Hausdorff continuous viscosity solutions of Hamilton-Jacobi equations and their numerical analysis

by

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Dedicated to the memory of my father and brothers

TO THE GLORY OF GOD

Hausdorff continuous viscosity solutions of Hamilton-Jacobi equations and their numerical analysis

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Abstract

The theory of viscosity solutions was developed for certain types of nonlinear first-order and second-order partial differential equations. It has been particularly useful in describing the solutions of partial differential equations associated with deterministic and stochastic optimal control problems [16], [53]. In its classical formulation, see [16], the theory deals with solutions which are continuous functions. The concept of continuous viscosity solutions was further generalized in various ways to include discontinuous solutions with the definition of Ishii given in [71] playing a pivotal role. In this thesis we propose a new approach for the treatment of discontinuous solutions of first-order Hamilton-Jacobi equations, namely, by involving Hausdorff continuous interval valued functions.

The advantages of the proposed approach are justified by demonstrating that the main ideas within the classical theory of continuous viscosity solutions can be extended almost unchanged to the wider space of Hausdorff continuous functions and the existing theory of discontinuous viscosity solutions is a particular case of that developed in this thesis in terms of Hausdorff continuous interval valued functions.

Two approaches to numerical solutions for Hamilton-Jacobi equations are presented. The first one is a monotone scheme for Hamilton-Jacobi equations while the second is based on preserving total variation diminishing property for conservation laws.

In the first approach, we couple the finite element method with the nonstandard finite difference method which is based on the Mickens' rule of nonlocal approximation [9]. The scheme obtained in this way is unconditionally monotone.

In the second approach, computationally simple implicit schemes are derived by using nonlocal approximation of nonlinear terms. Renormalization of the denominator of the discrete derivative is used for deriving explicit schemes of first or higher order. Unlike the standard explicit methods, the solutions of these schemes have diminishing total variation for any time step size.

DECLARATION

I, the undersigned, hereby declare that the thesis submitted herewith for the degree Philosophiae Doctor to the University of Pretoria contains my own, independent work and has not been submitted for any degree at any other university.

Signature:

Name: Froduald Minani **Date: 2007/December/06**

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$\mathbb{R}^n$	$n-$dimensional Euclidean space
$\mathbb{R} = \mathbb{R}^1$	set of all real numbers
$\mathbb{Z}$	set of all integers
$\mathbb{N}$	set of all positive integers
$\mathbb{Q}$	set of rational numbers
Ω	subset of $\mathbb{R}^n$ (open, in most cases)
$\partial\Omega$	boundary of the set Ω
$\overline{\Omega} = \Omega \cup \partial\Omega$	closure of the set Ω
$x = (x_1, ..., x_n)$	a typical point in $\mathbb{R}^n$
$x.y = \sum_{i=1}^{n} x_i\, y_i$	scalar product of elements x and y of $\mathbb{R}^n$
$\lvert x\rvert = \lvert x\rvert_2 = (\sum_{i=1}^{n} x_i^2)^{1/2}$	Euclidean norm of x in $\mathbb{R}^n$
$[a, b]$	closed real interval
(a, b)	open real interval
$B_r(a)$	open ball centered at $a \in \mathbb{R}^n$ with radius $r > 0$
$\overline{B_r}(a)$	closed ball centered at $a \in \mathbb{R}^n$ with radius $r > 0$
$\alpha = (\alpha_1, ..., \alpha_n)$	multi-index, $\alpha_i \in \mathbb{N}, i = 1, ..., n$
$\lvert\alpha\rvert = \sum_{j=1}^{n} \alpha_j$	length of multi-index α
$C^m(\Omega)$	space of functions having m continuous derivatives on $\Omega, m \in \mathbb{N}$
$C(\Omega) = C^0(\Omega)$	space of continuous functions on Ω
	and uniformly continuous on bounded subset of Ω
$D^\alpha u(x)$	partial derivative :$D^\alpha u(x) = \frac{\partial^{\lvert\alpha\rvert} u(x)}{\partial_{x_1}^{\alpha_1} ... \partial_{x_n}^{\alpha_n}}$
$D^m u(x)$	set of all partial derivatives of order m: $Du(x) = \{D^\alpha u(x) : \lvert\alpha\rvert = m\}$
$Du(x) = \nabla u(x)$	gradient of u at a point x in $\mathbb{R}^n$: $Du(x) = (u_{x_1}, ..., u_{x_n})$
$\nabla^2 u(x) = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$	Laplacian of u at x
$C^\infty(\Omega) = \bigcap_{k\geq 0} C^k(\Omega)$	space of infinitely differentiable functions on Ω

supp(u)	support of the function u
$C_0(\Omega)$	space of continuous functions having compact support in Ω
$C_0^m(\Omega)$	space $C_0(\Omega) \cap \mathcal{C}^m(\Omega)$
$C_0^\infty(\Omega)$	space $C_0(\Omega) \cap C^\infty(\Omega)$
$L^p(\Omega)$	usual space of measurable functions
	whose pth power is Lebesgue integrable on Ω
$L^\infty(\Omega)$	space of measurable functions
	which are bounded almost everywhere on Ω
$\|\|u\|\|_\infty = \{\sup_{x\in\Omega} \|u(x)\| < \infty\}$	norm of the function u in $L^\infty(\Omega)$
$\|u\|_{m,\infty} = \|\|D^m u\|\|_\infty$	norm of the function $D^m u$ in $L^\infty(\mathbb{R}^2)$
$f\|_A$	restriction of the function f to the set A
$C^{0,1}(\Omega)$	space of Lipshitz continuous functions on Ω
$USC(\Omega)$	set of upper semicontinuous functions on Ω
$LSC(\Omega)$	set of lower semicontinuous functions on Ω
$BUSC(\Omega)$	set of bounded upper semicontinuous functions on Ω
$BLSC(\Omega)$	set of bounded lower semicontinuous functions on Ω
$BUC(\Omega)$	set of bounded and uniformly continuous functions on Ω
$\mathbb{IR}$	set of finite closed real intervals: $\mathbb{IR} = \{[\underline{a}, \overline{a}] : \underline{a}, \overline{a} \in \mathbb{R}, \underline{a} \leq \overline{a}\}$
$\mathbb{A}(\Omega)$	set of locally bounded functions on Ω with values
	which are finite closed real intervals
$\mathcal{A}(\Omega)$	set of locally bounded functions with real values
$u = [\underline{u}, \overline{u}]$	function in $\mathbb{A}(\Omega)$, where $\underline{u}, \overline{u} \in \mathcal{A}(\Omega)$
$\mathbb{F}(\Omega)$	set of segment continuous functions on Ω
$\mathbb{H}(\Omega)$	set of Hausdorff continuous functions on Ω

Chapter 1

Introduction

1.1 The Hamilton-Jacobi Equations

The theory of viscosity solutions was developed for certain types of first and second order partial differential equations. It has been particularly useful in describing the solutions of partial differential equations associated with deterministic and stochastic optimal control problems [16], [53]. In this thesis, we are interested in the theory of viscosity solutions of first-order Hamilton-Jacobi equation in its general form associated with boundary condition, namely

$$H(x, u(x), Du(x)) = 0, x \in \Omega, \tag{1.1.1}$$

$$u(x) = g(x), x \in \partial\Omega. \tag{1.1.2}$$

Here,

Ω is an arbitrary nonempty open subset of $\mathbb{R}^n$
$H : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ (Hamiltonian) is jointly continuous function in all its arguments,
$g : \partial\Omega \to \mathbb{R}$ is a given (not necessarily continuous) function,
$u : \overline{\Omega} \to \mathbb{R}$ is the unknown function, and
$Du(x)$ is the gradient of the function u at a point $x = (x_1, ..., x_n) \in \Omega$.

The evolutionary Hamilton-Jacobi equations of the form

$$u_t(y, t) + H(y, t, u(y, t), D_y u(y, t)) = 0, \quad (y, t) \in D \times (0, T), \tag{1.1.3}$$

where $D \subset \mathbb{R}^n$, the t subscript denotes a temporal partial derivatives in the time variable t, and $D_y u$ is the gradient of u with respect to y, are reduced to the form (1.1.1) by the substitutions

$$x = (y,t) \in \Omega = D \times (0,T) \subseteq \mathbb{R}^{n+1}, \ \widetilde{H}(x,r,q) = H(x,r,q_1,...,q_n) + q_{n+1}$$

with

$$q = (q_1,...,q_n,q_{n+1}) \in \mathbb{R}^{n+1}.$$

The Hamilton-Jacobi equations play an important role in many fields of mathematics and physics, as for instance, calculus of variations [29], combustion [18], computer graphics [117], optimal control theory [93], differential games [55], image processing [122], quantum mechanics [114], and geometric optics [26]. For this reason, many theoretical and numerical studies have been devoted to solving the Hamilton-Jacobi equations.

Let us remark that the equation (1.1.1) is global nonlinear equation. Classical approach to the study of the problem (1.1.1)-(1.1.2) is the method of characteristics. This technique gives an elementary way a local existence result for smooth solutions, and at the same time shows that no global smooth solution exists in general.

By a classical solution of the problem (1.1.1)-(1.1.2), we mean a function $u \in C^1(\Omega) \cap C(\overline{\Omega})$ satisfying (1.1.1) and (1.1.2). Nonlinear partial differential equations of the form (1.1.1) do not, in general, possess classical solutions as can be seen in the following example.

Example 1.1.1 *Consider the following Dirichlet problem*

$$(u_x(x))^2 = 1, x \in (0,1), \tag{1.1.4}$$

$$u(0) = u(1) = 0. \tag{1.1.5}$$

There cannot exist a classical solution of (1.1.4) which satisfies (1.1.5). Indeed, assume there is $u \in C^1((0,1)) \cap C([0,1])$ satisfying (1.1.5). Then there exists $x_0 \in (0,1)$ such that $u_x(x_0) = 0$. This is a contradiction to (1.1.4).

For more general results about the method of characteristics applied to first-order Hamilton-Jacobi equations, we refer to R. Courant and D. Hilbert [35], L. C. Evans [46], F. John [75, 76], P. D. Lax [88] and H. Rund [118].

The equation (1.1.1) has been approached by looking for generalized solutions.

Definition 1.1.1 *A function $u : \overline{\Omega} \to \mathbb{R}$ is said to be a generalized solution of equation (1.1.1) if u is locally Lipschitz in Ω, continuous on $\overline{\Omega}$, and satisfies (1.1.1) almost everywhere on Ω.*

The generalized solution is almost everywhere differentiable by the well-known classical Rademacher's Theorem, see [47], on the almost everywhere differentiability of Lipschitz continuous functions, and (1.1.1)-(1.1.2) are to be understood as to hold almost everywhere. The existence results have been obtained by several authors, see for examples A. Douglis [43, 44], S. N. Kruzkov [82, 83, 84], W. H. Fleming [49, 50, 51], Hopf [68], E. D. Conway and Hopf [34], and A. Friedman [56]. For more complete references, we refer to some recent monographs by Benton [27] and P. -L. Lions [93].

The reader can also find an extensive presentation of the results on the solvability of the problem (1.1.1)-(1.1.2) in the books by Gilbarg and Trudinger [58, 59] and Ladyzhenskaya and Uraltseva [86].

The difficulty with the above concept of generalized solutions is that the equation (1.1.1), together with the boundary condition (1.1.2), typically has many generalized solutions. This is shown by Example 1.1.1. Obviously, the function

$$u_1(x) = \begin{cases} x & , \ 0 \leq x \leq \frac{1}{2} \\ 1 - x & , \ \frac{1}{2} \leq x \leq 1 \end{cases}$$

satisfies (1.1.5) and solves the equation (1.1.4) almost everywhere, in fact everywhere except on the point $x = \frac{1}{2}$. But u_1 is not unique and there exist infinitely many generalized solutions. In particular, we may build a sequence of generalized solutions as follows : for $m \geq 2$

$$u_m(x) = \begin{cases} x - \frac{2j}{2^m} & , \ \frac{2j}{2^m} \leq x \leq \frac{2j+1}{2^m} \\ \\ \frac{2j+2}{2^m} - x & , \ \frac{2j+1}{2^m} \leq x \leq \frac{2j+2}{2^m}, \end{cases} \tag{1.1.6}$$

for $j = 0, 1, ..., 2^{m-1} - 1$ and $x \in (0, 1)$.

It is evident that $u_m(0) = u_m(1) = 0$ and $((u_m)_x(x))^2 = 1$ everywhere on $(0, 1)$ except at the corners of its graph. The function u_m is bounded, Lipschitz continuous and piecewise analytic. Thus u_m is a generalized solution of (1.1.4) and satisfies (1.1.5).

One sees immediately that one has

$$0 \leq u_m(x) \leq \frac{1}{2^m}, \ x \in [0,1], \ \forall m \geq 1. \tag{1.1.7}$$

Thus, u_m converges to 0 uniformly as $m \to +\infty$, but $u \equiv 0$ does not satisfy (1.1.4) anywhere on $(0,1)$. Therefore, a stability property is false for generalized solutions.

Similarly, consider the equation $u_t + (u_y)^2 = 0$ for $y \in \mathbb{R}, t > 0$, coupled with the initial condition $u(y,0) = 0$. The function

$$v(y,t) = \begin{cases} 0 & , \quad |y| \geq t > 0 \\ -t + |y| & , \quad t \geq |y| \end{cases}$$

satisfies the initial condition, is continuous and has all the regularity one desires off the lines $y = 0, t = |y|$, and satisfies the equation off these lines. Thus $u \equiv 0$ and v are distinct generalized solutions of the above Cauchy problem and satisfy the initial condition.

The above two examples show that the notion of generalized solution, in terms of Definition 1.1.1, is too weak and in order to obtain an uniqueness result one needs to restrict the class of solutions by adding some suitable admissibility condition.

1.2 The Classical Theory of Viscosity Solutions

To solve uniqueness (and stability) question given in Section 1.1, in the early 1980s, M. G. Crandall and P.-L. Lions [38, 39] introduced a class of continuous generalized solutions of (1.1.1), called *viscosity solutions* (for reasons detailed below) which need not be differentiable anywhere, as the only regularity required in the definition is continuity.

To motivate the definition of continuous viscosity solution of equation (1.1.1), let us consider the approximate equation for (1.1.1), namely,

$$H(x, u_\varepsilon(x), Du_\varepsilon(x)) - \varepsilon\nabla^2 u_\varepsilon(x) = 0, \ x \in \Omega, \tag{1.2.1}$$

were $\varepsilon > 0$ is a small parameter. The equations (1.2.1) are quasilinear elliptic and have been studied for a long time (see in particular O. Ladyzenskaya and N. N. Uraltseva [87], D. Gilbarg and N. S. Trudinger [58], J. Serrin [121]). It is shown in [59, 85] that the equation (1.2.1) together with boundary condition has a unique classical solution. We hope that as $\varepsilon \to 0$ the solution $u_\varepsilon \in C^2(\Omega)$ of (1.2.1) will converge to some sort of weak solution of (1.1.1). This technique is the method of *vanishing viscosity.* It comes from a well known method in fluid dynamics where the coefficient ε represents physically the *viscosity* of the fluid and explains the name of solutions.

The vanishing viscosity method works as follows. Suppose that the family of solutions of (1.2.1), namely, $\{u_\varepsilon\}_{\varepsilon>0}$, is uniformly bounded and equicontinuous on compact set $\overline{\Omega}$. Consequently, the Arzela-Ascoli's [79] compactness criterion, ensures that there exists a function
$u \in C(\overline{\Omega})$ and a subsequence $\{u_{\varepsilon_j}\}_{j=1}^{\infty}$ of $\{u_\varepsilon\}_{\varepsilon>0}$ such that

$$u_{\varepsilon_j} \to u \text{ uniformly on } \overline{\Omega}, \text{ as } \varepsilon_j \to 0. \tag{1.2.2}$$

Fix any $\varphi \in C^1(\Omega)$ and suppose

$$u - \varphi \text{ has a local maximum at some point } x_0 \in \Omega. \tag{1.2.3}$$

For $\delta > 0$, consider the function $\psi \in C(\Omega)$ defined by $\psi(x) := \varphi(x) + \delta(|x - x_0|^2)$. Then we have $\psi \in C^1(\Omega)$ and $D\psi(x_0) = D\varphi(x_0)$.

Obviously, the function

$$u - \psi \text{ has a strict local maximum at } x_0. \tag{1.2.4}$$

By (1.2.2) and (1.2.4), for each sufficiently small $\varepsilon_j > 0$, there exists a point $x_{\varepsilon_j} \in \Omega$ such that $x_{\varepsilon_j} \to x_0$ as $\varepsilon_j \to 0$ and

$$u_{\varepsilon_j} - \psi \text{ has a local maximum at } x_{\varepsilon_j}. \tag{1.2.5}$$

Now owing to (1.2.5), by elementary calculus,

$$Du_{\varepsilon_j}(x_{\varepsilon_j}) = D\psi(x_{\varepsilon_j}) \tag{1.2.6}$$

and

$$\nabla^2(u_{\varepsilon_j} - \psi)(x_{\varepsilon_j}) \leq 0 \tag{1.2.7}$$

hold. We consequently can calculate

$$\begin{aligned} H(x_{\varepsilon_j}, u(x_{\varepsilon_j}), D\psi(x_{\varepsilon_j})) &= H(x_{\varepsilon_j}, u(x_{\varepsilon_j}), Du_{\varepsilon_j}(x_{\varepsilon_j})) \text{ by (1.2.6)} \\ &= \varepsilon_j \nabla^2 u_{\varepsilon_j}(x_{\varepsilon_j}) \qquad \text{by (1.2.1)} \\ &\leq \varepsilon_j \nabla^2 \psi(x_{\varepsilon_j}) \qquad \text{by (1.2.7).} \end{aligned} \tag{1.2.8}$$

Since $x_{\varepsilon_j} \to x_0$ as $\varepsilon_j \to 0$, we can pass to the limit in (1.2.8) using that $u_{\varepsilon_j}(x_{\varepsilon_j}) \to u(x_0)$, $D\psi(x_{\varepsilon_j}) \to D\psi(x_0) = D\varphi(x_0)$, $\varepsilon_j \nabla^2 \varphi(x_{\varepsilon_j}) \to 0$, and H is continuous to have

$$H(x_0, u(x_0), D\varphi(x_0)) \leq 0. \tag{1.2.9}$$

Consequently, condition (1.2.3) implies inequality (1.2.9). Similarly, we deduce the reverse inequality

$$H(x_0, u(x_0), D\varphi(x_0)) \geq 0 \tag{1.2.10}$$

provided

$$u - \varphi \text{ has a local minimum at } x_0. \tag{1.2.11}$$

The proof is exactly similar to that mentioned before, except that the inequalities (1.2.7) and thus in (1.2.8), are reversed. In summary, for any $x_0 \in \Omega$ and $\varphi \in C^1(\Omega)$ such that inequality (1.2.9) follows from (1.2.3), and (1.2.10) from (1.2.11). We have in effect put the derivatives onto φ, at the expense of certain inequalities holding.

The properties (1.2.9) and (1.2.10) of the limit u of the subsequence u_{ε_j} motivate the following concept of weak solution [36].

Definition 1.2.1 *A function $u \in C(\Omega)$ is a viscosity subsolution of (1.1.1) in Ω if, for any $\varphi \in C^1(\Omega)$, we have*

$$H(x_0, u(x_0), D\varphi(x_0)) \leq 0 \tag{1.2.12}$$

at any local maximum point $x_0 \in \Omega$ of $u - \varphi$.
Similarly, $u \in C(\Omega)$ is a viscosity supersolution of (1.1.1) in Ω if, for any $\varphi \in C^1(\Omega)$, we have

$$H(x_0, u(x_0), D\varphi(x_0)) \geq 0 \tag{1.2.13}$$

at any local minimum point $x_0 \in \Omega$ of $u - \varphi$.
Finally, $u \in C(\Omega)$ is a viscosity solution of (1.1.1) if it is both a viscosity subsolution and a viscosity supersolution of (1.1.1) in Ω.

The notion of viscosity solution is a notion of "weak " solution of Hamilton-Jacobi equation, since u is assumed to be only continuous and the existence of Du is not necessary. But, in some sense, at a point of maximum of $u - \varphi$, where $\varphi \in C^1(\Omega)$, a good candidate to replace Du is $D\varphi$.

Remark 1.2.1 *In the definition of viscosity subsolution one can always assume that $u-\varphi$ has a local strict maximum at x_0 (otherwise, replace $\varphi(x)$ by $\varphi(x)+|x-x_0|^2$). Moreover, since (1.2.12) depends only on the value of $D\varphi$ at x_0, it is not restrictive to assume that $u(x_0) = \varphi(x_0)$. Similar remarks apply of course to the definition of supersolution. Geometrically, this means that the validity of the subsolution condition (1.2.12) for u is tested on smooth functions "touching from above " the graph of u at x_0 and the validity of the supersolution condition (1.2.13) for u is tested on smooth functions "touching from below" the graph of u at x_0.*

There is an alternative way of defining viscosity solutions of (1.1.1) which is equivalent of Definition 1.2.1. Let us associate with a function $u \in C(\Omega)$ and $x \in \Omega$ the sets

$$D^+u(x) := \{p \in \mathbb{R}^n : \lim_{y \to x,\ y \in \Omega} \sup \frac{u(y) - u(x) - p.(y-x)}{|x-y|} \leq 0\},$$

$$D^-u(x) := \{p \in \mathbb{R}^n : \lim_{y \to x,\ y \in \Omega} \inf \frac{u(y) - u(x) - p.(y-x)}{|x-y|} \geq 0\}.$$

These sets are called, respectively, the superdifferential and the subdifferential (or semidifferentials) of a function u at a point x.

From the definition of subdifferentials and superdifferentials it follows that, for any $x \in \Omega$, $D^-(-u)(x) = -D^+u(x)$.

Some properties of semidifferentials are collected in the following lemma.

Lemma 1.2.1 [16] *Let $u \in C(\Omega)$ and $x \in \Omega$. Then,*
(a) $D^+u(x)$ and $D^-u(x)$ are closed convex (possibly empty) subsets of $\mathbb{R}^n$;
(b) if u is differentiable at x, then $D^+u(x) = D^-u(x) = \{Du(x)\}$;
(c) if for x both $D^+u(x), D^-u(x)$ are nonempty, then $D^+u(x) = D^-u(x) = \{Du(x)\}$.

The following new definition of continuous viscosity solution of (1.1.1) is equivalent of Definition 1.2.1, see [16].

Definition 1.2.2 *A function $u \in C(\Omega)$ is a viscosity subsolution of (1.1.1) in Ω if, for any $x \in \Omega$, it satisfies*

$$H(x, u(x), p) \leq 0, \ \forall p \in D^+u(x). \tag{1.2.14}$$

A function $u \in C(\Omega)$ is a viscosity supersolution of (1.1.1) in Ω if, for any $x \in \Omega$, we have

$$H(x, u(x), p) \geq 0, \ \forall p \in D^-u(x). \tag{1.2.15}$$

A function $u \in C(\Omega)$ is a viscosity solution of (1.1.1) if it is a viscosity subsolution and supersolution.

Example 1.2.1 *Consider the following equation*

$$(u_x(x))^2 - 1 = 0, \ x \in (-1, 1). \tag{1.2.16}$$

Then, the function $u(x) = 1 - |x|$ is a continuous viscosity solution of (1.2.16) but the function $v(x) = -u(x) = |x| - 1$ is a viscosity subsolution but not a viscosity supersolution of (1.2.16). To check this, notice first that if $x \neq 0$, u and v are classical solutions of (1.2.16). Therefore, at those points both the supersolution and the subsolution conditions (1.2.15) and (1.2.14), respectively, are trivially satisfied.

However, $D^+u(0) = [-1, 1]$ and $D^-u(0) = \emptyset$; thus, the requirement in (1.2.15) is empty, while (1.2.14) holds since $p^2 - 1 \leq 0$ for all $p \in D^+u(0)$. So u is a continuous viscosity solution of (1.2.16). On the other hand, since $p = 0$ belongs to $D^-v(0) = [-1, 1]$, then the condition of supersolution (1.2.15) is not satisfied at $x = 0$ and v is not a continuous viscosity solution of (1.2.16).

Note that viscosity solutions are not preserved by changing the sign in the equation. Indeed, $v(x) = |x| - 1$ is a viscosity solution of $-(u_x(x))^2 + 1 = 0$ in $(-1, 1)$ but in Example 1.2.1, v is not a continuous viscosity solution of $((u_x(x))^2 - 1 = 0$ in $(-1, 1)$.

In general, if $H(x, r, p)$ is nondecreasing in r, a function u is a viscosity subsolution of $H(x, u, Du) = 0$ if and only if $v = -u$ is a viscosity supersolution of $-H(x, -v, -Dv) = 0$ in Ω; similarly, u is a viscosity supersolution of $H(x, u, Du) = 0$ if and only if $v = -u$ is a viscosity subsolution of $-H(x, -v, -Dv) = 0$ in Ω.

The following theorem, given in [16], establishes the consistency of the notion of continuous viscosity solutions and classical solutions.

Theorem 1.2.1 *(i) If $u \in C^1(\Omega)$ is a classical solution of (1.1.1), then u is a viscosity solution of (1.1.1).*
(ii) If $u \in C(\Omega)$ is a viscosity solution of (1.1.1), then $H(x_0, u(x_0), Du(x_0)) = 0$ at any point $x_0 \in \Omega$ where u is differentiable.
(iii) If $u \in C(\Omega)$ is locally Lipschitz continuous and it is a viscosity solution of (1.1.1), then $H(x, u(x), Du(x)) = 0$ almost everywhere in Ω, and thus u is a generalized solution of (1.1.1) in terms of Definition 1.1.1.

The converse of property *(iii)* in Theorem 1.2.1 is false in general. There are many generalized solutions which are not viscosity solutions. As an example, observe that $v(x) = |x| - 1$ is a generalized solution of equation $(v_x(x))^2 = 1$ in $(-1, 1)$, because it satisfies the equation in $(-1, 1)$ except at $x = 0$, but it is shown in Example 1.2.1 that v is not a viscosity supersolution of the same equation in $(-1, 1)$.

The next theorem is a stability result for viscosity solutions.

Theorem 1.2.2 [16] *Let for every $m \in \mathbb{N}$ the function $u_m \in C(\Omega)$ be a viscosity solution of an equation of the form*

$$H_m(x, u(x), Du(x)) = 0, \ x \in \Omega,$$

where $\{H_m\}_{m\geq 1}$ is the sequence of Hamiltonians. If

$$\begin{aligned} u_m &\rightarrow u \text{ locally uniformly in } \Omega, \\ H_m(x, r, p) &\rightarrow H(x, r, p) \text{ locally uniformly in } \Omega \times \mathbb{R} \times \mathbb{R}^n, \end{aligned}$$

then u is a continuous viscosity solution of (1.1.1).

Theorem 1.2.2 does not hold for generalized solutions in general. As an example, the property (1.1.7) implies that the uniform limit of the sequence $\{u_m\}$, given in (1.1.6) for $m \geq 2$, is identically zero and does not satisfy the equation (1.1.4) at any point of $(0, 1)$. Therefore, the functions u_m, $m \geq 2$, are not viscosity solutions of (1.1.4).

The following theorem, given in [40], gives very general existence, uniqueness and continuous dependence results for viscosity solutions of the problem of the form (1.1.3)-(1.1.2).

Theorem 1.2.3 *Consider the equation*

$$u_t + H(Du) = 0, (x, t) \in \mathbb{R}^n \times (0, \infty) \tag{1.2.17}$$

with initial conditions

$$u(x, 0) = u_0(x), x \in \mathbb{R}^n. \tag{1.2.18}$$

Let $H \in C(\mathbb{R}^n)$ and u_0 be uniformly continuous in $\mathbb{R}^n$. Then there is a unique continuous function $u : \mathbb{R}^n \times [0, +\infty) \rightarrow \mathbb{R}$ with the following properties: u is uniformly continuous in x uniformly in t, u is a continuous viscosity solution of (1.2.17) and u satisfies (1.2.18).

The following theorem, given in [36], shows that continuous viscosity solution of the problem (1.2.17)-(1.2.18) depends monotonically on the initial value.

Theorem 1.2.4 *Let $0 < T < \infty$ and let u, v be bounded and uniformly continuous real functions in $(\mathbb{R}^n \times [0, T])$. If u and v are continuous viscosity solutions of the equation (1.2.17), with initial conditions u_0 and v_0, respectively, then*

$$\sup_{\mathbb{R}^n \times [0,T]} (\max(u - v, 0)) \leq \sup_{\mathbb{R}^n} (\max(u_0 - v_0, 0)) .$$

In general, for any given Hamilton-Jacobi equation of the form (1.1.3), where the Hamiltonian $H(y, t, u, Du)$ is continuous, nondecreasing in u, there exists a unique uniformly continuous viscosity solution if the initial data is bounded and uniformly continuous [128].

The theory of continuous viscosity solutions has been extensively studied in many relevant articles such as Crandall-Lions [39], Crandall-Evans-Lions [36]. For more complete references, we refer to some monographs by Lions [93], Bardi-Capuzzo-Dolcetta [16], Fleming-Soner [53], and Barles [19]. In [93], existence and uniqueness results for many classes of Hamilton-Jacobi equations are given . The existence and uniqueness of continuous viscosity solution of Hamilton-Jacobi equation in infinite dimensions are given in [41] and [42].

1.3 Discontinuous Viscosity Solutions

According to definition of viscosity solution, introduced by Crandall and Lions, when the value function of optimal control problem is uniformly continuous, it is then a viscosity solution of the associated partial differential equation [93].

However, many optimal control problems, such as the exit time problems, have discontinuous value functions [16]. For this reason, the concept of classical viscosity solutions was further generalized to include solutions that are not necessarily continuous.

The notion of viscosity solutions in the context of semicontinuous solutions has been introduced first by Ishii [71].

We recall that a function $u : \Omega \to \mathbb{R}$ is upper (respectively, lower) semicontinuous if for any $x \in \Omega$ and $\varepsilon > 0$ there is $\delta > 0$ such that $u(y) < u(x) + \varepsilon$ (respectively, $u(y) > u(x) - \varepsilon$) for all $y \in \Omega \cap B_\delta(x)$; Weierstrass' Theorem on the existence of maxima (respectively, minima) on compact sets holds for upper (respectively, lower) semicontinuous functions.

The definition of continuous viscosity subsolution and supersolution of (1.1.1) extends naturally to semicontinuous functions as follows [16].

Definition 1.3.1 *A function $u \in USC(\Omega)$ is called a viscosity subsolution of the Hamilton-Jacobi equation (1.1.1) if for any $\varphi \in C^1(\Omega)$ we have*

$$H(x, u(x), D\varphi(x)) \leq 0$$

at any local maximum point x of $u - \varphi$.
A function $u \in LSC(\Omega)$ is called a viscosity supersolution of the Hamilton-Jacobi equation (1.1.1) if for any $\varphi \in C^1(\Omega)$ we have

$$H(x, u(x), D\varphi(x)) \geq 0$$

at any local minimum point x of $u - \varphi$.

Notice that for any $\varphi \in C^1(\overline{\Omega})$ and $u \in USC(\overline{\Omega})$ the difference $u - \varphi \in USC(\overline{\Omega})$. Hence, $u - \varphi$ attains its maximum on $\overline{\Omega}$. This is an indication that the upper semicontinuous function is used in the definition of a viscosity subsolution. Similarly, if $u \in LSC(\overline{\Omega})$ then $u - \varphi \in LSC(\overline{\Omega})$ attains its minimum on $\overline{\Omega}$, and thus this explains the use of lower semicontinuous function in the definition of a viscosity supersolution.

Since $D^+u(x)$ and $D^-u(x)$ make sense for any real valued function u, see [14], we have the following result.

Lemma 1.3.1 [14] *(i) Let $u \in USC(\Omega)$. $p \in D^+u(x) \Leftrightarrow$ there exists $\varphi \in C^1(\Omega)$ such that $D\varphi(x) = p$ and $u - \varphi$ has a local maximum at a point $x \in \Omega$.*
(ii) Let $v \in LSC(\Omega)$. $p \in D^-v(x) \Leftrightarrow$ there exists $\varphi \in C^1(\Omega)$ such that $D\varphi(x) = p$ and $v - \varphi$ has a local minimum at a point $x \in \Omega$.

As for continuous viscosity subsolutions and supersolutions, there is an equivalent definition by means of semidifferentials of u instead of test functions.

Definition 1.3.2 [16] *A function $u \in USC(\Omega)$ is a viscosity subsolution of (1.1.1) in Ω if, for any $x \in \Omega$, it satisfies*

$$H(x, u(x), p) \leq 0, \ \forall p \in D^+u(x).$$

A function $u \in LSC(\Omega)$ is a viscosity supersolution of (1.1.1) in Ω if, for any $x \in \Omega$, we have

$$H(x, u(x), p) \geq 0, \ \forall p \in D^-u(x).$$

Naturally, a solution should be required somehow to incorporate the properties of both a subsolution and a supersolution. In the classical viscosity solutions theory, a viscosity solution is a function u which is both a subsolution and a supersolution.
Since $USC(\Omega) \cap LSC(\Omega) = C(\Omega)$ this clearly implies that the viscosity solutions defined in this way are all continuous functions.

The concept of viscosity solution for functions which are not necessarily continuous is introduced by using the upper and lower semicontinuous envelopes, see [71]. Let us recall that the upper semicontinuous envelope of a function u which we denote by $S(u)$ is the least upper semicontinuous function which is greater than or equal to u. In a similar way, the lower semicontinuous envelope of a function u, denoted by $I(u)$, is the largest lower semicontinuous function not greater than u.

