Chapter 5

Nonstandard Finite Difference Methods for Solutions of Hamilton-Jacobi Equations and Conservation Laws

5.1 Introduction

A major difficulty in the study of partial differential equations is, in general, the lack of exact analytical solutions. One way to proceed is to use numerical integration techniques to obtain useful information on the possible solution behaviors. A popular and important method is one based on the use of finite differences to construct discrete models of the differential equations of interest [54], [108].

A relevant question concerns stability. For problems with smooth solutions, usually a linear stability analysis is adequate. For problems with discontinuous solutions or discontinuous derivatives of solutions, a stronger measure of stability is usually required.

Almost all of the standard procedures yield schemes which are convergent with restriction on the step size. One response to this situation was the initiation by Mickens [103] of a research program for the investigation of new methods for constructing finite difference schemes which are convergent for any step size. These new procedures are called nonstandard difference methods, [101, 102, 104].

Throughout this chapter, we shall be concerned with two initial value problems. The fist problem is a Cauchy problem for Hamilton-Jacobi equation in the form

$$u_t(x,t) + H(Du(x,t)) = 0, (x,t) \in \mathbb{R}^n \times (0,+\infty) \quad (5.1.1)$$

$$u(x,0) = u_0(x), x \in \mathbb{R}^n. \quad (5.1.2)$$

The second problem is a Cauchy problem for conservation laws in the form

$$v_t(x,t) + (f(v(x,t))_x = 0, (x,t) \in \mathbb{R} \times (0,\infty) \quad (5.1.3)$$

$$v(x,0) = v_0, x \in \mathbb{R}, \quad (5.1.4)$$

where $f(v)$ is the nonlinear flux function.

Note that the equation (5.1.1) is closely related to equation (5.1.3), in fact in one dimension space, they are equivalent if one takes $v = u_x$. Thus the solution v to conservation laws is the derivative of a solution u to a Hamilton-Jacobi equations. Conversely, the solution u to a Hamilton-Jacobi equation is the integral of a solution v to conservation laws, see [77].

As typical for partial differential equations, problem (5.1.1)-(5.1.2) or (5.1.3)-(5.1.4) cannot be completely solved by analytic techniques. Consequently, numerical simulations are of fundamental importance in gaining some useful insights on the solutions. More precisely, it is crucial to design numerical methods, which replicate essential physical properties of the solutions, see [103], [104].

The precise way in which the properties are preserved is contained in the following definition of qualitative stability [8].

Definition 5.1.1 *Assume that the solution of (5.1.1)-(5.1.2) (resp. (5.1.3)-(5.1.4)) satisfies some property (P). A numerical method approximating (5.1.1)-(5.1.2) (resp. (5.1.3)-(5.1.4)) is called qualitatively stable with respect to (P) or P-stable if the numerical solutions for (5.1.1)-(5.1.2) (resp. (5.1.3)-(5.1.4)) satisfy property (P) for all values of the involved step sizes.*

It should be noted that standard finite difference schemes are generally not qualitatively stable with respect to essential physical properties of solution of interested problem [96].

The nonstandard finite difference methods introduced by R. E. Mickens in the late 1980s appear to be powerful in designing qualitatively stable schemes.

A formal definition is as follows [8]:

Definition 5.1.2 *A finite difference method for (5.1.1)-(5.1.2) or (5.1.3)-(5.1.4) is called nonstandard if at least one of the following is met*
(i) in the discrete derivatives the traditional denominator Δt *or* Δx *is replaced by a nonnegative function* $\psi(\Delta t)$ *or* $\psi(\Delta x)$ *such that*

$$\psi(z) = z + o(z^2), \text{ as } 0 < z \to 0; \tag{5.1.5}$$

(ii) nonlinear terms are approximated in a nonlocal way, i.e., by a suitable function of several points of the mesh.

Note that Mickens [103] set five rules for the construction of discrete models that have the capability to replicate the properties of the exact solution. The general rules for constructing such schemes are not precisely known at the present time, consequently, there exists a certain level of ambiguity in the practical implementation of nonstandard procedures to the formulation of finite difference schemes of differential equations. Here, Definition 5.1.2 of a nonstandard finite difference scheme is stated unambiguously making use of only two of Mickens rules. There: the renormalization of the denominator of the discrete derivative (part (i)) and the nonlocal approximation of nonlinear terms in the data (part (ii)). The other rules are expressed in terms of Definition 5.1.1.

One of the main advantages of the nonstandard finite difference method that in addition to the usual properties of consistency, stability and hence convergence, it produces numerical solutions which also exhibit essential properties of solution.

The following properties have received extensive attention in the design of qualitatively stable nonstandard finite difference schemes: fixed points and their stability [103], [8], oscillatory, conservation of energy, positivity and boundedness [104], [7], dissipation or dispersion, [105], etc.

This chapter is concerned with two physical properties, namely, monotonicity property and total variation diminishing property. The total variation diminishing property has not yet exploited in the context of the nonstandard finite difference method by the authors' best knowledge.

It is well known, see e.g. [40], that the solution of problem (5.1.1)-(5.1.2) depends monotonically on the initial value, that is, for any two solutions $u_1(x,t)$, $u_2(x,t)$ of (5.1.1),

$$u_1(x,0) \leq u_2(x,0),\ \forall\, x \Rightarrow u_1(x,t) \leq u_2(x,t),\ \forall\, t > 0,\ \forall\, x. \tag{5.1.6}$$

The property (5.1.6) is important from the physical point of view. One of the purposes of this chapter is to design monotone numerical schemes, that is, those that replicate this property. In [40], Crandall and Lions first studied the convergence of monotone scheme for Hamilton-Jacobi equation (5.1.1). They presented finite difference schemes on rectangular meshes. These methods are difficult to apply for complicated geometry where adaptive mesh refinement is often required. Finite volume and finite element schemes based on arbitrary triangulation, are thus attractive such cases. Monotone schemes on unstructured meshes for Hamilton-Jacobi equations were studied by Abgrall [1], Barth and Sethian [25], Kossioris et al. [80] and Li et al. [91]. Our general approach is along the lines of the many works, and specifically [91], where the finite element scheme discretization is coupled with the finite difference time discretization. However, we use the Mickens' nonstandard variant of the difference approach. The schemes employing standard finite difference technique are monotone under restrictive conditions on the time step size. On the contrary, the nonstandard finite difference scheme presented in this chapter preserves the monotonicity property unconditionally, improving therefore the results of [91].

In Section 5.2, firstly, we consider a space discretization of equation (5.1.1) using the finite element method. Secondly, we construct a nonstandard finite difference scheme for obtained system of ordinary differential equations. Finally, the convergence of this new scheme is proved and numerical results supporting the theory are presented.

The entropy solution $v(x,t)$ of (5.1.3)-(5.1.4) satisfies the property that the total variation with respect to x does not increase as t increases [123, Chapter 16], [69, Chapter 2]:

$$TV(v(x,t_1)) \geq TV(v(x,t_2)) \text{ for } 0 < t_1 \leq t_2, \tag{5.1.7}$$

where the total variation of a function $v(x,t)$ with respect to an one dimensional variable x is defined as

$$TV(v(.,t)) = \lim_{h\to 0} \frac{1}{h} \int_{-\infty}^{+\infty} |v(x+h,t) - v(x,t)|\, dx.$$

It is clear that $TV(v)$ is finite for any bounded increasing or decreasing function with respect to x, including functions with jump discontinuities. Moreover, if v is differentiable, then $TV(v)$ reduces to $TV(v) = \int_{-\infty}^{+\infty} |v_x|\, dx$.

We discuss here finite difference schemes which produce numerical solutions with diminishing total variation. The importance of schemes preserving the property (5.1.7) can not be overemphasized since it is an essential physical characteristic of the exact solution. Furthermore, it has been shown that schemes with such qualitative stability have the advantage of high-order accuracy in smooth regions while resolving discontinuities in the solutions without spurious oscillations which are often displayed by numerical solutions [124], [125, 126]. In [124], they are called total variation diminishing schemes. The preservation of the diminishing total variation property is also discussed in [61] within the context of the more general concept of strong stability.

One problem associated with the explicit total variation diminishing methods is a restriction on the time step-size which in some cases could be rather severe. This is particulary pronounced in high order methods, e.g., methods of Runge-Kutta type [60], [48]. On the other hand, the computational complexity of total variation diminishing implicit methods is significantly higher particularly when nonlinear functions are involved.

Following space discretization and time discretization, respectively, we impose our numerical method for (5.1.3) to be in conservation form. Our approach is to use the tools of the nonstandard finite difference method in constructing total variation diminishing schemes which have the advantages of being computationally simpler (in the case of implicit schemes) and have no step size restriction (in the case of explicit schemes).

Section 5.3 deals with total variation diminishing nonstandard finite difference schemes for conservation laws. We formulate an implicit nonstandard finite difference scheme using nonlocal approximations of nonlinear terms and explicit nonstandard finite difference schemes where renormalization of the denominator is used. Numerical results by both the implicit and explicit methods are presented in this section. At the end of this section, we use a discontinuous Galerkin finite difference method proposed by Hu and Shu [70] to solve the one dimensional Hamilton-Jacobi equation. They use the fact that the derivatives of the solution u of Hamilton-Jacobi equation satisfy a conservation laws, and apply the usual discontinuous Galerkin method on this conservation laws to advance the derivatives of u. Here, the solution u is recovered from these derivatives computed using nonstandard TVD method for conservation laws developed in Subsection 5.3.4. This will determine u up to a constant. The missing constant is obtained using combination of two ways given in [70].

5.2 A Monotone Scheme for Hamilton-Jacobi Equations via the Nonstandard Finite Difference Method

For simplicity, we consider the problem (5.1.1)-(5.1.2) in two space dimensions although a generalization to arbitrary space dimension is possible. It is well known that a problem (5.1.1)-(5.1.2) does not have classical solutions. Various kind of generalized solutions have been considered but may have discontinuous derivatives regardless of the smoothness of the initial condition $u_0(x)$. Here, we consider its continuous viscosity solution, which under the condition $H \in C^{0,1}(\mathbb{R}^2)$ and $u_0 \in C^{0,1}(\mathbb{R}^2)$ that we assume henceforth, is the uniform limit as $\varepsilon \to 0^+$ of the (classical) solutions of

$$u_t + H(\bigtriangledown u) - \varepsilon \nabla^2 u = 0, \ (x, y, t) \in \mathbb{R}^2 \times (0, +\infty), \tag{5.2.1}$$

where $\varepsilon > 0$ is a small parameter.

Due to the stated convergence, an approximation to the viscosity solution of (5.1.1) can be obtained by numerical schemes for (5.2.1), where ε is sufficiently small. We will use a finite element space discretization for equation (5.2.1) coupled with nonstandard finite difference time discretization for the obtained system of ordinary differential equations for constructing scheme which is qualitatively stable to respect the monotonicity property (5.1.6).

In the next subsection we consider a finite difference space discretization of equation (5.2.1), while Subsection 5.2.3 is devoted to a nonstandard finite difference scheme for the obtained system of differential equation. The convergence of this new scheme is proved in Subsection 5.2.4. Numerical results supporting the theory are presented in Subsection 5.2.5.

5.2.1 Finite element space discretization

In this subsection we refer essentially to [91]. Let $\mathcal{T}_h$ be a triangulation of $\mathbb{R}^2$ consisting of a countable set of triangles which satisfy the usual compatibility conditions. The generic triangle of $\mathcal{T}_h$ is denoted by T, h_T is the diameter (the largest side) of T, $h = \sup_{T \in \mathcal{T}_h} T$ and ρ_T is the diameter of the largest ball in T.

The triangulation is assumed to be *regular*, that is, there exists a constant $\gamma > 0$, independent of h, such that we have

$$\frac{h_T}{\rho_T} \leq \gamma \tag{5.2.2}$$

for all T on $\mathcal{T}_h$. The condition (5.2.2) is equivalent to Zlámal's condition [138] that there exists a constant $\theta_0 > 0$ such that

$$\forall T, \; \theta_T \geq \theta_0,$$

where for each triangle T, θ_T denotes the smallest angle of T. Let $\{X_i : i = 1, 2, ...\}$ be the set of nodes on $\mathcal{T}_h$. The edge connecting X_i and X_k is denoted by E_{ik} and its length is denoted by $|E_{ik}|$. For any node X_i, I_i is the index set of the triangles with common vertex X_i while N_i is the index set of the neighbor vertexes (vertexes connected to X_i by an edge). With each node X_i, we associate the basis function ϕ_i defined as a continuous piecewise linear function on $\mathbb{R}^2$ such that $\phi_i(X_i) = 1$ and $\phi_i(X_k) = 0$, $k \neq i$. Note that ϕ_i has "small" support in the sense that supp $\phi_i = V_i = \cup_{j \in I_i} T_j$, as shown in Figure 5.1.

