Transform analysis of affine jump diffusion processes with applications to asset pricing

by

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Summary

Title: Transform analysis of affine jump diffusion processes with applications to asset pricing
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This work presents a class of models in asset pricing, whose underlying has dynamics of Affine jump diffusion type. We first present Lévy processes with their properties. We then introduce Affine jump diffusion processes which are basically a particular class of Lévy processes. Our motivation for these is driven by the fact that many financial models are built on them. Affine jump diffusion processes present good analytical properties that allow one to get close form formulas for a wide range of option pricing.

The approach we use here is based on the paper by Duffie D, Pan J, and Singleton K. An example will show how incorporating parameters such as the volatility of the underlying asset in the model, can influence the resulting price of the financial instrument under consideration. We will also show how this class of models incorporate well known models, specially those used to model interest rates dynamics, like for instance the Vasicek model.

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INTRODUCTION

Affine jump diffusions are processes that are fairly general. They can accommodate many parameters such as stochastic volatility. These affine processes form a class of jump-diffusion processes that are constructed from Lévy processes. In turn, Lévy processes are stochastic processes that present convenient properties for modeling such as Markov properties, independence of increments and stochastic continuity. The only discontinuities they present are jumps and these jumps have random size and occur at random times. Lévy processes include particular cases like Brownian motion and the Poisson process. Roughly speaking Affine jump diffusion models are models whose state process X is a Lévy process with its parameters such as drift, covariance, interest rate etc, all being affine in X. One of the advantages these models have is that analytical formulas of asset prices are obtainable. In this project, we consider the state process X to be multidimensional.

The project is organized as follows, we first go through some preliminaries and then some generalities on Lévy processes. There we just state couple of definitions, some useful propositions and theorems whose proofs can be found in [1],[8] and many usual books of stochastic calculus. From then we introduce chapter 3 which explains Affine jump diffusion models as well as the transforms that will allow us to derive asset prices and options prices. To be more specific, if V_T is the pay-off at time T of a financial instrument and V_T is of the form:

$$V_T = \Big(v + w \cdot X_T\Big) e^{u \cdot X_T}$$

where u, w are real valued vectors in $\mathbb{R}^d$, $v \in \mathbb{R}$, then the transforms enable one to get a closed form expression of the price at time t of this financial instrument. That is,

$$V_t = E\left[e^{-\int_t^T R(X_s)ds}\Big(v + w \cdot X_T\Big)e^{u \cdot X_T}\Big|\mathcal{F}_t\right]$$

can be obtained in its closed form. Here, R denotes the interest rate and $\mathcal{F}_t$ is the $\sigma-$field generated by the process X up to time t. In the last section we show how Affine jump diffusion models actually extend model with affine term structures, as well derived by T. Björk [2]. An illustrative example is also treated for the case $d = 2$.

Chapter 1

Preliminaries

In this whole project the probability space under consideration is $(\Omega, \mathcal{F}, P)$ where $\mathcal{F}$ is the $\sigma-$algebra and P is the probability measure.

1.1 Random Measure

In order to introduce the definition of a random measure let first recall the definition of a measure.

Definition 1.1.1. *A mapping $M : \mathcal{F} \to \mathbb{R}_+$, $A \mapsto M(A)$ is called a* ***measure*** *if it satisfies:*

$$\begin{aligned} &(1) \qquad M(\emptyset) = 0 \\ &(2) \qquad M(\bigcup_{n\geq 1} A_n) = \sum_{n\geq 1} M(A_n) \end{aligned}$$

where (A_n) is any disjoint family of elements of $\mathcal{F}$.

Definition 1.1.2. *(Random Measure)*
Let $(\mathcal{S}, \mathcal{A})$ be a measurable space. A ***random measure*** *on $(\mathcal{S}, \mathcal{A})$ is a collection of random variables $\Big(M(B)\Big)_{B\in\mathcal{A}}$ such that*
(1) $M(\emptyset) = 0$;

(2) Given any sequence (B_n) of mutually disjoint sets in $\mathcal{A}$

$$M\Big(\bigcup_{n\in\mathbb{N}} B_n\Big) = \sum_{n\in\mathbb{N}} M(B_n) \qquad a.s.$$

(3) For each disjoint family (B_n) in $\mathcal{A}$, the random variable $M(B_j)$ is independent of $M(B_k)$ for all $j \neq k$.

Example

Let $T = (T_j)_{j\in\mathbb{N}}$ be an increasing sequence of stopping times that are iid. For any measurable set $A \subset \mathbb{R}$, let

$$M(\omega, A) = \#\{j \in \mathbb{N} : \ T_j(\omega) \in A\}.$$

In other words $M(\omega, A)$ counts the number of times the random variable $T_j(\omega)$ takes values in A; it is positive and integer valued. $M(\omega, A)$ is finite whenever A is bounded. $M(\omega, A)$ depends on the event ω, as result $M(\cdot, A)$ is a random variable and because T is a stopping time, $M(\cdot, A)$ is independent of $M(\cdot, B)$ whenever $A \cap B = \emptyset$. Therefore $M(\omega, \cdot)$ is a random measure.

The random measure defined above becomes the **Poisson random measure** with intensity μ if $M(\cdot, A)$ is a Poisson random variable with parameter $\mu(A)$ that is:

$$\forall k \in \mathbb{N} \qquad P\Big(M(A) = k\Big) = e^{\mu(A)} \frac{\Big(\mu(A)\Big)^k}{k!}.$$

The above example will be of important use for the rest of this chapter as well as chapter 2.

1.2 Poisson Processes

Definition 1.2.1. *Let (T_n) be a strictly increasing sequence of positive random variables with $T_0 = 0$. The process $N = (N_t)_{t\geq 0}$ defined by*

$$N_t = \sum_{n\geq 1} 1_{\{T_n \leq t\}},\ t \leq 0$$

is called a ***counting process*** *associated with the process (T_n).*

Definition 1.2.2. *An adapted counting process N is a* ***Poisson process*** *if it satisfies:*
$(P1)$ for all $s \leq t$, $(N_t - N_s)$ is independent of $\mathcal{F}_s$
$(P2)$ for all $s \leq t, \quad u \leq v$, if $t - s = v - u$ then $N_t - N_s$ and $N_v - N_u$ have same distribution.
$(P3)$ The increment $N_t - N_s$ follows a Poisson distribution, that is:

$$P(N_t - N_s \leq k) = \frac{1}{\lambda^k (t-s)^k} e^{\lambda(t-s)k}.$$

The parameter λ is called the ***intensity*** *of the process N_t.*

Usual properties of Poisson processes can be found in many books such as [3] and [8].

Proposition 1.2.3. *Let N be a Poisson process with intensity λ.*
Then $\tilde{N}_t := N_t - \lambda t$ and $\hat{N}_t := \tilde{N}_t^2 - \lambda t$ are both martingales and the characteristic function of $\tilde{N}$ is given by:

$$\phi_{\tilde{N}_t}(z) = e^{\lambda t(e^{iz} - 1 - iz)}.$$

$\tilde{N}$ is termed as the ***compensated Poisson process****.*

Chapter 2

Lévy Processes - Semimartingales

2.1 Definition and properties

Definition 2.1.1. *An adapted process X with $X_0 = 0$ (a.s) is called a* ***Lévy process*** *if:*
$(L1)$ X has increments independent from the past i.e. $X_t - X_s$ is independent of $\mathcal{F}_s$ for $0 \leq s < t < \infty$.
$(L2)$ X has stationary increments i.e. $X_t - X_s$ has same distribution as X_{t-s}.
$(L3)$ X is stochastically continuous i.e. for any $\epsilon > 0$,

$$\lim_{s \to t} P(|X_t - X_s| \geq \epsilon) = 0.$$

Property $(L3)$ implies that the number of jumps is countable. A jump of a process X will often be denoted by ΔX. Hence a jump occurring at time τ will simply be defined as

$$\Delta X_\tau = X_\tau - X_{\tau^-}$$

where

$$X_{\tau^-} = \lim_{s \to \tau, s < \tau} X_s.$$

Theorem 2.1.2. *Let A be a subset of $\mathbb{R}_+$. Let $N(t, A)$ be the random variable which counts the number of times the process X jumps with size in $A \subset \mathbb{R}$ and occurring within time 0 and t i.e.*

$$N(t, A) = \sum_{0 \le s \le t} 1_A(\Delta X_s),$$

then $\Big(N(t, A)\Big)$ is a Poisson process.

Definition 2.1.3. *For $I \subset \mathbb{R}_+$, the mapping $N(I, \cdot)$ defines a measure,* ***the Poisson Random measure*** *and if we take $I = [0, 1]$ then $\nu(\cdot) = E(N([0, 1], \cdot))$ also defines a measure, the* ***Lévy measure****.*

Note that ν is just a positive measure, not a probability measure!

For fixed (ω, t), it is not difficult to see that the set function $A \mapsto N_t(\omega, A)$ is increasing and non negative therefore we can always define a Stieltjes integral with respect to N.

Definition 2.1.4. *For fixed (ω, t) and for any C^1-function f we define the stochastic integral with respect to the process N as:*

$$\int_A f(x) N_t(\omega, dx) = \sum_{0 \le s \le t} f\big(\Delta X_s(\omega)\big) 1_A\big(\Delta X_s(\omega)\big).$$

The following process will be of important use for the next coming proofs. Define

$$\begin{aligned} J_t(\omega, A) &= \sum_{0 \le s \le t} \Delta X_s(\omega) 1_A(\Delta X_s(\omega)) \\ &= \int_A x N_t(\omega, dx). \end{aligned}$$

$J_t(\omega, A)$ or just simply J_t^A cumulates all the jumps of X that occurred before time t whose size are in A. As t varies J_t^A describes a stochastic process which is called the ***jump process*** associated with the lévy process X with sizes in A.

Corollary 2.1.5. *For fixed ω and for any C^1-function f, the stochastic process* $\left(\int_A f(x)N_t(\omega, dx)\right)_{t\geq 0}$ *is a Lévy process.*

Definition 2.1.6. *A subset $A \subset \mathbb{R}_+$ is* ***bounded from below*** *if there exists a neighborhood of 0 that is not in A. In other words*

$$0 \notin \bar{A}.$$

Definition 2.1.7. *We say that a process X is* ***of bounded jumps*** *if there exists a positive constant C such that*

$$\Delta X_s \leq C \text{ a.s for all } s \geq 0.$$

Definition 2.1.8. *The* ***variation*** *$V_T(X)$ of a process X on an interval $[0, T]$ is defined as*

$$V_T(X) = \sup_{n\in\mathbb{N}} \sum_{i=1}^{n} |X_{t_i} - X_{t_{i-1}}|$$

where $0 \leq t_0 < t_1 < t_2 < \ldots < t_n = T$ is a partition of $[0, T]$.

Before we proceed we will need the following lemmas whose proofs are quite technical see [8],[1].

Lemma 2.1.9. *If X is a Lévy process and $A \subset \mathbb{R}_+$ is bounded below, then*

$$N(t, A) < \infty \qquad \text{a.s for all } t \geq 0.$$

This means that the number of jumps of size in A is finite.

Lemma 2.1.10. *Let X be a Lévy process with bounded jumps, then $E(|X_t|^n) < \infty$ for all $n \in \mathbb{N}$.*

Lemma 2.1.11. *If M_t^1 and M_t^2 are two centered càdlàg processes and $E(|V_t(M_t^1)|^2) < \infty$, then*

$$E(M_t^1 M_t^2) = E\left[\sum_{0\leq s,t} \Delta M_s^1 \Delta M_s^2\right].$$

Lemma 2.1.12. *If A_1 and A_2 are two disjoint sets that are all bounded from below, then $\left(\int_{A_1} xN(t,dx)\right)_{t\geq 0}$ and $\left(\int_{A_2} xN(t,dx)\right)_{t\geq 0}$ are two independent stochastic processes.*

2.2 Lévy-Itô decomposition

Theorem 2.2.1. *The process $Y_t = X_t - J_t^A$ is a Lévy process with bounded jumps. Therefore it has finite moments.*

Theorem 2.2.2. *Suppose $A \in \mathcal{B}(\mathbb{R} - \{0\})$ and $f1_A \in L^2(A,\nu)$, then*

$$E\left(\int_A f(x)N_t(\cdot,dx)\right) = t\int_A f(x)d\nu(x).$$

$$E\left(\left(\int_A f(x)N_t(\cdot,dx) - t\int_A f(x)\nu(dx)\right)^2\right) = t\int_A f^2(x)d\nu(x)$$

where $\mathcal{B}(\mathbb{R} - \{0\})$ denotes the $\sigma-$algebra generated that the Borel sets of $\mathbb{R}$ not containing 0.

Theorem 2.2.3. *Let X be a Lévy process with jumps bounded by $(a > 0)$, then*

$$Z_t := X_t - E(X_t)$$

is a martingale in addition, Z_t admits the following decomposition
$Z_t = Z_t^c + Z_t^d$ where Z_t^c is a continuous martingale and

$$Z_t^d = \int_{|x|<a} xN_t(\cdot,dx) - t\int_{|x|<a} xd\nu(x).$$

Proof. Let A be a subset of $\mathbb{R}\backslash\{0\}$. Consider the random variable

$$M_t^A = \int_A xN_t(\cdot,dx) - t\int_A xd\nu(x).$$

Let (ϵ_n) be a decreasing sequence of positive numbers: $1 > \epsilon_1 > ... > \epsilon_n \to 0$ as $n \to \infty$.