For a locally bounded function $u : \Omega \to \mathbb{R}$, we have the following representations of $S(u)$ and $I(u)$:

$$S(u)(x) = \min\{f(x) : f \in USC(\Omega), u \leq f\} = \inf_{\delta>0} \sup\{u(y) : y \in B_\delta(x)\}, \quad (1.3.1)$$

$$I(u)(x) = \max\{f(x) : f \in LSC(\Omega), u \geq f\} = \sup_{\delta>0} \inf\{u(y) : y \in B_\delta(x)\}. \quad (1.3.2)$$

It is clear that $I(u) \leq u \leq S(u)$ in Ω. Note that

$$u \in USC(\Omega) \text{ iff } u = S(u) \text{ and } u \in LSC(\Omega) \text{ iff } u = I(u) \text{ in } \Omega.$$

Using the fact that for any locally bounded function $u : \Omega \to \mathbb{R}$, the functions $S(u)$ and $I(u)$ are always, respectively, upper semicontinuous and lower semicontinuous, see (2.2.15) and (2.2.14), a viscosity solution for u can be defined as follows, see [16].

Definition 1.3.3 *A locally bounded function $u : \Omega \to \mathbb{R}$ is a (discontinuous) viscosity solution of the Hamilton-Jacobi equation (1.1.1) if $S(u)$ is a viscosity subsolution of (1.1.1) and $I(u)$ is a viscosity supersolution of (1.1.1).*

Note that the definition of viscosity subsolution and viscosity supersolution for semicontinuous functions is consistent with the concept of continuous viscosity solutions, because a function that is simultaneously a viscosity subsolution and viscosity supersolution is automatically continuous. Thus, if $u \in C(\Omega)$, then $I(u) = S(u) = u$ and Definition 1.2.1 coincides with Definition 1.3.3. From now on the expressions viscosity subsolution and supersolution are used in the sense of Definition 1.3.1 and viscosity solution is used in the sense of Definition 1.3.3.

The direct method of proving the existence of viscosity solutions of the Hamilton-Jacobi equation (1.1.1) is the *Perron's method* [71]. The idea is to build a viscosity solution as the supremum of viscosity subsolutions. This is an analogue for Hamilton-Jacobi equations to the well-known method of finding solutions of the Laplace equation due to O. Perron [113]. The Perron method is the following:
Let $\mathcal{U}$ be a nonempty set of viscosity subsolutions of (1.1.1) having the following two properties:
(i) if $v \in \mathcal{U}$ is not a viscosity solution of (1.1.1), then there is a function $w \in \mathcal{U}$ such that $w(y) > v(y)$ for some $y \in \Omega$,
(ii) if $u(x) = \sup\{v(x) : v \in \mathcal{U}\}$ for $x \in \Omega$, then $u \in \mathcal{U}$.
Set $u(x) = \sup\{v(x) : v \in \mathcal{U}\}$ for $x \in \Omega$. Then u is a viscosity solution of (1.1.1). Moreover, if $S(u) \leq I(u)$ on Ω, then $u \in C(\Omega)$.

Thus the existence problem of viscosity solutions is reduced to that of finding an appropriate set $\mathcal{U}$ of viscosity subsolutions of (1.1.1) which satisfies (i) and (ii), depending on boundary conditions, assumptions on Hamiltonians H and so on. Also, the continuity of viscosity solutions may follow from the special structure of (1.1.1) or from the comparison principle between viscosity subsolutions and supersolutions.

We will use the following result which was provided in [16].

Theorem 1.3.1 *(a) Let $\mathcal{U}$ be a set of functions such that $S(w)$ is a viscosity subsolution of (1.1.1) for all $w \in \mathcal{U}$, and define*

$$u(x) := \sup_{w \in \mathcal{U}} w(x), \ x \in \Omega.$$

If u is locally bounded, then $S(u)$ is a viscosity subsolution of (1.1.1).
(b) Let $\mathcal{Z}$ be a set of functions such that $I(w)$ is a viscosity supersolution of (1.1.1) for all $w \in \mathcal{Z}$, and define

$$v(x) := \inf_{w \in \mathcal{Z}} w(x), \ x \in \Omega.$$

If v is locally bounded, then $I(v)$ is a viscosity supersolution of (1.1.1).

The following general existence theorem for equation (1.1.1) using Perron's method is given in [16].

Theorem 1.3.2 *Assume there exists a viscosity subsolution u_1 and a viscosity supersolution u_2 of the Hamilton-Jacobi equation (1.1.1) such that $u_1 \leq u_2$. Then the functions*

$$(a)\ u(x) := \sup\{w(x) : u_1 \leq w \leq u_2,\ S(w) \text{ viscosity subsolution of (1.1.1)}\},$$

$$(b)\ v(x) := \inf\{w(x) : u_1 \leq w \leq u_2,\ I(w) \text{ viscosity supersolution of (1.1.1)}\},$$

are (discontinuous) viscosity solutions of (1.1.1).

Since we deal, in some sense, with two functions $I(u)$ and $S(u)$, it is not immediately clear how to interpret uniqueness with discontinuous viscosity solutions. For example, if u is equal to 1 in Ω except at x_0 where is equal to 0, then $S(u) \equiv 1$ and the subsolution condition does not differentiate between the function u and a function constantly equal to $f \equiv 1$. The usual way to get an uniqueness result is to obtain a comparison principle between viscosity supersolutions and viscosity subsolutions [21].

For uniqueness of viscosity solutions, let us specify what is meant by comparison principle for semicontinuous functions.

Definition 1.3.4 *We say that the Dirichlet problem for equation (1.1.1) satisfies the comparison principle or the comparison principle holds for Hamiltonians H if for any function $u \in BUSC(\overline{\Omega})$ and any function $v \in BLSC(\overline{\Omega})$ which are, respectively, a viscosity subsolution and a viscosity supersolution of (1.1.1) such that $u \leq v$ on $\partial\Omega$, we have $u \leq v$ in Ω.*

The next theorem shows that the comparison principle between viscosity subsolutions and viscosity supersolutions, given in Definition 1.3.4, implies the continuity of the unique viscosity solution.

Theorem 1.3.3 *[16] Assume that the Dirichlet problem for (1.1.1) satisfies a comparison principle. If $u : \overline{\Omega} \to \mathbb{R}$ is a viscosity solution of (1.1.1) which is bounded on Ω and continuous at all $x \in \partial\Omega$, then $u \in C(\overline{\Omega})$. In particular, for any $g \in C(\partial\Omega)$ in (1.1.2), there is at most one such viscosity solution of (1.1.1) satisfying $u = g$ on $\partial\Omega$.*

Let u be a viscosity solution of (1.1.1) which is bounded on Ω and $u \in C(\partial\Omega)$. Then $S(u) \in BUSC(\overline{\Omega})$ is a viscosity subsolution and $I(u) \in BLSC(\overline{\Omega})$ is a viscosity supersolution of (1.1.1). Since u is continuous at $\partial\Omega$, we have $S(u)(x) = I(u)(x) = u(x), x \in \partial\Omega$. Therefore, by the comparison principle, we have $S(u) \leq I(u)$ in $\overline{\Omega}$. Since by construction $I(u) \leq S(u)$, we have $S(u) = I(u) = u$ is a continuous viscosity solution, but it is a too restrictive conclusion since a lot of applications has naturally discontinuous solutions. Therefore, the above comparison principle adds nothing to the problem of uniqueness of discontinuous viscosity solution.

Next we give some simple examples showing that the notion of viscosity solution, given in Definition 1.3.3, is rather weak. The first one shows that nowhere continuous functions can be solutions, in terms of Definition 1.3.3, of very simple equations such as

$$u_x = 0 \quad x \in \mathbb{R}. \tag{1.3.3}$$

Example 1.3.1 *Let u be function defined by*

$$u(x) = \begin{cases} 1 & , \quad x \in \mathbb{Q} \\ 0 & , \quad x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Then $S(u) \equiv 1$ and $I(u) \equiv 0$, so u is a viscosity solution of (1.3.3).

The second example explains the nonuniqueness of Ishii's results, namely a boundary value problem satisfying comparison principle, given in Definition 1.3.4, may have infinitely many discontinuous viscosity solutions satisfying boundary conditions.

Example 1.3.2 *The Dirichlet problem*

$$u_x(x) = 1,\ x \in (0,1) \tag{1.3.4}$$

$$u(0) = 1,\ u(1) = 2 \tag{1.3.5}$$

has the continuous viscosity solution $u(x) = x+1$ *by Theorem 1.3.3 and* u *satisfies (1.3.5). In addition, for any dense subset* X *of* $[0,1]$ *such that* 0 *and* 1 *belong to* X*, the function*

$$v(x) = \begin{cases} x & , x \in [0,1]\backslash X \\ x+1, & x \in [0,1] \cap X \end{cases}$$

is also a viscosity solution because $S(v)(x) = x+1$ *and* $I(v)(x) = x$ *are both classical solutions of the equation (1.3.4). Moreover,* v *satisfies the boundary conditions (1.3.5).*

Refer to Theorem 1.3.3, if we require continuity at all boundary points, then any function which is viscosity solution in the sense of Definition 1.3.3 is automatically continuous in $\overline{\Omega}$ and this rules out many interesting problems.

Then for Hamiltonians satisfying the structural assumptions of any comparison theorems, a viscosity solution of (1.1.1) may be a discontinuous function only if it is discontinuous at some boundary point. This fact is a general property of Hamilton-Jacobi equations.

Due to the nonuniqueness in Ishii's result, see Example 1.3.2, other notions of discontinuous solutions were proposed by various authors. Barles and Perthame [20], Barron and Jensen [23], Subbotin [131] made efforts in studying the discontinuous solutions. Their notions are in the context of semicontinuous solutions. But the definition of Ishii, given in [71], played a pivotal role. Other notions of discontinuous solutions are also introduced by Giga Sato [57], C. Guiqiang and S. Bo [62], Siconolfi [127], etc.

For presentation of discontinuous viscosity solution, the reader can consult Crandall-Ishii-Lions [37], Fleming-Soner [53], Bardi-Capuzzo-Dolcetta [16], Bardi et al. [17], and Barles [19] and the references there in.

The application of Perron's method for establishing an existence result for L^p-viscosity solutions of fully nonlinear second-order elliptic equations is given in [78].

1.4 Objectives of this Thesis

The first main goal of this thesis is to propose a new approach to the treatment of discontinuous solutions for first-order Hamilton-Jacobi equations by involving Hausdorff continuous interval valued functions. The only assumption is a continuity of Hamiltonian.

Applications of Hausdorff continuous functions are relevant because it has been shown recently by R. Anguelov and E. E. Rosinger [12] that the solutions of large classes of non-linear partial differential equations can be assimilated with Hausdorff continuous functions on the open domains.

To reach this goal, the following objectives are identified:

• to show that interval valued functions can be considered as discontinuous viscosity solutions;

• to prove that the main ideas within the classical theory of continuous viscosity solutions can be extended almost unchanged to the wider space of Hausdorff continuous functions;

• to prove that the Hausdorff continuous viscosity solutions have a more clear interpretation than the existing concepts of discontinuous solutions, e.g., envelope solutions;

• to show that the value function of optimal control problem as solution of associated Hamilton-Jacobi-Bellman equation typically belongs to the wider space of Hausdorff continuous functions.

The second main goal is the design of numerical schemes for Hamilton-Jacobi equations and for conservation laws which preserve essential properties of the exact solutions. This is typically expressed through the concept of qualitative stability. To achieve this we use the nonstandard finite difference method to design

• a scheme for Hamilton-Jacobi equation which is qualitatively stable with respect to monotonicity property;

• schemes for conservation laws which are qualitatively stable to respect to total variation diminishing property.

1.5 Outline of this Thesis

From Definition 1.3.3, interval valued functions appear naturally in the context of discontinuous viscosity solutions. Indeed, Definition 1.3.3 places requirements not on the function u itself but on its lower and upper semicontinuous envelopes or, in other words, on the interval valued function

$$F(u)(x) = [I(u)(x), S(u)(x)],\ x \in \Omega,$$

which is called the *graph completion* of u, see [15]. Clearly, Definition 1.3.3 treats functions which have the same upper and lower semicontinuous envelopes, that is, have the same graph completion, as identical functions. On the other hand, since different functions can have the same graph completion, a function can not in general be identified from its graph completion, that is, functions with the same graph completion are indistinguishable. Therefore, no generality will be lost if only interval valued functions representing graph completions are considered.

The second chapter introduces the concept of Hausdorff continuous (H-continuous) interval valued functions and discusses their properties. The applications of H-continuous functions to problems in Analysis [5] and to nonlinear partial differential equation [13] are based on the quite extraordinary fact that the set of all Hausdorff continuous functions on open domains is Dedekind order complete. This property is given in this chapter.

In Chapter 3, which contains the first main result of this thesis, we discuss Hausdorff continuous viscosity solutions of Hamilton-Jacobi equations and we prove that the notion of Hausdorff continuous viscosity solution is stronger than the notion of (discontinuous) viscosity solution, see Section 3.2. In the second place, we show that when the H-continuous viscosity solution is a supremum of viscosity subsolutions or infimum of viscosity supersolutions, it can be linked to an envelope viscosity solution. This is given in Section 3.3. In Section 3.4, we begin with some properties of Hausdorff continuous viscosity subsolutions and supersolutions. Moreover, we formulate and prove an existence theorem for Hamilton-Jacobi equations using Perron's method for H-continuous viscosity solutions. Section 3.5 deals with uniqueness of H-continuous viscosity solution. We formulate the comparison principle for Hausdorff continuous functions and give sufficient conditions implying it in a weaker form. The uniqueness result, see Theorem 3.5.1, is given under the assumption that comparison principle, given in Definition 3.5.1, is satisfied.

Finally we express the H-continuous viscosity solutions of Hamilton-Jacobi equation as solutions to an operator equation involving the extended Hamiltonian operator in the same way as the classical solutions of Hamilton-Jacobi equation are solutions of operator equation associated of this Hamilton-Jacobi equation.

In Chapter 4, the theory of Hausdorff continuous viscosity solutions is applied to optimal control problem in particular discounted minimum time problem. We show that the value function of discounted minimum time problem is an envelope viscosity solution of an associated Hamilton-Jacobi-Bellman equation. This is illustrated by a Zermelo navigation problem given in Section 4.3.

The second main result is given in Chapter 5. We consider two approaches to numerical solutions for Hamilton-Jacobi equations. The first one, given in Section 5.2, is a monotone scheme for Hamilton-Jacobi equation. The study of numerical approximation to multi-dimensional Hamilton-Jacobi equations was started by Crandall and Lions [40]. They presented monotone finite difference schemes on rectangular domains. The numerical solutions in [40, Theorem 1] indicate convergence to the viscosity solutions of Hamilton-Jacobi equations. Finite difference methods were developed for solving Hamilton-Jacobi equations [128, 92, 111]. However, the schemes applying standard finite difference techniques are typically monotone under some restriction on the time step sizes. We motivate by the paper [91] where a severe restriction on the time step size is imposed for the numerical scheme for Hamilton-Jacobi equations obtained through the coupling of the finite difference method (in space) and the finite element method (in time) to be monotone. In Section 5.2, we relax this restriction by using Micken's nonstandard finite difference method [103]. More precisely, Micken's rule of nonlocal approximation is exploited and this leads to a nonstandard scheme that replicates the monotonicity property of the Hamilton-Jacobi equations for all positive time step sizes. Furthermore, the superiority of the nonstandard method to the standard one is confirmed by numerical results at the end of this section.

The second approach is based on preserving total variation diminishing property of conservation laws. It has been shown that schemes with such qualitative stability resolve discontinuities in the solution without spurious oscillations which are often displayed by numerical solutions [124], [125]. These schemes are called total variation diminishing (TVD).

One problem associated with the explicit total variation diminishing methods is a restriction on the time step-size which in some cases could be rather severe. This is particularly pronounced in high-order methods, e.g., methods of Runge-Kutta type [60, 61]. On the other hand, the computational complexity of total variation diminishing implicit methods is significantly higher particularly when nonlinear functions are involved.

In Section 5.3, we construct (i) an implicit nonstandard finite difference scheme using nonlocal approximation of nonlinear terms and (ii) explicit nonstandard finite difference schemes where renormalization of the denominator is used. Numerical results demonstrating the properties of the methods are presented.

It is well known that the Hamilton-Jacobi equations are closely related to conservation laws [77], hence successful numerical methods for conservation laws are adapted for solving the Hamilton-Jacobi equations. A long this line, we mention the early work of Osher and Sethian [110] and Osher and Shu [111] in constructing high-order essentially non-oscillatory (ENO) schemes for solving the Hamilton-Jacobi equations. These ENO schemes for solving the Hamilton-Jacobi equations were based on ENO schemes for solving hyperbolic conservation laws in [65, 125, 126]. We mention also the weighted (ENO) (WENO) schemes for solving the Hamilton-Jacobi equations by Zhang and Shu [137] and by Jiang and Peng [72], based on the WENO schemes for solving conservation laws [95, 74]. Adapted from the discontinuous Galerkin methods for solving hyperbolic conservation laws [33], a discontinuous Galerkin method for solving Hamilton-Jacobi equations was developed by Hu and Shu in [70].

At the end of Section 5.3, we use a discontinuous Galerkin finite element method of Hu and Shu [70] to solve the one dimensional Hamilton-Jacobi equation which applies the discontinuous Galerkin framework on the conservations laws. Namely, since the derivative of the solution u of Hamilton-Jacobi equation satisfies a conservation law, we apply the usual discontinuous Galerkin method on this conservation law to advance the derivative of u. The solution u itself is then recovered from this derivative computed using nonstandard total variation diminishing method.

Finally, Chapter 6 summarizes the results that have been done in the thesis, highlights the most outstanding results and gives some directions for future research.

1.6 Summary of Contributions

The main contributions of this thesis are:

• The concept of Hausdorff continuous viscosity solution is given in Section 3.2, see Definition 3.2.1.

• Existence theorem for H-continuous viscosity solutions using Perron's method. The solution is constructed as a supremum of a set of viscosity subsolutions in the set of Hausdorff continuous functions, Theorem 3.4.2.

• The relation between the H-continuous viscosity solution and the existing theory of discontinuous viscosity solutions. Namely, any H-continuous viscosity solution is discontinuous viscosity solution as defined by Ishii, and it is typically also an envelope viscosity solution. The H-continuous viscosity solution is a stronger concept than the concept of discontinuous viscosity solution given by Ishii and as well as the concept of envelope viscosity solution. Yet, the existence is proved under the same assumptions, see Sections 3.2 and 3.3.

• In Section 3.5, the Hausdorff continuous viscosity solutions of Hamilton-Jacobi equations are expressed as the solutions to an operator equation involving the extended operator in the same way as the classical solutions of Hamilton-Jacobi equations are solutions of this operator equation.

• The value function of the optimal control problem can be considered as H-continuous viscosity solution of associated Hamilton-Jacobi-Bellman equation, see Chapter 4.

• The design of nonstandard finite difference scheme for Hamilton-Jacobi equation preserving the monotonicity property for any time step size is given in Section 5.2.

• The design of nonstandard finite difference schemes for conservation laws preserving the total variation diminishing property for all positive time step sizes is given in Section 5.3.

Chapter 2

The Space of Hausdorff Continuous Interval Valued Functions

2.1 Introduction

Historically, the interval analysis, or the analysis of interval valued functions, is associated with the so called validated computing where algorithms generating validated bounds for the exact solutions of mathematical problems are designed and investigated, see [2] and [81]. The Hausdorff continuous functions, being a particular class of interval valued functions, belong to interval analysis, see Moore [107]. However, interest in the interval valued functions comes also from other branches of mathematics such as nonlinear partial differential equations, see [13] which strengthens the results in [109], and approximation theory. In fact, Hausdorff continuous functions of one variable were introduced first by Sendov in connection with Hausdorff approximations of real functions of real argument, see [120]. The concept was further developed in [10] as part of the analysis of interval valued functions and extended to interval valued functions defined on a topological space [4]. The name Hausdorff continuous is due to a characterization of these functions in terms of the Hausdorff distance between the graphs of real functions as defined in [119]. Recently, it is shown by Anguelov and Rosinger [12] that very large classes of nonlinear partial differential equations have solutions which can be assimilated with Hausdorff continuous functions. Further applications of the concept of Hausdorff continuity are presented in [10]. For recent advances in theory of Hausdorff approximations, see [3] and the references therein.

This chapter will serve as an introduction to the Hausdorff continuous functions and the stated results will be used in the sequel. The Baire operators and the graph completion operator which are instrumental for the definition and the properties of Hausdorff continuous functions are discussed in Section 2.2. Section 2.3 deals with Hausdorff continuous functions. Two very important theorems are presented. The first theorem shows how to construct Hausdorff continuous functions, and the second gives useful necessary and sufficient conditions for an interval valued function to be Hausdorff continuous function. In Section 2.4, Theorem 2.4.1 shows that the set of Hausdorff continuous functions is Dedekind order complete while Theorem 2.4.2 gives a useful representation of supremum of a subset of a set of all Hausdorff continuous functions. The generalized lower, upper Baire and graph completion operators are given in Section 2.5.

2.2 Baire Operators and Graph Completion Operator

In this section, we recall the upper Baire operator S, lower Baire operator I and the graph completion operator F for interval valued function. Some properties of I, S, and F are given in this section.

Let X be an arbitrary topological space.

Identifying $a \in \mathbb{R}$ with the point interval $[a,\ a] \in \mathbb{IR}$, we consider $\mathbb{R}$ as a subset of $\mathbb{IR}$. From this, it follows that the set $\mathbb{A}(X)$ contains the set of functions with real $\mathcal{A}(X)$. Hence,

$$C(X) \subseteq \mathcal{A}(X) \subseteq \mathbb{A}(X).$$

A partial order which extends the total order on $\mathbb{R}$ can be defined on $\mathbb{IR}$ in more than one way. However, it will prove useful to consider on $\mathbb{IR}$ the partial order $\leq$ defined by

$$[\underline{a},\ \overline{a}] \leq [\underline{b},\ \overline{b}] \Longleftrightarrow \underline{a} \leq \underline{b},\ \overline{a} \leq \overline{b}. \tag{2.2.1}$$

The partial order, given in (2.2.1), is introduced and studied by Markov, see [99, 100]. The inclusion in $\mathbb{A}(X)$ is defined by

$$[\underline{a}, \overline{a}] \subseteq [\underline{b}, \overline{b}] \Longleftrightarrow \underline{b} \leq \underline{a} \leq \overline{a} \leq \overline{b}. \tag{2.2.2}$$

The partial order induced in $\mathbb{A}(X)$ by (2.2.1) in a point-wise way, i.e., for $u, v \in \mathbb{A}(X)$,

$$u \leq v \Longleftrightarrow u(x) \leq v(x),\ x \in X, \tag{2.2.3}$$

is an extension of the usual point-wise order in the set of real valued functions $\mathcal{A}(X)$. Let $u \in \mathbb{A}(X)$. For every $x \in X$, the value of u is interval $[\underline{u}(x), \overline{u}(x)] \in \mathbb{IR}$. Hence, the function u can be written in the form $u = [\underline{u}, \overline{u}]$, where $\underline{u}$, $\overline{u} \in \mathcal{A}(X)$ and $\underline{u}(x) \leq \overline{u}(x)$.

The inclusion induced in $\mathbb{A}(X)$ by (2.2.2) in a point-wise way is given by

$$u \subseteq v \Leftrightarrow \underline{v}(x) \leq \underline{u}(x) \leq \overline{u}(x) \leq \overline{v}(x),\ x \in X, \tag{2.2.4}$$

where $u = [\underline{u}, \overline{u}] \in \mathbb{A}(X)$ and $v = [\underline{v}, \overline{v}] \in \mathbb{A}(X)$.

The definition of upper semicontinuous envelope, the lower semicontinuous envelope given in (1.3.1) and (1.3.2), respectively, for $u \in \mathcal{A}(X)$, can be extended to functions $u = [\underline{u}, \overline{u}] \in \mathbb{A}(X)$ as follows [4]:

Definition 2.2.1 *The mappings $S, I : \mathbb{A}(X) \to \mathcal{A}(X)$, defined by*

$$S(u)(x) = \inf_{\delta>0} \sup\{z \in u(y) : y \in B_\delta(x) \cap X\},\ x \in X, \tag{2.2.5}$$

$$I(u)(x) = \sup_{\delta>0} \inf\{z \in u(y) : y \in B_\delta(x) \cap X\},\ x \in X, \tag{2.2.6}$$

are called upper Baire and lower Baire operators, respectively.

In [15], the mappings (2.2.5) and (2.2.6) were defined and studied in the particular case of functions $u \in \mathcal{A}(X)$ when X is a subset of $\mathbb{R}$.

The lower and upper Baire operators of $u = [\underline{u}, \overline{u}] \in \mathbb{A}(X)$ can be conveniently represented in terms of the functions $\underline{u}$ and $\overline{u}$. Indeed, from (2.2.5) and (2.2.6), it is easy to see that

$$I(u) = I(\underline{u}),\ S(u) = S(\overline{u}). \tag{2.2.7}$$

Clearly, for every $u \in \mathbb{A}(X)$, we have

$$I(u)(x) \leq u(x) \leq S(u)(x),\ x \in X. \tag{2.2.8}$$

Hence, the graph completion operator is defined on $\mathbb{A}(X)$ as follows [4]:

Definition 2.2.2 *The mapping*

$$F : \mathbb{A}(X) \to \mathbb{A}(X),$$

defined by

$$F(u)(x) := [I(u)(x),\ S(u)(x)],\ u \in \mathbb{A}(X),\ x \in X,$$

is called a graph completion operator.

The name of this operator, given by Sendov [119], is derived from the fact that, considering the graphs of u and $F(u)$ as subsets of the topological space $X \times \mathbb{R}$, the graph of $F(u)$ is the minimal closed set, which is a graph of interval valued function on X and contains the graph of u.

For every $u = [\underline{u}, \overline{u}] \in \mathbb{A}(X)$, the property (2.2.7) implies that $F(u)$ can be written in the form

$$F(u) = [I(\underline{u}), S(\overline{u})], \tag{2.2.9}$$

thus

$$F(u) = u \Leftrightarrow \underline{u} = I(\underline{u}),\ \overline{u} = S(\overline{u})].$$

In other words, $u = [\underline{u}, \overline{u}]$ is a fixed point of the operator F if and only if $\underline{u}$ is a fixed point of the operator I, while $\overline{u}$ is a fixed point of the operator S.

By (2.2.9) and (2.2.8), we have

$$u(x) \subseteq F(u)(x),\ x \in X.$$

There is an alternative characterization of lower and upper semicontinuity that involves the fixed points of the lower and upper Baire operators for $u \in \mathcal{A}(X)$. Indeed, we have

$$u \in LSC(X) \Leftrightarrow I(u) = u, \tag{2.2.10}$$

$$u \in USC(X) \Leftrightarrow S(u) = u. \tag{2.2.11}$$

In the sequel, we will use the following properties of operators I, S, and F, which were proved in [4].

Theorem 2.2.1 *(a) The operators I, S, and F are all monotone increasing with respect to the partial order* (2.2.3)*, that is, for any two functions u, $v \in \mathbb{A}(X)$,*

$$u \leq v \Longrightarrow I(u) \leq I(v),\ S(u) \leq S(v),\ F(u) \leq F(v). \tag{2.2.12}$$

(b) The operator F is monotone with respect to the relation inclusion (2.2.4), that is, for any two functions u, $v \in \mathbb{A}(X)$,

$$u(x) \subseteq v(x),\ x \in X \Rightarrow F(u)(x) \subseteq F(v)(x),\ x \in X.$$

(c) The operators I, S and F are all idempotent, that is, for any function $u \in \mathbb{A}(X)$, we have

$$I(I(u)) = I(u),\ S(S(u)) = S(u),\ F(F(u)) = F(u). \tag{2.2.13}$$

The property (c) of Theorem 2.2.1 and the properties (2.2.10) and (2.2.11) imply that for $u \in \mathbb{A}(X)$, we have

$$I(u) \in LSC(X), \tag{2.2.14}$$

$$S(u) \in USC(X). \tag{2.2.15}$$

Upon an obvious extension of the respective result in [15] we have the following lemma about semicontinuous functions.

Lemma 2.2.1 *We have the following:*
(i) Let $\mathcal{L} \subset LSC(X)$, $\mathcal{L} \neq \emptyset$ be bounded from above at each $x \in \Omega$. Then the function l defined by

$$l(x) = \sup\{v(x) : v \in \mathcal{L}\}$$

is lower semicontinuous.
(ii) Let $\mathcal{U} \subset USC(X)$, $\mathcal{U} \neq \emptyset$ be bounded from below at each $x \in \Omega$. Then the function u defined by

$$u(x) = \inf\{v(x) : v \in \mathcal{U}\}$$

is upper semicontinuous.

2.3 Hausdorff Continuous Functions

This section introduces the concept of Hausdorff continuous interval valued functions and discusses their properties.

Definition 2.3.1 *A function $u \in \mathbb{A}(X)$ is called Hausdorff continuous, or for short, H-continuous, if and only if for every function $v \in \mathbb{A}(X)$, we have satisfied the following minimality condition on u*

$$v(x) \subseteq u(x),\ x \in X \Rightarrow F(v)(x) = u(x),\ x \in X. \tag{2.3.1}$$

The minimality condition in (2.3.1) with respect to their graph completion operator F plays a fundamental role. At first it may appear that it applies to each individual point x of X, not involving neighborhoods. However, the graph completion operator F does appear in this condition and this operator according to definition of F and therefore to definition of I and S does certainly refer to neighborhoods of points in X, a situation typical, among others, for the concept of continuity.

We recall here the concept of segment continuity associated with the graph completion operator, [120].

Definition 2.3.2 *A function $u \in \mathbb{A}(X)$ is called segment continuous, or S-continuous, if $F(u) = u$.*

It follows from the idempotence of a function F, see (2.2.13), that for any function $u \in \mathbb{A}(X)$

$$F(u) \text{ is an S-continuous function.} \tag{2.3.2}$$

Furthermore,

$$u \text{ is S-continuous } \Leftrightarrow F(u) = u. \tag{2.3.3}$$

It is easy to see that on the set $\mathbb{A}(X)$

$$\text{H-continuity } \Rightarrow \text{ S-continuity.} \tag{2.3.4}$$

Indeed, if u is H-continuous from the inclusion $u(x) \subseteq u(x)$, $x \in X$, and Definition 2.3.1, it follows that $F(u)(x) = u(x)$, $x \in X$. Thus u is S-continuous on X.

The inverse implication in (2.3.4) is not true. Indeed, consider the example when $X = [0, 1]$. Then, the function u, defined by

$$u(x) = \begin{cases} [-1, 1], & x = 0 \\ 1 \qquad , & 0 < x \leq 1, \end{cases}$$

is S-continuous on $[0, 1]$, but it is not H-continuous on $[0, 1]$. Indeed, it is clear that $F(u)(x) = u(x), x \in [0, 1]$. Now consider interval valued function v defined on $[0, 1]$ by

$$v(x) = \begin{cases} [0, 1], & x = 0 \\ 1 \qquad , & 0 < x \leq 1. \end{cases}$$

Clearly, the inclusion $v(x) \subseteq u(x)$ holds for every $x \in [0, 1]$ and v is S-continuous on $[0, 1]$ because $F(v)(x) = v(x), x \in [0, 1]$. But $F(v)(0) \neq u(0)$ so u cannot be H-continuous function on $[0, 1]$.

However, through the concept of S-continuity, H-continuous functions can be characterized in the following way, [120].

Theorem 2.3.1 *A function $u \in \mathbb{A}(X)$ is H-continuous if and only if u satisfies the following two conditions:*
(i) u is S-continuous on X,
(ii) for every S-continuous function v, the inclusion $v(x) \subseteq u(x)$, $x \in X$ implies $v(x) = u(x)$, $x \in X$.

In general, H-continuity is not preserved on subsets of the domain X. Precisely, if $K \subset X$ and $u \in \mathbb{H}(X)$, then we cannot conclude that $u|_K \in \mathbb{H}(K)$, where $u|_K$ is the restriction of u to K. For example, consider $X = \mathbb{R}, K = [0, +\infty)$, and u defined by

$$u(x) = \begin{cases} -1 \quad , x < 0 \\ [-1, 1], \ x = 0 \\ 1 \qquad , x > 0. \end{cases} \tag{2.3.5}$$

Obviously, $u \in \mathbb{H}(\mathbb{R})$ and $u|_K$ is defined on K by

$$u|_K(x) = \begin{cases} [-1, 1], & x = 0 \\ 1 \qquad , & x > 0. \end{cases}$$

To prove that $u|_K \notin \mathbb{H}(K)$, we consider the interval valued function w defined on K by

$$w(x) = \begin{cases} [0, 1], & x = 0 \\ 1 \quad , & x > 0. \end{cases}$$

Clearly, the inclusion $w(x) \subseteq u|_K(x)$ holds for every $x \in K$ and w is S-continuous on K. Therefore $F(w)(x) = w(x), x \in K$. But $w(0) \neq u|_K(0)$, so $u|_K$ cannot be H-continuous function in K although $K \subset X$.

We know that every S-continuous function is defined in terms of a lower semicontinuous function and an upper semicontinuous function, but there is more.

The following theorem shows how to construct an H-continuous function.

