Chapter 3

Hausdorff Continuous Viscosity Solutions of Hamilton-Jacobi Equations

3.1 Introduction

Consider again the problem (1.1.1)-(1.1.2). As shown in the introduction the concept of viscosity solution, see Definition 1.3.3, has an implicit interval character. Clearly Definition 1.3.3 treats functions which have the same upper and lower semicontinuous envelopes, that is, have the same graph completion, as identical functions. On the other hand, since different functions can have the same graph completion, a function can not in general be identified from its graph completion, that is, functions with the same graph completion are indistinguishable. Using the properties of the lower and upper semicontinuous envelopes one can easily see that the graph completion operator maps $\mathcal{A}(\Omega)$ into $\mathbb{F}(\Omega)$.

Following the above discussion we define the concept of viscosity solutions for the interval valued functions in $\mathbb{F}(\Omega)$ as follows.

Definition 3.1.1 *A function* $u = [\underline{u}, \overline{u}] \in \mathbb{F}(\Omega)$ *is called a viscosity solution of (1.1.1) if* $\overline{u}$ *is a viscosity subsolution of (1.1.1) and* $\underline{u}$ *is a viscosity supersolution of (1.1.1).*

Definition 3.1.1 shows that a local bounded function $u \in \mathcal{A}(\Omega)$ is a viscosity solution of (1.1.1) in the sense of Definition 1.3.3 if and only if the interval valued function $F(u)$ is a viscosity solution of (1.1.1) in the sense of Definition 3.1.1. In this way the level of the regularity of a solution u is manifested through the width of the interval valued function $F(u)$.

It is well known that without any additional restrictions the concept of viscosity solution given in Definition 1.3.3 and by implication the concept of viscosity solution given in Definition 3.1.1 is rather weak, [16]. This is demonstrated by the following example.

Example 3.1.1 *Consider the following equation*

$$u_x(x) = 1, x \in (0,1). \tag{3.1.1}$$

Then the functions

$$v(x) = \begin{cases} x+1 & if \quad x \in (0,1) \cap \mathbb{Q} \\ x & if \quad x \in (0,1) \setminus \mathbb{Q} \end{cases}$$

and

$$w(x) = \begin{cases} x & if \quad x \in (0,1) \cap \mathbb{Q} \\ x+1 & if \quad x \in (0,1) \setminus \mathbb{Q} \end{cases}$$

are both viscosity solutions of equation (3.1.1) in terms of Definition 1.3.3, because $S(v)(x) = S(w)(x) = x+1$ *and* $I(v)(x) = I(w)(x) = x$ *are classical solutions of equation (3.1.1).*
The interval valued function

$$z = F(v) = F(w) \tag{3.1.2}$$

given by

$$z(x) = [x, x+1], x \in (0,1)$$

is a viscosity solution of (3.1.1) in terms of Definition 3.1.1.

With the interval approach adopted here it becomes apparent that the distance between $I(u)$ and $S(u)$ is an essential measure of the regularity of any solution u, irrespective of whether it is given as a point valued function or as an interval valued function. If no restriction is placed on the distance between $I(u)$ and $S(u)$ we will have some quite meaningless solutions like the solutions in Example 3.1.1. On the other hand, a strong restriction like $I(u) = S(u)$ gives only solutions which are continuous.

3.2 Hausdorff Continuous Viscosity Solution of Hamilton-Jacobi Equations

In this section, we consider solutions u in the sense of Definition 3.1.1 for which the Hausdorff distance, as defined by (2.3.6), between the functions $I(u)$ and $S(u)$ is zero, a condition represented through the concept of Hausdorff continuity.

Definition 3.2.1 *Let $u = [\underline{u}, \overline{u}] \in \mathbb{H}(\Omega)$. Then u is called a Hausdorff continuous, or H-continuous, viscosity subsolution of the Hamilton-Jacobi equation (1.1.1) if $\overline{u}$ is a viscosity subsolution of (1.1.1).*
Similarly u is called a Hausdorff continuous, or H-continuous, viscosity supersolution of the Hamilton-Jacobi equation (1.1.1) if $\underline{u}$ is a viscosity supersolution of (1.1.1).
Finally, u is called a Hausdorff continuous or H-continuous viscosity solution of (1.1.1) if it is simultaneously an H-continuous viscosity subsolution and H-continuous viscosity supersolution of (1.1.1).

One of the advantages of the method in this thesis is that the notion of H-continuous viscosity solution is stronger than the notion of viscosity solution in the sense of Definition 3.1.1 and by implication the notion of viscosity solution in the sense of Definition 1.3.3. This is shown by the following theorem.

Theorem 3.2.1 *Let $u \in \mathcal{A}(\Omega)$ be locally bounded. If u is an H-continuous viscosity solution of (1.1.1), then u is a viscosity solution of (1.1.1) in terms of Definition 1.3.3.*

Proof. *Since u is H-continuous viscosity solution of (1.1.1), by Definition 3.2.1, $\overline{u} = S(u)$ is a viscosity subsolution and $\underline{u} = I(u)$ is a viscosity supersolution of (1.1.1). Thus u is a viscosity solution of (1.1.1) in terms of Definition 1.3.3.* ■

Remark 3.2.1 *The converse of Theorem 3.2.1 is false in general. Indeed, consider the Example 3.1.1. The function z given in (3.1.2) is a viscosity solution of (3.1.1) in terms of Definition 3.1.1 because $S(v)(x) = x + 1$ and $I(v)(x) = x$ are both classical solutions of equation (3.1.1). But z is not an H-continuous function and thus it is not an H-continuous viscosity solution of the equation (3.1.1). Hence, the requirement that a viscosity solution is Hausdorff continuous function has a direct interpretation which we find clearer than the requirements related to some other concepts of discontinuous viscosity solutions.*

3.3 The Envelope Viscosity Solutions and Hausdorff Continuous Viscosity Solutions

Recognizing that the concept of viscosity solution given in Definition 1.3.3 is rather weak, the authors of [16] introduce the concept of envelope viscosity solution. The concept is defined in [16] for the equation (1.1.1) with Dirichlet boundary conditions. In order to keep the exposition as general as possible we give the definition without explicitly involving the boundary condition.

Definition 3.3.2 *A function $u \in \mathcal{A}(\Omega)$ is called envelope viscosity solution of the Hamilton-Jacobi equation (1.1.1) if there exists a nonempty set $\mathcal{Z}_1(u)$ of viscosity subsolutions of (1.1.1) and a nonempty set $\mathcal{Z}_2(u)$ of viscosity supersolutions of (1.1.1) such that*

$$u(x) = \sup_{f \in \mathcal{Z}_1(u)} f(x) = \inf_{f \in \mathcal{Z}_2(u)} f(x), \ x \in \Omega.$$

Remark 3.3.1 *Let $u \in \mathcal{A}(\Omega)$ be an envelope viscosity solution of (1.1.1). Then u is also a viscosity solution of (1.1.1). Indeed, observe that $S(u)$ is a viscosity subsolution of (1.1.1) by Theorem 1.3.1(a) and $I(u)$ is a viscosity supersolution of (1.1.1) by Theorem 1.3.1(b). Therefore, u is a viscosity solution of (1.1.1).*

Considering the concept from geometrical point of view, one can expect that by 'squeezing' the envelope viscosity solution u between a set of viscosity subsolutions and a set of viscosity supersolutions the gap between $I(u)$ and $S(u)$ would be small. But under some strong condition on u, namely,

$$I(S(u)) = I(u), S(I(u)) = S(u)$$

the Hausdorff distance between $I(u)$ and $S(u)$ is zero. However, in general this is not the case. The following example shows that the concept of envelope viscosity solution does not address the problem of the distance between $I(u)$ and $S(u)$. Hence one can have an envelope viscosity solution of little practical meaning similar to the viscosity solution in Example 3.1.1.