Define,

$$B_m = \{x \in \mathbb{R} : \epsilon_{m+1} < |x| \leq \epsilon_m\},$$

the set of all jumps sizes within the interval $[\epsilon_{m+1}, \epsilon_m)$. Consider

$$A_n = \bigcup_{m=1}^{n} B_m$$

Step 1

<u>Claim</u>: $(M_t^{A_n})$ converges in L^2 to Z_t^d as $n \longrightarrow \infty$.

First of all, observe that since (B_m) are disjoint subsets that are all bounded from below, then lemma 2.1.12 tells that the processes $\int_{B_{m_1}} xN_t(\cdot, dx)$ and $\int_{B_{m_2}} xN_t(\cdot, dx)$ are independent for all $m_1 \neq m_2$.
Next, we show that for all $n > 0$, $(M_t^{A_n})$ are centered martingales in addition we have

$$E\Big((M_t^{A_n})^2\Big) = \sum_{m=1}^{n} E\Big((M_t^{B_m})^2\Big). \tag{2.1}$$

It is not difficult to see that $E(M_t^{B_i}) = 0$ for all $i > 0$.

Let us prove it by induction.
For the case $n = 2$,

$$\begin{aligned} E\Big((M_t^{B_1 \cup B_2})^2\Big) &= E\Big[(M_t^{B_1} + M_t^{B_2})^2\Big] \qquad \text{since } B_1 \cap B_2 = \emptyset \\ &= E\Big[(M_t^{B_1})^2 + (M_t^{B_2})^2\Big] + 2E[M_t^{B_1} M_t^{B_2}] \\ &= E\Big[(M_t^{B_1})^2 + (M_t^{B_2})^2\Big] + 2E\Big[\sum_{0<s\leq t} \Delta M_s^{B_1} \Delta M_s^{B_2}\Big], \end{aligned}$$

since $(M_t^{B_1})$ and $(M_t^{B_2})$ do not jump at same times (a.s.) due to lemma 2.1.12

$$\Delta M_s^{B_1} \Delta M_s^{B_2} = 0 \qquad a.s., \text{ for all } s > 0.$$

Suppose (2.1) holds up to $n = k$, the same arguments used in the case $n = 2$ will still make (2.1) to be true up to $n = k + 1$. Therefore (2.1) is proved.

We are now ready to look at the convergence of $M_t^{A_n}$.

Observe that $Z_t - M_t^{A_n}$ and $M_t^{A_n}$ are two independent processes, hence:

$$Var(Z_t - M_t^{A_n} + M_t^{A_n}) = Var(Z_t - M_t^{A_n}) + Var(M_t^{A_n})$$

i.e.

$$Var(Z_t) = Var(Z_t - M_t^{A_n}) + Var(M_t^{A_n}).$$

Since Z_t has bounded jumps, it has finite moment of all order
(cf. lemma 2.1.10), which means that $Var(Z_t) < \infty$. Consequently

$$Var(M_t^{A_n}) < \infty.$$

As result of (2.1), the sequence $\Big(E((M_t^{A_n})^2)\Big), n > 0$ is increasing. It is bounded, therefore it converges. Consequently $(M_t^{A_n})$ converges in L^2 to a certain limit and Z_t^d tends to be a good candidate for that limit.
Well,

$$\begin{aligned}\Big|Z_t^d - M_t^{A_n}\Big|^2 &= \Big|\int_{0<|x|<\epsilon_n} x\tilde{N}(\cdot, dx)\Big|^2 \\ &\leq \Big|\int_{0<|x|<\epsilon_n} |x|^2\tilde{N}(\cdot, dx)\Big|\end{aligned}$$

so,

$$\begin{aligned}E\Big[\Big|Z_t^d - M_t^{A_n}\Big|^2\Big] &\leq \epsilon_n^2 E\Big[\Big|\int_{[0,\epsilon_n]} \tilde{N}(\cdot, dx)\Big|\Big] \\ &< \epsilon_n \to 0 \qquad (n \to \infty).\end{aligned}$$

Which confirms that Z_t^d is indeed the L^2-limit of $M_t^{A_n}$.
By definition of Z_t^c from the proposition,i.e.

$$Z_t^c := Z_t - Z_t^d$$

it is clear that Z_t^c would be L^2-limit of $Z_t - M_t^{A_n}$. Just write

$$\begin{aligned}\|(Z_t - M_t^{A_n}) - Z_t^c\| &= \|(Z_t - M_t^{A_n}) - Z_t^c + (M_t^{A_n} - Z_t^d) - (M_t^{A_n} - Z_t^d)\| \\ &\leq \|Z_t - M_t^{A_n} - Z_t^c + M_t^{A_n} - Z_t^d\| + \|M_t^{A_n} + Z_t^d\| \\ &= \|Z_t - Z_t^c - Z_t\| + \|M_t^{A_n} + Z_t^d\| \\ &\leq 0 \qquad + \epsilon \qquad \text{for } n \text{ large enough.}\end{aligned}$$

Note that the norm under consideration here is the L^2-norm.

Step 2

Here we shall prove that Z_t^c is continuous.
First of all and as we did it for $(Z_t - M_t^{A_n})$ and $(M_t^{A_n})$, in the same way it can be shown that Z_t^c and Z_t^d are both independent.
Now, suppose there exists $B \subset \Omega$ such that Z_t^c is discontinuous on B, we shall prove that B has measure 0. To see that, let assume $P(B) \neq 0$ then there exists a positive number b such that Z_t has jumps larger than b.
Define

$$B_n = \{t \geq 0 : |\Delta Z_t| > \frac{1}{n}\}.$$

It is trivial that $B_n \uparrow B$ hence $P(B_n) \uparrow P(B)$ which is strictly larger than 0 by assumption. Define T_n as the fist time that ΔZ_t gets larger than $\frac{1}{n}$ i.e.

$$T_n = \inf\{t \geq 0 : \Delta Z_t > \frac{1}{n}\}$$

$T = (T_n)$ is a sequence of stopping times and

$$P(|\Delta Z_T| > b) \neq 0.$$

Let $A = \{x \in \mathbb{R} : |x| > b\}$ then Z_t^c and $\int_A f(x)\tilde{N}_t(\cdot, dx)$ must jump at same times, for any continuous function f. From lemma 2.1.11,

$$\begin{aligned}
E\Big[Z_t^c \cdot \int_A f(x)\tilde{N}(\cdot, dx)\Big] &= E\Big[\sum_{0\leq s<t} \Delta Z_t^c \Delta\Big(\int_A f(x)\tilde{N}_t(\cdot, dx)\Big)\Big] \\
&\geq E\Big[\Delta Z_T^c \Delta\Big(\int_A f(x)\tilde{N}_T(\cdot, dx)\Big)\Big] \\
&> 0.
\end{aligned}$$

The above implies that

$$E\Big[Z_t^c \cdot \int_A f(x)\tilde{N}(\cdot, dx)\Big] \neq 0. \tag{2.2}$$

But

$$E\bigg(Z_t^c \cdot \int_A f(x)\tilde{N}_t(\cdot, dx)\bigg) = E\bigg[\lim_{n\to\infty}(Z_t - M_t^{A_n}) \cdot \int_A f(x)\tilde{N}(\cdot, dx)\bigg].$$

Remembering that the limit under consideration here is the L^2-limit and since the L^2 convergence implies convergence in expectation therefore

$$\begin{aligned} E\bigg(Z_t^c \cdot \int_A f(x)\tilde{N}(\cdot, dx)\bigg) &= \lim_{n\to\infty} E\bigg[(Z_t - M_t^{A_n}) \cdot \int_A f(x)\tilde{N}(\cdot, dx)\bigg] \\ &= 0 \qquad \text{which contradicts (2.2).} \end{aligned}$$

Therefore B has measure 0 as required to prove. □

Proposition 2.2.4. *Z_t^c is Gaussian.*

Theorem 2.2.5. *Let X be a Lévy process. Then $X_t = Y_t + Z_t$ where Y_t and Z_t are both Lévy processes and*
$\circ Y_t$ is a martingale with bounded jumps
$\circ Z_t$ has paths of finite variation on compact sets.

Theorem 2.2.6. *(Lévy-Itô decomposition)*
Let X be a Lévy process and ν its Lévy measure, then
$\circ$ ν is a random measure such that:

$$\int_{|x|\leq 1} |x^2|\nu(dx) < \infty$$

and

$$\int_{|x|\geq 1} \nu(dx) < \infty.$$

$\circ$ The jump measure $N_t(\cdot, dx)$ is a Poisson random measure on $[0,\infty] \times \mathbb{R}^d$ with intensity $d\nu(x)dt$.
$\circ$ There exists a vector γ and a Brownian motion (B_t) with covaviance matrix A such that:

$$X_t = \gamma t + B_t + X_t^d + \hat{X}_t^d$$

where

$$\hat{X}_t^d = \int_{|x|\geq 1} x N_t(\cdot, dx)$$

and

$$X_t^d = \int_{|x|\leq 1} x N_t(\cdot, dx) - t\int_{|x|\leq 1} \nu(dx).$$

The triplet (A, ν, γ) is called the ***characteristic triplet*** *of the lévy process X.*

2.3 Semimartingales

Definition 2.3.1. *(Semimartingale)*
An adapted càdlàg process X is called a ***semimartingale*** *if the stochastic integral of simple predictable processes, say on $[0, t]$ with respect to X:*

$$\phi = \phi_0 1_{[t=0]} + \sum_{i\geq 0} \phi_i \mathbf{1}_{(T_i, T_{i+1}]} \mapsto \int_0^t \phi(X) dX = \phi_0 X_0 + \sum_{i\geq 0} \phi_i (X_{T_{i+1}} - X_{T_i})$$

verifies the following continuity property:
for any sequence (ϕ^n) of simple functions on $[0, t]$,

$$\sup_{(s,\omega)\in[0,t]\times\Omega} |\phi_s^n(\omega) - \phi_s(\omega)| \to 0, \textit{ then } \int_0^t \phi^n dX \to \int_0^t \phi dX$$

in probability as $n \to \infty$.

The above definition implies the following proposition.

Proposition 2.3.2. *Let $S = (S_t)$, $t \in [0, T]$ be a semimartingale. Consider ϕ and a sequence (ϕ^n), all in the set of simple functions defined on $[0, T]$, say $\mathbb{S}_T$.*
If

$$\sup_{(t,\omega)\in[0,T]\times\Omega} |\phi_t^n(\omega) - \phi_t(\omega)| \longrightarrow 0 \qquad uniformally,$$

then

$$\sup_{t\in[0,T]} \left| \int_0^t \phi^n dS - \int_0^t \phi dS \right| \longrightarrow 0 \qquad \textit{in probability.}$$

A straightforward consequence of the above proposition is that
(1) Every process with finite variation is a semimartingale.
In fact, let $V_T(S)$ be the variation of S on $[0,T]$ then,

$$\sup_{t\in[0,T]}\int_0^t \phi dS \leq \sup_{(t,\omega)\in[0,T]\times\Omega} |\phi_t(\omega)\cdot V_T(S)|$$

according to definition 2.1.8 Another consequence is that
(2) Any square integrable martingale is a semimartingale.
To see that let M be a square integrable martingale then,

$$E\left[\left(\int_0^t \phi dM\right)^2\right] = E\left[\left(\phi_0 M_0 + \sum_{i=0}^{n}\phi_i(M_{T_{i+1}\wedge t} - M_{T_i\wedge t})\right)^2\right]$$

by Doob sampling theorem,

$$\begin{aligned}
&= E\left[\phi_0^2 M_0^2 + \sum_{i=0}^{n}\phi_i^2(M_{T_{i+1}\wedge t} - M_{T_i\wedge t})^2\right] \\
&\leq \sup_{s,\omega}|\phi_s(\omega)|E\left[M_0^2 + \sum_{i=0}^{n}(M_{T_{i+1}\wedge t} - M_{T_i\wedge t})^2\right] \\
&\leq \sup_{s,\omega}|\phi_s(\omega)|E\left[M_0^2 + \sum_{i=0}^{n}(M_{T_{i+1}\wedge t}^2 - M_{T_i\wedge t}^2)\right] \\
&\leq \sup_{s,\omega}|\phi_s(\omega)|E[M_s^2].
\end{aligned}$$

Therefore

$$\begin{aligned}
E\left[\left(\int_0^t \phi^n dM - \int_0^t \phi dM\right)^2\right] &\leq \sup_{s,\omega}|\phi_s^n - \phi_s(\omega)|E[M_s^2] \\
&\leq \epsilon E[M_s^2] \to 0 \qquad as\ n \to \infty.
\end{aligned}$$

This implies that a Brownian motion is a semimartingale. It can be shown that any linear combination of semimartingales is again a semimartingale; they define a vector space. Since any Lévy process can be decomposed into a Brownian motion with drift plus a process of finite variation, then all Lévy processes are semimartingales.

Definition 2.3.3. *(quadratic variation of a semimartingale)*

*The **quadratic variation** process of a semimartingale X is an adapted càdlàg process and it is defined as:*

$$[X,X]_t := |X_t|^2 - 2\int_0^t X_{u^-} dX_u.$$

and if (π^n) is a sequence of partitions of $[0,t]$ such that

$$\sup |t_k^n - t_{k-1}^n| \to 0 \text{ as } (n \to \infty), \text{ then } \sum_{i \geq 0} (X_{t_{i+1}} - X_{t_i})^2 \to [X,X]_t$$

in probability as $n \to \infty$.