Theorem 2.3.2 [4] *Every pair of a lower semicontinuous function $\underline{u}$ and an upper semicontinuous function $\overline{u}$ such that $\underline{u} \leq \overline{u}$ on X defines an S-continuous function $u(x) = [\underline{u}(x), \overline{u}(x)]$, $x \in X$. Furthermore, if the set*

$$\{\varphi \in \mathbb{A}(X) : \underline{u} \leq \varphi \leq \overline{u}\}$$

does not contain any lower or upper semicontinuous functions, except for $\underline{u}$, respectively $\overline{u}$, then the function u is H-continuous.

The minimality condition given in the above theorem associated with the Hausdorff continuous functions can also be formulated in terms of semicontinuous functions, namely, if $u = [\underline{u}, \overline{u}]$ is S-continuous, then u is H-continuous if and only if

$$\{\varphi \in \mathcal{A}(\Omega) : \varphi \text{ is semicontinuous, } \underline{u} \leq \varphi \leq \overline{u}\} = \{\underline{u}, \overline{u}\}.$$

Theorem 2.3.2 and property (2.3.4) imply that

$$C(X) = USC(X) \cap LSC(X) \subset \mathbb{H}(X) \subset \mathbb{F}(X).$$

Let us note that the set $\mathbb{H}(X)$ is certainly wider than $C(X)$. An example of H-continuous function which is not continuous is given in (2.3.5).

The concept of H-continuity can be considered as a generalization of the concept of continuity of real functions in the sense that the only real (point valued) functions contained in $\mathbb{H}(X)$ are the continuous functions, that is,

$$u \in \mathcal{A}(X) \cap \mathbb{H}(X) \Rightarrow u \in C(X).$$

The following theorem, proved in [4], generalizes some results discussed in [120] to the case when $X \subseteq \mathbb{R}$. It gives useful necessary and sufficient conditions for an interval valued function to be H-continuous.

Theorem 2.3.3 *Let $u = [\underline{u}, \overline{u}] \in \mathbb{A}(X)$. The following conditions are equivalent :*
(a) $u \in \mathbb{H}(X)$,
(b) $F(\overline{u}) = F(\underline{u}) = u$,
(c) $S(\underline{u}) = \overline{u}$, $I(\overline{u}) = \underline{u}$, and $u \in \mathbb{F}(X)$.

With every interval function u one can associate H-continuous functions as stated in the next theorem which is proved in [4].

Theorem 2.3.4 *Let $u \in \mathbb{A}(X)$. Both functions $F(S(I(u)))$ and $F(I(S(u)))$ are Hausdorff continuous and we have*

$$F(S(I(u))) \leq F(I(S(u))).$$

This theorem is illustrated by the following example.

Example 2.3.1 *Consider the function $u \in \mathbb{A}(\mathbb{R})$ given by*

$$u(x) = \begin{cases} [-1,1] & , \quad x \in \mathbb{Z} \\ 0 & , \quad x \in (-\infty, 0) \setminus \mathbb{Z} \\ [0,1] & , \quad x \in (0, +\infty) \setminus \mathbb{Z}, \end{cases}$$

where $\mathbb{Z}$ denotes the set of integers. We have $F(u) = u$ meaning that u is S-continuous. We have the H-continuous functions

$$F(S(I(u)))(x) = 0, x \in \mathbb{R}$$

and

$$F(I(S(u)))(x) = \begin{cases} 0 & , x \in (-\infty, 0) \\ [0,1], & x = 0 \\ 1 & , x \in (0, +\infty). \end{cases}$$

It is quite interesting that pairing a lower semicontinuous function $\underline{u}$ with an upper semicontinuous function $\overline{u}$ such that $\underline{u} \leq \overline{u}$ produces a completely new concept from both algebraic and topological points of view, namely the concept of S-continuous interval functions. The concept of Hausdorff continuity is closely connected with the Hausdorff distance between functions as introduced by Sendov [120]. The Hausdorff distance $\rho(u, v)$ between two functions $u,\ v \in \mathbb{A}(X)$ is defined as the Hausdorff distance between the graphs of the functions $F(u)$ and $F(v)$ considered as subsets of $\mathbb{R}^{n+1}$. More precisely, we have

$$\begin{aligned}\rho(u,v) \quad &= \max\{\sup_{x_1\in X}\sup_{y_1\in F(u)(x_1)}\inf_{x_2\in X}\inf_{y_2\in F(v)(x_2)}||(x_1-x_2,y_1-y_2)||,\\ &\qquad \sup_{x_2\in X}\sup_{y_2\in F(v)(x_2)}\inf_{x_1\in X}\inf_{y_1\in F(u)(x_1)}||(x_1-x_2,y_1-y_2)||\},\end{aligned} \tag{2.3.6}$$

where $||.||$ is a given norm in $\mathbb{R}^{n+1}$.

Condition (b) in Theorem 2.3.3 implies that for any H-continuous function $u = [\underline{u}, \overline{u}]$, the Hausdorff distance between the functions $\underline{u}$ and $\overline{u}$ is zero. More precisely, we have

$$u = [\underline{u}, \overline{u}] \in \mathbb{H}(X) \Leftrightarrow \begin{cases} u \in \mathbb{F}(X) \\ \rho(\underline{u}, \overline{u}) = 0. \end{cases}$$

We should note that any Hausdorff continuous function is "essentially" point valued in the sense that it assumes point values everywhere except on a small set. The next theorem shows that this set is a set of first Baire category, that is, a countable union of closed and nowhere dense sets.

Theorem 2.3.5 [4] *Let $u = [\underline{u}, \overline{u}]$ be an H-continuous function on X. The set*

$$W_u = \{x \in X : \overline{u}(x) - \underline{u}(x) > 0\} \tag{2.3.7}$$

is of first Baire category.

Through an application of the Baire category theorem [132], the above theorem implies that the complement of W_u in X is a set of second category. Hence,

$$D_u = X \setminus W_u = \{x \in X : \overline{u}(x) = \underline{u}(x)\} \text{ is dense in } X.$$

Since a finite or countable union of sets of first Baire category is also a set of first Baire category [130], we have that for every finite or countable set $\mathcal{F}$ of Hausdorff continuous functions, the set

$$\begin{aligned} D_{\mathcal{F}} &= \{x \in X : \underline{u}(x) = \overline{u}(x),\ u = [\underline{u}, \overline{u}] \in \mathcal{F}\} \\ &= X \setminus \left(\bigcup_{u \in \mathcal{F}} W_u \right) \end{aligned}$$

is dense in X.

In the following theorem, it is shown that for H-continuous functions interval values are used in an 'economical' way, namely only at points of discontinuity.

Theorem 2.3.6 [5] *Let $u = [\underline{u}, \overline{u}]$ be an H-continuous function on X.*
(a) If $\underline{u}$ or $\overline{u}$ is continuous at a point $a \in X$, then $\underline{u}(a) = \overline{u}(a)$.
(b) If $\underline{u}(a) = \overline{u}(a)$ for some $a \in X$, then both $\underline{u}$ and $\overline{u}$ are continuous at a.

The above theorem implies that for every $u \in \mathbb{A}(X)$ the set W_u defined by (2.3.7) has the following representations:

$$\begin{aligned} W_u &= \{x \in X : I(u) \text{ is discontinuous at } x\} \\ &= \{x \in X : S(u) \text{ is discontinuous at } x\}. \end{aligned}$$

In important similarity between continuous, and on other hand, H-continuous functions is that both of them are determined uniquely if they are known a dense subset of their domains. This property comes in spite of the fact that, as seen above, H-continuous functions can have discontinuities on sets of first Baire category, and such sets can have arbitrary large positive Lebesgue measure, see [112]. Indeed we have the following Theorem [5].

Theorem 2.3.7 *Let $u,\ v \in \mathbb{H}(X)$ and let D be a dense subset of X. Then*

$$(a)\ u(x) \leq v(x),\ x \in D \Rightarrow u(x) \leq v(x),\ x \in X, \tag{2.3.8}$$

and

$$(b)\ u(x) = v(x),\ x \in D \Rightarrow u(x) = v(x),\ x \in X.$$

2.4 The Set $\mathbb{H}(X)$ is Dedekind Order Complete

One of the most surprising and useful properties of the set $\mathbb{H}(X)$ of all H-continuous functions is its Dedekind order completeness with respect to the partial order given in (2.2.3). What makes this property so significant is the fact that, with very few exceptions, the usual spaces in Real Analysis or Functional Analysis, e.g., space of continuous functions, is not Dedekind order complete. In this way, the class of Hausdorff continuous functions can provide solutions to open problems or improve earlier results related to order. This section discusses Dedekind order completeness of $\mathbb{H}(X)$. The representation of supremum (resp. infimum) in $\mathbb{H}(X)$ trough the point-wise supremum (resp. infimum) is given.

Let us recall the concept of Dedekind order completeness [109].

Definition 2.4.1 *A partial ordered set $(X, \leq)$ is called Dedekind order complete, if and only if every nonempty subset A of X which is bounded from above has a supremum in X and every nonempty subset B of X which is bounded from below has an infimum in X.*

A general result on Dedekind order completion of partially ordered sets was established by MacNeille in 1937, see [98]. The problem of order completion of $C(X)$ is particulary addressed in [97].

The following theorem states one of the most amazing properties of the set $\mathbb{H}(X)$, namely its Dedekind order completeness.

Theorem 2.4.1 [5] *(a) For every nonempty subset $\mathcal{F}$ of $\mathbb{H}(X)$ which is bounded from above there exist $u \in \mathbb{H}(X)$ such that*

$$u = \sup \mathcal{F}. \tag{2.4.1}$$

(b) For every nonempty subset $\mathcal{Z}$ of $\mathbb{H}(X)$ which is bounded from below there exists $v \in \mathbb{H}(X)$ such that

$$v = \inf \mathcal{Z}. \tag{2.4.2}$$

The supremum in (2.4.1) and infimum in (2.4.2) are not taken point-wise but according the partial order (2.2.3). This result indicates that indeed the partial order (2.2.3) induced point-wise by (2.2.1) is an appropriate partial order to be associated with the Hausdorff continuous interval valued function.

Remark 2.4.1 *The concept of viscosity solution is defined through order. Hence the order properties of* $\mathbb{H}(X)$*, in particular its Dedekind completeness, play an important role. Topological structures have been defined on* $\mathbb{H}(X)$ *in different ways. For example, in [6] there is a convergence structure defined through the partial order (2.2.3) on the set* $\mathbb{H}(X)$ *which is generally not a topology. The supremum norm was considered in [11] on the set of bounded H-continuous functions.*

The following theorem gives useful representation of supremum (resp. infimum) of a subset of $\mathbb{H}(X)$ in terms of the point-wise supremum (resp. infimum).

Theorem 2.4.2 [5] *(a) Let the set* $\mathcal{F} \subseteq \mathbb{H}(X)$ *be bounded from above and let the function* $\psi \in \mathcal{A}(X)$ *be defined by*

$$\psi(x) := \sup\{\overline{f}(x) : f = [\underline{f}, \overline{f}] \in \mathcal{F}\}, \ x \in X.$$

Then

$$\sup \mathcal{F} = F(S(\psi)) = [I(S(\psi)), S(\psi)].$$

(b) Let the set $\mathcal{Z} \subseteq \mathbb{H}(X)$ *be bounded from below and let the function* $\varphi \in \mathcal{A}(X)$ *be defined by*

$$\varphi(x) := \inf\{\underline{f}(x) : f = [\underline{f}, \overline{f}] \in \mathcal{Z}\}, \ x \in X.$$

Then

$$\inf \mathcal{Z} = F(I(\varphi)) = [I(\varphi), S(I(\varphi))].$$

Theorem 2.4.1 establishes a close connection between the supremum (resp. infimum) in $\mathbb{H}(X)$ about the partial order (2.2.3) and the point-wise supremum (resp. infimum). These two functions are not the same, that is, for a set $\mathcal{F} \subseteq \mathbb{H}(X)$, in general, there exists $x \in X$ such that

$$\begin{aligned} (\inf \mathcal{F})(x) &= (\inf_{f \in \mathcal{F}} f)(x) \neq \inf_{f \in \mathcal{F}} (f(x)), \\ (\sup \mathcal{F})(x) &= (\sup_{f \in \mathcal{F}} f)(x) \neq \sup_{f \in \mathcal{F}} (f(x)). \end{aligned}$$

Example 2.4.1 *Let* $X = \mathbb{R}^n$*. Consider the set* $\mathcal{F} = \{f_\delta : \delta > 0\}$*, where*

$$f_\delta(x) = \begin{cases} 1 - \frac{|x|}{\delta}, & |x| \leq \delta \\ 0 \qquad , & \text{otherwise.} \end{cases}$$

The point-wise infimum of $\mathcal{F}$ *is*

$$\varphi(x) = \begin{cases} 1 \text{ if } x = 0 \\ 0 \text{ if } x \neq 0. \end{cases}$$

The function φ *is not an H-continuous function on* $\mathbb{R}^n$. *Indeed, consider the continuous function* $v(x) = 0, x \in \mathbb{R}^n$. *Obviously, we have* $\varphi(x) \supseteq v(x), x \in \mathbb{R}^n$ *and* $v(0) \neq \varphi(0)$ *so* φ *cannot be H-continuous function in* $\mathbb{R}^n$. *The infimum of* $\mathcal{F}$ *in* $\mathbb{H}(\mathbb{R}^n)$ *is* $u(x) = 0$, $x \in \mathbb{R}^n$.

Example 2.4.2 *Let* $X = \mathbb{R}$. *Consider the set* $\mathcal{F} = \{g_n : n \in \mathbb{N}\}$, *where*

$$g_n(x) = \begin{cases} x^{-2n-1} & , \quad x \neq 0 \\ [-\infty, +\infty] & , \quad x = 0. \end{cases}$$

The point-wise supremum of $\mathcal{F}$ *is*

$$\psi(x) = \begin{cases} 0 & , \quad x < -1 \\ x^{-1} & , \quad -1 \leq x < 0 \\ \infty & , \quad 0 \leq x < 1 \\ x^{-1} & , \quad x \geq 0. \end{cases}$$

Hence

$$(\sup \mathcal{F})(x) = \begin{cases} 0 & , \quad x < -1 \\ [-1, 0] & , \quad x = -1 \\ x^{-1} & , \quad -1 < x < 0 \\ [-\infty, +\infty] & , \quad x = 0 \\ \infty & , \quad 0 < x < 1 \\ [1, \infty] & , \quad x = 1 \\ x^{-1} & , \quad x > 1. \end{cases}$$

We will use the following theorem given in [4].

Theorem 2.4.3 *Let* D *be dense subset of* X, *we have the following:*

$$u \in \mathbb{H}(D) \Rightarrow F(u) \in \mathbb{H}(X).$$

Since X is always dense in $\overline{X}$, we have the following corollary.

Corollary 2.4.1 *Let* $u \in \mathbb{A}(X)$. *Then*

$$u \in \mathbb{H}(X) \Rightarrow F(u) \in \mathbb{H}(\overline{X}).$$

Further properties of the H-continuous functions are discussed in [120], [10], [4], where it is shown, among others, that they retain some of the essential characteristics of the usual continuous functions.

2.5 Generalized Baire Operators and Graph Completion Operator

Let D be a dense subset of X. For any $u \in \mathbb{A}(D)$ we can consider the following generalization of the upper and lower Baire operators as well as the graph completion operator, given in Section 2.2.

Definition 2.5.1 *The mapping $S(D,X,.), I(D,X,.) : \mathbb{A}(D) \to \mathcal{A}(X)$ defined for $\mathbb{A}(D)$ by*

$$S(D,X,u)(x) = \inf_{\delta>0} \sup\{z \in u(y) : y \in B_\delta(x) \cap D\},\ x \in X,$$
$$I(D,X,u)(x) = \sup_{\delta>0} \inf\{z \in u(y) : y \in B_\delta(x) \cap D\},\ x \in X,$$

are called generalized upper Baire and generalized lower Baire operators, respectively.

Clearly, for every $u \in \mathbb{A}(D)$, we have

$$I(D,X,u)(x) \leq S(D,X,u)(x),\ x \in X \tag{2.5.1}$$

and

$$I(D,X,u)(x) \leq u(x) \leq S(D,X,u)(x),\ x \in D.$$

Definition 2.5.2 *The mapping $F(D,X,.) : \mathbb{A}(D) \to \mathbb{A}(X)$, defined for $u \in \mathbb{A}(D)$ by*

$$F(D,X,u)(x) = [I(D,X,u)(x), S(D,X,u)(x)],\ x \in D,$$

is called generalized graph completion operator.

Note that the usual Baire operators and graph completion operator obtained from the above definitions using $D = X$, i.e., if $u \in \mathbb{A}(X)$ then

$$I(u) = I(X,X,u),\ S(u) = S(X,X,u),\ F(u) = F(X,X,u).$$

The generalized lower and upper Baire operators as well as the graph completion operator of an interval valued function $u = [\underline{u}, \overline{u}] \in \mathbb{A}(D)$ can be conveniently represented in terms of the functions $\underline{u}$ and $\overline{u}$:

$$I(D,X,u) = I(D,X,\underline{u}),\ S(D,X,u) = S(D,X,\overline{u}),\ F(D,X,u) = [I(D,X,\underline{u}), S(D,X,\overline{u})].$$

In the sequel, we will use the following properties of operators $I(D,X,.)$, S(D,X,.), and F(D,X,.), which were proved in [4].

Theorem 2.5.1 *(i) Let D be a dense subset of X. Then*

$$I(D,X,u) \in LSC(X) \text{ and } S(D,X,u) \in USC(X). \tag{2.5.2}$$

(ii) If $u, v \in \mathbb{A}(D)$, where D is dense in X, then

$$u(x) \leq v(x), x \in D \Rightarrow I(D,X,u)(x) \leq I(D,X,u)(x), x \in X; \tag{2.5.3}$$

$$u(x) \leq v(x), x \in D \Rightarrow S(D,X,u)(x) \leq S(D,X,u)(x), x \in X; \tag{2.5.4}$$

$$u(x) \leq v(x), x \in D \Rightarrow F(D,X,u)(x) \leq F(D,X,u)(x), x \in X. \tag{2.5.5}$$

(iii) The generalized graph completion operator is monotone about inclusion with respect to the functional argument, that is, if $u, v \in \mathbb{A}(D)$, where D is dense in X then

$$u(x) \subseteq v(x), x \in X \Rightarrow F(D,X,u)(x) \subseteq F(D,X,v)(x), x \in X.$$

(iv) If D_1 and D_2 are dense subsets of X and $u \in \mathbb{A}(D_1 \cup D_2)$, then

$$D_1 \subseteq D_2 \Rightarrow I(D_2,X,u)(x) \leq I(D_1,X,u)(x), x \in X, \tag{2.5.6}$$

$$D_1 \subseteq D_2 \Rightarrow S(D_1,X,u)(x) \leq S(D_2,X,u)(x), x \in X. \tag{2.5.7}$$

The generalized lower and upper Baire operators have the following optimality property.

Theorem 2.5.2 [4] *Let D be a dense subset of X. For any function $u \in \mathbb{A}(D)$ we have*

$$(i) \quad v \in LSC(X), v(x) \leq u(x), x \in D \Rightarrow v(x) \leq I(D,X,u)(x), x \in X, \tag{2.5.8}$$

$$(ii) \quad v \in USC(X), u(x) \leq v(x), x \in D \Rightarrow S(D,X,u)(x) \leq v(x), x \in X. \tag{2.5.9}$$

The above theorem explains why $I(D,X,.), S(D,X,.)$ are called, respectively, lower and upper semicontinuous envelope of the function u.

We will use the following Theorem [4]:

Theorem 2.5.4 *Let D be a dense subset of X. If $u \in C(D)$, then*

$$F(D,X,u) \in \mathbb{H}(X)$$

and

$$F(D,X,u)(x) = u(x), x \in D.$$

Chapter 3

Hausdorff Continuous Viscosity Solutions of Hamilton-Jacobi Equations

3.1 Introduction

Consider again the problem (1.1.1)-(1.1.2). As shown in the introduction the concept of viscosity solution, see Definition 1.3.3, has an implicit interval character. Clearly Definition 1.3.3 treats functions which have the same upper and lower semicontinuous envelopes, that is, have the same graph completion, as identical functions. On the other hand, since different functions can have the same graph completion, a function can not in general be identified from its graph completion, that is, functions with the same graph completion are indistinguishable. Using the properties of the lower and upper semicontinuous envelopes one can easily see that the graph completion operator maps $\mathcal{A}(\Omega)$ into $\mathbb{F}(\Omega)$.

Following the above discussion we define the concept of viscosity solutions for the interval valued functions in $\mathbb{F}(\Omega)$ as follows.

Definition 3.1.1 *A function $u = [\underline{u}, \overline{u}] \in \mathbb{F}(\Omega)$ is called a viscosity solution of (1.1.1) if $\overline{u}$ is a viscosity subsolution of (1.1.1) and $\underline{u}$ is a viscosity supersolution of (1.1.1).*

Definition 3.1.1 shows that a local bounded function $u \in \mathcal{A}(\Omega)$ is a viscosity solution of (1.1.1) in the sense of Definition 1.3.3 if and only if the interval valued function $F(u)$ is a viscosity solution of (1.1.1) in the sense of Definition 3.1.1. In this way the level of the regularity of a solution u is manifested through the width of the interval valued function $F(u)$.

It is well known that without any additional restrictions the concept of viscosity solution given in Definition 1.3.3 and by implication the concept of viscosity solution given in Definition 3.1.1 is rather weak, [16]. This is demonstrated by the following example.

Example 3.1.1 *Consider the following equation*

$$u_x(x) = 1, x \in (0,1). \tag{3.1.1}$$

Then the functions

$$v(x) = \begin{cases} x+1 & if \quad x \in (0,1) \cap \mathbb{Q} \\ x & if \quad x \in (0,1) \setminus \mathbb{Q} \end{cases}$$

and

$$w(x) = \begin{cases} x & if \quad x \in (0,1) \cap \mathbb{Q} \\ x+1 & if \quad x \in (0,1) \setminus \mathbb{Q} \end{cases}$$

are both viscosity solutions of equation (3.1.1) in terms of Definition 1.3.3, because $S(v)(x) = S(w)(x) = x+1$ *and* $I(v)(x) = I(w)(x) = x$ *are classical solutions of equation (3.1.1).*
The interval valued function

$$z = F(v) = F(w) \tag{3.1.2}$$

given by

$$z(x) = [x, x+1], x \in (0,1)$$

is a viscosity solution of (3.1.1) in terms of Definition 3.1.1.

With the interval approach adopted here it becomes apparent that the distance between $I(u)$ and $S(u)$ is an essential measure of the regularity of any solution u, irrespective of whether it is given as a point valued function or as an interval valued function. If no restriction is placed on the distance between $I(u)$ and $S(u)$ we will have some quite meaningless solutions like the solutions in Example 3.1.1. On the other hand, a strong restriction like $I(u) = S(u)$ gives only solutions which are continuous.

3.2 Hausdorff Continuous Viscosity Solution of Hamilton-Jacobi Equations

In this section, we consider solutions u in the sense of Definition 3.1.1 for which the Hausdorff distance, as defined by (2.3.6), between the functions $I(u)$ and $S(u)$ is zero, a condition represented through the concept of Hausdorff continuity.

Definition 3.2.1 *Let $u = [\underline{u}, \overline{u}] \in \mathbb{H}(\Omega)$. Then u is called a Hausdorff continuous, or H-continuous, viscosity subsolution of the Hamilton-Jacobi equation (1.1.1) if $\overline{u}$ is a viscosity subsolution of (1.1.1).*
Similarly u is called a Hausdorff continuous, or H-continuous, viscosity supersolution of the Hamilton-Jacobi equation (1.1.1) if $\underline{u}$ is a viscosity supersolution of (1.1.1).
Finally, u is called a Hausdorff continuous or H-continuous viscosity solution of (1.1.1) if it is simultaneously an H-continuous viscosity subsolution and H-continuous viscosity supersolution of (1.1.1).

One of the advantages of the method in this thesis is that the notion of H-continuous viscosity solution is stronger than the notion of viscosity solution in the sense of Definition 3.1.1 and by implication the notion of viscosity solution in the sense of Definition 1.3.3. This is shown by the following theorem.

Theorem 3.2.1 *Let $u \in \mathcal{A}(\Omega)$ be locally bounded. If u is an H-continuous viscosity solution of (1.1.1), then u is a viscosity solution of (1.1.1) in terms of Definition 1.3.3.*

Proof. *Since u is H-continuous viscosity solution of (1.1.1), by Definition 3.2.1, $\overline{u} = S(u)$ is a viscosity subsolution and $\underline{u} = I(u)$ is a viscosity supersolution of (1.1.1). Thus u is a viscosity solution of (1.1.1) in terms of Definition 1.3.3.* ■

Remark 3.2.1 *The converse of Theorem 3.2.1 is false in general. Indeed, consider the Example 3.1.1. The function z given in (3.1.2) is a viscosity solution of (3.1.1) in terms of Definition 3.1.1 because $S(v)(x) = x+1$ and $I(v)(x) = x$ are both classical solutions of equation (3.1.1). But z is not an H-continuous function and thus it is not an H-continuous viscosity solution of the equation (3.1.1). Hence, the requirement that a viscosity solution is Hausdorff continuous function has a direct interpretation which we find clearer than the requirements related to some other concepts of discontinuous viscosity solutions.*

3.3 The Envelope Viscosity Solutions and Hausdorff Continuous Viscosity Solutions

Recognizing that the concept of viscosity solution given in Definition 1.3.3 is rather weak, the authors of [16] introduce the concept of envelope viscosity solution. The concept is defined in [16] for the equation (1.1.1) with Dirichlet boundary conditions. In order to keep the exposition as general as possible we give the definition without explicitly involving the boundary condition.

Definition 3.3.2 *A function $u \in \mathcal{A}(\Omega)$ is called envelope viscosity solution of the Hamilton-Jacobi equation (1.1.1) if there exists a nonempty set $\mathcal{Z}_1(u)$ of viscosity subsolutions of (1.1.1) and a nonempty set $\mathcal{Z}_2(u)$ of viscosity supersolutions of (1.1.1) such that*

$$u(x) = \sup_{f \in \mathcal{Z}_1(u)} f(x) = \inf_{f \in \mathcal{Z}_2(u)} f(x), \ x \in \Omega.$$

Remark 3.3.1 *Let $u \in \mathcal{A}(\Omega)$ be an envelope viscosity solution of (1.1.1). Then u is also a viscosity solution of (1.1.1). Indeed, observe that $S(u)$ is a viscosity subsolution of (1.1.1) by Theorem 1.3.1(a) and $I(u)$ is a viscosity supersolution of (1.1.1) by Theorem 1.3.1(b). Therefore, u is a viscosity solution of (1.1.1).*

Considering the concept from geometrical point of view, one can expect that by 'squeezing' the envelope viscosity solution u between a set of viscosity subsolutions and a set of viscosity supersolutions the gap between $I(u)$ and $S(u)$ would be small. But under some strong condition on u, namely,

$$I(S(u)) = I(u), S(I(u)) = S(u)$$

the Hausdorff distance between $I(u)$ and $S(u)$ is zero. However, in general this is not the case. The following example shows that the concept of envelope viscosity solution does not address the problem of the distance between $I(u)$ and $S(u)$. Hence one can have an envelope viscosity solution of little practical meaning similar to the viscosity solution in Example 3.1.1.

Example 3.3.1 *Consider the following equation on* $\Omega = (0,1)$

$$-u(x)(u_x(x))^2 = 0, x \in \Omega. \tag{3.3.1}$$

For every $\alpha \in \Omega$ *we define the functions*

$$\phi_\alpha(x) = \begin{cases} 1 & , \quad x = \alpha \\ 0 & , \quad x \in \Omega \setminus \{\alpha\} \end{cases}$$

and

$$\psi_\alpha(x) = \begin{cases} 0 & , \quad x = \alpha \\ 1 & , \quad x \in \Omega \setminus \{\alpha\}. \end{cases}$$

We have

$$\phi_\alpha \in USC(\Omega),\ \psi_\alpha \in LSC(\Omega), \alpha \in \Omega.$$

Furthermore, for every $\alpha \in (0,1)$ *the functions* ϕ_α *is a viscosity subsolution of (3.3.1) while* ψ_α *is a viscosity supersolution of (3.3.1). Indeed, both functions satisfy the equation (3.3.1) for all* $x \in \Omega \setminus \{\alpha\}$ *and at* $x = \alpha$ *we have*

$$-\phi_\alpha(\alpha)p^2 = -p^2 \leq 0 \text{ for all } p \in D^+\phi_\alpha(\alpha) = (-\infty, +\infty),$$
$$-\psi_\alpha(\alpha)p^2 = 0 \geq 0 \text{ for all } p \in D^-\psi_\alpha(\alpha) = (-\infty, +\infty).$$

We will show that the function

$$u(x) = \begin{cases} 1 & , \quad x \in \Omega \setminus \mathbb{Q} \\ 0 & , \quad x \in \mathbb{Q} \cap \Omega \end{cases}$$

is an envelope viscosity solution of (3.3.1). Define

$$\begin{aligned} \mathcal{Z}_1 &= \{\phi_\alpha : \alpha \in \Omega \setminus \mathbb{Q}\} \\ \mathcal{Z}_2 &= \{\psi_\alpha : \alpha \in \Omega \cap \mathbb{Q}\}. \end{aligned}$$

Then u *satisfies*

$$u(x) = \sup_{w \in \mathcal{Z}_1} w(x) = \inf_{w \in \mathcal{Z}_2} w(x)$$

which implies that it is an envelope viscosity solution of (3.3.1). Clearly neither u *nor* $F(u)$ *is a Hausdorff continuous function. In fact we have* $F(u)(x) = [0,1], x \in \Omega$. *Thus,* u *and* $F(u)$ *are not H-continuous viscosity solutions of (3.3.1).*

The next interesting question is whether every H-continuous viscosity solution is an envelope viscosity solution. Since the concept of envelope viscosity solutions requires the existence of sets of viscosity subsolutions and viscosity supersolutions, respectively, below and above an envelope viscosity solution then an H-continuous viscosity solution is not in general an envelope viscosity solution, e.g., when the Hausdorff continuous viscosity solutions does not have any other viscosity subsolutions and viscosity supersolutions around it. However in the essential case when the H-continuous viscosity solution is a supremum of viscosity subsolutions or infimum of viscosity supersolutions it can be linked to an envelope viscosity solution as stated in the next theorem.

Theorem 3.3.1 *Let $u = [\overline{u}, \underline{u}]$ be an H-continuous viscosity solution of (1.1.1) and let*

$$\begin{aligned} \mathcal{Z}_1 &= \{w \in USC(\Omega) : w - \textit{viscosity subsolution of (1.1.1)}, w \leq \underline{u}\}, \\ \mathcal{Z}_2 &= \{w \in LSC(\Omega) : w - \textit{viscosity supersolution of (1.1.1)}, w \geq \overline{u}\}. \end{aligned}$$

(a) If $\mathcal{Z}_1 \neq \emptyset$ and $\underline{u}(x) = \sup_{w \in \mathcal{Z}_1} w(x)$, then $\underline{u}$ is an envelope viscosity solution of (1.1.1).
(b) If $\mathcal{Z}_2 \neq \emptyset$ and $\overline{u}(x) = \inf_{w \in \mathcal{Z}_2} w(x)$, then $\overline{u}$ is an envelope viscosity solution of (1.1.1).

Proof. *(a) We choose the sets $\mathcal{Z}_1(\underline{u})$ and $\mathcal{Z}_2(\underline{u})$ required in Definition 3.2.1 as follows*

$$\mathcal{Z}_1(\underline{u}) = \mathcal{Z}_1, \quad \mathcal{Z}_2(\underline{u}) = \{\underline{u}\}.$$

Then we have

$$\underline{u}(x) = \sup_{w \in \mathcal{Z}_1(\underline{u})} w(x) = \inf_{w \in \mathcal{Z}_2(\underline{u})} w(x)$$

which implies that $\underline{u}$ is an envelope viscosity solution of (1.1.1).
The proof of (b) is done in a similar way. ■

Let us note that if the conditions (a) and (b) in the above theorem are satisfied the both $\underline{u}$ and $\overline{u}$ are envelope viscosity solutions and in this case it makes even more sense to consider instead the H-continuous function $u = [\underline{u}, \overline{u}]$. More precisely, if conditions (a) and (b) are satisfied, an envelope viscosity solution can be considered as a particular case of Hausdorff continuous viscosity solution.

3.4 Existence of Hausdorff Continuous Viscosity Solutions

One of the primary virtues of the theory of viscosity solutions is that it provides very general existence and uniqueness theorems, [37]. In this section we will formulate and prove existence theorems for H-continuous viscosity solutions in a similar form to Theorem 1.3.1 and Theorem 1.3.2 given in Chapter 1.

Theorem 3.4.1 (Properties of H-continuous viscosity solutions)
(a) Let $\mathcal{U} \subset \mathbb{H}(\Omega)$ be a set of H-continuous viscosity subsolutions of the Hamilton-Jacobi equation (1.1.1) which is bounded from above. Then

$$u = \sup \mathcal{U}$$

is an H-continuous viscosity subsolution of (1.1.1).
(b) Let $\mathcal{Z} \subset \mathbb{H}(\Omega)$ be a set of H-continuous viscosity supersolutions of the Hamilton-Jacobi equation (1.1.1) which is bounded from below. Then

$$v = \inf \mathcal{Z}$$

is an H-continuous viscosity supersolution of (1.1.1).
(Both the supremum and the infimum are in the sense of the partial order (2.2.3) on $\mathbb{H}(\Omega)$).