Example 3.3.1 *Consider the following equation on $\Omega = (0, 1)$*

$$-u(x)(u_x(x))^2 = 0, x \in \Omega. \tag{3.3.1}$$

For every $\alpha \in \Omega$ we define the functions

$$\phi_\alpha(x) = \begin{cases} 1 & , \quad x = \alpha \\ 0 & , \quad x \in \Omega \setminus \{\alpha\} \end{cases}$$

and

$$\psi_\alpha(x) = \begin{cases} 0 & , \quad x = \alpha \\ 1 & , \quad x \in \Omega \setminus \{\alpha\}. \end{cases}$$

We have

$$\phi_\alpha \in USC(\Omega),\ \psi_\alpha \in LSC(\Omega), \alpha \in \Omega.$$

Furthermore, for every $\alpha \in (0, 1)$ the functions ϕ_α is a viscosity subsolution of (3.3.1) while ψ_α is a viscosity supersolution of (3.3.1). Indeed, both functions satisfy the equation (3.3.1) for all $x \in \Omega \setminus \{\alpha\}$ and at $x = \alpha$ we have

$$-\phi_\alpha(\alpha)p^2 = -p^2 \leq 0 \text{ for all } p \in D^+\phi_\alpha(\alpha) = (-\infty, +\infty),$$
$$-\psi_\alpha(\alpha)p^2 = 0 \geq 0 \text{ for all } p \in D^-\psi_\alpha(\alpha) = (-\infty, +\infty).$$

We will show that the function

$$u(x) = \begin{cases} 1 & , \quad x \in \Omega \setminus \mathbb{Q} \\ 0 & , \quad x \in \mathbb{Q} \cap \Omega \end{cases}$$

is an envelope viscosity solution of (3.3.1). Define

$$\begin{aligned} \mathcal{Z}_1 &= \{\phi_\alpha : \alpha \in \Omega \setminus \mathbb{Q}\} \\ \mathcal{Z}_2 &= \{\psi_\alpha : \alpha \in \Omega \cap \mathbb{Q}\}. \end{aligned}$$

Then u satisfies

$$u(x) = \sup_{w \in \mathcal{Z}_1} w(x) = \inf_{w \in \mathcal{Z}_2} w(x)$$

which implies that it is an envelope viscosity solution of (3.3.1). Clearly neither u nor $F(u)$ is a Hausdorff continuous function. In fact we have $F(u)(x) = [0, 1], x \in \Omega$. Thus, u and $F(u)$ are not H-continuous viscosity solutions of (3.3.1).

The next interesting question is whether every H-continuous viscosity solution is an envelope viscosity solution. Since the concept of envelope viscosity solutions requires the existence of sets of viscosity subsolutions and viscosity supersolutions, respectively, below and above an envelope viscosity solution then an H-continuous viscosity solution is not in general an envelope viscosity solution, e.g., when the Hausdorff continuous viscosity solutions does not have any other viscosity subsolutions and viscosity supersolutions around it. However in the essential case when the H-continuous viscosity solution is a supremum of viscosity subsolutions or infimum of viscosity supersolutions it can be linked to an envelope viscosity solution as stated in the next theorem.

Theorem 3.3.1 *Let $u = [\overline{u}, \underline{u}]$ be an H-continuous viscosity solution of (1.1.1) and let*

$$\begin{aligned}\mathcal{Z}_1 &= \{w \in USC(\Omega) : w - \textit{viscosity subsolution of (1.1.1)}, w \le \underline{u}\},\\ \mathcal{Z}_2 &= \{w \in LSC(\Omega) : w - \textit{viscosity supersolution of (1.1.1)}, w \ge \overline{u}\}.\end{aligned}$$

(a) If $\mathcal{Z}_1 \neq \emptyset$ and $\underline{u}(x) = \sup_{w \in \mathcal{Z}_1} w(x)$, then $\underline{u}$ is an envelope viscosity solution of (1.1.1).
(b) If $\mathcal{Z}_2 \neq \emptyset$ and $\overline{u}(x) = \inf_{w \in \mathcal{Z}_2} w(x)$, then $\overline{u}$ is an envelope viscosity solution of (1.1.1).

Proof. *(a) We choose the sets $\mathcal{Z}_1(\underline{u})$ and $\mathcal{Z}_2(\underline{u})$ required in Definition 3.2.1 as follows*

$$\mathcal{Z}_1(\underline{u}) = \mathcal{Z}_1, \quad \mathcal{Z}_2(\underline{u}) = \{\underline{u}\}.$$

Then we have

$$\underline{u}(x) = \sup_{w \in \mathcal{Z}_1(\underline{u})} w(x) = \inf_{w \in \mathcal{Z}_2(\underline{u})} w(x)$$

which implies that $\underline{u}$ is an envelope viscosity solution of (1.1.1).
The proof of (b) is done in a similar way. ■

Let us note that if the conditions (a) and (b) in the above theorem are satisfied the both $\underline{u}$ and $\overline{u}$ are envelope viscosity solutions and in this case it makes even more sense to consider instead the H-continuous function $u = [\underline{u}, \overline{u}]$. More precisely, if conditions (a) and (b) are satisfied, an envelope viscosity solution can be considered as a particular case of Hausdorff continuous viscosity solution.

3.4 Existence of Hausdorff Continuous Viscosity Solutions

One of the primary virtues of the theory of viscosity solutions is that it provides very general existence and uniqueness theorems, [37]. In this section we will formulate and prove existence theorems for H-continuous viscosity solutions in a similar form to Theorem 1.3.1 and Theorem 1.3.2 given in Chapter 1.

Theorem 3.4.1 (Properties of H-continuous viscosity solutions)
(a) Let $\mathcal{U} \subset \mathbb{H}(\Omega)$ be a set of H-continuous viscosity subsolutions of the Hamilton-Jacobi equation (1.1.1) which is bounded from above. Then

$$u = \sup \mathcal{U}$$

is an H-continuous viscosity subsolution of (1.1.1).
(b) Let $\mathcal{Z} \subset \mathbb{H}(\Omega)$ be a set of H-continuous viscosity supersolutions of the Hamilton-Jacobi equation (1.1.1) which is bounded from below. Then

$$v = \inf \mathcal{Z}$$

is an H-continuous viscosity supersolution of (1.1.1).
(Both the supremum and the infimum are in the sense of the partial order (2.2.3) on $\mathbb{H}(\Omega)$).

Proof. *We will prove (a). The proof of (b) can be done in a similar way. Since u is bounded from above, according to Theorem 2.4.1 (a), $u = \sup \mathcal{U} \in \mathbb{H}(\Omega)$ and by Theorem 2.4.2 (a), we have*

$$u = [I(S(\psi), S(\psi)],$$

where

$$\psi(x) := \sup\{\overline{w}(x) : w = [\underline{w}, \overline{w}] \in \mathcal{U}\},\ x \in \Omega.$$

Using that $S(\underline{w}) = \overline{w}$ is a viscosity subsolution of (1.1.1) for all $w = [\underline{w}, \overline{w}] \in \mathcal{U}$, it follows from Theorem 1.3.1 (a), that $\overline{u} = S(\psi)$ is a viscosity subsolution of (1.1.1).
By Definition 3.2.1, the function u is Hausdorff continuous viscosity subsolution of (1.1.1).
∎

Since the partially ordered set $\mathbb{H}(\Omega)$ is Dedekind order complete, it is an appropriate medium for such an application of Perron's method.