2.4 Itô Formula

Proposition 2.4.1. *(Itô formula for semimartingales)*
Let (X_t) be a semimartingale. For any $C^{1,2}$ function f:$[0,T] \times \mathbb{R} \to \mathbb{R}$,

$$\begin{aligned} f(t,X_t) - f(0,X_0) &= \int_0^t \frac{\partial f}{\partial s}(s,X_s)ds + \frac{\partial f}{\partial x}(s,X_{s^-})dX_s \\ &+ \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s,X_{s^-})d[X,X]_c^s \\ &+ \sum_{0 \leq s \leq t}^{\Delta X_s \neq 0} [f(s,X_s) - f(s,X_{s^-}) - \Delta X_s \frac{\partial f}{\partial x}(s,X_{s^-})]. \end{aligned}$$

Where $[X,X]_s^c$ is the continuous part of the process $[X,X]_s$.

Proof. Without loss of generality and to avoid heavy notations, here we suppose f is time independent. The proof will also hold in the case f is also time dependent. We use Taylor expansion of second order of f i.e. for any points x_1, x_2 in the domain of f,

$$f(x_2) = f(x_1) + f'(x_1)(x_2 - x_1) + \frac{1}{2} f''(x_1)(x_2 - x_1)^2 + r(x_2, x_1). \tag{2.3}$$

where $r(x_2, x_1)$ is a functions of the form:

$$(x_2 - x_1)^2 \xi(x_1, x_2) \;\; with \lim_{x_1 \to x_2} \xi(x_1, x_2) = 0. \tag{2.4}$$

Consider the following random partition $0 = T_0 < T_1 < \ldots < T_{n+1} = t$. Applying 2.3 to f and for n large enough we get

$$f(X_t) - f(X_0) = \sum_{i=0}^{n} [f(X_{T_{i+1}}) - f(X_{T_i})]$$

$$\begin{aligned} f(X_t) - f(X_0) = & \sum_{i=0}^{n} f'(X_{T_i})(X_{T_{i+1}} - X_{T_i}) + \frac{1}{2} f"(X_{T_i})(X_{T_{i+1}} - X_{T_i})^2 \\ & + \sum_{i=0}^{n} r(X_{T_{i+1}}, X_{T_i}). \end{aligned} \tag{2.5}$$

Step 1

If we let $sup|T_{i+1} - T_i| \to 0$ a.s. $\sum_{i=0}^{n} f'(X_{T_i})(X_{T_{i+1}} - X_{T_i})$ and $\sum_{i=0}^{n} f"(X_{T_i})(X_{T_{i+1}} - X_{T_i})^2$ become classical Riemann sums which converge to $\int_0^t f'(X_s) dX_s$ and $\int_0^t f"(X_s) d[X, X]_s$ respectively (see [3] proposition 8.4). Now let look at the last sum in equation (2.5)

The key here is to remember that X has finite quadratic variation so does $f(X)$, since f is continuous.

Take any $\epsilon > 0$ let $A \subset [0, t] \times \Omega$ such that $\sum_{0 \le s \le T} |\Delta X_s|^2 < \epsilon$ on A and let $B = \{(s, w) \notin A, \; such \; that \; \Delta X_s(\omega) \neq 0\}$. Roughly saying, A is the subset where X is continuous almost surely and B is the subset of $[0, T] \times \Omega$ where X visibly jumps. Though, the sum can be written as:

$$\sum_{i=0}^{n} r(X_{T_{i+1}}, X_{T_i}) = \sum_{B \cap (T_i, T_{i+1}] \neq \emptyset}^{n} r(X_{T_{i+1}}, X_{T_i}) + \sum_{B \cap (T_i, T_{i+1}] = \emptyset}^{n} r(X_{T_{i+1}}, X_{T_i}). \tag{2.6}$$

If $B \cap (T_i, T_{i+1}] = \emptyset, \quad \forall i \geq 0$, this implies that in those intervals X has no jumps (a.s) and hence continuous at T_i; so 2.4 applies, i.e.

$$r(X_{T_{i+1}}, X_{T_i}) = (X_{T_{i+1}} - X_{T_i})^2 \xi(X_{T_{i+1}}, X_{T_i}), \text{ with } \lim_{i \to \infty} \xi(X_{T_{i+1}}, X_{T_i}) = 0.$$

So for all $\epsilon > 0$ we can always make $\sup|T_{i+1} - T_i|$ small enough so that $\xi(X_{T_{i+1}}, X_{T_i})$ gets smaller than $\epsilon/[X, X]$.
Summing up over i from 1 up to n and letting $n \to \infty$ we get:

$$r(X_{T_{i+1}}, X_{T_i}) \leq (X_{T_{i+1}} - X_{T_i})^2 \frac{\epsilon}{[X, X]}, \qquad for\ i \geq 1$$

$$\begin{aligned}
\sum_{B\cap(T_i,T_{i+1}]=\emptyset}^{n} r(X_{T_{i+1}}, X_{T_i}) &\leq \sum_{B\cap(T_i,T_{i+1}]=\emptyset}^{n} (X_{T_{i+1}} - X_{T_i})^2 \frac{\epsilon}{[X, X]} \\
&= \frac{\epsilon}{[X, X]} \sum_{B\cap(T_i,T_{i+1}]\neq\emptyset}^{n} (X_{T_{i+1}} - X_{T_i})^2 \\
&\leq \frac{\epsilon}{[X, X]}[X, X] = \epsilon.
\end{aligned}$$

Therefore

$$\sum_{B\cap(T_i,T_{i+1}]=\emptyset}^{n} r(X_{T_{i+1}}, X_{T_i}) \to 0 \qquad as \quad \sup|T_{i+1} - T_i| \to 0 \qquad a.s.$$

For the second sum in 2.6 we recall the Taylor expansion in 2.3 and then rewrite $r(X_{T_{i+1}}, X_{T_i})$ as:

$$\begin{aligned}
\sum_{B\cap(T_i,T_{i+1}]\neq\emptyset}^{n} r(X_{T_{i+1}}, X_{T_i}) = &\sum_{B\cap(T_i,T_{i+1}]\neq\emptyset}^{n} f(X_{T_{i+1}}) - f(X_{T_i}) - f'(X_{T_i})|X_{T_{i+1}} - X_{T_i}| \\
&- \frac{1}{2} \sum_{B\cap(T_i,T_{i+1}]\neq\emptyset}^{n} f"(X_{T_i})|X_{T_{i+1}} - X_{T_i}|^2.
\end{aligned}$$

Observe that as the refinery gets smaller, the union of the family $\Big(B \cap (T_i, T_{i+1}] \neq \emptyset\Big)_{i\geq 0}$ increases to B. So

$$\begin{aligned}
&\sum_{B\cap(T_i,T_{i+1}]\neq\emptyset}^{n} \Big[f(X_{T_{i+1}}) - f(X_{T_i}) - f'(X_{T_i})|X_{T_{i+1}} - X_{T_i}| - \frac{1}{2}f"(X_{T_i})|X_{T_{i+1}} - X_{T_i}|^2\Big] \\
&\to \sum_{B} f(X_s) - f(X_{s^-}) - \Delta X_s f(X_{s^-}) - \frac{1}{2}f"(X_{s^-})|\Delta X_s|^2 \qquad (a.s).
\end{aligned}$$

Let see the mode of convergence. If the process X is bounded i.e. there exists a constant $K > 0$ such that $|X| \leq K$.

Since f" will be continuous on $[-K, K]$, there exists a positive constant c such that $|f"(x)| \leq c$ for all $|x| \leq K$. Moreover, we can always make $sup|T_{i+1} - T_i|$ small enough so that $\xi(X_{T_{i+1}}, X_{T_i}) < c/2$. Thus,

$$|f(s,X_s) - f(s,X_{s^-}) - \Delta X_s f'(X_{s^-})| \leq \frac{c}{2}|\Delta X_s|^2 + \frac{c}{2}|\Delta X_s|^2, \ (0 \leq s \leq t, \Delta X_s \neq 0)$$

$$\begin{aligned}\sum_{0\leq s\leq t} \Big| f(s,X_s) - f(s,X_{s^-}) - \Delta X_s f'(X_{s^-})| &\leq c \sum_{0\leq s\leq t} \Big| \Delta(X_s)|^2 \\ &\leq c[X,X]_t < \infty\end{aligned}$$

for the same reason,

$$c \sum_{0\leq s\leq t} |f"(X_{s^-})\Delta X_s|^2 \leq c[X,X]_t < \infty.$$

Now if X is not bounded, we can always replace it by $X1_{[0,T_K]}$ where $T_K = \inf\{t > 0, |X_t| \geq K\}$ and repeat the same analysis as for the case X is bounded. If we let K tend to infinity the convergence will not suffer we always get same result i.e. When eventually A covers $[0,t] \times \Omega \backslash B$ and as the refinery gets very small,

$$\begin{aligned}&\sum_{B\cap(T_i,T_{i+1}]\neq\emptyset} \left[f(X_{T_{i+1}}) - f(X_{T_i}) - f'(X_{T_i})(X_{T_{i+1}} - X_{T_i}) + \frac{1}{2}f"(X_{T_i})(X_{T_{i+1}} - X_{T_i})^2 \right] \\ &\to \sum_{0\leq s\leq t} \left[f(s,X_s) - f(s,X_{s^-}) - \Delta X_s f'(X_{s^-}) - f"(X_{s^-})|\Delta(X_s)|^2 \right] \ uniformally.\end{aligned}$$

therefore

$$\begin{aligned}f(X_t) - f(X_0) &= \int_0^t f'(X_{s^-})dX_s + \frac{1}{2}\int_0^t f"(X_s)d[X,X]_s \\ &+ \sum_{0\leq s\leq t} f(s,X_s) - f(s,X_{s^-}) - \Delta X_s f'(X_{s^-}) - f"(X_{s^-})|\Delta X_s|^2.\end{aligned}$$

Since,

$$\int_0^t f"(X_{s^-})d[X,X]_s = \int_0^t f"(X_{s^-})d[X,X]^c_s + \sum_{0\leq s\leq t} f"(X_{s^-})|\Delta X_s|^2.$$

Hence,

$$\begin{aligned}
f(X_t) - f(X_0) &= \int_0^t f'(X_{s-})dX_s + \frac{1}{2}\int_0^t f"(X_s)d[X,X]_s^c \\
&+ \frac{1}{2}\sum_{0\leq s\leq t} f"(X_{s-})|\Delta(X_s)|^2 \\
&+ \sum_{0\leq s\leq t} f(s,X_s) - f(s,X_{s-}) - \Delta X_s f'(X_{s-}) \\
&- \frac{1}{2}\sum_{0\leq s\leq t} f"(X_{s-})|\Delta X_s|^2. \\
f(X_t) - f(X_0) &= \int_0^t f'(X_{s-})dX_s + \frac{1}{2}\int_0^t f"(X_s)d[X,X]_s^c \\
&+ \sum_{0\leq s\leq t} f(s,X_s) - f(s,X_{s-}) - \Delta X_s f'(X_{s-}).
\end{aligned}$$

□

Proposition 2.4.2. *(Itô formula for multidimensional Lévy processes) Let $X_t = (X_t^1, ..., X_t^d)$ be a multidimensional Lévy process with characteristic triplet (A, ν, γ). Then for any $C^{1,2}$ function $f : [0,T] \times \mathbb{R}^d \to \mathbb{R}$,*

$$\begin{aligned}
f(t,X_t) - f(0,X_0) &= \int_0^t \sum_{i=1}^d \frac{\partial f}{\partial x^i}(s,X_s)dX_s^i + \int_0^t \frac{\partial f}{\partial s}(s,X_s)ds \\
&+ \frac{1}{2}\int_0^t \sum_{i,j=1}^d A_{ij}\frac{\partial^2 f}{\partial x_i \partial x_j}(s,X_s)ds \\
&+ \sum_{0\leq s\leq t}\left[f(X_{s^-} + \Delta X_i) - f(X_{s_k^-}) - \sum_{i=1}^d \Delta X_{s^-}^i \frac{\partial f}{\partial x^i}(s,X_s)\right].
\end{aligned}$$

Having set this background we may move on to the next level where we look at a particular class of Lévy processes which are called affine jump-diffusion type and also their application in finance.

Chapter 3

Affine jump diffusions

3.1 Definition

Affine jump diffusions is a class of processes that include many of the most commonly used dynamics in the modeling of the underlying asset price process of financial instruments. They are quite flexible since they can accommodate many parameters and mostly, their analysis allows us to derive closed form formulas for asset prices. In order to introduce Affine jump diffusion let first recall the usual stochastic differential equation (SDE) that drive the dynamics of the underlying asset X. In presence of jumps, the SDE is of the form:

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t + dZ_t, \tag{3.1}$$

where $\mu(X_t, t)$ is the drift term, $\sigma(X_t, t)$ is the volatility of the asset X, (W_t) is a standard Brownian motion and (Z_t) is a pure jump process, usually a Poisson process with intensity λ under the risk neutral measure.