Proof. *We will prove (a). The proof of (b) can be done in a similar way. Since u is bounded from above, according to Theorem 2.4.1 (a), $u = \sup \mathcal{U} \in \mathbb{H}(\Omega)$ and by Theorem 2.4.2 (a), we have*

$$u = [I(S(\psi), S(\psi)],$$

where

$$\psi(x) := \sup\{\overline{w}(x) : w = [\underline{w}, \overline{w}] \in \mathcal{U}\},\ x \in \Omega.$$

Using that $S(\underline{w}) = \overline{w}$ is a viscosity subsolution of (1.1.1) for all $w = [\underline{w}, \overline{w}] \in \mathcal{U}$, it follows from Theorem 1.3.1 (a), that $\overline{u} = S(\psi)$ is a viscosity subsolution of (1.1.1).
By Definition 3.2.1, the function u is Hausdorff continuous viscosity subsolution of (1.1.1).
■

Since the partially ordered set $\mathbb{H}(\Omega)$ is Dedekind order complete, it is an appropriate medium for such an application of Perron's method.

The technical lemma, sometimes called the Bump Lemma [16], showing that in some cases the supremum of viscosity subsolution or the infimum of viscosity supersolution are indeed viscosity solutions, can be formulated for Hausdorff continuous functions as follows.

Lemma 3.4.1 *Let $u = [\underline{u}, \overline{u}] \in \mathbb{H}(\Omega)$ be such that $\overline{u}$ is a viscosity subsolution of (1.1.1) and $\underline{u}$ fails to be a viscosity supersolution of (1.1.1) at some point $y \in \Omega$. Then, for any $\delta > 0$ there exists $\tau > 0$ such that, for all $r < \tau$, there exists a function $w = [\underline{w}, \overline{w}] \in \mathbb{H}(\Omega)$ with the following properties:*

$$
\begin{aligned}
&(i) && \overline{w} \text{ is a viscosity subsolution of (1.1.1),} \\
&(ii) && w \geq u \text{ in } \Omega, \\
&(iii) && w \neq u, \\
&(iv) && w(x) = u(x), \qquad\qquad x \in \Omega \backslash B_r(y), \\
&(v) && \underline{w}(x) \leq \max\{\underline{u}(x), \underline{u}(y) + \delta\}, \ x \in B_r(y).
\end{aligned}
$$

Proof. *Since $\underline{u}$ fails to be a viscosity supersolution of (1.1.1) at $y \in \Omega$, there exists $\varphi \in C^1(\Omega)$ such that*

$$h := H(y, \underline{u}(y), D\varphi(y)) < 0, \ \underline{u}(y) = \varphi(y), \ \varphi(x) \leq \underline{u}(x), \ x \in B_r(y) \text{ and some } r > 0.$$

For $\varepsilon > 0$, consider the function $v \in C^1(\Omega)$ defined by

$$v(x) := \varphi(x) + \varepsilon - |x - y|^2.$$

We can choose r small enough to have, in addition,

$$v(x) \leq \underline{u}(y) + \frac{\delta}{2} + \varepsilon, \ x \in B_r(y). \tag{3.4.1}$$

Note that $(v - \underline{u})(x) \leq (v - \varphi)(x) = -|x - y|^2 + \varepsilon \leq 0, |x - y| \geq \sqrt{\varepsilon}$, and thus

$$v(x) \leq \underline{u}(x), \text{ for } |x - y| \geq \frac{r}{2} \tag{3.4.2}$$

if we choose $\varepsilon < \frac{r^2}{4}$. Moreover, if $x_n \to y$ is such that $\underline{u}(x_n) \to \underline{u}(y)$, we have $\lim_{n\to\infty}(v - \underline{u})(x_n) = \varepsilon > 0$, so, for all $r > 0$,

$$\sup_{B_r(y)} (v - \underline{u}) > 0. \tag{3.4.3}$$

Let us prove that v is a classical viscosity subsolution of equation $H(x,v,Dv)=0$ in $B_r(y)$, i.e., $H(x,v(x),Dv(x))\leq 0$, $x\in B_r(y)$, for sufficiently small ε, $r>0$. For this purpose, a local uniform continuity argument shows that

$$\begin{aligned} |v(x)-v(y)| &= |\varphi(x)-|x-y|^2-\varphi(y)|\leq \omega_1(r)+r^2, \\ |Dv(x)-Dv(y)| &= |D\varphi(x)-2|x-y|-D\varphi(y)|\leq \omega_2(r)+2r \end{aligned}$$

for any $x\in B_r(y)$, where $\omega_i(i=1,2)$, are the moduli of continuity of φ and $D\varphi$. We recall that if $\varphi\in C(\Omega)$, then the function $\omega:[0,+\infty)\to[0,+\infty)$, defined by, $\omega(\delta)=\sup\{|\varphi(s)-\varphi(t)|, s,t\in\Omega, |s-t|\leq\delta\}$ for $\delta\geq 0$, is called a modulus of continuity of φ. Now, $H(x,v(x),Dv(x))=h+H\left(x,v(x),D\varphi(x)-2(x-y)\right)-H(x,v(x),D\varphi(x))$. If ω is a modulus of continuity for H, then $H(x,v(x),Dv(x))\leq h+\omega(r,\omega_1(r)+r^2,\omega_2(r)+2r)$, for all $x\in\overline{B}_r(y)$. Since $h<0$, the proceeding proves that $H(x,v(x),Dv(x))\leq 0$, $x\in B_r(y)$.

Now we define the interval valued function

$$w(x)=\begin{cases} \max(u(x),v(x)), & x\in B_r(y) \\ u(x) & , x\in\Omega\backslash B_r(y). \end{cases}$$

It is clear that $w\in\mathbb{H}(\Omega)$ since u and v are Hausdorff continuous functions in Ω and we can apply Theorem 2.4.1.

We claim that w has the desired properties. In fact, $w(x)=u(x)$ for $|x-y|\geq r/2$ by (3.4.2) and $w(x)\geq u(x)$, $x\in\Omega$. Then (iv) holds and w is an H-continuous viscosity subsolution of (1.1.1), because it coincides with u for $|x-y|>\frac{r}{2}$, while for $x\in B_r(y)$ we can apply Theorem 3.4.1. Moreover, (iii) follows from (3.4.3), and (v) follows from (3.4.1) if we choose $\varepsilon\leq\frac{\delta}{2}$. ■

Note that the proof of lemma 3.4.1 is similar to the proof of the Bump lemma in [16, Lemma V.2.12] for real function with some obvious changes due to interval character of the functions u and w.

Remark 3.4.1 *There is an analogue of Lemma 3.4.1 for the case when $I(u)$ is a viscosity supersolution and $S(u)$ fails to be a viscosity subsolution of (1.1.1).*

For a consequence of Theorem 3.4.1 and Lemma 3.4.1, we obtain the following very general existence theorem for equation (1.1.1).

Theorem 3.4.2 (Existence of H-continuous viscosity solutions by Perron's method) *Assume that there exist Hausdorff continuous functions $u_1 = [\underline{u_1}, \overline{u_1}]$ and $u_2 = [\underline{u_2}, \overline{u_2}]$ such that u_1 is a Hausdorff continuous viscosity subsolution of the Hamilton-Jacobi equation (1.1.1), u_2 is a Hausdorff continuous viscosity supersolution of (1.1.1) and $u_1 \leq u_2$. Then there exists a Hausdorff continuous viscosity solution u of (1.1.1) satisfying the inequalities*

$$u_1 \leq u \leq u_2.$$

Proof. *Consider the set*

$$\mathcal{F} = \{w = [\underline{w}, \overline{w}] \in \mathbb{H}(\Omega) : w \leq u_2,\ \overline{w} \text{ is a viscosity subsolution of } (1.1.1)\}.$$

Clearly the set $\mathcal{F}$ is not empty since $u_1 \in \mathcal{F}$. Let $u = \sup \mathcal{F}$, where the supremum is taken in the set $\mathbb{H}(\Omega)$, i.e., $u \in \mathbb{H}(\Omega)$. We will show that u is the required Hausdorff continuous viscosity solution of (1.1.1). Obviously, we have the inequalities

$$u_1 \leq u \leq u_2. \tag{3.4.4}$$

Furthermore, according to Theorem 2.4.1 (a) and Theorem 2.4.2 (a), u is given by

$$u = \sup \mathcal{F} = [I(S(\psi)), S(\psi)] \in \mathbb{H}(\Omega),$$

where

$$\psi(x) := \sup\{\overline{w}(x) : w = [\underline{w}, \overline{w}] \in \mathcal{F}\},\ x \in \Omega.$$

Using that $\mathcal{F}$ is the set of H-continuous viscosity subsolutions of (1.1.1) and $\mathcal{F}$ is bounded from above it follows by Theorem 3.4.1(a) that u is an H-continuous viscosity subsolution of (1.1.1). It remains to show that u is H-continuous viscosity supersolution of (1.1.1), i.e., $\underline{u} = I(S(\psi)))$ is a viscosity supersolution of (1.1.1). To this end, let us fix $y \in \Omega$. Consider first the case when

$$\underline{u}(y) = \underline{u_2}(y).$$

Let $\varphi \in C^1(\Omega)$ be such that $\underline{u} - \varphi$ has a local minimum at y and $\underline{u}(y) = \varphi(y)$. Then, for x in a neighborhood of y, we have

$$(\underline{u_2} - \varphi)(x) \geq (\underline{u} - \varphi)(x) \geq (\underline{u} - \varphi)(y) = (\underline{u_2} - \varphi)(y).$$

Therefore, the function $\underline{u_2} - \varphi$ also has a local minimum at y.

Using that $\underline{u_2}$ is a viscosity supersolution of (1.1.1), we obtain

$$H(y, \underline{u_2}(y), D\varphi(y)) \geq 0.$$

Since $\underline{u}(y) = \underline{u_2}(y)$, the above inequality shows that the function $\underline{u}$ satisfies at the point y the conditions of supersolutions as stated in Definition 1.3.1. Consider now the case when $\underline{u}(y) \neq \underline{u_2}(y)$. In view of (3.4.4), the only other possible case is

$$\underline{u}(y) < \underline{u_2}(y).$$

In this situation, there exists $\delta > 0$ such that

$$\underline{u}(y) + \delta \leq \underline{u_2}(y) - \delta. \tag{3.4.5}$$

Assume that $\underline{u}$ fails to be a supersolution of (1.1.1) at the point y. Then, according to Lemma 3.4.1, there exists an H-continuous function $w = [\underline{w}, \overline{w}]$ with the properties (i)-(v). Moreover, since $\underline{u_2}$ is lower semicontinuous, we can choose $r > 0$ small enough such that

$$\underline{u_2}(y) - \delta \leq \underline{u_2}(x),\ x \in B_r(y). \tag{3.4.6}$$

Using (3.4.5) and (3.4.6), we obtain

$$\underline{u}(y) + \delta \leq \underline{u_2}(y) - \delta \leq \underline{u_2}(x),\ x \in B_r(y).$$

Hence, from property (v) of Lemma 3.4.1, for $x \in B_r(y)$, we have

$$\underline{w}(x) \leq \max\{\underline{u}(x), \underline{u}(y) + \delta\} \leq \underline{u_2}(x). \tag{3.4.7}$$

Due to property (iv) of Lemma 3.4.1, the inequality (3.4.7) can be extended to all $x \in \Omega$ and we have

$$\underline{w} \leq \underline{u_2}. \tag{3.4.8}$$

Using Theorem 2.3.3 (b) and the monotonicity of a graph completion function F, see (2.2.12), the inequality (3.4.8) can be transferred over to the Hausdorff continuous functions w and u_2 as follows

$$w = F(\underline{w}) \leq F(\underline{u_2}) = u_2.$$

Then

$$w \leq u_2.$$

This implies that $w \in \mathcal{F}$. Then $u = \sup \mathcal{F} \geq w$ which contradicts conditions (ii) and (iii) in Lemma 3.4.1. The obtain contradiction shows that $\underline{u}$ is a viscosity supersolution of (1.1.1). Therefore u is Hausdorff continuous viscosity solution of (1.1.1). ■

3.5 Uniqueness of H-Continuous Viscosity Solution

As in the traditional theory of viscosity solutions, uniqueness results can be proved under the assumption that a comparison principle is satisfied. Here we formulate the comparison principle between H-continuous viscosity subsolutions and H-continuous viscosity supersolutions of Hamilton-Jacobi equations.

Definition 3.5.1 *We say that the Dirichlet problem (1.1.1)-(1.1.2) satisfies the comparison principle if for any $u \in \mathbb{H}(\Omega)$ and $v \in \mathbb{H}(\Omega)$ which are bounded and, respectively, H-continuous viscosity subsolution and supersolution of (1.1.1) and $u \leq v$ on $\partial\Omega$, we have $u \leq v$ in Ω.*

The following theorem of uniqueness of solution shows that if H-continuous viscosity subsolution and H-continuous viscosity supersolution of (1.1.1) are equal on the boundary $\partial\Omega$ and g given in (1.1.2) is assumed to be H-continuous function on Ω, then there exists a unique an H-continuous solution of (1.1.1) satisfying (1.1.2).

Theorem 3.5.1 *Assume that there exist Hausdorff continuous viscosity subsolution φ and Hausdorff continuous viscosity supersolution ψ of (1.1.1) on Ω and assume that the definition of both functions is extended on $\partial\Omega$ in such a way that the obtained functions are H-continuous on $\overline{\Omega}$. Suppose that (1.1.1) satisfies the comparison principle and that*

$$\varphi(x) = \psi(x) = g(x), \ x \in \partial\Omega, \tag{3.5.1}$$

where the function g may assume interval values. Then, there exists a unique Hausdorff continuous viscosity solution u of (1.1.1) such that

$$u(x) = g(x), x \in \partial\Omega.$$

Proof. *We extend φ and ψ by setting $\varphi = \psi = g$ on $\partial\Omega$. We can apply Theorem 3.4.2 to get an H-continuous viscosity solution u of (1.1.1) such that $\varphi \leq u \leq \psi$ in Ω.*
By monotonicity of a graph completion F, see (2.2.12) , we have

$$F(\varphi) \leq F(u) \leq F(\psi) \text{ in } \Omega. \tag{3.5.2}$$

By Corollary 2.4.1, we have that all functions $F(\varphi)$, $F(u)$, $F(\psi)$ belong to $\mathbb{H}(\overline{\Omega})$. Therefore, by property (2.3.8) in Theorem 2.3.7, the inequalities in (3.5.2) imply that $F(\varphi) \leq F(u) \leq F(\psi)$ on $\overline{\Omega}$. By the virtue of Theorem 2.3.3, we have $\varphi \leq u \leq \psi$ on $\overline{\Omega}$. In particular, $\varphi \leq u \leq \psi$ on $\partial\Omega$. Since (3.5.1) holds, we have $g(x) = u(x)$, $x \in \partial\Omega$. Assume there exist two H-continuous viscosity solutions u_1, u_2 of (1.1.1).

Since u_1 is an H-continuous viscosity subsolution and u_2 is an H-continuous viscosity supersolution of (1.1.1), and $u_1(x) = u_2(x) = g(x)$, $x \in \partial\Omega$, by comparison principle, we have

$$u_1 \leq u_2 \ \text{ in } \overline{\Omega}. \tag{3.5.3}$$

Since u_2 is an H-continuous viscosity subsolution and u_1 is an H-continuous supersolution, and $u_2(x) = u_1(x) = g(x)$, $x \in \partial\Omega$, by comparison principle, we have

$$u_2 \leq u_1 \ \text{ in } \overline{\Omega}. \tag{3.5.4}$$

Combining (3.5.3) and (3.5.4), we obtain $u_1 = u_2$ in $\overline{\Omega}$. ■

The comparison principle, given by Definition 3.5.1, is stronger than the comparison principle used in connection with upper semicontinuous viscosity subsolutions and lower semicontinuous viscosity supersolutions because it gives the existence of solutions under conditions as same as for existence of discontinuous solutions.

The following theorem gives sufficient conditions for a weaker form of the comparison principle given by Definition 3.5.1.

Theorem 3.5.2 *Let Ω be a bounded open subset of $\mathbb{R}^n$, $H \in C(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ be such that $H(x, r, p) \leq H(x, s, p)$ whenever $r \leq s$ and the following two assumptions hold:*

$$\exists\, \gamma > 0 : \gamma(r - s) \leq H(x, r, p) - H(x, s, p), \forall r \geq s, (x, p) \in \overline{\Omega} \times \mathbb{R}^n \tag{3.5.5}$$

and there exists $\omega : [0, +\infty] \to [0, +\infty]$ such that $\omega(0+) = 0$ and

$$H(y, r, \alpha(x - y)) - H(x, r, \alpha(x - y)) \leq \omega(\alpha|x - y|^2 + |x - y|), \tag{3.5.6}$$

whenever $x, y \in \Omega, r \in \mathbb{R}, \alpha > 0$.

Let $u = [\underline{u}, \overline{u}] \in \mathbb{H}(\Omega)$ and $v = [\underline{v}, \overline{v}] \in \mathbb{H}(\Omega)$ be respectively, H-continuous viscosity subsolution and H-continuous viscosity supersolution of (1.1.1) in Ω and

$$\overline{u} \leq \underline{v} \ \text{ on } \partial\Omega.$$

Then $\overline{u} \leq \underline{v}$ in Ω.

For the proof of Theorem 3.5.2, we need the following lemma given in [37].

Lemma 3.5.1 *Let X be a subset of $\mathbb{R}^n$, $u \in USC(X), v \in LSC(X)$ and*

$$M_\alpha = \sup_{X\times X}\left(u(x) - v(y) - \frac{\alpha}{2}|x-y|^2\right)$$

for $\alpha > 0$. Let $M_\alpha < \infty$ for large α and (x_α, y_α) be such that

$$\lim_{\alpha\to\infty}\left(M_\alpha - (u(x_\alpha) - v(y_\alpha) - \frac{\alpha}{2}|x_\alpha - y_\alpha|^2)\right) = 0.$$

Then the following hold:

$$\lim_{\alpha\to\infty} \alpha|x_\alpha - y_\alpha|^2 = 0 \tag{3.5.7}$$

and

$$\lim_{\alpha\to\infty} M_\alpha = u(z) - v(z) = \sup_{x\in X}(u(x) - v(x)) \tag{3.5.8}$$

whenever $z \in X$ is a limit point of x_α as $\alpha \to \infty$.

Proof of Theorem 3.5.2. *Define for $\alpha > 0$, an upper semicontinuous function ϕ_α on a set $\overline{\Omega}\times\overline{\Omega}$ by setting $\phi_\alpha(x,y) = \overline{u}(x) - \underline{u}(y) - \frac{\alpha}{2}|x-y|^2$ and let (x_α, y_α) be a maximum point for ϕ_α on $\overline{\Omega}\times\overline{\Omega}$ (the maximum is achieved in view of upper semicontinuity and compactness). Then $M_\alpha = \sup_{\overline{\Omega}\times\overline{\Omega}} \phi_\alpha(x,y) = \left(\overline{u}(x_\alpha) - \underline{v}(y_\alpha) - \frac{\alpha}{2}|x_\alpha - y_\alpha|^2\right)$ is finite. It follows from properties (3.5.7) and (3.5.8) of Lemma 3.5.1 and $\overline{u} \le \underline{v}$ on $\partial\Omega$ that $(x_\alpha, y_\alpha) \in \Omega\times\Omega$ for α large.*

Since we seek to prove that $\overline{u} \le \underline{v}$ in Ω, we assume to the contrary that $\overline{u}(z) > \underline{v}(z)$ for some $z \in \Omega$, it follows that

$$M_\alpha \ge \overline{u}(z) - \underline{v}(z) = \delta > 0 \text{ for } \alpha > 0. \tag{3.5.9}$$

Writing (x', y') in place of (x_α, y_α) for simplicity and set $\varphi_1(x) = \underline{v}(y') - \frac{\alpha}{2}|x-y'|^2$, $\varphi_2(y) = \overline{u}(x') - \frac{\alpha}{2}|x'-y|^2$. It is clear that $\varphi_i \in C^2(\Omega)(i = 1, 2)$. Since (x', y') is a maximum point of ϕ_α, then it is clear that x' is a local maximum point for $\underline{u} - \varphi_2$, whereas, y' is a local maximum point for $\overline{u} - \varphi_1$. Moreover, $D\varphi_1(x') = \alpha(x'-y') = D\varphi_2(y')$. Then we can exploit the fact that u is an H-continuous viscosity subsolution of (1.1.1) and we obtain

$$H(x', \overline{u}(x'), \alpha(x'-y')) \le 0 \tag{3.5.10}$$

Similar, since u is an H-continuous viscosity supersolution of (1.1.1), we obtain

$$H(x', \underline{v}(x'), \alpha(x'-y')) \ge 0. \tag{3.5.11}$$

Combining (3.5.10) and (3.5.11) we obtain

$$H(x', \overline{u}(x'), \alpha(x' - y')) \leq 0 \leq H(x', \underline{v}(x'), \alpha(x' - y')). \tag{3.5.12}$$

The next step is to use the assumption (3.5.5) and (3.5.6) and the condition (3.5.12) to estimate M_α *and contradict (3.5.9) for large* α.

Using the definition of δ, *see (3.5.9), and the fact that* (x', y') *is a maximum point for* ϕ_α, *we have*

$$\gamma\delta \leq \gamma(\overline{u}(z) - \underline{v}(z)) \leq \gamma(\overline{u}(x') - \underline{v}(y')). \tag{3.5.13}$$

Proceeding, we deduce from (3.5.13), (3.5.5) and (3.5.6) that

$$\begin{aligned} 0 \leq \gamma\delta &\leq \gamma(\overline{u}(x') - \underline{v}(y')) \\ &\leq H(x', \overline{u}(x'), \alpha(x' - y')) - H(x', \underline{v}(x'), \alpha(x' - y')) \\ &= H(x', \overline{u}(x'), \alpha(x' - y')) - H(y', \underline{v}(x'), \alpha(x' - y')) \\ &+ H(y', \underline{v}(x'), \alpha(x' - y')) - H(x', \underline{v}(x'), \alpha(x' - y')) \\ &\leq \omega(\alpha|x' - y'|^2 + |x' - y'|). \end{aligned}$$

Here we used (3.5.12) to estimate the first term on the right by 0 *and (3.5.6) on the second term.*

Since $\omega(\alpha|x' - y'|^2 + |x' - y'|) \to 0$ *as* $\alpha \to \infty$ *by (3.5.7), we have a contradiction with (3.5.9).* ■

3.6 Extending the Hamiltonian Operator over the Set $\mathbb{H}(\Omega)$

In this section, we consider the equation (1.1.1) in the following more general form

$$H(x, u(x), Du(x)) = f(x), x \in \Omega, \tag{3.6.1}$$

where $f \in C^0(\Omega)$. We call the mapping $\mathcal{H} : C^1(\Omega) \to C^0(\Omega)$ given by

$$\mathcal{H}(u)(x) = H(x, u(x), Du(x)), x \in \Omega$$

a Hamiltonian operator. Then, equation (3.6.1) can be written as

$$\mathcal{H}(u) = f. \tag{3.6.2}$$

It is well known that the mapping $\mathcal{H}$ is in general not surjective. This means of course, that there exists $f \in C^0(\Omega)$ in (3.6.2) such that the set $\mathcal{H}^{-1}(f) = \{u \in C^1(\Omega) : \mathcal{H}(u) = f\} = \emptyset$ which implies that (3.6.1) does not have classical solution, as illustrated by a variety of well known examples, some of them rather simple ones, see [109, Chapter 6]. Clearly, the function u is a classical solution of (3.6.1) iff $u \in \mathcal{H}^{-1}(f)$. Hence, the need for generalized solutions like the viscosity solutions considered here.

Let us note that equation (3.6.1) does not really generalize equation (1.1.1). Since function f can always be moved to the left hand side reducing the equation to the form (1.1.1).

The usual way of defining generalized solutions is by extending the operator $\mathcal{H}$ to a larger domain. This extension can be done in different ways: functional analytic method [46], algebraic method [116], order completion method [109].

Here, we extend the operator $\mathcal{H}$ to the set of Hausdorff continuous functions using the viscosity approach. Our aim is to express the H-continuous viscosity solutions of (3.6.1) as solutions to an operator equation involving the extended operator in the same way as the classical solutions of (3.6.1) are solutions of (3.6.2). We use subdifferentials and superdifferentials.

For $u = [\underline{u}, \overline{u}] \in \mathbb{H}(\Omega)$, consider the sets

$$G^+(u) = \{x \in \Omega : D^+\overline{u}(x) \neq \emptyset\} \text{ and } G^-(u) = \{x \in \Omega : D^-\underline{u}(x) \neq \emptyset\}.$$

For $u \in C(\Omega)$, it was proved in [16] that the sets $G^+(u)$ and $G^-(u)$ are each dense in Ω. This result can be extended to Hausdorff continuous functions using a similar argument and is given in the following lemma.

Lemma 3.6.1 *Let $u = [\underline{u}, \overline{u}] \in \mathbb{H}(\Omega)$. The sets $G^+(u)$ and $G^-(u)$ are each dense in Ω*

Proof. *Let $y \in \Omega$ and let $\delta > 0$ be such that $\overline{B}_\delta(y) = \{z \in \mathbb{R}^n : |z-y| \leq \delta\} \subset \Omega$. Consider the smooth function $\varphi_\varepsilon(x) = \frac{1}{2\varepsilon}|x-y|^2, \varepsilon > 0$. Since $\overline{u} - \varphi_\varepsilon$ is an upper semicontinuous function on Ω, it attains its maximum over $\overline{B} = \overline{B}_\delta(y)$ at some point x_ε. Then we have*

$$(\overline{u} - \varphi_\varepsilon)(x_\varepsilon) \geq (\overline{u} - \varphi_\varepsilon)(y). \tag{3.6.3}$$

From the inequality (3.6.3), for all $\varepsilon > 0$, we get

$$|x_\varepsilon - y|^2 \leq 2\varepsilon(\overline{u}(x_\varepsilon) - \overline{u}(y)) \leq 4\varepsilon \sup_{x \in \overline{B}} |\overline{u}(x)|.$$

Thus x_ε is not on the boundary of $\overline{B}$ for ε small enough, and by Lemma 1.3.1 (i), $D\varphi_\varepsilon(x_\varepsilon) = \frac{1}{2}(x_\varepsilon - y)$ belong to $D^+(\overline{u}(x_\varepsilon))$. This proves that $G^+(u)$ is dense in Ω, and similar argument shows that $G^-(u)$ is dense in Ω too. ■

Now, for $u = [\underline{u}, \overline{u}] \in \mathbb{H}(\Omega)$, define the following two functions

$$\psi(x) = \sup_{p \in D^+\overline{u}(x)} H(x, \overline{u}(x), p), x \in G^+(u), \tag{3.6.4}$$

$$\varphi(x) = \inf_{p \in D^-\underline{u}(x)} H(x, \underline{u}(x), p), x \in G^-(u). \tag{3.6.5}$$

For the extension of the operator $\mathcal{H}$, we need the following operators:
$T^+ : \mathbb{H}(\Omega) \to USC(\Omega), T^- : \mathbb{H}(\Omega) \to LSC(\Omega)$ defined by

$$T^+u(x) = S(G^+(u), \Omega, \psi)(x), x \in \Omega, \tag{3.6.6}$$

$$T^-u(x) = I(G^-(u), \Omega, \varphi)(x), x \in \Omega. \tag{3.6.7}$$

Since $G^+(u)$ and $G^-(u)$ are dense in Ω for every $u \in \mathbb{H}(\Omega)$, the operators T^+ and T^-, defined by (3.6.6) and (3.6.7), respectively, are well defined on $\mathbb{H}(\Omega)$ and by (2.5.2) $T^+u \in USC(\Omega)$ and $T^-u \in LSC(\Omega)$. The following theorem gives a new characterization of H-continuous viscosity subsolution and H-continuous viscosity supersolution of (3.6.1) in terms of the operators T^+ and T^-.

Theorem 3.6.1 *Let $u = [\underline{u}, \overline{u}] \in \mathbb{H}(\Omega)$. Then*
(a) u is an H-continuous viscosity subsolution of (3.6.1) if and only if

$$T^+u \leq f \text{ in } \Omega; \tag{3.6.8}$$

(b) u is an H-continuous viscosity supersolution of (3.6.1) if and only if

$$T^-u \geq f \text{ in } \Omega. \tag{3.6.9}$$

Proof. *We will prove only point (a). Point (b) is proved in a similar way. By Definition 3.2.1, u is an H-continuous viscosity subsolution of (3.6.1) if and only if*

$$H(x, \overline{u}(x), p) \leq f(x), p \in D^{+}\overline{u}(x), x \in \Omega. \tag{3.6.10}$$

In view of definition of ψ, see (3.6.4), the inequality (3.6.10) is equivalent to

$$\psi(x) \leq f(x), x \in G^{+}(u). \tag{3.6.11}$$

Since $f \in C^0(\Omega) \subset USC(\Omega)$, using the minimality property of upper semicontinuous envelopes, see (2.5.9), the inequality (3.6.11) implies that

$$S(G^{+}(u), \Omega, \psi)(x) \leq f(x), x \in \Omega. \tag{3.6.12}$$

To complete the proof, we show that (3.6.12) implies (3.6.11). Indeed, since $S(G^{+}(u), \Omega, \psi)$ is an upper bound of ψ on $G^{+}(u)$, we have

$$\psi(x) \leq S(G^{+}(u), \Omega, \psi)(x) \leq f(x), x \in G^{+}(u).$$

■

As a consequence of Theorem 3.6.1, we obtain the following.

Theorem 3.6.2 *Let $u = [\underline{u}, \overline{u}] \in \mathbb{H}(\Omega)$. Then there exists $f \in C^0(\Omega)$ such that u is an H-continuous viscosity solution of (3.6.1) if and only if*

$$T^{+}u \leq T^{-}u \text{ in } \Omega. \tag{3.6.13}$$

Proof. *Let there exists $f \in C^0(\Omega)$ such that u is an H-continuous viscosity solution of (3.6.1). Combining (3.6.8) and (3.6.9) we have the inequality (3.6.13). In order to prove the inverse implication, we use a well known Theorem of Hahn [130] which states that if a lower semicontinuous function majorates an upper semicontinuous function then there exists a continuous function between them. Since $T^{+}u \in USC(\Omega), T^{-}u \in LSC(\Omega)$ and $T^{+}(u) \leq T^{-}(u)$ in Ω, then there exists $f \in C^0(\Omega)$ such that $T^{+}(u) \leq f \leq T^{-}(u)$ in Ω. Therefore, u is an H-continuous viscosity solution of (3.6.1).* ■

There is an interesting question here. *Is the function f in Theorem 3.6.2 unique?* If f is not unique, then this means that two functions can be viscosity solutions of (3.6.1) for two different right hand terms and from practical consideration this is an undesirable situation. This issue has not been addressed in the existence theory of viscosity solutions, since the Hamilton-Jacobi equation is not considered in the operator form (3.6.2).

In view of Theorem 3.6.2, consider the operator $\hat{H} : \mathbb{H}(\Omega) \to \mathcal{P}(C^0(\Omega))$, defined by

$$\hat{H}(u) = \{f \in C^0(\Omega) : T^+u(x) \leq f(x) \leq T^-u(x), x \in \Omega\}, \tag{3.6.14}$$

where $\mathcal{P}(C^0(\Omega))$ is the set of all subsets of $C^0(\Omega)$. We can reformulate Theorem 3.6.2 as follows.

Theorem 3.6.3 *A function $u \in \mathbb{H}(\Omega)$ is an H-continuous viscosity solution of (3.6.1) if and only if*

$$f \in \hat{H}(u).$$

Then, the earlier question can be equivalently formulated as:
Can the set $\hat{H}(u)$ contain more than one element?

In general, this is an open problem. However, when Ω is an open interval of $\mathbb{R}$ we obtained an answer, namely that $\hat{H}(u)$ contains one element as shown by the following theorem.

Theorem 3.6.4 *Let Ω be a nonvoid open interval of $\mathbb{R}$ and $u = [\underline{u}, \overline{u}] \in \mathbb{H}(\Omega)$. Then*

$$T^-u(x) \leq T^+u(x), x \in \Omega. \tag{3.6.15}$$

The following three lemmas will be instrument in the proof of Theorem 3.6.4.

Lemma 3.6.2 *Let Ω be a nonvoid open interval of $\mathbb{R}$ and $u = [\underline{u}, \overline{u}] \in \mathbb{H}(\Omega)$. If there exists a dense subset G_0 of Ω such that $G_0 \subseteq G^+(u) \bigcap G^-(u)$, then (3.6.15) holds.*

Proof. *Since $G_0 \subset G^+(u)$, by the monotonicity of the generalized upper Baire operator about inclusion with respect to the dense subset of Ω, see (2.5.7), and property (2.5.1), we have*

$$T^+u = S(G^+(u), \Omega, \psi) \geq S(G_0, \Omega, \psi) \geq I(G_0, \Omega, \psi). \tag{3.6.16}$$

In view of the fact that $\psi \geq \varphi, x \in G_0$, by monotonicity of $I(G_0, \Omega, .)$, see (2.5.3), the inequality (3.6.16) implies that

$$T^+u \geq I(G_0, \Omega, \varphi). \tag{3.6.17}$$

Since $G_0 \subset G^-(u)$, by monotonicity of generalized lower Baire operator about inclusion to respect to the dense subset of Ω, see (2.5.6), from inequality (3.6.17) we have

$$T^+u \geq I(G_0, \Omega, \varphi) \geq I(G^-(u), \Omega, \varphi) = T^-u. \tag{3.6.18}$$

The property (3.6.18) implies that (3.6.15) holds. ■

Lemma 3.6.3 *Let $u = [\underline{u}, \overline{u}] \in \mathbb{H}(\Omega), \Omega = (a, b) \subset \mathbb{R}$. If $\overline{u}$ has no local maximum at any point of Ω and $\underline{u}$ has no local minimum at any point of Ω, then u is monotone on Ω, i.e, the function $\overline{u}$ and $\underline{u}$ are increasing or decreasing on Ω.*

Proof. *Let $D = \{x \in \Omega : \underline{u}(x) = \overline{u}(x) = u(x)\}$. It was shown in [132] that u is continuous on D and that D is dense in Ω. We will show that the function u is monotone on D. Indeed, if $u(x) = u(y), x, y \in D$, then u is a constant function on D.*

Let there exists $p, q \in D, p < q$ such that $u(p) \neq u(q)$. Then either $u(p) > u(q)$ or $u(p) < u(q)$.