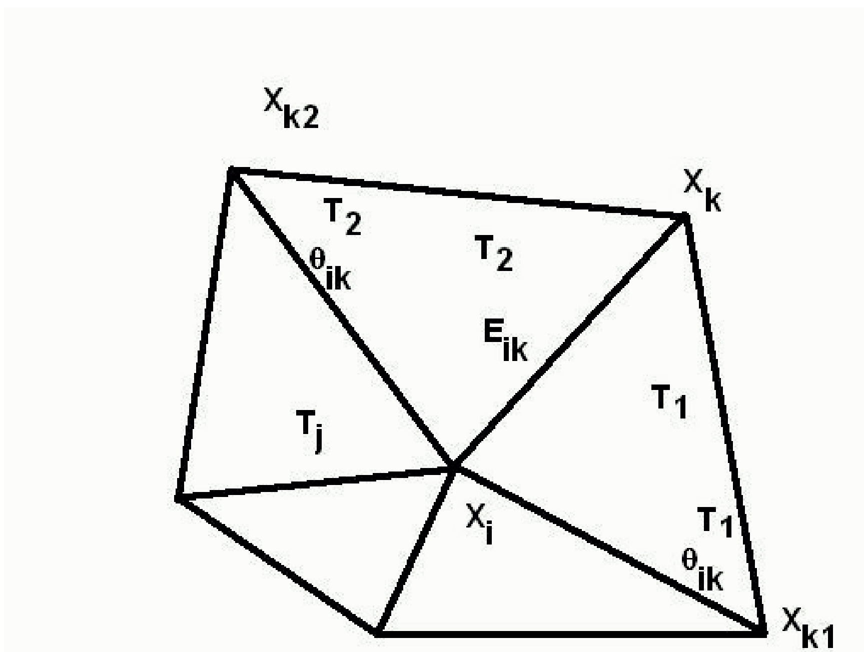

Figure 5.1: *The structure of* V_i.

We denote by $\mathcal{V}_h$ the finite element space which is spanned by the basis functions $(\phi_i)_i$.

An approximation $v_h(x, y, t)$ to the solution of (5.2.1) is sought such that $v_h(., ., t) \in \mathcal{V}_h$, i.e.,

$$v_h(x, y, t) = \sum_{i=1}^{\infty} v_{h,i}(t)\phi_i(x, y),$$

where $v_{h,i}(t) = v_h(X_i, t)$ and v_h satisfies the variational equation

$$\frac{d}{dt}\iint_{\mathbb{R}^2} v_h w \, dx \, dy + \iint_{\mathbb{R}^2} H(\nabla v_h) w \, dx \, dy - \varepsilon \iint_{\mathbb{R}^2} \nabla^2 v_h w \, dx \, dy = 0, \qquad (5.2.3)$$

for all functions $w(x, y) \in \mathcal{V}_h$. In the sequel $v_{h,i}(t)$ is abbreviated to $v_i(t)$. Replacing the test function w by the basis functions $\phi_i(x, y)$, $i = 1, 2, ...,$ in (5.2.3) and using that $\phi_i = 0$ in exterior of V_i, we have

$$\frac{d}{dt}\iint_{V_i} v_h \phi_i \, dx \, dy + \iint_{V_i} H(\nabla v_h)\phi_i \, dx \, dy - \varepsilon \iint_{V_i} \nabla^2 v_h \phi_i \, dx \, dy = 0.$$

Using Green's formula and the fact that ϕ_i vanishes on ∂V_i, we have

$$\frac{d}{dt}\iint_{V_i} v_h \phi_i \, dx \, dy + \iint_{V_i} H(\nabla v_h)\phi_i \, dx \, dy = -\varepsilon \iint_{V_i} \nabla v_h \nabla \phi_i \, dx \, dy. \qquad (5.2.4)$$

Approximating the integral in the first term of (5.2.4) by the "mass lumping" quadrature we obtain

$$\frac{d}{dt} v_i(t) \iint_{V_i} \phi_i \, dx \, dy + \int\int_{V_i} H(\nabla v_h)\phi_i dx dy = -\varepsilon \iint_{V_i} \nabla v_h \nabla \phi_i \, dx \, dy. \qquad (5.2.5)$$

Moreover, let

$$\begin{aligned} \iint_{T_i} \phi_i \, dx \, dy &= \frac{1}{3}\mu(T_j), \; j \in I_i, \\ \iint_{V_i} \phi_i \, dx \, dy &= \frac{1}{3}\mu(V_i), \\ \gamma_{ij} &= \frac{\mu(T_j)}{\mu(V_i)}, \end{aligned}$$

where μ denotes the area. It is clear that $0 < \gamma_{ij} < 1$ and $\sum_{j \in I_i} \gamma_{ij} = 1$.

Lemma 5.2.1 *Let T_1 and T_2 be triangles with common edge E_{ik} on V_i; $\theta_{ik}^{T_1}$ and $\theta_{ik}^{T_2}$ are angles of the triangles T_1 and T_2 opposite to the edge E_{ik}, respectively, as shown in Figure 5.1. We have the following formulation*

$$\iint_{V_i} \nabla v_h \nabla \phi_i \, dx\, dy = -\sum_{k\in N_i} a_{ik}(v_k - v_i), \tag{5.2.6}$$

where

$$a_{ik} = a_{ik}^{T_1} + a_{ik}^{T_2}, \quad a_{ik}^{T_1} = \frac{1}{2}\cot\theta_{ik}^{T_1}, \quad a_{ik}^{T_2} = \frac{1}{2}\cot\theta_{ik}^{T_2}.$$

Proof. *Since $\phi_i + \sum\limits_{k\in N_i} v_k\phi_k = 1$, by computation we have*

$$\begin{aligned}
\iint_{V_i} \nabla v_h \nabla \phi_i \, dx\, dy &= \iint_{V_i} \nabla(\sum_{k\in N_i} v_k\phi_k + v_i\phi_i).\nabla\phi_i \, dx\, dy \\
&= \iint_{V_i} \left(\sum_{k\in N_i} v_k \nabla\phi_k + v_i(-\sum_{k\in N_i}\nabla\phi_k)\right).\nabla\phi_i \, dx\, dy \\
&= \sum_{k\in N_i}\left((v_k - v_i)\iint_{V_i}\nabla\phi_k.\nabla\phi_i \, dx\, dy\right). \qquad (5.2.7)
\end{aligned}$$

It remains to prove that

$$\iint_{V_i} \nabla\phi_k.\nabla\phi_i \, dx\, dy = -a_{ik}. \tag{5.2.8}$$

Since $\phi_k = 0$ on any triangle of V_i which do not contain X_k, we have

$$\iint_{V_i} \nabla\phi_k.\nabla\phi_i \, dx\, dy = \iint_{T_1} \nabla\phi_k.\nabla\phi_i \, dx\, dy + \iint_{T_2} \nabla\phi_k.\nabla\phi_i \, dx\, dy.$$

Let X_{k1}, X_{k2} be the two nodes opposite to E_{ik} in triangle T_1 and T_2, respectively, see Figure 5.1.

$$\begin{aligned}
\iint_{T_1} \nabla\phi_k.\nabla\phi_i \, dx\, dy &= \frac{|E_{i\,k1}|}{2\mu(T_1)}.\frac{|E_{k1\,k}|}{2\mu(T_1)}\cos(180^0 - \theta_{ik}^{T_1}).\mu(T_1) \\
&= -\frac{|E_{i\,k1}|}{4\mu(T_1)}.|E_{k1\,k}|\cos(\theta_{ik}^{T_1}) \\
&= -\frac{|E_{i\,k1}|.|E_{k1\,k}|\cos(\theta_{ik}^{T_1})}{4.\frac{1}{2}|E_{i\,k1}|.|E_{k1\,k}|\sin(\theta_{ik}^{T_1})} = -\frac{1}{2}\cot(\theta_{ik}^{T_1}) = -a_{ik}^{T_1}.
\end{aligned}$$

Similarly,

$$\iint_{T_2} \nabla\varphi_k.\nabla\varphi_i \, dx\, dy = -\frac{1}{2}\cot(\theta_{ik}^{T_2}) = -a_{ik}^{T_2}.$$

Thus (5.2.8) is proved. Substituting (5.2.8) into (5.2.7), we have (5.2.6). ■

Substituting (5.2.6) into (5.2.5), and using that $H(\nabla v_h|_{T_j})$ is a constant, the equation (5.2.5) can be written in the following equivalent form

$$\frac{d}{dt}v_i = -\sum_{j\in I_i} H(\nabla v_h|_{T_j})\gamma_{ij} + \frac{3\varepsilon}{\mu(V_i)}\sum_{k\in N_i} a_{ik}(v_k - v_i). \qquad (5.2.9)$$

In [91], it is shown that the scheme (5.2.9) is consistent.

For the monotonicity of the scheme discussed in the next subsection the coefficients a_{ik} need to be bounded away from zero, that is, we have the following Lemma containing the additional assumption on the triangulation $\mathcal{T}_h$.

Lemma 5.2.2 [91] *If the triangulation $\mathcal{T}_h$ is such that there exists a constant c_1, $(0 < c_1 < \frac{\pi}{2})$, independent of h such that*

$$\theta_{ik}^{T_1} + \theta_{ik}^{T_2} \leq \pi - c_1, \qquad (5.2.10)$$

for every edge E_{ik} on $\mathcal{T}_h$, then there exists a positive constant C_0 such that for the scheme defined by (5.2.9) we have

$$a_{ik} \geq C_0, i = 1, 2, ..., \text{ and } k \in N_i. \qquad (5.2.11)$$

5.2.2 A nonstandard finite difference scheme

We consider a mesh $\{t_n, n = 0, 1, 2, ...\}$ in the time dimension with constant time step Δt, that is we have $\{t_n = n\Delta t\}$. As usual $v^n = (v_i^n)_{i=1}^{\infty}$ denotes the approximation of the solution of (5.2.9) at $t = t_n$.

Our aim in this subsection is to design a scheme for (5.2.9) that is qualitatively stable with respect to the monotonicity on initial values, as stated in the following definition:

Definition 5.2.1 *A finite difference scheme (5.2.9) is monotone if*

$$v_i^0 \leq w_i^0 \Longrightarrow v_i^n \leq w_i^n,$$

where v^n and w^n are discrete solutions initiated at v^0 and w^0, respectively.

For simplicity, we ignore for the moment the space index i and we assume that we are dealing with a scalar problem the discrete solution of which is given by an explicit scheme of the form

$$v^{n+1} = g(\Delta t; v^n). \qquad (5.2.12)$$

The following result is proved in [9].

Theorem 5.2.1 *The difference scheme (5.2.12) is qualitatively stable with respect to the monotonicity on initial values if and only if*

$$\frac{\partial v^{n+1}}{\partial v^n} \equiv \frac{\partial g(\Delta t; v)}{\partial v} \geq 0,\ \Delta t > 0,\ v \in \mathbb{R}. \tag{5.2.13}$$

Since we are reduced to checking the positivity condition (5.2.13), we will in what follows adapt and exploit the favorable situation described in the following theorem:

Theorem 5.2.2 *Let w be the solution of the problem*

$$Lw = f(w),$$

where L is either the differential operator $Lz = z'$ or the identity operator $Lz = z$. Assume that the solution w is nonnegative and that the function f admits the decomposition

$$f(z) = p(z) - q(z)z,$$

where $p(z) \geq 0$ and $q(z) \geq 0$. Then the difference scheme

$$\frac{w^{n+1} - w^n}{\Delta t} = p(w^n) - q(w^n)w^{n+1} \tag{5.2.14}$$

for $Lz = z'$ or

$$w^{n+1} = p(w^n) - q(w^n)w^{n+1} \tag{5.2.15}$$

for $Lz = z$ is qualitatively stable with respect to the positivity property of the solution w.

Proof. *Obvious by re-writing* (5.2.14) *and* (5.2.15) *as*

$$w^{n+1} = \frac{w^n + \Delta t\, p(w^n)}{1 + \Delta t\, q(w^n)} \ \ \text{and}\ \ w^{n+1} = \frac{p(w^n)}{1 + q(w^n)},$$

respectively.■

Remark 5.2.1 *The situation described in Theorem 5.2.2 was introduced in a more specific form by the authors in [7] in order to design schemes that preserve the positivity property of the solutions of reaction diffusion equations. The idea is also exploited for the approximation of differential models in population biology and mathematical epidemiology where the positivity of the involved species is essential (see, for instance, [63, 106, 105]). The underlining point of these schemes is, as it can be seen from* (5.2.14) *and* (5.2.15), *that one of Mickens' rules of constructing nonstandard finite difference schemes is reinforced: the nonlinear term $q(w)w$ is approximated in a nonlocal way, i.e., by $q(w^n)w^{n+1}$ and not by $q(w^n)w^n$ or $q(w^{n+1})w^{n+1}$.*

Coming back to the problem (5.2.9). In view of the form of the right-hand side of (5.2.9) and of Theorem 5.2.1, which requires to show the positivity condition $\frac{\partial v^{n+1}}{\partial v^n} \geq 0$, we propose in the spirit of the nonlocal approximation in Theorem 5.2.2, the following non-standard finite difference scheme for the system of equations (5.2.9):

$$v_i^{n+1} = v_i^n - \Delta t \sum_{j\in I_i} H(\nabla v_h^n|_{T_j})\gamma_{ij} + \frac{3\varepsilon\Delta t}{\mu(V_i)} \sum_{k\in N_i} a_{ik}(v_k^n - v_i^{n+1}). \tag{5.2.16}$$

Observe that the last sum in (5.2.16) is approximated in a nonlocal way.

