that $p_{t_i}(\xi) = \xi(x_i)$ for $i \in \mathbb{N}$. The set C is compact and hence the sequence $\{(x_i, t_i)\}$ has a subsequence $\{(x_{i_k}, t_{i_k})\}$ converging to some point $(x, t_0) \in C$. Let x_0 be the point in C_{t_0} at which ξ attains its maximum. Now $x \in C_{t_0}$ and hence $\xi(x) \leq \xi(x_0)$. On the other hand, the concavity of p_t implies that for every $k \in \mathbb{N}$, $\xi(x_{\lambda_{i_k} t_0}) \geq \lambda_{i_k}\xi(x_0)$. Taking limits as $k \to \infty$, yields $\xi(x) \geq \xi(x_0)$ and hence $p_{t_0}(\xi) = \lim_{t\to t_0} p_t(\xi)$. The mapping $t \mapsto p_t(\xi)$ is therefore continuous at t_0. A similar line of reasoning can be used to prove the continuity of p_t at the left endpoint of I.

The mapping $\xi \mapsto p_t(\xi)$:

Let $t \in I$, $\xi_1, \xi_2 \in C_t$ and $\lambda \in [0,1]$ be given. By definition, $p_t(\xi_1) \geq \xi_1(x)$ and $p_t(\xi_2) \geq \xi_2(x)$ for all $x \in C_t$. Thus

$$\lambda p_t(\xi_1) + (1-\lambda)p_t(\xi_2) \geq \lambda\xi_1(x) + (1-\lambda)\xi_2(x) \quad \text{for all} \quad x \in C_t$$

and hence $\lambda p_t(\xi_1) + (1-\lambda)p_t(\xi_2)$ is an upper bound of the set $\{\lambda\xi_1(x) + (1-\lambda)\xi_2(x) : x \in C_t\}$. But $p_t(\lambda\xi_1 + (1-\lambda)\xi_2)$ is the least upper bound and therefore

$$p_t(\lambda\xi_1 + (1-\lambda)\xi_2) \leq \lambda p_t(\xi_1) + (1-\lambda)p_t(\xi_2)$$

Note that $p_t(\xi) < \infty$ for any linear functional in $(\mathbb{R}^n)^*$, since C_t is compact. Thus int$D(p_t(.)) = D(p_t(.)) = (\mathbb{R}^n)^*$. Corollary A.1.3 now implies that $p_t(.)$ must be continuous at all $\xi \in (\mathbb{R}^n)^*$.

□

Lemma 5.2.7. *Let $\mathcal{H}$ be an n-dimensional subspace of $\mathbb{R}^{n+1}$ and let $C \subset \mathcal{H}\times\mathbb{R}$ a centrally symmetric convex body. Also denote the Minkowski functional of $(C_t)^o$ by p_t. Then:*

1. *For any* $\gamma \in \mathbb{R}$, *the* γ*-section* $(C^o)_\gamma := \{\xi \in \mathcal{H} : (\xi, \gamma) \in C^o\}$ *of* C^o, *satisfies* $(C^o)_\gamma \subseteq (C_0)^o$.
2. *For any* $\xi \in (C_0)^o$, *the* ξ*-section* $(C^o)_\xi := \{\lambda \in \mathbb{R} : (\xi, \lambda) \in C^o\}$ *of* C^o *is an interval in* $\mathbb{R}$ *of length* $r(\xi) + r(-\xi)$ *where* $r(\xi) = \inf\{\frac{1-p_t(\xi)}{t} : t > 0\}$

Proof. .
The first assertion can be easily proved. Let $\xi \in (C^o)_\gamma$ for some $\gamma \in \mathbb{R}$. This implies $(\xi, \gamma) \in C^o$ and hence

$$\xi(x) + \gamma t \leq 1 \quad \text{for all } (x, t) \in C$$

In particular, for any $x \in C_0$ it follows that

$$(x, 0) \in C \Rightarrow \xi(x) \leq 1 \Rightarrow \xi \in (C_0)^o$$

and therefore $(C^o)_\gamma \subseteq (C_0)^o$ for all $\gamma \in \mathbb{R}$.

In order to prove the second assertion, it is necessary to make a construction.

Construction

If $\xi \in (C_0)^o$ then

$$\xi(x) \leq 1, \forall x \in C_0 \Rightarrow p_0(\xi) := \sup\{\xi(x) : x \in C_0\} \leq 1$$

Also, the function $t \mapsto p_t(\xi)$ is concave and continuous for any fixed $\xi \in \mathcal{H}^*$. According to lemma A.2.3, both the right and the left derivatives exist at $t = 0 \in \text{int}I$ and $(p_0(\xi))_+ \leq (p_0(\xi))_-$. Moreover, it follows from lemma A.2.3 that

$$\begin{aligned} & \frac{p_t(\xi) - p_0(\xi)}{t} \leq (p_0(\xi))_+ \quad \forall t > 0 \\ \text{and} \quad & \frac{p_t(\xi) - p_0(\xi)}{t} \geq (p_0(\xi))_- \quad \forall t < 0 \end{aligned}$$

Let α be any element of the interval $[(p_0(\xi))_+, (p_0(\xi))_-]$. Then

$$\begin{aligned} \frac{p_t(\xi) - p_0(\xi)}{t} \leq (p_0(\xi))_+ \leq \alpha \Rightarrow p_t(\xi) - \alpha t \leq p_0(\xi) \quad \text{for all } t > 0 \\ \frac{p_t(\xi) - p_0(\xi)}{t} \geq (p_0(\xi))_- \geq \alpha \Rightarrow p_t(\xi) - \alpha t \leq p_0(\xi) \quad \text{for all } t < 0 \end{aligned}$$

