# Chapter 4

# The Value Functions of Optimal Control Problem as Envelope Viscosity Solutions

By the classical Hamilton-Jacobi theory, if the value function (i.e. the optimal value function associated with the optimal control problems as a function of the initial time and state) is smooth, then it is a classical solution of the Hamilton-Jacobi-Bellman equation.

In general, however, the value function is nonsmooth and therefore can not satisfy the Hamilton-Jacobi-Bellman equation in the classical sense. In the control theory literature, several approaches have been developed to cope with this difficulty. Boltyanski [28] (see also [52]) restricted the class of controls so that the value function becomes piecewise smooth. Vinter and Lewis [133] characterized optimality through a sequence of continuously differentiable subsolutions of the Hamilton-Jacobi equation. If the value function is Lipschitz continuous, Havelock [66] and Clarke [32] characterized the value function as a generalized solutions of the Hamilton-Jacobi equation involving the Clarke generalized gradient.

Many of the works on viscosity solutions were devoted to application in the Dynamic Programming approach to deterministic optimal control problems, e.g, [94], [45], [16], [30]. According to Definition 1.2.1, introduced by Crandall and Lions, when the value function is uniformly continuous, it is then a viscosity solution of the Hamilton-Jacobi-Bellman equation [16].



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According to Definition 1.3.3, introduced by Ishii, the value function is characterized as a viscosity solution through its lower and upper semicontinuous envelopes. The reader can also refer to [24], [53], and [67].

In this chapter, we study a particular optimal control problem, namely discounted minimum time problem. Since an envelope viscosity solution is typically Hausdorff continuous viscosity solution, we will show that the value functions associated with this problem are the envelope viscosity solutions of associated Hamilton-Jacobi-Bellman equation.

## 4.1 Discounted Minimum Time Problem

Consider the optimal control problem with the state equation

$$\begin{cases} y'(t) = f(y(t), a(t)), \ t > 0 \\ y(0) = x \in \mathbb{R}^n. \end{cases} \tag{4.1.1}$$

Here, the control $a(.) \in \mathcal{A}$:=$\{a : [0, \infty) \to A \text{ measurable}\}$, where $\mathcal{A}$ is the *set of admissible controls*, $A \subset \mathbb{R}^m$ is a given *control space.*

We now list some basic assumptions on our control system which are made for most of the results of this chapter. We will assume:

$$A \text{ is compact metric space} \tag{4.1.2}$$

and the *dynamics* $f : \mathbb{R}^n \times A \to \mathbb{R}^n$ satisfies

$$f(x, a) \text{ is continuous on } \mathbb{R}^n \times A, \tag{4.1.3}$$

$$|f(x, a)| \leq L, \ \forall x \in \mathbb{R}^n, \ a \in A, \tag{4.1.4}$$

$$|f(x, a) - f(y, a)| \leq C|x - y|, \ \forall x, \ y \in \mathbb{R}^n, \ a \in A. \tag{4.1.5}$$

for some constants $L > 0$ and $C > 0$.

It is well known [134] from the theory of ordinary differential equations that assumptions (4.1.3), (4.1.4), and (4.1.5), ensure that for each continuous control $a$ and $x \in \mathbb{R}^n$, the state equation (4.1.1) has a unique solution $y_x(t, a)$, existing for $t \geq 0$.



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Given

$$\mathcal{T} \subseteq \mathbb{R}^n \text{ nonempty closed target with compact boundary } \partial\mathcal{T}, \tag{4.1.6}$$

as usual we denote by $t_x(a)$ the first time the trajectory associated with $a \in \mathcal{A}$ and starting at $x$ hits $\mathcal{T}$, i.e., the exit time of the trajectory, that is

$$t_x(a) = \begin{cases} \min\{s \geq 0 : y_x(s,a) \in \mathcal{T}\} \text{ if } \{s : y_x(s,a) \in \mathcal{T}\} \neq \emptyset \\ +\infty \qquad \text{otherwise.} \end{cases}$$