Theorem 5.2.3 *Let the triangulation $\mathcal{T}_h$ be regular and satisfy the condition* (5.2.11). *Let $H \in C^{0,1}(\mathbb{R}^2)$. Then there exist a constant C independent of Δt and h such that if*

$$\varepsilon \geq Ch, \tag{5.2.17}$$

the scheme (5.2.16) *is monotone.*

proof. *Let $m \in N_i$ and let $X_{m'}$ and $X_{m''}$ be the nodes opposite to E_{im} in the two adjacent triangles T' and T'' containing E_{im}. We have*

$$\begin{aligned}\frac{\partial v_i^{n+1}}{\partial v_m^n} &= -\Delta t \left[\nabla H.\nabla\phi_m|_{T'}\frac{\mu(T')}{\mu(V_i)} + \nabla H.\nabla\phi_m|_{T''}\frac{\mu(T'')}{\mu(V_i)}\right] \\ &+ \frac{3\varepsilon\Delta t}{\mu(V_i)}a_{im} - \frac{3\varepsilon\Delta t}{\mu(V_i)}\sum_{k\in N_i} a_{ik}\frac{\partial v_i^{n+1}}{\partial v_m^n}.\end{aligned}$$

Hence

$$\begin{aligned}\left(1 + \frac{3\varepsilon\Delta t}{\mu(V_i)}\sum_{k\in N_i} a_{ik}\right)\frac{\partial v_i^{n+1}}{\partial v_m^n} &\geq \frac{\Delta t}{\mu(V_i)}\left[3\varepsilon a_{im} - \frac{1}{2}|H|_{1,\infty}(|E_{im'}| + |E_{im''}|)\right] \\ &\geq \frac{\Delta t}{\mu(V_i)}[3\varepsilon C_0 - h|H|_{1,\infty}].\end{aligned}$$

Setting $C = \frac{|H|_{1,\infty}}{3C_0}$, we obtain $\frac{\partial v_i^{n+1}}{\partial v_m^n} \geq 0$ whenever $\varepsilon \geq Ch$.

In similar way

$$\frac{\partial v_i^{n+1}}{\partial v_i^n} = 1 - \Delta t \sum_{j\in I_i} \nabla H.\nabla\phi_i|_{T_j}\frac{\mu(T_j)}{\mu(V_i)} - \frac{3\varepsilon\Delta t}{\mu(V_i)}\sum_{k\in N_i} a_{ik}\frac{\partial v_i^{n+1}}{\partial v_i^n}.$$

Hence

$$\left(1 + \frac{3\varepsilon\Delta t}{\mu(V_i)}\sum_{k\in N_i} a_{ik}\right)\frac{\partial v_i^{n+1}}{\partial v_i^n} = 1 - \frac{\Delta t}{\mu(V_i)}\nabla H\,.\sum_{j\in I_i}\nabla\phi_i|_{T_j}\mu(T_j). \tag{5.2.18}$$

It is easy to see that in any triangulation $\mathcal{T}_h$ we have

$$\sum_{j\in I_i} \nabla\phi_i|_{T_j}\mu(T_j) = \vec{0}. \tag{5.2.19}$$

Indeed, for any constant vector in $z = (z_1, z_2) \in \mathbb{R}^2$ we have

$$z\sum_{j\in I_i} \nabla\phi_i|_{T_j}\mu(T_j) = \iint_{V_i} z\nabla\phi_i\,dx\,dy = -\iint_{V_i} \nabla z\phi_i\,dx\,dy = \vec{0},$$

which implies (5.2.19). *Substituting* (5.2.19) *in* (5.2.18) *we obtain* $\frac{\partial v_i^{n+1}}{\partial v_i^n} > 0$. *This completes the proof.* ■

Remark 5.2.2 *With the notations of the proof of Theorem 5.2.3 in mind, we assume that the second term in the right hand side of (5.2.16) is increasing with respect to v_m^n. In this case, Theorem 5.2.3 is a straightforward consequence of Theorem 5.2.2 and the monotonicity of the scheme (5.2.16) occurs then without the relation $\varepsilon \geq Ch$. This relation is essential in the general setting of Theorem 5.2.3 where the monotonicity of the mentioned term cannot be monitored.*

Mickens'rule of nonlocal approximation is normally applied to nonlinear terms, see [103], [9]. Here we apply it to a linear term. Usually, the two conditions in Definition 5.1.2 are considered independently. It is interesting that in our case the scheme formulated in (5.2.16) through nonlocal approximations admits an equivalent formulation using a renormalization of the denominator of the discrete derivative. More precisely, (5.2.16) is equivalent to

$$\frac{v_i^{n+1} - v_i^n}{\psi_i(\Delta t)} = -\sum_{j\in I_i} H(\nabla v_h^n|_{T_j})\gamma_{ij} + \frac{3\varepsilon}{\mu(V_i)}\sum_{k\in N_i} a_{ik}(v_k^n - v_i^n), \tag{5.2.20}$$

where

$$\psi_i(\Delta t) = \frac{\Delta t}{1 + \frac{3\varepsilon\Delta t}{\mu(V_i)}\sum\limits_{k\in N_i} a_{ik}}.$$

Observe that the denominator function $\psi_i(\Delta t)$ has the following asymptotic behavior stated in (5.1.5).

Remark 5.2.3 *The more complex denominator function $\psi_i(\Delta t)$ captures the intrinsic property of the solution of the problem (5.2.16) of being monotone dependent on initial values under the condition (5.2.17) stated in Theorem 5.2.3. It would be interesting to investigate whether there are other physical properties of (5.2.16) that are captured by $\psi_i(\Delta t)$. For instance, when $H = 0$, can we say that the scheme (5.2.16) preserves the Lyapunov stability properties of the differential equation (5.2.16)?*

Remark 5.2.4 *At every $t = t_n$ the solution of the problem (5.1.1)-(5.1.2) is approximated by the function $v_h^n = \sum_i^{\infty} v_i^n \phi_i(x, y) \in \mathcal{V}_h$. Hence the scheme (5.2.16) can equivalently be considered as a mapping $G(\Delta t, .)$ from $\mathcal{V}_h$ such that $v_h^{n+1} = G(v_h^n)$. Due to the explicit formulation (5.2.20) of (5.2.16), the mapping G can also be given in an explicit form. More precisely, for any $w_h = \sum_{i=1}^{\infty} w_i \phi_i \in \mathcal{V}_h$, we have*

$$G(\Delta t, w_h) = \sum_{i=1}^{\infty} \alpha_i \phi_i(x, y), \tag{5.2.21}$$

where

$$\begin{aligned} \alpha_i &= w_i - \psi_i(\Delta t) \sum_{j \in I_i} H(\nabla w_h|_{T_j}) \gamma_{ij} \\ &\quad + \psi_i(\Delta t) \frac{3\varepsilon}{\mu(V_i)} \sum_{k \in N_i} a_{ik}(w_k - w_i), \quad i = 1, 2, ... \end{aligned} \tag{5.2.22}$$

It is clear that under the conditions in Theorem 5.2.3 the mapping $G(\Delta t, .)$ is monotone with respect to the usually point-wise partial order on $\mathcal{V}_h$.

Remark 5.2.5 *Numerical schemes using the standard finite difference method are typically monotone only under a restriction on the time step size. This might be disadvantage in applications. For example in [91] the restriction is*

$$\Delta t \leq C \frac{\min_{j \in N_i} \mu(T_j)}{\varepsilon}.$$

Since the bound of Δt involves the size of the smallest triangle in the triangulation the above inequality implies that even when the triangulation is refined only locally, Δt need to be adjusted as well. Through the nonstandard approach the scheme (5.2.16) is monotone for any time step size, because in (5.2.17) the time step size Δt is not involved.

5.2.3 Convergence

The convergence of the scheme is obtained through an abstract convergence result of Barles and Souganidis, [22], which is detailed in its consequences in [80] for the equation (5.1.1) as stated below. The function spaces and notations are defined in these references.

For $\tau > 0$, let a mapping $S(\tau) : L^\infty(\mathbb{R}^2) \to L^\infty(\mathbb{R}^2)$ be given. The following conditions are considered in connection with the mapping $S(\tau)$:

- monotonicity, i.e., $u \leq v \Longrightarrow S(\tau)u \leq S(\tau)v,$ (5.2.23)
- invariance under translation, i.e., $S(\tau)(u+k) = S(\tau)(u) + k, k \in \mathbb{R},$(5.2.24)
- consistency, i.e.,$\dfrac{S(\tau)\varphi - \varphi}{\tau} + H(\nabla\varphi) \to 0,$ as $\tau \to 0, \forall \varphi \in C_0^\infty(\mathbb{R}^2),$(5.2.25)
- rate of approximation:
$$|\frac{S(\tau)\varphi - \varphi}{\tau} + H(\nabla\varphi)| \leq o(\tau(|\varphi|_{1,\infty} + |\varphi|_{2,\infty})), \forall \varphi \in C_0^\infty(\mathbb{R}^2). \quad (5.2.26)$$

An approximation $u_{\Delta t}$ to the solution of (5.1.1)-(5.1.2) in two space dimension is constructed by using a grid $t_n = n\Delta t$ in time as follows:

$$u_{\Delta t}(x,y,t) = \begin{cases} S(t-t_n)u_{\Delta t}(.,.,t_n)(x,y) & , \quad t \in (t_n, t_{n+1}], n = 0, 1, ... \\ u_0(x) & , \quad t = 0. \end{cases}$$

The following conclusion holds.

Theorem 5.2.4 [22] *If a mapping* $S(\tau)$ *satisfies* (5.2.23), (5.2.24) *and* (5.2.25) *and* $H \in C^{0,1}(\mathbb{R}^2)$, $u_0 \in C^{0,1}(\mathbb{R}^2)$, *then for any* $t' > 0$ *we have* $u_{\Delta t} \to u$ *uniformly on* $\mathbb{R}^2 \times [0, t']$ *as* $\Delta t \to 0$. *Furthermore, if* $S(\tau)$ *satisfies also* (5.2.26) *then there exists a positive constant* C_3 *independent of* Δt *such that*

$$||u_{\Delta t} - u||_\infty \leq C_3\sqrt{\Delta t}.$$

Let $\mathcal{I}_h$ denote the piece-wise interpolation operator at the nodes of the triangulation $\mathcal{T}_h$, that is, for any real function φ on $\mathbb{R}^2$ the function $\mathcal{I}_h\varphi$ is linear on any triangle $T \in \mathcal{T}_h$ and $(\mathcal{I}_h\varphi)(X_i) = \varphi(X_i)$ at every node X_i of $\mathcal{T}_h$. Note that we have $\mathcal{I}_h\varphi \in \mathcal{V}_h$. We consider the mapping $S(\tau) : L^\infty(\mathbb{R}^2) \to L^\infty(\mathbb{R}^2)$ defined as a composition of the operator $\mathcal{I}_h$ and the scheme (5.2.16). More precisely using the mapping G given in (5.2.21)-(5.2.22) we have

$$S(\tau)\varphi = G(\tau, \mathcal{I}_h\varphi). \quad (5.2.27)$$

Then the numerical scheme (5.2.16) is equivalent to the scheme (5.2.27) where the numerical solution is evaluated only at the points of the mesh. Therefore the convergence of the scheme (5.2.16) can be obtained through Theorem 5.2.4, where the mapping $S(\tau)$ is given by (5.2.27). To this end we only need to verify the conditions (5.2.23)-(5.2.26) for $S(\tau)$ given by (5.2.27). The essential property of the monotonicity of $S(\tau)$ follows from the monotonicity of G and $\mathcal{I}_h$. The condition (5.2.24) follows trivially from the form (5.2.21)-(5.2.22) of the mapping G. Now we consider condition (5.2.25) and its stronger form (5.2.26).

By the polynomial approximation theory [31], there exist a positive constant K such that for all $\varphi \in C_0^\infty(\mathbb{R}^2)$,

$$|\mathcal{I}_h\varphi - \varphi|_{1,\infty} \leq Kh|\varphi|_{2,\infty}. \tag{5.2.28}$$

Substituting φ into scheme (5.2.16), we have

$$\frac{S(\Delta t)\varphi|_{(x,y)=X_i} - \varphi_i}{\Delta t} = -\sum_{j\in I_i} H(\nabla\varphi_h|_{T_j})\gamma_{ij} + \frac{3\varepsilon}{\mu(V_i)}\sum_{k\in N_i} a_{ik}(\varphi_k - \varphi_i). \tag{5.2.29}$$

In view of the fact that $\sum\limits_{j\in I_i}\gamma_{ij} = 1$ and (5.2.28)

$$\left| H(\nabla\varphi(X_i)) - \sum_{j\in I_i} H(\nabla\varphi_h|_{T_j})\gamma_{ij} \right| = 0(h). \tag{5.2.30}$$

By Lemma 5.1.1, for a polynomial v_h of degree ≤ 1, $\nabla v_h = \vec{C}$ and we have

$$\begin{aligned}
\frac{3\varepsilon}{\mu(V_i)}\sum_{k\in N_i} a_{ik}(v_k - v_i) &= -\frac{3\varepsilon}{\mu(V_i)}\int\int_{V_i} \nabla v_h \nabla\varphi_i \, dx\, dy \\
&= -\frac{3\varepsilon}{\mu(V_i)}\int\int_{V_i} \vec{C}.\nabla\varphi_i \, dx\, dy \\
&= 0.
\end{aligned}$$

Thus the bilinear form vanishes other the space of polynomial of degree ≤ 1. Therefore, using Bramble-Hilbert Theorem [31] and the fact that $\varepsilon = o(h)$, see (5.2.17), we obtain the conclusion that for $\varphi \in C_0^\infty(\mathbb{R}^2)$

$$\frac{3\varepsilon}{\mu(V_i)}\sum_{k\in N_i} a_{ik}(\varphi_k - \varphi_i) = 0(h). \tag{5.2.31}$$

Combining (5.2.29) with (5.2.30) and (5.2.31), we have

$$\begin{aligned}
&\left| \left[\frac{S(\Delta t)\varphi - \varphi}{\Delta t} + H(\nabla\varphi) \right]_{(x,y)=X_i} \right| \\
&\leq \left| \sum_{j\in I_i} H(\nabla\varphi_h|_{T_j})\gamma_{ij} + \frac{3\varepsilon}{\mu(V_i)} \sum_{k\in N_i} a_{ik}(\varphi_k - \varphi_i) + H(\nabla\varphi) \right|_{(x,y)=X_i} \qquad (5.2.32) \\
&\leq 0(h(|\varphi|_{1,\infty} + |\varphi|_{2,\infty})).
\end{aligned}$$

For convergence we assume that both Δt and h approach to zero. Hence the consistent condition (5.2.25) follows from (5.2.32). Moreover, if we assume that $\Delta t = o(h)$ and $h = o(\Delta t)$ the estimate (5.2.32) implies (5.2.26). Hence we have the following convergence result.