The technical lemma, sometimes called the Bump Lemma [16], showing that in some cases the supremum of viscosity subsolution or the infimum of viscosity supersolution are indeed viscosity solutions, can be formulated for Hausdorff continuous functions as follows.

Lemma 3.4.1 *Let $u = [\underline{u}, \overline{u}] \in \mathbb{H}(\Omega)$ be such that $\overline{u}$ is a viscosity subsolution of (1.1.1) and $\underline{u}$ fails to be a viscosity supersolution of (1.1.1) at some point $y \in \Omega$. Then, for any $\delta > 0$ there exists $\tau > 0$ such that, for all $r < \tau$, there exists a function $w = [\underline{w}, \overline{w}] \in \mathbb{H}(\Omega)$ with the following properties:*

$$\begin{array}{ll} (i) & \overline{w} \textit{ is a viscosity subsolution of (1.1.1)}, \\ (ii) & w \geq u \text{ in } \Omega, \\ (iii) & w \neq u, \\ (iv) & w(x) = u(x), \qquad\qquad x \in \Omega \backslash B_r(y), \\ (v) & \underline{w}(x) \leq \max\{\underline{u}(x), \underline{u}(y) + \delta\},\ x \in B_r(y). \end{array}$$

Proof. *Since $\underline{u}$ fails to be a viscosity supersolution of (1.1.1) at $y \in \Omega$, there exists $\varphi \in C^1(\Omega)$ such that*

$$h := H(y, \underline{u}(y), D\varphi(y)) < 0,\ \underline{u}(y) = \varphi(y),\ \varphi(x) \leq \underline{u}(x),\ x \in B_r(y) \textit{ and some } r > 0.$$

For $\varepsilon > 0$, consider the function $v \in C^1(\Omega)$ defined by

$$v(x) := \varphi(x) + \varepsilon - |x - y|^2.$$

We can choose r small enough to have, in addition,

$$v(x) \leq \underline{u}(y) + \frac{\delta}{2} + \varepsilon,\ x \in B_r(y). \tag{3.4.1}$$

Note that $(v - \underline{u})(x) \leq (v - \varphi)(x) = -|x - y|^2 + \varepsilon \leq 0, |x - y| \geq \sqrt{\varepsilon}$, and thus

$$v(x) \leq \underline{u}(x),\ \textit{for } |x - y| \geq \frac{r}{2} \tag{3.4.2}$$

if we choose $\varepsilon < \frac{r^2}{4}$. Moreover, if $x_n \to y$ is such that $\underline{u}(x_n) \to \underline{u}(y)$, we have $\lim_{n\to\infty}(v - \underline{u})(x_n) = \varepsilon > 0$, so, for all $r > 0$,

$$\sup_{B_r(y)} (v - \underline{u}) > 0. \tag{3.4.3}$$

Let us prove that v is a classical viscosity subsolution of equation $H(x, v, Dv) = 0$ in $B_r(y)$, i.e., $H(x, v(x), Dv(x)) \leq 0$, $x \in B_r(y)$, for sufficiently small ε, $r > 0$. For this purpose, a local uniform continuity argument shows that

$$\begin{aligned} |v(x) - v(y)| &= |\varphi(x) - |x-y|^2 - \varphi(y)| \leq \omega_1(r) + r^2, \\ |Dv(x) - Dv(y)| &= |D\varphi(x) - 2|x-y| - D\varphi(y)| \leq \omega_2(r) + 2r \end{aligned}$$

for any $x \in B_r(y)$, where $\omega_i (i = 1, 2)$, are the moduli of continuity of φ and $D\varphi$.
We recall that if $\varphi \in C(\Omega)$, then the function $\omega : [0, +\infty) \to [0, +\infty)$, defined by, $\omega(\delta) = \sup\{|\varphi(s) - \varphi(t)|, s, t \in \Omega, |s-t| \leq \delta\}$ for $\delta \geq 0$, is called a modulus of continuity of φ. Now, $H(x, v(x), Dv(x)) = h + H(x, v(x), D\varphi(x) - 2(x-y)) - H(x, v(x), D\varphi(x))$. If ω is a modulus of continuity for H, then $H(x, v(x), Dv(x)) \leq h + \omega(r, \omega_1(r) + r^2, \omega_2(r) + 2r)$, for all $x \in \overline{B}_r(y)$. Since $h < 0$, the proceeding proves that $H(x, v(x), Dv(x)) \leq 0$, $x \in B_r(y)$.

Now we define the interval valued function

$$w(x) = \begin{cases} \max(u(x), v(x)), & x \in B_r(y) \\ u(x) & , x \in \Omega \backslash B_r(y). \end{cases}$$

It is clear that $w \in \mathbb{H}(\Omega)$ since u and v are Hausdorff continuous functions in Ω and we can apply Theorem 2.4.1.

We claim that w has the desired properties. In fact, $w(x) = u(x)$ for $|x - y| \geq r/2$ by (3.4.2) and $w(x) \geq u(x)$, $x \in \Omega$. Then (iv) holds and w is an H-continuous viscosity subsolution of (1.1.1), because it coincides with u for $|x - y| > \frac{r}{2}$, while for $x \in B_r(y)$ we can apply Theorem 3.4.1. Moreover, (iii) follows from (3.4.3), and (v) follows from (3.4.1) if we choose $\varepsilon \leq \frac{\delta}{2}$. ■

Note that the proof of lemma 3.4.1 is similar to the proof of the Bump lemma in [16, Lemma V.2.12] for real function with some obvious changes due to interval character of the functions u and w.

Remark 3.4.1 *There is an analogue of Lemma 3.4.1 for the case when $I(u)$ is a viscosity supersolution and $S(u)$ fails to be a viscosity subsolution of (1.1.1).*

For a consequence of Theorem 3.4.1 and Lemma 3.4.1, we obtain the following very general existence theorem for equation (1.1.1).

Theorem 3.4.2 (Existence of H-continuous viscosity solutions by Perron's method) *Assume that there exist Hausdorff continuous functions* $u_1 = [\underline{u_1}, \overline{u_1}]$ *and* $u_2 = [\underline{u_2}, \overline{u_2}]$ *such that* u_1 *is a Hausdorff continuous viscosity subsolution of the Hamilton-Jacobi equation (1.1.1),* u_2 *is a Hausdorff continuous viscosity supersolution of (1.1.1) and* $u_1 \leq u_2$*. Then there exists a Hausdorff continuous viscosity solution* u *of (1.1.1) satisfying the inequalities*

$$u_1 \leq u \leq u_2.$$

Proof. *Consider the set*

$$\mathcal{F} = \{w = [\underline{w}, \overline{w}] \in \mathbb{H}(\Omega) : w \leq u_2,\ \overline{w} \text{ is a viscosity subsolution of } (1.1.1)\}.$$

Clearly the set $\mathcal{F}$ *is not empty since* $u_1 \in \mathcal{F}$*. Let* $u = \sup \mathcal{F}$*, where the supremum is taken in the set* $\mathbb{H}(\Omega)$*, i.e.,* $u \in \mathbb{H}(\Omega)$*. We will show that* u *is the required Hausdorff continuous viscosity solution of (1.1.1). Obviously, we have the inequalities*

$$u_1 \leq u \leq u_2. \tag{3.4.4}$$

Furthermore, according to Theorem 2.4.1 (a) and Theorem 2.4.2 (a), u *is given by*

$$u = \sup \mathcal{F} = [I(S(\psi)), S(\psi)] \in \mathbb{H}(\Omega),$$

where

$$\psi(x) := \sup\{\overline{w}(x) : w = [\underline{w}, \overline{w}] \in \mathcal{F}\},\ x \in \Omega.$$