If $t_x(a) < +\infty$, then $y_x(t_x(a))$ denotes the point where the trajectory reaches the target $\mathcal{T}$. We denote by $\mathcal{R}$ the set of all $x$ such that $t_x(a) < +\infty$ for some control $a$ and call $\mathcal{R}$ the controllable set. Thus,

$$\mathcal{R} = \{x \in \mathbb{R}^n : t_x(a) < +\infty\}.$$

We are concerned with the following optimal control problem called discounted minimum time problem:

$$\inf_{a\in\mathcal{A}} \begin{cases} \int_0^{t_x(a)} e^{-s} ds \text{ if } t_x(a) < +\infty \\ 1 \qquad \text{otherwise.} \end{cases} \tag{4.1.7}$$

The value function associated with the problem (4.1.7) is defined by

$$u(x) = \begin{cases} 1 - e^{-T(x)} \text{ if } T(x) < +\infty \\ 1 \qquad \text{if } T(x) = +\infty, \end{cases} \tag{4.1.8}$$

where the function $T$ is the minimum time function and is defined by

$$T(x) = \inf_{a\in\mathcal{A}} t_x(a), \;\; x \in \mathcal{R}. \tag{4.1.9}$$

If $f(x, A) = \{f(x, a) : a \in A\}$ is a convex set for all $x \in \mathbb{R}^n$, then $T$ is lower semicontinuous function (see e.g. [67]), and so $u$ is lower semicontinuous.

It is well known [16] that, the Hamilton-Jacobi-Bellman equation associated to value function given in (4.1.8) is

$$u(x) + H(x, Du(x)) - 1 = 0, \;\; x \in \mathcal{R}, \tag{4.1.10}$$

with

$$H(x, p) = \sup_{a(t)\in\mathcal{A}} \{-f(x, a).p)\}. \tag{4.1.11}$$



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The following theorem, given in [16], gives the dynamic programming property.

**Theorem 4.1.1** *For all $s > 0$ the function $u$ define by (4.1.8) is*

$$u(x) = \inf_{a(.)\in\mathcal{A}} \{\int_0^s e^{-t}\,dt + u(y_x(s,a))e^{-s}\}, \text{ if } s \le T(x), \; x \in \mathcal{R}. \tag{4.1.12}$$

**Remark 4.1.1** *The above dynamic programming property implies that the function*

$$s \longmapsto \int_0^s e^{-t}\,dt + u(y_x(s,a))e^{-s}, \; s \in [0, t_x(a)]$$

*is nondecreasing.*

## 4.2 The Value Function as an Envelope Viscosity Solution

In this section we prove that the value function $u$, given in (4.1.12), is an envelope viscosity solution of associated Hamilton-Jacobi-Bellman equation (4.1.10).

Our main results are stated in the next theorems.

**Theorem 4.2.1** *Assume that (4.1.2), (4.1.3), (4.1.4), (4.1.5), and (4.1.6) hold and the set $f(x, A)$ is convex for all $x \in \mathbb{R}^n$. Assume that we have $u : \Omega \to \mathbb{R}$ is locally bounded in open set $\Omega \subset \mathbb{R}^n$ and that for all $x$ there exists $\tau > 0$ such that for $0 < t < \tau$, we have*

$$u(x) = \inf_{a\in\mathcal{A}} \left( \int_0^t e^{-s}\,ds + u(y_x(t,a))e^{-t} \right). \tag{4.2.1}$$

*Then $u$ is an envelope viscosity solution of the equation*

$$u + H(x, Du) - 1 = 0 \text{ in } \Omega = \mathbb{R}^n \backslash \mathcal{T}, \tag{4.2.2}$$

*with Hamiltonian $H$ is defined in (4.1.11).*

Before we prove Theorem 4.2.1, let us remark that the minimum time function does not change if we add the null vector field to the system (4.1.1).