Theorem 5.2.5 *Let the family of triangulation* $(\mathcal{T}_h)$ *be regular and satisfy the condition* (5.2.11). *Let* $H \in C^{0,1}(\mathbb{R}^2)$ *and* $u_0 \in C^{0,1}(\mathbb{R}^2)$. *Then the numerical solution* v_h^n *obtained by* (5.2.16) *with* $\varepsilon \geq Ch$ *and* $\varepsilon = o(h)$ *converges to the exact solution* u *of the problem* (5.1.1)-(5.1.2) *, i.e., for any* $t' > 0$ *we have*

$$\sup_{i,n\leq t'/\Delta t} |u(X_i, t_n) - v_i^n| \to 0, \ \ as\ \Delta t \to 0, h \to 0.$$

Moreover, if $0 < \inf \frac{\Delta t}{h} \leq \sup \frac{\Delta t}{h} < \infty$, *then there exists a positive constant* C_3 *independent of* Δt *and* h *such that*

$$\sup_{i,n\leq t'/\Delta t} |u(X_i, t_n) - v_i^n| \leq C_3\sqrt{h}.$$

Remark 5.2.6 *As Xu and Zikatanov [136] have shown that a formulation similar to (5.2.6) holds in higher dimensional space and Theorem 5.2.5 also holds for arbitrary spatial dimension, the generalization to higher dimensional space is straight forward.*

5.2.4 Numerical results

We present the results of our numerical experiments for the nonstandard monotone schemes for Hamilton-Jacobi equations with convex and nonconvex Hamiltonians and with smooth and discontinuous initial conditions. We compare them with the standard techniques.

Example 5.2.1 *We consider the combustion equation, where the Hamiltonian is convex function, which is often used in testing numerical methods, [129], [91]:*

$$\frac{\partial v}{\partial t} - \sqrt{1 + v_x^2 + v_y^2} = 0, \quad (x, y) \in (0, 1) \times (0, 1), t > 0 \tag{5.2.33}$$

$$v(x, y, 0) = \cos(2\pi y) - \cos(2\pi x), \quad (x, y) \in (0, 1) \times (0, 1) \tag{5.2.34}$$

with periodic boundary conditions.

We use a triangulation with 6240 elements on $[0, 1] \times [0, 1]$*, which satisfies condition (5.2.10). The numerical solution obtained with* $\varepsilon = 0.01$ *and* $\Delta t = 0.01$ *is presented on Figure 5.2. For comparison we consider the standard Euler scheme for the equation (5.2.9) which is monotone only for sufficiently small values of* Δt*, see [91] for details. The numerical solution obtained for the same value of the parameters, that is,* $\varepsilon = 0.01$ *and* $\Delta t = 0.01$*, is presented on Figure 5.3. The advantage of the considered nonstandard method with regard to preserving the qualitative behavior of the solution apparent. One observes, for instance, that the nonstandard scheme is stable with respect to the monotonicity of solution contrary to the standard scheme.*

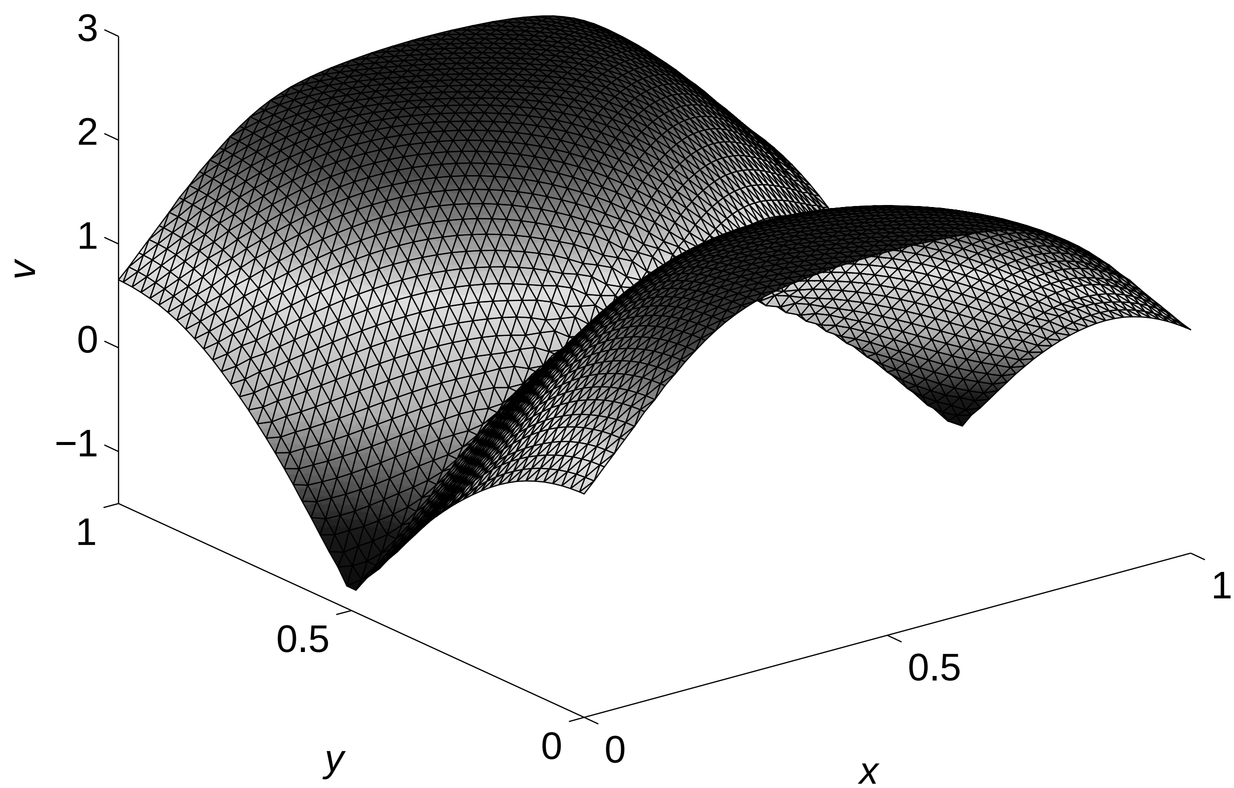

Figure 5.2: *Numerical solution of (5.2.33)-(5.2.34) using the nonstandard method (5.2.16) with* $\varepsilon = 0.01, \Delta t = 0.01$.

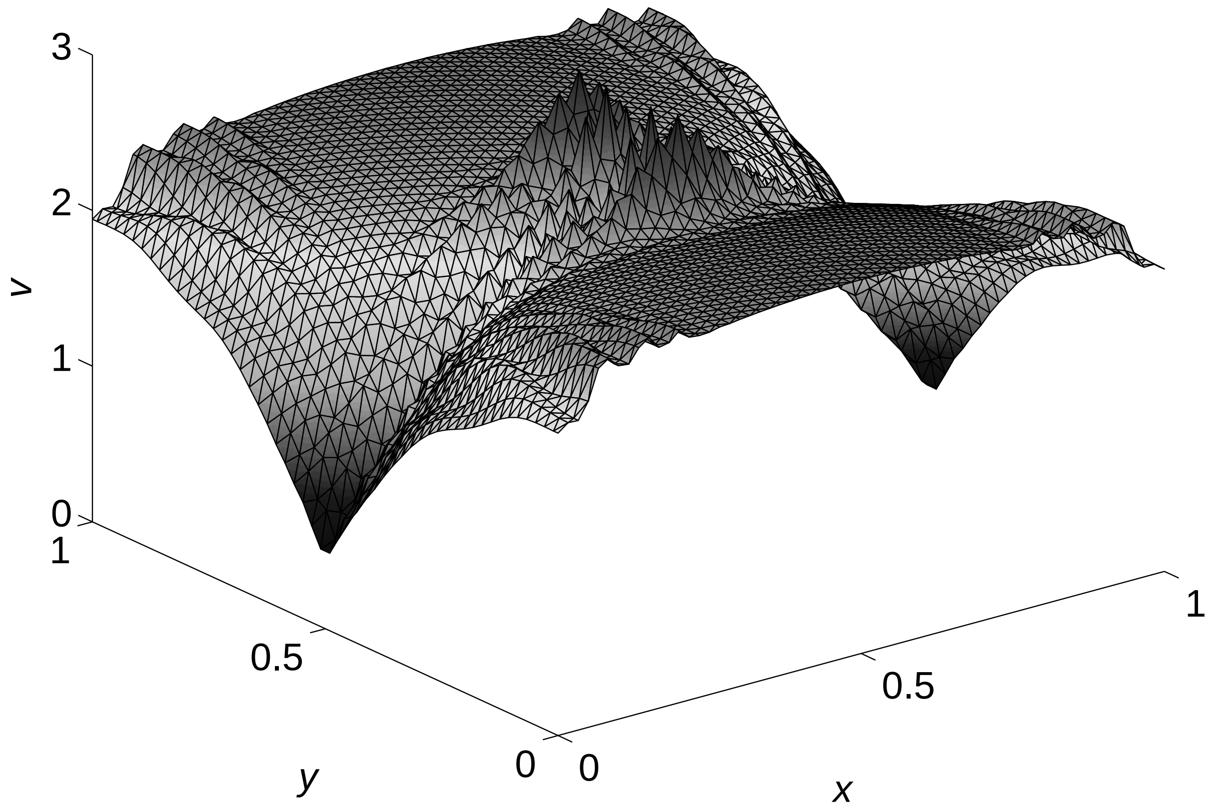

Figure 5.3: *Numerical solution of (5.2.33)-(5.2.34) using the standard Euler time discretization to (5.2.9) with* $\varepsilon = 0.01, \Delta t = 0.01$.

Example 5.2.2 *We consider the same equation as in Example 5.2.1 but with discontinuous initial condition where* $y \in [0,1]$*:*

$$\frac{\partial v}{\partial t} - \sqrt{1 + v_x^2 + v_y^2} = 0, \quad (x,y) \in (0,1) \times (0,1), t > 0 \tag{5.2.35}$$

$$v(x,y,0) = \begin{cases} (1 - 0.1x(1-x))(\cos(2\pi y) - \cos(2\pi y)) & , \quad 0 \leq x < 0.5 \\ (1 - 0.1x(1-x))(\cos(2\pi y) - \cos(2\pi y)) + 0.1 & , \quad 0.5 < x \leq 1. \end{cases} \tag{5.2.36}$$

with periodic boundary conditions.

We consider the same triangulation as in Example 5.2.1. The numerical solution obtained by nonstandard technique with $\varepsilon = 0.01$ and $\Delta t = 0.01$ is presented n Figure 5.4. It is also clear that the monotonicity of numerical solution with respect to the initial value is preserved.

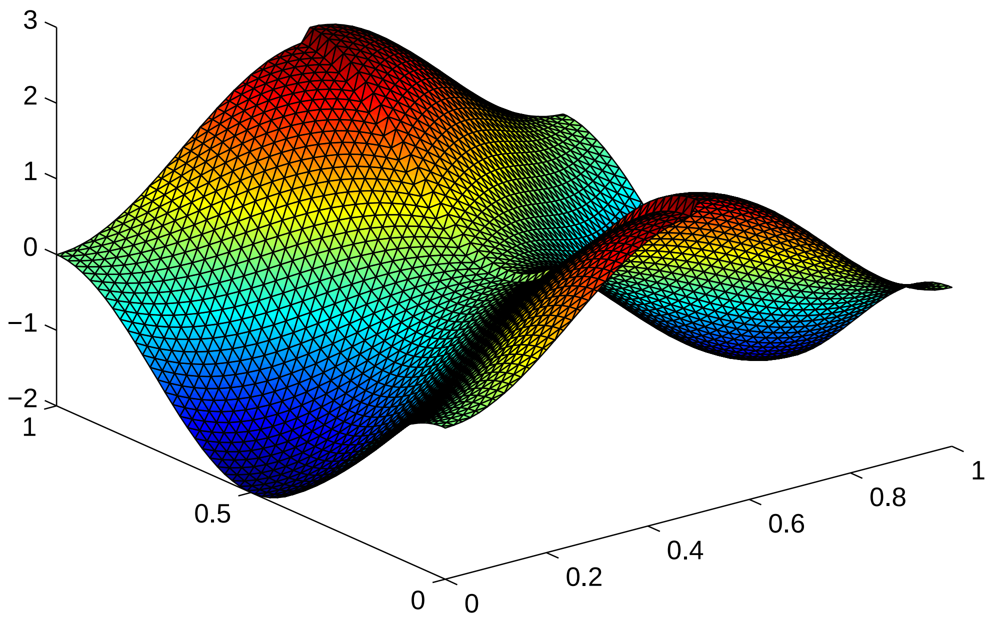

Figure 5.4: *Numerical solution of (5.2.35)-(5.2.36) using the nonstandard method (5.2.16) with* $\varepsilon = 0.01, \Delta t = 0.01$.

Example 5.2.3 *We consider the following equation, where the Hamiltonian is nonconvex function:*

$$\begin{aligned} \frac{\partial v}{\partial t} + \cos(v_x + v_y + 1) &= 0, \quad (x,y) \in (0,1)\times(0,1), t > 0 && (5.2.37) \\ v(x,y,0) &= \cos(2\pi y) - \cos(2\pi x), \quad (x,y) \in (0,1)\times(0,1) && (5.2.38) \end{aligned}$$

with periodic boundary conditions.