Using that $\mathcal{F}$ *is the set of H-continuous viscosity subsolutions of (1.1.1) and* $\mathcal{F}$ *is bounded from above it follows by Theorem 3.4.1(a) that* u *is an H-continuous viscosity subsolution of (1.1.1). It remains to show that* u *is H-continuous viscosity supersolution of (1.1.1), i.e.,* $\underline{u} = I(S(\psi))$ *is a viscosity supersolution of (1.1.1). To this end, let us fix* $y \in \Omega$*. Consider first the case when*

$$\underline{u}(y) = \underline{u_2}(y).$$

Let $\varphi \in C^1(\Omega)$ *be such that* $\underline{u} - \varphi$ *has a local minimum at* y *and* $\underline{u}(y) = \varphi(y)$*. Then, for* x *in a neighborhood of* y*, we have*

$$(\underline{u_2} - \varphi)(x) \geq (\underline{u} - \varphi)(x) \geq (\underline{u} - \varphi)(y) = (\underline{u_2} - \varphi)(y).$$

Therefore, the function $\underline{u_2} - \varphi$ *also has a local minimum at* y*.*

Using that $\underline{u_2}$ is a viscosity supersolution of (1.1.1), we obtain

$$H(y, \underline{u_2}(y), D\varphi(y)) \geq 0.$$

Since $\underline{u}(y) = \underline{u_2}(y)$, the above inequality shows that the function $\underline{u}$ satisfies at the point y the conditions of supersolutions as stated in Definition 1.3.1. Consider now the case when $\underline{u}(y) \neq \underline{u_2}(y)$. In view of (3.4.4), the only other possible case is

$$\underline{u}(y) < \underline{u_2}(y).$$

In this situation, there exists $\delta > 0$ such that

$$\underline{u}(y) + \delta \leq \underline{u_2}(y) - \delta. \tag{3.4.5}$$

Assume that $\underline{u}$ fails to be a supersolution of (1.1.1) at the point y. Then, according to Lemma 3.4.1, there exists an H-continuous function $w = [\underline{w}, \overline{w}]$ with the properties (i)-(v). Moreover, since $\underline{u_2}$ is lower semicontinuous, we can choose $r > 0$ small enough such that

$$\underline{u_2}(y) - \delta \leq \underline{u_2}(x),\ x \in B_r(y). \tag{3.4.6}$$

Using (3.4.5) and (3.4.6), we obtain

$$\underline{u}(y) + \delta \leq \underline{u_2}(y) - \delta \leq \underline{u_2}(x),\ \ x \in B_r(y).$$

Hence, from property (v) of Lemma 3.4.1, for $x \in B_r(y)$, we have

$$\underline{w}(x) \leq \max\{\underline{u}(x), \underline{u}(y) + \delta\} \leq \underline{u_2}(x). \tag{3.4.7}$$

Due to property (iv) of Lemma 3.4.1, the inequality (3.4.7) can be extended to all $x \in \Omega$ and we have

$$\underline{w} \leq \underline{u_2}. \tag{3.4.8}$$

Using Theorem 2.3.3 (b) and the monotonicity of a graph completion function F, see (2.2.12), the inequality (3.4.8) can be transferred over to the Hausdorff continuous functions w and u_2 as follows

$$w = F(\underline{w}) \leq F(\underline{u_2}) = u_2.$$

Then

$$w \leq u_2.$$

This implies that $w \in \mathcal{F}$. Then $u = \sup \mathcal{F} \geq w$ which contradicts conditions (ii) and (iii) in Lemma 3.4.1. The obtain contradiction shows that $\underline{u}$ is a viscosity supersolution of (1.1.1). Therefore u is Hausdorff continuous viscosity solution of (1.1.1). ■

3.5 Uniqueness of H-Continuous Viscosity Solution

As in the traditional theory of viscosity solutions, uniqueness results can be proved under the assumption that a comparison principle is satisfied. Here we formulate the comparison principle between H-continuous viscosity subsolutions and H-continuous viscosity supersolutions of Hamilton-Jacobi equations.

Definition 3.5.1 *We say that the Dirichlet problem (1.1.1)-(1.1.2) satisfies the comparison principle if for any $u \in \mathbb{H}(\Omega)$ and $v \in \mathbb{H}(\Omega)$ which are bounded and, respectively, H-continuous viscosity subsolution and supersolution of (1.1.1) and $u \leq v$ on $\partial\Omega$, we have $u \leq v$ in Ω.*

The following theorem of uniqueness of solution shows that if H-continuous viscosity subsolution and H-continuous viscosity supersolution of (1.1.1) are equal on the boundary $\partial\Omega$ and g given in (1.1.2) is assumed to be H-continuous function on Ω, then there exists a unique an H-continuous solution of (1.1.1) satisfying (1.1.2).

Theorem 3.5.1 *Assume that there exist Hausdorff continuous viscosity subsolution φ and Hausdorff continuous viscosity supersolution ψ of (1.1.1) on Ω and assume that the definition of both functions is extended on $\partial\Omega$ in such a way that the obtained functions are H-continuous on $\overline{\Omega}$. Suppose that (1.1.1) satisfies the comparison principle and that*

$$\varphi(x) = \psi(x) = g(x), \; x \in \partial\Omega, \tag{3.5.1}$$

where the function g may assume interval values. Then, there exists a unique Hausdorff continuous viscosity solution u of (1.1.1) such that

$$u(x) = g(x), x \in \partial\Omega.$$

Proof. *We extend φ and ψ by setting $\varphi = \psi = g$ on $\partial\Omega$. We can apply Theorem 3.4.2 to get an H-continuous viscosity solution u of (1.1.1) such that $\varphi \leq u \leq \psi$ in Ω.*
By monotonicity of a graph completion F, see (2.2.12) , we have

$$F(\varphi) \leq F(u) \leq F(\psi) \; in \; \Omega. \tag{3.5.2}$$

By Corollary 2.4.1, we have that all functions $F(\varphi)$, $F(u)$, $F(\psi)$ belong to $\mathbb{H}(\overline{\Omega})$. Therefore, by property (2.3.8) in Theorem 2.3.7, the inequalities in (3.5.2) imply that $F(\varphi) \leq F(u) \leq F(\psi)$ *on $\overline{\Omega}$. By the virtue of Theorem 2.3.3, we have* $\varphi \leq u \leq \psi$ on $\overline{\Omega}$. *In particular, $\varphi \leq u \leq \psi$ on $\partial\Omega$. Since (3.5.1) holds, we have $g(x) = u(x)$, $x \in \partial\Omega$. Assume there exist two H-continuous viscosity solutions u_1, u_2 of (1.1.1).*

Since u_1 is an H-continuous viscosity subsolution and u_2 is an H-continuous viscosity supersolution of (1.1.1), and $u_1(x) = u_2(x) = g(x)$, $x \in \partial\Omega$, by comparison principle, we have

$$u_1 \leq u_2 \quad in \ \overline{\Omega}. \tag{3.5.3}$$

Since u_2 is an H-continuous viscosity subsolution and u_1 is an H-continuous supersolution, and $u_2(x) = u_1(x) = g(x)$, $x \in \partial\Omega$, by comparison principle, we have

$$u_2 \leq u_1 \ in \ \overline{\Omega}. \tag{3.5.4}$$

Combining (3.5.3) and (3.5.4), we obtain $u_1 = u_2$ in $\overline{\Omega}$. ■

The comparison principle, given by Definition 3.5.1, is stronger than the comparison principle used in connection with upper semicontinuous viscosity subsolutions and lower semicontinuous viscosity supersolutions because it gives the existence of solutions under conditions as same as for existence of discontinuous solutions.