**Theorem 4.2.2** [16] *Let $\widetilde{A} = A \cup \{\widetilde{a}\}$, $\widetilde{a} \notin A$, $\widetilde{f}(x, a) = f(x, a)$ if $a \in A$, $\widetilde{f}(x, \widetilde{a}) = 0$ for all $x$. Then the minimum time function $\widetilde{T}$ associated with the system*

$$\begin{cases} y' = \widetilde{f}(y, \alpha) + h\beta, \; h > 0, \\ y(0) = x, \end{cases} \tag{4.2.3}$$

*coincides with the minimum time function $T$ given in (4.1.9) associated with (4.1.1).*


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We approximate the system (4.1.1) with the controllable system (4.2.3). The control functions are $(\alpha, \beta) \in \widetilde{\mathcal{A}} \times \mathcal{B}$, where $\mathcal{B} := \{\beta : [0, \infty) \to \overline{B}_1(0) \text{ measurable}\}$, and the trajectories are denoted by $y_x^h(., \alpha, \beta)$.

The value functions are

$$T_h(x) \quad = \quad \inf_{\widetilde{\mathcal{A}} \times \mathcal{B}} t_x^h$$

and

$$u_h(x) \quad : \quad = \inf_{\widetilde{\mathcal{A}} \times \mathcal{B}} \int_0^{t_x^h} e^{-t}\, dt = 1 - e^{-T_h(x)}, \tag{4.2.4}$$

where $t_x^h = t_x^h(\alpha, \beta)$ is the entry time in $\mathcal{T}$ of the trajectory of (4.2.3).

For the proof of Theorem 4.2.1, we will also use a representation of a value function as supremum of viscosity subsolutions as stated in the next theorem.

**Theorem 4.2.3** [16] *Under the assumptions (4.1.2), (4.1.3), (4.1.4), (4.1.5), and (4.1.6), for all $h > 0$, a function $u_h$, given in (4.2.4), is bounded continuous on $\mathbb{R}^n$ and it is continuous viscosity solution of (4.2.2). Moreover, we have the representation formula for the function $u$ given in* (4.2.1)

$$u(x) = \sup_{h>0} u_h(x) \textit{ for all } x \in \mathbb{R}^n. \tag{4.2.5}$$

**Proof of Theorem 4.2.1.** *Prove that $u$ given by (4.2.1) is an envelope viscosity solution of (4.2.2). Indeed, by Theorem 4.2.3, the function $u$ is a supremum of viscosity subsolutions of (4.2.2). It is enough to prove that $u$ is viscosity supersolution of (4.2.2). For that, we take $\varphi \in C^1(\Omega)$ and $z \in \Omega$ such that $u(z) = \varphi(z)$ and $u(x) \geq \varphi(x)$ for all $x$ in a neighborhood of $z$. We used that $u$ is lower semicontinuous, i.e., $u(x) = \underline{u}(x), x \in \Omega$*

*We assume by contradiction $\varphi(z) + H(z, D\varphi(z)) - 1 < 0$. Then for some $\varepsilon > 0$*

$$\varphi(x) + H(x, D\varphi(x)) - 1 \leq -\varepsilon \textit{ for all } x \in B_\varepsilon(z) \subseteq \Omega.$$

*By the assumption on $f$ in* (4.1.3)-(4.1.5) *and $A$ is compact, see* (4.1.2), *there is $t \in ]0, \tau[$ such that $y_x(s, \alpha) \in B_\varepsilon(z)$ for all $x \in B_{\varepsilon/2}(z)$, $0 < s \leq t$, and all $\alpha \in \mathcal{A}$. We fix such a $t$ and set*

$$\delta := \varepsilon(1 - e^{-t})/2. \tag{4.2.6}$$

*By the inequality $\geq$ in (4.2.1), for any $x$ there is $\alpha \in \mathcal{A}$ such that*

$$u(x) > -\delta + \int_0^t e^{-s}\, ds + u(y_x(t, s))e^{-t}.$$



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*Since* $u \geq \varphi$ *and*