To get an Affine jump diffusion process we simply impose an affine structure on $\mu(X_t, t)$, $\sigma(X_t, t)\sigma(X_t, t)^\top$, λ and the discount rate factor R in the dynamics of X. That is, we assume they are all affine functions of X on $\mathbb{R}^d$. In other words they are of the form:

$\mu(X_t, t) = K_0 + K_1 x$ where $K_0 \in \mathbb{R}^d, K_1 \in \mathbb{R}^{d\times d}$

$\left(\sigma(X_t,t)\sigma(X_t,t)^\top\right)_{ij} = (H_0)_{ij} + (H_1)_{ij}x$ where $H_0 \in \mathbb{R}^{d\times d}, H_1 \in \mathbb{R}^{d\times d\times d}$
$\lambda(x) = l_0 + l_1 \cdot x$ where $l_0 \in \mathbb{R}, l_1 \in \mathbb{R}^d$
$R(x) = \rho_0 + \rho_1 \cdot x$ where $\rho_0 \in \mathbb{R}, \rho_1 \in \mathbb{R}^d$
Note that these coefficients K, H, λ, R may be time-dependent.
X can be viewed as the state vector of the portfolio with d components ($d > 1$), say $X^{(1)}, X^{(2)}, \ldots, X^{(d)}$. For example one of them can be the volatility. The drift term $\mu^{(i)}$of a specific component $X^{(i)}$ is the i^{th} component of $\mu(X)$ and it is an affine combination of the $X^{(j)}$'s $(1 \leq j \leq d)$ i.e.

$$\mu^{(i)} = (\mu(X_t,t))^{(i)} = K_0^{(i)} + \sum_{j=1}^{d} K_1^{(ij)} X^{(j)}.$$

The same comment applies to $\sigma(X_t,t)\sigma(X_t,t)^\top$. The tensor matrix H_1 can be seen as a d-dimensional vector whose entries are $d \times d$ matrices. We will denote by $H_1^{(ij),k}$ the component (i,j) of the matrix at position k in the vector. For our application we treat the case $d = 2$, we introduce the following representation for H_1:

$$H_1 = \begin{pmatrix} a_1 & a_2 & \vdots & b_1 & b_2 \\ a_3 & a_4 & \vdots & b_3 & b_4 \end{pmatrix}$$

In this case, the first component and second component are:

$$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \qquad a_i, b_i \in \mathbb{R},\ i = 1,2,3,4.$$

The product $H_1 x$ with $x = (x_1, x_2) \in \mathbb{R}^2$ is simply defined as:

$$H_1 x = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} x_1 + \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} x_2.$$

Examples of Affine jump diffusion models are models with affine term structures such as Vasicek model, Ho-Lee model as well described in [2]. For instance in the Vasicek model the underlying process X is exactly the interest rate itself denoted by r. Its dynamics are given by:

$$dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))dW_t$$

where

$$\mu(t, r(t)) = \alpha(t)r(t) + \beta(t)$$

$$\sigma^2(t, r(t)) = \gamma(t)r(t) + \delta(t).$$

Comparing this with the above affine settings we see that we are in the one dimensional case i.e. $d = 1$ and also,

$$R = r,\ X = r \text{ which implies } \rho_0 = 0,\ \rho_1 = 1$$

$$K_0 = \beta(t),\ K_1 = \alpha(t),\ H_0 = \delta(t),\ H_1 = \gamma(t),\ l_0 = 0,\ l_1 = 0.$$

We will get back to these affine term structure models in the next section.
Define

$$\theta : \mathbb{C}^d \to \mathbb{R}_+, c \mapsto \int_{\mathbb{R}^d} e^{c \cdot z} d\nu(z).$$

The function θ is called the ***jump transform*** of X. Technically it is the moment generating function of the jumps, hence it characterizes the "jump size" distribution.
The coefficients (K, H, l, θ, ρ) characterizes completely the distribution of X and R given $X(0)$. For this reason we introduce a characteristic $\chi = (K, H, l, \theta, \rho)$.

We are now in the position to introduce our first transform.

3.2 Transform analysis

Let us consider the function ψ^χ defined as:

$$\psi^\chi(u, X_t, t, T) = E^\chi\left[e^{-\int_t^T R(X_s)ds} e^{u \cdot X_T} \mid \mathcal{F}_t\right] \tag{3.2}$$

where E^{χ} is the expectation under the distribution of X and R determined by χ, $u \in \mathbb{R}^d$ is a vector.
Before we move forward, let us define

$$\Psi_t = e^{-\int_0^t R(X_s)ds} e^{\alpha(t)+\beta(t)\cdot X_t}$$

where α is a deterministic function of $\mathbb{R} \to \mathbb{R}$ and β a function $\mathbb{R} \to \mathbb{R}^d$
Notation: From now onwards we will use sometimes the notation $\dot{a}(t)$ to denote $\frac{\partial a}{\partial t}(t)$ given that a is a function of one variable t. Whenever there is no ambiguity we will often use notation f_s instead of $f(X_s, s)$.

Proposition 3.2.1. *(Transform 1)*
The transform ψ^{χ} of X defined in (3.2) *satisfies:*

$$\psi^{\chi}(u, X_t, t, T) = e^{\alpha(t)+\beta(t)\cdot x} \tag{3.3}$$

provided that
(i) α and β are unique solutions of:

$$\frac{\partial \alpha}{\partial t}(t) = \rho_0 - K_0 \cdot \beta(t) - \frac{1}{2}\beta(t)^{\top} H_0 \beta(t) - l_0\Big[\theta(\beta(t)) - 1\Big] \tag{3.4}$$

$$\frac{\partial \beta}{\partial t}(t) = \rho_1 - K_1^{\top} \beta(t) - \frac{1}{2}\beta(t)^{\top} H_1 \beta(t) - l_1\Big[\theta(\beta(t)) - 1\Big] \tag{3.5}$$

with boundary condition $\beta(T) = u$ and $\alpha(T) = 0$.

$$(ii) E\left(\int_0^T |\gamma_t| dt\right) < \infty \text{ where } \gamma_t = \Psi_t\Big[\theta(\beta(t)) - 1\Big]\lambda(X_t) \tag{3.6}$$

$$(iii) E\left[\left(\int_0^T \eta_t \cdot \eta_t dt\right)^{1/2}\right] < \infty, \text{ where } \eta_t = \Psi_t \beta(t)^{\top} \sigma(X_t) \tag{3.7}$$

$$(iv) E(|\Psi_T|) < \infty \tag{3.8}$$

.

Proof. Let first prove that Ψ is a martingale.
To see that let us apply Itô formula as in proposition 2.4.2 to the function

$\Psi_t = f(t, X_t)$ we get

$$\Psi(X_t, t) - \Psi(X_0, 0) = \int_0^t \frac{\partial \Psi}{\partial s}(X_s, s) + \mu(X_s)\frac{\partial \Psi}{\partial X_s}(X_s, s) + \frac{\sigma(X_s)\sigma(X_s)^\top}{2}\frac{\partial^2 \Psi}{\partial {X_s}^2}(X_s, s)ds + \int_0^t \sigma(X_s)\frac{\partial \Psi}{\partial X_s}(X_s, s)dW_s + \sum_{0\le\tau_i\le t} \Psi(X_{\tau_i^-} + \Delta X_i) - \Psi(X_{\tau_i^-}) \tag{3.9}$$

Define a new process $J(X_t)$ as

$$J(X_t) = \sum_{0\le\tau_i\le t} \Psi(X_{\tau_i^-} + \Delta X_i) - \Psi(X_{\tau_i^-})$$

$J(X_t)$ is in fact the jump process associated with the process $\big(\Psi(X_t)\big)$.
Then

$$\begin{aligned} J(X_t) = &\int_0^t \int_{\mathbb{R}\backslash 0} \Psi(x_s + z) - \Psi(x_s)d\nu(z, s) + \sum_{0\le\tau_i\le t} \Psi(X_{\tau_i^-} + \Delta X_i) - \Psi(X_{\tau_i^-}) \\ &- \int_0^t \int_{\mathbb{R}\backslash 0} \Psi(x_s + z) - \Psi(x_s)d\nu(z, s) \\ = &\int_0^t \int_{\mathbb{R}\backslash 0} e^{-\int_0^s R(x_u)du + \alpha(s) + \beta(s)\cdot(x_s+z)} - e^{-\int_0^s R(x_u)du + \alpha(s) + \beta(s)\cdot x_s} d\nu(z, s) + \tilde{J}(X_t) \end{aligned}$$

where
$\tilde{J}(X_t) = \sum_{0\le\tau_i\le t} \Psi(X_{\tau_i^-} + \Delta X_i) - \Psi(X_{\tau_i^-}) - \int_0^t \int_{\mathbb{R}\backslash 0} \Psi(x_s + z) - \Psi(x_s)d\nu(z, s)$.
Now,

$$\begin{aligned} J(X_t) &= \int_0^t \int_{\mathbb{R}\backslash 0} e^{-\int_0^s R(x_u)du + \alpha(s) + \beta(s)\cdot x_s}[e^{\beta(s)z} - 1]d\nu(z)ds + \tilde{J}(X_t) \\ &= \int_0^t e^{-\int_0^s R(x_u)du + \alpha(s) + \beta(s)\cdot x_s} \int_{\mathbb{R}} e^{\beta(s)z} - 1 d\nu(z)ds + \tilde{J}(X_t) \\ &= \int_0^t \Psi(X_s) \int_{\mathbb{R}} [e^{\beta(s)z} - 1]\lambda(X_s)d\nu_0(z)ds + \tilde{J}(X_t) \end{aligned}$$

where ν_0 is the Lévy probability measure associated with the process (X_t).

So,

$$\begin{aligned}J(X_t) &= \int_0^t \Psi(X_s)\lambda(X_s)\left[\int_R e^{\beta(s)z}d\nu_0(z) - \int_R d\nu_0(z)\right]ds + \tilde{J}(X_t)\\ &= \int_0^t \Psi(X_s)\lambda(X_s)\Big[\theta(\beta(s)) - 1\Big]ds + \tilde{J}(X_t).\end{aligned}$$

In the same way it can be shown that

$$\tilde{J}(X_t) = \sum_{0\leq\tau_i\leq t} \Psi(X_{\tau_i^-} + \Delta X_i) - \Psi(X_{\tau_i^-}) - \int_0^t \Psi(X_s)\lambda(X_s)[\theta(\beta(s)) - 1]ds.$$

$\tilde{J}$ is a martingale since its expectation is equal to 0 and its increments are function of those of X so, they are $\mathcal{F}_t$-measurable and independent as well.

So equation (3.9) becomes then:

$$\begin{aligned}\Psi(X_t,t) - \Psi(X_0,0) &= \int_0^t \frac{\partial\Psi}{\partial s}(X_s,s) + \mu(X_s)\frac{\partial\Psi}{\partial X_s}(X_s,s) + \frac{\sigma(X_s)\sigma(X_s)^\top}{2}\frac{\partial^2\Psi}{\partial {X_s}^2}(X_s,s)ds\\ &+ \int_0^t \sigma(X_s)\frac{\partial\Psi}{\partial X_s}(X_s,s)dW_s + \int_0^t \Psi(X_s)\lambda(X_s)[\theta(\beta(s)) - 1]ds\\ &+ \tilde{J}(X_t).\\ &= \int_0^t \frac{\partial\Psi}{\partial s}(X_s,s) + \mu(X_s)\frac{\partial\Psi}{\partial X_s}(X_s,s) + \frac{\sigma(X_s)\sigma(X_s)^\top}{2}\frac{\partial^2\Psi}{\partial {X_s}^2}(X_s,s)\\ &\quad + \lambda(X_s)[\theta(\beta(s)) - 1]ds + \int_0^t \sigma(X_s)\frac{\partial\Psi}{\partial X_s}(X_s,s)dW_s + \tilde{J}(X_t).\end{aligned}$$

So, Ψ becomes martingale if and only if the drift term becomes zero, this means the following equation must be satisfied:

$$\begin{aligned}\frac{\partial\Psi}{\partial s}(X_s,s) + \mu(X_s)\frac{\partial\Psi}{\partial X_s}(X_s,s) + \frac{\sigma(X_s)\sigma(X_s)^\top}{2}\frac{\partial^2\Psi}{\partial {X_s}^2}(X_s,s)\\ + \lambda(X_s)\Psi(X_s)[\theta(\beta(s)) - 1] = 0.\end{aligned} \tag{3.10}$$

Given $\Psi(X_t,t) = e^{-\int_0^t R(X_s)ds}\cdot e^{\alpha(t)+\beta(t)\cdot X_t}$, then

$$
\begin{aligned}
\frac{\partial \Psi}{\partial t}(X_t,t) &= [-R(X_t)+\dot{\alpha}(t)+\dot{\beta}(t)X_t]\Psi(X_t,t)\\
\frac{\partial \Psi}{\partial X}(X_t,t) &= \beta(t)\Psi(X_t,t)\\
\frac{\partial^2 \Psi}{\partial X^2}(X_t,t) &= \beta(t)\Psi(X_t,t)\beta(t)^\top.
\end{aligned}
$$

Then the above equation (3.10) becomes

$$
\begin{aligned}
&\big[-R(X_t)+\dot{\alpha}(t)+\dot{\beta}(t)X_t]\Psi(X_t,t)+\mu(X_t)\beta(t)\Psi(X_t,t)\\
&\quad+\beta(t)\sigma(X_t)\sigma(X_t)^\top\frac{\Psi(X_t,t)}{2}\beta(t)^\top+\lambda(X_t)\Psi(X_t)\big[\theta(\beta(t))-1\big]=0
\end{aligned}
$$

hence

$$
-R(X_t)+\dot{\alpha}(t)+\dot{\beta}(t)X_t+\mu(X_t)\beta(t)+\frac{1}{2}\beta(t)\sigma(X_t)\sigma(X_t)^\top\beta(t)^\top+\lambda(X_t)[\theta(\beta(t))-1]=0
$$