Suppose that $u(p) > u(q)$. Since $\overline{u}$ has no local maximum on (a, b), we have

$$\sup_{x \in [p,q] \cap D} u(x) \leq \max_{x \in [p,q] \cap D} \overline{u}(x) = \overline{u}(p) = u(p).$$

Therefore, the maximum of u on $[p, q] \cap D$ exists and

$$\max_{x \in [p,q] \cap D} u(x) = \overline{u}(x) = u(p).$$

Similarly, the maximum of u on $[p, q] \cap D$ exists and

$$\min_{x \in [p,q] \cap D} u(x) = \overline{u}(x) = u(q).$$

Next we prove that u is strictly decreasing function on $D \cap (a, b)$. For that, it suffices to show that u is strictly decreasing on $(a, p) \cap D$, on $[p, q] \cap D$, and on $(q, b) \cap D$.

Let $x, y \in (a, p) \cap D$ such that $x < y$. If $\max_{(a,p) \cap D} \overline{u}(z) = \overline{u}(p) = u(p)$, then $p \in (x, q)$ is a local maximum of $\overline{u}$ and this contradicts that $\overline{u}$ has no local maximum on (a,b). Then $\max_{(a,p) \cap D} \overline{u}(z) = u(x)$. Since $y \in (x, p \cap D)$, we have $u(x) > u(y)$ and thus u is strictly decreasing on $(a, p) \cap D$.

Now, let $x, y \in [p, q] \cap D$ such that $x < y$. Then $u(x) > u(y)$, since if $u(x) < u(y)$, then $x \in (p, y)$ is a local maximum of u and this contradicts the fact that $\underline{u}$ has a no local minimum on (a, b). Thus u is a decreasing function on $[p, q] \cap D$.

Finally, let $x, y \in (q, b) \cap D$ such that $x < y$. Then $u(q) > u(p)$, otherwise if $u(q) < u(x)$, then $q \in (p, x)$ is a local minimum of $\underline{u}$ and this contradicts that $\underline{u}$ has no local minimum on (a,b). If $u(x) < u(y)$, then x is a local minimum for $\underline{u}$ and this is a contradiction of assumption. Then $u(q) > u(x) > u(y)$ and u is a decreasing function on $(q, b) \cap D$.

In summary, u is a decreasing function on D, i.e.,

$$\forall x, y \in D,\ x < y \Rightarrow u(x) > u(y). \tag{3.6.19}$$

By monotonicity of $F(D, \Omega, .)$, see (2.5.5), and the fact that $u \in \mathbb{H}(\Omega)$, (3.6.19) implies that for $x, y \in \Omega$ such that $x < y$, we have $u(x) = F(D, \Omega, u)(x) > F(D, \Omega, u)(y) = u(y)$ and thus u is a decreasing function on Ω.

Similarly, if $u(p) < u(q)$, then the function u is increasing on D and by the monotonicity of $F(D, \Omega, .)$, the function u is increasing on (a, b). ■

Lemma 3.6.4 *Let $u = [\underline{u}, \overline{u}] \in \mathbb{H}(V)$, where V is an open subset of $\mathbb{R}$. If u is not monotone on any interval $(\alpha, \beta) \subset V$, then the sets*

$$D_1 = \{x \in V : \overline{u} \text{ has a local maximum at } x\}$$

and

$$D_2 = \{x \in V : \underline{u} \text{ has a local minimum at } x\}$$

are dense in V.

Proof. *We need to show that for any interval $(a, b) \subset V$, we have $(a, b) \cap D_1 \neq \emptyset$ and $(a, b) \cap D_2 \neq \emptyset$.*

Let $(a, b) \subset V$. It is given that u is not monotone on (a, b). Therefore, by Lemma 3.6.3, there exists either local maximum of $\underline{u}$ at any interior point of (a, b). Let $\overline{u}$ has a local maximum at $c \in (a, b)$. We have $c \in (a, b) \cap D_1$. Therefore $(a, b) \cap D_1 \neq \emptyset$. The case when $\underline{u}$ has a local minimum on (a, b) is treated similarly.

Now consider intervals (a, c) and (c, b). Using the same argument in each one, either $\overline{u}$ has a local maximum or $\underline{u}$ has a local minimum. If $\overline{u}$ has a local minimum, then $D_2 \cap (a, b) \neq \emptyset$. Let $\overline{u}$ has a local maximum at $d \in (c, b)$. Without loss of generality, we can consider

$$\overline{u}(c) \leq \overline{u}(d).$$

Since c is a local maximum of $\overline{u}$, there exists $\delta > 0$ such that

$$u(p) < \overline{u}(c), x \in (c - \delta, c + \delta) \cap D,$$

where $D = \{x \in V : \underline{u}(x) = \overline{u}(x) = u(x)\}$ which is dense in V, see [130]. There exists $p \in (c + \frac{\delta}{2}, c + \delta) \cap D$ such that

$$u(p) < \overline{u}(c), \tag{3.6.20}$$

otherwise u is a constant on $(c+\delta/2, c+\delta)$ and this contradicts the fact that u is not monotone on any interval of V. Now, there exists $q \in (c-\delta/2, c+\delta/2) \cap D$ which implies $\overline{u}(c) \geq u(p)$ contradicting (3.6.20). Then we have

$$\overline{u}(d) \geq \overline{u}(c) > u(q) > u(p). \tag{3.6.21}$$

There exists $r \in (p,b) \cap D$ such that $u(r) > u(p)$, otherwise if $u(r) \leq u(p), r \in (p,b) \cap D$, then $\overline{u}(x) \leq u(p), x \in (p,b) \cap D$. In particular $\overline{u}(d) \leq u(p)$ which contradicts (3.6.21).

Consider an interval $[r,q]$. We have

$$\min_{x \in [q,r]} \underline{u}(x) \leq u(p) < \min\{u(q), u(r)\}.$$

Therefore, there exists $s \in (q,r)$ such that $\underline{u}(s) = \min_{x \in [q,r]} \underline{u}(x)$. Hence $s \in D_2 \cap (a,b)$ and this implies that $D_2 \cap (a,b) \neq \emptyset$. ■

Proof of Theorem 3.6.3 *Let $u = [\underline{u}, \overline{u}] \in \mathbb{H}(\Omega)$. Consider $W-$the union of all intervals of Ω, where the function u is monotone. Then by the theorem for differentiability of monotone real functions [115], $\overline{u}$ and $\underline{u}$ are differentiable almost everywhere in W and therefore $\underline{u}$ is differentiable on a dense subset W_1 of W and $\overline{u}$ is differentiable on a dense subset W_2 of W, where W_1, W_2 are sets of full measure (i.e., their complements are null sets). Let $W_0 = W_1 \bigcap W_2$. Then by Lemma 3.6.2, inequality (3.6.15) holds in $\overline{W}$.*

We have two cases: either the set $\overline{W} = \Omega$ or $\overline{W} \neq \Omega$. If $\overline{W} = \Omega$, then the proof is completed. Now suppose that $\overline{W} \neq \Omega$. Let $V = \Omega \setminus \overline{W}$. It is clear that V is nonempty open subset of Ω. Let $D_1 = \{x \in V : \overline{u}$ has a local maximum at $x\}$ and
$D_2 = \{x \in V : \underline{u}$ has a local minimum at $x\}$. Since u is not monotone on any subinterval of V, by Lemma 3.6.4, D_1, D_2 are each dense in V.

Suppose that $y \in D_1$. Then $0 \in D^+\overline{u}(y)$. Hence $D_1 \subseteq G^+(u)$ and we have

$$\psi(y) = \sup_{p \in D^+\overline{u}(y)} H(y, \overline{u}(y), p) \geq H(y, \overline{u}(y), 0). \tag{3.6.22}$$

Let $D = \{x \in \Omega : \underline{u}(x) = \overline{u}(x) = u(x)\}$. It was shown in [132] that D is dense in Ω, hence it is dense in V as well. Let $\lim_{y \to x} \overline{u}(y) = \lim_{y \to x} \underline{u}(y) = u(x)$ and inequality (3.6.22) implies that

$$\begin{aligned} T^+u(x) &= S(D_1, V, \psi)(x) \\ &\geq S(D_1, V, H(., \overline{u}(.), 0))(x) \\ &= \lim_{y \to x} H(y, \overline{u}(y), 0) \\ &= H(x, u(x), 0). \end{aligned} \tag{3.6.23}$$

Similarly, we have $\varphi(y) \leq H(y, \overline{u}(y), 0)$ *for* $y \in D_2$ *which implies*

$$T^+u(x) \geq H(x, u(x), 0), x \in D \cap V. \tag{3.6.24}$$

Combining (3.6.23) and (3.6.24), we get

$$T^-u(x) \leq T^+u(x), x \in D \cap V.$$

Let $x \in V$. *Then by the monotonicity of generalized lower Baire operator with respect to the first argument and with respect to the last argument and by the inequality between lower Baire and upper Baire operators, see (2.5.6) and (2.5.3), we have*

$$\begin{aligned} T^-u(x) &= I(\Omega, \Omega, T^-u)(x) \\ &\leq I(D, \Omega, T^-u)(x) \\ &\leq I(D, \Omega, T^+u)(x) \\ &\leq S(D, \Omega, T^+u)(x) \\ &= T^+u(x), x \in V. \end{aligned}$$

Hence (3.6.15) holds on V. *Since (3.6.15) holds also in* $\overline{W}$, *then it holds on* $\Omega = V \bigcup \overline{W}$.

■

We prove next that the operator $\hat{H}$, given in (3.6.14), is an extension of a Hamiltonian operator $\mathcal{H}$ defined by (3.6.2). Indeed, let $u \in C^1(\Omega)$. It suffices to prove that $\hat{H}(u) = \mathcal{H}(u)$, since $C^1(\Omega) \subset C(\Omega) \subset \mathbb{H}(\Omega)$. By Lemma 1.2.1 (b), we have $D^+u(x) = D^-u(x), x \in \Omega$ and this implies that $G^+(u) = G^-(u) = \Omega$ and the functions ψ and φ defined by (3.6.4) and (3.6.5), respectively, are identical. Therefore we have

$$T^+u(x) = T^-u(x), x \in \Omega. \tag{3.6.25}$$

The property (3.6.25) implies that

$$\hat{H}(u) = \{f\}, f \in C^0(\Omega). \tag{3.6.26}$$

In view of (3.6.2), if we identify a singleton by an element, (3.6.26) implies that

$$\hat{H}(u) = f = \mathcal{H}(u).$$

Hence $\hat{H}$ is an extension of $\mathcal{H}$.

Chapter 4

The Value Functions of Optimal Control Problem as Envelope Viscosity Solutions

By the classical Hamilton-Jacobi theory, if the value function (i.e. the optimal value function associated with the optimal control problems as a function of the initial time and state) is smooth, then it is a classical solution of the Hamilton-Jacobi-Bellman equation.

In general, however, the value function is nonsmooth and therefore can not satisfy the Hamilton-Jacobi-Bellman equation in the classical sense. In the control theory literature, several approaches have been developed to cope with this difficulty. Boltyanski [28] (see also [52]) restricted the class of controls so that the value function becomes piecewise smooth. Vinter and Lewis [133] characterized optimality through a sequence of continuously differentiable subsolutions of the Hamilton-Jacobi equation. If the value function is Lipschitz continuous, Havelock [66] and Clarke [32] characterized the value function as a generalized solutions of the Hamilton-Jacobi equation involving the Clarke generalized gradient.

Many of the works on viscosity solutions were devoted to application in the Dynamic Programming approach to deterministic optimal control problems, e.g, [94], [45], [16], [30]. According to Definition 1.2.1, introduced by Crandall and Lions, when the value function is uniformly continuous, it is then a viscosity solution of the Hamilton-Jacobi-Bellman equation [16].

According to Definition 1.3.3, introduced by Ishii, the value function is characterized as a viscosity solution through its lower and upper semicontinuous envelopes. The reader can also refer to [24], [53], and [67].

In this chapter, we study a particular optimal control problem, namely discounted minimum time problem. Since an envelope viscosity solution is typically Hausdorff continuous viscosity solution, we will show that the value functions associated with this problem are the envelope viscosity solutions of associated Hamilton-Jacobi-Bellman equation.

4.1 Discounted Minimum Time Problem

Consider the optimal control problem with the state equation

$$\begin{cases} y'(t) = f(y(t), a(t)), \ t > 0 \\ y(0) = x \in \mathbb{R}^n. \end{cases} \tag{4.1.1}$$

Here, the control $a(.) \in \mathcal{A}$:=$\{a : [0, \infty) \to A$ measurable$\}$, where $\mathcal{A}$ is the *set of admissible controls*, $A \subset \mathbb{R}^m$ is a given *control space*.

We now list some basic assumptions on our control system which are made for most of the results of this chapter. We will assume:

$$A \text{ is compact metric space} \tag{4.1.2}$$

and the *dynamics* $f : \mathbb{R}^n \times A \to \mathbb{R}^n$ satisfies

$$f(x, a) \text{ is continuous on } \mathbb{R}^n \times A, \tag{4.1.3}$$

$$|f(x, a)| \leq L, \ \forall\, x \in \mathbb{R}^n, \ a \in A, \tag{4.1.4}$$

$$|f(x, a) - f(y, a)| \leq C|x - y|, \ \forall\, x, \ y \in \mathbb{R}^n, \ a \in A. \tag{4.1.5}$$

for some constants $L > 0$ and $C > 0$.

It is well known [134] from the theory of ordinary differential equations that assumptions (4.1.3), (4.1.4), and (4.1.5), ensure that for each continuous control a and $x \in \mathbb{R}^n$, the state equation (4.1.1) has a unique solution $y_x(t, a)$, existing for $t \geq 0$.

Given

$$\mathcal{T} \subseteq \mathbb{R}^n \text{ nonempty closed target with compact boundary } \partial\mathcal{T}, \tag{4.1.6}$$

as usual we denote by $t_x(a)$ the first time the trajectory associated with $a \in \mathcal{A}$ and starting at x hits $\mathcal{T}$, i.e., the exit time of the trajectory, that is

$$t_x(a) = \begin{cases} \min\{s \geq 0 : y_x(s,a) \in \mathcal{T}\} \text{ if } \{s : y_x(s,a) \in \mathcal{T}\} \neq \emptyset \\ +\infty \qquad\qquad\qquad\qquad \text{otherwise.} \end{cases}$$

If $t_x(a) < +\infty$, then $y_x(t_x(a))$ denotes the point where the trajectory reaches the target $\mathcal{T}$. We denote by $\mathcal{R}$ the set of all x such that $t_x(a) < +\infty$ for some control a and call $\mathcal{R}$ the controllable set. Thus,

$$\mathcal{R} = \{x \in \mathbb{R}^n : t_x(a) < +\infty\}.$$

We are concerned with the following optimal control problem called discounted minimum time problem:

$$\inf_{a\in\mathcal{A}} \begin{cases} \int_0^{t_x(a)} e^{-s} ds \text{ if } t_x(a) < +\infty \\ 1 \qquad\qquad\qquad\qquad \text{otherwise.} \end{cases} \tag{4.1.7}$$

The value function associated with the problem (4.1.7) is defined by

$$u(x) = \begin{cases} 1 - e^{-T(x)} \text{ if } T(x) < +\infty \\ 1 \qquad\qquad\quad \text{if } T(x) = +\infty, \end{cases} \tag{4.1.8}$$

where the function T is the minimum time function and is defined by

$$T(x) = \inf_{a\in\mathcal{A}} t_x(a), \ x \in \mathcal{R}. \tag{4.1.9}$$

If $f(x, A) = \{f(x, a) : a \in A\}$ is a convex set for all $x \in \mathbb{R}^n$, then T is lower semicontinuous function (see e.g. [67]), and so u is lower semicontinuous.

It is well known [16] that, the Hamilton-Jacobi-Bellman equation associated to value function given in (4.1.8) is

$$u(x) + H(x, Du(x)) - 1 = 0, \ x \in \mathcal{R}, \tag{4.1.10}$$

with

$$H(x,p) = \sup_{a(t)\in\mathcal{A}} \{-f(x,a).p)\}. \tag{4.1.11}$$

The following theorem, given in [16], gives the dynamic programming property.

Theorem 4.1.1 *For all $s > 0$ the function u define by (4.1.8) is*

$$u(x) = \inf_{a(.)\in\mathcal{A}} \{\int_0^s e^{-t}\,dt + u(y_x(s,a))e^{-s}\},\ \text{if } s \le T(x),\ x \in \mathcal{R}. \tag{4.1.12}$$

Remark 4.1.1 *The above dynamic programming property implies that the function*

$$s \longmapsto \int_0^s e^{-t}\,dt + u(y_x(s,a))e^{-s},\ s \in [0, t_x(a)]$$

is nondecreasing.

4.2 The Value Function as an Envelope Viscosity Solution

In this section we prove that the value function u, given in (4.1.12), is an envelope viscosity solution of associated Hamilton-Jacobi-Bellman equation (4.1.10).

Our main results are stated in the next theorems.

Theorem 4.2.1 *Assume that (4.1.2), (4.1.3), (4.1.4), (4.1.5), and (4.1.6) hold and the set $f(x,A)$ is convex for all $x \in \mathbb{R}^n$. Assume that we have $u : \Omega \to \mathbb{R}$ is locally bounded in open set $\Omega \subset \mathbb{R}^n$ and that for all x there exists $\tau > 0$ such that for $0 < t < \tau$, we have*

$$u(x) = \inf_{a\in\mathcal{A}} \left(\int_0^t e^{-s}\,ds + u(y_x(t,a))e^{-t}\right). \tag{4.2.1}$$

Then u is an envelope viscosity solution of the equation

$$u + H(x, Du) - 1 = 0 \text{ in } \Omega = \mathbb{R}^n \backslash \mathcal{T}, \tag{4.2.2}$$

with Hamiltonian H is defined in (4.1.11).

Before we prove Theorem 4.2.1, let us remark that the minimum time function does not change if we add the null vector field to the system (4.1.1).

Theorem 4.2.2 [16] *Let $\widetilde{A} = A \cup \{\widetilde{a}\}$, $\widetilde{a} \notin A$, $\widetilde{f}(x,a) = f(x,a)$ if $a \in A$, $\widetilde{f}(x,\widetilde{a}) = 0$ for all x. Then the minimum time function $\widetilde{T}$ associated with the system*

$$\begin{cases} y' = \widetilde{f}(y,\alpha) + h\beta,\ h > 0, \\ y(0) = x, \end{cases} \tag{4.2.3}$$

coincides with the minimum time function T given in (4.1.9) associated with (4.1.1).

We approximate the system (4.1.1) with the controllable system (4.2.3). The control functions are $(\alpha, \beta) \in \widetilde{\mathcal{A}} \times \mathcal{B}$, where $\mathcal{B} := \{\beta : [0, \infty) \to \overline{B}_1(0) \text{ measurable}\}$, and the trajectories are denoted by $y_x^h(., \alpha, \beta)$.

The value functions are

$$T_h(x) \quad = \quad \inf_{\widetilde{\mathcal{A}} \times \mathcal{B}} t_x^h$$

and

$$u_h(x) \quad : \quad = \inf_{\widetilde{\mathcal{A}} \times \mathcal{B}} \int_0^{t_x^h} e^{-t}\, dt = 1 - e^{-T_h(x)}, \tag{4.2.4}$$

where $t_x^h = t_x^h(\alpha, \beta)$ is the entry time in $\mathcal{T}$ of the trajectory of (4.2.3).

For the proof of Theorem 4.2.1, we will also use a representation of a value function as supremum of viscosity subsolutions as stated in the next theorem.

Theorem 4.2.3 [16] *Under the assumptions (4.1.2), (4.1.3), (4.1.4), (4.1.5), and (4.1.6), for all $h > 0$, a function u_h, given in (4.2.4), is bounded continuous on $\mathbb{R}^n$ and it is continuous viscosity solution of (4.2.2). Moreover, we have the representation formula for the function u given in* (4.2.1)

$$u(x) = \sup_{h>0} u_h(x) \text{ for all } x \in \mathbb{R}^n. \tag{4.2.5}$$

Proof of Theorem 4.2.1. *Prove that u given by (4.2.1) is an envelope viscosity solution of (4.2.2). Indeed, by Theorem 4.2.3, the function u is a supremum of viscosity subsolutions of (4.2.2). It is enough to prove that u is viscosity supersolution of (4.2.2). For that, we take $\varphi \in C^1(\Omega)$ and $z \in \Omega$ such that $u(z) = \varphi(z)$ and $u(x) \geq \varphi(x)$ for all x in a neighborhood of z. We used that u is lower semicontinuous, i.e., $u(x) = \underline{u}(x), x \in \Omega$*

We assume by contradiction $\varphi(z) + H(z, D\varphi(z)) - 1 < 0$. Then for some $\varepsilon > 0$

$$\varphi(x) + H(x, D\varphi(x)) - 1 \leq -\varepsilon \text{ for all } x \in B_\varepsilon(z) \subseteq \Omega.$$

By the assumption on f in (4.1.3)-(4.1.5) *and A is compact, see* (4.1.2), *there is $t \in]0, \tau[$ such that $y_x(s, \alpha) \in B_\varepsilon(z)$ for all $x \in B_{\varepsilon/2}(z)$, $0 < s \leq t$, and all $\alpha \in \mathcal{A}$. We fix such a t and set*

$$\delta := \varepsilon(1 - e^{-t})/2. \tag{4.2.6}$$

By the inequality $\geq$ in (4.2.1), for any x there is $\alpha \in \mathcal{A}$ such that

$$u(x) > -\delta + \int_0^t e^{-s}\, ds + u(y_x(t, s))e^{-t}.$$

Since $u \geq \varphi$ *and*

$$\frac{d}{ds}\left(\varphi(y_x(s))e^{-s}\right) = e^{-s}\left(-\varphi(y_x) + D\varphi(y_x).y'_x\right),$$

a.e., *we get, for* $x \in B_{\varepsilon/2}(z)$,

$$\begin{aligned} u(x) - \varphi(x) &> -\delta + \int_0^t e^{-s}[1 - \varphi(y_x) + D\varphi(y_x).f(y_x,\alpha)](s)\,ds \\ &\geq -\delta - \int_0^t e^{-s}[-1 + \varphi(y_x) + H(y_x, D\varphi(y_x)](s)\,ds \\ &\geq -\delta + \int_0^t \varepsilon e^{-s}\,ds = \delta, \end{aligned}$$

where in the last inequality, we used (4.2.6) and the choice of t. *Then*

$$u(z) = \liminf_{x\to z} u(x) \geq \varphi(z) + \delta.$$

Thus

$$u(z) \geq \varphi(z) + \delta.$$

This is a contradiction to the choice of φ *because* $\delta > 0$. *Thus, since* u *is supremum of viscosity subsolutions, see (4.2.5) and it is a viscosity supersolution of (4.2.2), then* u *is an envelope viscosity solution of (4.2.2).* ■

To illustrate the above results, we consider the following optimal control problem.

4.3 Zermelo Navigation Problem

One of the most classical problems in optimal control theory, namely, Zermelo navigation problem, gives an example of discontinuous viscosity solution of an Hamilton-Jacobi-Bellman equation.

Example 4.3.1 *(Zermelo Navigation Problem) [16]*
Consider a boat moving with velocity of constant magnitude, which we normalize to 1, relative to a stream of constant velocity $\sigma \geq 1$. *We want to reach in minimum time a given compact target* $\mathcal{T} = \{0\}$.

We choose the axes in $\mathbb{R}^2$ *so that the stream velocity is* $(\sigma, 0)$. *Then, the dynamic system is*

$$\begin{cases} y_1' = \sigma + a_1, \\ y_2' = a_2, \end{cases}$$

with $a_1^2 + a_2^2 = 1, a_1, a_2$ *belong to* $\mathbb{R}$.
By an elementary geometrical argument, the reachable set $\mathcal{R}=\{x : T(x) < +\infty\}$ *is*

$$\begin{aligned} \mathcal{R} &= \{x : x_1 < 0 \text{ or } x_1 = x_2 = 0\} && \text{if } \sigma = 1 \\ \mathcal{R} &= \{x : x_1 \leq 0,\ |x_2| \leq -x_1(\sigma^2 - 1)^{-1/2}\} && \text{if } \sigma > 1. \end{aligned}$$

By standard control-theoretic methods we get

$$T(x) = \begin{cases} -\frac{|x|^2}{2x_1} & \text{if } \sigma = 1, \\ \\ \frac{-x_1\sigma - [x_2^2(1-\sigma^2) + x_1^2]^{1/2}}{\sigma^2 - 1} & \text{if } \sigma > 1. \end{cases}$$

An explicit form of Hamiltonian is given by

$$H(x, Du) = \sup_{a \in A}\{-f(x, a).Du\}, \tag{4.3.1}$$

where $f(x, a) = (\sigma + a_1, a_2)$ *and* $a = (a_1, a_2) \in A = \{(y_1, y_2) : y_1^2 + y_2^2 = 1\}$.

By calculation, (4.3.1) becomes

$$H(x, Du) = |Du| - \sigma u_{x_1}.$$

By Theorem 4.2.1, the value function

$$u(x) = 1 - e^{-T(x)}$$

is an envelope viscosity solution of the associated Hamilton-Jacobi-Bellman equation

$$u(x) + |Du| - \sigma u_{x_1} - 1 = 0, x = (x_1, x_2) \in \mathbb{R}^2 \setminus \mathcal{T}.$$

Chapter 5

Nonstandard Finite Difference Methods for Solutions of Hamilton-Jacobi Equations and Conservation Laws

5.1 Introduction

A major difficulty in the study of partial differential equations is, in general, the lack of exact analytical solutions. One way to proceed is to use numerical integration techniques to obtain useful information on the possible solution behaviors. A popular and important method is one based on the use of finite differences to construct discrete models of the differential equations of interest [54], [108].

A relevant question concerns stability. For problems with smooth solutions, usually a linear stability analysis is adequate. For problems with discontinuous solutions or discontinuous derivatives of solutions, a stronger measure of stability is usually required.

Almost all of the standard procedures yield schemes which are convergent with restriction on the step size. One response to this situation was the initiation by Mickens [103] of a research program for the investigation of new methods for constructing finite difference schemes which are convergent for any step size. These new procedures are called nonstandard difference methods, [101, 102, 104].

Throughout this chapter, we shall be concerned with two initial value problems. The fist problem is a Cauchy problem for Hamilton-Jacobi equation in the form

$$u_t(x,t) + H(Du(x,t)) = 0, (x,t) \in \mathbb{R}^n \times (0,+\infty) \quad (5.1.1)$$

$$u(x,0) = u_0(x), x \in \mathbb{R}^n. \quad (5.1.2)$$

The second problem is a Cauchy problem for conservation laws in the form

$$v_t(x,t) + (f(v(x,t))_x = 0, (x,t) \in \mathbb{R} \times (0,\infty) \quad (5.1.3)$$

$$v(x,0) = v_0, x \in \mathbb{R}, \quad (5.1.4)$$

where $f(v)$ is the nonlinear flux function.

Note that the equation (5.1.1) is closely related to equation (5.1.3), in fact in one dimension space, they are equivalent if one takes $v = u_x$. Thus the solution v to conservation laws is the derivative of a solution u to a Hamilton-Jacobi equations. Conversely, the solution u to a Hamilton-Jacobi equation is the integral of a solution v to conservation laws, see [77].

As typical for partial differential equations, problem (5.1.1)-(5.1.2) or (5.1.3)-(5.1.4) cannot be completely solved by analytic techniques. Consequently, numerical simulations are of fundamental importance in gaining some useful insights on the solutions. More precisely, it is crucial to design numerical methods, which replicate essential physical properties of the solutions, see [103], [104].

The precise way in which the properties are preserved is contained in the following definition of qualitative stability [8].

Definition 5.1.1 *Assume that the solution of (5.1.1)-(5.1.2) (resp. (5.1.3)-(5.1.4)) satisfies some property (P). A numerical method approximating (5.1.1)-(5.1.2) (resp. (5.1.3)-(5.1.4)) is called qualitatively stable with respect to (P) or P-stable if the numerical solutions for (5.1.1)-(5.1.2) (resp. (5.1.3)-(5.1.4)) satisfy property (P) for all values of the involved step sizes.*

It should be noted that standard finite difference schemes are generally not qualitatively stable with respect to essential physical properties of solution of interested problem [96].

The nonstandard finite difference methods introduced by R. E. Mickens in the late 1980s appear to be powerful in designing qualitatively stable schemes.

A formal definition is as follows [8]:

Definition 5.1.2 *A finite difference method for (5.1.1)-(5.1.2) or (5.1.3)-(5.1.4) is called nonstandard if at least one of the following is met*
(i) in the discrete derivatives the traditional denominator Δt or Δx is replaced by a nonnegative function $\psi(\Delta t)$ or $\psi(\Delta x)$ such that

$$\psi(z) = z + o(z^2), \text{ as } 0 < z \to 0; \tag{5.1.5}$$

(ii) nonlinear terms are approximated in a nonlocal way, i.e., by a suitable function of several points of the mesh.

Note that Mickens [103] set five rules for the construction of discrete models that have the capability to replicate the properties of the exact solution. The general rules for constructing such schemes are not precisely known at the present time, consequently, there exists a certain level of ambiguity in the practical implementation of nonstandard procedures to the formulation of finite difference schemes of differential equations. Here, Definition 5.1.2 of a nonstandard finite difference scheme is stated unambiguously making use of only two of Mickens rules. There: the renormalization of the denominator of the discrete derivative (part (i)) and the nonlocal approximation of nonlinear terms in the data (part (ii)). The other rules are expressed in terms of Definition 5.1.1.

One of the main advantages of the nonstandard finite difference method that in addition to the usual properties of consistency, stability and hence convergence, it produces numerical solutions which also exhibit essential properties of solution.

The following properties have received extensive attention in the design of qualitatively stable nonstandard finite difference schemes: fixed points and their stability [103], [8], oscillatory, conservation of energy, positivity and boundedness [104], [7], dissipation or dispersion, [105], etc.

This chapter is concerned with two physical properties, namely, monotonicity property and total variation diminishing property. The total variation diminishing property has not yet exploited in the context of the nonstandard finite difference method by the authors' best knowledge.

It is well known, see e.g. [40], that the solution of problem (5.1.1)-(5.1.2) depends monotonically on the initial value, that is, for any two solutions $u_1(x,t)$, $u_2(x,t)$ of (5.1.1),

$$u_1(x,0) \leq u_2(x,0),\ \forall\, x \Rightarrow u_1(x,t) \leq u_2(x,t),\ \forall\, t > 0,\ \forall\, x. \tag{5.1.6}$$

The property (5.1.6) is important from the physical point of view. One of the purposes of this chapter is to design monotone numerical schemes, that is, those that replicate this property. In [40], Crandall and Lions first studied the convergence of monotone scheme for Hamilton-Jacobi equation (5.1.1). They presented finite difference schemes on rectangular meshes. These methods are difficult to apply for complicated geometry where adaptive mesh refinement is often required. Finite volume and finite element schemes based on arbitrary triangulation, are thus attractive such cases. Monotone schemes on unstructured meshes for Hamilton-Jacobi equations were studied by Abgrall [1], Barth and Sethian [25], Kossioris et al. [80] and Li et al. [91]. Our general approach is along the lines of the many works, and specifically [91], where the finite element scheme discretization is coupled with the finite difference time discretization. However, we use the Mickens' nonstandard variant of the difference approach. The schemes employing standard finite difference technique are monotone under restrictive conditions on the time step size. On the contrary, the nonstandard finite difference scheme presented in this chapter preserves the monotonicity property unconditionally, improving therefore the results of [91].

In Section 5.2, firstly, we consider a space discretization of equation (5.1.1) using the finite element method. Secondly, we construct a nonstandard finite difference scheme for obtained system of ordinary differential equations. Finally, the convergence of this new scheme is proved and numerical results supporting the theory are presented.

The entropy solution $v(x,t)$ of (5.1.3)-(5.1.4) satisfies the property that the total variation with respect to x does not increase as t increases [123, Chapter 16], [69, Chapter 2]:

$$TV(v(x,t_1)) \geq TV(v(x,t_2)) \text{ for } 0 < t_1 \leq t_2, \tag{5.1.7}$$

where the total variation of a function $v(x,t)$ with respect to an one dimensional variable x is defined as

$$TV(v(.,t)) = \lim_{h\to 0} \frac{1}{h} \int_{-\infty}^{+\infty} |v(x+h,t) - v(x,t)|\, dx.$$

It is clear that $TV(v)$ is finite for any bounded increasing or decreasing function with respect to x, including functions with jump discontinuities. Moreover, if v is differentiable, then $TV(v)$ reduces to $TV(v) = \int_{-\infty}^{+\infty} |v_x|\, dx$.