Once again we consider the same triangulation as an example 5.2.1. The numerical solution obtained by nonstandard technique with $\varepsilon = 0.01$ and $\Delta t = 0.01$ is presented n Figure 5.5. The numerical solution obtained by the standard method with the same value of parameters ε and Δt is presented on Figure 5.6. We can observe that the result by nonstandard scheme has a good resolution.

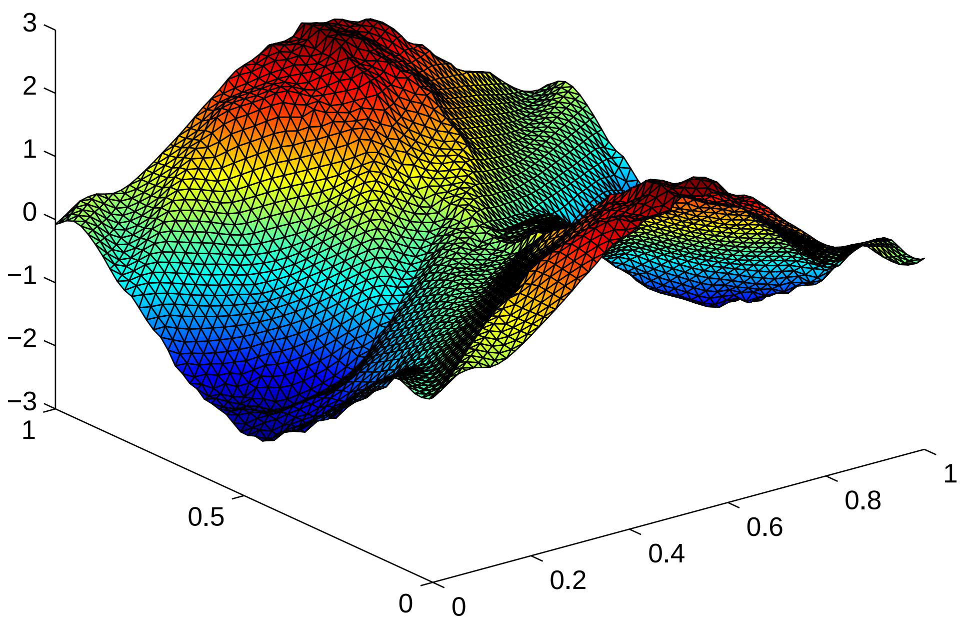

Figure 5.5: *Numerical solution of (5.2.37)-(5.2.38) using the nonstandard method (5.2.16) with* $\varepsilon = 0.01, \Delta t = 0.01$.

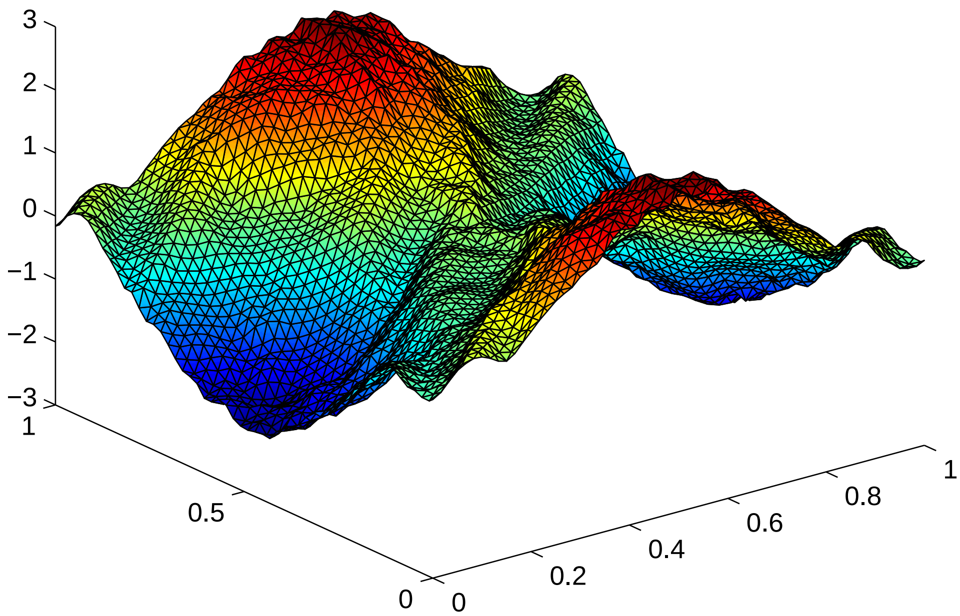

Figure 5.6: *Numerical solution of (5.2.37)-(5.2.38) using the standard Euler time discretization to (5.2.9) with* $\varepsilon = 0.01, \Delta t = 0.01$.

5.3 Total Variation Diminishing Nonstandard Finite Difference Schemes for Conservation Laws

In this section, we consider the Cauchy problem (5.1.3)-(5.1.4) for conservation laws. The equation (5.1.3) describe the behavior of many different physical phenomena including traffic flow [135]. It is well known [90] that the solution of (5.1.3) may become discontinuous as time evolves, even for smooth $v_0(x)$ given in (5.1.4). Thus it turn requires an entropy condition to select the physical relevant discontinuous solutions, called the entropy solutions.

Here, we assume that the data functions f and v_0 are such that equations (5.1.3)-(5.1.4) has unique entropy solution, e.g., f smooth and uniformly convex and $v_0 \in L^\infty(\mathbb{R})$, see [46, Section 3.4]. There is a well known theory regarding the existence and uniqueness of an entropy solution, using special integral structure of the equation (5.1.3), [88].

A very successful class of schemes for solving (5.1.3)-(5.1.4) is the class of total variation diminishing conservative schemes which resolve discontinuities in the solutions without spurious oscillations displayed by numerical solutions. We construct nonstandard implicit and explicit conservative schemes, including Runge-Kutta higher order schemes, which are total variation diminishing without a restriction on the time step size. Moreover, the implicit schemes have the advantages of being computationally simpler.

In Subsection 5.3.1 we give some preliminaries settings and results including the Harten's lemma. In Subsection 5.3.2 we formulate an implicit nonstandard finite difference scheme using nonlocal approximation of nonlinear terms. Subsection 5.3.3 deals with explicit and Runge-Kutta nonstandard finite difference schemes where renormalization of the denominator is used. Numerical results by both implicit and the explicit methods are presented in Subsection 5.3.4. The numerical solution of Hamilton-Jacobi equation in one space dimension obtained using a discontinuous Galerkin method approximating its derivatives using nonstandard TVD method is given in Section 5.3.5.

5.3.1 Preliminaries

For simplicity, we assume that the grid points $\{x_j\}_{j=1}^{\infty}$ are uniformly distributed with the cell size $x_{j+1} - x_j = \Delta x$.

Following this space discretization, the equation for conservation laws (5.1.3) is written as a system of ordinary differential equations of the form

$$w_t = L(w), \tag{5.3.1}$$

where $w = (w_j)$ and $w_j(t) \approx v(x_j, t)$. The operator L in (5.3.1) is obtained from the following spacial discretization

$$(L(w))_j = -\frac{1}{\Delta x}(\hat{f}_{j+\frac{1}{2}} - \hat{f}_{j-\frac{1}{2}}),$$

where $\hat{f}_{j+\frac{1}{2}} = \hat{f}(w_j, w_{j+1})$ is a numerical flux function which is nondecreasing in the first argument and nonincreasing in the second; Lipschitz continuous in all its arguments, and satisfies the consistency condition

$$\hat{f}(\widetilde{w}, \widetilde{w},) = f(\widetilde{w}).$$

Conservative scheme for equation (5.1.3) has the form

$$\frac{d}{dt}w_j = -\frac{1}{\Delta x}(\hat{f}_{j+\frac{1}{2}} - \hat{f}_{j-\frac{1}{2}}). \tag{5.3.2}$$

For choices of numerical flux, we refer to, e.g., [111]. Notice that (5.3.2) is written in a semidiscrete method of lines form, while in practice the time variable t must also be discretized.

Let a mesh $t_n = n\Delta t$, $n = 0, 1, 2, ...$, in the time direction be given. As usual w^n denotes an approximation of w at $t = t_n$. Below we propose a scheme of explicit Euler-type, which is the familiar form of conservative scheme

$$w_j^{n+1} = w_j^n - \frac{\Delta t}{\Delta x}\left(\hat{f}(w_j^n, w_{j+1}^n) - \hat{f}(w_{j-1}^n, w_j^n)\right). \tag{5.3.3}$$

Conservative schemes are especially suitable for computing entropy solutions with shocks, because of the important Lax-Wendroff Theorem [89], which states that solutions to conservative schemes if convergent, would converge to a weak solution of (5.1.3).

The Lax-Wendroff Theorem does not say anything about whether the method converges, only that if a sequence of approximations converges then the limit is a weak solution of (5.1.3). For that, we need some form of strong stability to guarantee convergence. A very successful method which guarantees convergence is total variation diminishing method.

Definition 5.3.1 *A numerical scheme (5.3.3) is called total variation diminishing (TVD) if $TV(w^n)$ is decreasing with respect to n, that is,*

$$TV(w^{n+1}) \leq TV(w^n),\ n = 0, 1, 2, ...,$$

where the total variation of a grid function w^n is defined by

$$TV(w^n) = \sum_{j=-\infty}^{+\infty} |w_{j+1}^n - w_j^n|.$$

The TVD property of numerical methods is often proved by using the Harten's Lemma. We give below a version dealing with both the explicit and the implicit cases [64], [61].

Lemma 5.3.1 *(Harten) Consider the explicit scheme*

$$w_j^{n+1} = w_j^n - \left(-C_{j+\frac{1}{2}}(w_{j+1}^n - w_j^n) + D_{j-\frac{1}{2}}(w_j^n - w_{j-1}^n)\right), \tag{5.3.4}$$

and the implicit scheme

$$w_j^{n+1} = w_j^n - \left(-C_{j+\frac{1}{2}}(w_{j+1}^{n+1} - w_j^{n+1}) + D_{j-\frac{1}{2}}(w_j^{n+1} - w_{j-1}^{n+1})\right), \tag{5.3.5}$$

where $C_{j+\frac{1}{2}}$ and $D_{j-\frac{1}{2}}$ are functions of w^n and /or w^{n+1} at various (usually neighboring) grid points. If $C_{j+\frac{1}{2}} \geq 0$ and $D_{j-\frac{1}{2}} \geq 0$, then the scheme (5.3.5) is TVD. If in addition to these conditions we have $C_{j+\frac{1}{2}} + D_{j+\frac{1}{2}} \leq 1$, then the scheme (5.3.4) is TVD.

5.3.2 Implicit nonstandard schemes by nonlocal approximation

Here, we design nonstandard schemes by exploiting the nonlocal approximation of nonlinear terms as stated in (ii) of Definition 5.1.2. We consider nonlocal approximation of the function L, given in (5.3.1), for deriving nonstandard total variation diminishing schemes for equation (5.1.3). The next stated techniques are discussed in the case when L is obtained from spacial discretization using Lax-Friedrichs numerical flux [90]

$$\begin{aligned} \hat{f}_{j+\frac{1}{2}} &= \frac{1}{2}\left(f(w_{j+1}) + f(w_j) - \alpha(w_{j+1} - w_j)\right), && (5.3.6)\\ \alpha &= \max_w |f'(w)|, && (5.3.7) \end{aligned}$$

the maximum being taken over the relevant range of w.

We consider nonlocal approximation of the function L for deriving nonstandard TVD schemes for equation (5.1.3). Below, we propose an implicit scheme of Euler type

$$w_j^{n+1} = w_j^n + \frac{\Delta t}{2\Delta x}\left(\alpha(w_{j+1}^{n+1} - 2w_j^{n+1} + w_{j-1}^{n+1}) - (w_{j+1}^{n+1} - w_{j-1}^{n+1})\frac{f(w_{j+1}^n) - f(w_{j-1}^n)}{w_{j+1}^n - w_{j-1}^n}\right). \quad (5.3.8)$$

We should note that the linear terms in (5.3.1) are evaluated at $t = t_{n+1}$. The expression $f(w_{j+1}) - f(w_{j-1})$ is multiplied and divided by $w_{j+1}^n - w_{j-1}^n$, where the multiplier is evaluated at $t = t_{n+1}$ and the remaining part of the expression evaluated at $t = t_n$.

One of our main result in this section is the following theorem:

Theorem 5.3.1 *The scheme* (5.3.8) *is qualitatively stable with respect to the total variation diminishing property (5.1.7).*

Proof. *The scheme (5.3.8) can be written as*

$$\begin{aligned} w_j^{n+1} = w_j^n &+ \frac{\Delta t}{2\Delta x}\left(\alpha - \frac{f(w_{j+1}^n) - f(w_{j-1}^n)}{w_{j+1}^n - w_{j-1}^n}\right)(w_{j+1}^{n+1} - w_j^{n+1}) \\ &- \frac{\Delta t}{2\Delta x}\left(\alpha + \frac{f(w_{j+1}^n) - f(w_{j-1}^n)}{w_{j+1}^n - w_{j-1}^n}\right)(w_j^{n+1} - w_{j-1}^{n+1}). \end{aligned}$$

Therefore the scheme (5.3.8) can be represented in the form (5.3.5) with

$$C_{j+\frac{1}{2}} = \frac{\Delta t}{2\Delta x}\left(\alpha - \frac{f(w_{j+1}^n) - f(w_{j-1}^n)}{w_{j+1}^n - w_{j-1}^n}\right), \; D_{j-\frac{1}{2}} = \frac{\Delta t}{2\Delta x}\left(\alpha + \frac{f(w_{j+1}^n) - f(w_{j-1}^n)}{w_{j+1}^n - w_{j-1}^n}\right).$$

Using (5.3.7), we obtain that $C_{j+1} \geq 0$ and $D_{j-\frac{1}{2}} \geq 0$ for all j. Hence it follows from Lemma 5.3.1 that the scheme (5.3.8) is TVD. ■

Using standard techniques of numerical analysis [90], one can easily obtain that for linear systems the scheme (5.3.8) is consistent and unconditionally stable. Moreover, the qualitative stability of the scheme (5.3.8) also does not impose any condition on Δx and /or Δt.