Using the affine structure of R, μ, σ and λ as mentioned in 3.1, we get

$$
-\Big(\rho_0+\rho_1X_t\Big)+\dot{\alpha}(t)+\dot{\beta}(t)X(t)+\Big(K_0+K_1\cdot X_t\Big)\beta(t)+\frac{1}{2}\beta(t)\Big[H_0+H_1X(t)\Big]\beta(t)^\top
$$

$$
+\Big(l_0+l_1X_t\Big)\Big[\theta(\beta(t))-1\Big]=0.
$$

$$
\begin{aligned}
-\rho_0+\dot{\alpha}(t)+K_0\beta(t)+\frac{1}{2}\beta(t)^{\cdot}H_0\beta(t)+l_0\Big[\theta(\beta(t))-1\Big]&=0.\\
-\rho_1+\dot{\alpha}(t)+K_1^\top\beta(t)+\frac{1}{2}\beta(t)^\top H_1\beta(t)+l_1\Big[\theta(\beta(t))-1\Big]&=0.\\
\dot{\alpha}(t)=\rho_0-K_0\beta(t)-\frac{1}{2}\beta(t)H_0\beta(t)^\top-l_0\Big[\theta(\beta(t))-1\Big]&.\\
\dot{\beta}(t)=\rho_1-K_1^\top\beta(t)-\frac{1}{2}\beta(t)H_1\beta(t)^\top-l_1\Big[\theta(\beta(t))-1\Big]&.
\end{aligned}
$$

So under these ordinary differential equations Ψ is a martingale. From then we write,

$$
\Psi_t=E\Big(\Psi_T|\mathcal{F}_t\Big).
$$

$$
\text{Then,}\qquad \Psi_te^{\int_0^t R(X_s)ds}=e^{\int_0^t R(X_s)ds}E\Big(\Psi_T|\mathcal{F}_t\Big)
$$

$$
e^{\int_0^t R(X_s)ds}e^{-\int_0^t R(X_s)ds}e^{\alpha(t)+\beta(t)\cdot x}=E\Big[e^{\int_0^t R(X_s)ds}\Psi_T|\mathcal{F}_t\Big]
$$

$$\begin{aligned}
e^{\alpha(t)+\beta(t)\cdot X_t} &= E\Big[e^{\int_0^t R(X_s)ds}e^{-\int_0^T R(X_s)ds}e^{\alpha(T)+\beta(T)\cdot X_T}|\mathcal{F}_t\Big] \\
&= E\Big[e^{-\int_t^T R(X_s)ds}e^{\alpha(T)+\beta(T)\cdot X_T}|\mathcal{F}_t\Big] \\
&= E\Big[e^{-\int_t^T R(X_s)ds}e^{u\cdot X_T}|\mathcal{F}_t\Big], \text{ since } \alpha(T)=0 \text{ and } \beta(T)=u
\end{aligned}$$

on the boundary. Therefore

$$\Psi_t e^{\int_0^t R(X_s)ds} = \psi^\chi(u, X_t, t, T).$$

□

We next consider another transform ϕ^χ defined as a conditional expection of the product of a linear function of X and the exponential of X. That is, we define ϕ^χ as:

$$\phi^\chi(v, u, X_t, t, T) = E^\chi\Big[e^{-\int_t^T R(X_s)ds} v\cdot X_T e^{u\cdot X_T}|\mathcal{F}_t\Big]$$

Proposition 3.2.2. *(Transform 2)*
Under some technical conditions the transform ϕ^χ is given by,

$$\phi^\chi(v, u, x, t, T) = \psi^\chi(u, x, t, T)\Big(A(t) + B(t)\cdot x\Big)$$

where ψ^χ is given by (3.3) *and where A and B satisfy the linear differential equations:*

$$\begin{aligned}
-\dot{B}(t) &= -K_1^\top B(t) + \beta(t)^\top H_1 B(t) + l_1\nabla\theta(\beta(t))B(t) \qquad (3.11)\\
-\dot{A}(t) &= -K_0\cdot B(t) + \beta(t)^\top H_0 B(t) + l_0\nabla\theta(\beta(t))B(t) \qquad (3.12)
\end{aligned}$$

Where $\nabla\theta(c)$ is the gradient of $\theta(c)$ with respect to c.

Proof. This proof picks up its foundations in the proof of Transform 1.
We recall

$$\Psi_t = \Psi(X_t, t) = e^{-\int_0^t R(X_s)ds}e^{\alpha(t)+\beta(t)\cdot X_t}$$

and equation (3.10) :

$$\frac{\partial \Psi_t}{\partial t} + \mu \frac{\partial \Psi_t}{\partial X} + \frac{\sigma\sigma^\top}{2}\frac{\partial^2 \Psi_t}{\partial X^2} + \lambda\Big[\theta(\beta) - 1\Big] = 0 \tag{3.13}$$

provided that α and β satisfy (3.4) and (3.5) uniquely.
Let us apply Itô formula to the process Φ_t defined by:

$$\Phi_t = \Big(A(t) + B(t)\cdot X_t\Big)\Psi_t. \tag{3.14}$$

i.e,

$$\begin{aligned}\Phi_t - \Phi_0 &= \int_0^t \frac{\partial \Phi_s}{\partial s} + \mu_s \frac{\partial \Phi_s}{\partial X} + \frac{\sigma_s\sigma_s^\top}{2}\frac{\partial^2 \Phi_s}{\partial X^2} ds \\ &+ \int_0^t \sigma_s \frac{\partial \Phi_s}{\partial X} dW_s + \sum_{0\le\tau_i\le t} \Phi(X_{\tau_i^-} + \Delta X_i) - \Phi(X_{\tau_i^-})\end{aligned} \tag{3.15}$$

where μ_t, σ_t stand for $\mu(X_t,t), \sigma(X_t,t)$ and Let

$$J_{\Phi_t} = \sum_{0\le\tau_i\le t} \Phi(X_{\tau_i^-} + \Delta X_i) - \Phi(X_{\tau_i^-})$$

Using the same arguments as in the proof of Transform 1 we get

$$\begin{aligned}J_{\Phi_t} &= \int_0^t \int_{\mathbb{R}} \Phi(x_s + z) - \Phi(x_s) d\nu(z,s) + \sum_{0\le\tau_i\le t} \Phi(X_{\tau_i^-} + \Delta X_i) - \Phi(X_{\tau_i^-}) \\ &\quad - \int_0^t \int_{\mathbb{R}\backslash 0} \Phi(x_s + z) - \Phi(x_s) d\nu(z,s). \\ &= \int_0^t \int_{\mathbb{R}\backslash 0} e^{-\int_0^s R(x_u)du}\Big(A(s) + B(s)\cdot(x_s + z)\Big)e^{\alpha(s)+\beta(s)\cdot(x_s+z)} \\ &+ \qquad -e^{-\int_0^s R(x_u)du}\Big(A(s) + B(s)\cdot x_s\Big)e^{\alpha(s)+\beta(s)\cdot x_s} d\nu(z)ds + \tilde{J_{\Phi_t}}. \\ &= \int_0^t e^{-\int_0^s R(x_u)du} e^{\alpha(s)+\beta(s)\cdot x_s}\Bigg[\int_{\mathbb{R}\backslash 0}\Big(A(s) + B(s)\cdot(x_s+z)\Big)e^{\beta(s)\cdot z} \\ &\quad - \Big(A(s) + B(s)\cdot(x_s)\Big)d\nu(z)\Bigg] ds + \tilde{J_{\Phi_t}}.\end{aligned}$$

$$
\begin{aligned}
J_{\Phi_t} &= \int_0^t \Psi_s \int_{\mathbb{R}\backslash 0} \Big(A(s) + B(s)\cdot(x_s+z)\Big)e^{\beta(s)\cdot z} - \Big(A(s) + B(s)\cdot(x_s)\Big)d\nu(z)ds \\
&\quad + \tilde{J_{\Phi_t}}. \\
&= \int_0^t \Psi_s \Big(\int_{\mathbb{R}\backslash 0} A(s)e^{\beta\cdot z} - A(s)d\nu(z) + B(s)\cdot X_s \int_{\mathbb{R}\backslash 0} e^{\beta\cdot z} - 1d\nu(z) \\
&\quad + \int_{\mathbb{R}\backslash 0} B(t)\cdot ze^{\beta\cdot z}d\nu(z)\Big)ds + \tilde{J_{\Phi_t}}. \\
&= \int_0^s \Psi_s \Big[\Big(A(s) + B(s)\cdot X_s\Big)\int_{\mathbb{R}\backslash 0} e^{\beta\cdot z} - 1d\nu(z) + B(s)\int_{\mathbb{R}\backslash 0} ze^{\beta\cdot z}d\nu(z)\Big]ds \\
&\quad + \tilde{J_{\Phi_t}}. \\
&= \int_0^t \Psi_s \Big[\Big(A(s) + B(s)\cdot X_s\Big)\lambda_s\int_{\mathbb{R}\backslash 0} e^{\beta\cdot z} - 1d\nu_0(z) + B(s)\int_{\mathbb{R}\backslash 0} ze^{\beta\cdot z}d\nu(z)\Big]ds \\
&\quad + \tilde{J_{\Phi_t}}.
\end{aligned}
$$

where ν_0 is the Lévy probability measure associated with the process (X_t) and λ_s just stands for $\lambda(X_s, s)$.
So,

$$
\begin{aligned}
J_{\Phi_t} &= \int_0^t \Psi_s \Big[\Big(A(s) + B(s)\cdot X_s\Big)\lambda_s[\theta(\beta) - 1] + B(s)\int_{\mathbb{R}\backslash 0} \nabla e^{\beta\cdot z}d\nu(z)\Big]ds + \tilde{J_{\Phi_t}}. \\
&= \int_0^t \Psi_s \Big[\Big(A(s) + B(s)\cdot X_s\Big)\lambda_s[\theta(\beta(s)) - 1] + \nabla\int_{\mathbb{R}\backslash 0} e^{\beta\cdot z}d\nu(z)\cdot B(s)\Big]ds + \tilde{J_{\Phi_t}} \\
&= \int_0^t \Psi_s \Big[\Big(A(s) + B(s)\cdot X_s\Big)\lambda_s[\theta(\beta(s)) - 1] + \nabla(\beta(s))\cdot B(s)\Big]ds + \tilde{J_{\Phi_t}}.
\end{aligned}
$$

From (3.14) We have

$$
\begin{aligned}
\dot{\Phi}_t &= \Big(\dot{A}(t) + \dot{B}(t)\cdot X_t)\Big)\Psi_t + \Big(A(t) + B(t)\cdot X_t\Big)\dot{\Psi}_t \\
&= \dot{A}(t)\Psi_t + \dot{B}(t)\cdot X_t)\Psi_t + A(t)\dot{\Psi}_t + B(t)\cdot X_t\dot{\Psi}_t \\
\frac{\partial\Phi_t}{\partial X} &= B(t)\Psi_t + \Big(A(t) + B(t)\cdot X_t\Big)\frac{\partial\Psi_t}{\partial X} \\
&= B(t)\Psi_t + A(t)\frac{\partial\Psi_t}{\partial X} + B(t)\cdot X_t\frac{\partial\Psi_t}{\partial X}
\end{aligned}
$$

and

$$\begin{aligned}\frac{\partial^2\Phi_t}{\partial X^2} &= B(t)\Psi_t + B(t)\Psi_t + \Big(A(t) + B(t)\cdot X_t\Big)\frac{\partial^2\Psi_t}{\partial X^2}\\ &= 2B(t)\Psi_t\frac{\partial\Psi_t}{\partial X} + A(t)\frac{\partial^2\Psi_t}{\partial X^2} + B(t)\cdot X_t\frac{\partial^2\Psi_t}{\partial X^2}.\end{aligned}$$

Hence, equation (3.15) becomes

$$\begin{aligned}\Phi_t - \Phi_0 = &\int_0^t \Big(\dot{A}(s)\Psi_s + \dot{B}(s)\cdot X_s\Psi_s + A(s)\dot{\Psi}_s + B(s)\cdot X_s\dot{\Psi}_s\Big)\\ &+ \mu_s\Big(B(s)\Psi_s + A(s)\frac{\partial\Psi_s}{\partial X} + B(s)\cdot X_s\frac{\partial\Psi_s}{\partial X}\Big)\\ &+ \frac{\sigma_s\sigma_s^\top}{2}\Big(2B(s)\Psi_s\frac{\partial\Psi_s}{\partial X} + A(s)\frac{\partial^2\Psi_s}{\partial X^2} + B(s)\cdot X_s\frac{\partial^2\Psi_s}{\partial X^2}\Big)ds\\ &+ \int_0^t \sigma_s\frac{\partial\Phi_s}{\partial X}dW_s\\ &+ \int_0^t \Psi_s\Big[\Big(A(s) + B(s)\cdot X_s\Big)\lambda_s[\theta(\beta(s)) - 1] + \nabla\theta(\beta(s))\cdot B(s)\Big]ds + \tilde{J_{\Phi_t}}.\end{aligned}$$

Φ is martingale if the drift term is zero that is,

$$\begin{aligned}&\dot{A}(t)\Psi_t + A(t)\dot{\Psi}_t + \mu_t A(t)\frac{\partial\Psi_t}{\partial X} + \frac{\sigma_t\sigma_t^\top}{2}A(t)\frac{\partial^2\Psi_t}{\partial X^2} + A(t)\Psi_t\lambda_t[\theta(\beta(t)) - 1]\\ &+ B(t)\cdot X_t\dot{\Psi}_t + \mu_t B(t)\cdot X_t\frac{\partial\Psi_t}{\partial X} + \frac{\sigma_t\sigma_t^\top}{2}B(t)\cdot X_t\frac{\partial^2\Psi_t}{\partial X^2} + B(t)\cdot X_t\Psi_t\lambda_t[\theta(\beta(t)) - 1]\\ &+ \mu_t B(t)\Psi_t + \frac{\sigma_t\sigma_t^\top}{2}2B(t)\frac{\partial\Psi_t}{\partial X} + \dot{B}(t)\cdot X_t\Psi_t + \Psi_t\lambda_t\nabla\theta(\beta(t))B(t) = 0.\end{aligned}$$

i.e.