The following theorem gives sufficient conditions for a weaker form of the comparison principle given by Definition 3.5.1.

Theorem 3.5.2 *Let Ω be a bounded open subset of $\mathbb{R}^n$, $H \in C(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ be such that $H(x, r, p) \leq H(x, s, p)$ whenever $r \leq s$ and the following two assumptions hold:*

$$\exists \gamma > 0 : \gamma(r - s) \leq H(x, r, p) - H(x, s, p), \forall r \geq s, (x, p) \in \overline{\Omega} \times \mathbb{R}^n \tag{3.5.5}$$

and there exists $\omega : [0, +\infty] \to [0, +\infty]$ such that $\omega(0+) = 0$ and

$$H(y, r, \alpha(x - y)) - H(x, r, \alpha(x - y)) \leq \omega(\alpha|x - y|^2 + |x - y|), \tag{3.5.6}$$

whenever $x, y \in \Omega, r \in \mathbb{R}, \alpha > 0$.

Let $u = [\underline{u}, \overline{u}] \in \mathbb{H}(\Omega)$ and $v = [\underline{v}, \overline{v}] \in \mathbb{H}(\Omega)$ be respectively, H-continuous viscosity subsolution and H-continuous viscosity supersolution of (1.1.1) in Ω and

$$\overline{u} \leq \underline{v} \ on \ \partial\Omega.$$

Then $\overline{u} \leq \underline{v}$ in Ω.

For the proof of Theorem 3.5.2, we need the following lemma given in [37].

Lemma 3.5.1 *Let X be a subset of $\mathbb{R}^n$, $u \in USC(X), v \in LSC(X)$ and*

$$M_\alpha = \sup_{X\times X}\Big(u(x) - v(y) - \frac{\alpha}{2}|x-y|^2\Big)$$

for $\alpha > 0$. Let $M_\alpha < \infty$ for large α and (x_α, y_α) be such that

$$\lim_{\alpha\to\infty}\Big(M_\alpha - (u(x_\alpha) - v(y_\alpha) - \frac{\alpha}{2}|x_\alpha - y_\alpha|^2)\Big) = 0.$$

Then the following hold:

$$\lim_{\alpha\to\infty}\alpha|x_\alpha - y_\alpha|^2 = 0 \tag{3.5.7}$$

and

$$\lim_{\alpha\to\infty} M_\alpha = u(z) - v(z) = \sup_{x\in X}(u(x) - v(x)) \tag{3.5.8}$$

whenever $z \in X$ is a limit point of x_α as $\alpha \to \infty$.

Proof of Theorem 3.5.2. *Define for $\alpha > 0$, an upper semicontinuous function ϕ_α on a set $\overline{\Omega}\times\overline{\Omega}$ by setting $\phi_\alpha(x,y) = \overline{u}(x) - \underline{u}(y) - \frac{\alpha}{2}|x-y|^2$ and let (x_α, y_α) be a maximum point for ϕ_α on $\overline{\Omega}\times\overline{\Omega}$ (the maximum is achieved in view of upper semicontinuity and compactness). Then $M_\alpha = \sup_{\overline{\Omega}\times\overline{\Omega}} \phi_\alpha(x,y) = \big(\overline{u}(x_\alpha) - \underline{v}(y_\alpha) - \frac{\alpha}{2}|x_\alpha - y_\alpha|^2\big)$ is finite. It follows from properties (3.5.7) and (3.5.8) of Lemma 3.5.1 and $\overline{u} \le \underline{v}$ on $\partial\Omega$ that $(x_\alpha, y_\alpha) \in \Omega\times\Omega$ for α large.*

Since we seek to prove that $\overline{u} \le \underline{v}$ in Ω, we assume to the contrary that $\overline{u}(z) > \underline{v}(z)$ for some $z \in \Omega$, it follows that

$$M_\alpha \ge \overline{u}(z) - \underline{v}(z) = \delta > 0 \text{ for } \alpha > 0. \tag{3.5.9}$$

Writing (x', y') in place of (x_α, y_α) for simplicity and set $\varphi_1(x) = \underline{v}(y') - \frac{\alpha}{2}|x-y'|^2$, $\varphi_2(y) = \overline{u}(x') - \frac{\alpha}{2}|x'-y|^2$. It is clear that $\varphi_i \in C^2(\Omega)(i = 1,2)$. Since (x', y') is a maximum point of ϕ_α, then it is clear that x' is a local maximum point for $\underline{u} - \varphi_2$, whereas, y' is a local maximum point for $\overline{u} - \varphi_1$. Moreover, $D\varphi_1(x') = \alpha(x'-y') = D\varphi_2(y')$. Then we can exploit the fact that u is an H-continuous viscosity subsolution of (1.1.1) *and we obtain*

$$H(x', \overline{u}(x'), \alpha(x'-y')) \le 0 \tag{3.5.10}$$

Similar, since u is an H-continuous viscosity supersolution of (1.1.1), we obtain

$$H(x', \underline{v}(x'), \alpha(x'-y')) \ge 0. \tag{3.5.11}$$

Combining (3.5.10) and (3.5.11) we obtain

$$H(x', \overline{u}(x'), \alpha(x'-y')) \leq 0 \leq H(x', \underline{v}(x'), \alpha(x'-y')). \tag{3.5.12}$$

The next step is to use the assumption (3.5.5) and (3.5.6) and the condition (3.5.12) to estimate M_α and contradict (3.5.9) for large α.

Using the definition of δ, see (3.5.9), and the fact that (x', y') is a maximum point for ϕ_α, we have

$$\gamma\delta \leq \gamma(\overline{u}(z) - \underline{v}(z)) \leq \gamma(\overline{u}(x') - \underline{v}(y')). \tag{3.5.13}$$

Proceeding, we deduce from (3.5.13), (3.5.5) and (3.5.6) that

$$\begin{aligned} 0 \leq \gamma\delta &\leq \gamma(\overline{u}(x') - \underline{v}(y')) \\ &\leq H(x', \overline{u}(x'), \alpha(x'-y')) - H(x', \underline{v}(x'), \alpha(x'-y')) \\ &= H(x', \overline{u}(x'), \alpha(x'-y')) - H(y', \underline{v}(x'), \alpha(x'-y')) \\ &+ H(y', \underline{v}(x'), \alpha(x'-y')) - H(x', \underline{v}(x'), \alpha(x'-y')) \\ &\leq \omega(\alpha|x'-y'|^2 + |x'-y'|). \end{aligned}$$

Here we used (3.5.12) to estimate the first term on the right by 0 *and (3.5.6) on the second term.*

Since $\omega(\alpha|x'-y'|^2 + |x'-y'|) \to 0$ as $\alpha \to \infty$ by (3.5.7), we have a contradiction with (3.5.9). ■

3.6 Extending the Hamiltonian Operator over the Set $\mathbb{H}(\Omega)$

In this section, we consider the equation (1.1.1) in the following more general form

$$H(x, u(x), Du(x)) = f(x), x \in \Omega, \tag{3.6.1}$$

where $f \in C^0(\Omega)$. We call the mapping $\mathcal{H} : C^1(\Omega) \to C^0(\Omega)$ given by

$$\mathcal{H}(u)(x) = H(x, u(x), Du(x)), x \in \Omega$$

a Hamiltonian operator. Then, equation (3.6.1) can be written as

$$\mathcal{H}(u) = f. \tag{3.6.2}$$

It is well known that the mapping $\mathcal{H}$ is in general not surjective. This means of course, that there exists $f \in C^0(\Omega)$ in (3.6.2) such that the set $\mathcal{H}^{-1}(f) = \{u \in C^1(\Omega) : \mathcal{H}(u) = f\} = \emptyset$ which implies that (3.6.1) does not have classical solution, as illustrated by a variety of well known examples, some of them rather simple ones, see [109, Chapter 6]. Clearly, the function u is a classical solution of (3.6.1) iff $u \in \mathcal{H}^{-1}(f)$. Hence, the need for generalized solutions like the viscosity solutions considered here.