We should note that one step in the time dimension requires the solutions of a tridiagonal linear system. Hence the computation effort is similar to the one for explicit methods. Furthermore, the suggested scheme is not unique. One may use a different kind of nonlocal approximation to obtain a TVD scheme. For example, the scheme

$$\begin{aligned} w_j^{n+1} &= w_j^n + \frac{\Delta t}{2\Delta x}\left(\alpha(w_{j+1}^{n+1} - 2w_j^{n+1} + w_{j-1}^{n+1}) - (w_{j+1}^{n+1} - w_j^{n+1})\frac{f(w_{j+1}^n) - f(w_j^n)}{w_{j+1}^n - w_j^n}\right) \\ &\quad - \frac{\Delta t}{2\Delta x}\left((w_j^{n+1} - w_{j-1}^{n+1})\frac{f(w_j^n) - f(w_{j-1}^n)}{w_j^n - w_{j-1}^n}\right) \end{aligned}$$

is also TVD.

5.3.3 Explicit nonstandard schemes by renormalization

The schemes in this subsection are based on the renormalization of the denominator of the discrete derivatives, see Definition 5.1.2 (i). This means that the denominator Δt in the discrete time derivative is replaced by a function $\psi(\Delta t)$ satisfying (5.1.5). In order to obtain elementary stable schemes, that is, schemes which are qualitatively stable with respect to fixed points of the differential equations and their stability, the following renormalization was considered, [103], [8]:

$$\psi(\Delta t) = \frac{\phi(q\Delta t)}{q}, \tag{5.3.9}$$

where the function ϕ is such that

$$\phi(z) \quad = \quad z + o(z^2) \text{ as } z \to 0, \tag{5.3.10}$$

$$0 \quad < \quad \phi(z) < 1 \text{ for } z > 0, \tag{5.3.11}$$

and $q = \max\{|\lambda|\}$, λ tracing the eigenvalues of the Jacobian $J(\tilde{v})$ of the right hand side of equation (5.3.1) at the fixed points $\tilde{v}$ of the equation. The choice of the number q is not so critial. In practice, one may take $q = \max ||J(\tilde{v})||_\infty$, where $||.||_\infty$ is the matrix norm associated with the supremum norm on $\mathbb{R}^n$. We will show that similar renormalization also ensures the TVD property of the scheme. We consider function ψ as given by (5.3.9) where the value of q is suitably determined by the function L.

Let us consider first the Euler scheme

$$\frac{w^{n+1} - w^n}{\psi(\Delta t)} = L(w^n), \tag{5.3.12}$$

where L is also obtained from special discretization using Lax-Friedrichs numerical flux given in (5.3.6)

The second main result in this section is the following theorem.

Theorem 5.3.2 *The scheme (5.3.12) where*

$$\psi(z) = \frac{\phi(\frac{\alpha z}{\Delta x})}{\frac{\alpha}{\Delta x}}, \; z > 0,$$

and ϕ satisfies conditions (5.3.10)-(5.3.11) is qualitatively stable with respect to total variation diminishing property (5.1.7).

Proof. *The scheme (5.3.12) can be written in the form*

$$\begin{aligned} w_j^{n+1} &= w_j^n + \frac{\psi(\Delta t)}{2\Delta x}\left(\alpha(w_{j+1}^n - 2w_j^n + w_{j-1}^n) - f(w_{j+1}^n) + f(w_{j-1}^n)\right) \\ &= w_j^n + \frac{\psi(\Delta t)}{2\Delta x}\left(\alpha - \frac{f(w_{j+1}^n) - f(w_j^n)}{w_{j+1}^n - w_j^n}\right)(w_{j+1}^n - w_j^n) \\ & \quad -\frac{\psi(\Delta t)}{2\Delta x}\left(\alpha + \frac{f(w_j^n) - f(w_{j-1}^n)}{w_j^n - w_{j-1}^n}\right)(w_j^n - w_{j-1}^n). \end{aligned}$$

Therefore (5.3.12) can be represented in the form (5.3.4) with

$$\begin{aligned} C_{j+\frac{1}{2}} &= \frac{\psi(\Delta t)}{2\Delta x}\left(\alpha - \frac{f(w_{j+1}^n) - f(w_j^n)}{w_{j+1}^n - w_j^n}\right) \\ D_{j-\frac{1}{2}} &= \frac{\psi(\Delta t)}{2\Delta x}\left(\alpha + \frac{f(w_j^n) - f(w_{j-1}^n)}{w_j^n - w_{j-1}^n}\right). \end{aligned}$$

Using the definition of α*, see (5.3.7), it is easy to see that* $C_{j+1} \geq 0$*, and* $D_{j-\frac{1}{2}} \geq 0$*. Furthermore, we have*

$$\begin{aligned} C_{j+1} + D_{j+\frac{1}{2}} &= \frac{\psi(\Delta t)}{2\Delta x}\left(\alpha - \frac{f(w_{j+1}^n) - f(w_j^n)}{w_{j+1}^n - w_j^n}\right) + \frac{\psi(\Delta t)}{2\Delta x}\left(\alpha + \frac{f(w_{j+1}^n) - f(w_j^n)}{w_{j+1}^n - w_j^n}\right) \\ &= \frac{\phi(\frac{\alpha\Delta t}{\Delta x})}{\frac{\alpha}{\Delta x}2\Delta x}(2\alpha) = \phi(\frac{\alpha\Delta t}{\Delta x}). \end{aligned}$$

Then it follows from (5.3.11) that $C_{j+1} + D_{j+\frac{1}{2}} \leq 1$*. Hence we can apply the Harten's Lemma, see Lemma 5.3.2, and obtain that the scheme (5.3.12) is TVD.* ■

Renormalization can also be used in higher order methods, e.g., Runge-Kutta methods. For investigation of TVD properties a Runge-Kutta method is typically written in the so called Shu-Osher form, [48], namely

$$\begin{aligned} y^{(0)} &= w^n \\ y^{(i)} &= \sum_{j=1}^{i-1}(\lambda_{ij}y^{(j)} + \Delta t\mu_{ij}L(y^{(j)})),\ i = 1, 2, ..., m \\ w^{n+1} &= y^{(m)}. \end{aligned}$$

By consistency

$$\sum_{j=0}^{i-1}\lambda_{ij} = 1,\ i = 1, ..., m.$$

Therefore in each intermediate step of the method $y^{(i)}$ is a convex combination of Euler forward operators:

$$y^{(i)} = \sum_{j=1}^{i-1}\lambda_{ij}\left(y^{(j)} + \Delta t\frac{\mu_{ij}}{\lambda_{ij}}L(y^{(j)})\right).$$

If these operators are TVD then the Runge-Kutta method is also TVD [48]. Following the result of Theorem 5.3.2 we will obtain a TVD scheme if the Euler operator involving $y^{(j)}$ above is renormalized by

$$\psi_{ij}(\Delta t) = \frac{\phi(\frac{\alpha\mu_{ij}\Delta t}{\lambda_{ij}\Delta x})}{\frac{\alpha\mu_{ij}}{\lambda_{ij}\Delta x}},$$

where the function ϕ satisfies conditions (5.3.10)-(5.3.11). Note that this function might have to satisfy additional conditions for the scheme to be accurate of particular order. We will illustrate this by an example.

Following the discussion above, the following two stage scheme is TVD:

$$y^{(1)} = w^n + \phi\left(\frac{\alpha\Delta t}{\Delta x}\right)\frac{\Delta x}{\alpha}L(w^n) \tag{5.3.13}$$

$$w^{n+1} = \frac{1}{2}w^n + \frac{1}{2}y^{(1)} + \frac{1}{2}\phi\left(\frac{\alpha\Delta t}{\Delta x}\right)\frac{\Delta x}{\alpha}L(y^{(1)}). \tag{5.3.14}$$

Using standard techniques one can also obtain that it is of order two provided $\phi(z) = z + o(z^3)$.

Remark 5.3.1 *Since the schemes (5.3.8) and (5.3.12) are TVD, the convergence follows from [90, Theorem 15.2].*

5.3.4 Numerical results

We present the results of our numerical experiments for the implicit and explicit nonstandard total variation diminishing schemes considered in Subsection 5.3.2 and Subsection 5.3.3, respectively, to Burger's equation. We compare them with the standard techniques given in [60].

Example 5.3.1 *We apply the schemes considered in Subsection 5.3.2 and Subsection 5.3.3 to the Burger's equation*

$$v_t + \left(\frac{1}{2}v^2\right)_x = 0, \quad -1 \le x \le 9, \quad t > 0. \tag{5.3.15}$$

Here the flux is $f(v) = \frac{1}{2}v^2$.

It is well known [90] that the entropy solution of equation (5.3.15) develops discontinuities (shocks) even for smooth initial condition. To simplify the matters we take the Riemann initial data

$$v(x,0) = \begin{cases} 1.2, & -1 \le x < 0 \\ 0, & 0 \le x \le 9. \end{cases} \tag{5.3.16}$$

The entropy solution of the problem (5.3.15)-(5.3.16) is given by

$$v(x,t) = \begin{cases} 1.2 & , \quad x < 0.6t \\ 0 & , \quad x \ge 0.6t. \end{cases}$$

For the considered problem (5.3.15)-(5.3.16), after some obvious transformations, the scheme (5.3.8) can be written in the form

$$w_j^{n+1} = w_j^n + \frac{\Delta t}{2\Delta x}\left(\alpha(w_{j+1}^{n+1} - 2w_j^{n+1} + w_{j-1}^{n+1}) - \frac{1}{2}(w_{j+1}^{n+1} - w_{j-1}^{n+1})(w_{j+1}^n + w_{j-1}^n)\right). \tag{5.3.17}$$

Below, the solid line is the exact solution $v(x,t)$, the points joined by a dashed line are numerical solutions. It was shown in [61] that non TVD methods typically produce oscillations around the points of discontinuity. Figure 5.7 shows such oscillations around the point $x = 0$ produced by the standard Euler method applied to problem (5.3.15)-(5.3.16). Figures 5.8, 5.9 and 5.10 show the numerical solution of (5.3.15)-(5.3.16) obtained by nonstandard implicit scheme (5.3.17) for various time steps Δt. One can observe that while an increase in Δt affects the accuracy of the solution it nevertheless remains TVD and free of spurious oscillations.

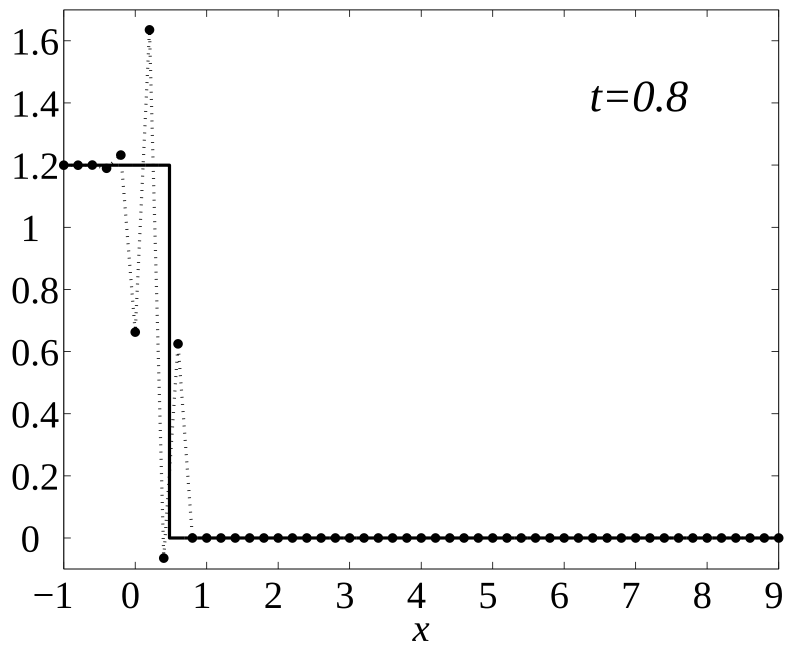

Figure 5.7: *Numerical solution of (5.3.15)-(5.3.16) by the standard explicit Euler method (5.3.3) with* $\Delta x = \Delta t = 0.2$.