$$\dot{A}(t)\Psi_t+\dot{B}(t)\cdot X_t\Psi_t+\Big(A(t)+B(t)\cdot X_t\Big)\Big[\dot{\Psi}_t+\mu_t\frac{\partial\Psi_t}{\partial X}+\frac{\sigma_t\sigma_t^\top}{2}\frac{\partial^2\Psi_t}{\partial X^2}+\lambda_t[\theta(\beta)-1]\Big]$$

$$+\mu_t B(t)\Psi_t + \sigma_t\sigma_t^\top B(t)\frac{\partial\Psi_t}{\partial X} + \Psi_t\lambda_t\nabla\theta(\beta(t))B(t) = 0$$

$$\dot{A}(t)\Psi_t + \dot{B}(t)\cdot X_t\Psi_t + \mu_t B(t)\Psi_t + \sigma_t\sigma_t^\top B(t)\frac{\partial\Psi_t}{\partial X} + \Psi_t\lambda_t\nabla\theta(\beta(t))B(t) = 0$$

$$\begin{aligned}&\dot{A}(t)\Psi_t + \dot{B}(t)\cdot X_t\Psi_t + K_0 B(t)\Psi_t + K_1 B(t)\cdot X_t\Psi_t + \beta^\top H_0 B(t)\Psi_t\\ &\quad + l_0\nabla\theta(\beta(t))B(t)\Psi_t + \beta^\top H_1\cdot X_t B(t)\Psi_t + l_1 X_t\nabla\theta(\beta(t))B(t)\Psi_t = 0.\end{aligned}$$

Thus

$$\dot{A}(t) + \dot{B}(t) \cdot X_t + K_0 B(t) + K_1^\top B(t) \cdot X_t + \beta^\top H_1 \cdot X_t B(t) + l_0 \nabla\theta(\beta(t))B(t)$$
$$+ l_1 X_t \nabla\theta(\beta(t))B(t) = 0$$

$$\dot{A}(t) + K_0 B(t) + \beta^\top H_0 B(t) + l_0 \nabla\theta(\beta(s)) = 0.$$
$$\dot{B}(t) + K_1^\top B(t) + \beta^\top H_1 B(t) + l_1 \nabla\theta(\beta(s)) = 0.$$

□

The last transform we introduce is the one that is more specific to option pricing in the sense that it reflects the pay-off of an european call option. This last is based on the Fourier-Stieltjes transform.
Define G to be

$$G_{a,b}(y; X_0, T, \chi) := E^\chi \left(e^{-\int_0^T R(X_s)ds} e^{a \cdot X_T} 1_{b \cdot X_T \leq y} \right) \tag{3.16}$$

Proposition 3.2.3. *The Fourier-Stieltjes transform $\mathcal{G}_{a,b}(\cdot; X_0, T, \chi)$ of $G_{a,b}(\cdot; X_0, T, \chi)$ exists and it is given by:*

$$\mathcal{G}_{a,b}(v; X_0, T, \chi) = \psi^\chi(a + ibv, X_0, 0, T) \tag{3.17}$$

Proof. Well,

$$\begin{aligned} \mathcal{G}_{a,b}(\cdot; X_0, T, \chi) &= \int_{\mathbb{R}} e^{ivy} dG_{a,b}(y; X_0, T, \chi) \\ &= E^\chi \left(e^{-\int_0^T R(X_s)ds} e^{(a+ivb)\cdot X_T} \right) \\ &= \psi^\chi(a + ibv, X_0, 0, T). \end{aligned}$$

□

Proposition 3.2.4. *Suppose T is fixed in $[0, \infty), b, a \in \mathbb{R}$ and that*

$$\int_{\mathbb{R}} |\psi^\chi(a + ibv, X_0, 0, T)| dv < \infty, \tag{3.18}$$

then $G_{a,b}(\cdot; X_0, T, \chi)$ *given by* (3.16) *is well defined and we have,*

$$G_{a,b}(y; X_0, T, \chi) = \frac{\psi^\chi(a, X_0, 0, T)}{2} - \frac{1}{\pi}\int_0^\infty \frac{Im[\psi^\chi(a+ibv, X_0, 0, T)e^{-ivy}]}{v}dv$$

Proof. From prop 3.2.3 we know that $\psi^\chi(a+ibv) = \int_{\mathbb{R}} e^{ivy}dG_{a,b}(y; X_0, T, \chi)$. It is also important to notice that:

$$\overline{e^{-ivy}\psi^\chi(a+ibv, x, 0, T)} = e^{ivy}\psi^\chi(a-ibv, x, 0, T)$$

hence

$$\mathrm{Im}[\psi^\chi(a+ibv, X_0, 0, T)e^{-ivy}] = -\mathrm{Im}[\psi^\chi(a-ibv, X_0, 0, T)e^{ivy}]$$

$$\frac{\mathrm{Im}[\psi^\chi(a+ibv, X_0, 0, T)e^{-ivy}]}{v} = \frac{\mathrm{Im}[\psi^\chi(a-ibv, X_0, 0, T)e^{ivy}]}{-v}.$$

So the function $v :\longmapsto \frac{\mathrm{Im}[\psi^\chi(a+ibv, X_0, 0, T)e^{-ivy}]}{v}$ is an even function, hence

$$\int_0^\infty \frac{\mathrm{Im}[\psi^\chi(a+ibv, X_0, 0, T)e^{-ivy}]}{v}dv = \frac{1}{2}\int_{-\infty}^\infty \frac{\mathrm{Im}[\psi^\chi(a+ibv, X_0, 0, T)e^{-ivy}]}{v}dv.$$

$$\frac{\mathrm{Im}[\psi^\chi(a+ibv, X_0, 0, T)e^{-ivy}]}{v} = \frac{1}{2iv}\Big[e^{-ivy}\psi^\chi(a+ibv, X_0, 0, T) - e^{ivy}\psi^\chi(a-ibv, X_0, 0, T)\Big]$$

$$= \frac{1}{2iv}\Big[e^{-ivy}\int_{\mathbb{R}} e^{ivz}dG_{a,b}(z; X_0, T, \chi) - e^{ivy}\int_{\mathbb{R}} e^{-ivz}dG_{a,b}(z; x, T, \chi)\Big]$$

$$= \frac{1}{2iv}\int_{\mathbb{R}} e^{-iv(y-z)} - e^{iv(y-z)}dG_{a,b}(z).$$

Therefore

$$-\frac{1}{\pi}\int_0^\infty \frac{\mathrm{Im}[\psi^\chi(a+ibv, X_0, 0, T)e^{-ivy}]}{v}dv = -\frac{1}{2\pi}\int_{-\infty}^\infty \frac{\mathrm{Im}[\psi^\chi(a+ibv, X_0, 0, T)e^{-ivy}]}{v}dv$$

$$= -\frac{1}{2\pi}\int_{-\infty}^\infty \frac{1}{2iv}\int_{\mathbb{R}} e^{-iv(y-z)} - e^{iv(y-z)}dG_{a,b}(z)dv$$

i.e.

$$-\frac{1}{\pi}\int_0^\infty \frac{\mathrm{Im}[\psi^\chi(a+ibv, X_0, 0, T)e^{-ivy}]}{v}dv = -\frac{1}{4i\pi}\int_{-\infty}^\infty\int_{\mathbb{R}} \frac{e^{-iv(y-z)}}{v} - \frac{e^{-iv(z-y)}}{v}dG_{a,b}(z)dv. \tag{3.19}$$

Since

$$\left|\frac{\operatorname{Im}[\psi^{\chi}(a+ibv,X_0,0,T)e^{-ivy}]}{v}\right| \leq \left|\frac{\operatorname{Im}[\psi^{\chi}(a+ibv,X_0,0,T)e^{-ivy}]}{v}\right|$$
$$\leq \left|\frac{\operatorname{Im}[\psi^{\chi}(a+ibv,X_0,0,T)]}{v}\right|$$
$$\leq \left|\frac{\psi^{\chi}(a+ibv,X_0,0,T)}{v}\right| \leq \left|\psi^{\chi}(a+ibv,X_0,0,T)\right|,$$

whenever $|v| > 1$.
Assumption (3.18) says that $|\psi^{\chi}(a+ibv,X_0,0,T)|$ is integrable hence,

$$\left|\frac{\operatorname{Im}[\psi^{\chi}(a+ibv,X_0,0,T)e^{-ivy}]}{v}\right|$$

is dominated by a function which is integrable and more, Fubini theorem applies. Thus equation (3.19) becomes:

$$-\frac{1}{\pi}\int_0^{\infty}\frac{\operatorname{Im}[\psi^{\chi}(a+ibv,X_0,0,T)e^{-ivy}]}{v}dv = -\frac{1}{4i\pi}\int_{\mathbb{R}}\int_{-\infty}^{\infty}\frac{e^{-iv(y-z)}}{v}-\frac{e^{-iv(z-y)}}{v}dvdG_{a,b}(z)$$

$$= -\frac{1}{4i\pi}\int_{\mathbb{R}}\left[\int_{-\infty}^{\infty}\frac{e^{-iv(y-z)}}{v}dv - \int_{-\infty}^{\infty}\frac{e^{-iv(z-y)}}{v}dv\right]dG_{a,b}(z)$$
$$= -\frac{1}{4i\pi}\int_{\mathbb{R}}\left[i\pi sign(y-z) - i\pi sign(z-y)\right]dG_{a,b}(z)$$

where $sign(x) = -1\ if\ x < 0,\ 0\ if\ x = 0,$ and $1\ if\ x > 0.$ In addition it is clear that $sign(x) = -sign(-x), \quad \forall x.$ Therefore

$$-\frac{1}{\pi}\int_0^{\infty}\frac{\operatorname{Im}[\psi^{\chi}(a+ibv,X_0,0,T)e^{-ivy}]}{v}dv = -\frac{1}{4i\pi}\int_{\mathbb{R}}2i\pi sign(y-z)dG_{a,b}(z)$$

$$= -\frac{1}{2}\left[\psi^{\chi}(a,X_0,0,T) - \lim_{z\to y, z>y}G(z;x,T,\chi) - \lim_{z\to y, z<y}G(z;x,T,\chi)\right]$$
$$= -\frac{\psi^{\chi}(a,X_0,0,T)}{2} + \frac{G(y;x,T,\chi)+G(y^-;x,T,\chi)}{2}$$

i.e.

$$-\frac{1}{\pi}\int_0^{\infty}\frac{\operatorname{Im}[\psi^{\chi}(a+ibv,X_0,0,T)e^{-ivy}]}{v}dv = -\frac{\psi^{\chi}(a,X_0,0,T)}{2} + G(y;x,T,\chi)$$

as required to prove. □

Corollary 3.2.5. *Let $p(d,c,T,\chi)$ be the price of an European call option with strike price c and maturing at time T. Then*

$$p(d,c,T,\chi) = E^{\chi}[e^{-\int_t^T R(X_s)ds}(e^{d\cdot X_T} - c)^+]$$

$$= E^{\chi}[e^{-\int_t^T R(X_s)ds}(e^{d\cdot X_T} - c)1_{d\cdot X_T \geq \ln c}]$$

$$p(d,c,T,\chi) = G_{d,-d}(-\ln c; X_0, T, \chi) - cG_{0,-d}(-\ln c; X_0, T, \chi).$$

This result takes us to the next section where the above transforms are applied to asset and option pricing.

3.3 Asset pricing

In all that follows, S represents the price process of the asset underlying the option. We consider it to be of the form

$$S_t = (\bar{a}_t + \bar{b}_t \cdot X_t)e^{a_t + b_t \cdot X_t}$$

for deterministic $a, \bar{a}, b, \bar{b}$. This is the case for many applications in affine settings such as bond prices.
The approach we use for modeling price process is this: we first model the "risk-neutral" behaviour of X under an equivalent martingale measure Q. That means we find a martingale measure Q such that the process X will be of AJD type. Then we apply equation (3.2) accompanied with the solution given by (3.4) and (3.5).

Practically, let take Q to be an equivalent martingale measure associated with the short term interest rate process $R(X_t) = \rho_0 + \rho_1 \cdot X_t$ such that under Q the state vector X_t behaves as an AJD with coefficient $(K^Q, H^Q, l^Q, \theta^Q)$. The characteristic $\chi^Q = (K^Q, H^Q, l^Q, \theta^Q, \rho)$ fully determines both stock price and the interest rate dynamics.
Thus, the market value V_t at time t of any claim that pays off V_T at maturity

T is given by:

$$V_t = E^Q\left[e^{-\int_t^T R(X_s)ds} \cdot V_T|\mathcal{F}_t\right]. \tag{3.20}$$

For simplicity let take the case $\ln S_t$ be a the i^{th} component of the state vector X for a certain integer $i \in [0, d]$ i.e $\ln S_t = X_t^{(i)}$. Assume also that the asset S has a dividend paying process $(\zeta(X_t))_{t>0}$ defined by:

$$\zeta(x) = q_0 + q_1 \cdot x$$

for given $q_0 \in \mathbb{R}, q_1 \in \mathbb{R}^d$.

Proposition 3.3.1. *Under the martingale measure Q defined as above, the risk neutral drift μ_t satisfies:*

$$(K_0^Q)_i = \rho_0 - q_0 - \frac{1}{2}(H_0^Q)_{ii} - l_0^Q[\theta(\beta_i(t)) - 1] \tag{3.21}$$

$$(K_1^Q)_i = \rho_1 - q_1 - \frac{1}{2}(H_1^Q)_{ii} - l_1^Q[\theta(\beta_i(t)) - 1] \tag{3.22}$$

Where $\beta_i(t) \in \mathbb{R}^n$ has 1 at his i^{th} component and 0 otherwise.