Let us note that equation (3.6.1) does not really generalize equation (1.1.1). Since function f can always be moved to the left hand side reducing the equation to the form (1.1.1).

The usual way of defining generalized solutions is by extending the operator $\mathcal{H}$ to a larger domain. This extension can be done in different ways: functional analytic method [46], algebraic method [116], order completion method [109].

Here, we extend the operator $\mathcal{H}$ to the set of Hausdorff continuous functions using the viscosity approach. Our aim is to express the H-continuous viscosity solutions of (3.6.1) as solutions to an operator equation involving the extended operator in the same way as the classical solutions of (3.6.1) are solutions of (3.6.2). We use subdifferentials and superdifferentials.

For $u = [\underline{u}, \overline{u}] \in \mathbb{H}(\Omega)$, consider the sets

$$G^+(u) = \{x \in \Omega : D^+\overline{u}(x) \neq \emptyset\} \text{ and } G^-(u) = \{x \in \Omega : D^-\underline{u}(x) \neq \emptyset\}.$$

For $u \in C(\Omega)$, it was proved in [16] that the sets $G^+(u)$ and $G^-(u)$ are each dense in Ω. This result can be extended to Hausdorff continuous functions using a similar argument and is given in the following lemma.

Lemma 3.6.1 *Let $u = [\underline{u}, \overline{u}] \in \mathbb{H}(\Omega)$. The sets $G^+(u)$ and $G^-(u)$ are each dense in Ω*

Proof. *Let $y \in \Omega$ and let $\delta > 0$ be such that $\overline{B}_\delta(y) = \{z \in \mathbb{R}^n : |z-y| \leq \delta\} \subset \Omega$. Consider the smooth function $\varphi_\varepsilon(x) = \frac{1}{2\varepsilon}|x-y|^2, \varepsilon > 0$. Since $\overline{u} - \varphi_\varepsilon$ is an upper semicontinuous function on Ω, it attains its maximum over $\overline{B} = \overline{B}_\delta(y)$ at some point x_ε. Then we have*

$$(\overline{u} - \varphi_\varepsilon)(x_\varepsilon) \geq (\overline{u} - \varphi_\varepsilon)(y). \tag{3.6.3}$$

From the inequality (3.6.3), for all $\varepsilon > 0$, we get

$$|x_\varepsilon - y|^2 \leq 2\varepsilon(\overline{u}(x_\varepsilon) - \overline{u}(y)) \leq 4\varepsilon \sup_{x \in \overline{B}} |\overline{u}(x)|.$$

Thus x_ε is not on the boundary of $\overline{B}$ for ε small enough, and by Lemma 1.3.1 (i), $D\varphi_\varepsilon(x_\varepsilon) = \frac{1}{2}(x_\varepsilon - y)$ belong to $D^+(\overline{u}(x_\varepsilon))$. This proves that $G^+(u)$ is dense in Ω, and similar argument shows that $G^-(u)$ is dense in Ω too. ■

Now, for $u = [\underline{u}, \overline{u}] \in \mathbb{H}(\Omega)$, define the following two functions

$$\psi(x) = \sup_{p \in D^+\overline{u}(x)} H(x, \overline{u}(x), p), x \in G^+(u), \tag{3.6.4}$$

$$\varphi(x) = \inf_{p \in D^-\underline{u}(x)} H(x, \underline{u}(x), p), x \in G^-(u). \tag{3.6.5}$$

For the extension of the operator $\mathcal{H}$, we need the following operators:
$T^+ : \mathbb{H}(\Omega) \to USC(\Omega), T^- : \mathbb{H}(\Omega) \to LSC(\Omega)$ defined by

$$T^+u(x) = S(G^+(u), \Omega, \psi)(x), x \in \Omega, \tag{3.6.6}$$

$$T^-u(x) = I(G^-(u), \Omega, \varphi)(x), x \in \Omega. \tag{3.6.7}$$

Since $G^+(u)$ and $G^-(u)$ are dense in Ω for every $u \in \mathbb{H}(\Omega)$, the operators T^+ and T^-, defined by (3.6.6) and (3.6.7), respectively, are well defined on $\mathbb{H}(\Omega)$ and by (2.5.2) $T^+u \in USC(\Omega)$ and $T^-u \in LSC(\Omega)$. The following theorem gives a new characterization of H-continuous viscosity subsolution and H-continuous viscosity supersolution of (3.6.1) in terms of the operators T^+ and T^-.

Theorem 3.6.1 *Let $u = [\underline{u}, \overline{u}] \in \mathbb{H}(\Omega)$. Then*
(a) u is an H-continuous viscosity subsolution of (3.6.1) if and only if

$$T^+u \leq f \text{ in } \Omega; \tag{3.6.8}$$

(b) u is an H-continuous viscosity supersolution of (3.6.1) if and only if

$$T^-u \geq f \text{ in } \Omega. \tag{3.6.9}$$

Proof. *We will prove only point (a). Point (b) is proved in a similar way. By Definition 3.2.1, u is an H-continuous viscosity subsolution of (3.6.1) if and only if*

$$H(x,\overline{u}(x),p) \leq f(x), p \in D^+\overline{u}(x), x \in \Omega. \tag{3.6.10}$$

In view of definition of ψ*, see (3.6.4), the inequality (3.6.10) is equivalent to*

$$\psi(x) \leq f(x), x \in G^+(u). \tag{3.6.11}$$

Since $f \in C^0(\Omega) \subset USC(\Omega)$*, using the minimality property of upper semicontinuous envelopes, see (2.5.9), the inequality (3.6.11) implies that*

$$S(G^+(u),\Omega,\psi)(x) \leq f(x), x \in \Omega. \tag{3.6.12}$$

To complete the proof, we show that (3.6.12) implies (3.6.11). Indeed, since $S(G^+(u),\Omega,\psi)$ *is an upper bound of* ψ *on* $G^+(u)$*, we have*

$$\psi(x) \leq S(G^+(u),\Omega,\psi)(x) \leq f(x), x \in G^+(u).$$

■

As a consequence of Theorem 3.6.1, we obtain the following.

Theorem 3.6.2 *Let* $u = [\underline{u},\overline{u}] \in \mathbb{H}(\Omega)$*. Then there exists* $f \in C^0(\Omega)$ *such that u is an H-continuous viscosity solution of (3.6.1) if and only if*

$$T^+u \leq T^-u \text{ in } \Omega. \tag{3.6.13}$$

Proof. *Let there exists* $f \in C^0(\Omega)$ *such that u is an H-continuous viscosity solution of (3.6.1). Combining (3.6.8) and (3.6.9) we have the inequality (3.6.13). In order to prove the inverse implication, we use a well known Theorem of Hahn [130] which states that if a lower semicontinuous function majorates an upper semicontinuous function then there exists a continuous function between them. Since* $T^+u \in USC(\Omega), T^-u \in LSC(\Omega)$ *and* $T^+(u) \leq T^-(u)$ *in* Ω*, then there exists* $f \in C^0(\Omega)$ *such that* $T^+(u) \leq f \leq T^-(u)$ *in* Ω*. Therefore, u is an H-continuous viscosity solution of (3.6.1).* ■

There is an interesting question here. *Is the function f in Theorem 3.6.2 unique?* If f is not unique, then this means that two functions can be viscosity solutions of (3.6.1) for two different right hand terms and from practical consideration this is an undesirable situation. This issue has not been addressed in the existence theory of viscosity solutions, since the Hamilton-Jacobi equation is not considered in the operator form (3.6.2).