We discuss here finite difference schemes which produce numerical solutions with diminishing total variation. The importance of schemes preserving the property (5.1.7) can not be overemphasized since it is an essential physical characteristic of the exact solution. Furthermore, it has been shown that schemes with such qualitative stability have the advantage of high-order accuracy in smooth regions while resolving discontinuities in the solutions without spurious oscillations which are often displayed by numerical solutions [124], [125, 126]. In [124], they are called total variation diminishing schemes. The preservation of the diminishing total variation property is also discussed in [61] within the context of the more general concept of strong stability.

One problem associated with the explicit total variation diminishing methods is a restriction on the time step-size which in some cases could be rather severe. This is particulary pronounced in high order methods, e.g., methods of Runge-Kutta type [60], [48]. On the other hand, the computational complexity of total variation diminishing implicit methods is significantly higher particularly when nonlinear functions are involved.

Following space discretization and time discretization, respectively, we impose our numerical method for (5.1.3) to be in conservation form. Our approach is to use the tools of the nonstandard finite difference method in constructing total variation diminishing schemes which have the advantages of being computationally simpler (in the case of implicit schemes) and have no step size restriction (in the case of explicit schemes).

Section 5.3 deals with total variation diminishing nonstandard finite difference schemes for conservation laws. We formulate an implicit nonstandard finite difference scheme using nonlocal approximations of nonlinear terms and explicit nonstandard finite difference schemes where renormalization of the denominator is used. Numerical results by both the implicit and explicit methods are presented in this section. At the end of this section, we use a discontinuous Galerkin finite difference method proposed by Hu and Shu [70] to solve the one dimensional Hamilton-Jacobi equation. They use the fact that the derivatives of the solution u of Hamilton-Jacobi equation satisfy a conservation laws, and apply the usual discontinuous Galerkin method on this conservation laws to advance the derivatives of u. Here, the solution u is recovered from these derivatives computed using nonstandard TVD method for conservation laws developed in Subsection 5.3.4. This will determine u up to a constant. The missing constant is obtained using combination of two ways given in [70].

5.2 A Monotone Scheme for Hamilton-Jacobi Equations via the Nonstandard Finite Difference Method

For simplicity, we consider the problem (5.1.1)-(5.1.2) in two space dimensions although a generalization to arbitrary space dimension is possible. It is well known that a problem (5.1.1)-(5.1.2) does not have classical solutions. Various kind of generalized solutions have been considered but may have discontinuous derivatives regardless of the smoothness of the initial condition $u_0(x)$. Here, we consider its continuous viscosity solution, which under the condition $H \in C^{0,1}(\mathbb{R}^2)$ and $u_0 \in C^{0,1}(\mathbb{R}^2)$ that we assume henceforth, is the uniform limit as $\varepsilon \to 0^+$ of the (classical) solutions of

$$u_t + H(\bigtriangledown u) - \varepsilon \nabla^2 u = 0,\ (x, y, t) \in \mathbb{R}^2 \times (0, +\infty), \tag{5.2.1}$$

where $\varepsilon > 0$ is a small parameter.

Due to the stated convergence, an approximation to the viscosity solution of (5.1.1) can be obtained by numerical schemes for (5.2.1), where ε is sufficiently small. We will use a finite element space discretization for equation (5.2.1) coupled with nonstandard finite difference time discretization for the obtained system of ordinary differential equations for constructing scheme which is qualitatively stable to respect the monotonicity property (5.1.6).

In the next subsection we consider a finite difference space discretization of equation (5.2.1), while Subsection 5.2.3 is devoted to a nonstandard finite difference scheme for the obtained system of differential equation. The convergence of this new scheme is proved in Subsection 5.2.4. Numerical results supporting the theory are presented in Subsection 5.2.5.

5.2.1 Finite element space discretization

In this subsection we refer essentially to [91]. Let $\mathcal{T}_h$ be a triangulation of $\mathbb{R}^2$ consisting of a countable set of triangles which satisfy the usual compatibility conditions. The generic triangle of $\mathcal{T}_h$ is denoted by T, h_T is the diameter (the largest side) of T, $h = \sup_{T \in \mathcal{T}_h} T$ and ρ_T is the diameter of the largest ball in T.

The triangulation is assumed to be *regular*, that is, there exists a constant $\gamma > 0$, independent of h, such that we have

$$\frac{h_T}{\rho_T} \leq \gamma \tag{5.2.2}$$

for all T on $\mathcal{T}_h$. The condition (5.2.2) is equivalent to Zlámal's condition [138] that there exists a constant $\theta_0 > 0$ such that

$$\forall T,\ \theta_T \geq \theta_0,$$

where for each triangle T, θ_T denotes the smallest angle of T. Let $\{X_i : i = 1, 2, ...\}$ be the set of nodes on $\mathcal{T}_h$. The edge connecting X_i and X_k is denoted by E_{ik} and its length is denoted by $|E_{ik}|$. For any node X_i, I_i is the index set of the triangles with common vertex X_i while N_i is the index set of the neighbor vertexes (vertexes connected to X_i by an edge). With each node X_i, we associate the basis function ϕ_i defined as a continuous piecewise linear function on $\mathbb{R}^2$ such that $\phi_i(X_i) = 1$ and $\phi_i(X_k) = 0$, $k \neq i$. Note that ϕ_i has "small" support in the sense that $\operatorname{supp} \phi_i = V_i = \cup_{j \in I_i} T_j$, as shown in Figure 5.1.

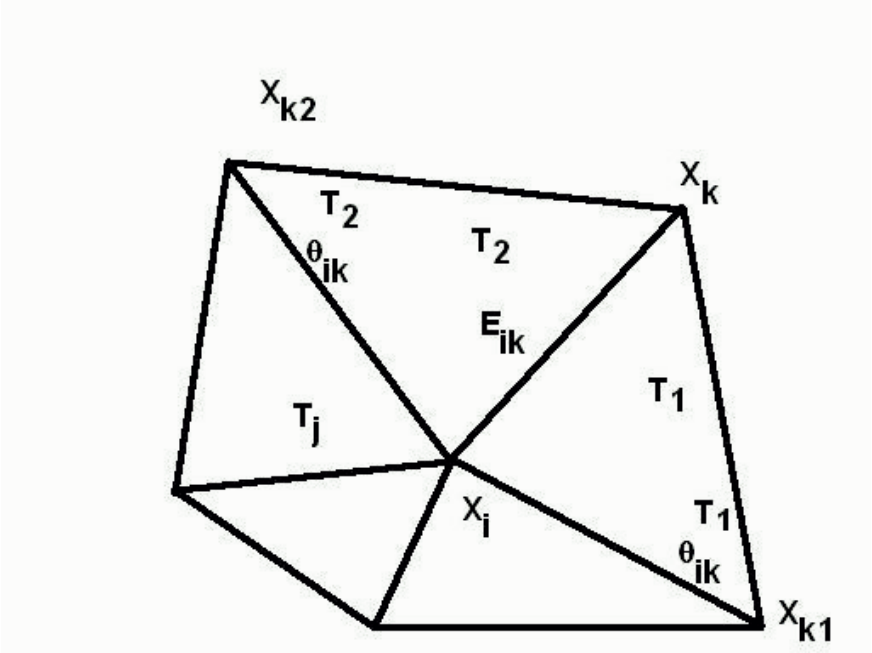

Figure 5.1: *The structure of V_i.*

We denote by $\mathcal{V}_h$ the finite element space which is spanned by the basis functions $(\phi_i)_i$.

An approximation $v_h(x, y, t)$ to the solution of (5.2.1) is sought such that $v_h(., ., t) \in \mathcal{V}_h$, i.e.,

$$v_h(x, y, t) = \sum_{i=1}^{\infty} v_{h,i}(t)\phi_i(x, y),$$

where $v_{h,i}(t) = v_h(X_i, t)$ and v_h satisfies the variational equation

$$\frac{d}{dt}\iint_{\mathbb{R}^2} v_h w \, dx \, dy + \iint_{\mathbb{R}^2} H(\nabla v_h) w \, dx \, dy - \varepsilon \iint_{\mathbb{R}^2} \nabla^2 v_h w \, dx \, dy = 0, \tag{5.2.3}$$

for all functions $w(x, y) \in \mathcal{V}_h$. In the sequel $v_{h,i}(t)$ is abbreviated to $v_i(t)$. Replacing the test function w by the basis functions $\phi_i(x, y)$, $i = 1, 2, ...$, in (5.2.3) and using that $\phi_i = 0$ in exterior of V_i, we have

$$\frac{d}{dt}\iint_{V_i} v_h \phi_i \, dx \, dy + \iint_{V_i} H(\nabla v_h)\phi_i \, dx \, dy - \varepsilon \iint_{V_i} \nabla^2 v_h \phi_i \, dx \, dy = 0.$$

Using Green's formula and the fact that ϕ_i vanishes on ∂V_i, we have

$$\frac{d}{dt}\iint_{V_i} v_h \phi_i \, dx \, dy + \iint_{V_i} H(\nabla v_h)\phi_i \, dx \, dy = -\varepsilon \iint_{V_i} \nabla v_h \nabla \phi_i \, dx \, dy. \tag{5.2.4}$$

Approximating the integral in the first term of (5.2.4) by the "mass lumping" quadrature we obtain

$$\frac{d}{dt}v_i(t)\iint_{V_i} \phi_i \, dx \, dy + \int\int_{V_i} H(\nabla v_h)\phi_i dx dy = -\varepsilon \iint_{V_i} \nabla v_h \nabla \phi_i \, dx \, dy. \tag{5.2.5}$$

Moreover, let

$$\begin{aligned} \iint_{T_i} \phi_i \, dx \, dy &= \frac{1}{3}\mu(T_j), \ j \in I_i, \\ \iint_{V_i} \phi_i \, dx \, dy &= \frac{1}{3}\mu(V_i), \\ \gamma_{ij} &= \frac{\mu(T_j)}{\mu(V_i)}, \end{aligned}$$

where μ denotes the area. It is clear that $0 < \gamma_{ij} < 1$ and $\sum_{j \in I_i} \gamma_{ij} = 1$.

Lemma 5.2.1 *Let T_1 and T_2 be triangles with common edge E_{ik} on V_i; $\theta_{ik}^{T_1}$ and $\theta_{ik}^{T_2}$ are angles of the triangles T_1 and T_2 opposite to the edge E_{ik}, respectively, as shown in Figure 5.1. We have the following formulation*

$$\iint_{V_i} \nabla v_h \nabla \phi_i \, dx \, dy = -\sum_{k \in N_i} a_{ik}(v_k - v_i), \tag{5.2.6}$$

where

$$a_{ik} = a_{ik}^{T_1} + a_{ik}^{T_2}, \quad a_{ik}^{T_1} = \frac{1}{2}\cot\theta_{ik}^{T_1}, \quad a_{ik}^{T_2} = \frac{1}{2}\cot\theta_{ik}^{T_2}.$$

Proof. *Since $\phi_i + \sum_{k \in N_i} v_k \phi_k = 1$, by computation we have*

$$\begin{aligned}
\iint_{V_i} \nabla v_h \nabla \phi_i \, dx \, dy &= \iint_{V_i} \nabla\Big(\sum_{k \in N_i} v_k \phi_k + v_i \phi_i\Big).\nabla \phi_i \, dx \, dy \\
&= \iint_{V_i} \left(\sum_{k \in N_i} v_k \nabla \phi_k + v_i\Big(-\sum_{k \in N_i} \nabla \phi_k\Big)\right).\nabla \phi_i \, dx \, dy \\
&= \sum_{k \in N_i} \left((v_k - v_i) \iint_{V_i} \nabla \phi_k . \nabla \phi_i \, dx \, dy\right).
\end{aligned} \tag{5.2.7}$$

It remains to prove that

$$\iint_{V_i} \nabla \phi_k . \nabla \phi_i \, dx \, dy = -a_{ik}. \tag{5.2.8}$$

Since $\phi_k = 0$ on any triangle of V_i which do not contain X_k, we have

$$\iint_{V_i} \nabla \phi_k . \nabla \phi_i \, dx \, dy = \iint_{T_1} \nabla \phi_k . \nabla \phi_i \, dx \, dy + \iint_{T_2} \nabla \phi_k . \nabla \phi_i \, dx \, dy.$$

Let X_{k1}, X_{k2} be the two nodes opposite to E_{ik} in triangle T_1 and T_2, respectively, see Figure 5.1.

$$\begin{aligned}
\iint_{T_1} \nabla \phi_k . \nabla \phi_i \, dx \, dy &= \frac{|E_{i\,k1}|}{2\mu(T_1)} . \frac{|E_{k1\,k}|}{2\mu(T_1)} \cos(180^0 - \theta_{ik}^{T_1}).\mu(T_1) \\
&= -\frac{|E_{i\,k1}|}{4\mu(T_1)} . |E_{k1\,k}| \cos(\theta_{ik}^{T_1}) \\
&= -\frac{|E_{i\,k1}| . |E_{k1\,k}| \cos(\theta_{ik}^{T_1})}{4 . \frac{1}{2} |E_{i\,k1}| . |E_{k1\,k}| \sin(\theta_{ik}^{T_1})} = -\frac{1}{2}\cot(\theta_{ik}^{T_1}) = -a_{ik}^{T_1}.
\end{aligned}$$

Similarly,

$$\iint_{T_2} \nabla \varphi_k . \nabla \varphi_i \, dx \, dy = -\frac{1}{2}\cot(\theta_{ik}^{T_2}) = -a_{ik}^{T_2}.$$

Thus (5.2.8) is proved. Substituting (5.2.8) into (5.2.7), we have (5.2.6).■

Substituting (5.2.6) into (5.2.5), and using that $H(\nabla v_h|_{T_j})$ is a constant, the equation (5.2.5) can be written in the following equivalent form

$$\frac{d}{dt}v_i = -\sum_{j\in I_i} H(\nabla v_h|_{T_j})\gamma_{ij} + \frac{3\varepsilon}{\mu(V_i)}\sum_{k\in N_i} a_{ik}(v_k - v_i). \tag{5.2.9}$$

In [91], it is shown that the scheme (5.2.9) is consistent.

For the monotonicity of the scheme discussed in the next subsection the coefficients a_{ik} need to be bounded away from zero, that is, we have the following Lemma containing the additional assumption on the triangulation $\mathcal{T}_h$.

Lemma 5.2.2 [91] *If the triangulation $\mathcal{T}_h$ is such that there exists a constant c_1, $(0 < c_1 < \frac{\pi}{2})$, independent of h such that*

$$\theta_{ik}^{T_1} + \theta_{ik}^{T_2} \leq \pi - c_1, \tag{5.2.10}$$

for every edge E_{ik} on $\mathcal{T}_h$, then there exists a positive constant C_0 such that for the scheme defined by (5.2.9) we have

$$a_{ik} \geq C_0, i = 1, 2, ..., \text{ and } k \in N_i. \tag{5.2.11}$$

5.2.2 A nonstandard finite difference scheme

We consider a mesh $\{t_n,\ n = 0, 1, 2, ...\}$ in the time dimension with constant time step Δt, that is we have $\{t_n = n\Delta t\}$. As usual $v^n = (v_i^n)_{i=1}^{\infty}$ denotes the approximation of the solution of (5.2.9) at $t = t_n$.

Our aim in this subsection is to design a scheme for (5.2.9) that is qualitatively stable with respect to the monotonicity on initial values, as stated in the following definition:

Definition 5.2.1 *A finite difference scheme (5.2.9) is monotone if*

$$v_i^0 \leq w_i^0 \Longrightarrow v_i^n \leq w_i^n,$$

where v^n and w^n are discrete solutions initiated at v^0 and w^0, respectively.

For simplicity, we ignore for the moment the space index i and we assume that we are dealing with a scalar problem the discrete solution of which is given by an explicit scheme of the form

$$v^{n+1} = g(\Delta t; v^n). \tag{5.2.12}$$

The following result is proved in [9].

Theorem 5.2.1 *The difference scheme (5.2.12) is qualitatively stable with respect to the monotonicity on initial values if and only if*

$$\frac{\partial v^{n+1}}{\partial v^n} \equiv \frac{\partial g(\Delta t; v)}{\partial v} \geq 0,\ \Delta t > 0,\ v \in \mathbb{R}. \tag{5.2.13}$$

Since we are reduced to checking the positivity condition (5.2.13), we will in what follows adapt and exploit the favorable situation described in the following theorem:

Theorem 5.2.2 *Let w be the solution of the problem*

$$Lw = f(w),$$

where L is either the differential operator $Lz = z'$ or the identity operator $Lz = z$. Assume that the solution w is nonnegative and that the function f admits the decomposition

$$f(z) = p(z) - q(z)z,$$

where $p(z) \geq 0$ and $q(z) \geq 0$. Then the difference scheme

$$\frac{w^{n+1} - w^n}{\Delta t} = p(w^n) - q(w^n)w^{n+1} \tag{5.2.14}$$

for $Lz = z'$ or

$$w^{n+1} = p(w^n) - q(w^n)w^{n+1} \tag{5.2.15}$$

for $Lz = z$ is qualitatively stable with respect to the positivity property of the solution w.

Proof. *Obvious by re-writing* (5.2.14) *and* (5.2.15) *as*

$$w^{n+1} = \frac{w^n + \Delta t\, p(w^n)}{1 + \Delta t\, q(w^n)} \text{ and } w^{n+1} = \frac{p(w^n)}{1 + q(w^n)},$$

respectively.■

Remark 5.2.1 *The situation described in Theorem 5.2.2 was introduced in a more specific form by the authors in [7] in order to design schemes that preserve the positivity property of the solutions of reaction diffusion equations. The idea is also exploited for the approximation of differential models in population biology and mathematical epidemiology where the positivity of the involved species is essential (see, for instance, [63, 106, 105]). The underlining point of these schemes is, as it can be seen from* (5.2.14) *and* (5.2.15), *that one of Mickens' rules of constructing nonstandard finite difference schemes is reinforced: the nonlinear term $q(w)w$ is approximated in a nonlocal way, i.e., by $q(w^n)w^{n+1}$ and not by $q(w^n)w^n$ or $q(w^{n+1})w^{n+1}$.*

Coming back to the problem (5.2.9). In view of the form of the right-hand side of (5.2.9) and of Theorem 5.2.1, which requires to show the positivity condition $\frac{\partial v^{n+1}}{\partial v^n} \geq 0$, we propose in the spirit of the nonlocal approximation in Theorem 5.2.2, the following non-standard finite difference scheme for the system of equations (5.2.9):

$$v_i^{n+1} = v_i^n - \Delta t \sum_{j \in I_i} H(\nabla v_h^n|_{T_j})\gamma_{ij} + \frac{3\varepsilon\Delta t}{\mu(V_i)} \sum_{k \in N_i} a_{ik}(v_k^n - v_i^{n+1}). \tag{5.2.16}$$

Observe that the last sum in (5.2.16) is approximated in a nonlocal way.

Theorem 5.2.3 *Let the triangulation $\mathcal{T}_h$ be regular and satisfy the condition* (5.2.11). *Let $H \in C^{0,1}(\mathbb{R}^2)$. Then there exist a constant C independent of Δt and h such that if*

$$\varepsilon \geq Ch, \tag{5.2.17}$$

the scheme (5.2.16) *is monotone.*

proof. *Let $m \in N_i$ and let $X_{m'}$ and $X_{m''}$ be the nodes opposite to E_{im} in the two adjacent triangles T' and T'' containing E_{im}. We have*

$$\begin{aligned} \frac{\partial v_i^{n+1}}{\partial v_m^n} &= -\Delta t \left[\nabla H.\nabla\phi_m|_{T'} \frac{\mu(T')}{\mu(V_i)} + \nabla H.\nabla\phi_m|_{T''} \frac{\mu(T'')}{\mu(V_i)} \right] \\ &+ \frac{3\varepsilon\Delta t}{\mu(V_i)} a_{im} - \frac{3\varepsilon\Delta t}{\mu(V_i)} \sum_{k \in N_i} a_{ik} \frac{\partial v_i^{n+1}}{\partial v_m^n}. \end{aligned}$$

Hence

$$\begin{aligned} \left(1 + \frac{3\varepsilon\Delta t}{\mu(V_i)} \sum_{k \in N_i} a_{ik}\right) \frac{\partial v_i^{n+1}}{\partial v_m^n} &\geq \frac{\Delta t}{\mu(V_i)} \left[3\varepsilon a_{im} - \frac{1}{2}|H|_{1,\infty}(|E_{im'}| + |E_{im''}|) \right] \\ &\geq \frac{\Delta t}{\mu(V_i)} [3\varepsilon C_0 - h|H|_{1,\infty}]. \end{aligned}$$

Setting $C = \frac{|H|_{1,\infty}}{3C_0}$, we obtain $\frac{\partial v_i^{n+1}}{\partial v_m^n} \geq 0$ whenever $\varepsilon \geq Ch$.

In similar way

$$\frac{\partial v_i^{n+1}}{\partial v_i^n} = 1 - \Delta t \sum_{j \in I_i} \nabla H.\nabla\phi_i|_{T_j} \frac{\mu(T_j)}{\mu(V_i)} - \frac{3\varepsilon\Delta t}{\mu(V_i)} \sum_{k \in N_i} a_{ik} \frac{\partial v_i^{n+1}}{\partial v_i^n}.$$

Hence

$$\left(1 + \frac{3\varepsilon\Delta t}{\mu(V_i)} \sum_{k \in N_i} a_{ik}\right) \frac{\partial v_i^{n+1}}{\partial v_i^n} = 1 - \frac{\Delta t}{\mu(V_i)} \nabla H . \sum_{j \in I_i} \nabla\phi_i|_{T_j} \mu(T_j). \tag{5.2.18}$$

It is easy to see that in any triangulation $\mathcal{T}_h$ we have

$$\sum_{j\in I_i} \nabla\phi_i|_{T_j}\mu(T_j) = \vec{0}. \tag{5.2.19}$$

Indeed, for any constant vector in $z = (z_1, z_2) \in \mathbb{R}^2$ we have

$$z\sum_{j\in I_i} \nabla\phi_i|_{T_j}\mu(T_j) = \iint_{V_i} z\nabla\phi_i\, dx\, dy = -\iint_{V_i} \nabla z\phi_i\, dx\, dy = \vec{0},$$

which implies (5.2.19). *Substituting* (5.2.19) *in* (5.2.18) *we obtain* $\frac{\partial v_i^{n+1}}{\partial v_i^n} > 0$. *This completes the proof.* ■

Remark 5.2.2 *With the notations of the proof of Theorem 5.2.3 in mind, we assume that the second term in the right hand side of (5.2.16) is increasing with respect to v_m^n. In this case, Theorem 5.2.3 is a straightforward consequence of Theorem 5.2.2 and the monotonicity of the scheme (5.2.16) occurs then without the relation $\varepsilon \geq Ch$. This relation is essential in the general setting of Theorem 5.2.3 where the monotonicity of the mentioned term cannot be monitored.*

Mickens'rule of nonlocal approximation is normally applied to nonlinear terms, see [103], [9]. Here we apply it to a linear term. Usually, the two conditions in Definition 5.1.2 are considered independently. It is interesting that in our case the scheme formulated in (5.2.16) through nonlocal approximations admits an equivalent formulation using a renormalization of the denominator of the discrete derivative. More precisely, (5.2.16) is equivalent to

$$\frac{v_i^{n+1} - v_i^n}{\psi_i(\Delta t)} = -\sum_{j\in I_i} H(\nabla v_h^n|_{T_j})\gamma_{ij} + \frac{3\varepsilon}{\mu(V_i)}\sum_{k\in N_i} a_{ik}(v_k^n - v_i^n), \tag{5.2.20}$$

where

$$\psi_i(\Delta t) = \frac{\Delta t}{1 + \frac{3\varepsilon\Delta t}{\mu(V_i)}\sum\limits_{k\in N_i} a_{ik}}.$$

Observe that the denominator function $\psi_i(\Delta t)$ has the following asymptotic behavior stated in (5.1.5).

Remark 5.2.3 *The more complex denominator function $\psi_i(\Delta t)$ captures the intrinsic property of the solution of the problem (5.2.16) of being monotone dependent on initial values under the condition (5.2.17) stated in Theorem 5.2.3. It would be interesting to investigate whether there are other physical properties of (5.2.16) that are captured by $\psi_i(\Delta t)$. For instance, when $H = 0$, can we say that the scheme (5.2.16) preserves the Lyapunov stability properties of the differential equation (5.2.16)?*

Remark 5.2.4 *At every $t = t_n$ the solution of the problem (5.1.1)-(5.1.2) is approximated by the function $v_h^n = \sum_i^{\infty} v_i^n \phi_i(x, y) \in \mathcal{V}_h$. Hence the scheme (5.2.16) can equivalently be considered as a mapping $G(\Delta t, .)$ from $\mathcal{V}_h$ such that $v_h^{n+1} = G(v_h^n)$. Due to the explicit formulation (5.2.20) of (5.2.16), the mapping G can also be given in an explicit form. More precisely, for any $w_h = \sum_{i=1}^{\infty} w_i \phi_i \in \mathcal{V}_h$, we have*

$$G(\Delta t, w_h) = \sum_{i=1}^{\infty} \alpha_i \phi_i(x, y), \tag{5.2.21}$$

where

$$\begin{aligned} \alpha_i &= w_i - \psi_i(\Delta t) \sum_{j \in I_i} H(\nabla w_h|_{T_j}) \gamma_{ij} \\ &\quad + \psi_i(\Delta t) \frac{3\varepsilon}{\mu(V_i)} \sum_{k \in N_i} a_{ik}(w_k - w_i), \quad i = 1, 2, ... \end{aligned} \tag{5.2.22}$$

It is clear that under the conditions in Theorem 5.2.3 the mapping $G(\Delta t, .)$ is monotone with respect to the usually point-wise partial order on $\mathcal{V}_h$.

Remark 5.2.5 *Numerical schemes using the standard finite difference method are typically monotone only under a restriction on the time step size. This might be disadvantage in applications. For example in [91] the restriction is*

$$\Delta t \leq C \frac{\min_{j \in N_i} \mu(T_j)}{\varepsilon}.$$

Since the bound of Δt involves the size of the smallest triangle in the triangulation the above inequality implies that even when the triangulation is refined only locally, Δt need to be adjusted as well. Through the nonstandard approach the scheme (5.2.16) is monotone for any time step size, because in (5.2.17) the time step size Δt is not involved.

5.2.3 Convergence

The convergence of the scheme is obtained through an abstract convergence result of Barles and Souganidis, [22], which is detailed in its consequences in [80] for the equation (5.1.1) as stated below. The function spaces and notations are defined in these references.

For $\tau > 0$, let a mapping $S(\tau) : L^\infty(\mathbb{R}^2) \to L^\infty(\mathbb{R}^2)$ be given. The following conditions are considered in connection with the mapping $S(\tau)$:

- monotonicity, i.e., $u \leq v \Longrightarrow S(\tau)u \leq S(\tau)v$, (5.2.23)
- invariance under translation, i.e., $S(\tau)(u + k) = S(\tau)(u) + k, k \in \mathbb{R}$, (5.2.24)
- consistency, i.e., $\frac{S(\tau)\varphi - \varphi}{\tau} + H(\nabla\varphi) \to 0$, as $\tau \to 0, \forall\varphi \in C_0^\infty(\mathbb{R}^2)$, (5.2.25)
- rate of approximation:
  $|\frac{S(\tau)\varphi - \varphi}{\tau} + H(\nabla\varphi)| \leq o(\tau(|\varphi|_{1,\infty} + |\varphi|_{2,\infty})), \forall\varphi \in C_0^\infty(\mathbb{R}^2)$. (5.2.26)

An approximation $u_{\Delta t}$ to the solution of (5.1.1)-(5.1.2) in two space dimension is constructed by using a grid $t_n = n\Delta t$ in time as follows:

$$u_{\Delta t}(x, y, t) = \begin{cases} S(t - t_n)u_{\Delta t}(., ., t_n)(x, y) & , \quad t \in (t_n, t_{n+1}], n = 0, 1, ... \\ u_0(x) & , \quad t = 0. \end{cases}$$

The following conclusion holds.

Theorem 5.2.4 [22] *If a mapping $S(\tau)$ satisfies* (5.2.23), (5.2.24) *and* (5.2.25) *and $H \in C^{0,1}(\mathbb{R}^2)$, $u_0 \in C^{0,1}(\mathbb{R}^2)$, then for any $t' > 0$ we have $u_{\Delta t} \to u$ uniformly on $\mathbb{R}^2 \times [0, t']$ as $\Delta t \to 0$. Furthermore, if $S(\tau)$ satisfies also* (5.2.26) *then there exists a positive constant C_3 independent of Δt such that*

$$||u_{\Delta t} - u||_\infty \leq C_3\sqrt{\Delta t}.$$

Let $\mathcal{I}_h$ denote the piece-wise interpolation operator at the nodes of the triangulation $\mathcal{T}_h$, that is, for any real function φ on $\mathbb{R}^2$ the function $\mathcal{I}_h\varphi$ is linear on any triangle $T \in \mathcal{T}_h$ and $(\mathcal{I}_h\varphi)(X_i) = \varphi(X_i)$ at every node X_i of $\mathcal{T}_h$. Note that we have $\mathcal{I}_h\varphi \in \mathcal{V}_h$. We consider the mapping $S(\tau) : L^\infty(\mathbb{R}^2) \to L^\infty(\mathbb{R}^2)$ defined as a composition of the operator $\mathcal{I}_h$ and the scheme (5.2.16). More precisely using the mapping G given in (5.2.21)-(5.2.22) we have

$$S(\tau)\varphi = G(\tau, \mathcal{I}_h\varphi). \tag{5.2.27}$$

Then the numerical scheme (5.2.16) is equivalent to the scheme (5.2.27) where the numerical solution is evaluated only at the points of the mesh. Therefore the convergence of the scheme (5.2.16) can be obtained through Theorem 5.2.4, where the mapping $S(\tau)$ is given by (5.2.27). To this end we only need to verify the conditions (5.2.23)-(5.2.26) for $S(\tau)$ given by (5.2.27). The essential property of the monotonicity of $S(\tau)$ follows from the monotonicity of G and $\mathcal{I}_h$. The condition (5.2.24) follows trivially from the form (5.2.21)-(5.2.22) of the mapping G. Now we consider condition (5.2.25) and its stronger form (5.2.26).

By the polynomial approximation theory [31], there exist a positive constant K such that for all $\varphi \in C_0^\infty(\mathbb{R}^2)$,

$$|\mathcal{I}_h\varphi - \varphi|_{1,\infty} \leq Kh|\varphi|_{2,\infty}. \tag{5.2.28}$$

Substituting φ into scheme (5.2.16), we have

$$\frac{S(\Delta t)\varphi|_{(x,y)=X_i} - \varphi_i}{\Delta t} = -\sum_{j\in I_i} H(\nabla\varphi_h|_{T_j})\gamma_{ij} + \frac{3\varepsilon}{\mu(V_i)}\sum_{k\in N_i} a_{ik}(\varphi_k - \varphi_i). \tag{5.2.29}$$

In view of the fact that $\sum\limits_{j\in I_i}\gamma_{ij} = 1$ and (5.2.28)

$$\left| H(\nabla\varphi(X_i)) - \sum_{j\in I_i} H(\nabla\varphi_h|_{T_j})\gamma_{ij} \right| = 0(h). \tag{5.2.30}$$

By Lemma 5.1.1, for a polynomial v_h of degree ≤ 1, $\nabla v_h = \vec{C}$ and we have

$$\begin{aligned}
\frac{3\varepsilon}{\mu(V_i)}\sum_{k\in N_i} a_{ik}(v_k - v_i) &= -\frac{3\varepsilon}{\mu(V_i)}\int\int_{V_i} \nabla v_h \nabla\varphi_i \, dx\, dy \\
&= -\frac{3\varepsilon}{\mu(V_i)}\int\int_{V_i} \vec{C}.\nabla\varphi_i \, dx\, dy \\
&= 0.
\end{aligned}$$

Thus the bilinear form vanishes other the space of polynomial of degree ≤ 1. Therefore, using Bramble-Hilbert Theorem [31] and the fact that $\varepsilon = o(h)$, see (5.2.17), we obtain the conclusion that for $\varphi \in C_0^\infty(\mathbb{R}^2)$

$$\frac{3\varepsilon}{\mu(V_i)}\sum_{k\in N_i} a_{ik}(\varphi_k - \varphi_i) = 0(h). \tag{5.2.31}$$

Combining (5.2.29) with (5.2.30) and (5.2.31), we have

$$\begin{aligned}
&\left|\left[\frac{S(\Delta t)\varphi-\varphi}{\Delta t}+H(\nabla\varphi)\right]_{(x,y)=X_i}\right| \\
&\leq \left|\sum_{j\in I_i} H(\nabla\varphi_h|_{T_j})\gamma_{ij}+\frac{3\varepsilon}{\mu(V_i)}\sum_{k\in N_i} a_{ik}(\varphi_k-\varphi_i)+H(\nabla\varphi)\right|_{(x,y)=X_i} \qquad (5.2.32)\\
&\leq 0(h(|\varphi|_{1,\infty}+|\varphi|_{2,\infty})).
\end{aligned}$$

For convergence we assume that both Δt and h approach to zero. Hence the consistent condition (5.2.25) follows from (5.2.32). Moreover, if we assume that $\Delta t = o(h)$ and $h = o(\Delta t)$ the estimate (5.2.32) implies (5.2.26). Hence we have the following convergence result.