Proof.

$$\begin{aligned} \ln S_t &= X_t^{(i)} \\ S_t &= e^{X_t^{(i)}} \\ &= e^{\beta_i \cdot X_t} \end{aligned}$$

Under the martingale measure Q the payoff is expressed as

$$\Psi(X_t, t) = e^{-\int_0^t R(X_s)+\zeta(X_s)ds} \cdot e^{\beta_i \cdot X_t}$$

Then applying (3.4) and (3.5) to that payoff we get

$$0 = \rho_0 - q_0 - K_0\beta_i(t) - \frac{1}{2}\beta_i(t)^\top H_0\beta_i(t) - l_0\left[\theta(\beta_i(t)) - 1\right] \tag{3.23}$$

and

$$0 = \rho_1 - K_1^\top \beta_i(t) - \frac{1}{2}\beta_i(t)^\top H_1 \beta_i(t) - l_1 \Big[\theta(\beta_i(t)) - 1\Big]. \tag{3.24}$$

The first equation becomes

$$K_0^Q \beta_i(t) = \rho_0 - q_0 - \frac{1}{2}\beta_i(t)^\top H_0^Q \beta_i(t) - l_0^Q [\theta^Q(\beta_i(t)) - 1]$$

i.e.

$$K_0^Q (0, ..., 1, 0, ...) = \rho_0 - q_0 - \frac{1}{2}(0, ..., 1, 0...) H_0^Q \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \end{pmatrix} - l_0^Q [\theta^Q(\beta_i(t)) - 1]$$

which implies that

$$(K_0^Q)_i = \rho_0 - q_0 - \frac{1}{2}(H_0^Q)_{ii} - l_0^Q [\theta^Q(\beta_i(t)) - 1]$$

In the same way we get equation (3.22), except that the equation is derived from the ith component of the vector ${K_1^Q}^\top \beta_i(t)$, the other components are 0's. $\square$

Chapter 4

Model comparison and application

4.1 Model comparison

Here we show how the transforms derived in the previous chapter can be used to get well known formulas of bond prices that are of affine term structure type.

In models with affine term structure the underlying process X is exactly the interest rate itself denoted by r. The dynamics of r are given by:

$$dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))dW_t$$

where

$$\mu(t, r(t)) = \alpha(t)r(t) + \beta(t)$$

$$\sigma^2(t, r(t)) = \gamma(t)r(t) + \delta(t).$$

where $\alpha(t), \beta(t), \gamma(t), \delta(t)$ are deterministic functions of t. Using the same notation as [2], Björk derives the price $p(t, T)$ at time t of a bond that pays off 1 at time T as:

$$p(t, T) = e^{-A(t,T)-B(t,T)\cdot r(t)}$$

where A and B are smooth and deterministic functions of t that satisfy the ordinary differential equations:

$$\begin{cases} \dot{B}(t,T) + \alpha(t)B(t,T) - \frac{1}{2}\gamma(t)B^2(t,T) = -1 \\ B(T,T) = 0. \end{cases} \tag{4.1}$$

and

$$\begin{cases} \dot{A}(t,T) = \beta(t)B(t,T) - \frac{1}{2}\delta(t)B^2(t,T) \\ A(T,T) = 0 \end{cases} \tag{4.2}$$

Note that the function $A(t,T)$ and $A(t)$ mentioned in chapter 3 are a priori not the same.

Let use transform 1.
Observe that the underlying process X is again the interest rate r.
The price $p(t,T)$ at time t of an asset that pays off 1 at time T is given by:

$$\begin{aligned} p(t,T) &= E^{\chi}\Big(e^{-\int_t^T r\,ds}1\mathcal{F}_t\Big) \\ &= E^{\chi}\Big(e^{-\int_t^T r\,ds}e^{0\cdot X_T}\Big|\mathcal{F}_t\Big) \end{aligned}$$

We are right in the framework of affine settings where in this case we see that $u = 0$. We recall the observations made in the beginning of section 3 that is: $d = 1$ in addition

$$R = r,\ X = r \text{ which implies } \rho_0 = 0,\ \rho_1 = 1$$

$$K_0 = \beta(t),\ K_1 = \alpha(t),\ H_0 = \delta(t),\ H_1 = \gamma(t),\ l_0 = 0, \text{ and } l_1 = 0.$$

Applying transform 1 we know that $p(t,T)$ has a closed form formula, namely:

$$p(t,T) = \psi_t^{\chi}(0,x,t,T) = e^{\bar{\alpha}(t)+\bar{\beta}(t)\cdot x},$$

where $\bar{\alpha}$ and $\bar{\beta}$ solve uniquely the following ordinary differential equations:

$$\dot{\bar{\alpha}}(t) = \rho_0 - K_0\cdot\bar{\beta}(t) - \frac{1}{2}\bar{\beta}(t)^{\top}H_0\bar{\beta}(t) - l_0\Big[\theta(\bar{\beta}(t)) - 1\Big] \tag{4.3}$$

$$\dot{\bar{\beta}}(t) = \rho_1 - K_1^\top \bar{\beta}(t) - \frac{1}{2}\bar{\beta}(t)^\top H_1 \bar{\beta}(t) - l_1 \Big[\theta(\bar{\beta}(t)) - 1\Big] \tag{4.4}$$

with boundary condition $\bar{\beta}(T) = u = 0$ and $\bar{\alpha}(T) = 0$.
The equation (4.3) and (4.4) then become

$$\begin{cases} \dot{\bar{\beta}}(t) = 1 - \alpha(t)\bar{\beta}(t) - \dfrac{1}{2}\gamma(t)\bar{\beta}^2(t) \\ \bar{\beta}(T) = 0 \end{cases}$$

and

$$\begin{cases} \dot{\bar{\alpha}}(t) = -\beta(t) \cdot \bar{\beta}(t) - \dfrac{1}{2}\delta(t)\bar{\beta}^2(t) \\ \bar{\beta}(T) = 0 \end{cases}$$

If we let $\bar{\alpha}(t) = A(t,T)$, $x = r$ and $B(t,T) = -\bar{\beta}(t)$ we get exactly the equations derived by Björk for affine term structure models.
In the case of the Vasicek model the dynamics of r is given by:

$$dr(t) = \big(-a + br(t)\big)dt + \sigma dW_t$$

where a, b and σ are constants. It is clear in this case that $K_0 = -a,\ H_0 = \sigma^2,\ K_1 = b, H_1 = 0$ and $u = 0$ using the notations in [2].
In the same way, we get the other models by just matching parameters K, H and u.
Note that here there is no jump involved that is why λ is absent in the differential equations. In the next section we illustrate the methodology on a two dimensional affine jump-diffusion model.

4.2 Application

Here we suppose S is the price process of a security that pays dividends at a constant proportional rate ζ and let $Y = \ln S$. Consider then the state process $X = (Y, V)^\top$ where V is the variance process.
We suppose for simplicity that the short rate is a constant r, and that there exists an equivalent martingale measure Q, under which S follows the dy-

namics:

$$dS_t = (r-\zeta)dt + \sqrt{V_t}dW_t^Q + dZ_t$$

where $\sqrt{V_t}$ is the volatility of S_t, W_t^Q is a one dimensional standard Brownian and Z_t a one dimensional pure jump process with constant mean jump-arrival rate λ.
So the state price process X will have the dynamics described by:

$$dX_t = d\begin{pmatrix} Y_t \\ V_t \end{pmatrix} = \begin{pmatrix} r-\zeta-\lambda\mu-\frac{1}{2}V_t \\ \kappa(\upsilon - V_t) \end{pmatrix} dt + \sqrt{V_t}\begin{pmatrix} 1 & 0 \\ \rho\sigma_v & \sqrt{1-\rho^2}\sigma_v \end{pmatrix} dW_t^Q + dZ_t, \tag{4.5}$$

where ρ is the correlation between S and V, σ_v is the volatility of the volatility V of X; κ, μ, υ are constants. W^Q is an $\mathcal{F}_t-$standard Brownian motion under Q in $\mathbb{R}^2$ and Z is a pure jump process in $\mathbb{R}^2$ with constant mean jump-arrival rate λ, whose bivariate jump-size distribution ν has the transform θ, as well defined in the begining of chapter 3. We use the same notation for the 2-dimensional and the one-dimensional Brownian motion just for simplicity in notation. The same remark applies for the jump process Z_t.

For this specific case let us apply equation (3.4) and (3.5) to derive the transform of the log-price state variable Y_T.
The transform at time t of Y_T can be written as:

$$\psi^{\chi}(u,(y,v),t,T) = E^{\chi}\left(e^{-\int_t^T R(X_s)ds}e^{U\cdot X_T}\Big|\mathcal{F}_t\right),$$

with $U = \begin{pmatrix} u & 0 \end{pmatrix}$.
Theorem (3.3) tells us that the transform above can be written as

$$\psi^{\chi}(u,(y,v),t,T) = e^{\alpha(t)+\beta(t)\cdot X_t}$$

with

$$\alpha(T) = 0 \text{ and } \beta(T) = U, \text{ i.e } \beta(T) = \begin{pmatrix} u & 0 \end{pmatrix}.$$

<u>Question</u>: How do we find α and β ?
<u>Solution</u>:

Equation (4.5) can be written as:

$$dX_t = \mu_t dt + \sigma_t dW_t^Q + dZ_t$$

where

$$\mu_t = \begin{pmatrix} r-\zeta-\lambda\mu \\ \kappa_v\upsilon \end{pmatrix} + \begin{pmatrix} 0 & -\frac{1}{2} \\ 0 & -\kappa_v \end{pmatrix} X_t,\ \sigma_t\sigma_t^\top = \begin{pmatrix} 1 & \rho\sigma_v \\ \rho\sigma_v & \sigma_v^2. \end{pmatrix} V_t$$

i.e.

$$\sigma_t\sigma_t^\top = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 & \rho\sigma_v \\ 0 & 0 & \rho\sigma_v & \sigma_v^2 \end{pmatrix} X_t$$

Referring to affine settings in chapter 3, we see that:

$$K_0 = \begin{pmatrix} r-\zeta-\lambda\mu \\ \kappa_v\upsilon \end{pmatrix}, K_1 = \begin{pmatrix} 0 & -\frac{1}{2} \\ 0 & -\kappa_v \end{pmatrix}, H_0 \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, H_1 = \begin{pmatrix} 0 & 0 & \vdots & 1 & \rho\sigma_v \\ 0 & 0 & \vdots & \rho\sigma_v & \sigma_v^2 \end{pmatrix},$$

$$\rho_0 = r,\ \rho_1 = (0,0),\ l_0 = \lambda,\ l_1 = (0,0)$$

Remark: The risk-neutral restriction in equation (3.21) is:

$$(K_0^Q)_i = \rho_0 - q_0 - \frac{1}{2}(H_0^Q)_{ii} - l_0^Q[\theta(\beta_i(t)) - 1].$$

Since $K_0 = r - \zeta - \lambda\mu \quad and\ H_0 = 0$ we get:

$$r - \zeta - \lambda\mu = \rho_0 - q_0 - 0 - l_0[\theta(\beta_i(t) - 1].$$

Because $R = r$ is constant, the dividends are constant so $\zeta = q_0$ and λ also is constant equal to l_0, we get

$$\lambda\mu = l_0[\theta(\beta_i(t)) - 1]$$
$$\mu = \theta(\beta_i(t)) - 1.$$

Since $Y = \ln S$ with $X = (Y, V)$ therefore $\beta_i(t) = (1, 0)$ by definition of β_i in proposition 3.3.1. That means

$$\mu = \theta(0,1) - 1.$$

For this chapter we will sometimes write β instead of $\beta(t)$, same for $\alpha(t)$.

Since α and β satisfy equations (3.4) and (3.5) simultaneously then we have:

$$\begin{pmatrix}\dot\beta_1\\ \dot\beta_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} - \begin{pmatrix}0 & 0\\ -\frac{1}{2} & -\kappa_v\end{pmatrix}\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix} - \frac{1}{2}\begin{pmatrix}\beta_1 & \beta_2\end{pmatrix}\begin{pmatrix}0 & 0 & \vdots & 1 & \rho\sigma_v\\ 0 & 0 & \vdots & \rho\sigma_v & \sigma_v^2\end{pmatrix}\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix} - \begin{pmatrix}0\\0\end{pmatrix}$$

$$\begin{pmatrix}\dot\beta_1\\ \dot\beta_2\end{pmatrix} = -\begin{pmatrix}0\\ -\frac{1}{2}\beta_1 - \kappa_v\beta_2\end{pmatrix} - \frac{1}{2}\begin{pmatrix}0\\ \beta_1^2 + 2\beta_1\beta_2\rho\sigma_v + \beta_2^2\sigma_v^2 + 0.\end{pmatrix} \tag{4.6}$$

The first line in equation (4.6) tells us that:

$$\dot\beta_1(t) = 0, \text{ for all } t$$

Hence

$$\beta_1(t) = Constant, \text{ for all } t.$$

Since $\beta(T) = (u,0)$ therefore $\beta_1(t) = u$.
From the equation of second components, we have that:

$$\begin{aligned}\dot\beta_2 &= \frac{1}{2}u + \kappa_v\beta_2 - \frac{u^2}{2} - \beta_2 u\sigma_v\rho - \frac{\beta_2^2\sigma_v^2}{2}.\\ &= \frac{1}{2}u(1-u) - \beta_2(\sigma_v\rho u - \kappa_v) - \frac{\beta_2^2\sigma_v^2}{2}.\end{aligned}$$

which implies that

$$\dot\beta_2 = \frac{a}{2} - b\beta_2 - \frac{\beta_2^2\sigma_v^2}{2} \quad \text{,with } \beta_2(T) = 0 \tag{4.7}$$

where $a = u(1-u)$ and $b = \sigma_v\rho u - \kappa_v$.
This is a Ricatti type of equation which is of the from.

$$y' = q_0(t) + q_1(t)y + q_2(t)y^2. \tag{4.8}$$

Using simple techniques of reduction process, the equation reduces to a second order differential equation

$$w" - P(t)w' + Q(t)w = 0 \tag{4.9}$$

where

$$P(t) = q_1 + (\frac{q_2'}{q_2}),\ Q(t) = q_2 q_0.$$

Since

$$q_0 = \frac{a}{2}, \quad q_1 = -b \quad q_2 = -\frac{\sigma_v^2}{2},$$

$$P(t) = -b \quad \text{and} \quad Q(t) = -\frac{a\sigma_v^2}{4}$$

hence (4.9) becomes

$$w" + bw' - \frac{a\sigma_v^2}{4} w = 0.$$

The characteristic equation is:

$$r^2 + br - \frac{a\sigma_v^2}{4} = 0$$

$$\Delta = b^2 - a\sigma_v^2$$

therefore the two roots are: $r_1 = \frac{1}{2}(-b-\gamma)$ and $r_2 = \frac{1}{2}(-b+\gamma)$ where we set $\gamma^2 = \Delta$.