Now define $q(t) := p_t(\xi) - \alpha t$. The concavity and continuity of $p_t(\xi)$ immediately imply that q is concave and continuous. Also

$$q(t) \leq q(0) \leq 1 \quad \text{for all } t \in \mathbb{R}.$$

It also follows directly from the definition of C^o that any real number γ is an element of $(C^o)_\xi$ if and only if

$$\forall t, \forall x \in C_t \quad \xi(x) + \gamma t \leq 1$$

This condition is equivalent to:

$$\begin{aligned} & p_t(\xi) + \gamma t \leq 1 && \forall t \in \mathbb{R} \\ \Leftrightarrow \;& \alpha t + q(t) + \gamma t \leq 1 && \forall t \in \mathbb{R} \\ \Leftrightarrow \;& (\alpha + \gamma)t \leq 1 - q(t) && \forall t \in \mathbb{R} \end{aligned} \tag{5.2.2}$$

It can easily be seen that if $\gamma_1, \gamma_2 \in \mathbb{R}$ are two numbers satisfying condition (5.2.2), then any convex combination of γ_1 and γ_2 will also satisfy (5.2.2) and hence $(C^o)_\xi$ is an interval in $\mathbb{R}$.
In addition, $-\alpha \in (C^o)_\xi$, since $(\alpha - \alpha)t = 0 \leq 1 - q(t)$ for all $t \in \mathbb{R}$.

The upper bound γ_0 of $(C^o)_\xi$

Define $\gamma_0 := \sup(C^o)_\xi$. Consider the mapping $t \mapsto (\gamma + \alpha)t$ where $\gamma \in (C^o)_\xi$. This function represents a line ℓ_1 through the origin which, according to inequality (5.2.2), is bounded above by the concave function $t \mapsto 1 - q(t)$. Note that $(\gamma_0 + \alpha)$ is defined as the supremum of all the slopes $(\gamma + \alpha)$ such that $(\gamma + \alpha)t \leq 1 - q(t)$ $\forall t \in \mathbb{R}$. Since $(\gamma + \alpha) \geq 0$ for all $\gamma \geq -\alpha$, it follows that the slope of ℓ_1 is non-negative and hence $(\gamma + \alpha)t \leq 0 \leq 1 - q(t)$ $\forall t \leq 0$. In order to find $(\gamma_o + \alpha)$, it thus suffices to consider only the case when $t > 0$. Therefore:

$$\begin{aligned} (\gamma_0 + \alpha) &= \sup\{(\gamma + \alpha) : (\gamma + \alpha)t \leq 1 - q(t) \ \forall t > 0, \gamma \in (C^o)_\xi\} \\ &= \inf_{t>0} \frac{1 - q(t)}{t} \\ \Rightarrow \gamma_0 &= \inf_{t>0} \frac{1 - q(t)}{t} - \alpha = \inf_{t>0} \frac{1 - q(t) - \alpha t}{t} \\ &= \inf_{t>0} \frac{1 - p_t(\xi)}{t} =: r(\xi) \end{aligned}$$

The lower bound γ_1 of $(C^o)_\xi$

Similarly, for all $\gamma \leq -\alpha$, the function $t \mapsto= (\gamma + \alpha)t$ represents a line through the origin with negative slope. Let $\gamma_1 := \inf(C^o)_\xi$.

Since $(\gamma+\alpha)t \leq 0 \leq 1-q(t)\ \forall t \geq 0$, it follows that

$$\begin{aligned}
\gamma_1+\alpha &= \inf\{\gamma+\alpha : (\gamma+\alpha)t \leq 1-q(t), \forall t<0\} \\
&= \inf\{\gamma+\alpha : (\gamma+\alpha) \geq \frac{1-q(t)}{t}, \forall t<0\} \\
&= \sup_{t<0} \frac{1-q(t)}{t} \\
\Rightarrow \gamma_1 &= \sup_{t<0} \frac{1-q(t)-\alpha t}{t} = \sup_{t<0} \frac{1-p_t(\xi)+\alpha t-\alpha t}{t} \\
&= \sup_{t<0} \frac{1-p_t(\xi)}{t} = -\inf_{-t>0} \frac{1-p_{-t}(-\xi)}{-t} =: -r(-\xi)
\end{aligned}$$

The length of the interval $(C^o)_\xi$ is thus given by $\gamma_0-\gamma_1 = r(\xi)+r(-\xi)$ for all $\xi \in (C_0)^o$. $\square$

5.3 Saint Raymond's proof of the Blaschke-Santaló Inequality for Convex Symmetric bodies

As was mentioned in the Introduction, this proof is essentially constructive. It will be shown that for every $C \in \mathcal{C}_s$ there exists a set $C' \in \mathcal{C}_s$ such that $P(C) < P(C')$ unless C satisfies the hypothesis of Corollary 4.4.4 from which it follows that C is an ellipsoid. The proof of Lemma 5.3.1 relies heavily on certain properties of concave functions, which are examined in Appendix A.