Theorem 5.2.5 *Let the family of triangulation $(\mathcal{T}_h)$ be regular and satisfy the condition* (5.2.11). *Let $H \in C^{0,1}(\mathbb{R}^2)$ and $u_0 \in C^{0,1}(\mathbb{R}^2)$. Then the numerical solution v_h^n obtained by* (5.2.16) *with $\varepsilon \geq Ch$ and $\varepsilon = o(h)$ converges to the exact solution u of the problem* (5.1.1)-(5.1.2) *, i.e., for any $t' > 0$ we have*

$$\sup_{i,n\leq t'/\Delta t} |u(X_i,t_n)-v_i^n| \to 0, \ \text{as } \Delta t\to 0, h\to 0.$$

Moreover, if $0 < \inf \frac{\Delta t}{h} \leq \sup \frac{\Delta t}{h} < \infty$, then there exists a positive constant C_3 independent of Δt and h such that

$$\sup_{i,n\leq t'/\Delta t} |u(X_i,t_n)-v_i^n| \leq C_3\sqrt{h}.$$

Remark 5.2.6 *As Xu and Zikatanov [136] have shown that a formulation similar to (5.2.6) holds in higher dimensional space and Theorem 5.2.5 also holds for arbitrary spatial dimension, the generalization to higher dimensional space is straight forward.*

5.2.4 Numerical results

We present the results of our numerical experiments for the nonstandard monotone schemes for Hamilton-Jacobi equations with convex and nonconvex Hamiltonians and with smooth and discontinuous initial conditions. We compare them with the standard techniques.

Example 5.2.1 *We consider the combustion equation, where the Hamiltonian is convex function, which is often used in testing numerical methods, [129], [91]:*

$$\frac{\partial v}{\partial t} - \sqrt{1 + v_x^2 + v_y^2} = 0, \quad (x, y) \in (0,1) \times (0,1), t > 0 \tag{5.2.33}$$

$$v(x, y, 0) = \cos(2\pi y) - \cos(2\pi x), \quad (x, y) \in (0,1) \times (0,1) \tag{5.2.34}$$

with periodic boundary conditions.

We use a triangulation with 6240 elements on $[0,1] \times [0,1]$*, which satisfies condition (5.2.10). The numerical solution obtained with* $\varepsilon = 0.01$ *and* $\Delta t = 0.01$ *is presented on Figure 5.2. For comparison we consider the standard Euler scheme for the equation (5.2.9) which is monotone only for sufficiently small values of* Δt*, see [91] for details. The numerical solution obtained for the same value of the parameters, that is,* $\varepsilon = 0.01$ *and* $\Delta t = 0.01$*, is presented on Figure 5.3. The advantage of the considered nonstandard method with regard to preserving the qualitative behavior of the solution apparent. One observes, for instance, that the nonstandard scheme is stable with respect to the monotonicity of solution contrary to the standard scheme.*

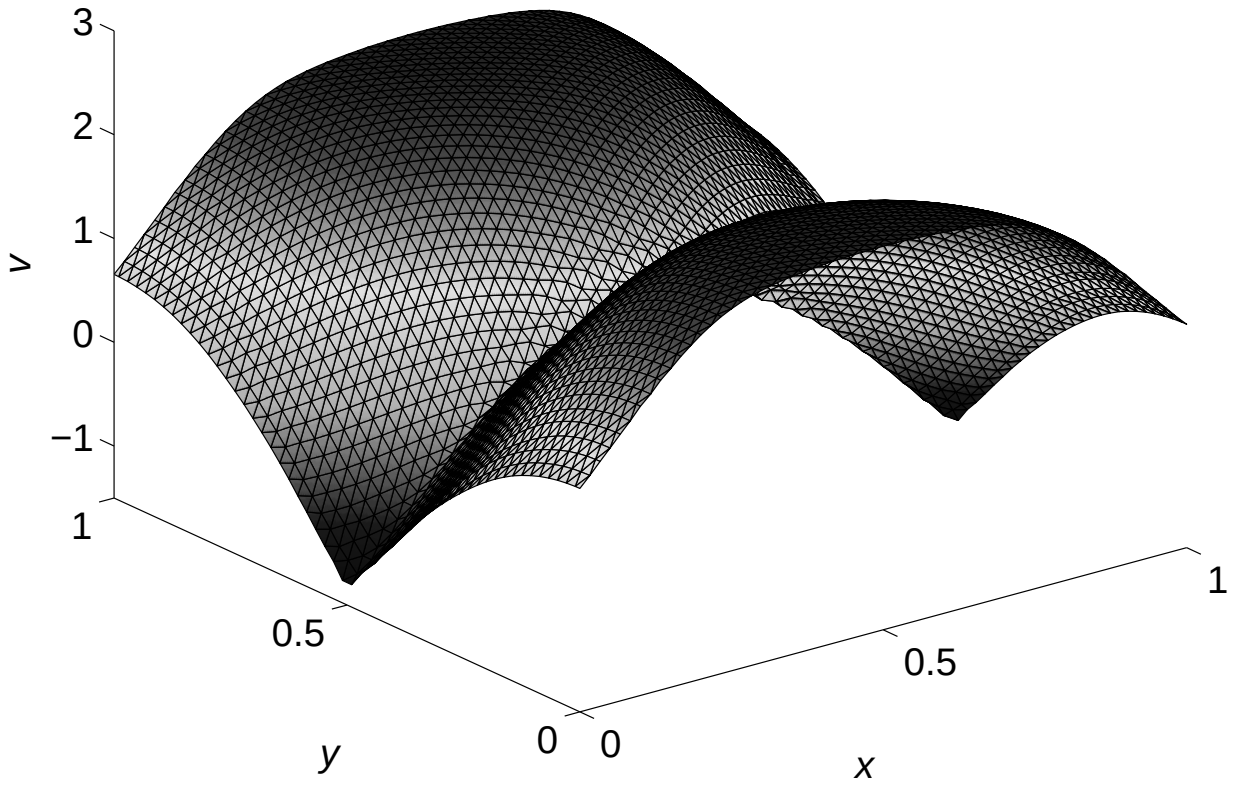

Figure 5.2: *Numerical solution of (5.2.33)-(5.2.34) using the nonstandard method (5.2.16) with* $\varepsilon = 0.01, \Delta t = 0.01$.

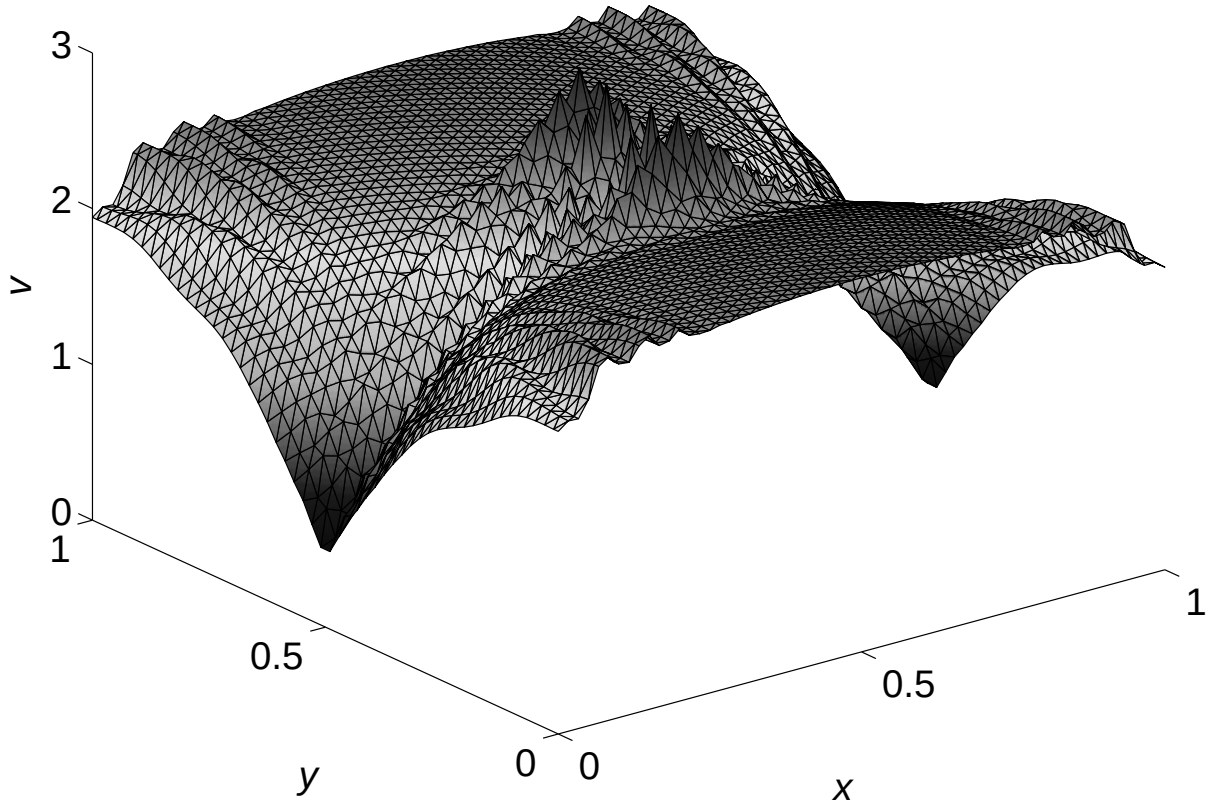

Figure 5.3: *Numerical solution of (5.2.33)-(5.2.34) using the standard Euler time discretization to (5.2.9) with* $\varepsilon = 0.01, \Delta t = 0.01$.

Example 5.2.2 *We consider the same equation as in Example 5.2.1 but with discontinuous initial condition where* $y \in [0,1]$*:*

$$\frac{\partial v}{\partial t} - \sqrt{1 + v_x^2 + v_y^2} = 0, \quad (x,y) \in (0,1) \times (0,1), t > 0 \tag{5.2.35}$$

$$v(x,y,0) = \begin{cases} (1 - 0.1x(1-x))(\cos(2\pi y) - \cos(2\pi y)) & , \quad 0 \leq x < 0.5 \\ (1 - 0.1x(1-x))(\cos(2\pi y) - \cos(2\pi y)) + 0.1 & , \quad 0.5 < x \leq 1. \end{cases} \tag{5.2.36}$$

with periodic boundary conditions.

We consider the same triangulation as in Example 5.2.1. The numerical solution obtained by nonstandard technique with $\varepsilon = 0.01$ and $\Delta t = 0.01$ is presented n Figure 5.4. It is also clear that the monotonicity of numerical solution with respect to the initial value is preserved.

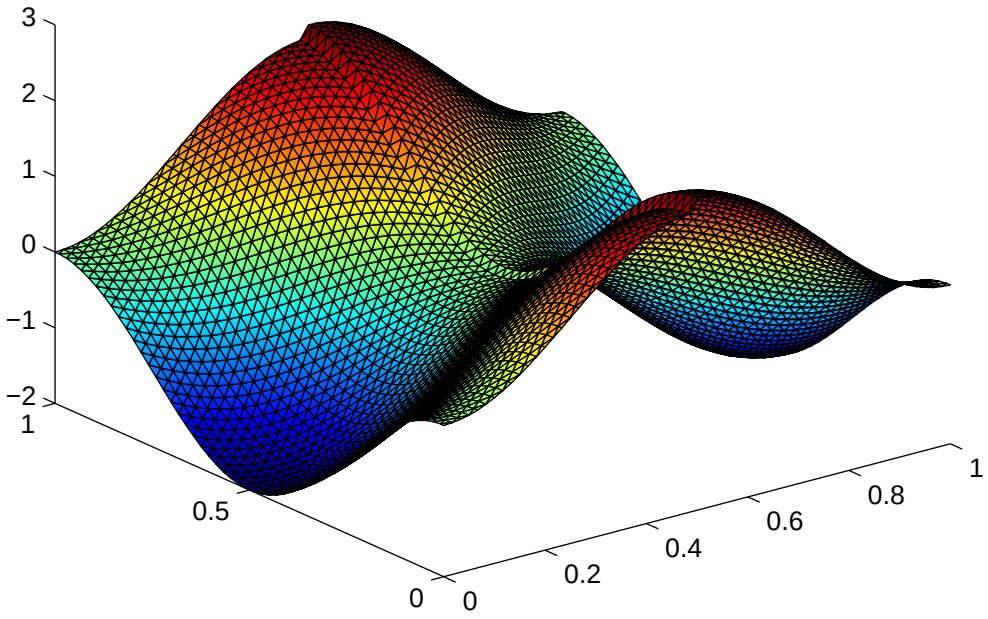

Figure 5.4: *Numerical solution of (5.2.35)-(5.2.36) using the nonstandard method (5.2.16) with* $\varepsilon = 0.01, \Delta t = 0.01$.

Example 5.2.3 *We consider the following equation, where the Hamiltonian is nonconvex function:*

$$\frac{\partial v}{\partial t} + \cos(v_x + v_y + 1) = 0, \quad (x, y) \in (0,1) \times (0,1), t > 0 \tag{5.2.37}$$

$$v(x, y, 0) = \cos(2\pi y) - \cos(2\pi x), \quad (x, y) \in (0,1) \times (0,1) \tag{5.2.38}$$

with periodic boundary conditions.

Once again we consider the same triangulation as an example 5.2.1. The numerical solution obtained by nonstandard technique with $\varepsilon = 0.01$ and $\Delta t = 0.01$ is presented n Figure 5.5. The numerical solution obtained by the standard method with the same value of parameters ε and Δt is presented on Figure 5.6. We can observe that the result by nonstandard scheme has a good resolution.

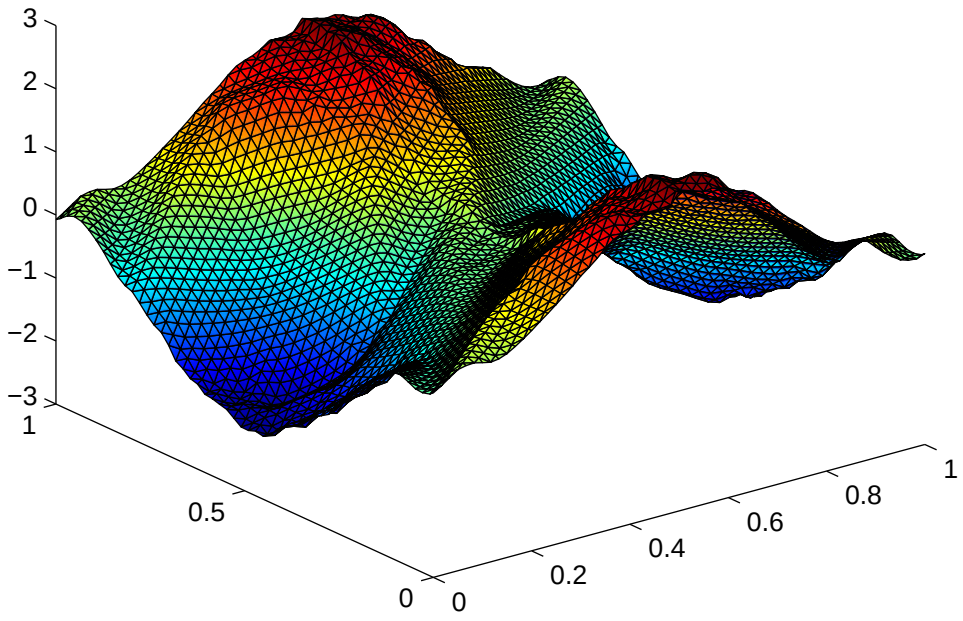

Figure 5.5: *Numerical solution of (5.2.37)-(5.2.38) using the nonstandard method (5.2.16) with* $\varepsilon = 0.01, \Delta t = 0.01$.

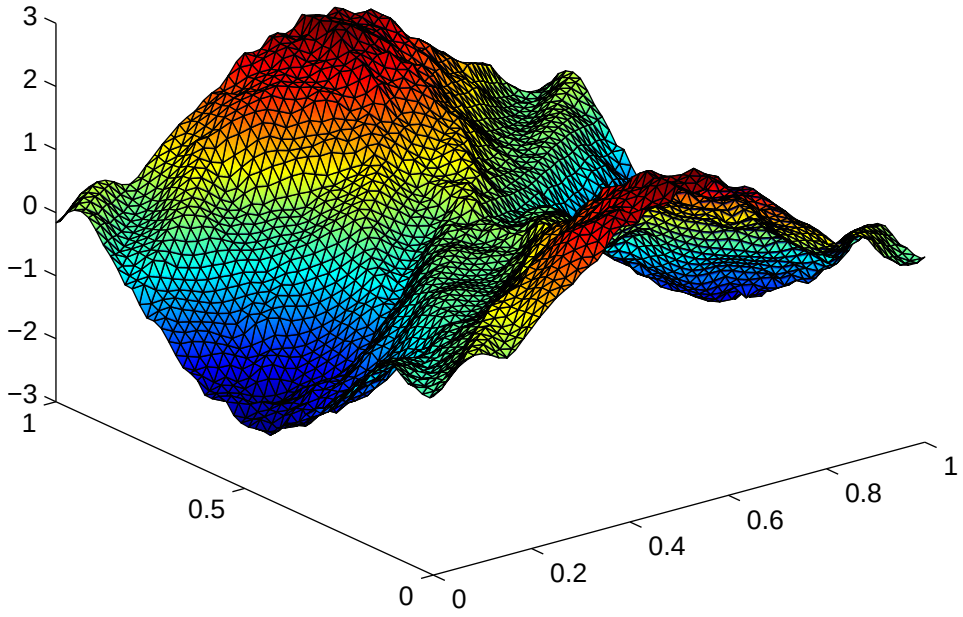

Figure 5.6: *Numerical solution of (5.2.37)-(5.2.38) using the standard Euler time discretization to (5.2.9) with* $\varepsilon = 0.01, \Delta t = 0.01$.

5.3 Total Variation Diminishing Nonstandard Finite Difference Schemes for Conservation Laws

In this section, we consider the Cauchy problem (5.1.3)-(5.1.4) for conservation laws. The equation (5.1.3) describe the behavior of many different physical phenomena including traffic flow [135]. It is well known [90] that the solution of (5.1.3) may become discontinuous as time evolves, even for smooth $v_0(x)$ given in (5.1.4). Thus it turn requires an entropy condition to select the physical relevant discontinuous solutions, called the entropy solutions.

Here, we assume that the data functions f and v_0 are such that equations (5.1.3)-(5.1.4) has unique entropy solution, e.g., f smooth and uniformly convex and $v_0 \in L^\infty(\mathbb{R})$, see [46, Section 3.4]. There is a well known theory regarding the existence and uniqueness of an entropy solution, using special integral structure of the equation (5.1.3), [88].

A very successful class of schemes for solving (5.1.3)-(5.1.4) is the class of total variation diminishing conservative schemes which resolve discontinuities in the solutions without spurious oscillations displayed by numerical solutions. We construct nonstandard implicit and explicit conservative schemes, including Runge-Kutta higher order schemes, which are total variation diminishing without a restriction on the time step size. Moreover, the implicit schemes have the advantages of being computationally simpler.

In Subsection 5.3.1 we give some preliminaries settings and results including the Harten's lemma. In Subsection 5.3.2 we formulate an implicit nonstandard finite difference scheme using nonlocal approximation of nonlinear terms. Subsection 5.3.3 deals with explicit and Runge-Kutta nonstandard finite difference schemes where renormalization of the denominator is used. Numerical results by both implicit and the explicit methods are presented in Subsection 5.3.4. The numerical solution of Hamilton-Jacobi equation in one space dimension obtained using a discontinuous Galerkin method approximating its derivatives using nonstandard TVD method is given in Section 5.3.5.

5.3.1 Preliminaries

For simplicity, we assume that the grid points $\{x_j\}_{j=1}^{\infty}$ are uniformly distributed with the cell size $x_{j+1} - x_j = \Delta x$.

Following this space discretization, the equation for conservation laws (5.1.3) is written as a system of ordinary differential equations of the form

$$w_t = L(w), \tag{5.3.1}$$

where $w = (w_j)$ and $w_j(t) \approx v(x_j, t)$. The operator L in (5.3.1) is obtained from the following spacial discretization

$$(L(w))_j = -\frac{1}{\Delta x}(\hat{f}_{j+\frac{1}{2}} - \hat{f}_{j-\frac{1}{2}}),$$

where $\hat{f}_{j+\frac{1}{2}} = \hat{f}(w_j, w_{j+1})$ is a numerical flux function which is nondecreasing in the first argument and nonincreasing in the second; Lipschitz continuous in all its arguments, and satisfies the consistency condition

$$\hat{f}(\widetilde{w}, \widetilde{w},) = f(\widetilde{w}).$$

Conservative scheme for equation (5.1.3) has the form

$$\frac{d}{dt}w_j = -\frac{1}{\Delta x}(\hat{f}_{j+\frac{1}{2}} - \hat{f}_{j-\frac{1}{2}}). \tag{5.3.2}$$

For choices of numerical flux, we refer to, e.g., [111]. Notice that (5.3.2) is written in a semidiscrete method of lines form, while in practice the time variable t must also be discretized.

Let a mesh $t_n = n\Delta t$, $n = 0, 1, 2, ...$, in the time direction be given. As usual w^n denotes an approximation of w at $t = t_n$. Below we propose a scheme of explicit Euler-type, which is the familiar form of conservative scheme

$$w_j^{n+1} = w_j^n - \frac{\Delta t}{\Delta x}\left(\hat{f}(w_j^n, w_{j+1}^n) - \hat{f}(w_{j-1}^n, w_j^n)\right). \tag{5.3.3}$$

Conservative schemes are especially suitable for computing entropy solutions with shocks, because of the important Lax-Wendroff Theorem [89], which states that solutions to conservative schemes if convergent, would converge to a weak solution of (5.1.3).

The Lax-Wendroff Theorem does not say anything about whether the method converges, only that if a sequence of approximations converges then the limit is a weak solution of (5.1.3). For that, we need some form of strong stability to guarantee convergence. A very successful method which guarantees convergence is total variation diminishing method.

Definition 5.3.1 *A numerical scheme (5.3.3) is called total variation diminishing (TVD) if $TV(w^n)$ is decreasing with respect to n, that is,*

$$TV(w^{n+1}) \leq TV(w^n), \ n = 0, 1, 2, ...,$$

where the total variation of a grid function w^n is defined by

$$TV(w^n) = \sum_{j=-\infty}^{+\infty} |w_{j+1}^n - w_j^n|.$$

The TVD property of numerical methods is often proved by using the Harten's Lemma. We give below a version dealing with both the explicit and the implicit cases [64], [61].

Lemma 5.3.1 *(Harten) Consider the explicit scheme*

$$w_j^{n+1} = w_j^n - \left(-C_{j+\frac{1}{2}}(w_{j+1}^n - w_j^n) + D_{j-\frac{1}{2}}(w_j^n - w_{j-1}^n)\right), \tag{5.3.4}$$

and the implicit scheme

$$w_j^{n+1} = w_j^n - \left(-C_{j+\frac{1}{2}}(w_{j+1}^{n+1} - w_j^{n+1}) + D_{j-\frac{1}{2}}(w_j^{n+1} - w_{j-1}^{n+1})\right), \tag{5.3.5}$$

where $C_{j+\frac{1}{2}}$ and $D_{j-\frac{1}{2}}$ are functions of w^n and /or w^{n+1} at various (usually neighboring) grid points. If $C_{j+\frac{1}{2}} \geq 0$ and $D_{j-\frac{1}{2}} \geq 0$, then the scheme (5.3.5) is TVD. If in addition to these conditions we have $C_{j+\frac{1}{2}} + D_{j+\frac{1}{2}} \leq 1$, then the scheme (5.3.4) is TVD.

5.3.2 Implicit nonstandard schemes by nonlocal approximation

Here, we design nonstandard schemes by exploiting the nonlocal approximation of nonlinear terms as stated in (ii) of Definition 5.1.2. We consider nonlocal approximation of the function L, given in (5.3.1), for deriving nonstandard total variation diminishing schemes for equation (5.1.3). The next stated techniques are discussed in the case when L is obtained from spacial discretization using Lax-Friedrichs numerical flux [90]

$$\hat{f}_{j+\frac{1}{2}} = \frac{1}{2}\left(f(w_{j+1}) + f(w_j) - \alpha(w_{j+1} - w_j)\right), \tag{5.3.6}$$

$$\alpha = \max_w |f'(w)|, \tag{5.3.7}$$

the maximum being taken over the relevant range of w.

We consider nonlocal approximation of the function L for deriving nonstandard TVD schemes for equation (5.1.3). Below, we propose an implicit scheme of Euler type

$$w_j^{n+1} = w_j^n + \frac{\Delta t}{2\Delta x}\left(\alpha(w_{j+1}^{n+1} - 2w_j^{n+1} + w_{j-1}^{n+1}) - (w_{j+1}^{n+1} - w_{j-1}^{n+1})\frac{f(w_{j+1}^n) - f(w_{j-1}^n)}{w_{j+1}^n - w_{j-1}^n}\right). \quad (5.3.8)$$

We should note that the linear terms in (5.3.1) are evaluated at $t = t_{n+1}$. The expression $f(w_{j+1}) - f(w_{j-1})$ is multiplied and divided by $w_{j+1}^n - w_{j-1}^n$, where the multiplier is evaluated at $t = t_{n+1}$ and the remaining part of the expression evaluated at $t = t_n$.

One of our main result in this section is the following theorem:

Theorem 5.3.1 *The scheme* (5.3.8) *is qualitatively stable with respect to the total variation diminishing property (5.1.7).*

Proof. *The scheme (5.3.8) can be written as*

$$w_j^{n+1} = w_j^n + \frac{\Delta t}{2\Delta x}\left(\alpha - \frac{f(w_{j+1}^n) - f(w_{j-1}^n)}{w_{j+1}^n - w_{j-1}^n}\right)(w_{j+1}^{n+1} - w_j^{n+1})$$
$$-\frac{\Delta t}{2\Delta x}\left(\alpha + \frac{f(w_{j+1}^n) - f(w_{j-1}^n)}{w_{j+1}^n - w_{j-1}^n}\right)(w_j^{n+1} - w_{j-1}^{n+1}).$$

Therefore the scheme (5.3.8) can be represented in the form (5.3.5) with

$$C_{j+\frac{1}{2}} = \frac{\Delta t}{2\Delta x}\left(\alpha - \frac{f(w_{j+1}^n) - f(w_{j-1}^n)}{w_{j+1}^n - w_{j-1}^n}\right),\ D_{j-\frac{1}{2}} = \frac{\Delta t}{2\Delta x}\left(\alpha + \frac{f(w_{j+1}^n) - f(w_{j-1}^n)}{w_{j+1}^n - w_{j-1}^n}\right).$$

Using (5.3.7), we obtain that $C_{j+1} \geq 0$ and $D_{j-\frac{1}{2}} \geq 0$ for all j. Hence it follows from Lemma 5.3.1 that the scheme (5.3.8) is TVD. ■

Using standard techniques of numerical analysis [90], one can easily obtain that for linear systems the scheme (5.3.8) is consistent and unconditionally stable. Moreover, the qualitative stability of the scheme (5.3.8) also does not impose any condition on Δx and /or Δt.

We should note that one step in the time dimension requires the solutions of a tridiagonal linear system. Hence the computation effort is similar to the one for explicit methods. Furthermore, the suggested scheme is not unique. One may use a different kind of nonlocal approximation to obtain a TVD scheme. For example, the scheme

$$w_j^{n+1} = w_j^n + \frac{\Delta t}{2\Delta x}\left(\alpha(w_{j+1}^{n+1} - 2w_j^{n+1} + w_{j-1}^{n+1}) - (w_{j+1}^{n+1} - w_j^{n+1})\frac{f(w_{j+1}^n) - f(w_j^n)}{w_{j+1}^n - w_j^n}\right)$$
$$-\frac{\Delta t}{2\Delta x}\left((w_j^{n+1} - w_{j-1}^{n+1})\frac{f(w_j^n) - f(w_{j-1}^n)}{w_j^n - w_{j-1}^n}\right)$$

is also TVD.

5.3.3 Explicit nonstandard schemes by renormalization

The schemes in this subsection are based on the renormalization of the denominator of the discrete derivatives, see Definition 5.1.2 (i). This means that the denominator Δt in the discrete time derivative is replaced by a function $\psi(\Delta t)$ satisfying (5.1.5). In order to obtain elementary stable schemes, that is, schemes which are qualitatively stable with respect to fixed points of the differential equations and their stability, the following renormalization was considered, [103], [8]:

$$\psi(\Delta t) = \frac{\phi(q\Delta t)}{q}, \tag{5.3.9}$$

where the function ϕ is such that

$$\phi(z) \quad = \quad z + o(z^2) \text{ as } z \to 0, \tag{5.3.10}$$

$$0 \quad < \quad \phi(z) < 1 \text{ for } z > 0, \tag{5.3.11}$$

and $q = \max\{|\lambda|\}$, λ tracing the eigenvalues of the Jacobian $J(\tilde{v})$ of the right hand side of equation (5.3.1) at the fixed points $\tilde{v}$ of the equation. The choice of the number q is not so critial. In practice, one may take $q = \max ||J(\tilde{v})||_\infty$, where $||.||_\infty$ is the matrix norm associated with the supremum norm on $\mathbb{R}^n$. We will show that similar renormalization also ensures the TVD property of the scheme. We consider function ψ as given by (5.3.9) where the value of q is suitably determined by the function L.

Let us consider first the Euler scheme

$$\frac{w^{n+1} - w^n}{\psi(\Delta t)} = L(w^n), \tag{5.3.12}$$

where L is also obtained from special discretization using Lax-Friedrichs numerical flux given in (5.3.6)

The second main result in this section is the following theorem.

Theorem 5.3.2 *The scheme (5.3.12) where*

$$\psi(z) = \frac{\phi(\frac{\alpha z}{\Delta x})}{\frac{\alpha}{\Delta x}}, \ z > 0,$$

and ϕ satisfies conditions (5.3.10)-(5.3.11) is qualitatively stable with respect to total variation diminishing property (5.1.7).

Proof. *The scheme (5.3.12) can be written in the form*

$$\begin{aligned} w_j^{n+1} &= w_j^n + \frac{\psi(\Delta t)}{2\Delta x}\left(\alpha(w_{j+1}^n - 2w_j^n + w_{j-1}^n) - f(w_{j+1}^n) + f(w_{j-1}^n)\right) \\ &= w_j^n + \frac{\psi(\Delta t)}{2\Delta x}\left(\alpha - \frac{f(w_{j+1}^n) - f(w_j^n)}{w_{j+1}^n - w_j^n}\right)(w_{j+1}^n - w_j^n) \\ &\quad -\frac{\psi(\Delta t)}{2\Delta x}\left(\alpha + \frac{f(w_j^n) - f(w_{j-1}^n)}{w_j^n - w_{j-1}^n}\right)(w_j^n - w_{j-1}^n). \end{aligned}$$

Therefore (5.3.12) can be represented in the form (5.3.4) with

$$\begin{aligned} C_{j+\frac{1}{2}} &= \frac{\psi(\Delta t)}{2\Delta x}\left(\alpha - \frac{f(w_{j+1}^n) - f(w_j^n)}{w_{j+1}^n - w_j^n}\right) \\ D_{j-\frac{1}{2}} &= \frac{\psi(\Delta t)}{2\Delta x}\left(\alpha + \frac{f(w_j^n) - f(w_{j-1}^n)}{w_j^n - w_{j-1}^n}\right). \end{aligned}$$

Using the definition of α, see (5.3.7), it is easy to see that $C_{j+1} \geq 0$, and $D_{j-\frac{1}{2}} \geq 0$. Furthermore, we have

$$\begin{aligned} C_{j+1} + D_{j+\frac{1}{2}} &= \frac{\psi(\Delta t)}{2\Delta x}\left(\alpha - \frac{f(w_{j+1}^n) - f(w_j^n)}{w_{j+1}^n - w_j^n}\right) + \frac{\psi(\Delta t)}{2\Delta x}\left(\alpha + \frac{f(w_{j+1}^n) - f(w_j^n)}{w_{j+1}^n - w_j^n}\right) \\ &= \frac{\phi(\frac{\alpha\Delta t}{\Delta x})}{\frac{\alpha}{\Delta x}2\Delta x}(2\alpha) = \phi(\frac{\alpha\Delta t}{\Delta x}). \end{aligned}$$

Then it follows from (5.3.11) that $C_{j+1} + D_{j+\frac{1}{2}} \leq 1$. Hence we can apply the Harten's Lemma, see Lemma 5.3.2, and obtain that the scheme (5.3.12) is TVD. ■

Renormalization can also be used in higher order methods, e.g., Runge-Kutta methods. For investigation of TVD properties a Runge-Kutta method is typically written in the so called Shu-Osher form, [48], namely

$$\begin{aligned} y^{(0)} &= w^n \\ y^{(i)} &= \sum_{j=1}^{i-1}(\lambda_{ij}y^{(j)} + \Delta t\mu_{ij}L(y^{(j)})),\ i = 1, 2, ..., m \\ w^{n+1} &= y^{(m)}. \end{aligned}$$

By consistency

$$\sum_{j=0}^{i-1}\lambda_{ij} = 1,\ i = 1, ..., m.$$

Therefore in each intermediate step of the method $y^{(i)}$ is a convex combination of Euler forward operators:

$$y^{(i)} = \sum_{j=1}^{i-1}\lambda_{ij}\left(y^{(j)} + \Delta t\frac{\mu_{ij}}{\lambda_{ij}}L(y^{(j)})\right).$$

If these operators are TVD then the Runge-Kutta method is also TVD [48]. Following the result of Theorem 5.3.2 we will obtain a TVD scheme if the Euler operator involving $y^{(j)}$ above is renormalized by

$$\psi_{ij}(\Delta t) = \frac{\phi(\frac{\alpha\mu_{ij}\Delta t}{\lambda_{ij}\Delta x})}{\frac{\alpha\mu_{ij}}{\lambda_{ij}\Delta x}},$$

where the function ϕ satisfies conditions (5.3.10)-(5.3.11). Note that this function might have to satisfy additional conditions for the scheme to be accurate of particular order. We will illustrate this by an example.