Therefore the general solution of (4.9) is of the form

$$w(t) = C_2 e^{\frac{-b-\gamma}{2}t} + C_1 e^{\frac{-b+\gamma}{2}t} \qquad C_1, C_2 \in \mathbb{R}.$$

The solution of (4.8) is given by

$$y = -\frac{w'}{q_2 w}.$$

But

$$w' = C_2(\frac{-b-\gamma}{2})e^{\frac{-b-\gamma}{2}t} + C_1(\frac{-b+\gamma}{2})e^{\frac{-b+\gamma}{2}t}$$

so,

$$\frac{w'}{w} = \frac{1}{2}\frac{C_2(-b-\gamma)e^{\frac{-b-\gamma}{2}t} + C_1(-b+\gamma)e^{\frac{-b+\gamma}{2}t}}{C_2 e^{\frac{-b-\gamma}{2}t} + C_1 e^{\frac{-b+\gamma}{2}t}}$$

hence,

$$2\frac{w'}{2\sigma_v^2 w} = \frac{1}{\sigma_v^2}\frac{C_2(-b-\gamma)e^{\frac{-\gamma}{2}t} + C_1(-b+\gamma)e^{\frac{\gamma}{2}t}}{C_2 e^{\frac{-\gamma}{2}t} + C_1 e^{\frac{\gamma}{2}t}}$$

and

$$\beta_2(t) = \frac{1}{\sigma_v^2}\frac{C_1(\gamma-b)e^{\gamma t} - C_2(b+\gamma)}{C_1 e^{\gamma t} + C_2}.$$

Since $\beta_2(T) = 0$, then

$$C_1(\gamma-b)e^{\gamma T} - C_2(\gamma+b) = 0$$

meaning

$$C_1 = C_2\left(\frac{\gamma+b}{\gamma-b}\right)e^{-\gamma T}.$$

This yields

$$\begin{aligned}
\beta_2(t) &= \frac{1}{\sigma_v^2}\frac{-C_2(\gamma+b) + C_2\left(\frac{\gamma+b}{\gamma-b}\right)(\gamma-b)e^{\gamma t}e^{-\gamma T}}{C_2 + C_2\left(\frac{\gamma+b}{\gamma-b}\right)e^{\gamma t}e^{-\gamma T}} \\
&= \frac{1}{\sigma_v^2}\frac{-(\gamma+b) + (\gamma+b)e^{-\gamma(T-t)}}{1 + \frac{\gamma+b}{\gamma-b}e^{-\gamma(T-t)}} \\
&= -\frac{(\gamma-b)(\gamma+b)}{\sigma_v^2}\frac{1-e^{-\gamma(T-t)}}{\gamma-b+(\gamma+b)e^{-\gamma(T-t)}} \\
&= -\frac{(\gamma^2-b^2)}{\sigma_v^2}\frac{1-e^{-\gamma(T-t)}}{2\gamma-\gamma-b+(\gamma+b)e^{-\gamma(T-t)}} \\
&= -\frac{(\gamma^2-b^2)}{\sigma_v^2}\frac{1-e^{-\gamma(T-t)}}{2\gamma-(\gamma+b)(1-e^{-\gamma(T-t)})}.
\end{aligned}$$

Since $\gamma^2 = b^2 - a\sigma_v^2$, we get

$$\beta_2(t) = \frac{-a(1-e^{-\gamma(T-t)})}{2\gamma-(\gamma+b)(1-e^{-\gamma(T-t)})}.$$

On the other hand

$$\dot{\alpha}(t) = \rho_0 - K_0\cdot\beta(t) - \frac{1}{2}\beta(t)^\top H_0\beta(t) - l_0\Big[\theta(\beta(t)) - 1\Big]$$

i.e.

$$\begin{aligned}
\dot{\alpha} &= r - \Big(r - \zeta - \lambda\mu \quad \kappa_v\beta_2\Big)\begin{pmatrix} u \\ \beta_2 \end{pmatrix} - \frac{1}{2}0 - \lambda\Big[\theta(\beta) - 1\Big] \\
&= r - ru + \zeta u - \lambda\mu u - \kappa_v\beta_2 - \lambda\Big[\theta(\beta) - 1\Big] \\
\dot{\alpha} &= r - (r - \zeta)u + \lambda(1 + \mu u) - \kappa_v\beta_2 - \lambda\theta(\beta);
\end{aligned}$$

hence

$$\alpha(T) - \alpha(t) = \int_t^T r - (r - \zeta)u + \lambda(1 + \mu u)ds - \lambda\int_t^T \theta(\beta(s))ds - \kappa_v\int_t^T \beta_2(s)ds.$$

Since $\alpha(T) = 0$,

$$-\alpha(t) = r(T - t) - (r - \zeta)u(T - t) + \lambda(1 + \mu u)(T - t) - \lambda\int_t^T \theta(\beta_2(s))ds - \kappa_v\int_t^T \beta_2(s)ds.$$

Using the change of variable

$$u = T - s \Rightarrow du = -ds \text{ and for } s = t,\ u = T - t$$

hence

$$\begin{aligned}
\int_t^T \theta(\beta_2(s))ds &= \int_{T-t}^0 \theta(\beta_2(T - u))(-du) \\
&= \int_0^{T-t} \theta(\beta_2(T - u))(-du).
\end{aligned}$$

In addition,

$$\begin{aligned}
\int_t^T \beta_2(s)ds &= \int_{T-t}^0 \beta_2(T - u)(-du) \\
&= \int_0^{T-t} \beta_2(T - u)du \\
&= \int_0^{T-t} \frac{-a(1 - e^{-\gamma u})}{2\gamma - (\gamma + b)(1 - e^{-\gamma u})}du \\
&= \int_0^{T-t} \frac{-a}{2\gamma}\frac{1 - e^{-\gamma u}}{1 - \frac{(\gamma+b)}{2\gamma}(1 - e^{-\gamma u})}du \\
&= \int_0^{T-t} \frac{-a}{2\gamma}\left(\frac{2\gamma}{\gamma - b} - \frac{\frac{2\gamma}{\gamma-b}e^{-\gamma u}}{1 - \frac{\gamma+b}{2\gamma}(1 - e^{-\gamma u})}\right)du
\end{aligned}$$

$$\begin{aligned}
\int_t^T \beta_2(s)ds &= \int_0^{T-t} \frac{-a}{2\gamma}\left(\frac{2\gamma(\gamma+b)}{\gamma^2-b^2} - \frac{\frac{2\gamma}{\gamma-b}e^{-\gamma u}}{1-\frac{\gamma+b}{2\gamma}(1-e^{-\gamma u})}\right)du \\
&= \int_0^{T-t} \frac{a(\gamma+b)}{a\sigma^2} + \frac{a2\gamma}{2\gamma(\gamma-b)}\frac{e^{-\gamma u}}{1-\frac{\gamma+b}{2\gamma}(1-e^{-\gamma u})}du \\
&= \int_0^{T-t} -\frac{\gamma+b}{\sigma^2} + \frac{a}{\gamma-b}\frac{e^{-\gamma u}}{1-\frac{\gamma+b}{2\gamma}(1-e^{-\gamma u})}du \\
&= \int_0^{T-t} -\frac{\gamma+b}{\sigma^2} + \frac{a}{\gamma-b}\left(\frac{1}{\gamma}\frac{-2\gamma}{\gamma+b}\right)\frac{\left(\frac{1}{\gamma}\frac{-2\gamma}{\gamma+b}\right)\gamma e^{-\gamma u}}{1-\frac{\gamma+b}{2\gamma}(1-e^{-\gamma u})}du \\
&= \int_0^{T-t} -\frac{\gamma+b}{\sigma^2} - \left(\frac{2a}{\gamma^2-b^2}\right)\frac{\gamma\left(\frac{-(\gamma+b)}{2\gamma}\right)e^{-\gamma u}}{1-\frac{\gamma+b}{2\gamma}(1-e^{-\gamma u})}du \\
&= \int_0^{T-t} -\frac{\gamma+b}{\sigma^2} - \frac{2}{\sigma^2}\frac{-\gamma\left(\frac{\gamma+b}{2\gamma}\right)e^{-\gamma u}}{1-\frac{\gamma+b}{2\gamma}(1-e^{-\gamma u})}du.
\end{aligned}$$

Therefore

$$\begin{aligned}
\int_0^{T-t} \beta_2(u)du &= -\int_0^{T-t}\frac{\gamma+b}{\sigma_v^2}du - \frac{2}{\sigma_v^2}\int_0^{T-t}\frac{-\gamma\left(\frac{\gamma+b}{2\gamma}\right)e^{-\gamma u}}{1-\frac{\gamma+b}{2\gamma}(1-e^{-\gamma u})}du \\
&= \frac{\gamma+b}{\sigma_v^2}(T-t) - \frac{2}{\sigma_v^2}\ln\left(1-\frac{\gamma+b}{2\gamma}(1-e^{-\gamma u})\right)\Bigg|_0^{T-t} \\
&= \frac{\gamma+b}{\sigma_v^2}(T-t) - \frac{2}{\sigma_v^2}\ln\left(1-\frac{\gamma+b}{2\gamma}(1-e^{-\gamma(T-t)})\right).
\end{aligned}$$

Set

$$\bar{\beta}(\tau) = \beta_2(T-t) \text{ and } \alpha(\tau) = \alpha(T-t)$$

we get

$$\begin{aligned}
\alpha(\tau) = -r\tau + (r-\zeta)u\tau - \kappa_v\upsilon\left[\frac{\gamma+b}{\sigma_v^2}\tau + \frac{2}{\sigma_v^2}\ln\left(1-\frac{\gamma+b}{2\gamma}(1-e^{-\gamma\tau})\right)\right] \\
-\lambda(1+\mu u)\tau + \lambda\int_t^T \theta(\beta(s))ds.
\end{aligned}$$

The price at time 0 of an European call option on S with strike price c and maturity T is given by:

$$\begin{aligned} C(T,\chi) &= E^{\chi}[e^{-\int_t^T R(S_s)ds}(S_T - c)^+] \\ &= E^{\chi}[e^{-\int_t^T R(X_s)ds}(e^{Y_T} - c)^+]. \end{aligned}$$

By corollary 3.2.5 (with $d = 1$) this price is known in its closed form and it is given by:

$$C(t,\chi) = G_{1,-1}(-\ln c; Y_0, T, \chi) - cG_{0,-1}(-\ln c; Y_0, T, \chi).$$

where $G_{d,-d}(x; Y_0, T, \chi)$ is given by (3.17) in proposition 3.2.3 and X_0 is replaced by Y_0.

Index of Notation

Here we use the format: symbol, page number then definition.

$(\Omega, \mathcal{F}, P)$,7	probability space
X_t,	state process
$\Delta(X_t)$, 10	jump process of X_t
$V_T(X)$, 12	total variation of X_t on $[0, T]$
$[X, X]_t$, 20	quadratic variation process of X_t
$\mu(X_t, t)$ 25,	drift term of X_t
$\sigma(X_t, t)$, 25	volatility of X_t
$\lambda(X_t, t)$, 25	mean arrivals of the jumps of X_t
ν, 11	Lévy measure associated with X_t
ζ_t, 45	dividend process
χ, 27	characteristic of X_t
E^χ, 27	expectation under χ
T, 8	stopping time
W_t, 25	Brownian motion
N_t, 9	counting process
Z_t, 25	Poisson process
J_t, 11	auxiliary process
S_t, 25	stock price process
V_t, 5	variance process
R, 6	interest rate process
θ, 27	jump transform of X_t
Ψ_t, 27	auxiliary transform

ψ^χ, 27	transform 1
ϕ^χ, 32	transform 2
δ, 10	constant coefficient
K_0, H_0, K_1, H_1, 26	coefficients
l_0, l_1, ρ_0, ρ_1, 26	coefficients
u, 27	coefficient vector
$M(\omega, \cdot)$, 7	random measure
$G_{a,b}$, 36	Fourier transform of X_t
$p(d, t, T)$, 39	option price
$p(t, T)$, 42	bond price

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