In view of Theorem 3.6.2, consider the operator $\hat{H} : \mathbb{H}(\Omega) \to \mathcal{P}(C^0(\Omega))$, defined by

$$\hat{H}(u) = \{f \in C^0(\Omega) : T^+u(x) \le f(x) \le T^-u(x), x \in \Omega\}, \tag{3.6.14}$$

where $\mathcal{P}(C^0(\Omega))$ is the set of all subsets of $C^0(\Omega)$. We can reformulate Theorem 3.6.2 as follows.

Theorem 3.6.3 *A function $u \in \mathbb{H}(\Omega)$ is an H-continuous viscosity solution of (3.6.1) if and only if*

$$f \in \hat{H}(u).$$

Then, the earlier question can be equivalently formulated as:
Can the set $\hat{H}(u)$ contain more than one element?

In general, this is an open problem. However, when Ω is an open interval of $\mathbb{R}$ we obtained an answer, namely that $\hat{H}(u)$ contains one element as shown by the following theorem.

Theorem 3.6.4 *Let Ω be a nonvoid open interval of $\mathbb{R}$ and $u = [\underline{u}, \overline{u}] \in \mathbb{H}(\Omega)$. Then*

$$T^-u(x) \le T^+u(x), x \in \Omega. \tag{3.6.15}$$

The following three lemmas will be instrument in the proof of Theorem 3.6.4.

Lemma 3.6.2 *Let Ω be a nonvoid open interval of $\mathbb{R}$ and $u = [\underline{u}, \overline{u}] \in \mathbb{H}(\Omega)$. If there exists a dense subset G_0 of Ω such that $G_0 \subseteq G^+(u) \bigcap G^-(u)$, then (3.6.15) holds.*

Proof. *Since $G_0 \subset G^+(u)$, by the monotonicity of the generalized upper Baire operator about inclusion with respect to the dense subset of Ω, see (2.5.7), and property (2.5.1), we have*

$$T^+u = S(G^+(u), \Omega, \psi) \ge S(G_0, \Omega, \psi) \ge I(G_0, \Omega, \psi). \tag{3.6.16}$$

In view of the fact that $\psi \ge \varphi, x \in G_0$, by monotonicity of $I(G_0, \Omega, .)$, see (2.5.3), the inequality (3.6.16) implies that

$$T^+u \ge I(G_0, \Omega, \varphi). \tag{3.6.17}$$

Since $G_0 \subset G^-(u)$, by monotonicity of generalized lower Baire operator about inclusion to respect to the dense subset of Ω, see (2.5.6), from inequality (3.6.17) we have

$$T^+u \ge I(G_0, \Omega, \varphi) \ge I(G^-(u), \Omega, \varphi) = T^-u. \tag{3.6.18}$$

The property (3.6.18) implies that (3.6.15) holds. ■

Lemma 3.6.3 *Let $u = [\underline{u}, \overline{u}] \in \mathbb{H}(\Omega), \Omega = (a,b) \subset \mathbb{R}$. If $\overline{u}$ has no local maximum at any point of Ω and $\underline{u}$ has no local minimum at any point of Ω, then u is monotone on Ω, i.e, the function $\overline{u}$ and $\underline{u}$ are increasing or decreasing on Ω.*

Proof. *Let $D = \{x \in \Omega : \underline{u}(x) = \overline{u}(x) = u(x)\}$. It was shown in [132] that u is continuous on D and that D is dense in Ω. We will show that the function u is monotone on D. Indeed, if $u(x) = u(y), x, y \in D$, then u is a constant function on D.*

Let there exists $p, q \in D, p < q$ such that $u(p) \neq u(q)$. Then either $u(p) > u(q)$ or $u(p) < u(q)$.

Suppose that $u(p) > u(q)$. Since $\overline{u}$ has no local maximum on (a,b), we have

$$\sup_{x \in [p,q] \cap D} u(x) \leq \max_{x \in [p,q] \cap D} \overline{u}(x) = \overline{u}(p) = u(p).$$

Therefore, the maximum of u on $[p,q] \cap D$ exists and

$$\max_{x \in [p,q] \cap D} u(x) = \overline{u}(x) = u(p).$$

Similarly, the maximum of u on $[p,q] \cap D$ exists and

$$\min_{x \in [p,q] \cap D} u(x) = \overline{u}(x) = u(q).$$

Next we prove that u is strictly decreasing function on $D \cap (a,b)$. For that, it suffices to show that u is strictly decreasing on $(a,p) \cap D$, on $[p,q] \cap D$, and on $(q,b) \cap D$.

Let $x, y \in (a,p) \cap D$ such that $x < y$. If $\max_{(a,p) \cap D} \overline{u}(z) = \overline{u}(p) = u(p)$, then $p \in (x,q)$ is a local maximum of $\overline{u}$ and this contradicts that $\overline{u}$ has no local maximum on (a,b). Then $\max_{(a,p) \cap D} \overline{u}(z) = u(x)$. Since $y \in (x, p \cap D)$, we have $u(x) > u(y)$ and thus u is strictly decreasing on $(a,p) \cap D$.

Now, let $x, y \in [p,q] \cap D$ such that $x < y$. Then $u(x) > u(y)$, since if $u(x) < u(y)$, then $x \in (p,y)$ is a local maximum of u and this contradicts the fact that $\underline{u}$ has a no local minimum on (a,b). Thus u is a decreasing function on $[p,q] \cap D$.

Finally, let $x, y \in (q,b) \cap D$ such that $x < y$. Then $u(q) > u(p)$, otherwise if $u(q) < u(x)$, then $q \in (p,x)$ is a local minimum of $\underline{u}$ and this contradicts that $\underline{u}$ has no local minimum on (a,b). If $u(x) < u(y)$, then x is a local minimum for $\underline{u}$ and this is a contradiction of assumption. Then $u(q) > u(x) > u(y)$ and u is a decreasing function on $(q,b) \cap D$.

In summary, u is a decreasing function on D, i.e.,

$$\forall x, y \in D,\ x < y \Rightarrow u(x) > u(y). \tag{3.6.19}$$

By monotonicity of $F(D,\Omega,.)$, *see (2.5.5), and the fact that* $u \in \mathbb{H}(\Omega)$, *(3.6.19) implies that for* $x, y \in \Omega$ *such that* $x < y$, *we have* $u(x) = F(D,\Omega,u)(x) > F(D,\Omega,u)(y) = u(y)$ *and thus u is a decreasing function on* Ω.

Similarly, if $u(p) < u(q)$, *then the function u is increasing on D and by the monotonicity of* $F(D,\Omega,.)$, *the function u is increasing on* (a,b). ■

Lemma 3.6.4 *Let* $u = [\underline{u}, \overline{u}] \in \mathbb{H}(V)$, *where V is an open subset of* $\mathbb{R}$. *If u is not monotone on any interval* $(\alpha,\beta) \subset V$, *then the sets*

$$D_1 = \{x \in V : \overline{u} \text{ has a local maximum at } x\}$$

and

$$D_2 = \{x \in V : \underline{u} \text{ has a local minimum at } x\}$$

are dense in V.