Similar results are obtained using the explicit schemes (5.3.12). For the considered problem (5.3.15)-(5.3.16), Euler's method (5.3.12) can be written as

$$w_j^{n+1} = w_j^n + \frac{1}{2\alpha}\phi\left(\frac{\alpha\Delta t}{\Delta x}\right)\left(\alpha(w_{j+1}^n - 2w_j^n + w_{j-1}^n) - (w_{j+1}^n)^2 + (w_{j-1}^n)^2\right), \quad (5.3.18)$$

where we take $\phi(z) = 1 - e^{-z}$. *The numerical solution given by the scheme (5.3.18) computed with* $\Delta x = \Delta t = 0.2$ *is presented on Figure 5.11. Figure 5.12 represents the solution produced by the Runge-Kutta method (5.3.13)-(5.3.14) with renormalizing function* $\phi(z) = \frac{1-e^{-z^2}}{z}$ *so that the method is of order two. Let us note that since the exact solution is discontinuous, a higher order method does not necessarily give a better approximation. Naturally, the accuracy can be improved by decreasing the step sizes. However, the major point here is that irrespective of the step sizes the numerical solutions is free of spurious oscillations and its total variation does not increase with time, for this particular equation it is in fact constant.*

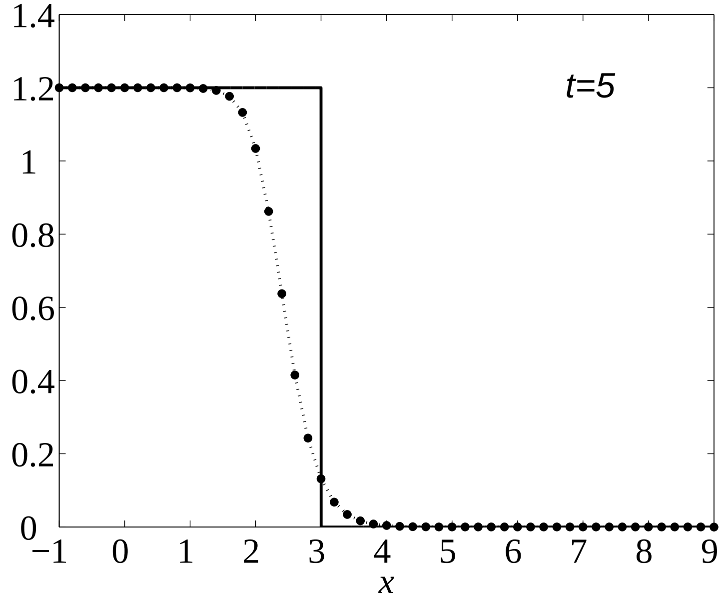

Figure 5.8: *Numerical solution of (5.3.15)-(5.3.16) by the implicit nonstandard scheme (5.3.17) with* $\Delta x = \Delta t = 0.2$.

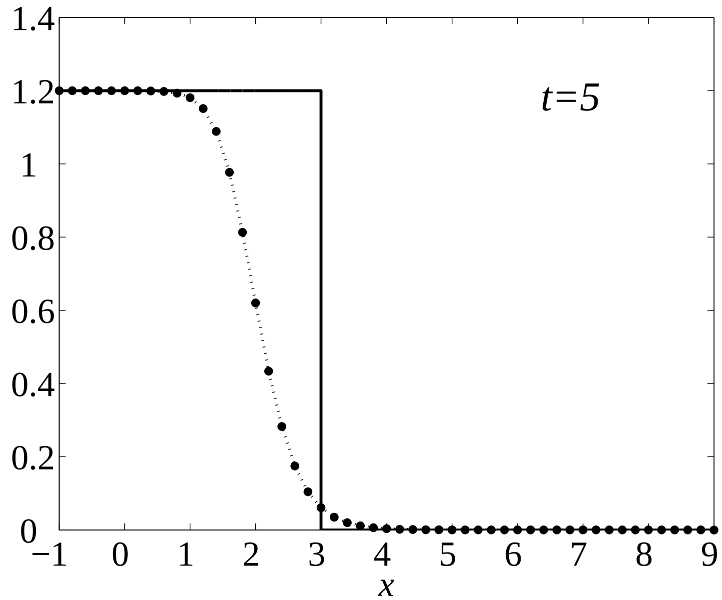

Figure 5.9: *Numerical solution of (5.3.15)-(5.3.16) by the implicit nonstandard scheme (5.3.17) with* $\Delta x = 0.2$, $\Delta t = 0.5$.

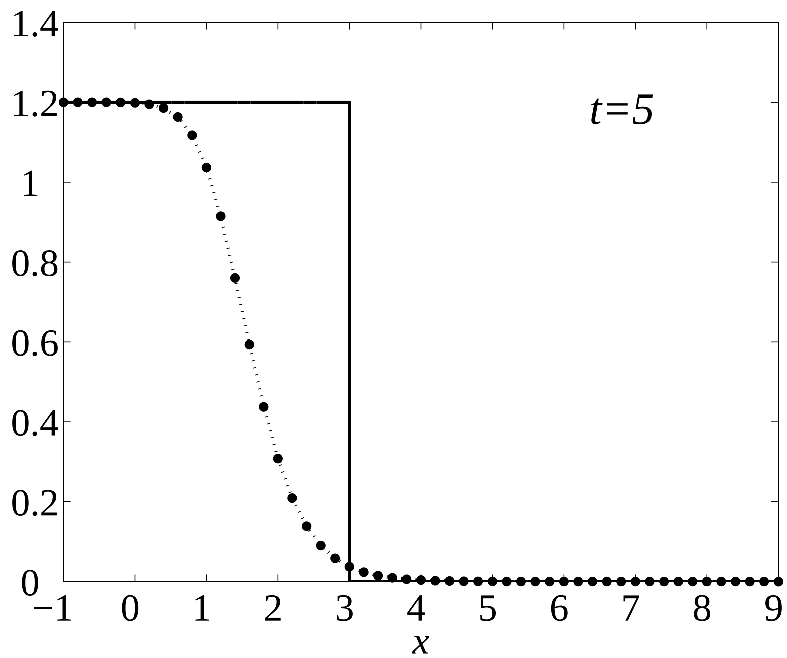

Figure 5.10: *Numerical solution of (5.3.15)-(5.3.16) by the implicit nonstandard scheme (5.3.17) with* $\Delta x = 0.2$, $\Delta t = 1.0$.

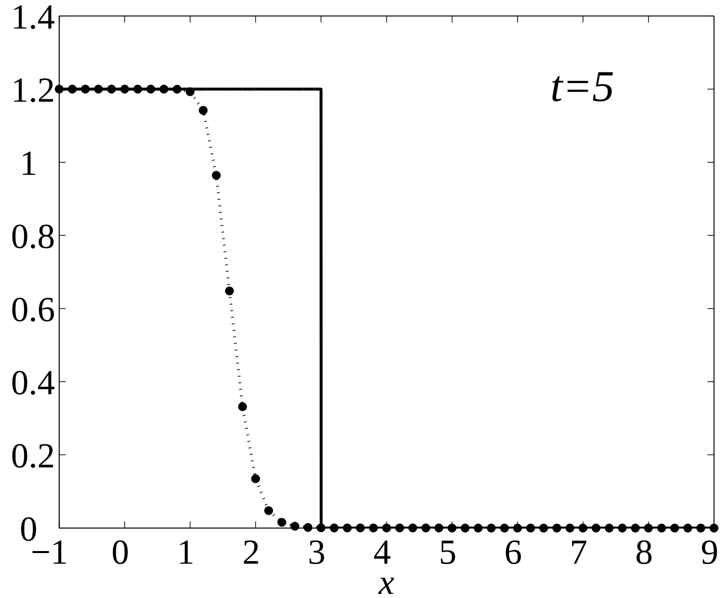

Figure 5.11: *Numerical solution of (5.3.15)-(5.3.16) by the explicit nonstandard scheme (5.3.18) with* $\Delta x = \Delta t = 0.2$.

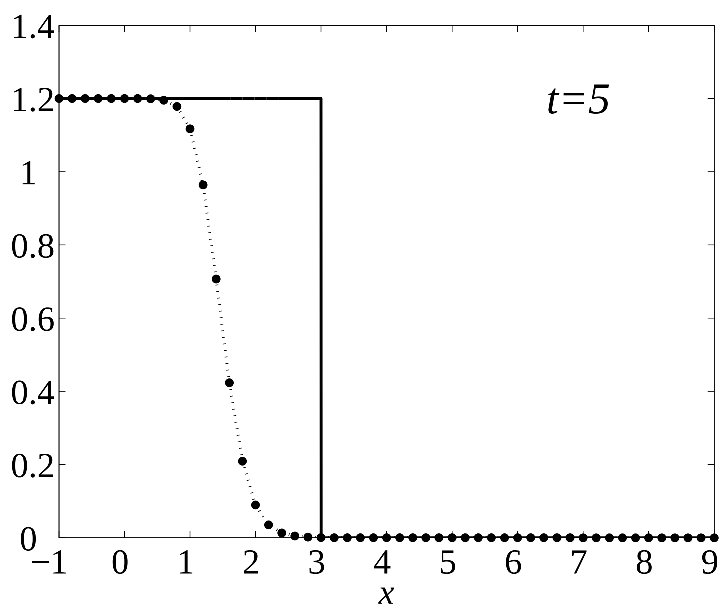

Figure 5.12: *Numerical solution of (5.3.15)-(5.3.16) by the Runge-Kutta nonstandard method (5.3.13)-(5.3.14) with* $\Delta x = \Delta t = 0.2$

5.3.5 Numerical solution of Hamilton-Jacobi equation via TVD method for conservation laws

Consider the one-space dimensional Dirichlet problem for Hamilton-Jacobi equation:

$$\begin{aligned} u_t + H(u_x) &= 0 && (5.3.19)\\ u(x,0) &= u_0(x), && (5.3.20)\end{aligned}$$

where H is smooth function of u_x. Differentiate (5.3.19) with respect to x and let $v = u_x$. Then a problem (5.3.19)-(5.3.20) is equivalent to the following problem for conservation laws:

$$\begin{aligned} v_t + H(v)_x &= 0 && (5.3.21)\\ v(x,0) &= \frac{d}{dx}u_0(x). && (5.3.22)\end{aligned}$$

Thus, the question of viscosity solution of (5.3.19) is equivalent to the question of entropy solution of (5.3.21). More precisely, if u is the unique viscosity solution of (5.3.19) satisfying (5.3.20), then $v = u_x$ is the unique entropy solution of (5.3.21) satisfying (5.3.22). Conversely, if v is the unique entropy solution of (5.3.21) satisfying (5.3.22), then u defined by

$$u(x,t) = \int^x v(y,t)dy$$

is the unique viscosity solution of (5.3.19) satisfying (5.3.20). Furthermore, if u_0 is bounded uniformly continuous function in $\mathbb{R}$, then $u_x(x,t) = v(x,t)$ is satisfied almost everywhere [77].

We recall that this equivalence has been exploited by many authors trying to translate successful numerical methods for conservation laws to methods for the Hamilton-Jacobi equations.

If we want to solve the problem (5.3.19)-(5.3.20) on the finite interval $[a,b]$, first we divide it into N cells as follows:

$$a = x_{\frac{1}{2}} < x_{\frac{3}{2}} < ... < x_{N+\frac{1}{2}} = b.$$

We denote

$$I_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}),\ x_j = \frac{1}{2}(x_{j-\frac{1}{2}} + x_{j+\frac{1}{2}}),\ \Delta x_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}},\ h = \max_j \Delta x_j,\ j = 1, 2, ..., N.$$

Now, we define the following approximation space

$$V_h^k = \{w : w|_{I_j} \in P^k(I_j), \quad j = 1, ..., N\},$$

where $P^k(I_j)$ is the set of all polynomials of degree $\leq k$ on the cell I_j.

A kth-order discontinuous Galerkin scheme for the problem (5.3.19)-(5.3.20) can be defined as follows [70]: find $u \in V_h^k$, such that

$$\int_{I_j} \frac{d}{dt}(u_x) f\, dx - \int_{I_j} H(u_x) f_x\, dx + \hat{H}_{j+\frac{1}{2}} f^-_{j+\frac{1}{2}} - \hat{H}_{j-\frac{1}{2}} f^+_{j-\frac{1}{2}} = 0, \quad j = 1, ..., N \quad (5.3.23)$$

holds for any $f \in V_h^{k-1}$. Here

$$\hat{H}_{j+\frac{1}{2}} = \hat{H}\left((u_x)^-_{j+\frac{1}{2}}, (u_x)^+_{j+\frac{1}{2}}\right)$$

is a monotone flux, $(u_x)^\pm_{j+\frac{1}{2}}$ and $f^\pm$ are the numerical approximations, respectively, to the point values of $u_x(x_{j+\frac{1}{2}}, t)$, and $f(x_{j+\frac{1}{2}})$ from left and right and $\alpha = \max_v |H'(v)|$ with maximum taken over the range covered by $(u_x)^-_{j+\frac{1}{2}}$ and $(u_x)^+_{j+\frac{1}{2}}$. We will mainly use the Lax-Friedrichs flux

$$\hat{H}\left((u_x)^-_{j+\frac{1}{2}}, (u_x)^+_{j+\frac{1}{2}}\right) = \frac{1}{2}\left(H((u_x)^-_{j+\frac{1}{2}}) + H((u_x)^+_{j+\frac{1}{2}}) - \alpha\left((u_x)^+_{j+\frac{1}{2}} - (u_x)^-_{j+\frac{1}{2}}\right)\right). \quad (5.3.24)$$

Notice that the method described above is exactly the discontinuous Galerkin method for the conservation law equation (5.3.21) satisfied by the derivative $v = u_x$, see [70]. This only determines u for each element up to a constant, since it is only a scheme for u_x. In [70], the missing constant can be obtained in one of the following two ways:
The first way is to require that

$$\int_{I_j} (u_t + H(u_x)) f\, dx = 0, \quad j = 1, ..., N,$$

for all $f \in V_h^0$, that is

$$\int_{I_j} (u_t + H(u_x))\, dx = 0, \quad j = 1, ..., N. \quad (5.3.25)$$

The second one is to use

$$u(x_j, t) = u(x_1, t) + \int_{x_1}^{x_j} u_x(x, t)\, dx \quad (5.3.26)$$

to determine the missing constant for the cell I_j.

About the stability of the method proposed above, we can quote the following result of Jiang and Shu [73]. Here we assume compact support or periodic boundary condition for the solution u.