Lemma 5.3.1. *Let $\mathcal{H} \subset \mathbb{R}^n$ be an n-dimensional hyperplane containing the origin and let $C \subset H \times \mathbb{R}$ be a convex symmetric body with t-sections C_t. Define $C' \subset \mathcal{H} \times \mathbb{R}$ in terms of its t-sections as follows:*

$$C'_t := \frac{1}{2}(C_t - C_t) = \frac{1}{2}(C_t + C_{-t}) \qquad \because \textit{ by symmetry}$$

Then:

1. *C' is a convex symmetric body*
2. *The volume of C' is greater than that of C with equality occurring only if all t-sections C_t of C have a center of symmetry.*
3. *The volume of C'^o is greater than that of C^o with equality occurring only if for all $\xi \in \mathcal{H}^*$ there exists an $\alpha_\xi \in \mathbb{R}$ such that*

$$p_t(\xi) = t\alpha_\xi + p_t(-\xi) \qquad \forall t > 0$$

Proof. .
1. The proof that $C' \in \mathcal{C}_s$ is based on the results in section 6.2.1:

Convexity

Suppose $(x_1, t_1), (x_2, t_2) \in C'$ and $\alpha \in [0, 1]$. It now follows from Lemma 5.2.4 that

$$\begin{aligned}
& x_1 \in \frac{1}{2}C_{t_1} + \frac{1}{2}C_{-t_1} \quad \text{and} \quad x_2 \in \frac{1}{2}C_{t_2} + \frac{1}{2}C_{-t_2} \\
\Rightarrow\ & \alpha x_1 + (1-\alpha)x_2 \in \alpha(\frac{1}{2}C_{t_1} + \frac{1}{2}C_{-t_1}) + (1-\alpha)(\frac{1}{2}C_{t_2} + \frac{1}{2}C_{-t_2}) \\
=\ & \frac{1}{2}(\alpha C_{t_1} + (1-\alpha)C_{t_2}) + \frac{1}{2}(\alpha C_{-t_1} + (1-\alpha)C_{-t_2}) \\
\subseteq\ & \frac{1}{2}C_{\alpha t_1 + (1-\alpha)t_2} + \frac{1}{2}C_{-(\alpha t_1 + (1-\alpha)t_2)} \\
\Rightarrow\ & \alpha(x_1, t_1) + (1-\alpha)(x_2, t_2) \in C'
\end{aligned}$$

Symmetry

Suppose $(x,t) \in -C'$ then $x \in -\frac{1}{2}(C_t - C_t) = \frac{1}{2}(C_t - C_t) \Rightarrow (x,t) \in C'$ and hence C' is symmetrical.

Compactness

Let $\{(x_i,t_i)\}_{i=1}^{\infty}$ be a sequence in C'. According to the definition of C', any x_i can be written as:

$$x_i = \frac{1}{2}y_i + \frac{1}{2}z_i \quad \text{where } y_i \in C_{t_i} \text{ and } z_i \in C_{-t_i}$$

The sequences $\{(y_i,t_i)\}_{i=1}^{\infty}$ and $\{(z_i,-t_i)\}_{i=1}^{\infty}$ are both contained in C and must hence have subsequences converging to some points (y,t) and $(z,-t)$ in C respectively. The corresponding subsequence $\{t_{i_k}\}$ of $\{t_i\}$ must, by the continuity of the projection mapping and the uniqueness of limits converge to $t \in I$ whereas $\{x_{i_k}\}$ converges to $x = \frac{1}{2}y + \frac{1}{2}z$ where $y \in C_t$ and $z \in C_{-t}$. Therefore $(x,t) \in C'$ and C' is sequentially compact.

2. Let λ^{n+1} denote the $(n+1)$-dimensional Lebesgue measure and consider the volumes of C and C'. According to Fubini's Theorem

$$\lambda^{n+1}(C) = \int_I \lambda^n(C_t)dt \quad \text{and} \quad \lambda^{n+1}(C') = \int_I \lambda^n(C'_t)dt$$

where $I \subset \mathbb{R}$ is the interval on which $C_t \neq \emptyset$. It can readily be seen that the function $\psi : t \mapsto [\lambda^n(C_t)]^{\frac{1}{n}}$ is concave. Indeed, for any $t_1, t_2 \in I$ and any $\alpha \in [0,1]$

$$\begin{aligned}
\psi(\alpha t_1 + (1-\alpha)t_2) &= [\lambda^n(C_{\alpha t_1 + (1-\alpha)t_2})]^{\frac{1}{n}} \\
&\geq [\lambda^n(\alpha C_{t_1} + (1-\alpha)C_{t_2})]^{\frac{1}{n}} && \because \text{Lemma 5.2.4} \\
&\geq \alpha[\lambda^n(C_{t_1}]^{\frac{1}{n}} + (1-\alpha)[\lambda^n(C_{t_2})]^{\frac{1}{n}} && \because \text{Brunn-Minkowski theorem} \\
&= \alpha\psi(t_1) + (1-\alpha)\psi(t_2)
\end{aligned}$$

Since λ^n and hence ψ is finite for any $t \in \mathbb{R}$, Corollary A.1.4 guarantees that ψ is and hence the mapping $t \mapsto \lambda^n(C_t)$ is continuous on $\mathbb{R}$. Consider the comparison of the volume of an arbitrary t-section of C with the corresponding t-section of C', where $t \in I$. Let t be an arbitrary element of I. By the Brunn-Minkowski Theorem

$$\begin{aligned}
[\lambda^n(C'_t)]^{\frac{1}{n}} &= [\lambda^n(\frac{1}{2}(C_t - C_t))]^{\frac{1}{n}} \geq \frac{1}{2}[\lambda^n(C_t)]^{\frac{1}{n}} + \frac{1}{2}[\lambda^n(-C_t)]^{\frac{1}{n}} \\
&= \frac{1}{2}[\lambda^n(C_t)]^{\frac{1}{n}} + \frac{1}{2}|(-1)|\ [\lambda^n(C_t)]^{\frac{1}{n}} = [\lambda^n(C_t)]^{\frac{1}{n}}
\end{aligned} \tag{5.3.1}$$

This inequality is strict unless

$$C_t = kC_{-t} + 2x_t \quad \text{ where } k \in \mathbb{R},\ x_t \in \mathbb{R}^n.$$

In the case when equality does occur we have

$$\lambda^n(C_t) = \lambda^n(-kC_t) = |-k|\lambda^n(C_t) \Rightarrow k = 1 \quad \because C_{-t} = -C_t$$

and hence

$$C_t = -C_t + 2x_t$$

in which case x_t is the center of symmetry of C_t. By virtue of Lemma 2.2.6a, the continuity of the mapping $t \mapsto \lambda^n(C_t)$ together with inequality (5.3.1) imply that

$$\lambda^{n+1}(C') := \int_I \lambda^n(C'_t)dt > \int_I \lambda^n(C_t)dt =: \lambda^{n+1}(C)$$

unless C_t has a center of symmetry for all $t \in I$.