Proof. *We need to show that for any interval* $(a,b) \subset V$, *we have* $(a,b) \cap D_1 \neq \emptyset$ *and* $(a,b) \cap D_2 \neq \emptyset$.

Let $(a,b) \subset V$. *It is given that u is not monotone on* (a,b). *Therefore, by Lemma 3.6.3, there exists either local maximum of* $\underline{u}$ *at any interior point of* (a,b). *Let* $\overline{u}$ *has a local maximum at* $c \in (a,b)$. *We have* $c \in (a,b) \cap D_1$. *Therefore* $(a,b) \cap D_1 \neq \emptyset$. *The case when* $\underline{u}$ *has a local minimum on* (a,b) *is treated similarly.*

Now consider intervals (a,c) *and* (c,b). *Using the same argument in each one, either* $\overline{u}$ *has a local maximum or* $\underline{u}$ *has a local minimum. If* $\overline{u}$ *has a local minimum, then* $D_2 \cap (a,b) \neq \emptyset$. *Let* $\overline{u}$ *has a local maximum at* $d \in (c,b)$. *Without loss of generality, we can consider*

$$\overline{u}(c) \leq \overline{u}(d).$$

Since c is a local maximum of $\overline{u}$, *there exists* $\delta > 0$ *such that*

$$u(p) < \overline{u}(c), x \in (c-\delta, c+\delta) \cap D,$$

where $D = \{x \in V : \underline{u}(x) = \overline{u}(x) = u(x)\}$ *which is dense in V, see [130]. There exists* $p \in (c + \frac{\delta}{2}, c+\delta) \cap D$ *such that*

$$u(p) < \overline{u}(c), \tag{3.6.20}$$

otherwise u *is a constant on* $(c + \delta/2, c + \delta)$ *and this contradicts the fact that* u *is not monotone on any interval of* V*. Now, there exists* $q \in (c - \delta/2, c + \delta/2) \cap D$ *which implies* $\overline{u}(c) \geq u(p)$ *contradicting (3.6.20). Then we have*

$$\overline{u}(d) \geq \overline{u}(c) > u(q) > u(p). \tag{3.6.21}$$

There exists $r \in (p, b) \cap D$ *such that* $u(r) > u(p)$*, otherwise if* $u(r) \leq u(p), r \in (p, b) \cap D$*, then* $\overline{u}(x) \leq u(p), x \in (p, b) \cap D$*. In particular* $\overline{u}(d) \leq u(p)$ *which contradicts (3.6.21).*

Consider an interval $[r, q]$*. We have*

$$\min_{x \in [q,r]} \underline{u}(x) \leq u(p) < \min\{u(q), u(r)\}.$$

Therefore, there exists $s \in (q, r)$ *such that* $\underline{u}(s) = \min_{x \in [q,r]} \underline{u}(x)$*. Hence* $s \in D_2 \cap (a, b)$ *and this implies that* $D_2 \cap (a, b) \neq \emptyset$*.* ■

Proof of Theorem 3.6.3 *Let* $u = [\underline{u}, \overline{u}] \in \mathbb{H}(\Omega)$*. Consider* $W-$*the union of all intervals of* Ω*, where the function* u *is monotone. Then by the theorem for differentiability of monotone real functions [115],* $\overline{u}$ *and* $\underline{u}$ *are differentiable almost everywhere in* W *and therefore* $\underline{u}$ *is differentiable on a dense subset* W_1 *of* W *and* $\overline{u}$ *is differentiable on a dense subset* W_2 *of* W*, where* W_1, W_2 *are sets of full measure (i.e., their complements are null sets). Let* $W_0 = W_1 \bigcap W_2$*. Then by Lemma 3.6.2, inequality (3.6.15) holds in* $\overline{W}$*.*

We have two cases: either the set $\overline{W} = \Omega$ *or* $\overline{W} \neq \Omega$*. If* $\overline{W} = \Omega$*, then the proof is completed. Now suppose that* $\overline{W} \neq \Omega$*. Let* $V = \Omega \setminus \overline{W}$*. It is clear that* V *is nonempty open subset of* Ω*. Let* $D_1 = \{x \in V : \overline{u}$ *has a local maximum at* $x\}$ *and*
$D_2 = \{x \in V : \underline{u}$ *has a local minimum at* $x\}$*. Since* u *is not monotone on any subinterval of* V*, by Lemma 3.6.4,* D_1, D_2 *are each dense in* V*.*

Suppose that $y \in D_1$*. Then* $0 \in D^+\overline{u}(y)$*. Hence* $D_1 \subseteq G^+(u)$ *and we have*

$$\psi(y) = \sup_{p \in D^+\overline{u}(y)} H(y, \overline{u}(y), p) \geq H(y, \overline{u}(y), 0). \tag{3.6.22}$$

Let $D = \{x \in \Omega : \underline{u}(x) = \overline{u}(x) = u(x)\}$*. It was shown in [132] that* D *is dense in* Ω*, hence it is dense in* V *as well. Let* $\lim_{y \to x} \overline{u}(y) = \lim_{y \to x} \underline{u}(y) = u(x)$ *and inequality (3.6.22) implies that*

$$\begin{aligned} T^+u(x) &= S(D_1, V, \psi)(x) \\ &\geq S(D_1, V, H(., \overline{u}(.), 0))(x) \\ &= \lim_{y \to x} H(y, \overline{u}(y), 0) \\ &= H(x, u(x), 0). \end{aligned} \tag{3.6.23}$$

Similarly, we have $\varphi(y) \leq H(y, \overline{u}(y), 0)$ *for* $y \in D_2$ *which implies*

$$T^+u(x) \geq H(x, u(x), 0), x \in D \cap V. \qquad (3.6.24)$$

Combining (3.6.23) and (3.6.24), we get

$$T^-u(x) \leq T^+u(x), x \in D \cap V.$$

Let $x \in V$*. Then by the monotonicity of generalized lower Baire operator with respect to the first argument and with respect to the last argument and by the inequality between lower Baire and upper Baire operators, see (2.5.6) and (2.5.3), we have*

$$\begin{aligned} T^-u(x) &= I(\Omega, \Omega, T^-u)(x) \\ &\leq I(D, \Omega, T^-u)(x) \\ &\leq I(D, \Omega, T^+u)(x) \\ &\leq S(D, \Omega, T^+u)(x) \\ &= T^+u(x), x \in V. \end{aligned}$$

Hence (3.6.15) holds on V*. Since (3.6.15) holds also in* $\overline{W}$*, then it holds on* $\Omega = V \bigcup \overline{W}$*.*

■

We prove next that the operator $\hat{H}$, given in (3.6.14), is an extension of a Hamiltonian operator $\mathcal{H}$ defined by (3.6.2). Indeed, let $u \in C^1(\Omega)$. It suffices to prove that $\hat{H}(u) = \mathcal{H}(u)$, since $C^1(\Omega) \subset C(\Omega) \subset \mathbb{H}(\Omega)$. By Lemma 1.2.1 (b), we have $D^+u(x) = D^-u(x), x \in \Omega$ and this implies that $G^+(u) = G^-(u) = \Omega$ and the functions ψ and φ defined by (3.6.4) and (3.6.5), respectively, are identical. Therefore we have

$$T^+u(x) = T^-u(x), x \in \Omega. \qquad (3.6.25)$$

The property (3.6.25) implies that

$$\hat{H}(u) = \{f\}, f \in C^0(\Omega). \qquad (3.6.26)$$

In view of (3.6.2), if we identify a singleton by an element, (3.6.26) implies that

$$\hat{H}(u) = f = \mathcal{H}(u).$$

Hence $\hat{H}$ is an extension of $\mathcal{H}$.