$$\frac{d}{ds}\left(\varphi(y_x(s))e^{-s}\right) = e^{-s}\left(-\varphi(y_x) + D\varphi(y_x).y_x'\right),$$

a.e., *we get, for* $x \in B_{\varepsilon/2}(z)$,

$$\begin{aligned} u(x) - \varphi(x) &> -\delta + \int_0^t e^{-s}[1 - \varphi(y_x) + D\varphi(y_x).f(y_x, \alpha)](s)\,ds \\ &\geq -\delta - \int_0^t e^{-s}[-1 + \varphi(y_x) + H(y_x, D\varphi(y_x)](s)\,ds \\ &\geq -\delta + \int_0^t \varepsilon e^{-s}\,ds = \delta, \end{aligned}$$

*where in the last inequality, we used (4.2.6) and the choice of* $t$*. Then*

$$u(z) = \liminf_{x \to z} u(x) \geq \varphi(z) + \delta.$$

*Thus*

$$u(z) \geq \varphi(z) + \delta.$$

*This* is *a contradiction to the choice of* $\varphi$ *because* $\delta > 0$. *Thus, since* $u$ *is supremum of viscosity subsolutions, see (4.2.5) and it is a viscosity supersolution of (4.2.2), then* $u$ *is an envelope viscosity solution of (4.2.2).* ■

To illustrate the above results, we consider the following optimal control problem.

## 4.3 Zermelo Navigation Problem

One of the most classical problems in optimal control theory, namely, Zermelo navigation problem, gives an example of discontinuous viscosity solution of an Hamilton-Jacobi-Bellman equation.

**Example 4.3.1** *(Zermelo Navigation Problem) [16]*
*Consider a boat moving with velocity of constant magnitude, which we normalize to 1, relative to a stream of constant velocity* $\sigma \geq 1$*. We want to reach in minimum time a given compact target* $\mathcal{T} = \{0\}$.



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*We choose the axes in* $\mathbb{R}^2$ *so that the stream velocity is* $(\sigma, 0)$. *Then, the dynamic system is*

$$\begin{cases} y_1' = \sigma + a_1, \\ y_2' = a_2, \end{cases}$$

*with* $a_1^2 + a_2^2 = 1, a_1, a_2$ *belong to* $\mathbb{R}$.
*By an elementary geometrical argument, the reachable set* $\mathcal{R}$=$\{x : T(x) < +\infty\}$ *is*

$$\begin{aligned} \mathcal{R} &= \{x : x_1 < 0 \text{ or } x_1 = x_2 = 0\} && \text{if } \sigma = 1 \\ \mathcal{R} &= \{x : x_1 \leq 0,\ |x_2| \leq -x_1(\sigma^2 - 1)^{-1/2}\} && \text{if } \sigma > 1. \end{aligned}$$

*By standard control-theoretic methods we get*

$$T(x) = \begin{cases} -\frac{|x|^2}{2x_1} & \text{if } \sigma = 1, \\ \\ \frac{-x_1\sigma - [x_2^2(1-\sigma^2) + x_1^2]^{1/2}}{\sigma^2 - 1} & \text{if } \sigma > 1. \end{cases}$$

*An explicit form of Hamiltonian is given by*

$$H(x, Du) = \sup_{a \in A}\{-f(x, a).Du\}, \tag{4.3.1}$$

*where* $f(x, a) = (\sigma + a_1, a_2)$ *and* $a = (a_1, a_2) \in A = \{(y_1, y_2) : y_1^2 + y_2^2 = 1\}$.

*By calculation, (4.3.1) becomes*

$$H(x, Du) = |Du| - \sigma u_{x_1}.$$

*By Theorem 4.2.1, the value function*

$$u(x) = 1 - e^{-T(x)}$$

*is an envelope viscosity solution of the associated Hamilton-Jacobi-Bellman equation*

$$u(x) + |Du| - \sigma u_{x_1} - 1 = 0, x = (x_1, x_2) \in \mathbb{R}^2 \setminus \mathcal{T}.$$



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