3. Consider the volume of the polar body C^o:

$$\lambda^{n+1}(C^o) = \int_{\mathcal{H}\times\mathbb{R}} \chi_{C^o} d\lambda^{n+1} = \int_{\mathcal{H}} \left[\int_{\mathbb{R}} \chi_{\{\gamma\in\mathbb{R}:(\xi,\gamma)\in C^o\}} d\gamma \right] d\lambda^n$$

According to Lemma 5.2.7(1), $(C^o)_\gamma \subseteq (C_0)^o$ for all $\gamma \in \mathbb{R}$. Therefore, the set $\{\gamma \in \mathbb{R} : (\xi, \gamma) \in C^o\}$ is empty whenever $\xi \notin (C_0)^o$. The above integral therefore reduces to:

$$\int_{(C_0)^o} \left[\int_{\mathbb{R}} \chi_{\{\gamma\in\mathbb{R}:(\xi,\gamma)\in C^o\}} d\lambda \right] d\lambda^n$$

Moreover, Lemma 5.2.7(2) implies that for any $\xi \in (C_0)^o$, the interval $(C^o)_\gamma := \{\gamma \in \mathbb{R} : (\xi, \gamma) \in C^o\}$ has length $r(\xi) + r(-\xi)$. A suitable change of variables thus yields

$$\begin{aligned}
\lambda^{n+1}(C^o) &= \int_{(C_0)^o} r(\xi)d\lambda^n + \int_{-(C_0)^o} r(-\xi)d\lambda^n \\
&= \int_{(C_0)^o} r(\xi)d\lambda^n + \int_{(C_0)^o} r(\xi)d\lambda^n = 2\int_{(C_0)^o} r(\xi)d\lambda^n \qquad (5.3.2)
\end{aligned}$$

where $r(\xi) = \inf_{t>0} \frac{1-p_t(\xi)}{t}$.
Since $-C_0 = C_{-0} = C_0$ and C_0 is convex, $C_0 = C'_0$. Hence

$$\lambda^{n+1}(C'^o) = \int_{(C_0)^o} 2r'(\xi)d\lambda^n \qquad (5.3.3)$$

where $r'(\xi) = \inf_{t>0} \frac{1-p'_t(\xi)}{t}$ and

$$\begin{aligned} p'_t(\xi) = \sup_{C'_t} \xi(x) &= \frac{1}{2}(\sup_{C_t} \xi(x) + \sup_{C_{-t}} \xi(x)) \\ &= \frac{1}{2}[p_t(\xi) + p_{-t}(\xi)] \end{aligned}$$

Since

$$\begin{aligned} r(\xi) + r(-\xi) &= \inf_{t>0} \frac{1-p_t(\xi)}{t} + \inf_{t>0} \frac{1-p_{-t}(\xi)}{t} \\ &\leq \inf_{t>0} \frac{1-p_t(\xi)+1-p_{-t}(\xi)}{t} = 2\inf_{t>0} \frac{1-p'_t(\xi)}{t} \\ &= 2r'(\xi) \end{aligned}$$

for all $\xi \in (C_0)^o$, it follows that $\lambda^{n+1}(C^o) \leq \lambda^{n+1}(C'^o)$. Moreover, the continuity of the mapping $\xi \mapsto p_t(\xi)$ ensures that equality is only obtained if the integrands in equations (5.3.2) and (5.3.3) are equal for every $\xi \in (C_0)^o$. In other words

$$\begin{aligned} &\inf_{t>0} \frac{1-p_t(\xi)}{t} + \inf_{t>0} \frac{1-p_{-t}(\xi)}{t} = 2\inf_{t>0} \frac{1-p_t(\xi)}{t} \\ \Rightarrow\ &\inf_{t>0} \frac{1-p_t(\xi)}{t} = \inf_{t>0} \frac{1-p_{-t}(\xi)}{t} \text{ for all } \xi \in (C_0)^o. \end{aligned} \tag{5.3.4}$$

Equation (5.3.4) now implies that for any $\xi \in \text{int}(C_0)^o$, both functions $t \mapsto \frac{1-p_t(\xi)}{t}$ and $t \mapsto \frac{1-p_{-t}(\xi)}{t}$ attain their minima at the same value of $t > 0$. Indeed, let $\xi \in \text{int}(C_0)^o$ be fixed. It follows directly from the definition that $p_0(\xi) < 1$. Moreover, according to Lemma A.2.3, the concavity of the mapping $t \mapsto p_t(\xi)$ (Lemma 5.2.6) ensures that $\frac{p_0(\xi)-p_t(\xi)}{t}$ is decreasing as $t \to 0^+$ and that

$$\lim_{t\to 0^+} \frac{p_0(\xi)-p_t(\xi)}{t} = -(p_0(\xi))_+.$$

Therefore

$$\begin{aligned} &\frac{1-p_t(\xi)}{t} = \frac{1-p_0(\xi)}{t} + \frac{p_0(\xi)-p_t(\xi)}{t} \\ \Rightarrow\ &\frac{1-p_t(\xi)}{t} \geq \frac{1-p_0(\xi)}{t} + (p_0(\xi))_+ \\ \Rightarrow\ &\lim_{t\to 0^+} \frac{1-p_t(\xi)}{t} = \infty \end{aligned}$$