Following the discussion above, the following two stage scheme is TVD:

$$y^{(1)} = w^n + \phi\left(\frac{\alpha\Delta t}{\Delta x}\right)\frac{\Delta x}{\alpha}L(w^n) \tag{5.3.13}$$

$$w^{n+1} = \frac{1}{2}w^n + \frac{1}{2}y^{(1)} + \frac{1}{2}\phi\left(\frac{\alpha\Delta t}{\Delta x}\right)\frac{\Delta x}{\alpha}L(y^{(1)}). \tag{5.3.14}$$

Using standard techniques one can also obtain that it is of order two provided $\phi(z) = z + o(z^3)$.

Remark 5.3.1 *Since the schemes (5.3.8) and (5.3.12) are TVD, the convergence follows from [90, Theorem 15.2].*

5.3.4 Numerical results

We present the results of our numerical experiments for the implicit and explicit nonstandard total variation diminishing schemes considered in Subsection 5.3.2 and Subsection 5.3.3, respectively, to Burger's equation. We compare them with the standard techniques given in [60].

Example 5.3.1 *We apply the schemes considered in Subsection 5.3.2 and Subsection 5.3.3 to the Burger's equation*

$$v_t + \left(\frac{1}{2}v^2\right)_x = 0, \quad -1 \le x \le 9, \quad t > 0. \tag{5.3.15}$$

Here the flux is $f(v) = \frac{1}{2}v^2$.

It is well known [90] that the entropy solution of equation (5.3.15) develops discontinuities (shocks) even for smooth initial condition. To simplify the matters we take the Riemann initial data

$$v(x,0) = \begin{cases} 1.2, & -1 \le x < 0 \\ 0, & 0 \le x \le 9. \end{cases} \tag{5.3.16}$$

The entropy solution of the problem (5.3.15)-(5.3.16) is given by

$$v(x,t) = \begin{cases} 1.2 & , \quad x < 0.6t \\ 0 & , \quad x \ge 0.6t. \end{cases}$$

For the considered problem (5.3.15)-(5.3.16), after some obvious transformations, the scheme (5.3.8) can be written in the form

$$w_j^{n+1} = w_j^n + \frac{\Delta t}{2\Delta x}\left(\alpha(w_{j+1}^{n+1} - 2w_j^{n+1} + w_{j-1}^{n+1}) - \frac{1}{2}(w_{j+1}^{n+1} - w_{j-1}^{n+1})(w_{j+1}^n + w_{j-1}^n)\right). \tag{5.3.17}$$

Below, the solid line is the exact solution $v(x,t)$, the points joined by a dashed line are numerical solutions. It was shown in [61] that non TVD methods typically produce oscillations around the points of discontinuity. Figure 5.7 shows such oscillations around the point $x = 0$ produced by the standard Euler method applied to problem (5.3.15)-(5.3.16). Figures 5.8, 5.9 and 5.10 show the numerical solution of (5.3.15)-(5.3.16) obtained by nonstandard implicit scheme (5.3.17) for various time steps Δt. One can observe that while an increase in Δt affects the accuracy of the solution it nevertheless remains TVD and free of spurious oscillations.

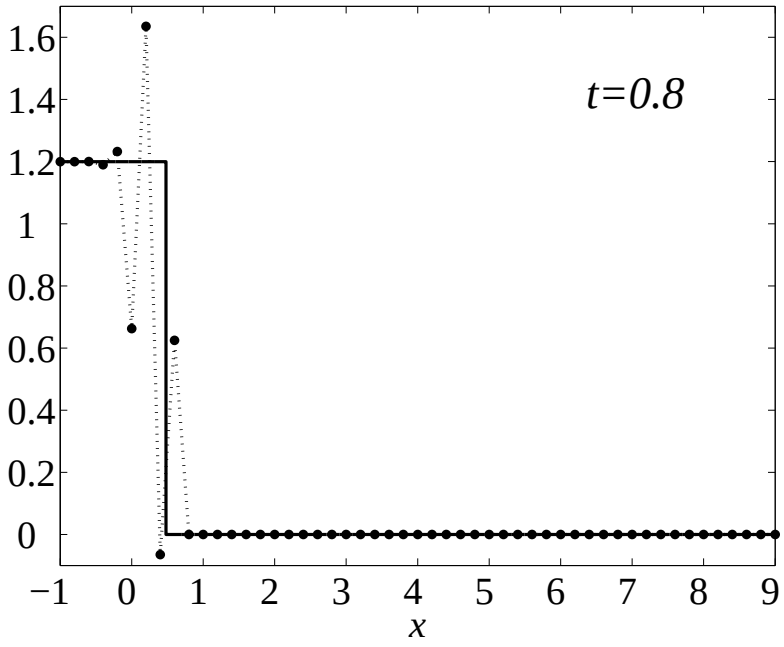

Figure 5.7: *Numerical solution of (5.3.15)-(5.3.16) by the standard explicit Euler method (5.3.3) with* $\Delta x = \Delta t = 0.2$.

Similar results are obtained using the explicit schemes (5.3.12). For the considered problem (5.3.15)-(5.3.16), Euler's method (5.3.12) can be written as

$$w_j^{n+1} = w_j^n + \frac{1}{2\alpha}\phi\left(\frac{\alpha\Delta t}{\Delta x}\right)\left(\alpha(w_{j+1}^n - 2w_j^n + w_{j-1}^n) - (w_{j+1}^n)^2 + (w_{j-1}^n)^2\right), \quad (5.3.18)$$

where we take $\phi(z) = 1 - e^{-z}$. *The numerical solution given by the scheme (5.3.18) computed with* $\Delta x = \Delta t = 0.2$ *is presented on Figure 5.11. Figure 5.12 represents the solution produced by the Runge-Kutta method (5.3.13)-(5.3.14) with renormalizing function* $\phi(z) = \frac{1-e^{-z^2}}{z}$ *so that the method is of order two. Let us note that since the exact solution is discontinuous, a higher order method does not necessarily give a better approximation. Naturally, the accuracy can be improved by decreasing the step sizes. However, the major point here is that irrespective of the step sizes the numerical solutions is free of spurious oscillations and its total variation does not increase with time, for this particular equation it is in fact constant.*

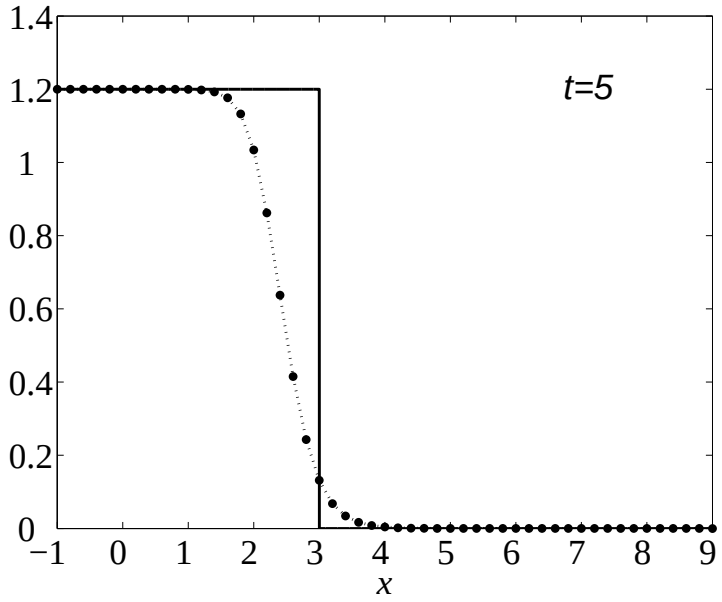

Figure 5.8: *Numerical solution of (5.3.15)-(5.3.16) by the implicit nonstandard scheme (5.3.17) with* $\Delta x = \Delta t = 0.2$.

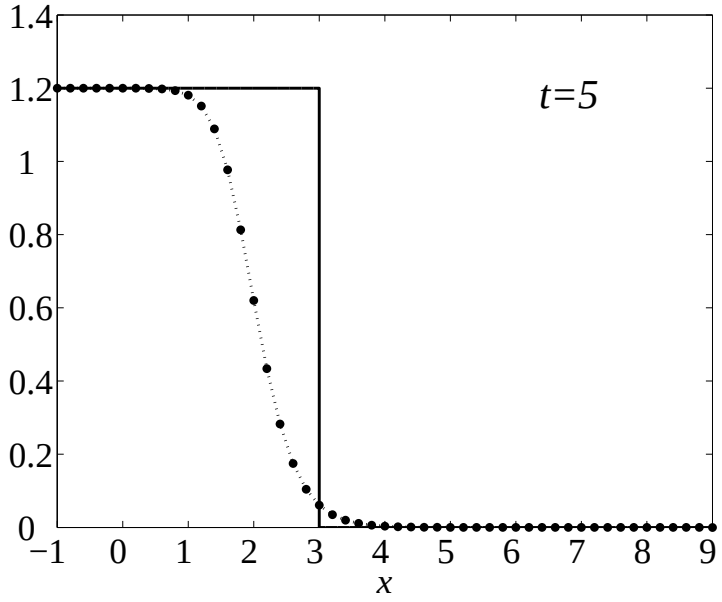

Figure 5.9: *Numerical solution of (5.3.15)-(5.3.16) by the implicit nonstandard scheme (5.3.17) with* $\Delta x = 0.2$, $\Delta t = 0.5$.

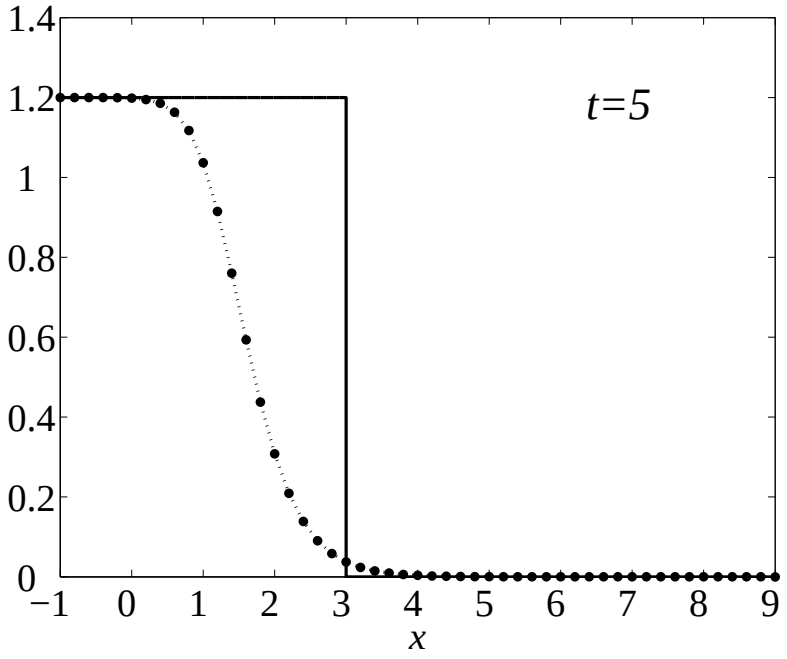

Figure 5.10: *Numerical solution of (5.3.15)-(5.3.16) by the implicit nonstandard scheme (5.3.17) with* $\Delta x = 0.2$, $\Delta t = 1.0$.

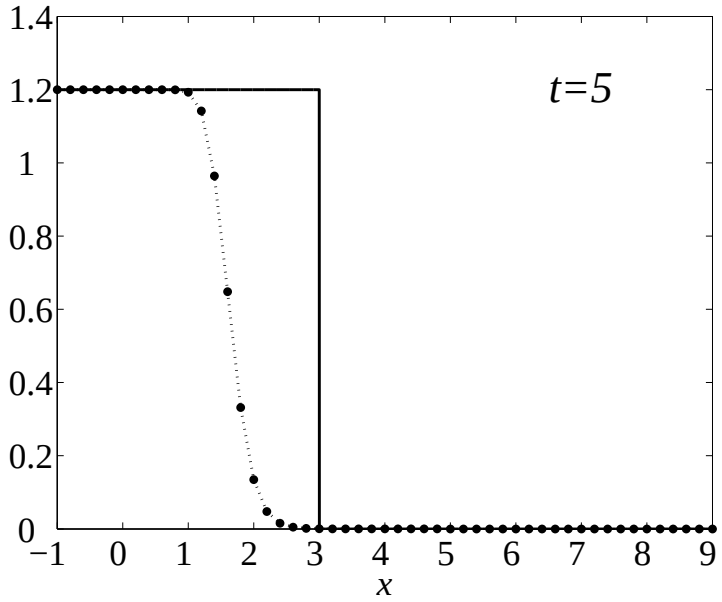

Figure 5.11: *Numerical solution of (5.3.15)-(5.3.16) by the explicit nonstandard scheme (5.3.18) with* $\Delta x = \Delta t = 0.2$.

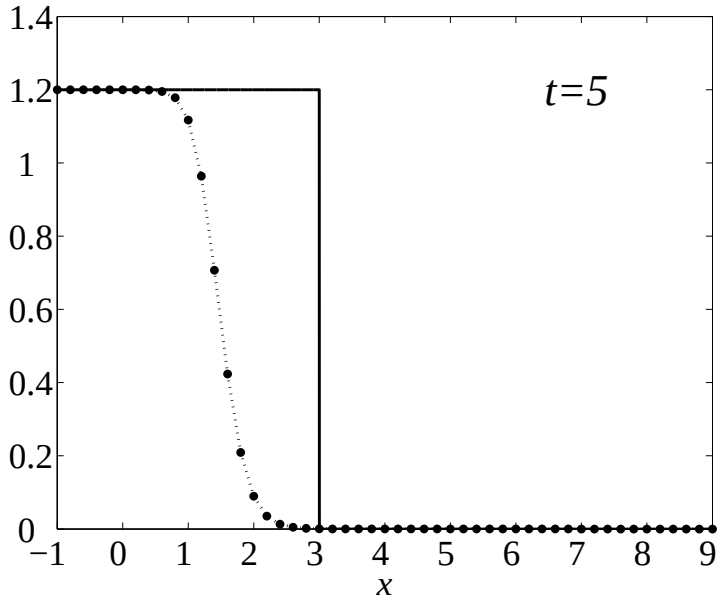

Figure 5.12: *Numerical solution of (5.3.15)-(5.3.16) by the Runge-Kutta nonstandard method (5.3.13)-(5.3.14) with* $\Delta x = \Delta t = 0.2$

5.3.5 Numerical solution of Hamilton-Jacobi equation via TVD method for conservation laws

Consider the one-space dimensional Dirichlet problem for Hamilton-Jacobi equation:

$$u_t + H(u_x) = 0 \tag{5.3.19}$$

$$u(x,0) = u_0(x), \tag{5.3.20}$$

where H is smooth function of u_x. Differentiate (5.3.19) with respect to x and let $v = u_x$. Then a problem (5.3.19)-(5.3.20) is equivalent to the following problem for conservation laws:

$$v_t + H(v)_x = 0 \tag{5.3.21}$$

$$v(x,0) = \frac{d}{dx}u_0(x). \tag{5.3.22}$$

Thus, the question of viscosity solution of (5.3.19) is equivalent to the question of entropy solution of (5.3.21). More precisely, if u is the unique viscosity solution of (5.3.19) satisfying (5.3.20), then $v = u_x$ is the unique entropy solution of (5.3.21) satisfying (5.3.22). Conversely, if v is the unique entropy solution of (5.3.21) satisfying (5.3.22), then u defined by

$$u(x,t) = \int^x v(y,t)dy$$

is the unique viscosity solution of (5.3.19) satisfying (5.3.20). Furthermore, if u_0 is bounded uniformly continuous function in $\mathbb{R}$, then $u_x(x,t) = v(x,t)$ is satisfied almost everywhere [77].

We recall that this equivalence has been exploited by many authors trying to translate successful numerical methods for conservation laws to methods for the Hamilton-Jacobi equations.

If we want to solve the problem (5.3.19)-(5.3.20) on the finite interval $[a,b]$, first we divide it into N cells as follows:

$$a = x_{\frac{1}{2}} < x_{\frac{3}{2}} < ... < x_{N+\frac{1}{2}} = b.$$

We denote

$$I_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}),\ x_j = \frac{1}{2}(x_{j-\frac{1}{2}} + x_{j+\frac{1}{2}}),\ \Delta x_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}},\ h = \max_j \Delta x_j,\ j = 1,2,...,N.$$

Now, we define the following approximation space

$$V_h^k = \{w : w|_{I_j} \in P^k(I_j), \quad j = 1, ..., N\},$$

where $P^k(I_j)$ is the set of all polynomials of degree $\leq k$ on the cell I_j.

A kth-order discontinuous Galerkin scheme for the problem (5.3.19)-(5.3.20) can be defined as follows [70]: find $u \in V_h^k$, such that

$$\int_{I_j} \frac{d}{dt}(u_x) f \, dx - \int_{I_j} H(u_x) f_x \, dx + \hat{H}_{j+\frac{1}{2}} f^-_{j+\frac{1}{2}} - \hat{H}_{j-\frac{1}{2}} f^+_{j-\frac{1}{2}} = 0, \quad j = 1, ..., N \quad (5.3.23)$$

holds for any $f \in V_h^{k-1}$. Here

$$\hat{H}_{j+\frac{1}{2}} = \hat{H}\left((u_x)^-_{j+\frac{1}{2}}, (u_x)^+_{j+\frac{1}{2}}\right)$$

is a monotone flux, $(u_x)^{\pm}_{j+\frac{1}{2}}$ and $f^{\pm}$ are the numerical approximations, respectively, to the point values of $u_x(x_{j+\frac{1}{2}}, t)$, and $f(x_{j+\frac{1}{2}})$ from left and right and $\alpha = \max_v |H'(v)|$ with maximum taken over the range covered by $(u_x)^-_{j+\frac{1}{2}}$ and $(u_x)^+_{j+\frac{1}{2}}$. We will mainly use the Lax-Friedrichs flux

$$\hat{H}\left((u_x)^-_{j+\frac{1}{2}}, (u_x)^+_{j+\frac{1}{2}}\right) = \frac{1}{2}\left(H((u_x)^-_{j+\frac{1}{2}}) + H((u_x)^+_{j+\frac{1}{2}}) - \alpha\left((u_x)^+_{j+\frac{1}{2}} - (u_x)^-_{j+\frac{1}{2}}\right)\right). \quad (5.3.24)$$

Notice that the method described above is exactly the discontinuous Galerkin method for the conservation law equation (5.3.21) satisfied by the derivative $v = u_x$, see [70]. This only determines u for each element up to a constant, since it is only a scheme for u_x. In [70], the missing constant can be obtained in one of the following two ways:
The first way is to require that

$$\int_{I_j} (u_t + H(u_x)) f \, dx = 0, \quad j = 1, ..., N,$$

for all $f \in V_h^0$, that is

$$\int_{I_j} (u_t + H(u_x)) \, dx = 0, \quad j = 1, ..., N. \quad (5.3.25)$$

The second one is to use

$$u(x_j, t) = u(x_1, t) + \int_{x_1}^{x_j} u_x(x, t) \, dx \quad (5.3.26)$$

to determine the missing constant for the cell I_j.

About the stability of the method proposed above, we can quote the following result of Jiang and Shu [73]. Here we assume compact support or periodic boundary condition for the solution u.

Lemma 5.3.2 [73] *The following L^2-stability result for the derivative u_x holds for the discontinuous Galerkin method (5.3.23), for any order of accuracy k applied to problem (5.3.19)-(5.3.20):*

$$\frac{d}{dt}\int_a^b u_x^2\,dx \leq 0. \tag{5.3.27}$$

It is clear that the function $\varphi(t) = \int_a^b u_x^2(x,t)dx$ is non-increasing in t. The relation (5.3.27) implies total variation bounded (TVB) property for the numerical solution of u:

$$\begin{aligned} TV(u) = \int_a^b |u_x(x,t)|\,dx &\leq \sqrt{b-a}\sqrt{\int_a^b (u_x(x,t))^2\,dx} \\ &\leq \sqrt{b-a}\sqrt{\int_a^b \left(\frac{d}{dx}u_0(x)\right)^2 dx}. \end{aligned}$$

This is a rather strong stability result, considering that it applies even if the derivative of the solution u_x develops discontinuities, and the scheme (5.3.23) can be of arbitrary high order in accuracy. It also implies convergence of at least a subsequences of the numerical solution u when $h \to 0$. However, this stability result is not strong enough to imply that the limit solution is the viscosity solution of (5.3.19). However, in case when $k = 1$, the scheme (5.3.23) becomes

$$\frac{d}{dt}u_x(x_j,t) = -\frac{1}{\Delta x_j}\left(\hat{H}_{j+\frac{1}{2}} - \hat{H}_{j-\frac{1}{2}}\right) \tag{5.3.28}$$

which is the conservative scheme for $v = u_x$. We know [89] that solutions of such schemes, if convergent, would converge to an entropy solutions of (5.3.21). Moreover, the numerical solution for u obtained from the relation $v = u_x$ would converge to a viscosity solution of (5.3.19). Here we consider only the case when $k = 1$.

We adopt a local orthogonal basis of V_h^1 over I_j, namely, $\{\phi_0^{(j)}, \phi_1^{(j)}\}$, where $\phi_0^{(j)}(x) = 1$ and $\phi_1^{(j)} = x - x_j, x \in I_j$. The solution $u(x,t) \in V_h^1$ of (5.3.19)-(5.3.20) can be written as

$$u(x,t) = w_j(t) + v_j(t)(x - x_j), \quad x \in I_j, \quad j = 1, ..., N. \tag{5.3.29}$$

From (5.3.29), it is clear that

$$\frac{\partial}{\partial x}u(x,t) = v_j(t) \tag{5.3.30}$$

and

$$u(x_j,t) = w_j(t).$$

Substituting u given by (5.3.29) into (5.3.28) and remember (5.3.30) and (5.3.24), we have the following scheme

$$\frac{d}{dt}v_j(t) = -\frac{1}{2\Delta x_j}\left(H(v_{j+1}(t)) - H(v_{j-1}(t)) - \alpha(v_{j+1}(t) - 2v_j(t) + v_{j-1}(t))\right). \tag{5.3.31}$$

The scheme (5.3.31) is the same as scheme given by (5.3.2) in Subsection 5.3.2 with Lax-Friedrichs flux given by (5.3.24) for the conservation law (5.3.21) satisfied by the derivatives $v = u_x$.

The function $w_j(t)$ (missing constant) is obtained from (5.3.25) as follows

$$\int_{I_j}\left(\frac{d}{dt}\left(w_j(t) + v_j(t)(x - x_j)\right) + H(v_j(t))\right)dx = 0, \quad j = 1, ..., N.$$

Since $\int_{I_j}(x - x_j)\,dx = 0$, the above equation gives

$$\frac{d}{dt}w_j(t) = -H(v_j(t)). \tag{5.3.32}$$

For the time discretization, let Δt be the constant time step and v_j^n, w_j^n, denote, respectively, the approximation solutions v_j and w_j at time $t = n\Delta t$. The way for finding the function $u(x,t)$ can be carried out in the following steps:
Firstly, we evaluate v_j^n in every I_j, using the nonstandard total variation diminishing methods for (5.3.21)-(5.3.22) developed in Section 5.5.4, namely

$$v_j^{n+1} = v_j^n - \frac{1}{2\alpha}\phi\left(\frac{\alpha\Delta t}{\Delta x_j}\right)\left(H(v_{j+1}^n) - H(v_{j-1}^n) - \alpha(v_{j+1}^n - 2v_j^n + v_{j-1}^n)\right),$$

where $\phi(z) = 1 - e^{-z}$.
Secondly, w_j^n is given by average of two solutions given by (5.3.32) and (5.3.26).
Finally, we update $u(x, n\Delta t) = w_j^n + v_j^n(x - x_j), x \in I_j$.

Next, we provide numerical experimental result to demonstrate the behavior of our scheme.

Example 5.3.2 *Consider the initial-boundary value problem.*

$$\begin{cases} u_t + \frac{(u_x)^2}{2} = 0 & , \quad 0 \le x \le 2\pi, t > 0 \\ u(x,0) = \sin(x) & , \quad 0 \le x \le 2\pi \\ u(0,t) = u(2\pi,t) & , \quad t \ge 0. \end{cases} \tag{5.3.33}$$

Here, the Lax-Friedrichs flux given by (5.3.6) and uniform meshes of $N = 70$ elements are used. The exact solution when u is still smooth is obtained by the characteristic methods. First solve x_0 from $x = x_0 + \cos(x_0)t$ then get u as $u(x,t) = \sin(x_0) + \frac{(\cos(x_0))^2}{2}t$. In Figure 5.13, the asterisks joined by lines is numerical solution at $t = 2$ with $\Delta t = 0.01, \Delta x = 2\pi/70$ for the P^1 case while the solid line is exact solution. It is clear that at $t = 2$ the exact solution develops a discontinuous derivative and the numerical solution approximates the viscosity solution very well.

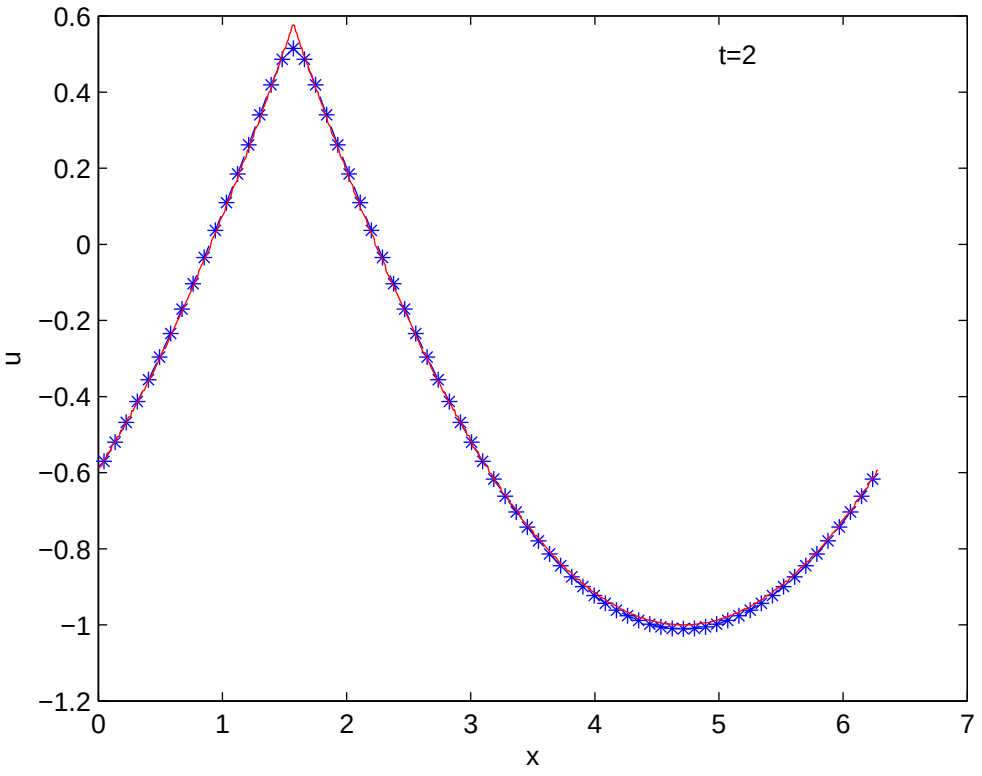

Figure 5.13: *Numerical solution of (5.3.33) with $\Delta x = \frac{2\pi}{70}$ and $\Delta t = 0.01$.*

Chapter 6

Conclusion

The Hausdorff continuous functions, being a particular class of interval valued functions, belong to what is usually called Interval Analysis, see [107]. Nevertheless, recent results have shown that they can provide exact solutions to problems formulated in terms of point valued functions [2]. A long outstanding problem related to the Dedekind order completion of spaces $C(X)$ of real valued continuous functions on rather arbitrary topological spaces X was solved through Hausdorff continuous [4]. Following this breakthrough, a significant improvement of the regularity properties of the solutions of large classes of nonlinear partial differential equations obtained through the Dedekind order completion method, see [109], was discussed in [12, 13]. Namely, it was shown that these solutions can be assimilated with the class of Hausdorff continuous functions on the open domains Ω. The applications of the class of Hausdorff continuous functions [12, 13, 10] show that this class may play an important role in what is typically called Real Analysis. In particular, one may note that one of the main engines behind the development of the various spaces in Real and Abstract Analysis are the partial differential equations with the need to assimilate the various types of "weak" solutions. Moreover, the set of Hausdorff continuous functions might be a viable alternative to some of the presently used functions (e.g. Lebesgue spaces, Sobolev spaces) with the advantage of being both more regular and universal.

In this work, the Hausdorff continuous functions are linked with the concept of discontinuous viscosity solutions. As shown in the Chapter 1 the definition of viscosity solution, see Definition 1.3.3, has an implicit interval character since it places requirements only on the upper semicontinuous envelope $S(u)$ and the lower semicontinuous envelope $I(u)$.

For a Hausdorff continuous viscosity solution u, the functions $I(u)$ and $S(u)$ are as close as they can be in the sense of the distance ρ defined by (2.3.6), namely, we have $\rho(I(u), S(u)) = 0$.

Hence, the requirements that a viscosity solution is Hausdorff continuous has a direct interpretation, which we find clearer than the requirements related to other concepts of discontinuous viscosity solutions, e.g., envelope solutions.

In the theoretical study of this thesis, we come up with the following main results. First, we define the concept of viscosity solution for interval valued functions in $\mathbb{H}(\Omega)$. We proved an existence theorem for Hausdorff continuous viscosity solution using Perron's method. The solution is constructed as a supremum of a subset of viscosity subsolutions in the set of Hausdorff continuous functions. It is shown that there is the relation between the Hausdorff continuous viscosity solutions and the existing theory of discontinuous viscosity solutions. Namely, any H-continuous viscosity solution is discontinuous viscosity solution as defined by Ishii, and it is typically also an envelope viscosity solution. Moreover, the H-continuous viscosity solutions is stronger concept than the concept of discontinuous viscosity solution given by Ishii and as well as under wild the concept of envelope viscosity solution. Uniqueness result have been shown using comparison principle between H-continuous viscosity subsolutions and supersolutions. This comparison principle is stronger than comparison principle used in connection with the lower semicontinuous viscosity supersolutions and upper semicontinuous viscosity subsolutions in the sense that the existence result holds under the same conditions. Sufficient conditions for a weaker form of this comparison principle are given. However, it could be interesting to give sufficient conditions for a strong comparison principle for Hausdorff continuous viscosity subsolutions and supersolutions. Finally, we expressed the Hausdorff viscosity solution of Hamilton-Jacobi equations as solutions to an operator equation involving the extended a Hamiltonian operator in the same way as the classical solution of Hamilton-Jacobi equations are solutions of operator equation associated of this Hamilton-Jacobi equations. It is shown also that the value function of discounted minimum time problem is an envelope viscosity solution of associated Hamilton-Jacobi-Bellman equation.

Numerical study deals with two approaches to numerical solutions for Hamilton-Jacobi equations. First approach is a finite difference space discretization coupled with a non-standard difference time discretization for constructing monotone scheme for any time step size, see Section 5.2.

This is motivated by the paper [91], where a severe restriction on the time step size is imposed for the numerical scheme for Hamilton-Jacobi equations obtained through the coupling of the finite difference method (in space) and the finite difference method (in time) to be monotone.

We have relaxed this restriction by using Micken's nonstandard finite difference method [103]. More precisely, Micken's rule of nonlocal approximation is exploited and this leads to a nonstandard scheme that replicates the monotonicity property of the Hamilton-Jacobi equations for all positive step sizes. Furthermore, the superiority of the nonstandard method to the standard one is confirmed by numerical results.

The second approach is on total variation diminishing scheme for conservation laws. The schemes preserving the essential physical property of diminishing total variation for conservation laws are studied in Section 5.3. Such schemes are free of spurious oscillations around discontinuities. We have discussed nonstandard finite difference schemes, which have this qualitative stability property. We used Micken's rules of approximating nonlinear terms in a nonlocal way and of renormalizing denominators. The obtained schemes are computationally simple. Furthermore, the require no restriction on the time step-size as typical for qualitatively stable nonstandard schemes.

We exploited the fact that in one space dimension the derivatives of the solutions u of Hamilton-Jacobi equations satisfy conservation laws and applied the discontinuous Galerkin method of Hu and Shu [70] to get the scheme for u_x. This determines u for each element up to a constant, since it is only the scheme for u_x. The missing constant is obtained combining two ways developed in [70].

We think that the results presented in this thesis provide a foundation for future research in the following areas:

• Discontinuous viscosity solution of second order Hamilton-Jacobi equation in the sense of Hausdorff continuous function;

• the validity of Theorem 3.6.4 in $\mathbb{R}^n, n \geq 2$;

• TVD property for conservation laws in higher dimensional space;

• conditions implying strong comparison principle between H-continuous viscosity subsolutions and supersolutions.

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