Lemma 5.3.2 [73] *The following L^2-stability result for the derivative u_x holds for the discontinuous Galerkin method (5.3.23), for any order of accuracy k applied to problem (5.3.19)-(5.3.20):*

$$\frac{d}{dt}\int_a^b u_x^2\,dx \leq 0. \tag{5.3.27}$$

It is clear that the function $\varphi(t) = \int_a^b u_x^2(x,t)dx$ is non-increasing in t. The relation (5.3.27) implies total variation bounded (TVB) property for the numerical solution of u:

$$\begin{aligned} TV(u) = \int_a^b |u_x(x,t)|\,dx &\leq \sqrt{b-a}\sqrt{\int_a^b (u_x(x,t))^2\,dx} \\ &\leq \sqrt{b-a}\sqrt{\int_a^b \left(\frac{d}{dx}u_0(x)\right)^2 dx}. \end{aligned}$$

This is a rather strong stability result, considering that it applies even if the derivative of the solution u_x develops discontinuities, and the scheme (5.3.23) can be of arbitrary high order in accuracy. It also implies convergence of at least a subsequences of the numerical solution u when $h \to 0$. However, this stability result is not strong enough to imply that the limit solution is the viscosity solution of (5.3.19). However, in case when $k = 1$, the scheme (5.3.23) becomes

$$\frac{d}{dt}u_x(x_j,t) = -\frac{1}{\Delta x_j}\left(\hat{H}_{j+\frac{1}{2}} - \hat{H}_{j-\frac{1}{2}}\right) \tag{5.3.28}$$

which is the conservative scheme for $v = u_x$. We know [89] that solutions of such schemes, if convergent, would converge to an entropy solutions of (5.3.21). Moreover, the numerical solution for u obtained from the relation $v = u_x$ would converge to a viscosity solution of (5.3.19). Here we consider only the case when $k = 1$.

We adopt a local orthogonal basis of V_h^1 over I_j, namely, $\{\phi_0^{(j)}, \phi_1^{(j)}\}$, where $\phi_0^{(j)}(x) = 1$ and $\phi_1^{(j)} = x - x_j, x \in I_j$. The solution $u(x,t) \in V_h^1$ of (5.3.19)-(5.3.20) can be written as

$$u(x,t) = w_j(t) + v_j(t)(x - x_j), \quad x \in I_j, \quad j = 1, ..., N. \tag{5.3.29}$$

From (5.3.29), it is clear that

$$\frac{\partial}{\partial x}u(x,t) = v_j(t) \tag{5.3.30}$$

and

$$u(x_j, t) = w_j(t).$$

Substituting u given by (5.3.29) into (5.3.28) and remember (5.3.30) and (5.3.24), we have the following scheme

$$\frac{d}{dt}v_j(t) = -\frac{1}{2\Delta x_j}\left(H(v_{j+1}(t)) - H(v_{j-1}(t)) - \alpha(v_{j+1}(t) - 2v_j(t) + v_{j-1}(t))\right). \tag{5.3.31}$$

The scheme (5.3.31) is the same as scheme given by (5.3.2) in Subsection 5.3.2 with Lax-Friedrichs flux given by (5.3.24) for the conservation law (5.3.21) satisfied by the derivatives $v = u_x$.

The function $w_j(t)$ (missing constant) is obtained from (5.3.25) as follows

$$\int_{I_j}\left(\frac{d}{dt}\left(w_j(t) + v_j(t)(x - x_j)\right) + H(v_j(t))\right)dx = 0, \quad j = 1, ..., N.$$

Since $\int_{I_j}(x - x_j)\,dx = 0$, the above equation gives

$$\frac{d}{dt}w_j(t) = -H(v_j(t)). \tag{5.3.32}$$

For the time discretization, let Δt be the constant time step and v_j^n, w_j^n, denote, respectively, the approximation solutions v_j and w_j at time $t = n\Delta t$. The way for finding the function $u(x,t)$ can be carried out in the following steps:

Firstly, we evaluate v_j^n in every I_j, using the nonstandard total variation diminishing methods for (5.3.21)-(5.3.22) developed in Section 5.5.4, namely

$$v_j^{n+1} = v_j^n - \frac{1}{2\alpha}\phi\left(\frac{\alpha\Delta t}{\Delta x_j}\right)\left(H(v_{j+1}^n) - H(v_{j-1}^n) - \alpha(v_{j+1}^n - 2v_j^n + v_{j-1}^n)\right),$$

where $\phi(z) = 1 - e^{-z}$.

Secondly, w_j^n is given by average of two solutions given by (5.3.32) and (5.3.26).

Finally, we update $u(x, n\Delta t) = w_j^n + v_j^n(x - x_j), x \in I_j$.

Next, we provide numerical experimental result to demonstrate the behavior of our scheme.

Example 5.3.2 *Consider the initial-boundary value problem.*

$$\begin{cases} u_t + \frac{(u_x)^2}{2} = 0 & , \quad 0 \leq x \leq 2\pi, t > 0 \\ u(x,0) = \sin(x) & , \quad 0 \leq x \leq 2\pi \\ u(0,t) = u(2\pi,t) & , \quad t \geq 0. \end{cases} \tag{5.3.33}$$

Here, the Lax-Friedrichs flux given by (5.3.6) and uniform meshes of $N = 70$ *elements are used. The exact solution when* u *is still smooth is obtained by the characteristic methods. First solve* x_0 *from* $x = x_0 + \cos(x_0)t$ *then get* u *as* $u(x,t) = \sin(x_0) + \frac{(\cos(x_0))^2}{2}t$*. In Figure 5.13, the asterisks joined by lines is numerical solution at* $t = 2$ *with* $\Delta t = 0.01, \Delta x = 2\pi/70$ *for the* P^1 *case while the solid line is exact solution. It is clear that at* $t = 2$ *the exact solution develops a discontinuous derivative and the numerical solution approximates the viscosity solution very well.*

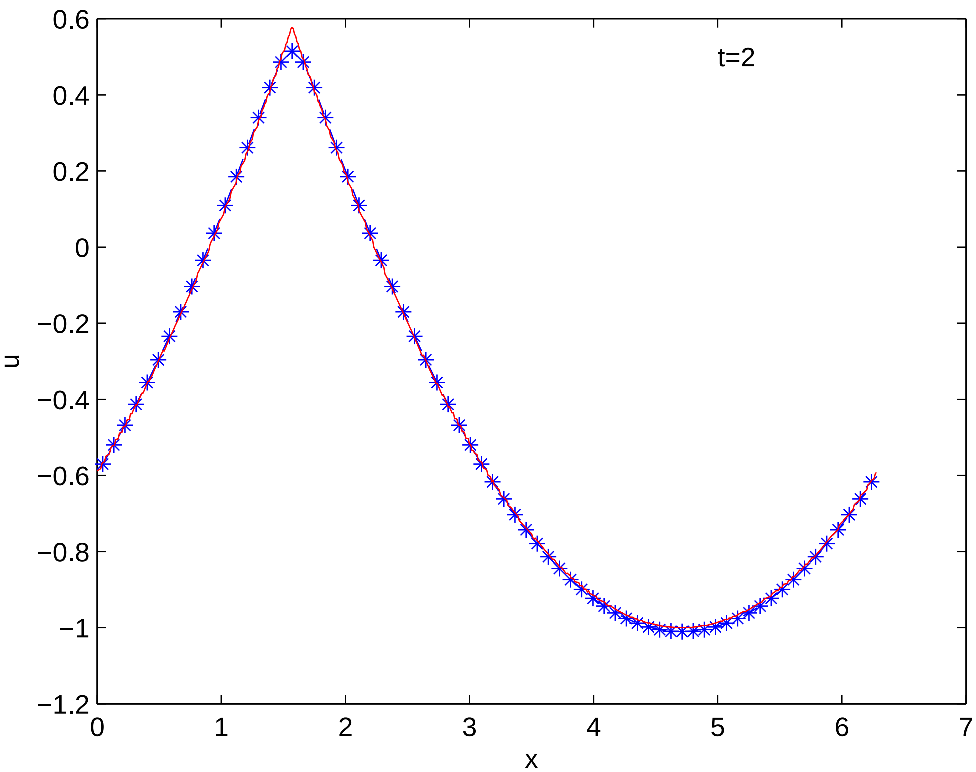

Figure 5.13: *Numerical solution of (5.3.33) with* $\Delta x = \frac{2\pi}{70}$ *and* $\Delta t = 0.01$.

Chapter 6

Conclusion

The Hausdorff continuous functions, being a particular class of interval valued functions, belong to what is usually called Interval Analysis, see [107]. Nevertheless, recent results have shown that they can provide exact solutions to problems formulated in terms of point valued functions [2]. A long outstanding problem related to the Dedekind order completion of spaces $C(X)$ of real valued continuous functions on rather arbitrary topological spaces X was solved through Hausdorff continuous [4]. Following this breakthrough, a significant improvement of the regularity properties of the solutions of large classes of nonlinear partial differential equations obtained through the Dedekind order completion method, see [109], was discussed in [12, 13]. Namely, it was shown that these solutions can be assimilated with the class of Hausdorff continuous functions on the open domains Ω. The applications of the class of Hausdorff continuous functions [12, 13, 10] show that this class may play an important role in what is typically called Real Analysis. In particular, one may note that one of the main engines behind the development of the various spaces in Real and Abstract Analysis are the partial differential equations with the need to assimilate the various types of "weak" solutions. Moreover, the set of Hausdorff continuous functions might be a viable alternative to some of the presently used functions (e.g. Lebesgue spaces, Sobolev spaces) with the advantage of being both more regular and universal.

In this work, the Hausdorff continuous functions are linked with the concept of discontinuous viscosity solutions. As shown in the Chapter 1 the definition of viscosity solution, see Definition 1.3.3, has an implicit interval character since it places requirements only on the upper semicontinuous envelope $S(u)$ and the lower semicontinuous envelope $I(u)$.

For a Hausdorff continuous viscosity solution u, the functions $I(u)$ and $S(u)$ are as close as they can be in the sense of the distance ρ defined by (2.3.6), namely, we have $\rho(I(u), S(u)) = 0$.

Hence, the requirements that a viscosity solution is Hausdorff continuous has a direct interpretation, which we find clearer than the requirements related to other concepts of discontinuous viscosity solutions, e.g., envelope solutions.

In the theoretical study of this thesis, we come up with the following main results. First, we define the concept of viscosity solution for interval valued functions in $\mathbb{H}(\Omega)$. We proved an existence theorem for Hausdorff continuous viscosity solution using Perron's method. The solution is constructed as a supremum of a subset of viscosity subsolutions in the set of Hausdorff continuous functions. It is shown that there is the relation between the Hausdorff continuous viscosity solutions and the existing theory of discontinuous viscosity solutions. Namely, any H-continuous viscosity solution is discontinuous viscosity solution as defined by Ishii, and it is typically also an envelope viscosity solution. Moreover, the H-continuous viscosity solutions is stronger concept than the concept of discontinuous viscosity solution given by Ishii and as well as under wild the concept of envelope viscosity solution. Uniqueness result have been shown using comparison principle between H-continuous viscosity subsolutions and supersolutions. This comparison principle is stronger than comparison principle used in connection with the lower semicontinuous viscosity supersolutions and upper semicontinuous viscosity subsolutions in the sense that the existence result holds under the same conditions. Sufficient conditions for a weaker form of this comparison principle are given. However, it could be interesting to give sufficient conditions for a strong comparison principle for Hausdorff continuous viscosity subsolutions and supersolutions. Finally, we expressed the Hausdorff viscosity solution of Hamilton-Jacobi equations as solutions to an operator equation involving the extended a Hamiltonian operator in the same way as the classical solution of Hamilton-Jacobi equations are solutions of operator equation associated of this Hamilton-Jacobi equations. It is shown also that the value function of discounted minimum time problem is an envelope viscosity solution of associated Hamilton-Jacobi-Bellman equation.

Numerical study deals with two approaches to numerical solutions for Hamilton-Jacobi equations. First approach is a finite difference space discretization coupled with a non-standard difference time discretization for constructing monotone scheme for any time step size, see Section 5.2.

This is motivated by the paper [91], where a severe restriction on the time step size is imposed for the numerical scheme for Hamilton-Jacobi equations obtained through the coupling of the finite difference method (in space) and the finite difference method (in time) to be monotone.

We have relaxed this restriction by using Micken's nonstandard finite difference method [103]. More precisely, Micken's rule of nonlocal approximation is exploited and this leads to a nonstandard scheme that replicates the monotonicity property of the Hamilton-Jacobi equations for all positive step sizes. Furthermore, the superiority of the nonstandard method to the standard one is confirmed by numerical results.

The second approach is on total variation diminishing scheme for conservation laws. The schemes preserving the essential physical property of diminishing total variation for conservation laws are studied in Section 5.3. Such schemes are free of spurious oscillations around discontinuities. We have discussed nonstandard finite difference schemes, which have this qualitative stability property. We used Micken's rules of approximating nonlinear terms in a nonlocal way and of renormalizing denominators. The obtained schemes are computationally simple. Furthermore, the require no restriction on the time step-size as typical for qualitatively stable nonstandard schemes.

We exploited the fact that in one space dimension the derivatives of the solutions u of Hamilton-Jacobi equations satisfy conservation laws and applied the discontinuous Galerkin method of Hu and Shu [70] to get the scheme for u_x. This determines u for each element up to a constant, since it is only the scheme for u_x. The missing constant is obtained combining two ways developed in [70].

We think that the results presented in this thesis provide a foundation for future research in the following areas:

• Discontinuous viscosity solution of second order Hamilton-Jacobi equation in the sense of Hausdorff continuous function;

• the validity of Theorem 3.6.4 in $\mathbb{R}^n, n \geq 2$;

• TVD property for conservation laws in higher dimensional space;

• conditions implying strong comparison principle between H-continuous viscosity subsolutions and supersolutions.