In order to find $\inf_{t>0} \frac{1-p_t(\xi)}{t}$ it therefore suffices to consider values of $t > 0$ in some positive compact interval $J \subset I$. The function $\frac{1-p_t(\xi)}{t}$ is continuous on this interval (Lemma 5.2.6) and hence attains its minimum at some $t_1 > 0$. It can similarly be be shown that there exists a $t_2 > 0$ such that $\inf_{t>0} \frac{1-p_{-t}(\xi)}{t} = \frac{1-p_{-t_2}(\xi)}{t_2}$. Equation (5.3.4) now implies that

$$\begin{aligned} \frac{1-p_{t_1}(\xi)}{t_1} &= \frac{1-p_{-t_2}(\xi)}{t_2} \\ \Rightarrow\ p_{t_1}(\xi) &= 1 - \frac{t_1}{t_2}\left(1 - p_{-t_2}(\xi)\right) \end{aligned}$$

Moreover, by the definition of p_{t_1},

$$\begin{aligned} &\frac{1-p_{t_1}(\xi)}{t_1} \leq \frac{1-p_{t_2}(\xi)}{t_2} \\ \Rightarrow\ &\frac{1-(1-\frac{t_1}{t_2}(1-p_{-t_2}(\xi)))}{t_1} \leq \frac{1-p_{t_2}(\xi)}{t_2} \\ \Rightarrow\ &p_{-t_2}(\xi) \leq p_{t_2}(\xi) \end{aligned}$$

Similarly, it follows from the definition of p_{-t_2} that

$$\begin{aligned} &\frac{1-p_{-t_2}(\xi)}{t_2} \leq \frac{1-p_{-t_1}(\xi)}{t_1} \\ \Rightarrow\ &p_{t_2}(\xi) \leq p_{-t_2}(\xi) \end{aligned}$$

and hence

$$p_{-t_2}(\xi) = p_{t_2}(\xi) \ \Rightarrow\ \frac{1-p_{t_2}(\xi)}{t_2} = \frac{1-p_{-t_2}(\xi)}{t_2} = \frac{1-p_{t_1}(\xi)}{t_1} \quad \Rightarrow t_1 = t_2.$$

For any $\gamma > 0$ and any $\xi \in (C_0)^o$ it follows that $\frac{\xi}{\gamma} \in \mathrm{int}(C_0)^o$ and that

$$\frac{1-p_t(\frac{\xi}{\gamma})}{t} = \frac{1}{\gamma} \cdot \frac{\gamma - p_t(\xi)}{t}.$$

According to Lemma A.2.4 there must therefore exists an α_ξ such that

$$p_t(\xi) = \alpha_\xi \cdot t + p_{-t}(\xi) \quad \forall t > 0$$

$\square$

Theorem 5.3.2. *Let $n \in \mathbb{N}$ and let $C \subset \mathbb{R}^{n+1}$ be a convex symmetric body such that*

$$P(C) = \max\{P(B) : B \in \mathcal{C}_s\}.$$

Then C is an ellipsoid.

Proof. Let $\mathcal{H}$ be an arbitrary hyperplane passing through the origin. A suitable application of a volume preserving map ensures that $\mathcal{H} \times \mathbb{R}$ is isomorphic to $\mathbb{R}^{n+1}$. In order to prove that C is an ellipsoid, it suffices to show that there exists a line ℓ, not contained in $\mathcal{H}$, that satisfies the conditions in Corollary 4.4.4. According to Lemma 5.3.1 there exists a convex symmetric body $C' \subset \mathbb{R}^{n+1}$ such that

$$\lambda^{n+1}(C) \leq \lambda^{n+1}(C') \text{ and } \lambda^{n+1}(C^o) \leq \lambda^{n+1}(C'^o) \Rightarrow P(C) \leq P(C').$$

However, since $P(C) = \max\{P(B) : B \in \mathcal{C}_s\}$, it follows that

$$\lambda^{n+1}(C) = \lambda^{n+1}(C') \quad \text{and} \quad \lambda^{n+1}(C^o) = \lambda^{n+1}(C'^o)$$

which, by Lemma 5.3.1, in turn implies that for every $t \in I$ there exists a center $x_t \in C_t$ such that $C_t = C_{-t} + 2x_t$. Therefore, for any $\xi \in (C_0)^o$

$$\begin{aligned} \sup_{C_t} \xi(x) &= \sup_{C_{-t}} \xi(x) + 2\xi(x_t) \\ \Rightarrow p_t(\xi) &= p_{-t}(\xi) + 2\xi(x_t) \end{aligned}$$

Lemma 5.3.1 also implies that $p_t(\xi) = \alpha_\xi \cdot t + p_{-t}(\xi)$ for every $t \in I$ and for some $\alpha_\xi \in \mathbb{R}$, and hence

$$\begin{aligned} \alpha_\xi \cdot t + p_{-t}(\xi) &= p_{-t}(\xi) + 2\xi(x_t) \\ \Rightarrow \quad \xi(x_t) &= \frac{1}{2}\alpha_\xi \cdot t \end{aligned}$$

It can readily be seen from the above expression that the mapping $t \mapsto x_t$ is linear. Indeed, let $t_1, t_2 \in I$ and suppose $\beta_1, \beta_2 \in \mathbb{R}$ are such that $\beta_1 t_1 + \beta_2 t_2 \in I$. Then

$$\begin{aligned} \xi(x_{\beta_1 t_1 + \beta_2 t_2}) &= \beta_1 \alpha_\xi t_1 + \beta_2 \alpha_\xi t_2 = \beta_1 \xi(x_{t_1}) + \beta_2 \xi(x_{t_2}) \\ &= \xi(\beta_1 x_{t_1} + \beta_2 x_{t_2}) \end{aligned}$$

For every $\xi \in (C_0)^o$ and hence for every $\xi \in (\mathbb{R}^n)^*$. The algebraic reflexivity of $\mathbb{R}^n$ now implies that $x_{\beta_1 t_1 + \beta_2 t_2} = \beta_1 x_{t_1} + \beta_2 x_{t_2}$. The set $\{(x_t, t) : t \in I\}$ is therefore contained in the straight line $\ell = \{\tau(x_{t_0}, t_0) : \tau \in \mathbb{R}^n\}$, where $t_0 \neq 0$ is some fixed element of I. Since $\mathcal{H}$ was arbitrary, the conditions for Corollary 4.4.4 are met and C must therefore be an ellipsoid. $\square$

Appendix A

Convex- and Concave Functions

A.1 Continuity of Convex-/Concave Functions

Definition A.1.1. *Consider the function $f : X \to (-\infty, \infty]$ defined on the Minkowski space X. The set $D(f) := \{x \in X : f(x) < \infty\}$ is called the effective domain of f and f is said to be a proper convex function if $D(f) \neq \emptyset$ and*

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \quad \text{for all } x, y \in X \text{ and } \lambda \in [0,1].$$

A function $f : X \to [-\infty, \infty)$ is said to be a proper concave function if $-f$ is a proper convex function

Theorem A.1.2 and Corollaries A.1.3 and A.1.4 aim to show that all proper convex- and concave functions are continuous on the interior of their effective domains.

Theorem A.1.2. *If a proper convex function f is bounded above on some neighborhood V of a point $x_0 \in \mathrm{int}D(f)$, then f is continuous on $\mathrm{int}D(f)$.*

Proof. **Continuity at x_0**

Let V be a neighborhood of x_0 and $M \geq 0$ such that $|f(x)| \leq M$ for all $x \in V$. Assume without loss of generality that $x_0 = 0$ and $f(x_0) = 0$. Indeed, let the function F be defined by $F(x) := f(x + x_0) - f(x_0) \quad \forall x \in X$. F is convex since f is convex and $F(0) = f(x_0) - f(x_0) = 0$. Now f is continuous at x_0 if and only if F is continuous at 0. In addition, note that $0 \in V$ if and only if 0 is an element of the symmetric open set $V \cap (-V)$ and hence we may assume without loss of generality that V is symmetric. For every $\epsilon \in (0,1)$ and every $x \in \epsilon V$, $\frac{x}{\epsilon} \in V$ and hence we have

$$f(x) = f\left(\epsilon \frac{x}{\epsilon} + (1-\epsilon)0\right) \leq \epsilon f(\frac{x}{\epsilon}) \leq \epsilon M$$

Conversely, the symmetry of V implies that $-\frac{x}{\epsilon} \in V$ and therefore

$$\begin{aligned} 0 = f(0) &= f\left(\frac{1}{1+\epsilon}x + (1 - \frac{1}{1+\epsilon})(-\frac{x}{\epsilon})\right) \\ &\leq \frac{1}{1+\epsilon}f(x) + \frac{\epsilon}{1+\epsilon}f\left(-\frac{x}{\epsilon}\right) \\ \Rightarrow \quad &- f(x) \leq \epsilon f\left(\frac{-x}{\epsilon}\right) \leq \epsilon M \end{aligned}$$

Combining these two inequalities, we obtain $|f(x)| \leq \epsilon M$ for all $x \in \epsilon V$. For every $\delta \in (0,1)$, there exists an $\epsilon = \frac{\delta}{M}$ such that $|f(x) - f(0)| \leq \delta$ for all $x \in \epsilon V$.

Continuity at an arbitrary point $y \in \text{int}D(f)$

In order to prove that f is continuous at an arbitrary $y \in \text{int}D(f)$, it suffices, by virtue of the above arguments, to prove that there exists a neighborhood V_y of y on which f is bounded. Since $y \in \text{int}D(f)$, there exists a $\rho > 1$ such that $\rho y \in D(f)$. Let V be the symmetric neighborhood of 0 which was used above and define $V_y := y + (1 - \frac{1}{\rho})V$. Any $x \in V_y$can be written as

$$x = \frac{1}{\rho}(\rho y) + (1 - \frac{1}{\rho})z \quad \text{where } z \in V \subset D(f) \text{ and } \rho y \in D(f)$$

and since $D(f)$ is convex, $V_y \subset D(f)$. Moreover, the convexity of f implies that for any $x \in V_y$:

$$f(x) = f\left(\frac{1}{\rho}(\rho y) + (1 - \frac{1}{\rho})z\right) \leq \frac{1}{\rho}f(\rho y) + (1 - \frac{1}{\rho})f(z) \leq \frac{1}{\rho}f(\rho y) + (1 - \frac{1}{\rho})\epsilon M$$

and hence f is bounded above on V_y. □

Corollary A.1.3. *A proper convex function defined on* $\mathbb{R}^n$ *is continuous on* $\text{int}D(f)$.

Proof. According to theorem A.1.2, it suffices to show that that f is bounded on some neighborhood V of 0. Choose an $\alpha > 0$ small enough so that the open set

$$V := \{(x_1, ..., x_n) \in \mathbb{R}^n : 0 < x_i < \frac{\alpha}{n} \quad \text{for } i = 1, ..., n\}$$

is contained in $D(f)$ and denote the standard basis of $\mathbb{R}^n$ by $\{e_1, ..., e_n\}$. Any $x = (x_1, ..., x_n) \in V$ can be written in the form

$$x = \sum_{i=1}^{n} x_i e_i = \sum_{i=1}^{n} \frac{x_i}{\alpha}(\alpha e_i) + \left(1 - \sum_{i=1}^{n} \frac{x_i}{\alpha}\right) 0$$

with $0 < \frac{x_i}{\alpha} < \frac{1}{n} < 1$ for $i = 1, ..., n$, and
$0 < \sum_{i=1}^{n} \frac{x_i}{\alpha} < \sum_{i=1}^{n} \frac{1}{n} = 1 \Rightarrow (1 - \sum_{i=1}^{n} \frac{x_i}{\alpha}) \in (0,1)$. Since f is convex, it follows that

$$\begin{aligned} f(x) &\leq \sum_{i=1}^{n} \frac{x_i}{\alpha} f(\alpha e_i) + \left(1 - \sum_{i=1}^{n} \frac{x_i}{\alpha}\right) f(0) \\ &< \frac{1}{n} \sum_{i=1}^{n} |f(\alpha e_i)| + |f(0)| \end{aligned}$$

□

This result can readily be extended to include proper concave functions.

Corollary A.1.4. *Any concave function $f : \mathbb{R}^n \to [-\infty, \infty)$ is continuous on the set $-\text{int}D(-f) = \{x \in \mathbb{R}^n : f(x) > -\infty\}$.*

Proof. By definition of f, $-f$ is a convex function which, according to corollary A.1.3, is continuous on $\text{int}D(-f)$. It therefore follows directly that f is continuous on $-\text{int}D(-f)$. □

A.2 Derivatives of Concave Functions

Definition A.2.1. *The right (left) derivative of a function $f : \mathbb{R} \to \mathbb{R}$ at a point $x \in \mathbb{R}$ is defined as:*

$$f'_+(x) = \lim_{t \to 0_+} \frac{f(x+t) - f(x)}{t}$$

$$f'_-(x) = \lim_{t \to 0_-} \frac{f(x+t) - f(x)}{t}$$

respectively, provided these limits exist.

Lemma A.2.2. *Let $f : \mathbb{R} \to [-\infty, \infty)$ be a concave function and let t_1, t_2, t_3 be real numbers such that $t_1 < t_2 < t_3$. Then*

$$\frac{f(t_2) - f(t_1)}{t_2 - t_1} \geq \frac{f(t_3) - f(t_1)}{t_3 - t_1} \geq \frac{f(t_3) - f(t_2)}{t_3 - t_2} \tag{A.2.1}$$

Proof. Since t_2 can be expressed as a convex combination of t_1 and t_2:

$$t_2 = \frac{t_3 - t_2}{t_3 - t_1} t_1 + \frac{t_2 - t_1}{t_3 - t_1} t_3$$

It now follows from the concavity of f that

$$f(t_2) \geq \frac{t_3 - t_2}{t_3 - t_1} f(t_1) + \frac{t_2 - t_1}{t_3 - t_1} f(t_3)$$

Thus

$$\begin{aligned} f(t_2) &\geq \frac{t_3 - t_2}{t_3 - t_1} f(t_1) + \frac{t_2 - t_1}{t_3 - t_1} f(t_1) - \frac{t_2 - t_1}{t_3 - t_1} f(t_1) + \frac{t_2 - t_1}{t_3 - t_1} f(t_3) \\ &= f(t_1) + \frac{t_2 - t_1}{t_3 - t_1}(f(t_3) - f(t_1)) \end{aligned}$$

Hence

$$\frac{f(t_2) - f(t_1)}{t_2 - t_1} \geq \frac{f(t_3) - f(t_1)}{t_3 - t_1}$$

The second inequality follows similarly. □

Lemma A.2.3. *For every concave function $f : \mathbb{R} \to [-\infty, \infty)$ both the right- and left sided derivatives exist at any point $x \in \operatorname{int}\{x : f(x) > -\infty\}$ and*

$$f'_+(x) \leq f'_-(x)$$

Moreover for all $t_1, t_2 \in \operatorname{int}\{x : f(x) > -\infty\}$ with $t_1 < t_2$ we have:

$$f'_-(t_2) \leq \frac{f(t_2) - f(t_1)}{t_2 - t_1} \leq f'_+(t_1)$$

Proof. Consider $0 < s_1 < s_2$. Then for any $t \in \operatorname{int} D(f)$ we have

$$t - s_2 < t - s_1 < t < t + s_1 < t + s_2$$

Hence by inequality (A.2.1):

$$\frac{f(t) - f(t - s_2)}{s_2} \geq \frac{f(t) - f(t - s_1)}{s_1} \geq \frac{f(t + s_1) - f(t)}{s_1} \geq \frac{f(t + s_2) - f(t)}{s_2}$$

Since this is true for any $0 < s_1 < s_2$ it follows that the function $s \mapsto \frac{f(t+s)-f(t)}{s}$ is non-decreasing as $s \to 0_+$ and bounded above (by the number $\frac{f(t)-f(t-s_1)}{s_1}$ for example). Hence the limit:

$$f_+(t) := \lim_{s \to 0_+} \frac{f(t + s) - f(t)}{s}$$

exists.
Similarly, the function $s \mapsto \frac{f(t)-f(t-s)}{s}$ is non-increasing as $s \to 0_+$ and bounded below by the number $\frac{f(t+s_1)-f(t)}{s_1}$. The limit

$$f'_-(t) := \lim_{s \to 0_-} \frac{f(t + s) - f(t)}{s} = \lim_{s \to 0_+} \frac{f(t) - f(t - s)}{s}$$

thus also exists. Moreover we have

$$f'_-(t) \geq f'_+(t) \quad \forall t \in \operatorname{int} D(f)$$

Furthermore, for any $s > 0, t_1 < t_2$:

$$f'_+(t_1) \geq \frac{f(t_1+s) - f(t_1)}{s}$$

and

$$f'_-(t_2) \leq \frac{f(t_2) - f(t_2 - s)}{s}$$

Particularly, letting $s = t_2 - t_2$ we obtain:

$$f'_+(t_1) \geq \frac{f(t_2) - f(t_1)}{t_2 - t_1} \geq f'_-(t_2)$$

□

Lemma A.2.4. *Let $f, h : (0, \infty) \to [-\infty, \infty)$ be two concave functions. If for all $\lambda \in \mathbb{R}$ the functions $F(t) = \frac{\lambda - f(t)}{t}$ and $H(t) = \frac{\lambda - h(t)}{t}$ attain their minimum at the same point, then there exists an $\alpha \in \mathbb{R}$ such that*

$$f(t) = h(t) + \alpha t \quad \forall t > 0$$

Proof. . We will show that

$$f(t) - tf'_+(t) = h(t) - th'_+(t) \quad \text{for all } t > 0. \tag{A.2.2}$$

By making use of the quotient rule for one-sided derivatives, this equation can be reformulated as

$$\left(\frac{(f-h)(t)}{t}\right)'_+ = 0 \quad \text{for all } t > 0$$

$$\Rightarrow \quad \frac{f(t) - h(t)}{t} = \alpha \quad \text{for some } \alpha \in \mathbb{R}, \forall t > 0$$

from which the result follows.
Suppose (by way of contradiction) that there is a $t_0 > 0$ such that

$$f(t_0) - t_0 f'_+(t_0) < h(t_0) - t_0 h'_+(t_0)$$

then, by completeness of reals, we can choose a λ such that:

$$f(t_0) - t_0 f'_+(t_0) < \lambda < h(t_0) - t_0 h'_+(t_0) \tag{A.2.3}$$

Now let s be a point where both F and H attain their maximum. Hence $H(t) \geq H(s)$ and $F(t) \geq F(s)$ for all $t > 0$. From this it follows that

$$\frac{F(t) - F(s)}{t - s} \geq 0 \quad \forall t > s$$

$$\Rightarrow \quad F'_+(s) := \lim_{t \to s_+} \frac{F(t) - F(s)}{t - s} \geq 0$$

Similarly

$$\frac{H(t)-H(s)}{t-s} \leq 0 \quad \forall t \in (0,s)$$
$$\Rightarrow \quad H'_{-}(s) := \lim_{t \to s_{-}} \frac{H(t)-H(s)}{t-s} \leq 0$$

Making use of the quotient rule for one-sided derivatives we have

$$F'_{+}(s) = \frac{-f'_{+}(s)+f(s)-\lambda}{s^2} \geq 0$$
$$H'_{-}(s) = \frac{-h'_{-}(s)+h(s)-\lambda}{s^2} \leq 0$$

From this it follows that:

$$f(s) - sf'_{+}(s) \geq \lambda \geq h(s) - sh_{-}(s) \tag{A.2.4}$$

Now inequality (A.2.3) together with (A.2.4) imply

$$f(t_0) - t_0 f'_{+}(t_0) < \lambda \leq f(s) - sf'_{+}(s) \tag{A.2.5}$$

from which it follows that $t_0 < s$. This assertion is proved in remark A.2.5, appended to this lemma.
Another consequence of inequalities (A.2.3) and (A.2.4) is:

$$h(s) - sh'_{-}(s) \leq \lambda < h(t_0) - t_0 h'_{+}(t_0) \tag{A.2.6}$$

According to remark A.2.6, inequality (A.2.6) implies

$$h(t) - th'_{+}(t) \leq \lambda \quad \text{for all } t \in (0,s) \tag{A.2.7}$$

But inequality (A.2.6) together with the fact that $t_0 < s$ contradicts (A.2.7). This proves that $f(t) - tf'_{+}(t) \geq h(t) - th'_{+}(t)$ for all $t > 0$. By interchanging the roles of f and h we obtain equation (A.2.2) and hence the result is proved. $\square$

Remark A.2.5. *Suppose* $0 < s < t_0$. *From (A.2.5) we know:*

$$0 < f(s) - f(t_0) + t_0 f'_{+}(t_0) - sf'_{+}(s)$$
$$\Rightarrow \quad 0 < f(s) - f(t_0) + t_0 f'_{+}(t_0) - sf_{+}(t_0) + sf'_{+}(t_0) - sf'_{+}(s)$$
$$\Rightarrow \quad \frac{f(t_0)-f(s)}{t_0-s} < f'_{+}(t_0) + \frac{s}{t_0-s}(f'_{+}(t_0) - f'_{+}(s))$$

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But f is concave and hence lemma A.2.3 asserts that $\frac{f(t_0)-f(s)}{t_0-s} \geq f'_+(t_0)$. *Therefore*

$$\begin{aligned} & \frac{s}{t_0 - s}(f'_+(t_0) - f'_+(s)) > 0 \\ \Rightarrow \quad & f'_+(s) < f'_+(t_0) \end{aligned}$$

But according to lemma A.2.3 $f'_+(s) \geq f'_+(t_0)$, *since* $s < t_0$. *This is clearly a contradiction.*

Remark A.2.6. *Suppose* $h(t) - th'_+(t) > \lambda$ *for some* $t \in (0, s)$. *Then*

$$\begin{aligned} & h(s) - sh'_-(s) < h(t) - th'_+(t) \\ \Rightarrow & h(s) - h(t) < sh'_-(s) - th'_+(t) + th_-(s) - th_-(s) \\ \Rightarrow & \frac{h(s) - h(t)}{s - t} < h'_-(s) + \frac{t}{s-t}(h'_-(s) - h'_+(t)) \end{aligned}$$

Again, the concavity of h implies that $\frac{h(s)-h(t)}{s-t} \geq h'_-(s)$, *from which it follows that* $h'_+(t) < h'_-(s)$, *which is impossible according to lemma A.2.3.*

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